Properties of the dot product

(i) $\vec{0} \cdot \vec{v} = \vec{v} \cdot \vec{0} = 0$

(ii) $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$ (Commutativity).

(iii) $(\lambda \vec{v}) \cdot \vec{w} = \vec{v} \cdot (\lambda \vec{w}) = \lambda (\vec{v} \cdot \vec{w})$

(iv) $\left.\begin{aligned} \vec{u} \cdot (\vec{v} + \vec{w}) &= \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \\ (\vec{v} + \vec{w}) \cdot \vec{u} &= \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u} \end{aligned}\right\}$

(Distributive law)

(v) $\vec{v} \cdot \vec{v} = \|\vec{v}\|^2$; relation with length.

The next theorem establises the relation between the dot product & the angle between two vectors. Hence the angle between two vectors is chosen to satisfy $0 \leq \theta \leq \pi$.

Theorem (Dot product & the Angle).

Let θ be the angle between two nonzero vectors $\vec{v}$ & $\vec{w}$. Then

$$\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta \quad \text{or}$$

$$\cos\theta = \frac{\vec{v}\cdot\vec{w}}{\|\vec{v}\|\,\|\vec{w}\|}$$

Proof:

The laws of cosines gives

$$\|\vec{v}-\vec{w}\|^2 = \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\cos\theta\,\|\vec{v}\|\,\|\vec{w}\| \quad (1)$$

Now

$$\|\vec{v}-\vec{w}\|^2 = (\vec{v}-\vec{w})\cdot(\vec{v}-\vec{w})$$
$$= \vec{v}\cdot\vec{v} - 2\vec{v}\cdot\vec{w} + \vec{w}\cdot\vec{w}$$
$$= \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\vec{v}\cdot\vec{w}$$

$\therefore$ Therefore by (1) $\Rightarrow$

$$\boxed{\vec{v}\cdot\vec{w} = \|\vec{v}\|\,\|\vec{w}\|\cos\theta.}$$

//

Ex. Find the angle θ between $\vec{v} = \langle 3, 6, 2 \rangle$ & $\vec{w} = \langle 4, 2, 4 \rangle$.

Sol^n

$\|\vec{v}\| = \sqrt{49} = 7$, $\|\vec{w}\| = \sqrt{36} = 6$.

$$\cos\theta = \frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\| \|\vec{w}\|} = \frac{\langle 3,6,2 \rangle \cdot \langle 4,2,4 \rangle}{7 \cdot 6}$$

$$= \frac{3 \cdot 4 + 6 \cdot 2 + 2 \cdot 4}{42} = \frac{32}{42} = \frac{16}{21}$$

$$\therefore \theta = \cos^{-1}(16/21)$$ //

Def^n (Perpendicular / Orthogonal).

Two non-zero vectors $\vec{v}$ & $\vec{w}$ are called perpendicular or orthogonal if the angle between them is $\pi/2$. We write $\vec{v} \perp \vec{w}$ if $\vec{v}$ & $\vec{w}$ are orthogonal. Thus we have

$$\vec{v} \perp \vec{w} \quad \text{if and only if} \quad \vec{v} \cdot \vec{w} = 0$$

Ex. Determine if $\vec{v} = \langle 2,6,1 \rangle$ is orthogonal to $\vec{u} = \langle 2,-1,1 \rangle$ or $\vec{w} = \langle -4,1,2 \rangle$

Sol^n

$\vec{v} \cdot \vec{u} = \langle 2,6,1 \rangle \cdot \langle 2,-1,1 \rangle = 2 \cdot 2 + 6(-1) + 1 \cdot 1$

$= -1$ (not orthogonal).

$\vec{v} \cdot \vec{w} = \langle 2,6,1 \rangle \cdot \langle -4,1,2 \rangle = 2(-4) + 6(1) + 1(2)$

$= 0$ (orthogonal)

(*) The cross product or vector product is used in physics and engineering to describe quantities involving rotation such as torque and angular momentum. In electro-magnetic theory, magnatic forces, etc...

(*) The cross product $\vec{v} \times \vec{w}$ is ~~agia~~ again a vector.

(*) It is defined using determinant,

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For 2x2

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

For 3x3

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$- a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Def^{n} (The Cross Product / Vector Product).

The cross product of vectors $\vec{v} = \langle a_1, b_1, c_1 \rangle$ & $\vec{w} = \langle a_2, b_2, c_2 \rangle$ is the vector

$$\vec{v} \times \vec{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

$$\vec{v} \times \vec{w} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \underline{i} - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \underline{j}$$

$$+ \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \underline{k}$$

__Ex__ Calculate $\vec{v} \times \vec{w}$, where $\vec{v} = \langle -2, 1, 4 \rangle$ & $\vec{w} = \langle 3, 2, 5 \rangle$.

