Electric Field Notes

Alastair Gosnay

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Theory

The electric field, $\mathbf{E}$, is related to the 2D electrostatic scalar potential, V, by,

$$\mathbf{E} = -\nabla V = -\left(\frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}\right).$$

To allow numerical determination of $\mathbf{E}$ the spacial partial derivatives of V must be approximated, which can be accomplished through the use of finite difference methods.

The derivative of a single variable function is determined analytically by the limit,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

Finite difference methods assume that h is small such that the derivative may be approximated without the use of the limit,

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

This result may also be derived by examination of the Taylor expansion of $f(x+h)$,

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + ...$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x)}{h} - \frac{h}{2}f''(x) - \frac{h^2}{3!}f'''(x)... = \frac{f(x+h) - f(x)}{h} + O(h).$$

$$\Rightarrow f'(x) \approx \frac{f(x+h) - f(x)}{h}.$$

Equation 1 demonstrates the approximation of the derivative being determined using the point to the *right*. Similarly, the derivative may be approximated using the point to the left and hence the Taylor expansion of $f(x-h)$,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{3!}f'''(x)...$$

$$\Rightarrow f'(x) = \frac{f(x) - f(x-h)}{h} + \frac{h}{2}f''(x) - \frac{h^2}{3!}f'''(x)... = \frac{f(x) - f(x-h)}{h} + O(h),$$

$$\Rightarrow f'(x) \approx \frac{f(x) - f(x-h)}{h}.$$

Notice that both of these approximations have an error term of order h. This the order of this error may be reduced by combining the Taylor expansions of $f(x+h)$ and $f(x-h)$ in an attempt to cancel off higher order terms. This can be done by,

$$f(x+h) - f(x-h) = 2hf'(x) + \frac{2h^2}{3!}f'''(x) + ...$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x-h)}{2h} - \frac{2h^2}{3!}f'''(x) + ... = \frac{f(x+h) - f(x-h)}{2h} + O(h^2),$$

$$\Rightarrow f'(x) = \frac{f(x+h) - f(x-h)}{2h}.$$

The above result is known as the *central* finite difference method. Notice the central method has an error or order h^2 whereas the right and left approximation methods have error of order h. Hence, the central method is the main tool to determine $\mathbf{E}$ numerically but the other two methods will still prove useful. The above results can easily be generalised to multivariable functions such as V, where the partial derivatives with respect to x may be approximated as,

$$\frac{\partial V}{\partial x} \approx \frac{V(x+h, y) - V(x, y)}{h} \tag{1}$$

$$\frac{\partial V}{\partial x} \approx \frac{V(x, y) - V(x-h, y)}{h} \tag{2}$$

$$\frac{\partial V}{\partial x} \approx \frac{V(x+h, y) - V(x-h, y)}{2h} \tag{3}$$

and the partial derivatives with respect to y may be approximated as,

$$\frac{\partial V}{\partial y} \approx \frac{V(x, y+h) - V(x, y)}{h} \tag{4}$$

$$\frac{\partial V}{\partial y} \approx \frac{V(x, y) - V(x, y-h)}{h} \tag{5}$$

$$\frac{\partial V}{\partial y} \approx \frac{V(x, y+h) - V(x, y-h)}{2h}. \tag{6}$$

These sets of equations allow $\mathbf{E}$ to be determined from V.

Implmentation of Finite Difference in C++