

# Transfer function

(1)

Transfer function for a single input single output system.

The transfer function is defined as the ratio of the Laplace transform of the output to the Laplace transform of input with all initial conditions are zero.

Consider a linear system having input  $r(t)$  &  $c(t)$  is the output of the system.

The input-output relationship can be described by the following  $n$ th order differential equation.

$$a_n \frac{d^n c(t)}{dt^n} + a_{n-1} \frac{d^{n-1} c(t)}{dt^{n-1}} + \dots + a_1 \frac{dc(t)}{dt} + a_0 c(t) = b_m \frac{d^m r(t)}{dt^m} + b_{m-1} \frac{d^{m-1} r(t)}{dt^{m-1}} + \dots + b_1 \frac{dr(t)}{dt} + b_0 r(t)$$

we can define the transfer function as,

$$G(s) = \frac{C(s)}{R(s)}$$

$G(s)$  = So taking the Laplace transform, we get,

$$(a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0) C(s) = (b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0) R(s)$$

$$\therefore G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

In the equation, if the order of the denominator polynomial is greater than the order of the numerator polynomial, then the transfer function is said to be STRICTLY PROPER.

If the order of both polynomials are same, then transfer function is said to be PROPER.

The transfer function is said to be IMPROPER, if the order of the numerator polynomial is greater than the order of the denominator polynomial.

### Procedure

The following steps are involved to obtain the transfer function of a given system.

Step 1: Write the differential equations for the given system.

Step 2: Take the Laplace transform of the equations obtained in Step 1, with all initial conditions are set to zero.

Step 3: Take the ratio of transformed output to input.

Step 4: The ratio of transformed output to the input, in Step 3 is the required transfer function.

(3)

## Characteristic equation of a transfer function

The characteristic equation of a linear system can be obtained by equating the denominator polynomial of the transfer function to zero.

$$\therefore a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0 = 0.$$

## Poles and zeros of a transfer function.

Consider the transfer function,  $G(s) = C(s)/R(s)$ .

If the numerator & denominator can be factored as below,

$$\frac{C(s)}{R(s)} = G(s) = \frac{k(s+z_1)(s+z_2)(as^2+bs+c)}{(s+p_1)(s+p_2)(As^2+Bs+C)}.$$

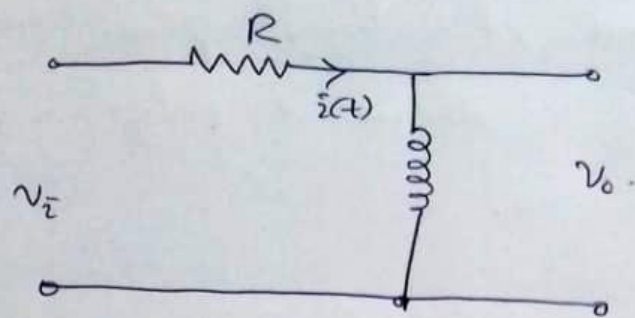
Poles: The poles of  $G(s)$  are those values of  $s$  which make  $G(s)$  tend to infinity,

$$s = -p_1, s = -p_2, \text{ and } s = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

Zeros: Zeros of  $G(s)$  are those values of  $s$  which make  $G(s)$  to zero.

$$\text{eg: } s_1 = -z_1, s_2 = -z_2, s = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Find the transfer function of the given network. (9)



Apply KVL.

$$v_i = Ri + L \frac{di}{dt} \quad \text{--- (1)}$$

$$v_o = L \frac{di}{dt} \quad \text{--- (2)}$$

By taking the Laplace transform on equations (1) & (2), and with assumptions that all initial conditions are zero,

$$V_i(s) = R I(s) + s L I(s)$$

$$V_o(s) = s L I(s)$$

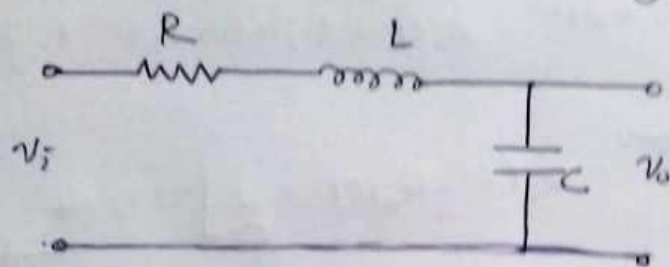
Calculating the transfer function;

$$\frac{V_o(s)}{V_i(s)} = \frac{s L I(s)}{(R + L s) I(s)}$$

$$\frac{V_o(s)}{V_i(s)} = \frac{s L}{(R + L s)}$$

(5)

Determine the transfer function of the electrical network shown in the following figure.



Apply KVL,

$$v_i(t) = iR + L \frac{di}{dt} + \frac{1}{C} \int i dt$$

$$v_o(t) = \frac{1}{C} \int i dt$$

$$V_i(s) = RI(s) + sLI(s) + \frac{1}{sC} I(s)$$

$$= I(s) \left[ R + sL + \frac{1}{sC} \right] = I(s) \left[ \frac{RCs + s^2LC + 1}{s} \right]$$

$$V_o(s) = \frac{1}{sC} I(s) \quad \text{--- (2)}$$

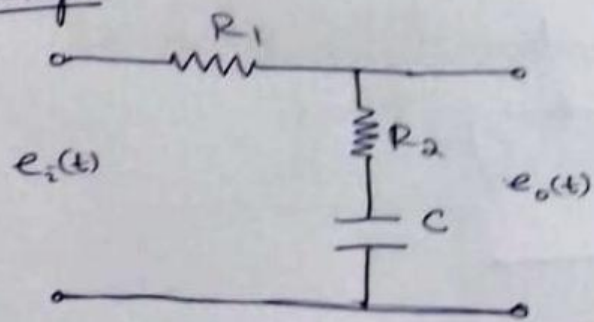
Determine the transfer function,

$$\frac{V_o(s)}{V_i(s)} = \frac{I(s)}{sC} \cdot \frac{Cs}{I(s)[s^2LC + sRC + 1]}$$

$$= \frac{1}{s^2LC + sRC + 1}$$

example

(6)



Apply KVL,

$$e_i(t) = R_1 i(t) + R_2 i(t) + \frac{1}{C} \int i(t) dt \quad (1)$$

$$e_o(t) = R_2 i(t) + \frac{1}{C} \int i(t) dt \quad (2)$$

Apply Laplace transform on eq<sup>n</sup>s (1) & (2),

$$E_i(s) = \left[ R_1 + R_2 + \frac{1}{sC} \right] I(s)$$

$$E_o(s) = \left[ R_2 + \frac{1}{sC} \right] I(s)$$

Calculating transfer function,

$$\begin{aligned} \frac{E_o(s)}{E_i(s)} &= \frac{\left[ R_2 + \frac{1}{sC} \right] I(s)}{\left[ \frac{R_1 Cs + R_2 Cs + 1}{Cs} \right] I(s)} \\ &= \frac{1 + R_2 Cs}{1 + R_1 Cs + R_2 Cs} \end{aligned}$$

## Mechanical Systems

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Mechanical Systems can be categorized into two systems based on the type of motion.

