

# THE LEGENDRIAN KNOT COMPLEMENT PROBLEM

MARC KEGEL

ABSTRACT. We prove that every Legendrian knot in the tight contact structure of the 3-sphere is determined by the contactomorphism type of its exterior. Moreover, by giving counterexamples we show this to be not true for Legendrian links in the tight 3-sphere and for Legendrian knots in arbitrary contact manifolds. On the way a new user-friendly formula for computing the Thurston-Bennequin invariant of a Legendrian knot in a surgery diagram is given.

## 1. INTRODUCTION

The *Knot Complement Problem*, first proposed in 1908 by Heinrich Tietze [40, Section 15], asks when a knot in a given 3-manifold is determined by its complement. It had been open for eighty years until in 1989 Gordon and Luecke [24] proved it to be true for every knot in  $S^3$ . A first step in this proof was to translate this problem into a problem concerning *Dehn surgery*. In Sections 2-4 we recall the basic facts about the Knot Complement Problem and Dehn surgery.

Here I want to consider the same problem for *Legendrian knots in contact 3-manifolds*. The main result is a generalization of the result by Gordon and Luecke, roughly speaking it says the following (see Theorem 3 in Section 5 for the precise statement).

**Main Result** (Legendrian Knot Exterior Theorem).

A Legendrian knot in the tight contact structure of  $S^3$  is determined by the contactomorphism type of its exterior.

This result implies that all invariants of Legendrian knots are actually invariants of the contactomorphism type of its exterior. The Legendrian Knot Complement Problem was also mentioned in [16].

In Sections 5-7 I recall the definition of *contact Dehn surgery* and explain how to generalize the proof of Gordon-Luecke to this setting. The main ingredient for this is the fact that if there is a non-trivial contact Dehn surgery along a Legendrian knot in  $(S^3, \xi_{st})$  resulting again in  $(S^3, \xi_{st})$ , then this Legendrian knot has to be a Legendrian unknot of a very special type. An easy way to prove this fact (given in Section 9) is looking at all other contact surgeries resulting topologically again in  $S^3$  and finding a Legendrian knot in its exterior that violates the Bennequin inequality after the surgery. Therefore, I present in Section 8 a new formula to compute the Thurston-Bennequin invariant of such Legendrian knots in the surgered manifold.

Moreover, in Section 10 I explain how to do a crossing change in a Legendrian knot diagram with help of a contact Dehn surgery (a so called *contact Rolfsen twist*). With that it is easy to construct counterexamples to the Legendrian Link Exterior Problem in the tight contact structure of  $S^3$ , i.e. two non-equivalent Legendrian links with contactomorphic exteriors.

Finally, in Section 11 and 12 I give a short discussion about the Legendrian Knot Exterior Problem in other manifolds. It turns out that in the topological setting the Knot Complement Problem in a general manifold is equivalent to the non-existence of an *exotic cosmetic Dehn surgery* resulting in this manifold. In the contact setting it is not clear if this equivalence is true. But in Section 12 I present examples of Legendrian knots in lens spaces not determined by their complements and of *exotic cosmetic contact Dehn surgeries*.

## 2. THE KNOT COMPLEMENT PROBLEM

All links are assumed to be tame and considered up to (coarse) equivalence.

**Definition** (Coarse equivalence).

Let  $L_1$  and  $L_2$  be two links in an oriented closed 3-manifold  $M$ . Then  $L_1$  is **(coarse) equivalent** to  $L_2$ , if there exists a homeomorphism  $f$  of  $M$

$$\begin{aligned} f: M &\longrightarrow M \\ L_1 &\longmapsto L_2 \end{aligned}$$

that maps  $L_1$  to  $L_2$ . Then one writes  $L_1 \sim L_2$ .

**Remark** (Coarse equivalence vs. oriented coarse equivalence vs. isotopy).

The (coarse) equivalence is a weaker condition than the equivalence of knots up to isotopy. For example there is a reflection of  $S^3$  that maps the left handed trefoil to the right handed trefoil, so this two knots are (coarse) equivalent, but one can show that the left handed trefoil is not isotopic to the right handed trefoil (i.e. there is no such homeomorphism isotopic to the identity).

One also can consider the **oriented (coarse) equivalence**, that means equivalence where only orientation preserving homeomorphisms of  $M$  are allowed. In  $S^3$  oriented equivalence is equivalent to isotopy (because in  $S^3$  every orientation preserving homeomorphism is isotopic to the identity), but in general this is a weaker condition than isotopy.

A first observation is, that if two links  $L_1$  and  $L_2$  are equivalent, then the complements are homeomorphic. The following question is called the **Link Complement Problem** (or for one component links, the **Knot Complement Problem**):

**Problem** (Link Complement Problem).

Are two links in the same manifold with homeomorphic complements equivalent?

Link complements are non-compact, but often it is much easier to work with compact manifolds. Therefore, pick some regular (closed) neighborhood  $\nu L$  of a link  $L$  in  $M$  and call the complement  $M \setminus \overset{\circ}{\nu} L$  of the interior  $\overset{\circ}{\nu} L$  of these neighborhood the **exterior** of  $L$ .

The corresponding Problem whether the equivalence class of a link is determined by the homeomorphism type of its exterior is called the **Link Exterior Problem**. (Actually, this was the problem asked by Tietze in [40, Section 15].) By work of Edwards [12, Theorem 3] this two problems are equivalent.

**Example** (The Whitehead links).

The first counterexample was given in 1937 by Whitehead [42]. He considered the following two links  $L_1$  and  $L_2$  in  $S^3$ , now called **Whitehead links**, see Figure 1.

If one deletes one component out of  $L_1$  then the remaining knot is an unknot. But if one deletes the unknot  $U$  out of  $L_2$  then the remaining knot is a trefoil. So  $L_1$

cannot be equivalent to  $L_2$ . But the exteriors of this links are homeomorphic, as one can see as follows: Consider the exteriors  $S^3 \setminus \nu U$  of the unknot  $U$ . Then cut open this 3-manifold along the Seifert disk of the unknot  $U$ , make a full  $2\pi$ -twist and re-glue the two disks together again. This is called a **Rolfsen twist** and describes a homeomorphism of the link exteriors.

By twisting several times along  $U$  one can even get infinitely many non-equivalent links all with homeomorphic exteriors.

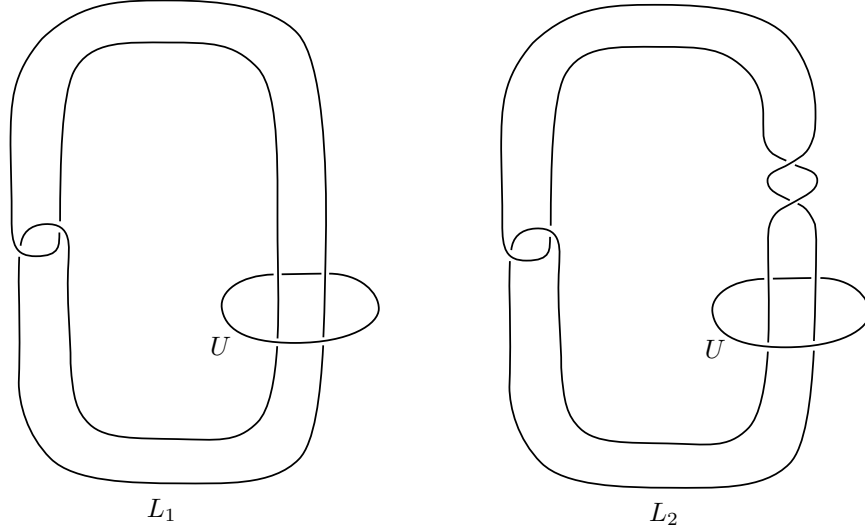


FIGURE 1. Two non-equivalent links with homeomorphic complements

The next natural question would be to ask if this holds on the level of knots. This is the so called Knot Complement Theorem by Gordon-Luecke [24].

**Theorem 1** (Knot Complement Theorem by Gordon-Luecke).

*Let  $K_1$  and  $K_2$  be two knots in  $S^3$  with homeomorphic complements, then  $K_1$  is equivalent to  $K_2$ .*

A starting point for proving this theorem, was to translate it into a problem concerning Dehn surgery.

### 3. DEHN SURGERY

In this section I recall the definition of Dehn surgery, which is a very effective construction method for 3-manifolds (for more information see [34, Chapter VI] or [37, Chapter 9]). Roughly speaking one cuts out the neighborhood of a knot and glues a solid torus back in a different way to obtain a new 3-manifold. More precisely:

**Definition** (Dehn surgery).

Let  $K$  be a knot in a closed oriented 3-manifold  $M$ . Take a non-trivial simple closed curve  $r$  on  $\partial(\nu K)$  and a homeomorphism  $\varphi$ , such that

$$\begin{aligned} \varphi: \partial(S^1 \times D^2) &\longrightarrow \partial(\nu K) \\ \{\text{pt}\} \times \partial D^2 := \mu_0 &\longmapsto r. \end{aligned}$$

Then define

$$M_K(r) := \frac{S^1 \times D^2}{\partial(S^1 \times D^2) \ni p} + \frac{M \setminus \nu \mathring{K}}{\sim \varphi(p) \in \partial(\nu K)} \bigg/ \sim,$$

One says that  $M_K(r)$  is obtained out of  $M$  by **Dehn surgery** along  $K$  with **slope**  $r$ .

One can easily show that  $M_K(r)$  is again a 3-manifold independent of the choice of  $\varphi$  (see [37, Chapter 9.F]). So to specify  $M_K(r)$  one only has to describe the knot  $K$  (in the 3-manifold  $M$ ) together with the slope  $r$ . To do this effectively one observes that there are two special kinds of curves on  $\partial \nu K$ :

- The **meridian**  $\mu$ : A simple closed curve on  $\partial(\nu K)$ , that is non-trivial on  $\partial(\nu K)$ , but trivial in  $\nu K$ .
- The **longitudes**  $\lambda$ : Simple closed curves on  $\partial(\nu K)$ , that are non-trivial on  $\partial(\nu K)$  and intersect  $\mu$  transversely exactly once.

The curves shall be oriented in such a way, that the pair  $(\mu, \lambda)$  represents the positive orientation of  $\partial(\nu K)$  in  $M$ . One can show that the meridian  $\mu$  is up to isotopy uniquely determined. But for the longitudes there are different choices. For a given longitude  $\lambda$  there are infinitely many other longitudes given by  $\tilde{\lambda} = \lambda + q\mu$ , for  $q \in \mathbb{Z}$ . Given such a longitude  $\lambda$  one can write  $r$  uniquely as

$$r = p\mu + q\lambda, \text{ for } p, q \text{ coprime.}$$

**Example** (Surgeries along the unknot).

I want to describe surgeries along the unknot  $U$  in  $S^3$ . In this case there is a special longitude on  $\partial(\nu U)$ . To see this observe that the exterior  $S^3 \setminus \nu \mathring{U}$  is again homeomorphic to a solid torus  $S^1 \times D^2$ , this corresponds to the trivial genus-1 Heegaard splitting of  $S^3$  (see for example [34, Example 8.5.]). Write

$$\begin{aligned} T_1 &:= \nu U \cong S^1 \times D^2, \\ T_2 &:= S^3 \setminus \nu \mathring{U} \cong S^1 \times D^2. \end{aligned}$$

Then choose the so called **surface longitude**  $\lambda_1$  of  $T_1$  to be the meridian  $\mu_2$  of  $T_2$  (see for example [34, Figure 8.7.]). So for surgeries along the unknot in  $S^3$  one can write the slope  $r$  uniquely as  $r = p\mu_1 + q\lambda_1$ .

