A Simple Numerical Solution Procedure for Statics Problems Involving Arbitrary-Shaped Surfaces

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Abstract—**A simple and efficient numerical procedure is presented for determining the static charge distribution on arbitrary-shaped surfaces. Surfaces are modeled by planar triangular subdomains in which the charge density is assumed to be constant. The method of moments is employed to calculate the charge distribution on the surface.**

I. INTRODUCTION

DESPITE the inherent advantages of patch modeling over wire grid modeling and the unique ability of the electric field integral equation (EFIE) to treat both open and closed surfaces, no numerical approaches using the EFIE and surface patch modeling to treat arbitrary-shaped surfaces are yet available. In the opinion of the authors, this is because of a number of seemingly conflicting practical and numerical constraints which must be simultaneously resolved before a reliable computer code can be developed. While several approaches have appeared [1]-[3] which one would expect could be extended to treat arbitrary geometries, the generalization of such techniques to arbitrary surfaces seems in each case to be either impractical, impossible, or unobvious. A procedure employing triangular subdomain regions has recently been proposed [4], however, which is simple and which applies directly to the treatment of arbitrary-shaped surfaces. The numerical solution procedure for static problems presented in this paper is equivalent to the static limit ($\omega \to 0$) of the proposed approach [4] and therefore, aside from its intrinsic usefulness in solving statics problems, represents a first step toward the development and validation of a method for treating truly arbitrary geometries in the electrodynamic case. Electrostatic problems have been treated previously by employing rectangular subdomain regions in which the charge density is assumed to be constant [5]. The necessity of using triangular subdomains for more complex surfaces has also been noted in [5]. Electrostatic problems have also been solved using so-called triangular isoparametric patches borrowed from the finite-element methods [6]. However, the extension of the finite-element approach to treat the vector fields which arise in the electrodynamic case is not immediately obvious and, furthermore, difficulties are likely to arise in the modeling of any edges and/or sharp bends in the surface due to the use of excessively smooth basis functions [7]. In this paper we demonstrate the effectiveness of a simpler approach wherein triangular subdomains of constant charge density are used with collocation to compute the charge distribution and capacitance for several representative geometries.

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II. FORMULATION

Consider a perfectly conducting surface which is charged to a potential V_0. The unknown surface charge density distribution $\rho(\bar{r})$ may then be determined by solving the integral equation

$$\frac{1}{4\pi\epsilon_0} \iint_{\text{surface}} \frac{\rho(\bar{r}')}{|\bar{r}-\bar{r}'|}\, dS' = V_0. \tag{1}$$

The exact solution for the charge distribution can be obtained only for a few very specialized geometries. In the general case the surface must be discretized and the charge distribution found by numerical methods. The simplest approach for arbitrary-shaped bodies is to approximate the surface by planar triangular subdomains. We therefore choose to approximate the surface in this manner and to assume that within each triangular subdomain the charge density is constant. The method of moments with collocation is then used to determine the approximate charge distribution. Since the numerical formulation of (1) via the method of moments is well-known [5], we consider only the evaluation of a single general element of the moment matrix. Each element corresponds to the potential at some point in space, $\bar{r} = (x, y, z)$, due to a triangular patch of surface charge of unit charge density. In general, the patch is arbitrarily positioned and oriented in space. The required potential is thus given by

$$V(\bar{r}) = \frac{1}{4\pi\epsilon_0} \iint_{\text{triangle}} \frac{1}{|\bar{r}-\bar{r}'|}\, dA', \tag{2}$$

where A' is the area of the source triangle. The integrations indicated in (2) may be performed either numerically, analytically, or one integral performed analytically and the other performed numerically. However, complete analytical integration eliminates the need for special treatment of the singular integrals which arise for the diagonal elements of the matrix and also sidesteps the question of what order numerical quadrature is required to integrate over any given triangular-shaped region. Preliminary tests have indicated that analytical evaluation of the integrals is also more efficient than numerical integration. To analytically perform the integrations in (2) we first define the positions of the corners of the source triangle by the vectors

$$\bar{r}_1 = (x_1, y_1, z_1), \qquad \bar{r}_2 = (x_2, y_2, z_2), \qquad \bar{r}_3 = (x_3, y_3, z_3).$$

