

Motivation

X filtered topological space:

$$\emptyset \subseteq X^0 \subseteq X^1 \subseteq \dots \subseteq X^n = X$$

- X is a "complicated" space, but each X^k is "just slightly more complicated" than X^{k-1} ,
- more precisely: We know how to compute $H_*(X^k, X^{k-1})$ for $k \in \mathbb{N}$
- Question: Can we use this to (more easily) compute $H_*(X)$?

Specific example

X CW-complex, the filtration is given by the k -skeletons,

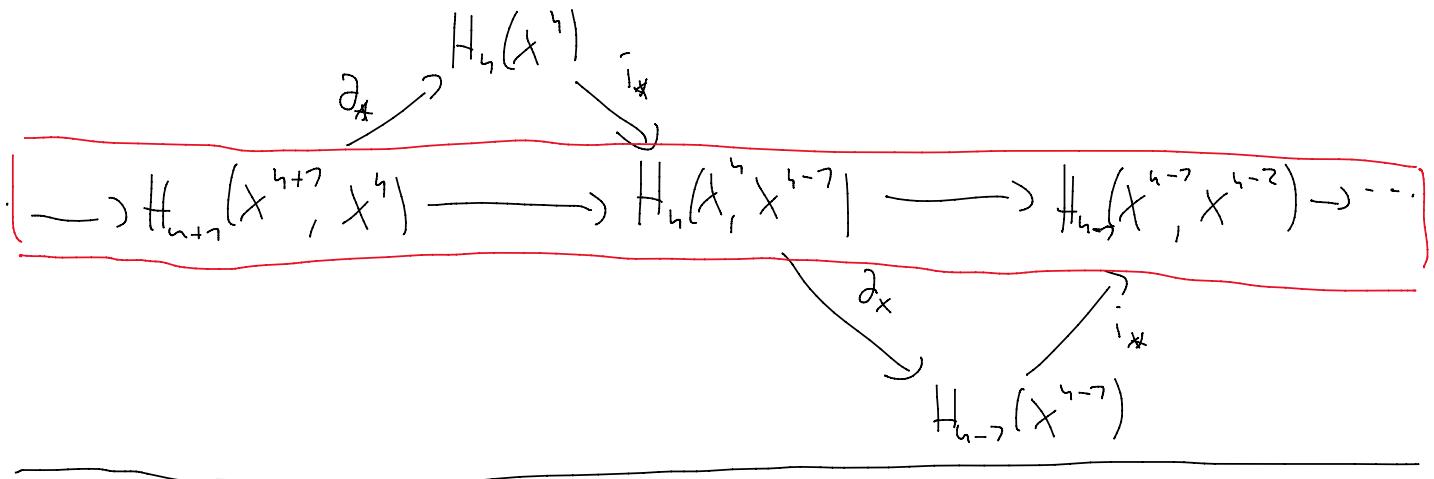
$$\emptyset \subseteq X^0 \subseteq X^1 \subseteq \dots \subseteq X^n \subseteq \dots$$

To compute $H_*(X)$; Prove

Lemma 2.34. If X is a CW complex, then:

- (a) $H_k(X^n, X^{n-1})$ is zero for $k \neq n$ and is free abelian for $k = n$, with a basis in one-to-one correspondence with the n -cells of X .
- (b) $H_k(X^n) = 0$ for $k > n$. In particular, if X is finite-dimensional then $H_k(X) = 0$ for $k > \dim X$.
- (c) The map $H_k(X^n) \rightarrow H_k(X)$ induced by the inclusion $X^n \hookrightarrow X$ is an isomorphism for $k < n$ and surjective for $k = n$.

(Hatcher)



Definitions

- A differential group is an abelian group C , together with a homomorphism $d: C \rightarrow C$ such that $d^2 = d \circ d = 0$.
 - A graded group is an abelian group C , together with a collection of subgroups $\{C_k \mid k \in \mathbb{Z}\}$ such that $C = \bigoplus_{k \in \mathbb{Z}} C_k$
 - A filtered group is an abelian group, together with a filtration $0 = F_1 C \subseteq F_0 C \subseteq F_1 C \subseteq \dots \subseteq F_n C = C$ of subgroups.
-

Main example:

X topological space: $C_{\text{sing}}(X) := \bigoplus_{k \in \mathbb{Z}} \underbrace{C_{k, \text{sing}}(X)}_{k\text{-chain in } X}$ is a differential graded group.