Now I want to show that  $S_U^3(\mu_1 + q\lambda_1)$  is homeomorphic to  $S^3$ . Therefore, one first considers the so called **trivial Dehn surgery**  $S_U^3(\mu_1)$ , where one cuts out a neighborhood of the knot and glues it back in the same way as before. So the manifold is not changing, and in this case it is again  $S^3$ . Then the idea is to do again a Rolfsen twist along the unknot to obtain a homeomorphism from  $S^3 \cong S_U^3(\mu_1)$  to  $S_K^3(\mu_1 + q\lambda_1)$ . Therefore, consider the diagram that will be specified in the following.

$$\begin{array}{ccc} S^3 \cong S_U^3(\mu_1) & := & \frac{S^1 \times D^2}{\quad} + \frac{T_2}{\quad} \bigg/ \sim \\ \downarrow \cong & & \downarrow \text{Id} \\ S_U^3(\mu_1 + q\lambda_1) & := & \frac{S^1 \times D^2}{\quad} + \frac{T_2}{\quad} \bigg/ \sim \end{array}$$

$\begin{array}{ccc} \mu_0 \mapsto \mu_1 & \xrightarrow{\varphi_1} & \mu_1 \\ \lambda_0 \mapsto \lambda_1 & \xrightarrow{\quad} & \lambda_1 \\ & \circlearrowleft & \\ \lambda_0 \mapsto \lambda_1 & \xrightarrow{\varphi_2} & \lambda_1 \\ \mu_0 \mapsto \mu_1 + q\lambda_1 & \xrightarrow{\quad} & \mu_1 + q\lambda_1 \end{array}$

First one chooses the gluing maps  $\varphi_i$  such that they map the meridians  $\mu_0$  as determined by the slope. By this the manifolds are fixed, but the maps  $\varphi_i$  are not. There are many possibilities to what a longitude  $\lambda_0$  can map, but the homeomorphism type of the resulting manifold is not affected by this. In this example one can choose the maps  $\varphi_i$  such that they map  $\lambda_0$  to  $\lambda_1$ .

To construct a homeomorphism between the two resulting manifolds, one uses on the  $S^1 \times D^2$ -factor the identity map. If one finds a homeomorphism  $h$  of the  $T_2$ -factor such that the diagram commutes, then this two maps fit together to a homeomorphism of the whole manifolds. For the map  $h$  one can choose a  $q$ -fold Dehn-twist of the solid torus  $T_2$ , i.e.

$$\begin{array}{ccc} h: & T_2 & \longrightarrow T_2 \\ & \mu_1 = \lambda_2 & \longmapsto \mu_1 + q\lambda_1 = \lambda_2 + q\mu_2 \\ & \lambda_1 = \mu_2 & \longmapsto \lambda_1 = \mu_2. \end{array}$$

So the diagram gives rise to a homeomorphism between the two manifolds.

**Remark** (The homology of the surgered manifolds).

For all other surgeries along the unknot one computes the homology as

$$H_1(S_U^3(p\mu_1 + q\lambda_1); \mathbb{Z}) = \mathbb{Z}_p.$$

So the surgeries from the above example are the only surgeries along the unknot that lead again to  $S^3$ . In fact, these are the only non-trivial surgeries along an arbitrary knot in  $S^3$  that yield again  $S^3$ , as the following deep theorem shows.

**Theorem 2** (Surgery Theorem by Gordon-Luecke).

*Let  $K$  be a knot in  $S^3$ . If  $S_K^3(r)$  is homeomorphic to  $S^3$  for  $r \neq \mu$ , then  $K$  is equivalent to the unknot  $U$ .*

*Proof.* see [24] □

#### 4. PROOF OF THE KNOT COMPLEMENT THEOREM 1

The Knot Complement Theorem 1 now follows from the Surgery Theorem 2. The idea is to see a Dehn surgery not as cutting out a solid torus and gluing it back in a different way. Now one sees a Dehn surgery as cutting out the complement of a solid torus, then doing a homeomorphism of this complement and then glue this complement back in a different way.

To be more precise choose a homeomorphism

$$h: S^3 \setminus \nu \mathring{K}_1 \longrightarrow S^3 \setminus \nu \mathring{K}_2$$

and write

$$S^3 \cong S_{K_1}^3(\mu_1) = S^1 \times D^2 + S^3 \setminus \nu \mathring{K}_1 / \sim,$$

where the gluing map is  $\varphi_1: \mu_0 \mapsto \mu_1$ . Then consider the surgery along  $K_2$  with respect to the composition of maps

$$\begin{array}{ccccc} \partial(S^1 \times D^2) & \xrightarrow{\varphi_1} & \partial(S^3 \setminus \nu \mathring{K}_1) & \xrightarrow{h} & \partial(S^3 \setminus \nu \mathring{K}_2) \\ \mu_0 & \longmapsto & \mu_1 & \longmapsto & h(\mu_1) := r_2. \end{array}$$

To determine the homeomorphism type of this new manifold  $S_{K_2}^3(r_2)$  look at the following diagram:

$$\begin{array}{ccccc}
S^3 \cong S_{K_1}^3(\mu_1) := S^1 \times D^2 & & + & & S^3 \setminus \nu \mathring{K}_1 \quad / \sim \\
\downarrow f & \swarrow \text{Id} & \mu_0 \xrightarrow{\varphi_1} \mu_1 & \searrow h & \\
& & \circlearrowleft & & \\
& \searrow \mu_0 & \xrightarrow{h \circ \varphi_1} r_2 = h(\mu_1) & \downarrow & \\
S_{K_2}^3(r_2) := S^1 \times D^2 & & + & & S^3 \setminus \nu \mathring{K}_2 \quad / \sim
\end{array}$$

With similar arguments as in the example in Section 3 this induces a homeomorphism  $f$  from  $S^3 \cong S_{K_1}^3(\mu_1)$  to  $S_{K_2}^3(r_2)$  and with the Surgery Theorem 2 it follows that  $r_2$  is equal to  $\mu_2$  or  $K_2$  is equivalent to the unknot  $U$ .

If  $r_2 = \mu_2$ , then the surgery  $S_{K_2}^3(r_2)$  is the trivial surgery, so the spines

$$S^1 \times \{0\} \subset S^1 \times D^2 \subset S^3$$

of the new solid tori are equal to the knots  $K_i$ . Therefore  $f$  sends  $K_1$  to  $K_2$ .

In the other case ( $K_2 \sim U$ ) one does the same thing again but with  $K_1$  and  $K_2$  reversed, then it follows that  $K_1 \sim U \sim K_2$ .  $\square$

**Remark** (Oriented Knot Complement Theorem).

Exactly the same works also with orientations. The Dehn Surgery Theorem 2 also holds for oriented homeomorphism from  $S_K^3(r)$  to  $S^3$  and oriented equivalence from  $K$  to  $U$ . Then exactly the same proof as before shows that two knots with orientation preserving homeomorphic complements are orientation preserving equivalent. For  $S^3$  this is the same as isotopic knots (see also Section 2).

## 5. THE LEGENDRIAN KNOT COMPLEMENT PROBLEM

Now I want to generalize this proof to the case of **Legendrian knots** in the unique tight contact structure  $\xi_{st}$  on  $S^3$ , i.e. the tangent line to the knots lies always in the 2-plane field given by the contact structure (see [20] for all basics about contact geometry and Legendrian knots). I want to consider Legendrian links up to (coarse) equivalence like in Section 2 and I only consider cooriented contact structures.

**Definition** (Coarse equivalence).

Let  $L_1$  and  $L_2$  be two Legendrian links in a closed contact 3-manifold  $(M, \xi)$ . Then  $L_1$  is **(coarse) equivalent** to  $L_2$  if there exists a contactomorphism  $f$  of  $M$

$$\begin{array}{ccc}
f: & (M, \xi) & \longrightarrow & (M, \xi) \\
& L_1 & \longmapsto & L_2,
\end{array}$$

that maps  $L_1$  to  $L_2$ . Then one writes  $L_1 \sim L_2$ .

**Remark** (Coarse equivalence vs Legendrian isotopy).

The (coarse) equivalence is in general a weaker condition than the equivalence given by Legendrian isotopy (for example in overtwisted contact structures on  $S^3$ ). But one can show that in  $(S^3, \xi_{st})$  this two concepts are the same (see [13, Section 4.3]).

It is a standard fact that every Legendrian knot  $K$  in a general contact 3-manifold  $(M, \xi)$  has a so called **standard neighborhood**  $\nu K$  in  $(M, \xi)$  which is contactomorphic to

$$(S^1 \times D^2, \ker(\cos \theta dx - \sin \theta dy)),$$

with  $S^1$ -coordinate  $\theta$  and Cartesian coordinates  $(x, y)$  on  $D^2$ . This contactomorphism maps  $K$  to  $S^1 \times \{0\}$  (see [20, Example 2.5.10]). It is easy to show that the boundary of this standard neighborhood is a convex surface. When I write  $\nu K$  for a Legendrian knot  $K$ , I always mean that  $\nu K$  is such a standard neighborhood and analogously to Section 2 I call the complement  $(M \setminus \mathring{\nu K}, \xi)$  of the interior of such a standard neighborhood the **exterior** of the Legendrian knot  $K$ .

But for contact manifolds there is the following problem.

**Problem** (Contactomorphisms of closed and open Legendrian knot complements). If  $(M \setminus \mathring{\nu K}_1, \xi)$  is contactomorphic to  $(M \setminus \mathring{\nu K}_2, \xi)$ , is then also  $(M \setminus K_1, \xi)$  contactomorphic to  $(M \setminus K_2, \xi)$ ?

Again, two equivalent Legendrian links have contactomorphic exteriors. I want to show that for Legendrian knots in  $(S^3, \xi_{st})$  the converse also holds. But this time it is not clear if this also holds for the complements.

**Theorem 3** (Legendrian Knot Exterior Theorem).

*Let  $K_1$  and  $K_2$  be two Legendrian knots in  $(S^3, \xi_{st})$  with contactomorphic exteriors. Then  $K_1$  is equivalent to  $K_2$ .*

**Remark** (Unoriented Legendrian links).

Here  $K_1$  and  $K_2$  are understood to be unoriented Legendrian knots, because the exterior of a knot cannot see its orientation. But if one fixes an oriented longitude of the knot in its exterior, the same result holds also for oriented Legendrian knots.

For Legendrian links the Theorem is in general wrong. In Section 10 I will give some examples of Legendrian links in  $(S^3, \xi_{st})$  that are **not** determined by the contactomorphism type of their exteriors.

## 6. CONTACT DEHN SURGERY

To generalize the proof from the first sections to Legendrian knots in contact manifolds, I first want to recall the definition of contact Dehn surgery along Legendrian knots developed by Ding and Geiges in [6].

**Definition** (Contact longitude).

Let  $K$  be a Legendrian knot in a contact 3-manifold  $(M, \xi)$ . Then there is a distinguished, so called **contact longitude**  $\lambda$  on  $\partial(\nu K)$ , given by pushing  $K$  in a direction transverse to the contact planes, for example in the direction of the Reeb vector field.

When nothing else is said, from now on  $\lambda$  denotes always this contact longitude. It is easy to show that for a slope  $r = p\mu + q\lambda$  the surgered manifold  $M_K(r)$  is determined by the rational number  $p/q \in \mathbb{Q} \cup \{\infty\}$ , called the **(contact) surgery coefficient** (see for example [37, Section 9.G]). Sometimes it will be necessary to express the slope  $r$  with respect to the surface longitude  $\lambda_s$ , i.e.  $r = p'\mu + q'\lambda_s$ , then the rational number  $p'/q' \in \mathbb{Q} \cup \{\infty\}$  is called the **(topological) surgery coefficient**. (Recall that the surface longitude for a nullhomologous knot is obtained

by pushing the knot into one of its Seifert surfaces.) From now on, depending on the context, I will denote by  $r$  the slope or the corresponding surgery coefficient.

Now one wants to do Dehn surgery along Legendrian links with respect to this contact longitude. One can show that the contact structure of the old manifold extends to a contact structure of the resulting manifold.

**Definition/Theorem** (Contact Dehn surgery).

Let  $K$  be a Legendrian knot in a contact 3-manifold  $(M, \xi)$ .

(1) Then  $M_K(r)$  carries a contact structure  $\xi_K(r)$ , which coincides with the old contact structure  $\xi$  on  $M \setminus \nu K$ .

(2) For  $r \neq \pm\lambda$  one can choose  $\xi_K(r)$  to be tight on the new glued-in solid torus.

(3) For  $r = \mu + q\lambda$  this tight contact structure on this new solid torus is unique.

One says  $(M_K(r), \xi_K(r))$  is obtained from  $(M, \xi)$  by **contact Dehn surgery** along the Legendrian knot  $K$  with slope  $r$ .