① Translational Systems: motion takes place along a straight line.

② Rotational system: motion takes place about a fixed axis.

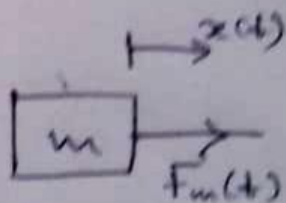
### Translational Systems

① Inertia force: Consider a body of mass 'm' & acceleration of 'a', then according to Newton's second law of motion, the inertia force is calculated by,

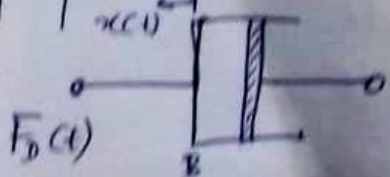
$$F_m(t) = m a(t)$$

$$= m \frac{dv(t)}{dt}$$

$$= m \frac{d^2 x(t)}{dt^2}$$



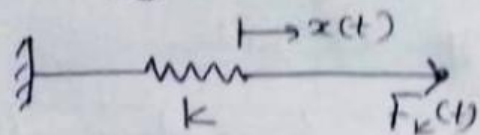
② Damping force: For viscous friction, the damping force is proportional to the velocity.



$$F_D(t) = B v(t) = B \frac{dx(t)}{dt}$$

B: damping coefficient.

② Spring Force: The spring force stores the potential energy.



$$F_k(t) \propto x(t) = kx(t)$$

$k$ : Spring constant or stiffness.

## Rotational System

There are three types of torques resist the rotational motion.

① Inertia torque: Let the inertia torque  $T$  and  $J$  the moment of inertia, and  $\ddot{\theta}(t)$ : the angular acceleration. Then,

$$\begin{aligned} T(t) &= J \ddot{\theta}(t) \\ &= J \frac{d}{dt} \dot{\theta}(t) \\ &= J \frac{d^2}{dt^2} \theta(t). \end{aligned}$$

② Damping torque: The damping torque  $T_D(t)$  can be written,

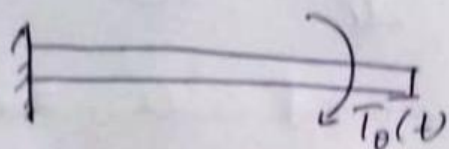
$$T_D(t) = B\omega(t) = B \frac{d}{dt} \theta(t).$$

$B$ : damping coefficient.

## Spring torque

Spring torque  $T_\theta(t)$  is the product of torsional stiffness and angular displacement.

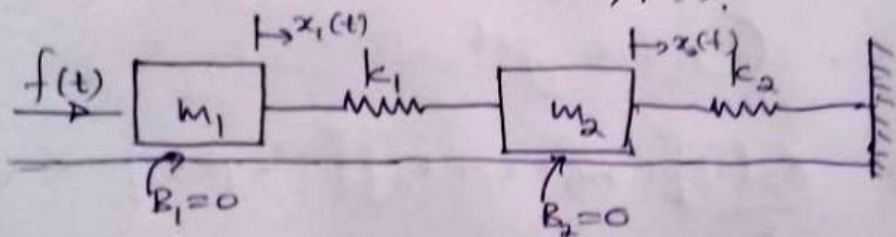
$$T_\theta(t) = k \theta(t).$$



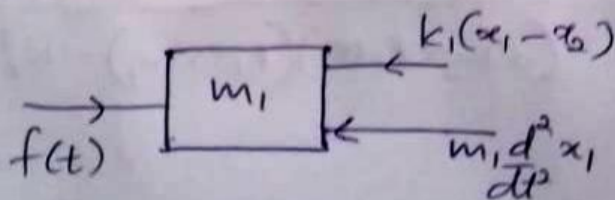
$$T_\theta(t) = k \theta(t).$$

Example: Write the differential equations describing the dynamics of the system shown below.

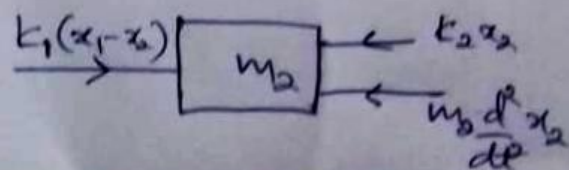
Find the ratio  $X_2(s)/F(s)$ .



Free body diagram for mass  $m_1$ ,



Free body diagram for mass  $m_2$



For mass  $m_1$ ,

$$f(t) = k_1(x_1 - x_2) + m_1 \frac{d^2 x_1}{dt^2} \quad \text{--- (1)}$$

for mass  $m_2$

$$k_1(x_1 - x_2) - k_2 x_2 = m_2 \frac{d^2 x_2}{dt^2} \quad \text{--- (2)}$$

Take Laplace transform, assume initial conditions zero,

$$F(s) = k_1 X_1(s) - k_1 X_2(s) + m_1 s^2 X_1(s) \quad \text{--- (3)}$$

$$k_1 X_1(s) - k_1 X_2(s) - k_2 X_2(s) = m_2 s^2 X_2(s) \quad \text{--- (4)}$$

$$\text{then } X_1(s) = \frac{X_2(s)}{k_1} [s^2 m_2 + k_1 + k_2] \quad \text{--- (5)}$$

put (5) in (3) we get,

$$F(s) = \frac{X_2(s)}{k_1} [s^2 m_2 + k_1 + k_2] [s^2 m_1 + k_1] - k_1 X_2(s)$$

$$\therefore \frac{X_2(s)}{F(s)} = \frac{k_1}{(s^2 m_2 + k_1 + k_2)(k_1 + s^2 m_1) - k_1^2}$$

## Block Diagram

(1)

Any system can be represented by a set of differential equations or can be represented by a schematic diagram containing all components and their connectors.

