APM1111 Statistical Theory

FA 5: Probability and Distributions

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Problem 1

An experiment consists of drawing three cards in succession from a well-shuffled ordinary deck. Let E_1 be a king on the first draw, E_2 be a king on the second draw, and E_3 be a king on the third draw.

(a) $P(\overline{E_1 E_2})$: The probability of not drawing a king on the first draw AND not drawing a king on the second draw.

(b) $P(E_1 + E_2)$: The probability of drawing a king on the first draw OR drawing a king on the second draw (or both).

(c) $E_1 + E_2$: The event of drawing a king on the first draw OR drawing a king on the second draw.

(d) $P(E_3|E_1\overline{E_2})$: The probability of drawing a king on the third draw, GIVEN THAT a king was drawn on the first draw AND a king was not drawn on the second draw.

(e) $\overline{E_1 E_2 E_3}$: The event of not drawing a king on the first draw, AND not drawing a king on the second draw, AND not drawing a king on the third draw.

(f) $P(E_1 E_2 + \overline{E_2} E_3)$: The probability of drawing a king on the first AND second draws, OR not drawing a king on the second draw AND drawing a king on the third draw.

Problem 2

Three dice are rolled and X is the sum of the three upper faces.

(a) Probability distribution of X

The total number of possible outcomes is $6 \times 6 \times 6 = 216$.

- $P(X = 3) = 1/216$
- $P(X = 4) = 3/216$
- $P(X = 5) = 6/216$
- $P(X = 6) = 10/216$

- $P(X = 7) = 15/216$
- $P(X = 8) = 21/216$
- $P(X = 9) = 25/216$
- $P(X = 10) = 27/216$
- $P(X = 11) = 27/216$
- $P(X = 12) = 25/216$
- $P(X = 13) = 21/216$
- $P(X = 14) = 15/216$
- $P(X = 15) = 10/216$
- $P(X = 16) = 6/216$
- $P(X = 17) = 3/216$
- $P(X = 18) = 1/216$

(b) Find $P(7 \leq X \leq 11)$

$$\begin{aligned} P(7 \leq X \leq 11) &= P(X = 7) + P(X = 8) + P(X = 9) + P(X = 10) + P(X = 11) \\ &= \frac{15 + 21 + 25 + 27 + 27}{216} \\ &= \frac{115}{216} \end{aligned}$$

Problem 3

Given the probability distribution $X \in \{-10, -20, 30\}$ with probabilities $p(X) \in \{1/5, 3/10, 1/2\}$.

(a) $E(X)$

$$\begin{aligned} E(X) &= \sum X \cdot p(X) \\ &= (-10)\left(\frac{1}{5}\right) + (-20)\left(\frac{3}{10}\right) + (30)\left(\frac{1}{2}\right) \\ &= -2 - 6 + 15 = 7 \end{aligned}$$

(b) $E(X^2)$

$$\begin{aligned} E(X^2) &= \sum X^2 \cdot p(X) \\ &= (-10)^2\left(\frac{1}{5}\right) + (-20)^2\left(\frac{3}{10}\right) + (30)^2\left(\frac{1}{2}\right) \\ &= 100(0.2) + 400(0.3) + 900(0.5) \\ &= 20 + 120 + 450 = 590 \end{aligned}$$

(c) $E[(X - \overline{X})^2]$

This represents the variance, $Var(X)$.

$$\begin{aligned} Var(X) &= E(X^2) - [E(X)]^2 \\ &= 590 - (7)^2 \\ &= 590 - 49 = 541 \end{aligned}$$

(d) $E(X^3)$

$$\begin{aligned} E(X^3) &= \sum X^3 \cdot p(X) \\ &= (-10)^3\left(\frac{1}{5}\right) + (-20)^3\left(\frac{3}{10}\right) + (30)^3\left(\frac{1}{2}\right) \\ &= -1000(0.2) - 8000(0.3) + 27000(0.5) \\ &= -200 - 2400 + 13500 = 10900 \end{aligned}$$

Problem 4

(a) No restriction is imposed

The number of circular permutations for 6 people is $(n-1)!$.

$$(6-1)! = 5! = 120 \text{ ways}$$

(b) Two particular women must not sit together

First, find the number of ways they do sit together by treating them as a single unit. We have 5 units to arrange in a circle: $(5-1)! = 4! = 24$. Since the two women can swap seats, multiply by 2!.

$$24 \times 2 = 48 \text{ ways they sit together}$$

Subtract this from the total unrestricted ways:

$$120 - 48 = 72 \text{ ways}$$

(c) Each woman is to be between two men

Seat the 3 men first at the round table in $(3-1)! = 2! = 2$ ways. This creates 3 empty spaces. The 3 women can be arranged into these 3 spaces in $3! = 6$ ways.

$$2 \times 6 = 12 \text{ ways}$$

Problem 5

(a) If no restrictions are imposed

Multiply the combinations of each selection:

$$\binom{6}{2} \times \binom{8}{4} \times \binom{4}{3} \times \binom{5}{3} = 15 \times 70 \times 4 \times 10 = 42,000 \text{ ways}$$

(b) If a particular man and woman must be selected

Select 1 more man from the remaining 5, and 3 more women from the remaining 7:

$$\binom{5}{1} \times \binom{7}{3} \times \binom{4}{3} \times \binom{5}{3} = 5 \times 35 \times 4 \times 10 = 7,000 \text{ ways}$$

Problem 6

Prove that for any events E_1 and E_2, $P(E_1 + E_2) \leq P(E_1) + P(E_2)$.

Proof: Using the Addition Rule of Probability, we know that:

$$P(E_1 \cup E_2) = P(E_1) + P(E_2) - P(E_1 \cap E_2)$$

In the notation of the problem, this is:

$$P(E_1 + E_2) = P(E_1) + P(E_2) - P(E_1 E_2)$$

By the axioms of probability, the probability of any event is non-negative, meaning $P(E_1 E_2) \geq 0$. Because we are subtracting a non-negative value from the sum of $P(E_1)$ and $P(E_2)$, the result must be less than or equal to the sum itself. Thus:

$$P(E_1 + E_2) \leq P(E_1) + P(E_2)$$