Yet Another TSP:

Problem:

Consider a special case of the travelling salesman problem (TSP) where the vertices correspond to points in the plane, with the cost defined for an edge for every pair (p, q) being the usual Euclidean distance between p and q. Prove that an optimal tour will not have any pair of crossing edges.

Solution:

Proposition: If a tour is optimal, it will not have any crossing edges.

In order to prove this proposition, I will advance the following argument:

1. Assume a TSP tour has a pair of crossing edges.
2. Denote the edges to be $e_{a,d}$ and $e_{c,b}$, connecting nodes a, b, c, d.
3. Denote the intersection to be, M.
4. $dist_{a,m} + dist_{m,d} = dist_{a,d}$.
5. Premise (4) is true because the intersection point, M, lies on the edge line of a to d.
6. Therefore, consider the following sub-proof:

$$\frac{\begin{array}{c} dist_{a,m} + dist_{m,b} > dist_{a,b} \\ dist_{c,m} + dist_{m,d} > dist_{c,d} \end{array}}{dist_{a,d} + dist_{c,b} > dist_{a,b} + dist_{c,d}}$$

Recall that the two lines above the addition bar are true due to fundamental properties of triangles. Particularly, the sum of the lengths of any two sides of a triangle is greater than the length of the third side.

This means that if we remove edges $e_{a,d}$ and $e_{c,b}$ and replace them with edges $e_{a,b}$ and $e_{c,d}$ (i.e., edges that do not cross), then the total distance between the nodes will be shorter.

Note that this is still a solution to the TSP, since removing edges that intersect can still produce a fully connected graph.

Since this approach produces a shorter path, the original tour was not optimal. Therefore, an optimal tour will not have a pair of crossing edges.