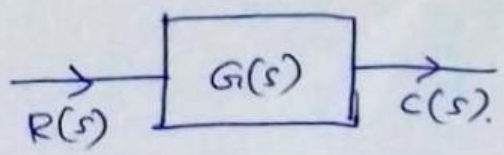
But for complicated systems these two methods are not suitable.

The block diagram representation is the combination of above two methods.

A block may represent a single component or a group of components, but each block is completely characterised by a transfer function.

The transfer function is an expression which relates output to input in  $s$ -domain. Transfer function does not give any information about the internal structure of the system.

Block diagrams are single line diagrams, that is the flow of system variables from one block to another block is represented by a single line.



$R(s)$  : input .

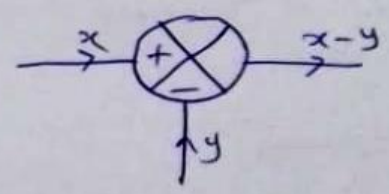
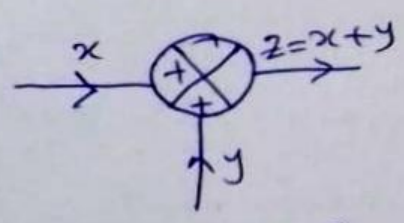
$C(s)$  = : Output

$G(s)$  : transfer function.

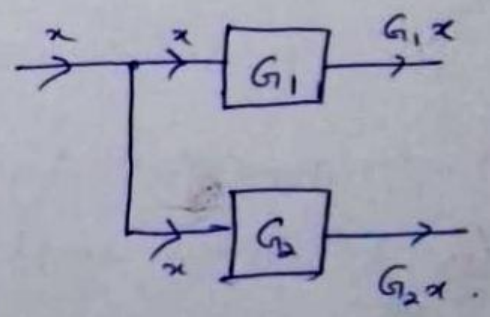
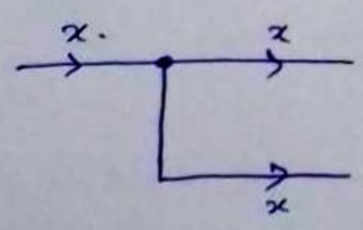
Then,  $\frac{C(s)}{R(s)} = G(s) \Leftrightarrow C(s) = G(s) \cdot R(s)$ .

In addition to this, the sum of the signals or the difference of the signals are represented by a summing point.

Application of one source to two or more blocks is represented by a take off point.

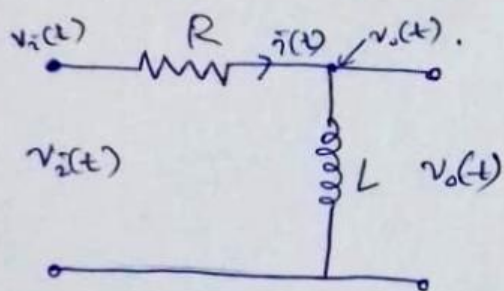


Summing point.



# How to draw the block diagram

(3)



Apply KVL

$$v_i(t) = R i(t) + L \frac{di(t)}{dt}$$

$$v_o(t) = L \frac{di(t)}{dt}$$

Apply Laplace transform with initial conditions zero,

$$V_i(s) = I(s)R + sL I(s)$$

$$= (R + sL) I(s)$$

$$V_o(s) = sL I(s)$$

from the diagram

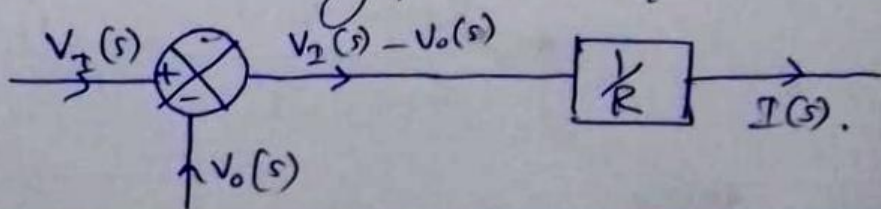
$$i(t) = \frac{v_i(t) - v_o(t)}{R}$$

$$v_o = L \frac{di(t)}{dt}$$

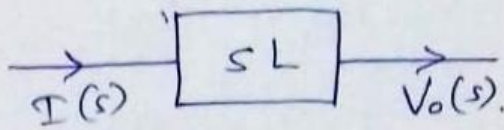
$$I(s) = \frac{1}{R} [V_i(s) - V_o(s)] \quad \text{--- (1)}$$

$$V_o(s) = sL I(s) \quad \text{--- (2)}$$

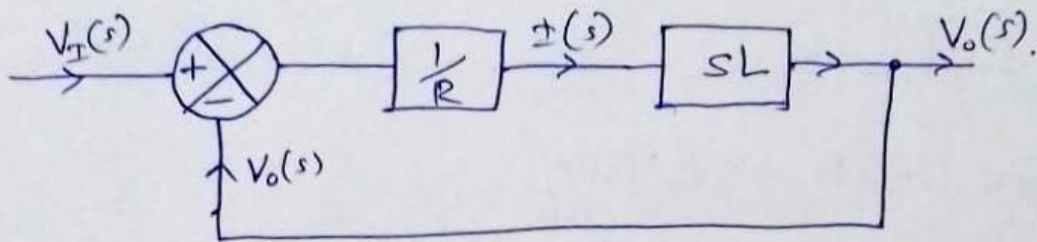
we can use a summing point for equation (1),



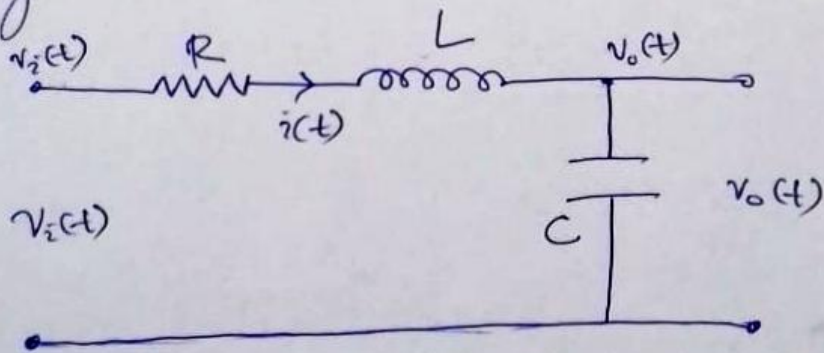
From equation (2), we can draw a block diagram,



