1 Lemma 3

Let $N = (V, E, s, t, c)$ be a network and f an st-preflow in N. Assume that an $n \in \mathbb{N}$ and an augmenting path $p = (v_1 w_1, ..., v_n w_n)$ of f exists.

Show that if f is a (pre-)flow than f' is also a (pre-)flow

First we show

$$\forall v \in V \backslash \{s, t\} : \varphi'_v = \varphi_v \tag{1}$$

We know by the definition of f' that for all $e = uw \in E$ for which neither uw nor wu are included in the augmenting path p

$$f'(e) = f(e)$$

and therefore $\varphi_v^{+\prime} = \varphi_v^+$, $\varphi_v^{-\prime} = \varphi_v^-$ and also $\varphi'_v = \varphi_v^{+\prime} - \varphi_v^{-\prime} = \varphi_v^+ - \varphi_v^- = \varphi_v$ for all $v \in V$ that are neither s or t or included in p.
Let $v \in V$ be a node from the augmented path p but not s or t. Since the path starts in s and ends in t, v must be an intermediate node, i.e. the path is of the form $..., uv, vw, ...$ for some u and w. Now we distinguish 4 cases:

1. $uv \in E$ and $vw \in E$
   Then by definition of f' we have $f'(uv) = f(uv) + \delta$ and $f'(vw) = f(vw) + \delta$. For all other edges including v, the flow remains the same, since p is a simple path. Thus
   $$\varphi'_v = (\varphi_v^+ + \delta) - (\varphi_v^- + \delta) = \varphi_v$$
2. $uv \in E$ and $wv \in E$ Then by definition of f' we have $f'(uv) = f(uv) + \delta$ and $f'(wv) = f(wv) - \delta$. For all other edges including v, the flow remains the same, since p is a simple path. Thus
   $$\varphi'_v = \varphi_v^+ - (\varphi_v^- + \delta - \delta) = \varphi_v$$
3. $vu \in E$ and $vw \in E$
   Then by definition of f' we have $f'(vu) = f(vu) - \delta$ and $f'(vw) = f(vw) + \delta$. For all other edges including v, the flow remains the same, since p is a simple path. Thus
   $$\varphi'_v = (\varphi_v^+ + \delta - \delta) - \varphi_v^- = \varphi_v$$
4. $vu \in E$ and $wv \in E$
   Then by definition of f' we have $f'(vu) = f(vu) - \delta$ and $f'(wv) = f(wv) - \delta$. For all other edges including v, the flow remains the same, since p is a simple path. Thus
   $$\varphi'_v = (\varphi_v^+ - \delta) - (\varphi_v^- - \delta) = \varphi_v$$

Thus, equation 1 holds. Now consider the last edge $v_n w_n$ in the path p. It must hold that $v_n w_n = v_n t$ since p ends in t. Also since p is a simple path, t does not appear in any other edge in p. By definition of f' the flow remains the same for all other edges including t except $v_n t$ (or $t v_n$ if $v_n t \in E^-$). Now distinguish 2 cases:

1. $v_n t \in E$
   Then by definition of f' we have $f'(v_n t) = f(v_n t) + \delta$. Therefore $\varphi'_t = \varphi_t - \delta$
2. $tv_n \in E$
   Then by definition of f' we have $f'(tv_n) = f(tv_n) - \delta$. Therefore $\varphi'_t = \varphi_t - \delta$

Thus it holds in general that

$$\varphi'_t = \varphi_t - \delta \tag{2}$$

Thus from 1 and 2 it also holds that

$$f \text{ is st-preflow} \Rightarrow \forall v \in V \backslash \{s\}. \quad \varphi'_v \begin{cases} = \varphi_v - \delta \leq 0, & \text{if } v = t \\ = \varphi_v \leq 0, & \text{else} \end{cases} \tag{3}$$

$$f \text{ is st-flow} \Rightarrow \forall v \in V \backslash \{s, t\}. \quad \varphi'_v = \varphi_v = 0 \tag{4}$$

As mentioned before $f'(e) = f(e)$ for all $e = uw \in E$ such that neither uw nor wu are included in the path by the definition of f'. For all other edges consider the following two cases:

1. $uw \in E$
   Then $f'(uw) = f(uw) + \delta$. We know that $f(uw) \geq 0$ and $\delta \in \mathbb{N}$. Thus the sum is greater equal 0 too. Also $f(uw) + \delta \leq f(uw) + c'_{uw} = f(uw) + c_{uw} - f(uw) = c_{uw}$ by the definition of δ. Together we have $0 \leq f'(uw) \leq c_{uw}$.
2. $wu \in E$
   Then $f'(uw) = f(uw) - \delta$. We know that $f(uw) \leq c_{uw}$ and $\delta \in \mathbb{N}$. Thus the difference is smaller equal c_{uw} too. Also $f(uw) - \delta \geq f(uw) - c'_{uw} = f(uw) - f(uw) = 0$ by the definition of δ. Together we have $0 \leq f'(uw) \leq c_{uw}$.

Thus it generally holds that

$$\forall e \in E : 0 \leq f'_e \leq c_e \tag{5}$$

From 3 and 5 we can now conclude that

$$f \text{is st-preflow} \Rightarrow f' \text{is st-preflow}$$

$$f \text{is st-flow} \Rightarrow f' \text{is st-flow}$$

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Show that $\varphi'_s = \varphi_s + \delta$

Similar to the proof of 2, we know that $v_1 w_1 = sw_1$. Again, since p is a simple path, s does not appear in any other edge in p. Hence, f' remains the same for all edges involving s except sw_1 (or $w_1 s$ if $sw_1 \in E^-$) by definition. We consider 2 cases:

1. $sw_1 \in E$
   By definition of f', we have $f'(sw_1) = f(sw_1) + \delta$. Therefore $\varphi'_s = \varphi_s + \delta$.
2. $w_1 s \in E$
   By definition of f', we have $f'(w_1 s) = f(w_1 s) - \delta$. Therefore $\varphi'_s = \varphi_s + \delta$.

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2 Lemma 5

$$\begin{aligned}
\varphi_s &= \varphi_s + \varphi_{X\setminus\{s\}} && \text{using } \varphi_{X\setminus\{s\}} = 0 \text{ because } f \text{ is st-flow and } t \notin X \\
&= \varphi_X && \text{by definition of } \varphi_X \\
&= \varphi_X^+ - \varphi_X^- && \text{by definition of flux} \\
&\leq \varphi_X^+ && \text{using } \varphi_X^- \geq 0 \text{ because } \forall e \in E. f_e \geq 0 \\
&= \sum_{vw \in XX^c} f_{vw} && \text{by definition of } \varphi_X^+ \\
&\leq \sum_{vw \in XX^c} c_{vw} && \text{because } \forall e \in E. f_e \leq c_e
\end{aligned}$$

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3 Discussion of Lemma 5

If f is just an st-preflow, the lemma does not hold. That is the case because nodes in X between s and t can "absorb" flow (flux can be ≤ 0). Counter-example:
Consider the following network: $s \rightarrow^{10} v \rightarrow^{1} t$ where the numbers at the edges are the capacities. A valid st-preflow is specified by $f(sv) = 10, f(vt) = 1$ with $\varphi_v = -9 \leq 0$. Choose as cutset $X = \{s, v\}$. Then $\sum_{vw \in XX^c} c_{vw} = 1 < 10 = \varphi_s$.