*Sketch of proof.*

The neighborhood  $\nu K$  is chosen such that the boundary is a convex surface. The germ of a contact structure along a convex surface is determined by some simple data on that surface. Therefore, it is easy to glue contact structures along convex surfaces. Tight contact structures on solid tori with prescribed boundary conditions have been classified by Honda [25]. In particular, such a contact structure always exists if  $r \neq \pm\lambda$  and is unique if  $r = \mu + q\lambda$ . But in general there are more possibilities to choose this contact structure, so for other slopes the result of contact Dehn surgery is in general not unique. For details see [6] or [28].  $\square$

**Example** (A unique +2 contact Dehn surgery).

Consider the contact surgery along the Legendrian unknot with  $tb = -1$  and surgery coefficient +2 (see Figure 2). The surgery coefficient +2 corresponds then to the slope  $r = 2\mu + \lambda$ , where  $\lambda$  denotes the contact longitude. By expressing this slope with respect to the surface longitude  $\lambda_1$  given by the meridian of the complementary solid torus one gets  $r = \mu + \lambda_1$ . So the resulting manifold is by the example from Section 3 topologically again  $S^3$ .

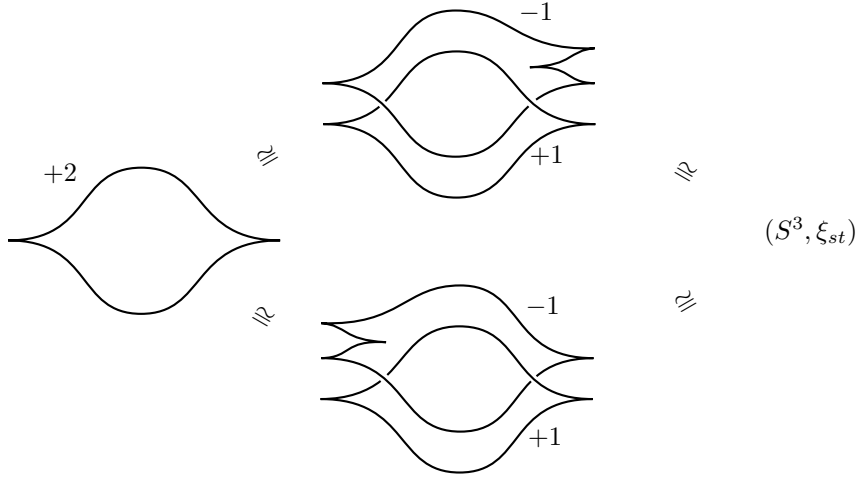


FIGURE 2. A unique +2 contact Dehn surgery resulting again in  $(S^3, \xi_{st})$



Next I want to show that the resulting contact structure is unique and leads again to  $\xi_{st}$  (if one requires the contact structure on the new glued-in solid torus to be tight). In general a contact Dehn surgery with surgery coefficient not of the form  $1/q$  is not unique. But actually in this example it is. To see this, one first uses the algorithm in [9, Section 1] (see also [7]) to change the contact surgery diagram into contact surgeries along a link with only  $\pm 1$  surgery coefficients (see Figure 2). Observe that different choices of stabilizations lead in general to different contact structures (and correspond exactly to the different contact structures on the glued-in solid torus), but in this case the resulting contact structures are contactomorphic. The contactomorphism of the resulting manifold is induced by the contactomorphism  $(x, y, z) \mapsto (-x, -y, z)$  of the old  $(S^3, \xi_{st})$ , that maps one link to the other (see also [8, Section 9]).

To see that this contact structure is really  $\xi_{st}$ , it is enough to show that the contact structure is Stein fillable. In [9, Lemma 4.2.] it is shown, that  $(+1)$ -contact Dehn surgery along the Legendrian unknot with  $tb = -1$  leads to  $S^1 \times S^2$  with the unique Stein fillable contact structure on it. Because  $(-1)$ -contact Dehn surgery preserves Stein fillability the claim follows.

To prove the Legendrian Knot Exterior Theorem 3 I generalize the Surgery Theorem by Gordon-Luecke 2.

**Theorem 4** (Contact Dehn Surgery Theorem).

*Let  $K$  be a Legendrian knot in  $(S^3, \xi_{st})$ . If  $(S_K^3(r), \xi_K(r))$  is contactomorphic to  $(S^3, \xi_{st})$  for  $r \neq \mu$  then  $K$  is equivalent to a Legendrian unknot  $U_n$  with  $tb(U_n) = -n$  and  $rot(U_n) = \pm(n-1)$ .*

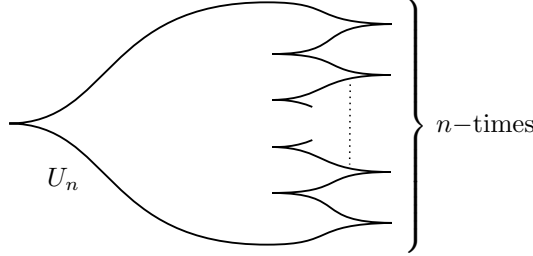


FIGURE 3. The front projection of a Legendrian unknot of type  $U_n$

**Remark** (Uniqueness of the Legendrian unknots of type  $U_n$ ).

(1) The classification of Legendrian unknots in  $(S^3, \xi_{st})$  by Eliashberg-Fraser [13] says, that two such Legendrian unknots are equivalent (with orientations of the knots) if and only if they have the same  $tb$  and  $rot$ . Of course the exterior of a knot cannot see its orientation, which for Legendrian knots is given by the sign of  $rot$ . So the knots  $U_n$  are unique up to equivalence (of unoriented Legendrian knots). A front projection of a Legendrian unknot of type  $U_n$  is shown in Figure 3.

(2) For general slopes the contact structure  $\xi_K(r)$  is not unique. So one should read Theorem 4 as follows: If there is a contact structure  $\xi_K(r)$  on  $S_K^3(r)$ , such that  $(S_K^3(r), \xi_K(r))$  is obtained from  $(M, \xi)$  by contact Dehn surgery along  $K$  with slope  $r$  and if  $(S_K^3(r), \xi_K(r))$  is contactomorphic to  $(S^3, \xi_{st})$  for  $r \neq \mu$ , then the conclusion holds.

The proof of this Contact Dehn Surgery Theorem 4 is given in Section 9.

## 7. PROOF OF THE LEGENDRIAN KNOT EXTERIOR THEOREM 3

The proof of the Legendrian Knot Exterior Theorem 3 is now similar to the topological case. Pick a contactomorphism

$$h: (S^3 \setminus \nu \mathring{K}_1, \xi_{st}) \longrightarrow (S^3 \setminus \nu \mathring{K}_2, \xi_{st}).$$

And then consider again the following diagram:

$$\begin{array}{ccc}
 (S^3, \xi_{st}) \cong (S_{K_1}^3(\mu_1), \xi_{K_1}(\mu_1)) := (S^1 \times D^2, \xi_1) & + & (S^3 \setminus \nu \mathring{K}_1, \xi_{st}) / \sim \\
 \downarrow f & \begin{array}{c} \text{Id} \quad \quad \quad \circlearrowright \quad \quad \quad h \\ \mu_0 \xrightarrow{\varphi_1} \mu_1 \\ \mu_0 \xrightarrow{h \circ \varphi_1} r_2 := h(\mu_1) \end{array} & \\
 (S_{K_2}^3(r_2), \xi_{K_2}(r_2)) := (S^1 \times D^2, \xi_1) & + & (S^3 \setminus \nu \mathring{K}_2, \xi_{st}) / \sim
 \end{array}$$

Here the contact structure  $\xi_1$  denotes the unique tight contact structure on  $S^1 \times D^2$  with convex boundary corresponding to the slope  $\mu_1$ . Because the contactomorphisms  $\text{Id}$  and  $h$  on the two factors agree on the boundary convex surfaces, which determine the germ of the contact structures, these two maps glue together to a contactomorphism  $f$  of the whole contact manifolds. From the Contact Dehn Surgery Theorem 4 it follows that  $r_2$  is equal to  $\mu_2$  or  $K_2$  is equivalent to  $U_n$  for some  $n$ .

If  $r_2 = \mu_2$  then this is the trivial contact Dehn surgery, and so the contactomorphism  $f$  maps  $K_1$  to  $K_2$ .

In the other case one makes the same argument with  $K_1$  and  $K_2$  reversed and concludes that  $K_1$  is equivalent to  $U_m$  for some  $m$ . Again the classification result of Eliashberg-Fraser [13] implies that  $K_1 \sim U_m$  is equivalent to  $U_n \sim K_2$  if and only if  $n = m$ . To show the last statement one observes that  $(S^3 \setminus \nu \mathring{K}_1, \xi_{st})$  is a solid torus with tight contact structure and convex boundary. Therefore, one can compute  $-n = \text{tb}(U_n)$  also as half the number of intersection points of the Seifert disk of  $U_n$  with the dividing set of the convex boundary (see [14, Theorem 2.30]). Because the Seifert disks of  $U_n$  and  $U_m$  are both given by the  $D^2$ -factors of the exterior solid tori and because the exteriors are contactomorphic, the number of intersection points stays the same.  $\square$

## 8. COMPUTING THE THURSTON-BENNEQUIN INVARIANT OF A LEGENDRIAN KNOT IN A SURGERED CONTACT MANIFOLD

To prove the Contact Dehn Surgery Theorem 4 one first determines with the Surgery Theorem by Gordon-Luecke 2 all surgeries that lead again to  $S^3$ . The main problem is then that there are always many different choices for extending the contact structure over the new glued-in solid torus. But a very simple proof can be given by finding some new Legendrian knots in the exteriors of the surgery knots, that violates the Bennequin inequality in the new contact manifold. For doing this I want to present in this section a formula for computing the Thurston-Bennequin invariant of a Legendrian knot in a surgered manifold.

The main problem when doing this, is that the Thurston-Bennequin invariant is only defined for nullhomologous (or rationally nullhomologous) knots and a surgery in general destroys this property. But as long as the resulting manifold is a homology sphere (or a rationally homology sphere) every knot has to be nullhomologous (or rationally nullhomologous). In this case formulas for computing the new Thurston-Bennequin invariants out of the old ones and out of the algebraic surgery dates are given in [29, Lemma 6.6], [21, Lemma 2] and [5, Lemma 6.4].

Here I first want to give an easy (and easy to check) condition (out of the algebraic surgery data) when such a knot is nullhomologous in the new manifold. If this is the case I secondly show how to compute out of this dates the new Thurston-Bennequin invariant.

### 8.1. Computing the homology class of a knot.

Let  $L = L_1 \sqcup \cdots \sqcup L_n \subset S^3$  be an oriented link (where the choice of orientation is not important). And let  $M$  be the 3-manifold obtained out of  $S^3$  by Dehn surgery along  $L$  with topological surgery coefficients  $r_i = p_i/q_i$ , for  $i = 1, \dots, n$ . Denote by  $L_0 \subset S^3 \setminus \nu L$  an oriented knot in  $S^3$  and  $M$  depending on the context. For simplicity write the linking numbers as  $l_{ij} := \text{lk}(L_i, L_j)$ , for  $i = 0, \dots, n$ . Set also

$$Q := \begin{pmatrix} p_1 & q_2 l_{12} & \cdots & q_n l_{1n} \\ q_1 l_{21} & p_2 & & \\ \vdots & & \ddots & \\ q_1 l_{n1} & & & p_n \end{pmatrix} \quad \text{and} \quad \mathbf{l} := \begin{pmatrix} l_{01} \\ \vdots \\ l_{0n} \end{pmatrix}.$$

The matrix  $Q$  is a generalization of the linking matrix, because for  $q_i = 1$  the matrix  $Q$  is the linking matrix.

The knot  $L_0$  is called **nullhomologous** in  $M$  if  $[L_0] = 0 \in H_1(M; \mathbb{Z})$ . One can show (see for example [22, Page 123]) that this is equivalent to the existence of a Seifert surface for the knot, so the surface longitude for a knot is defined if and only if the knot is nullhomologous.

With the following lemma one can decide from the algebraic surgery data if such a knot is nullhomologous in the surgered manifold.

**Lemma** (Nullhomologous knots).

$L_0$  is nullhomologous in  $M$  if and only if there exists an  $\mathbf{a} \in \mathbb{Z}^n$  such that  $\mathbf{l} = Q\mathbf{a}$ .