Combining both diagrams,



example: Draw the block diagram of RCL circuit given below.



$$v_i(t) - v_o(t) = i(t)R + L \frac{di(t)}{dt}$$

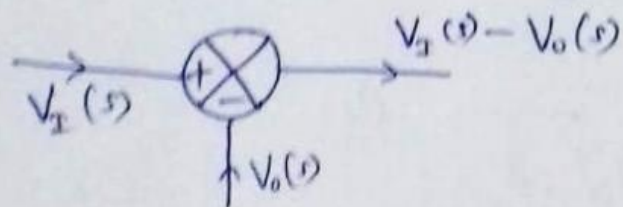
in Laplacian,

$$I(s) = \frac{V_i(s) - V_o(s)}{(R + sL)} \quad \text{--- (1)}$$

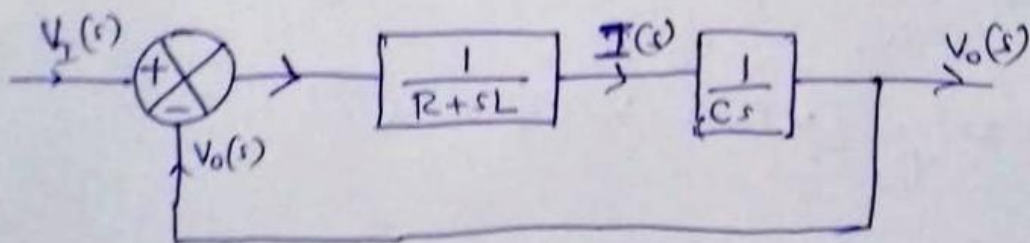
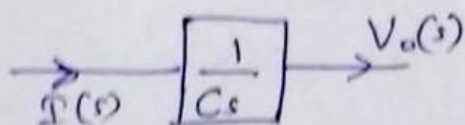
$$V_o(s) = \frac{1}{Cs} T(s) \quad \text{--- (2)}$$

(5)

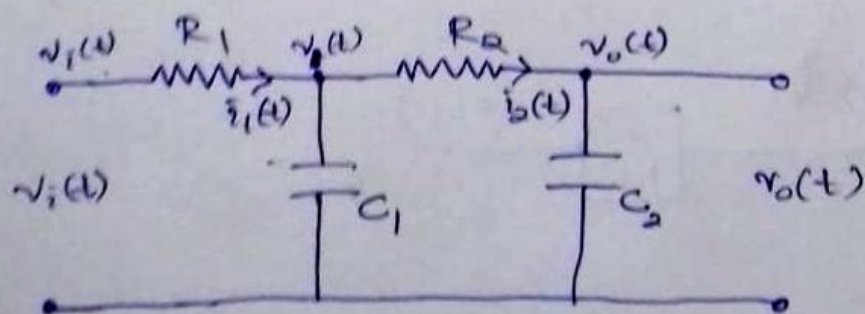
Equation (1) as a summing point,



Equation (2) as a block diagram,



Draw the block diagram of the circuit shown,



$$i_1(t) = \frac{v_2(t) - v_1(t)}{R_1} \quad \text{--- (1)}$$

$$v_1(t) = \frac{1}{C_1} \int [i_1(t) - i_2(t)] dt \quad \text{--- (2)}$$

$$i_2(t) = \frac{1}{R_2} [v_1(t) - v_o(t)] \quad \text{--- (3)}$$

$$v_o(t) = \frac{1}{C_2} \int i_2 dt \quad \text{--- (4)}$$

Take Laplace on above, we get,

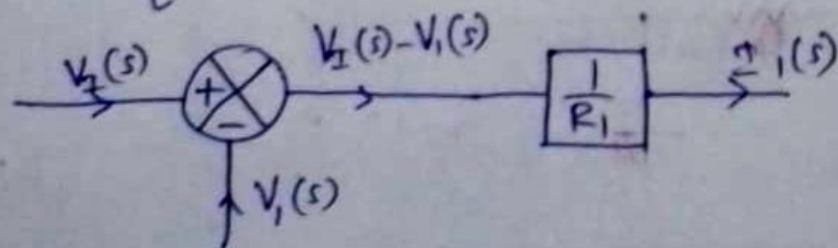
$$I_1(s) = \frac{1}{R_1} [V_1(s) - V_o(s)] \quad \text{--- (5)}$$

$$V_1(s) = \frac{1}{sC_1} [I_1(s) - I_2(s)] \quad \text{--- (6)}$$

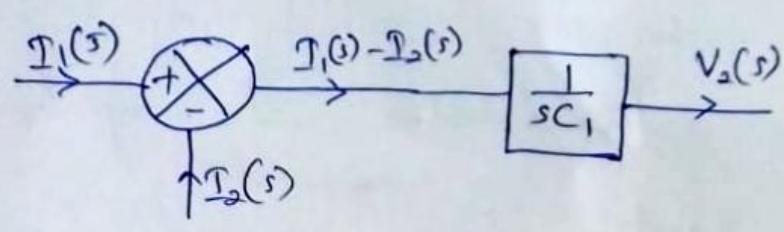
$$I_2(s) = \frac{1}{R_2} [V_1(s) - V_o(s)] \quad \text{--- (7)}$$