__Sol__

$$\vec{v} \times \vec{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ -2 & 1 & 4 \\ 3 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \underline{i} - \begin{vmatrix} -2 & 4 \\ 3 & 5 \end{vmatrix} \underline{j}$$

$$+ \begin{vmatrix} -2 & 1 \\ 3 & 2 \end{vmatrix} \underline{k}$$

$$\cancel{= -3\underline{i} - (-22)\underline{j} + (-7)\underline{k}}$$

$$\cancel{\vec{v} \times \vec{w} = \langle -3, 22, -7 \rangle}$$

$$= \underline{i}(5 - 8) - \underline{j}(-10 - 12) + \underline{k}(-4 - 3).$$

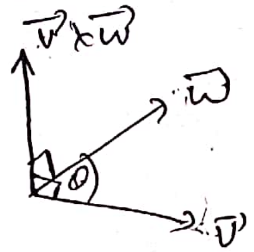
$$= \underline{i}(-3) - \underline{j}(-22) + \underline{k}(-7)$$

$$= -3\underline{i} + 22\underline{j} - 7\underline{k}$$

$$= \langle -3, 22, -7 \rangle$$

Note

(*) $\{\vec{v}, \vec{w}, \vec{v}\times\vec{w}\}$ forms a right handed system.



i.e. when the fingers of yours right hand curl from $\vec{v}$ to $\vec{w}$, your thumb points to the same side of the plane spaned by $\vec{v}$ & $\vec{w}$.

Theorem (Geometric Description of the Cross Product).

The cross product $\vec{v}\times\vec{w}$ is the unique vector with the following three properties:

(i) $\vec{v} \times \vec{w}$ is orthogonal to $\vec{v}$ & $\vec{w}$.

(ii) $\vec{v} \times \vec{w}$ has length $\|\vec{v}\| \|\vec{w}\| |\sin\theta|$,
θ = angle between $\vec{v}$ & $\vec{w}$.

(iii) $\{\vec{v}, \vec{w}, \vec{v} \times \vec{w}\}$ forms a right-handed system.

Basis Properties of the Cross Product

(i) $\vec{w} \times \vec{v} = -\vec{v} \times \vec{w}$

(ii) $\vec{v} \times \vec{v} = \vec{0}$

(iii) $\vec{v} \times \vec{w} = \vec{0}$ iff $\vec{w} = \lambda \vec{v}$ for some scalar λ or $\vec{v} = \vec{0}$

(iv) $(\lambda \vec{v}) \times \vec{w} = \vec{v} \times (\lambda \vec{w}) = \lambda (\vec{v} \times \vec{w})$.

(v) $(\vec{u} + \vec{v}) \times \vec{w} = (\vec{u} \times \vec{w}) + (\vec{v} \times \vec{w})$

$\vec{v} \times (\vec{u} + \vec{w}) = (\vec{v} \times \vec{u}) + (\vec{v} \times \vec{w})$

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Ex.

$\underline{i} \times \underline{j} = \underline{k}$, $\underline{j} \times \underline{k} = \underline{i}$, $\underline{k} \times \underline{j} = \underline{i}$

$\underline{i} \times \underline{i} = \underline{j} \times \underline{j} = \underline{k} \times \underline{k} = \vec{0}$.

Ex.

Compute $(2\underline{i} + \underline{k}) \times (3\underline{j} + 5\underline{k})$.

Sol.

$(2\underline{i} + \underline{k}) \times (3\underline{j} - 5\underline{k})$

$= (2\underline{i} \times 3\underline{j}) + (2\underline{i} \times 5\underline{k}) + (\underline{k} \times 3\underline{j})$

$\quad + (\underline{k} \times 5\underline{k})$

$= 6(\underline{i} \times \underline{j}) + 10(\underline{i} \times \underline{k}) + 3(\underline{k} \times \underline{j})$

