Turbulence Monte Carlo

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Abstract

This document describes some of the theory behind simulations performed in this project.

1 From Fourth to Second-Order Simulation

Performing simulations for quantities that are fourth-order in field is a formidable task when propagation under atmospheric turbulence is considered, so we try to reframe the model in terms of equivalent second-order beam propagation. This can be done for special cases detailed below, using, for the phase screens that emulate turbulent refractive index fields, a modified 'effective field'.

1.1 Delta Correlation in Near Field

We first consider a special case in which the fourth-order beam is delta correlated in the near-field, so that on the source-plane it has the form

$$E_0(\rho_1', \rho_2') = E_p\left(\frac{\rho_1' \pm \rho_2'}{2}\right) V_a(\rho_1' \mp \rho_2') \tag{1}$$

$$\text{, where } V_a(\rho) = \delta^2(\rho - a)\,.$$

Top and bottom sign refer to the case where there is and isn't a coordinate inversion scheme in place. We consider the former for concreteness and as it corresponds to the more interesting case. Hence its propagation reads:

$$
\begin{aligned}
E_a(\rho_1,\rho_2) &= \iint E_p\left(\frac{\rho_1'-\rho_2'}{2}\right) V_a(\rho_1'+\rho_2') h(\rho_1,\rho_1') h(\rho_2,\rho_2')\ d^2\rho_1' d^2\rho_2' \\
&= \int E_p\left(\rho_1'+\frac{a}{2}\right) h(\rho_1,\rho_1') h(\rho_2,-\rho_1'+a)\ d^2\rho_1' \\
&= \int E_p(\rho') h\left(\rho_1,\rho'+\frac{a}{2}\right) h\left(\rho_2,-\rho'+\frac{a}{2}\right)\ d^2\rho'. \qquad (2)
\end{aligned}
$$

The propagators $h(x,y)$ can be split in the following form:

$$
h(x,y) = h_0(x,y)\exp\left[\psi(x,y)\right] \qquad (3)
$$

$$
\text{, where } h_0(x,y) \propto \exp\left[\frac{ik(x-y)^2}{2z}\right]. \qquad (4)
$$

The term h_0 is the usual vaccuum propagator, while ψ, which describes the contribution from refractive index turbulence, is written

$$
\begin{aligned}
\psi(\rho,\rho') \propto & \int_0^L dz' \int d^2\rho'' \frac{n_1(\rho'',z')}{z'(L-z')} \\
& \times \exp\left\{\frac{ik}{2}\left[\frac{(\rho-\rho'')^2}{L-z'} + \frac{(\rho''-\rho')^2}{z'} - \frac{(\rho-\rho')^2}{L}\right]\right\}. \qquad (5)
\end{aligned}
$$

Close inspection of (4) reveals that

$$
h_0(\rho_1,\rho'+a/2) = h_0(\rho_1-a/2,\rho') \qquad (6)
$$

$$
h_0(\rho_2,-\rho'+a/2) = h_0(-\rho_2+a/2,\rho')\ . \qquad (7)
$$

As for the contribution from atmospheric turbulence, one can see that:

$$
\begin{aligned}
\psi\left(\rho_1,\rho'+\frac{a}{2}\right) \propto & \int_0^L dz' \int d^2\rho'' \frac{n_1(\rho'',z')}{z'(L-z')} \cdots \\
& \times \exp\left\{\frac{ik}{2}\left[\frac{(\rho_1-\rho'')^2}{L-z'} + \frac{\left(\rho''-\rho'-\frac{a}{2}\right)^2}{z'} - \frac{\left(\rho_1-\rho'-\frac{a}{2}\right)^2}{L}\right]\right\} \\
= & \int_0^L dz' \int d^2\tau \frac{n_1\left(\tau+\frac{a}{2},z'\right)}{z'(L-z')} \cdots \\
& \times \exp\left\{\frac{ik}{2}\left[\frac{\left(\rho_1-\frac{a}{2}-\tau\right)^2}{L-z'} + \frac{(\tau-\rho')^2}{z'} - \frac{\left(\rho_1-\frac{a}{2}-\rho'\right)^2}{L}\right]\right\}, \qquad (8)
\end{aligned}
$$

that is, it corresponds to $\psi\left(\rho_1 - \frac{a}{2}, \rho'\right)$ with an effective refractive index $n_{eff}(\rho'') = n_1\left(\rho'' + \frac{a}{2}, z'\right)$. Equivalently, we have

$$\begin{aligned}
\psi\left(\rho_2, -\rho' + \frac{a}{2}\right) \propto & \int_0^L dz' \int d^2\rho'' \frac{n_1(\rho'', z')}{z'(L-z')} \cdots \\
& \times \exp\left\{\frac{ik}{2}\left[\frac{(\rho_2 - \rho'')^2}{L - z'} + \frac{\left(\rho'' + \rho' - \frac{a}{2}\right)^2}{z'} - \frac{\left(\rho_2 + \rho' - \frac{a}{2}\right)^2}{L}\right]\right\} \\
= & \int_0^L dz' \int d^2\tau \frac{n_1\left(-\tau + \frac{a}{2}, z'\right)}{z'(L-z')} \cdots \\
& \times \exp\left\{\frac{ik}{2}\left[\frac{(\rho_2 - \frac{a}{2} + \tau)^2}{L - z'} + \frac{(\rho' - \tau)^2}{z'} - \frac{\left(\rho_2 - \frac{a}{2} + \rho'\right)^2}{L}\right]\right\},
\end{aligned} \tag{9}$$

corresponding to $\psi\left(-\rho_2 + \frac{a}{2}, \rho'\right)$ with an effective refractive index $n_{eff}(\rho'') = n_1\left(-\rho'' + \frac{a}{2}, z'\right)$.

If we choose detectors 1 and 2 to be positioned at $\rho_1 = \rho + \frac{a}{2}$ and $\rho_2 = -\rho + \frac{a}{2}$, by putting the above equations together we see that both propagators h become identical except for the effective refractive indexes. In this case the propagated two-photon amplitude $E_a\left(\rho + \frac{a}{2}, -\rho + \frac{a}{2}\right)$ turns out to be formally equivalent to a second-order beam with the pump profile and wavenumber (that is, $k_p = 2k$) propagating in a medium with refractive index given by

$$n_p(\rho'', z') = n_1\left(\rho'' + \frac{a}{2}, z'\right) + n_1\left(-\rho'' + \frac{a}{2}, z'\right) . \tag{10}$$