$$V_o(s) = \frac{1}{sC_2} I_2(s) \quad \text{--- (8)}$$

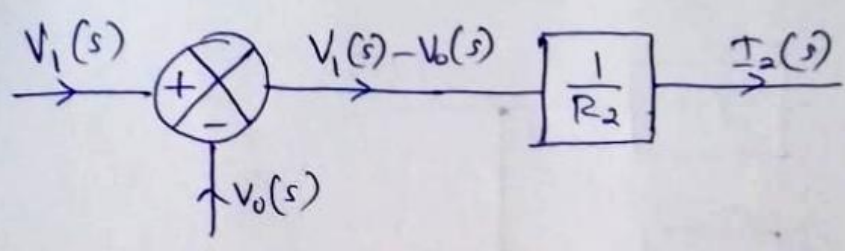
from eq<sup>n</sup> (5),



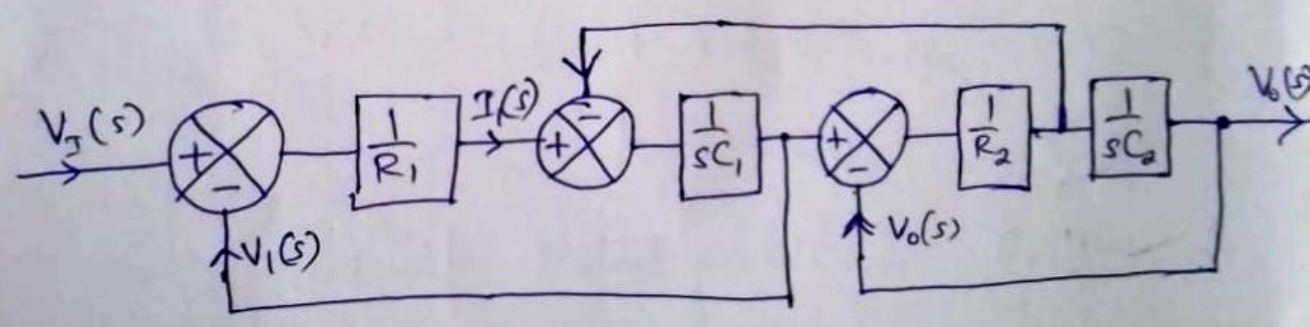
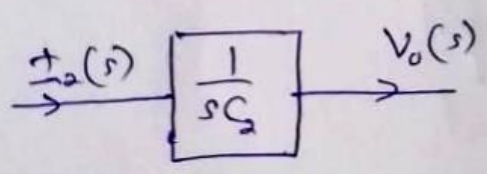
from eq<sup>n</sup> (6),



from eq<sup>n</sup> (7),



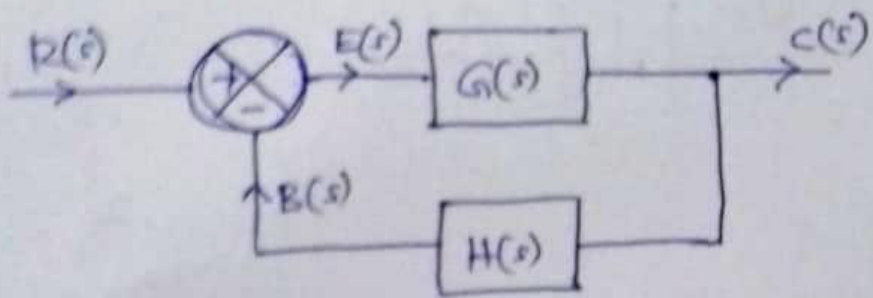
from eq<sup>n</sup> (8),



# Closed loop Control System

A closed loop system is one which output is fed back into an error detector and compared with the reference input.

The feedback may be negative or positive.



$R(s)$  : Reference input.

$E(s)$  : Error signal

$G(s)$  : forward path transfer function.

$C(s)$  : Output signal

$H(s)$  : feedback transfer function.

$B(s)$  : Feedback signal.

$$C(s) = G(s)E(s) \quad \text{--- (1)}$$

$$B(s) = H(s) \cdot C(s) \quad \text{--- (2)}$$

$$E(s) = R(s) - B(s) \quad \text{--- (3)}$$

① in ②,  $B(s) = H(s) G(s) E(s)$ .

$$\frac{B(s)}{E(s)} = G(s) H(s)$$

$$\begin{aligned} C(s) &= G(s) [R(s) - B(s)] \\ &= R(s) G(s) - G(s) B(s) \\ &= R(s) G(s) - G(s) H(s) C(s) \end{aligned}$$

Then  $C(s) [1 + G(s) H(s)] = R(s) G(s)$

$$C(s) = \left[ \frac{G(s)}{1 + G(s) H(s)} \right] R(s),$$

So  $\frac{C(s)}{R(s)}$  : closed loop transfer function

and that is equal to  $\frac{G(s)}{1 + G(s) H(s)}$ .

If the feedback is positive, then,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 - G(s) H(s)}$$

$$\frac{E(s)}{R(s)} : \text{error ratio} = \frac{1}{1 + G(s)H(s)}$$

for positive feedback,

$$\frac{E(s)}{R(s)} = \frac{1}{1 - G(s)H(s)}.$$

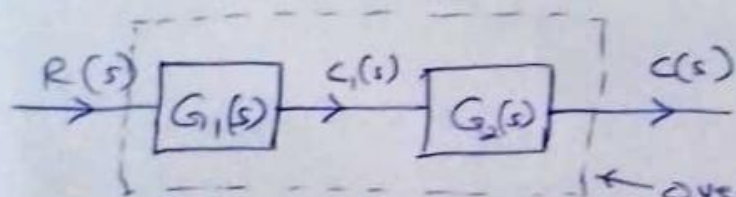
## Block Diagram Reduction

①

When a number of blocks are connected, the overall transfer function can be obtained by block diagram reduction.

For block diagram reduction following ~~techniques~~ rules should be followed.

Rule 1: Blocks in cascade,



← overall block representation,  
and the transfer function is  $G(s)$ .

$$\frac{C_1(s)}{R(s)} = G_1(s) \quad \text{--- (1)}$$

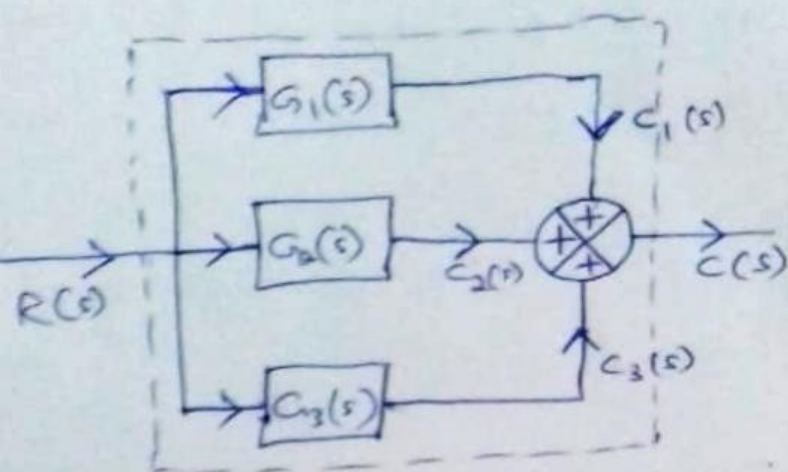
$$\frac{C(s)}{C_1(s)} = G_2(s) \quad \text{--- (2)}$$

$$\textcircled{1} \times \textcircled{2}, \quad \frac{C(s)}{C_1(s)} \times \frac{C_1(s)}{R(s)} = G_1(s) G_2(s)$$

$$\frac{C(s)}{R(s)} = G_1(s) G_2(s)$$

$$\therefore G(s) = G_1(s) G_2(s) \quad \rightarrow \quad \boxed{G_1(s) G_2(s)} \quad \rightarrow \quad C(s)$$