$\quad + 5(\underline{k} \times \underline{k})$

$= 6\underline{k} - 10\underline{j} - 3\underline{i} + 5 \times \vec{0}$

$= \langle -3, -10, 6 \rangle$ //

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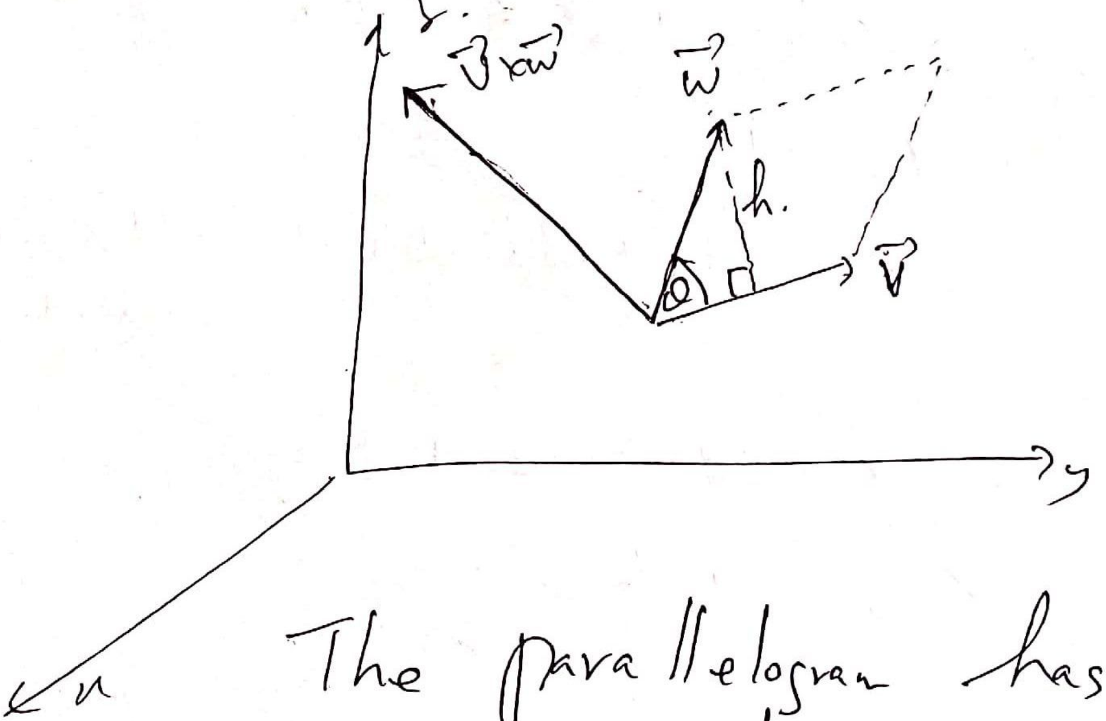
Relation between the Cross Product, Area and Volume

(i) **Area:**

Consider the parallelogram P spaned by non-zero vectors $\vec{v}$ & $\vec{w}$ with a common basepoint.



The parallelogram has a base $b = \|\vec{v}\|$ and height $h = \|\vec{w}\| \sin\theta$,

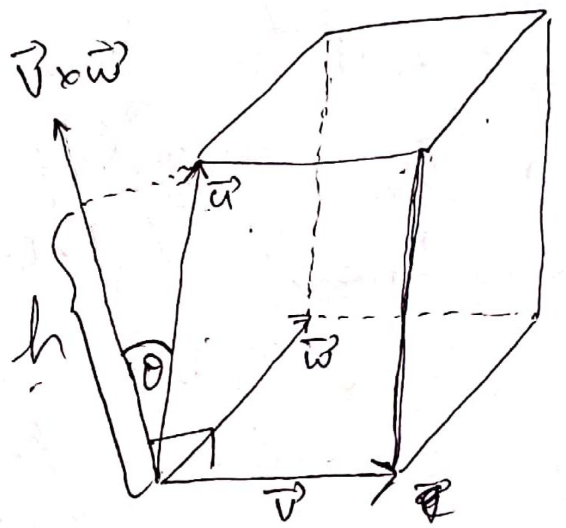
$\therefore$ Therefore area of $P = bh$

$$= \|\vec{v}\| \|\vec{w}\| \sin\theta.$$

$$= \|\vec{v} \times \vec{w}\|$$

(II) Volume.

Consider the parallelepiped P spaned by three non-zero vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$.



The base of P is the parallelogram spaned by $\vec{v}$ & $\vec{w}$, so the area of the base is $\|\vec{v} \times \vec{w}\|$. The height of P is, $h = \|\vec{u}\| \cdot |\cos\theta|$ where θ is the angle between $\vec{u}$ & $\vec{v} \times \vec{w}$.

$\therefore$ Volume of P = (area of base) $\times$ height

$$= \|\vec{v} \times \vec{w}\| \cdot \|\vec{u}\| \cdot |\cos\theta|$$

$$= \left| \vec{u} \cdot (\vec{v} \times \vec{w}) \right|$$

Note

(*) The quantity $\vec{u} \cdot (\vec{v} \times \vec{w})$, called the <u>vector triple product</u>, can be expressed as a following way. Let $\vec{u} = \langle a_1, b_1, c_1 \rangle$, $\vec{v} = \langle a_2, b_2, c_2 \rangle$ & $\vec{w} = \langle a_3, b_3, c_3 \rangle$. Then.

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$