*Proof.*

It is easy to compute the homology of  $M$  (see for example [22, Proposition 5.3.11]) as

$$H_1(M; \mathbb{Z}) = \mathbb{Z}_{\mu_1} \oplus \cdots \oplus \mathbb{Z}_{\mu_n} / \langle p_i \mu_i + q_i \sum_{\substack{j=1 \\ j \neq i}}^n l_{ij} \mu_j = 0 \mid i = 1, \dots, n \rangle,$$

where the generators of the  $\mathbb{Z}$ -factors are given by right-handed meridians  $\mu_i$  corresponding to the components  $L_i$ . Next I express  $L_0$  as a linear combination of the  $\mu_i$ . One can show that the coefficients are the linking numbers  $l_{i0}$ , i.e.

$$[L_0] = \sum_{i=1}^n l_{i0} \mu_i.$$

So  $L_0$  is nullhomologous if and only if one can express  $[L_0] = \sum l_{i0}\mu_i$  as a linear combination of the relations, i.e. if there exists integers  $a_i$ ,  $i = 1, \dots, n$ , such that

$$\sum_{i=1}^n l_{i0}\mu_i = \sum_{i=1}^n a_i(p_i\mu_i + q_i \sum_{\substack{j=1 \\ j \neq i}}^n l_{ij}\mu_j) = \sum_{i=1}^n (a_i p_i + \sum_{\substack{j=1 \\ j \neq i}}^n q_j l_{ij} a_j) \mu_i.$$

By comparing the coefficients and by writing the corresponding equations in vector form one sees that this is true if and only if there exists a vector  $\mathbf{a} \in \mathbb{Z}^n$  such that  $\mathbf{l} = Q\mathbf{a}$ .  $\square$

### 8.2. Computing the Thurston-Bennequin Invariant of a Legendrian knot.

Now assume  $L_0$  is a Legendrian link in  $(S^3 \setminus \nu\tilde{L}, \xi_{st}) \subset (S^3, \xi_{st})$ . And let  $\xi$  be a contact structure on  $M$  that coincides with  $\xi_{st}$  outside a tubular neighborhood of  $L$ . For example if  $L$  is also a Legendrian link in  $(S^3, \xi_{st})$  and  $(M, \xi)$  is the result of a contact Dehn surgery along  $L$ . But it is important to notice that the setting here is a more general one, one can also use contact surgery along transversal knots or surgery along a knot that is not adapted to the contact structure.

Here all surgery coefficients are understood to be topological surgery coefficients, i.e. with respect to the surface longitude  $\lambda_s$  which has linking number zero with the knot. (So if one has a Legendrian surgery diagram one first has to change the contact surgery coefficients to topological surgery coefficients, for example with the formula  $r_{i,top} = r_{i,cont} + \text{tb}(L_i)$ .)

**Lemma** (Computing the Thurston-Bennequin invariant).

If  $L_0$  is nullhomologous in  $M$ , then one can compute the new Thurston-Bennequin number  $\text{tb}_{new}$  of  $L_0$  in  $(M, \xi)$  from the old one  $\text{tb}_{old}$  of  $L_0$  in  $(S^3, \xi_{st})$  as

$$\text{tb}_{new} = \text{tb}_{old} - \left\langle \mathbf{a}, \begin{pmatrix} q_1 l_{10} \\ \vdots \\ q_n l_{n0} \end{pmatrix} \right\rangle,$$

where  $\mathbf{a}$  is a vector given by the formula from the previous lemma.

*Proof.*

Let  $\lambda_s$  be the surface longitude of  $L_0$  in  $S^3$ , i.e.  $\text{lk}(L_0, \lambda_s) = 0$ . And let  $\lambda_c$  be the contact longitude of  $L_0$  in  $(S^3, \xi_{st})$ . Then  $\text{tb}_{old}$  is given by

$$\lambda_c = \text{tb}_{old} \mu_0 + \lambda_s \in H_1(\partial\nu L_0).$$

Because the contact longitude is defined by the contact structure along  $L_0$  and the contact structure does not change near  $L_0$  by doing the surgery along  $L$  the contact longitude  $\lambda_c$  represents also the contact longitude in  $(M, \xi)$ . But in general the surface longitude  $\lambda_s$  changes. Since the knot  $L_0$  is nullhomologous in  $M$ , there is a unique  $f \in \mathbb{Z}$  such that  $f\mu_0 + \lambda_s = 0 \in H_1(M \setminus \nu\tilde{L}_0; \mathbb{Z})$  (this is the new surface longitude). Then  $\text{tb}_{new}$  is given by

$$\lambda_c = \text{tb}_{new} \mu_0 + (f\mu_0 + \lambda_s) \in H_1(\partial\nu L_0).$$

Putting this together leads to

$$\text{tb}_{new} = \text{tb}_{old} - f.$$

So the only thing left is to compute  $f$ . As in the proof of the previous lemma one computes the homology of  $M \setminus \nu \mathring{L}_0$  as

$$H_1(M \setminus \nu \mathring{L}_0; \mathbb{Z}) = \mathbb{Z}_{\mu_0} \oplus \cdots \oplus \mathbb{Z}_{\mu_n} / \langle p_i \mu_i + q_i \sum_{\substack{j=0 \\ j \neq i}}^n l_{ij} \mu_j = 0 \mid i = 1, \dots, n \rangle$$

and expresses  $\lambda_s$  as

$$\lambda_s = \sum_{i=1}^n l_{i0} \mu_i.$$

So  $f\mu_0 + \lambda_s$  is zero in  $H_1(M \setminus \nu \mathring{L}_0; \mathbb{Z})$  if and only if there exists integers  $b_i \in \mathbb{Z}$ ,  $i = 1, \dots, n$ , such that

$$\begin{aligned} f\mu_0 + \sum_{i=1}^n l_{i0} \mu_i &= \sum_{i=1}^n b_i (p_i \mu_i + q_i \sum_{\substack{j=0 \\ j \neq i}}^n l_{ij} \mu_j) \\ &= \left( \sum_{j=1}^n b_j q_j l_{j0} \right) \mu_0 + \sum_{i=1}^n \left( b_i p_i + \sum_{\substack{j=1 \\ j \neq i}}^n q_j l_{ij} b_j \right) \mu_i. \end{aligned}$$

By comparing the coefficients this is equivalent to the existence of a vector  $\mathbf{b} \in \mathbb{Z}^n$  such that

$$\mathbf{l} = Q\mathbf{b} \text{ and}$$

$$f = \sum_{j=1}^n b_j q_j l_{j0}.$$

If one chooses for  $\mathbf{b}$  a solution  $\mathbf{a}$  from the first lemma the formula follows.  $\square$

**Example** (Computing tb-numbers with this formulas).

(1) Consider the surgery diagram from Figure 4 (i). The old Thurston-Bennequin invariant of  $L_0$  is  $-1$ . And because the surgery along the unknot  $L_1$  with topological framing  $1/n$  leads again to  $S^3$  the knot  $L_0$  is again nullhomologous in the new manifold. This can be checked also with the formula from above. For  $\mathbf{a}$  one gets the equation  $1 = l_{10} = Qa_1 = p_1 a_1 = a_1$ . And therefore  $tb_{new} = -1 - l_{10} q_1 a_1 = -1 - n$ . Observe again that this result does not depend on the explicit contact structure chosen for the surgery.

(2) Consider the surgery diagram from Figure 4 (ii). Again the surgery leads to  $S^3$  so the resulting knot is again nullhomologous. The new Thurston-Bennequin invariant can be computed as follows. For  $\mathbf{a}$  one gets the equation  $2 = l_{10} = Qa_1 = p_1 a_1 = a_1$ . And therefore  $tb_{new} = -1 - l_{10} q_1 a_1 = -1 - 4n$ .

In the next section we will see that the Contact Surgery Theorem is an easy corollary out of these two examples.

(3) There are also examples where the solution  $\mathbf{a}$  of  $\mathbf{l} = M\mathbf{a}$  is not unique. But the result of  $tb_{new}$ , of course, is not affected by this. For example consider the surgery diagram from Figure 5 (i). First one has to check if the knot  $L_0$  is again nullhomologous in the surgered manifold. For this, one has to look if there exists a

solution  $\mathbf{a} \in \mathbb{Z}$  of

$$\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \mathbf{l} = Q\mathbf{a} = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}.$$

Two obvious solutions are

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ or } \mathbf{a} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

So the new knot is nullhomologous and the new Thurston-Bennequin invariant can be computed out of the old one as follows

$$tb_{new} = tb_{old} - \langle \mathbf{a}, \begin{pmatrix} q_1 l_{10} \\ q_2 l_{20} \end{pmatrix} \rangle = tb_{old} - \langle \mathbf{a}, \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rangle = tb_{old} - 2.$$

Observe that this result does not depend on the choice of  $\mathbf{a}$ .

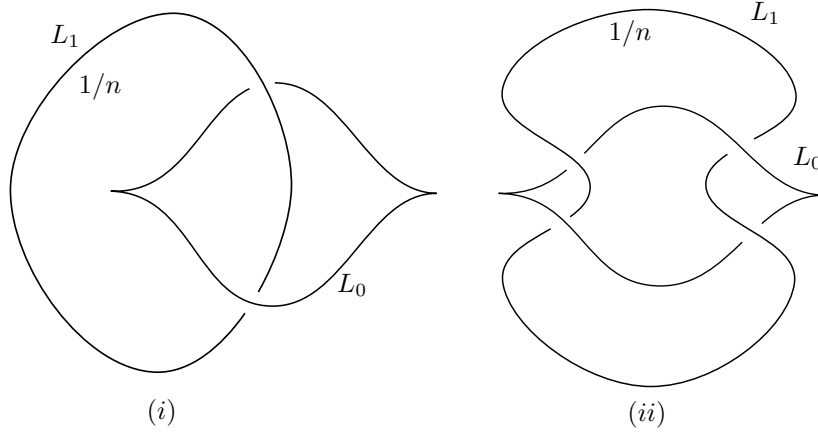


FIGURE 4. Computing tb-numbers in surgery diagrams

### 8.3. Rationally Nullhomologous Knots.

This results can be easily generalized to rationally nullhomologous knots. A knot  $L_0$  in  $M$  is called **rationally nullhomologous** if there exists a natural number  $k \in \mathbb{N}$  such that  $k[L_0] = 0 \in H_1(M; \mathbb{Z})$ . For rationally nullhomologous Legendrian knots in contact 3-manifolds one can generalize the Thurston-Bennequin Invariant to the so called **rational Thurston-Bennequin Invariant**  $tb_{\mathbb{Q}}$  (see for Example [2, Definition 6.2] or [21, Section 2 and 3], for the fact that this is in general well defined see [11, Section 5]).

With the same notation as from the first sections one gets the following generalizations of these results.

**Lemma** (Computing the rationally Thurston-Bennequin invariant).

- (1)  $L_0$  is rationally nullhomologous in  $M$  if and only if there exists a natural number  $k \in \mathbb{N}$  and a vector  $\mathbf{a} \in \mathbb{Z}^n$  such that  $k\mathbf{l} = Q\mathbf{a}$ .
- (2) If  $L_0$  is rationally nullhomologous in  $M$  then one can compute the new rationally

Thurston-Bennequin number  $\text{tb}_{\mathbb{Q},new}$  of  $L_0$  in  $(M, \xi)$  from the old one  $\text{tb}_{old}$  of  $L_0$  in  $(S^3, \xi_{st})$  as follows

$$\text{tb}_{\mathbb{Q},new} = \text{tb}_{old} - \frac{1}{k} \langle \mathbf{a}, \begin{pmatrix} q_1 l_{10} \\ \vdots \\ q_n l_{n0} \end{pmatrix} \rangle.$$

*Proof.*

The proofs are similar to the ones in the foregoing subsections. For the first part one has to change the condition  $[L_0] = 0$  to  $k[L_0] = 0$  and do the same computations again.