(2)

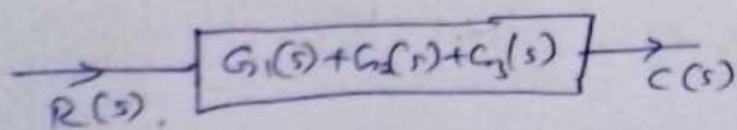
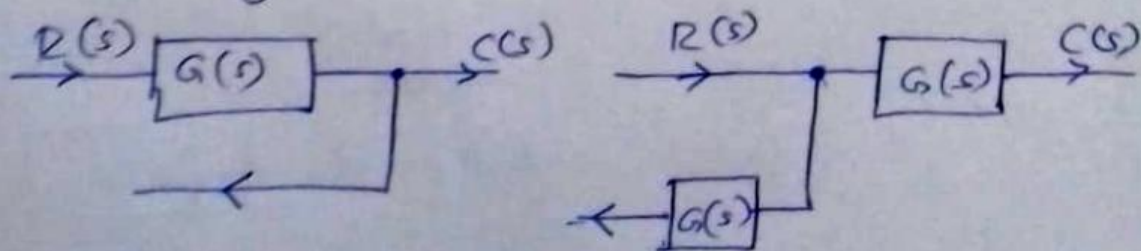
Rule 2: Blocks in Parallel

$$C(s) = C_1(s) + C_2(s) + C_3(s)$$

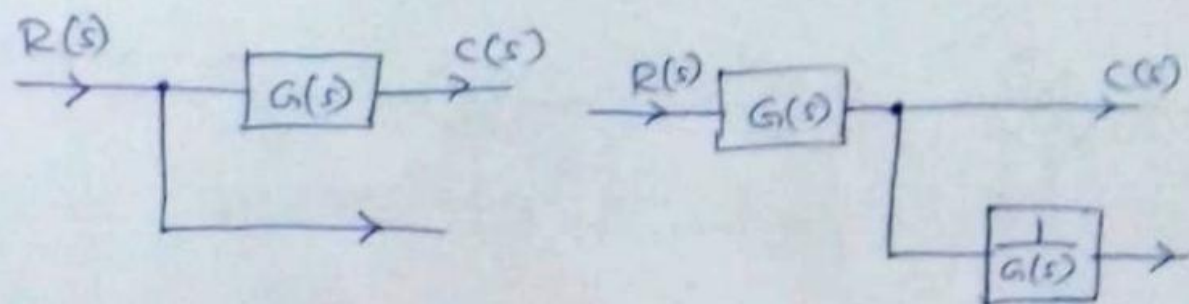
$$= R(s)G_1(s) + R(s)G_2(s) + R(s)G_3(s)$$

$$= [G_1(s) + G_2(s) + G_3(s)] R(s)$$

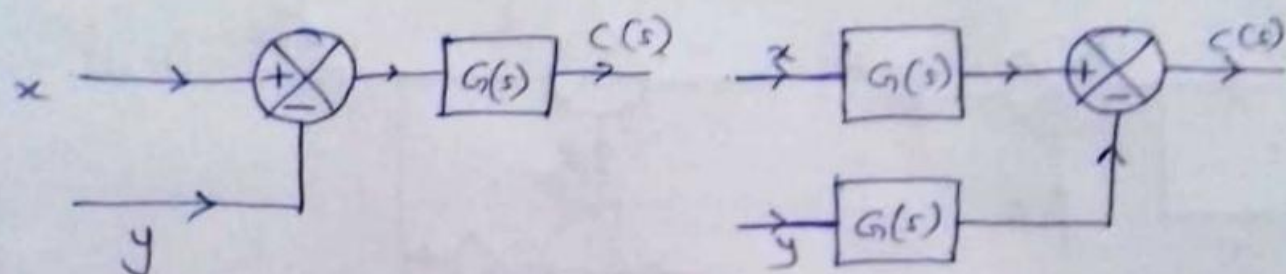
$$\frac{C(s)}{R(s)} = G(s) = [G_1(s) + G_2(s) + G_3(s)]$$

Rule 3: Moving a takeoff point ahead of a block.

Rule 4: Moving a take off point after the block. (3)



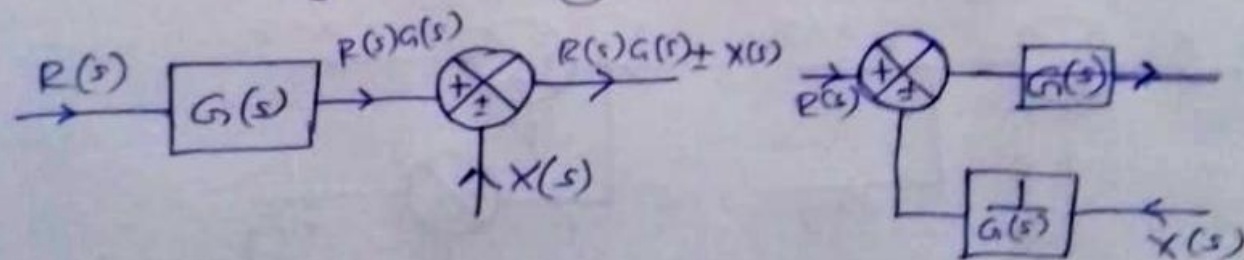
Rule 5: Moving a summing point beyond a block.



$$C(s) = G(s)[x - y]$$

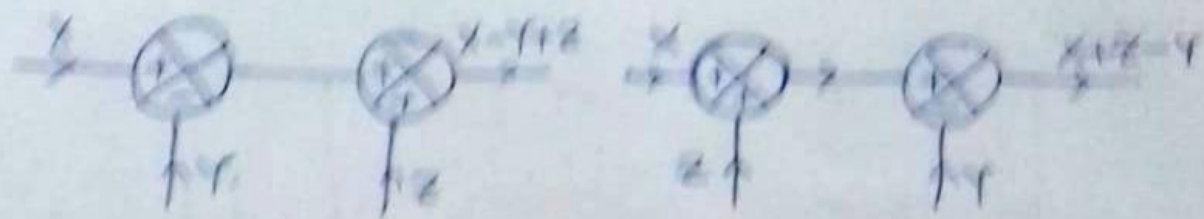
$$\hookrightarrow G(s)x - G(s)y$$

Rule 6: Moving a summing point ahead of a block.