If $X = X_n \supseteq X_{n-1} \supseteq \dots \supseteq X_0$ is filtered, then $C_{\text{sing}}(X)$ will become a filtered group by setting

$$F_p C_{\text{sing}}(X) := C_{\text{sing}}(X_p) = \bigoplus_{k \in \mathbb{Z}} C_{k, \text{sing}}(X_p)$$

Convention: Whenever an abelian group C has more than one of the structures defined above we require these to be compatible as follows:

- If C is differential graded group, we require the differential to be homogeneous:

$$\exists m \in \mathbb{Z} \text{ s.t. } \forall k \in \mathbb{Z}: d(C_k) \subseteq C_{k+m}$$

- If C is a differential filtered group, we require:

$$d(F_p C) \subseteq F_p C \quad \text{for all } p \in \mathbb{Z}.$$

- If C is a filtered graded group, we require

$$F_p C = \bigoplus_{k \in \mathbb{Z}} (F_p C \cap C_k)$$

- For a differential group C , define

$$H(C) := \frac{\text{Ker } d}{\text{Im } d}$$

If C is graded, then $H(C)$ inherits a grading:

$$H(C) = \bigoplus_{k \in \mathbb{Z}} H(C_k)$$

If C is filtered, then $H(C)$ inherits a filtration:

$F_p H(C) :=$ Image of $H(F_p C)$ under the induced morphism $H(F_p C) \rightarrow H(C)$.

- For a filtered group C , define

$$S_r(C) := \bigoplus_{p \in \mathbb{Z}} \frac{F_p C}{F_{p-r} C} \quad (\text{this is a graded group})$$

the associated graded group of C

If C is a differential group, then $S_r(C)$ will inherit a differential $d^r : S_r(C) \rightarrow S_r(C)$

$$(d_p^r : \frac{F_p C}{F_{p-r} C} \rightarrow \frac{F_p C}{F_{p-r} C} \text{ for all } p \in \mathbb{Z})$$

The Spectral Sequence of a Filtered Differential Group

Throughout, let C be a filtered differential group.

We want to relate $H(S_r(C))$ to $H(C)$.

Why aren't they isomorphic?

First of all: $H(S_r(C))$ is a graded group

$H(C)$ is a filtered group

Let's instead compare $H(Sr(C))$ to $Sr(H(C))$.
Why aren't these groups isomorphic?

- The cycles in $H_p(Sr(C))$ are represented by elements of $F_p C$ whose boundaries lie in $F_{p-1} C$.

The elements of $Sr(H(C))$ are represented by elements of $H_p(C)$. The cycles in $H_p(C) \subseteq H(F_p C)$ are elements of $F_p C$ whose boundary is \emptyset .

- The boundaries in $H_p(Sr(C))$ are boundaries of elements in $F_p C$.

The boundaries in $(Sr(H(C)))_p$ are boundaries of elements of C

↳ so in general, these groups are not isomorphic.

(Informal) idea/claim: These two problems are all you have to fix.

Idea: Find a gradual transition from $H(Sr(C))$ to $Sr(H(C))$ by gradually decreasing the number of cycles, and increasing the number of boundaries.

Rewrite the definition for $H_p(S(C))$:

$$H_p(S_r(C)) = \frac{F_p C \cap d^{-1}(F_{p-1} C)}{[F_{p-1} C \cap d^{-1}(F_{p-2} C)] + [F_p C \cap d(F_{p-1} C)]}$$

Definition: (Let C be a filtered differential group),

For $r \in \mathbb{Z}$, define

$$F_p^r := \frac{F_p C \cap d^{-1}(F_{p-r} C)}{[F_{p-1} C \cap d^{-1}(F_{p-r} C)] + [F_p C \cap d(F_{p+r-1} C)]} \quad \forall p \in \mathbb{Z}$$

$$\Gamma \vdash L_{p-r} C \cap d(L_{p-r} C) \vdash L_p C \cap d(L_{p+r-1} C) \vdash$$

$$E^r := \bigoplus_{p \in \mathbb{Z}} E_p^r$$

"the n -th page of the spectral sequence of C ".

If C is also graded, then E_p^r is also graded:

$$E_{pq}^r = \frac{F_p C_{p+q} \cap d^r(F_{p-r} C_{p+q+m})}{[F_{p-r} C \cap d^r(F_{p-r} C_{p+q+m})] + [F_p C \cap d(F_{p+r-1} C_{p+q+m})]} \quad \forall p \in \mathbb{Z}$$

then $E_p^r = \bigoplus_{q \in \mathbb{Z}} E_{pq}^r$

Observe

$$E_p^\infty = H_p(S_r(C)) \Rightarrow E^\infty = H(S_r(C)).$$

furthermore

$$E_p^0 = \frac{F_p C}{F_{p-r} C} \Rightarrow E^0 = S_r(C)$$

and: for $r \gg 0$: $F_{p-r} C = 0$, $F_{p+r-1} C = C$

(if the filtration is finite).

so

$$E_p^r = E_p^{r+1} = \dots = \frac{F_p C \cap d^r(0)}{[F_{p-r} C \cap d^r(0)] + [F_p C \cap d(C)]}$$

$$= \frac{H(F_p C)}{H(F_{p-r} C)} =: E_p^\infty,$$

$$\text{so } E^\infty := \bigoplus_{p \in \mathbb{Z}} E_p^\infty = S_r(H(C)).$$

"the spectral sequence stabilizes"

and induces a differential $d^r : E_p^r \rightarrow E_{p-r}^r$

d induces a differential $d_p^r: E_p^r \rightarrow E_{p-r}^r$
 (and hence $d: E^r \rightarrow E^r$ homogeneous of degree $-r$):