The second part works similar. If  $L_0$  is only rationally nullhomologous in  $M$ , then the Thurston-Bennequin numbers are related as follows (see also [21, Proof of Lemma 2]).

$$k\lambda_s + k \text{tb}_{alt} \mu_0 = k\lambda_c = k \text{tb}_{\mathbb{Q},new} \mu_0 + (f\mu_0 + k\lambda_s).$$

Then same computations as in the first part lead to the result.  $\square$

**Example** (Computing tb-numbers of rationally Legendrian unknots in lens spaces). This formula for computing the rational Thurston-Bennequin invariant is very useful to calculate  $\text{tb}_{\mathbb{Q}}$  in lens spaces. For example, consider the surgery diagram from Figure 5 (ii). The  $(-p/q)$ -surgery along  $L_1$  leads to the lens space  $L(p, q)$ . To check if the knot  $L_0$  is nullhomologous one has to solve the equation  $1 = \mathbf{l} = Q\mathbf{a} = -pa_1$ . For  $p \neq 1$  this equation has no solution in  $\mathbb{Z}$  and therefore  $L_0$  is not nullhomologous in the surgered manifold. But (for  $p \neq 0$ ) the equation  $k = k\mathbf{l} = Q\mathbf{a} = -pa_1$  has a solution, for example  $k = p$  and  $a_1 = -1$ . So  $L_0$  is rationally nullhomologous in  $L(p, q)$ . The rationally Thurston-Bennequin invariant is computed as follows

$$\text{tb}_{\mathbb{Q},new} = \text{tb}_{old} - \frac{1}{k} a_1 l_{01} q_1 = \text{tb}_{old} + \frac{q}{p}.$$

Observe again that the result is independent of the chosen solution  $\mathbf{a}$  and independent of the chosen contact structure on the new glued-in solid torus.

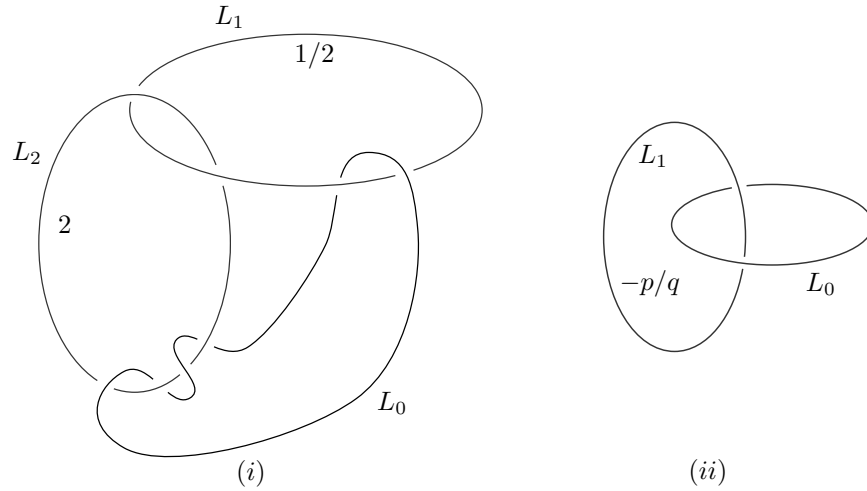


FIGURE 5. Computing rationally tb-numbers in surgery diagrams

#### 8.4. Extension to surgeries on general manifolds.

One can also study the same problem for a surgery in a general contact manifold (not on  $(S^3, \xi_{st})$ ). This is motivated by [5, Lemma 6.4].

Consider now  $L = L_1 \sqcup \cdots \sqcup L_n \subset N^3$  an oriented nullhomologous link in some contact 3-manifold  $(N, \xi_N)$ . Denote by  $(M, \xi)$  some result of contact surgery along  $L$  and by  $L_0 \subset (N \setminus \nu \mathring{L}, \xi_N)$  an oriented nullhomologous Legendrian knot in  $(N, \xi_N)$  and  $(M, \xi)$ , depending on the context. Then one gets exactly the same formulas as before. The only part changing in the proof is that the homologies are different, for example

$$H_1(M; \mathbb{Z}) = H_1(N; \mathbb{Z}) \oplus \mathbb{Z}_{\mu_1} \oplus \cdots \oplus \mathbb{Z}_{\mu_n} / \langle p_i \mu_i + q_i \sum_{j \neq i} l_{ij} \mu_j = 0 \mid i = 1, \dots, n \rangle.$$

For more details see [5, Proof of Lemma 6.4].

If one has a surgery diagram with also 1-handles included, then one can use the above methods as well. The first possibility is to change all 1-handles into 0-surgeries along unknots and the second possibility is to think of the surgery diagram as a surgery diagram in  $(\#_n S^1 \times S^2, \xi_{st})$  (represented by  $n$  1-handles) and then use the above extension.

In further work [10] there will also appear a formula like this for the rotation number, that generalizes the formulas from [29, Lemma 6.6], [21, Lemma 2] and [5, Lemma 6.4], together with a similar formula for the self-linking number of transversal knots.

### 9. PROOF OF THE CONTACT DEHN SURGERY THEOREM 4

Let  $K$  be a Legendrian knot in  $(S^3, \xi_{st})$  such that  $(S_K^3(r), \xi_K(r))$  is again a contact  $S^3$  (with any contact structure) for  $r \neq \mu$ . From the (topological) Surgery Theorem 2 it follows, that  $K$  is topologically equivalent to an unknot  $U$ . (So I will write  $U$  instead of  $K$ .) In the example from Section 3 and the following remark I explained that the topological surgery coefficient (with respect to the surface longitude  $\lambda_1$ ) has to be of the form  $1/n$ , for  $n \in \mathbb{Z}$ .

Now I want to show that every resulting contact structure  $\xi_U(r)$  is overtwisted if  $U$  is not coarse equivalent to an unknot of the form  $U_n$ . To do this I will show that in the resulting contact 3-spheres there exists Legendrian knots that cannot be realized in  $\xi_{st}$ .

For this, one considers the first two examples from Subsection 8.2. By doing a Rolfsen twists along  $U$  one sees that the Legendrian knot  $L_0$  from the first example remains an unknot in the new surgered manifold. For  $n < 0$  one gets  $\text{tb}_{new} = -n - 1 > 0$  (independent of the choice of the contact structure on the new glued-in solid torus). According to the Bennequin inequality this knot cannot lie in a tight contact structure. So for  $n < 0$  it is not possible to have a non-trivial contact surgery from  $(S^3, \xi_{st})$  to itself.

For  $n > 0$  one looks at the second example from Subsection 8.2. In Figure 6 it is shown (again by doing a Rolfsen twist) that  $L_0$  becomes in the new surgered manifold a negative  $(2, 2n + 1)$ -torus knot  $T_{2, 2n+1}$ . In [15, Section 1] it is shown that the maximal  $\text{tb}$ -number of such a knot in  $(S^3, \xi_{st})$  is given by  $-2 - 4n$ , which is smaller than  $\text{tb}_{new} = -1 - 4n$ . For  $U$  not coarse equivalent to an Legendrian unknot of the form  $U_n$  this example can be realized as a contact surgery along a Legendrian knot, which proves the result.  $\square$



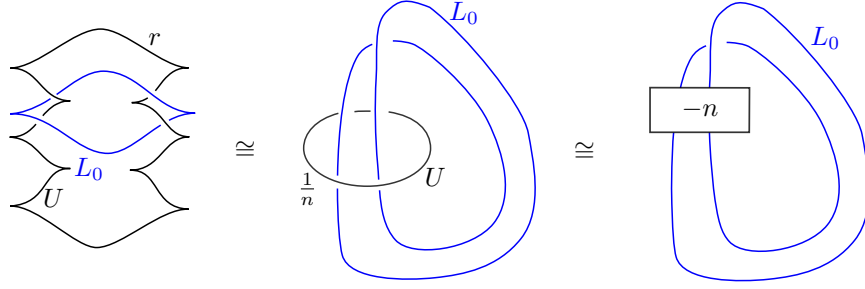


FIGURE 6. An unknot  $L_0$  that becomes a negative  $(2, 2n+1)$ -torus knot after surgery

#### 10. A CONTACT ROLFSEN TWIST AND COUNTEREXAMPLES TO THE LEGENDRIAN LINK EXTERIOR PROBLEM

The next natural question is if there exists (like in the topological case) Legendrian links not determined by the contactomorphism type of their exteriors. If one wants to generalize the examples of Whitehead one needs a contact analogon of a Rolfsen twist. In the Example from Section 6 I gave a contact surgery from  $(S^3, \xi_{st})$  to  $(S^3, \xi_{st})$ . Topologically this surgery represents a  $(+1)$ -Dehn surgery along an unknot. Deleting such components from contact surgery diagrams leads to a contact analogon of a  $(-1)$ -Rolfsen twist. But it is not clear how the rest of the diagram changes then.

**Lemma** (A contact Rolfsen twist).

Figure 7 shows two moves for contact Dehn surgery diagrams. There,  $L$  is an arbitrary Legendrian link with surgery coefficients  $\pm 1$ , that can go arbitrarily often through the unknot (even one knot several times is allowed).

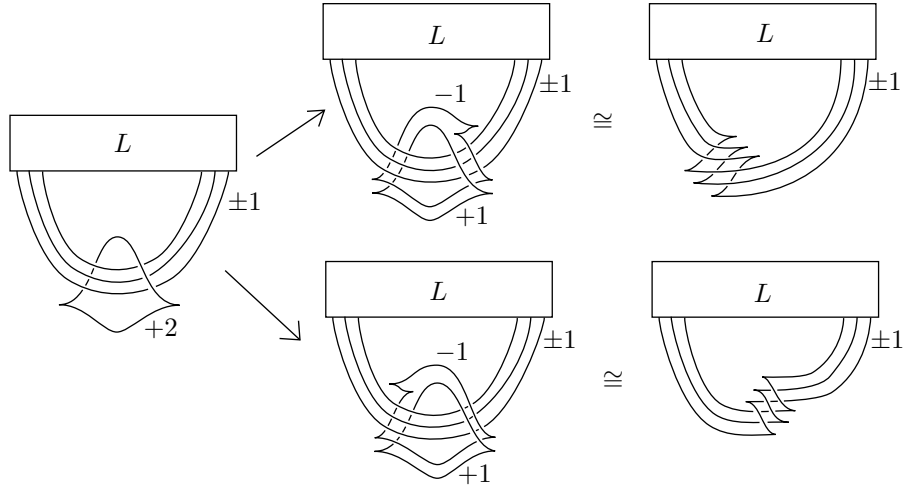


FIGURE 7. A contact Rolfsen twist

**Remark** (Doing  $n$ -fold contact Rolfsen twists).

Of course one can do such a contact Rolfsen twist more than once, this corresponds to doing  $(+2)$ -contact Dehn surgeries along  $n$  Legendrian push-offs of the unknot. It is easy to see that this is the same as doing a single contact Dehn surgery along the unknot with contact surgery coefficient  $1 + \frac{1}{n}$ .

*Proof of the Lemma.*

First of all one observes that the first row in Figure 7 maps to the second row under the contactomorphism  $(x, y, z) \mapsto (-x, -y, z)$ . So it is enough to prove the second row.

For this, one observes that one can put the whole Legendrian link  $L$  together with the Legendrian unknot  $U$  on the pages of an abstract open book for  $(S^3, \xi_{st})$  (see for example [1, Proof of Theorem 5.5]). A part of this abstract open book is shown in Figure 8. The page is pictured in gray and a part of the monodromy is described as a right handed Dehn twist along the blue curve. One has to read the red curve in the open book, as necessarily many parallel copies. Observe that the abstract open book determines the contact manifold only up to contactomorphism, as the surgery diagram does.

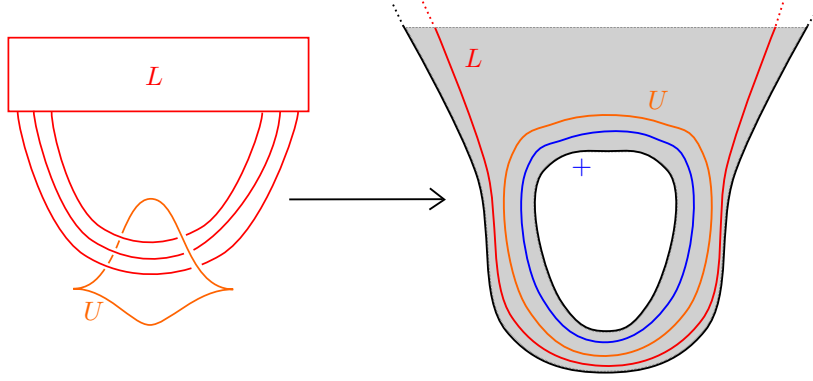


FIGURE 8. Putting  $L$  and  $U$  on an open book for  $(S^3, \xi_{st})$

Next one constructs from this an abstract open book for the surgered manifold as explained in [33]. First, one puts the Legendrian link  $L$  together with the Legendrian unknots  $U_1$  and  $U_2$  from the  $(\pm 1)$ -surgery diagram on the page of an abstract open book for  $(S^3, \xi_{st})$ . This is shown in Figure 9 on the upper right side. The additional stabilization of the Legendrian unknot  $U_2$  corresponds to a stabilization of the open book as shown in [33, Figure 1 and 2].