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Equation (2) can then be written in the following form:

$$V(\bar{r}) = \frac{|(\bar{r}_2 - \bar{r}_1) \times (\bar{r}_3 - \bar{r}_1)|}{4\pi\epsilon_0} \cdot \int_0^1 \left\{ \int_0^{1-\eta} \frac{1}{R}\, d\xi \right\} d\eta, \tag{3}$$

where

$$\begin{aligned} R &= |\bar{r} - \bar{r}'| \\ &= ([(x - x_1) - (x_2 - x_1)\xi - (x_3 - x_1)\eta]^2 \\ &\quad + [(y - y_1) - (y_2 - y_1)\xi - (y_3 - y_1)\eta]^2 \\ &\quad + [(z - z_1) - (z_2 - z_1)\xi - (z_3 - z_1)\eta]^2)^{1/2}. \end{aligned}$$

The integrations are quite tedious, but the final result is relatively simple. We have

$$\begin{aligned} V(\bar{r}) = {} & \frac{|(\bar{r}_2 - \bar{r}_1) \times (\bar{r}_3 - \bar{r}_1)|}{4\pi\epsilon_0 |\bar{r}_2 - \bar{r}_1|} \\ & \cdot \sum_{i=1}^{2} \frac{(-1)^i}{D_i} \left\{ u_i \ln (R_i + x_i) \right. \\ & - \frac{b_i}{2\sqrt{a_i}(a_i - 1)} \ln |2\sqrt{a_i} R_i + 2a_i x_i + b_i| \\ & \left. + d_i \tan^{-1} \left[\frac{u_i}{d_i} \right] - d_i \tan^{-1} \left[\frac{2d_i R_i (a_i - 1)}{b_i x_i + 2c_i} \right] \right\} \Bigg|_{\eta=0}^{\eta=1}, \end{aligned} \tag{4}$$

where

$$R_i = \sqrt{A_i \eta^2 + B_i \eta + C_i}$$

$$u_i = x_i + \frac{b_i}{2(a_i - 1)}$$

$$x_i = D_i \eta + E_i$$

$$A_i = |\bar{r}_3 - \bar{r}_i|^2$$

$$B_i = -2(\bar{r}_3 - \bar{r}_i) \cdot (\bar{r} - \bar{r}_i)$$

$$C_i = |\bar{r} - \bar{r}_i|^2$$

$$D_i = \frac{(\bar{r}_2 - \bar{r}_1) \cdot (\bar{r}_3 - \bar{r}_i)}{|\bar{r}_2 - \bar{r}_1|}$$

$$E_i = -\frac{(\bar{r}_2 - \bar{r}_1) \cdot (\bar{r} - \bar{r}_i)}{|\bar{r}_2 - \bar{r}_1|}$$

$$a_i = A_i / D_i^2$$

$$b_i = (B_i D_i - 2A_i E_i)/D_i^2$$

$$c_i = (C_i D_i^2 - B_i D_i E_i + A_i E_i^2)/D_i^2$$

$$d_i^2 = \frac{c_i}{(a_i - 1)} - \frac{b_i^2}{4(a_i - 1)^2}.$$

The following comments concerning the implementation of (4) are in order.

1) If either D_1 or D_2 is zero (which happens whenever the corner of the triangle at $\bar{r}_1$ or $\bar{r}_2$, respectively, forms a right angle), (4) cannot be evaluated in the form indicated. A simple way to circumvent this difficulty is to cyclically permute the assignment of the vectors $\bar{r}_1$, $\bar{r}_2$, and $\bar{r}_3$ to the corners of the triangle until neither D_1 nor D_2 is zero, i.e., until the right-angle corner is at $\bar{r}_3$. Under the new assignment (4) is again valid.

2) The argument of either of the logarithmic functions appearing in (4) may be zero. Whenever this situation occurs, however, the coefficient multiplying the logarithmic term is also zero and the term can easily be shown to vanish.

The moment matrix is generated by "matching" (1) at the centroid of each triangular surface patch. The elements of the moment matrix are determined via (4). Solution of the resulting set of simultaneous equations yields values for the surface charge density at the centroids of the subdomains. The capacitance of the body is then given by

$$C = \frac{Q}{V_0} \approx \frac{1}{V_0} \sum_{i=1}^{N} \rho_i A_i, \tag{5}$$

where N is the number of triangular patches (subdomains) which have been used to model the body, ρ_i is the charge density in subdomain i, and A_i is the area of subdomain i.

III. NUMERICAL RESULTS

A computer code using the preceding formulation has been developed to determine the charge distribution and capacitance for arbitrary-shaped surfaces. Numerical results have been obtained for four example cases; for three either an exact solution is known or other numerical data are available for comparison.

