Student seminar on the geometry of 3-manifolds

July 2022

This is an advertisement for a seminar on 3-manifolds in the WiSe22. It is planned as a weekly 2 hour talk, each given by a participant. The main goal is to paint an overview of the theory of 3-manifolds and, perhaps, introduce some research topics at the end.

Rough provisional outline The organization of topics is open to discussion and not every item listed below has to be a talk, some could be compressed into one glossing over more details and focusing on more on ideas or vice versa. As the seminar goes on, we can adapt material according to demand and background of the participants.

1. Topology and examples of 3-manifolds.
   - i) Prime decomposition,
   - ii) incompressible surfaces and Haken manifolds,
   - iii) Seifert manifolds I: Dehn filling and classificaiton of lens spaces,
   - iv) Seifert manifolds II: preliminaries and commensurability classes,
   - v) Seifert manifolds III: classification,
   - vi) Heegard splittings, Dehn surgery, knots and links I,
   - vii) knots and links II,
   - viii) surface bundles and torus decomposition.
2. Geometry of 3-manifolds.
   - i) The 8 geometries I: hyperbolic, elliptic and flat 3-manifolds (classification of the latter), and product geometries,
   - ii) the 8 geometries II: Nil, Sol and $\widetilde{SL_2(\mathbb{R})}$ geometries, geometries and Seifert manifolds. The geometrization conjecture,
   - iii) hyperbolic geometry I: basics and thick-thin decomposition.
   - iv) hyperbolic geometry II: Mostow rigidity,
   - v) hyperbolic geometry III: hyperbolic Dehn filling.
3. Possible extra topics (suggestions are super welcome).
   - i) An introduction to the virtual Haken conjecture,
   - ii) an overview of the Ricci flow in the geometrization conjecture,
   - iii) hyperbolic knot theory,
   - iv) geometrization of surfaces and Teichmüller theory,
   - v) homotopy properties of 3-manifolds (loop and sphere theorems),
   - vi) Thurston norm and related concepts,
   - vii) spin geometry and 3-manifolds,
   - viii) contact geometry of 3-manifolds (slightly introduced in the previous sections),
   - ix) topological, PL and smooth manifolds,
   - x) ...

Main goal The geometry of 2-manifolds is very well understood, we have a very simple topological classification and a good grasp on the 3 geometries of surfaces. The realm of 3-manifolds is more complicated, even though much is known, even more is not understood. This seminar aims to provide an overview of this world: we will have a firm grounding on what Thurston's geometrization conjecture is (proved by Perelman in 2002) and what it entails. For this, we will develop a basic and hopefully intuitive understanding of the eight geometries with a special emphasis on hyperbolic geometry, the most important of them all.

We would start by understanding basic topological facts about 3-manifolds, mainly by studying how surfaces can be contained in them. Particularly, we show that the connected sum of compact 3-manifolds behaves like the product for the integers: we have a decomposition of compact 3-manifolds into unique prime pieces. We would proceed to study several classes of 3-manifolds with a special emphasis on Seifert manifold ($\mathbb{S}^1$-bundles over 2-dimensional orbifolds, which can be seen to be equivalent to a folliation by circles), which we'd classify up to diffeomorphism.

Then, we'd study several ways to build 3-manifolds, mainly Heegard splittings. We'd touch on some knot theory and Dehn surgery in order to prove that every closed orientable 3-manifold can be described via an integral surgery along a link. After some more studying, the main goal would be to show that the 'prime' pieces (not exactly but essentially) of the prime decomposition can be decomposed along tori in a unique minimal way up to isotopy.

While for two dimensions every surface naturally admits one of three types of geometries (flat, spherical or hyperbolic), the same is not true for 3-manifolds. The geometrization conjecture states that the blocks of the torus decomposition of those 'prime' pieces admit one of eight geometries. This eight geometries can be understood as the most symmetric Riemannian metrics a simply connected 3-manifold admits (a transitive action of compact stablizers of a Lie group appears here). The manifolds admitting these geometries are relatively well understood via the Seifert 3-manifold classification except the hyperbolic case, and as you may have heard, most 3-manifolds are hyperbolic.

Finally, we'd go on to study hyperbolic geometry more closely in the general setting (here we'd study many results for $n \geq 2$ and some for $n = 3$). For example, we could study the thick-thin decomposition, a wonderful tool to understand non-compact finite-volume hyperbolic manifolds; Mostow's rigidity, which states that if two hyperbolic n-maniolds for $n \geq 3$ have isomorphic fundamental group, there is an isometry between them that induces that isomorphism (wow); and perhaps Thurston's hyperbolic Dehn filling theorem.

This string of thought is usually called geometric topology: the interplay between these two fields is fruitful, surprising and beautiful in the study of 3-manifolds. If time permits, we could cover extra topics on demand and have a couple of talks speaking about the use of the Ricci flow to prove the geometrization conjecture (from which the Poincare conjecture follows), the virtual Haken conjecture, one of the biggest advancements in the field in the last 15 years, and more.

After the main goals of the seminar, we would have enough language to play with some research ideas. Moreover, a follow-up official seminar is planned for the SoSe23, which will be official. Also, after this seminar, whoever is interested, can join for some research activities on 3-manifolds.

Who is this seminar for? Anyone who fancies themselves a geometer should definitely take this seminar, it is a great opportunity to learn a beautiful research-active theory with colleagues and friends. As for prerequisits, Topology I and Differential Geometry I would be enough if sufficient motivation exists, Topology II and Differential Geometry II are more than enough.

Contact details and more If you have ideas or are not sure if you would like to take this seminar, you should still email us and we could talk about it. Also, this is a first draft of our intention for the seminar so there will be updates and hence, if you send us an email, you will be updated.

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Bureaucracy Due to lack of time, this seminar can not be offered in exchange for credit. What can be promised is that the SoSe23 seminar will be official.

References The main reference is the book "An Introduction to Geometric Topology" by Bruno Martelli, which is freely available at `https://people.dm.unipi.it/martelli/geometric_topology.html`). While this book is the driving threat of the seminar we will provide specialized references to each topic.