The monodromy of the open book for  $(S^3, \xi_{st})$  is given by right handed Dehn twists along the blue curves. The monodromy of the surgered manifold is obtained by composing this old monodromy with right handed Dehn twists along the  $(-1)$ -surgery knot and left handed Dehn twists along the  $(+1)$ -surgery knot (see [33, Proposition 8]). Two Dehn twists cancel each other and one gets the open book in the bottom right corner of Figure 9. That open book represents the surgery diagram of the once stabilized Legendrian link  $L$  shown in the bottom left corner of Figure 9.

Observe also that the topological surgery coefficients of  $L$  change exactly as prescribed by the topological Rolfsen twists.  $\square$

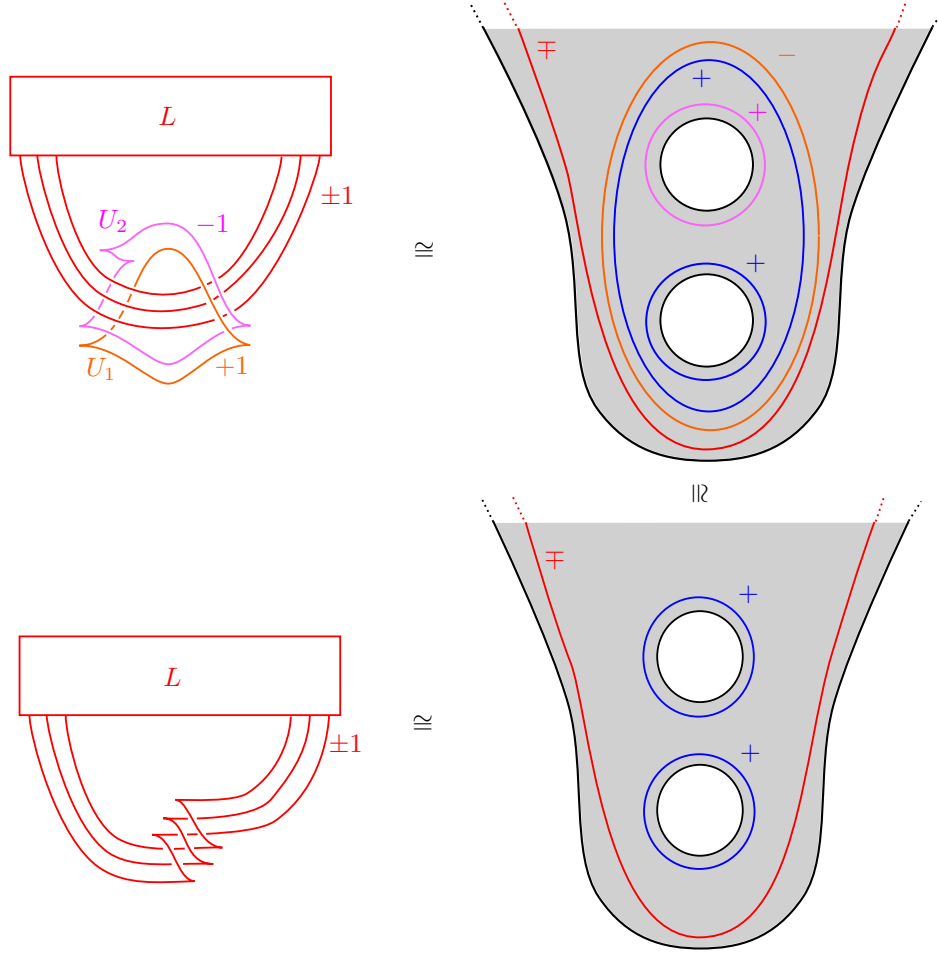


FIGURE 9. Proof of the contact Rolfsen twists via open book decompositions

**Example** (Counterexamples to the Legendrian Link Complement Problem).

With this contact Rolfsen twist one can give counterexamples to the Legendrian Link Complement Problem. But it is not as easy as in the topological setting. The main point there was that the new glued-in solid tori is again a tubular neighborhood of the spine of this torus. This is not the case in the contact setting.

To see this, consider a Legendrian knot  $K$  in a contact 3-manifold  $(M, \xi)$  and the result of contact surgery along this knot  $(M_K(r), \xi_K(r))$ . Then the slope of the new glued-in solid torus is  $r$ . But if this new glued-in solid torus would be the standard neighborhood of a Legendrian knot then the slope would be of the form  $\lambda + n\mu$ . So for general  $r$  this is not the case.

Therefore, one looks at the contact surgery diagram with one  $(+1)$ -surgery along  $U_1$  and one  $(-1)$ -surgery along  $U_2$ . The new glued-in solid tori are again in a canonical way standard neighborhoods of the Legendrian spines  $U'_1$  and  $U'_2$ . From the proof of the Lemma it follows that this spines  $U'_1$  and  $U'_2$  lie in the resulting surgery diagram as shown in Figure 10.

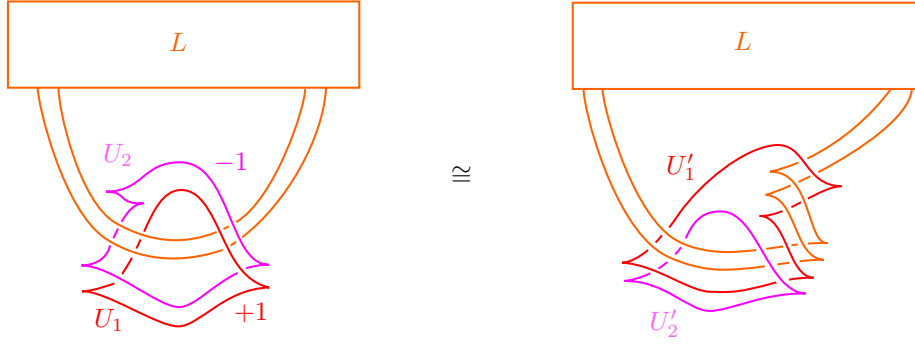


FIGURE 10. The spines of the new glued in solid tori

(1) Consider the two Legendrian links  $L \sqcup U_1 \sqcup U_2$  and  $L' \sqcup U_1' \sqcup U_2'$  in  $(S^3, \xi_{st})$  as depicted in Figure 11. These two links are not equivalent because their triples of tb-numbers are different:  $\text{tb}(L \sqcup U_1 \sqcup U_2) = (-1, -1, -2)$ ;  $\text{tb}(L' \sqcup U_1' \sqcup U_2') = (-2, -2, -1)$ .

But their exteriors are contactomorphic, as one can see as follows. One does a  $(+1)$ -surgery along  $U_1$  and a  $(-1)$ -surgery along  $U_2$ . The resulting manifold is again  $(S^3, \xi_{st})$ , in which the Legendrian knot  $L$  looks now like  $L'$  in Figure 11 on the right. So in the exteriors of  $U_1 \sqcup U_2$  and  $U_1' \sqcup U_2'$  the Legendrian knots  $L$  and  $L'$  are the same.

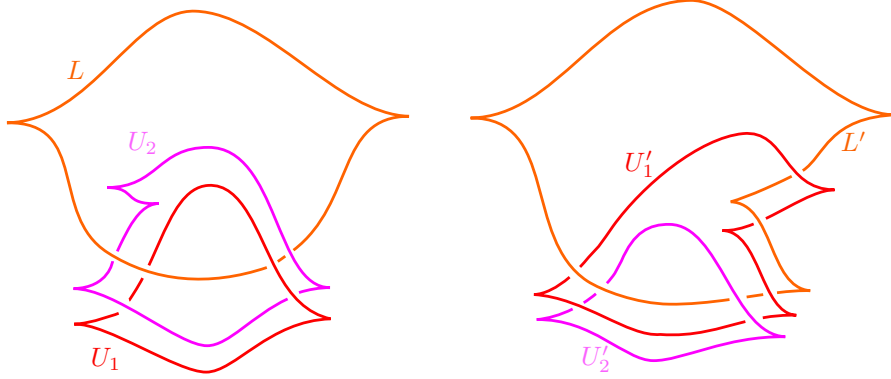


FIGURE 11. Two different Legendrian Links with contactomorphic exteriors

(2) One can also get examples with different topological types and the same tb-numbers. For that consider the Legendrian links in Figure 12 similar to the Whitehead links as in the example from Section 2. In the left Legendrian link in Figure 12 all three knots are unknots, but in the right link the knot  $L'$  is non-trivial, so they cannot be equivalent. But their exteriors are contactomorphic with the same argument as in the foregoing example.

(3) By doing contact Dehn surgeries corresponding to an  $n$ -fold Rolfsen twist, for  $n > 0$ , one gets in both foregoing examples infinitely many different Legendrian links with contactomorphic exteriors.

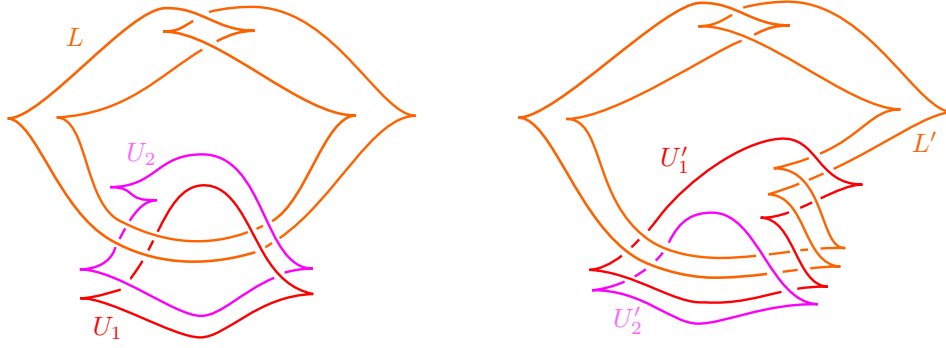


FIGURE 12. Two different Legendrian Links with contactomorphic exteriors

## 11. THE KNOT COMPLEMENT PROBLEM IN GENERAL MANIFOLDS

Instead of looking at Legendrian knots in  $(S^3, \xi_{st})$  one can also look at Legendrian knots in general contact 3-manifolds and study the Legendrian Knot Exterior Problem for these knots. Before studying the contact case in the next section, I first recall some basic facts about this problem in the topological setting.

Of course, one can transfer this problem in an arbitrary manifold to a problem concerning Dehn surgery like before in the proof of Theorem 1.

**Lemma** (Criterion for the Knot Complement Problem).

Let  $K$  be a knot in a 3-manifold  $M$ , such that there is no non-trivial Dehn surgery along  $K$  resulting again in  $M$ . Then the equivalence type of  $K$  is determined by the diffeomorphism type of its complement.

But to prove the hypothesis of this lemma turns out to be very difficult and in general, both, the conclusion and the hypothesis of the lemma are not true. To see this, consider the following counterexample (see also [38]).

**Example** (Two non-equivalent knots with the same complements).

Consider two different Dehn surgeries along the unknot  $U$  in  $S^3$  with the (topological) surgery coefficients  $r_1 = -5/2$  and  $r_2 = -5/3$ , leading to the lens spaces  $L(5, 2)$  and  $L(5, 3)$ . By the classification of lens spaces [22, Exercise 5.3.8. (b)] this two lens spaces are orientation preserving homeomorphic and the homeomorphism is given by interchanging the two solid tori (see also [21, Section 2]).

From this Dehn surgery example it is easy to find two non-equivalent knots with the same exteriors. For this write

$$L(5, 2) \cong S_U^3(r_i) := S^1 \times D^2 + S^3 \setminus \nu U \Big/_{\sim} \mu_0 \longmapsto r_i$$

and consider the knots

$$K_i := S^1 \times \{0\} \subset S^1 \times D^2 \subset L(5, 2).$$

The knots  $K_1$  and  $K_2$  given as the spines of the new glued-in solid tori represents the spines of the genus-1 Heegaard splitting of  $L(5, 2)$ . As tubular neighborhood of  $K_i$  one chooses the whole new glued-in solid tori  $S^1 \times D^2$ , therefore the exterior is

in both cases  $S^3 \setminus \nu \mathring{U}$ . It remains to show that these two knots are not equivalent. Therefore, assume that there is an orientation preserving homeomorphism

$$\begin{array}{ccc} f: & L(5, 2) & \longrightarrow & L(5, 2) \\ & K_1 & \longmapsto & K_2. \end{array}$$

By restricting  $f$  to the complementary solid tori

$$L(5, 2) \setminus \nu \mathring{K}_i = S^3 \setminus \nu \mathring{U} = T_2$$

one gets a homeomorphism

$$\begin{array}{ccc} T_2 & \longrightarrow & T_2 \\ r_1 & \longmapsto & r_2, \end{array}$$

which sends the slope  $r_1 = -5\lambda_2 + 2\mu_2$  to the slope  $r_2 = -5\lambda_2 + 3\mu_2$ . But such a map cannot exist because all orientation preserving homeomorphisms of solid tori are isotopic to Dehn twists along meridians. So  $K_1$  is not orientation preserving equivalent to  $K_2$  in  $L(5, 2)$ .