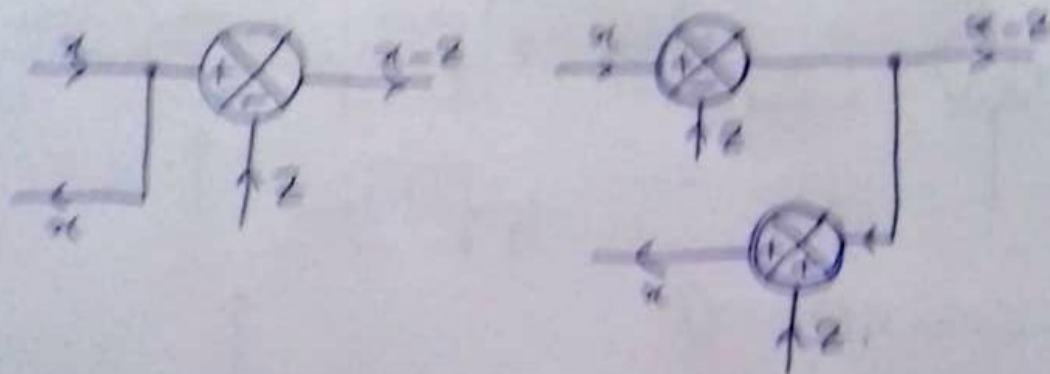


Rule 7: Interchange two summing points.

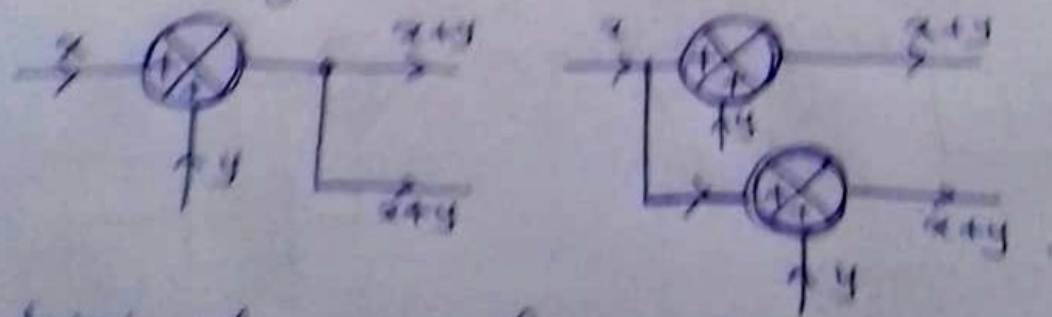
(4)



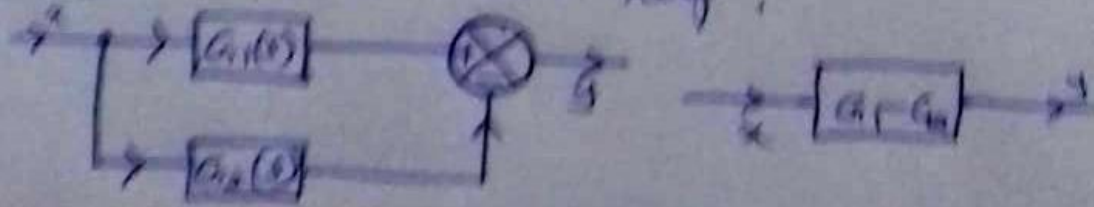
Rule 8: Moving a takeoff point beyond a summing point.



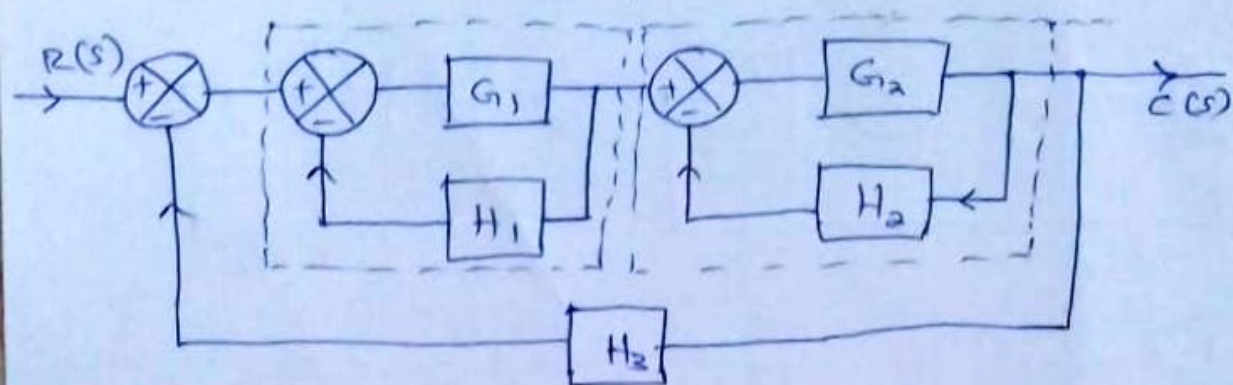
Rule 9: Moving a takeoff point ahead of a summing point.