We first consider calculation of the capacitance of a perfectly conducting square plate. No exact solution is available, but numerical data have been obtained via the method of moments using collocation and square subdomains in which the charge density is assumed constant [5]. For this case the use of square or rectangular subdomains is entirely satisfactory since the subdomains can be arranged to precisely fit the geometry under consideration. Fig. 1 illustrates the convergence rate of the capacitance when rectangular subdomains are used and when triangular subdomains are used. The procedure for dividing the plate into rectangular subdomains is obvious; a triangulation of the plate is easily obtained by adding a diagonal to each such rectangular patch. In the figure the capacitance is plotted versus $1/N$, where N is the number of unknowns. Results are shown for varying degrees of triangle skewness. We note that while increasing the skewness of the triangular sub-

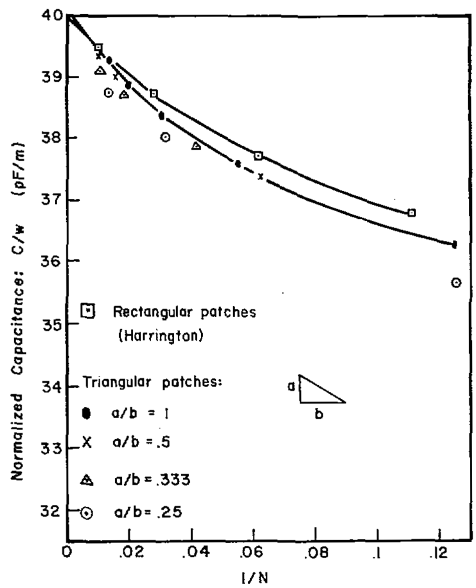

Fig. 1. Plot of capacitance of square plate of width w versus inverse of number of unknowns for rectangular and triangular subdomains.

domains slightly degrades the convergence rate, the difference is not appreciable.

Consider next the case of a sphere of radius a which is charged to a constant potential V_0. The surface charge density for this case is known to be $V_0\epsilon_0/a$ and the capacitance is $4\pi\epsilon_0 a$. For convenience the sphere is placed at the center of a spherical coordinate system. To approximate the sphere with triangular patches, the sphere is first divided into N_ϕ points equally spaced along constant coordinate lines for $N_\theta - 1$ equally spaced values of θ, excluding the poles. Connecting together by straight line segments adjacent points having common θ or ϕ coordinates results in triangular patches surrounding the poles and quadrilateral patches elsewhere. The addition of diagonals to all the quadrilaterals completes the triangulation of the sphere. The total number of unknowns generated in this manner is $N = 2N_\phi(N_\theta - 1)$. Table I presents the numerical results obtained for the capacitance of the sphere for various values of N_ϕ and N_θ. The calculated capacitance appears to approach the exact value with increasing N, as expected. We note in passing that for a numerical solution to the sphere problem, square (or rectangular) patches cannot be conveniently used to model the surface. In electrostatic problems arbitrary quadrilateral patches might be used to provide an adequate model of general surfaces, but it appears that such patches cannot be easily generalized to the dynamic case. Triangular-shaped subdomains, on the other hand, can be generalized to treat the dynamic problem as proposed in [4].

For the case of a flat disk of radius a which is charged to a constant potential V_0, the charge density distribution is also known [8] and is given as

$$\rho(r) = \frac{4V_0\epsilon_0}{\pi\sqrt{a^2 - r^2}}, \tag{6}$$

where r is the distance from the center of the disk. The capacitance of the disk is $8\epsilon_0 a$. For a numerical solution, the disk is divided into N_ϕ equiangular sections in the ϕ direction and N_r sections of equal length in the r direction, and the triangular subdomains are then formed in the same manner as was done for the sphere. The total number of unknowns is $N = 2N_\phi(N_r - \frac{1}{2})$. We note that a possibly more accurate model for the circular disk could be obtained by increasing the number of unknowns in the region near the edge of the disk in order to more accurately sample the rapid variation of the charge which is singular at the edge. Fig. 2 demonstrates excellent agreement between the computed and exact charge density distribution. Numerical results for the capacitance of the disk are presented in Table II for various subdivision schemes.

As a geometrically more complex example we consider the case in which the unit disk is bent through an angle of 90° along a diameter. The charge is now singular both around the edge of the disk and along the bend. The same subdomain scheme can be employed as for the flat disk since the bend is located along a diameter. Fig. 3 illustrates the computed charge distribution along the symmetry plane of the structure which is perpendicular to the bend. Note that the quantity N_ϕ indicated in the figure is still interpreted in the same manner as for the flat disk even though the division lines on each half

TABLE I
NORMALIZED CAPACITANCE OF A SPHERE (IN PICOFARADS/METER)

N_ϕ	N_θ	N	C/a
6	3	24	94.03
6	4	36	98.35
6	5	48	100.39
6	6	60	101.51
6	8	84	102.64
8	3	32	96.81
8	4	48	101.20
8	5	64	103.28
8	6	80	104.43
exact			111.26

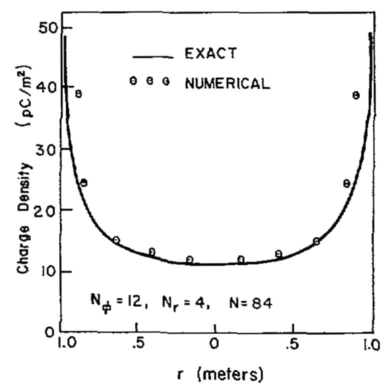

Fig. 2. Comparison of calculated charge distribution on disk of unit radius with exact distribution.