With the same methods as above it is easy to show (see [38]) that if  $K_1$  and  $K_2$  are the cores of the two solid tori in the standard Heegaard splitting of  $L(p, q)$ , then they have homeomorphic complements, but there is an orientation preserving (reversing) homeomorphism of  $L(p, q)$  sending  $K_1$  to  $K_2$  if and only if  $q^2 \equiv 1 \pmod{p}$  ( $q^2 \equiv -1 \pmod{p}$ ).

The key point in the foregoing examples is that there is a so called **exotic cosmetic surgery**, that means two surgeries along the same knot resulting in the same manifold but with different slopes, such that there is no homeomorphism of the knot exterior mapping one slope to the other (see [4]). One can show that every knot in a given 3-manifold is determined by its complement if and only if this manifold cannot be obtained by exotic cosmetic surgery from another manifold.

**Theorem 5** (Exotic cosmetic surgeries).

*The following two claims are equivalent.*

- (1) *Let  $K_1$  and  $K_2$  be knots in a closed 3-manifold  $M$  with homeomorphic complements. Then  $K_1$  is equivalent to  $K_2$ .*
- (2) *Let  $K'$  be a knot in a closed 3-manifold  $M'$ , such that  $M'_{K'}(r_1)$  and  $M'_{K'}(r_2)$  are both homeomorphic to  $M$  (for  $r_1 \neq r_2$ ). Then there exists a homeomorphism of the knot exterior*

$$\begin{array}{ccc} h: & M' \setminus \nu \mathring{K}' & \longrightarrow & M' \setminus \nu \mathring{K}' \\ & r_1 & \longmapsto & r_2, \end{array}$$

*mapping one slope to the other.*

**Remark** (Oriented exotic cosmetic surgeries).

- (1) The statement holds in the oriented and unoriented case.
- (2) One can take the manifold  $M'$  to be homeomorphic to  $M$ , this explains the name cosmetic surgery.

*Proof of Theorem 5.*

The proof of (1)  $\Rightarrow$  (2) works exactly as in the foregoing example. The implication (2)  $\Rightarrow$  (1) is similar to the proof of Theorem 1.  $\square$

So the study of the Knot Complement Problem is equivalent to the study of exotic cosmetic surgeries. In the topological setting some is known, but much remains open. Beside the discussed Knot Complement Theorem for knots in  $S^3$ , Gabai

showed in [17] that knots in  $S^1 \times S^2$  (or more generally in a connected sum of arbitrary  $T^2$ - or  $S^2$ -bundles over  $S^1$ ) are determined by their complements.

But in general manifolds this will not hold. Building up on work by Mathieu [30], Rong classified in [38] all knots in 3-manifolds with Seifert fibered complements, that are not determined by their complements. These knots are given by the spines of the solid tori in the standard Heegaard splitting of some special lens spaces  $L(p, q)$  as described at the beginning of this section or as exceptional fibers of index 2 in special Seifert fibered manifolds. But all these homeomorphisms sending one of the exceptional fibers to another one have to be orientation reversing.

Later Matignon [31] proved that all non-hyperbolic knots in atoroidal irreducible Seifert fibered 3-manifolds are determined by their complements (except the cores of the standard Heegaard splittings in Lens spaces).

For hyperbolic knots there is until now only one counterexample. In [4] Bleiler, Hodgson and Weeks construct two non-equivalent hyperbolic knots in  $L(49, 18)$  with orientation reversing homeomorphic complements. They also give very good reasons for the conjecture that all knots in hyperbolic 3-manifolds are determined by their oriented complements. Recently Ichihara and Jong [27] found examples of knots in hyperbolic manifolds with orientation reversing homeomorphic complements.

All together this leads to the still open Oriented Knot Complement Conjecture in general manifolds:

**Conjecture** (Oriented Knot Complement Conjecture).

If  $K_1$  and  $K_2$  are knots in a closed oriented 3-manifold  $M$  with orientation preserving homeomorphic complements (not homeomorphic to  $S^1 \times D^2$ ), then the knots are orientation preserving equivalent.

With Theorem 5 and the following remark this is equivalent to the Cosmetic Surgery Conjecture formulated in [4]:

**Conjecture** (Oriented Cosmetic Surgery Conjecture).

Exotic cosmetic surgeries (not resulting in a Lens space) are never orientation preserving (or **truly**) cosmetic.

Many of the mentioned results rely on the classification of all Dehn surgeries in a solid torus resulting again in a solid torus by Berge [3] and Gabai [18]. But in the last years also Heegaard Floer homology turned out to be a very useful tool to study such questions. For Example, Wang [41] showed that there are no exotic Dehn surgeries along Seifert genus-1 knots in  $S^3$ . The same holds for non-trivial algebraic knots in  $S^3$  by [35]. Finally, in [19] and [36] it is shown that knots in  $L$ -space homology spheres are determined by their complements. In particular, knots in the Poincaré sphere are determined up to orientation preserving equivalence by the oriented homeomorphism types of their complements.

## 12. THE LEGENDRIAN KNOT COMPLEMENT PROBLEM IN GENERAL MANIFOLDS

As far as I know, nothing is known about this in the contact setting. The question is which results from the topological setting generalize to the contact setting and where are the differences. In the foregoing section I presented examples of non-equivalent knots in Lens spaces with homeomorphic exteriors. The first interesting question is if one can generalize these examples to the contact setting.

**Example** (Non-equivalent Legendrian knots with the same complements).

Consider a Legendrian knot  $K$  with  $\text{tb}(K) = n$ . A standard neighborhood of  $K$  is given by

$$(S^1 \times D^2, \ker(\cos n\theta \, dx - \sin n\theta \, dy)).$$

The surface longitude is given by  $\lambda = S^1 \times \{p\}$  and the contact longitude by  $\lambda + n\mu$ . Take two copies  $(V_1, \xi_1)$  and  $(V_2, \xi_2)$  of this standard neighborhood and glue them together along their boundaries to obtain the lens space  $L(p, q)$  as follows:

$$\begin{array}{rcccl} L(p, q) & = & V_1 & + & V_2 & / \sim \\ & & \mu_1 \longmapsto & \longrightarrow & -q\mu_2 + p\lambda_2, & \\ & & \lambda_1 \longmapsto & \longrightarrow & r\mu_2 + s\lambda_2, & \end{array}$$

where  $-qs - pr = 1$ . It is a standard fact that the contact structures on the solid tori fit together to a contact structure in the new lens space  $L(p, q)$  if the gluing map sends the contact longitude of  $V_1$  to the contact longitude of  $V_2$ . This leads to the conditions  $r - qn = n$  and  $s + pn = 1$ . Putting this together it follows that  $q = -pn - 1$ .

In particular, it follows that if a contact lens space is obtained by gluing together two standard neighborhoods of Legendrian knots, then the lens space is of the form  $L(p, p - 1)$ . But in this case the spines  $K_1$  and  $K_2$  of the Heegaard tori are topologically equivalent, as was explained in the foregoing section.

To distinguish the Legendrian knots  $K_1$  and  $K_2$  one can compute their rational tb-numbers (see [2, Definition 6.2]). Therefore consider the following gluing:

$$\begin{array}{rcccl} L(p, p - 1) & = & V_1 & + & V_2 & / \sim \\ & & \mu_1 \longmapsto & \longrightarrow & (1 + pn)\mu_2 + p\lambda_2, & \\ & & \lambda_1 \longmapsto & \longrightarrow & -pn^2\mu_2 + (1 - pn)\lambda_2. & \end{array}$$

This gluing map sends the contact longitude of  $V_1$  to the contact longitude of  $V_2$  and leads therefore to a contact structure on the lens space. The Legendrian knots  $K_1$  and  $K_2$  represent the spines of the Heegaard tori in this lens space. Therefore they both have order  $p$  in  $H_1(L(p, p - 1); \mathbb{Z})$ . The extended Seifert longitude  $r_2$  of  $K_2$  is given by

$$r_2 = (1 + pn)\mu_2 + p\lambda_2.$$

(It is nullhomologous in the exterior of  $K_2$  and  $\mu_2 \bullet r_2 = p$ .) So one computes

$$\text{tb}_{\mathbb{Q}}(K_2) = \frac{1}{p}(\lambda_2 + n\mu_2) \bullet ((1 + pn)\mu_2 + p\lambda_2) = -\frac{1}{p}.$$

To compute  $\text{tb}_{\mathbb{Q}}(K_1)$  one first computes the inverse gluing map as:

$$\begin{array}{rcccl} L(p, p - 1) & = & V_2 & + & V_1 & / \sim \\ & & \mu_2 \longmapsto & \longrightarrow & (1 - pn)\mu_1 - p\lambda_1, & \\ & & \lambda_2 \longmapsto & \longrightarrow & pn^2\mu_1 + (1 + pn)\lambda_1. & \end{array}$$

Then one sees that the extended Seifert longitude of  $K_1$  is given by

$$r_1 = (pn - 1)\mu_1 + p\lambda_1$$



and so

$$\text{tb}_{\mathbb{Q}}(K_1) = \frac{1}{p}(\lambda_1 + n\mu_1) \bullet ((pn - 1)\mu_1 + p\lambda_1) = \frac{1}{p}.$$

So the rational Thurston-Bennequin invariants are different, therefore  $K_1$  and  $K_2$  are not Legendrian isotopic and then also not coarse equivalent.

So in contrast to the topological case one gets obvious counterexamples to the Legendrian knot exterior conjectures exactly in the lens spaces  $L(p, p - 1)$ . It would be interesting if one can generalize the examples from this section to a general classification result in lens spaces.

**Problem** (The Legendrian Knot Exterior Problem in lens spaces).

Can one classify all Legendrian knots in contact lens spaces (or general Seifert fibered spaces) that are not determined by their exteriors?

A first step to study the Legendrian knot exterior problem in general contact manifolds would be a generalization of Theorem 5. The generalization of the implication  $(2) \Rightarrow (1)$  one can prove similarly to the proof of Theorem 3.

**Lemma** (Criterion for the Legendrian Knot Exterior Problem).

Let  $K$  be a Legendrian knot in a contact manifold  $(M, \xi)$ , such that there is no non-trivial contact Dehn surgery along  $K$  resulting again in  $(M, \xi)$ . Then the equivalence type of  $K$  is determined by the contactomorphism type of its exterior.

But the other implication does not generalize. The problem is again that the new glued-in solid torus is in general not a standard neighborhood of a Legendrian knot as explained in Section 10.

So it is not directly clear if exotic cosmetic contact surgeries leads to counterexamples to the Legendrian Knot Exterior problem. But their existence is interesting in its own. The following example was obtained earlier by Geiges and Onaran (see also [21]).

**Example** (Exotic contact surgeries).

Consider the two different contact Dehn surgeries along the Legendrian unknot  $U$  with  $\text{tb} = -1$  in  $(S^3, \xi_{st})$  with respect to the contact surgery coefficients  $r_1 = -3/2$  and  $r_2 = -2/3$  (see Figure 13). By expressing the surgery coefficients with respect to the surface longitude  $\lambda_1$  given by the meridian of the complementary solid torus one sees that the resulting manifolds are topologically the manifolds from the example from the foregoing section, so the homeomorphic lens spaces  $L(5, 2)$  and  $L(5, 3)$ .