Take $\alpha \in E_p^r$. Then α is represented by

$$\alpha \in F_p C \cap d^{-1}(F_{p-r} C). \quad \text{Then}$$

$$d(\alpha) \in F_{p-r} C, \quad \text{and} \quad d(d(\alpha)) = d^2(\alpha) = 0 \in F_{p-2r} C, \quad \text{so}$$

$$d(\alpha) \in \underbrace{F_{p-r} C \cap d^{-1}(F_{(p-r)-r} C)}_{\text{numerator of } E_{p-r}^r}.$$

Define $d_p^r(\alpha)$ to be the equivalence class of $d(\alpha)$
 in E_{p-r}^r .

Main theorem

1) d_p^r is well-defined.

2) $d_{p-r}^r \circ d_p^r = 0$ (so $d^r \circ d^r = 0$, so E^r is
 canonically a differential group)

3) $\frac{\ker d_p^r}{\text{Im } d_{p+r}^r} \cong E_p^{r+1}$ (in part. $H(E^r) \cong E^{r+1}$)
 [canonically]

Proof diagram chase

Result: knowing E^r (or some other E^r) as well
 as d , we can compute E_1^r , then E_2^r, \dots , until
 the sequence stabilizes, to obtain E^∞

↳ we can compute $S_r(H(C))$ from $H(S_r(C))$
 (or from $f_r(C)$).

(or from $\text{fr}(C)$).

Back to our original motivation:

X CW-complex, $\emptyset = X^0 \subseteq X^1 \subseteq \dots \subseteq X^n = X$ skeletons

$$E_{pq}^\infty = C_q^{\text{tors}}(X^p) / C_{q-p}^{\text{tors}}(X^p) \xrightarrow{C_p(\lambda)}$$

$$E_{pq}^1 = H_{p+q}(X^p, X^{p-1}) = \begin{cases} \mathbb{Z} & \text{if } q=0 \\ 0 & \text{otherwise.} \end{cases}$$

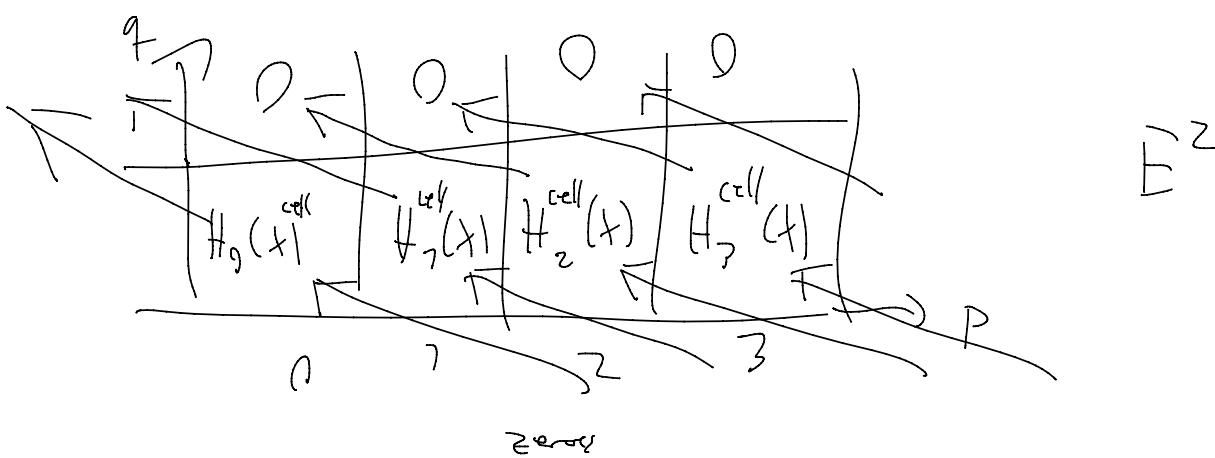
Draw a table:

	0	1	2	3	
0	0	0	0	0	
1	$C_0(X)$	$C_1(X)$	$C_2(X)$	$C_3(X)$	
	0	1	2	3	

E^1



(zeros)



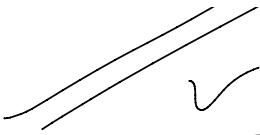
zeros

no more nontrivial homology, so the sequence stabilizes here.

$$\Rightarrow \frac{H_n(X^n)}{H_n(X^{n-1})} = E_{n,0}^\infty = E_{n,0}^2 = H_n^{\text{cell}}(X)$$



$$H_4(x^4)$$



Question

$$C = \bigoplus_{k \in \mathbb{Z}} C_k \quad \text{graded group}$$

$$\dots \subseteq \bigoplus_{k=-\infty}^1 C_k \subseteq \bigoplus_{k=-\infty}^{\infty} C_k \subseteq \bigoplus_{k=-\infty}^{\infty} C_k \subseteq \dots$$