TABLE II
NORMALIZED CAPACITANCE OF A DISK (IN PICOFARADS/METER)

N_ϕ	N_r	N	C/a
6	2	18	59.80
6	3	30	61.10
6	4	42	61.80
6	5	54	62.24
12	2	36	64.75
12	3	60	66.03
12	4	84	66.72
exact			70.83

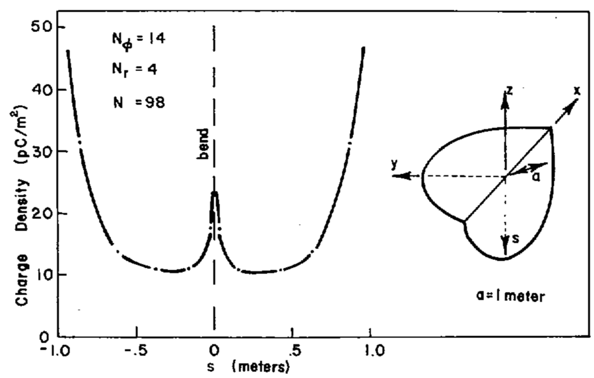

Fig. 3. Calculated charge density distribution on unit disk bent 90° along diameter. Distribution is plotted along symmetry plane perpendicular to bend.

TABLE III
NORMALIZED CAPACITANCE OF A DISK BENT 90° ALONG A DIAMETER (IN PICOFARADS/METER)

N_ϕ	N_r	N	C/a
8	3	40	60.55
8	4	56	61.25
8	5	72	61.69
8	6	88	61.99
12	3	60	62.61
12	4	84	63.29
14	4	98	63.75

of the bent disk do not lie in the same plane. The computed capacitance of the bent disk is presented in Table III for various values of N_ϕ and N_r. From elementary principles one can conclude that the capacitance of the bent disk should be lower than that of the flat disk and, indeed, a comparison of Table II and Table III shows that the computed values are always slightly lower for the bent disk than for the flat disk for a comparable number of unknowns.

IV. CONCLUSION

It has been demonstrated in this paper that the static charge distribution on a charged conductor of arbitrary shape can be numerically determined by a method of moments approach. In the approach used here, the surface of the conductor is approximated by planar triangular patches, which, in contrast to other patch shapes, have the advantages of simplicity, generality, and of complete flexibility in their placement. The use of triangular patches also facilitates using varying patch densities on different portions of the object. Thus, where the charge density needs to be resolved with greater fidelity, more patches may be used. Furthermore, no distinction needs to be made between edges and gradually curving surfaces, the role of surface curvature being played by the angle between the surface normals of adjacent patches in the model. It is also demonstrated that the solution is not highly sensitive to the actual shape of the triangles used.

In the model used here the charge is assumed to be constant within each planar triangle region, yielding a piecewise constant or pulse representation of the charge on the conducting surface. Although it is generally true that the use of smoother basis functions enhances the convergence of the solution for closed surfaces with continuous normals, for general abritrary-shaped surfaces having corners and/or edges where the charge may be singular, the lack of continuity of the simple pulse functions turns out to be an advantage since pulses actually model the singular behavior better. Finally, we note that the use of triangular subdomains in conjunction with a constant charge density in each patch forms the basis for a generalization to the corresponding electrodynamic problem. We intend to report on current progress in treating the dynamic case in the very near future.

A computer program employing the formulation reported here is available for general use. The program is capable of finding the static surface charge density distribution and capacitance of any open or closed charged body having no intersecting surfaces. The approach could easily be extended to treat intersecting surfaces and to treat the problem of determining the charge distribution when a neutrally charged object is immersed in a static electric field.

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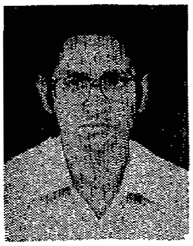

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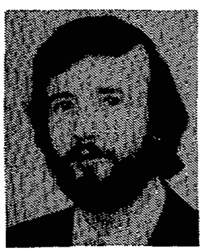

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