The next thing I want to show is that the two contact surgeries are unique and represent the same contact manifold. Therefore one uses the same approach as in the example in Section 6. The same argument as in that example shows that the two surgery diagrams both represent a unique contact structure  $\xi$  on  $L(5, 2)$ . The from the algorithm [9, Section 1] resulting  $(\pm 1)$ -contact surgery diagrams are shown in Figure 13 in the middle (for both contact surgeries is only one of the two possible stabilizations drawn). Both surgery diagrams contain only  $(-1)$ -contact surgeries, so the resulting contact structures have to be tight. But the classification of tight contact structures on lens spaces [25, Theorem 2.1] says that on  $L(5, 2)$  there are two non-contactomorphic tight contact structures, so in this case this is not enough to conclude that these surgery diagrams represent the same contact manifold.

To show that the contact structures of the two different surgeries are really contactomorphic one changes the contact Dehn surgery diagrams into compatible open

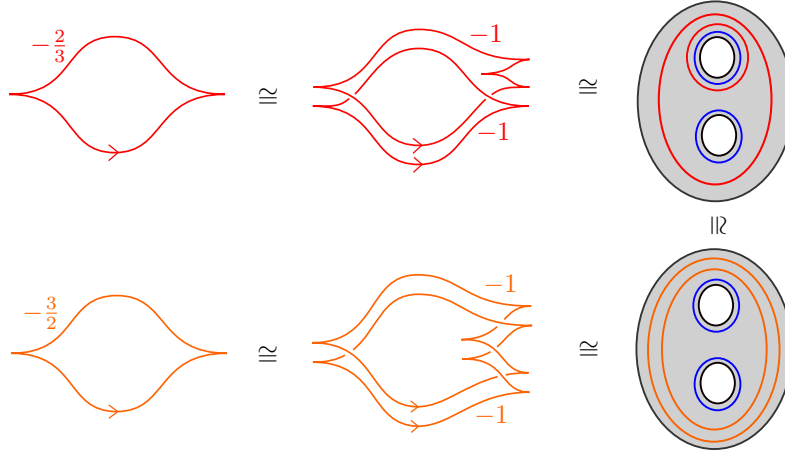


FIGURE 13. Two surgery diagrams of the same tight contact structure  $\xi$  on  $L(5, 2)$

book decompositions (for details see for example [33]). The resulting open books are shown in Figure 13 on the right. All colored curves represent right-handed Dehn twists, the blue ones correspond to the monodromy of the open book decomposition of  $(S^3, \xi_{st})$  and the red and orange curves represent the Dehn twists corresponding to the same colored Legendrian links. All Dehn twists together represent the monodromy of  $(L(5, 2), \xi)$ . By interchanging the holes, these two open books are the same, and therefore represent the same contact manifolds.

These cosmetic contact surgeries are exotic because the corresponding topological cosmetic surgeries are exotic (i.e. there is no homeomorphism of the exterior of the unknot that maps one slope to the other).

With exactly the same methods one can get many other examples of this kind. For example, by contact Dehn surgery along  $U$  with contact surgery coefficients  $r_1 = -2/5$  and  $r_2 = -3/4$  one gets contactomorphic contact structures on the lens spaces  $L(7, 5)$  and  $L(7, 3)$ .

An easy consequence of the Contact Dehn Surgery Theorem 4 is that the only possible candidates for Legendrian knots in  $(S^3, \xi_{st})$  admitting a (non-trivial) contact surgery resulting again in  $(S^3, \xi_{st})$  are the Legendrian unknots of type  $U_n$ .

That cosmetic contact surgeries along such knots really exists was shown in the example in Section 6. It is also possible to show that every Legendrian unknot of type  $U_n$  admits (infinitely many) cosmetic contact surgeries resulting again in  $(S^3, \xi_{st})$ . By computing explicitly the  $d_3$ -invariants (see [9]) of all other contact surgeries one can show that these are the only cosmetic contact surgeries from  $(S^3, \xi_{st})$  to  $(S^3, \xi_{st})$ .

**Problem** (Cosmetic contact surgeries).

Is it possible to classify cosmetic contact surgeries in other contact manifolds? Can one find examples of exotic cosmetic contact surgeries not resulting in Lens spaces?

Moreover, in Section 10 I showed that all topological examples of links not determined by their complements, where one does a composition of  $(-1)$ -Rolfsen twist

along unknot components, works also for the contact case. But in the topological category there are also examples that do not arise in this way. Teragaito [39] and Ichihara [26] construct, building up on unpublished work of Berge, links in  $S^3$  with no unknot components, such that non-trivial Dehn surgery along that link leads again to  $S^3$ . Similar to the arguments in Theorem 5 this leads to links not determined by their complements. Gordon proves in [23] that for every such link (not determined by its complement, but without unknot components) there exists only finitely many other links with homeomorphic complements. Finally in [32] the reverse question is considered. They study links that **are** determined by their complements.

**Problem** (The Legendrian Link Exterior Problem).

Which of these statements hold also for Legendrian links in contact manifolds?

*Acknowledgment.* I would like to thank Sebastian Durst, Hansjörg Geiges, Mirko Klukas and Sinem Onaran for reading earlier versions of this article and for many useful discussions and suggestions. Part of this work was supported by the DFG Graduiertenkolleg 1269 "Global Structures in Geometry and Analysis".

#### REFERENCES

- [1] R. AVDEK, Contact surgery and supporting open books, *Algebr. Geom. Topol.* **13** (2013), 1613-1660.
- [2] K. BAKER AND J. GRIGSBY, Grid diagrams and Legendrian lens space links, *J. Symplectic Geom.* **7** (2009), 415-448.
- [3] J. BERGE, The knots in  $D^2 \times S^1$  which have nontrivial Dehn surgeries that yield  $D^2 \times S^1$ , *Topology Appl.* **38** (1991), 1-19.
- [4] S. BLEILER, C. HODGSON AND J. WEEKS, Cosmetic Surgery on Knots, in *Proceedings of the Kirbyfest*, 23-34, Geometry and Topology Monographs **2**, Coventry (1999).
- [5] J. CONWAY, Transverse Surgery on Knots in Contact 3-Manifolds, preprint available on: <http://arxiv.org/abs/1409.7077>
- [6] F. DING AND H. GEIGES, Symplectic fillability of tight contact structures on torus bundles, *Algebr. Geom. Topol.* **1** (2001), 153-172.
- [7] F. DING AND H. GEIGES, A Legendrian surgery presentation of contact 3-manifolds, *Math. Proc. Cambridge Philos. Soc.* **136** (2004), 583-598.
- [8] F. DING AND H. GEIGES, The diffeotopy group of  $S^1 \times S^2$  via contact topology, *Compos. Math.* **146** (2010), 1096-1112.
- [9] F. DING, H. GEIGES AND A. I. STIPSICZ, Surgery diagrams for contact 3-manifolds, *Turkish J. Math.* **28** (2004), 41-74.
- [10] S. DURST AND M. KEGEL, Computing rotation and self-linking numbers, preprint
- [11] S. DURST, M. KEGEL AND M. KLUKAS, Computing the Thurston-Bennequin invariant in open books, preprint
- [12] J. EDWARDS, Concentricity in 3-manifolds, *Trans. Amer. Math. Soc.* **113** (1964), 406-423.
- [13] Y. ELIASBERG AND M. FRASER, Topologically trivial Legendrian knots, *J. Symplectic Geom.* **7** (2009), 77-127.
- [14] J. ETNYRE, Convex surfaces in contact geometry: class notes, Lecture notes available on: <http://people.math.gatech.edu/~etnyre/preprints/papers/surfaces.pdf>
- [15] J. ETNYRE AND K. HONDA, Knots and contact geometry I: torus knots and the figure eight knot, *J. Symplectic Geom.* **1** (2001), 63-120.
- [16] J. ETNYRE, Legendrian and transversal knots, in: *Handbook of Knot Theory* (W. Menasco and M. Thistlethwaite, eds.), Elsevier, Amsterdam (2005), 105-185
- [17] D. GABAI, Foliations and the Topology of 3-Manifolds II, *J. Differential Geom.* **26** (1987), 461-478.
- [18] D. GABAI, Surgery on knots in solid tori, *Topology* **28** (1989), 1-6.
- [19] F. GAINULLIN, Heegaard Floer homology and knots determined by their complements, preprint available on: <http://arxiv.org/abs/1504.06180>

- [20] H. GEIGES, *An Introduction to Contact Topology*, Cambridge Stud. Adv. Math. **109** (Cambridge University Press, 2008).
- [21] H. GEIGES AND S. ONARAN, Legendrian rational unknots in lens spaces, *J. Symplectic Geom.* **13** (2015), 17-50.
- [22] R. E. GOMPF AND A. I. STIPSICZ, *4-Manifolds and Kirby Calculus*, Grad. Stud. Math. **20** (American Mathematical Society, Providence, RI, 1999).
- [23] C. GORDON, Links and their complements, in: "Topology and Geometry: commemorating SISTAG", 71-82, Contemp. Math. **314** (American Mathematical Society, 2002).
- [24] C. GORDON AND J. LUECKE, Knots are determined by their complements, *J. Amer. Math. Soc.* **2** (1989), 371-415.
- [25] K. HONDA, On the classification of tight contact structures I, *Geom. Topol.* **4** (2000), 309-368.
- [26] K. ICHIHARA, On tunnel number one links with surgeries yielding the 3-sphere, *Osaka J. Math.* **47**, Number 1 (2010), 189-208.
- [27] K. ICHIHARA AND I. JONG, Cosmetic banding on knots and links, preprint available on: <http://arxiv.org/abs/1602.01542>
- [28] M. KEGEL, Kontakt-Dehn-Chirurgie entlang Legendre-Knoten, Bachelorarbeit, Universität zu Köln (2011), available online: <http://www.mi.uni-koeln.de/~mkegel/downloads.html>
- [29] P. LISCA, P. OZSVÁTH, A. I. STIPSICZ AND Z. SZABÓ, Heegaard Floer invariants of Legendrian knots in contact three-manifolds, *J. Eur. Math. Soc. (JEMS)* **11** (2009), 1307-1363.
- [30] Y. MATHIEU, Closed 3-manifolds unchanged by Dehn surgery, *J. Knot Theory Ramifications* **1** (1992), 279-296.
- [31] D. MATIGNON, On the knot complement problem for non-hyperbolic knots, *Topology Appl.* **157** (2010), 1900-1925.
- [32] H. MATSUDA, M. OZAWA AND K. SHIMOKAWA, On non-simple reflexive links, *J. Knot Theory Ramifications* **11** (2002), 787-791.
- [33] B. ÖZBAĞCI, An open book decomposition compatible with rational contact surgery, *Proceedings of Gökova Geometry-Topology Conference* (2005), 175-186.
- [34] V. V. PRASOLOV AND A. B. SOSSINSKY, *Knots, Links, Braids and 3-Manifolds*, Transl. math. Monogr. **154** (American Mathematical Society, Providence, 1997).
- [35] H. RAVELOMANANA, Exceptional Cosmetic surgeries on  $S^3$ , preprint available on: <http://arxiv.org/abs/1505.00238>
- [36] H. RAVELOMANANA, Knot Complement Problem for  $L$ -space  $\mathbb{Z}HS^3$ , preprint available on: <http://arxiv.org/abs/1505.00239>
- [37] D. ROLFSEN, *Knots and Links*, Mathematics Lecture Series **7** (Publish or Perish, Berkeley, 1976).
- [38] Y. RONG, Some knots not determined by their complements, in: "Quantum topology", Ser. Knots Everything **3**, (World Sci. Publishing, River Edge, NJ, 1993), 339-353.
- [39] M. TERAGAITO, Links with surgery yielding the 3-sphere, *J. Knot Theory Ramifications* **11** (2002), 105-108.
- [40] H. TIETZE, Über die topologischen Invarianten mehrdimensionaler Mannigfaltigkeiten, *Monatshefte für Mathematik und Physik* **19** (1908), 1-118.
- [41] J. WANG, Cosmetic surgeries on genus one knots, *Algebr. Geom. Topol.* **6** (2006), 1491-1517.
- [42] J. H. C. WHITEHEAD, On doubled knots, *J. London Math. Soc.* **12** (1937), 63-71.

MATHEMATISCHES INSTITUT, UNIVERSITÄT ZU KÖLN, WEYERTAL 86-90, 50931 KÖLN  
[HTTP://WWW.MI.UNI-KOELN.DE/~MKEGEL/](http://www.mi.uni-koeln.de/~mkegel/)  
*E-mail address:* [mkegel@math.uni-koeln.de](mailto:mkegel@math.uni-koeln.de)