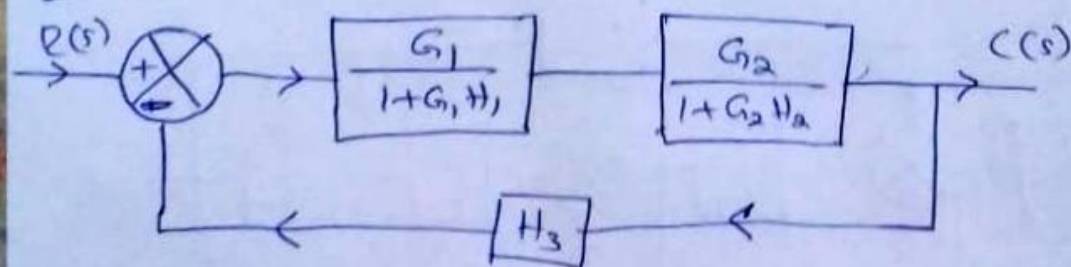
Rule 10: Eliminate a forward loop.



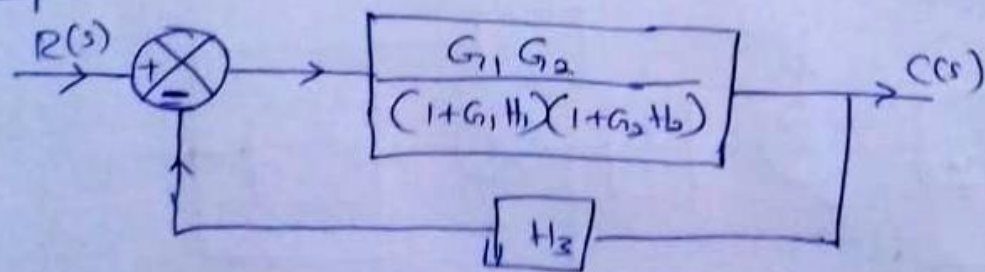
Derive the transfer function using block diagram reduction



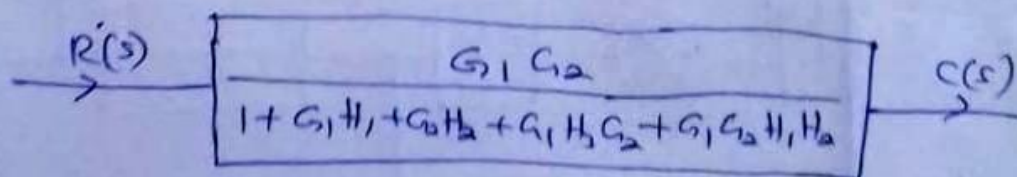
Step 1



Step 2



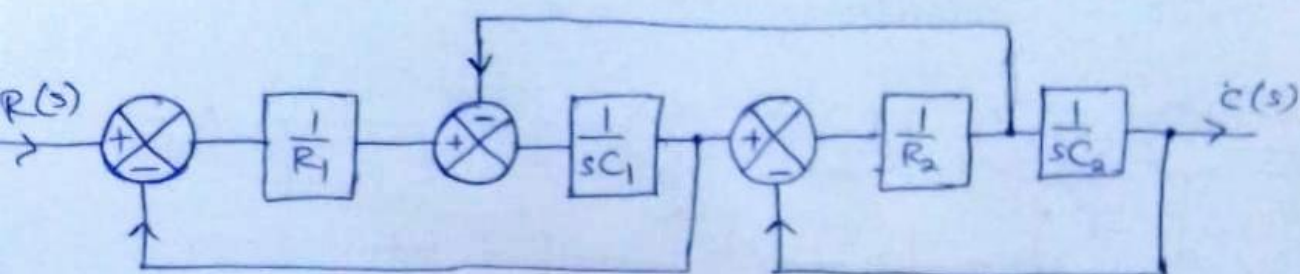
Step 3



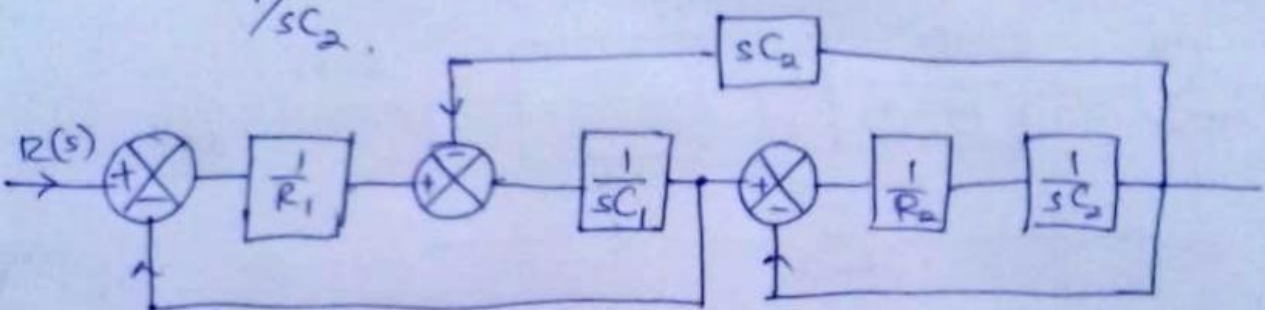
$$\therefore \frac{C(s)}{R(s)} = \frac{G_1 G_2}{1 + G_1 H_1 + G_2 H_2 + G_1 H_2 G_2 + G_1 G_2 H_1 H_2}$$

Find overall transfer function

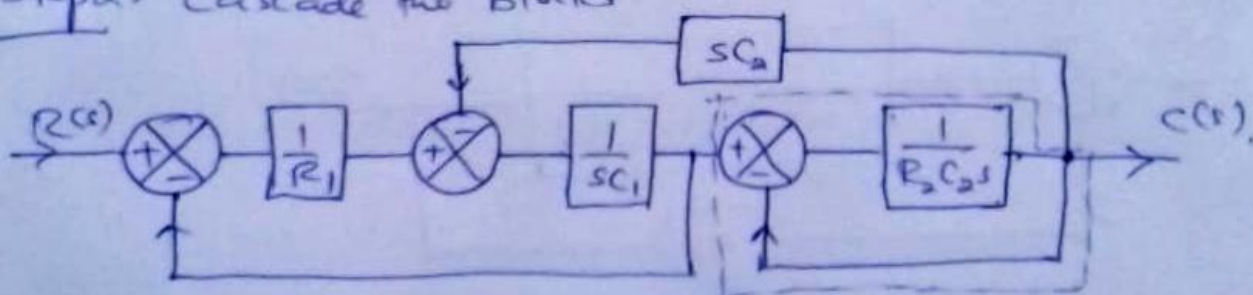
6



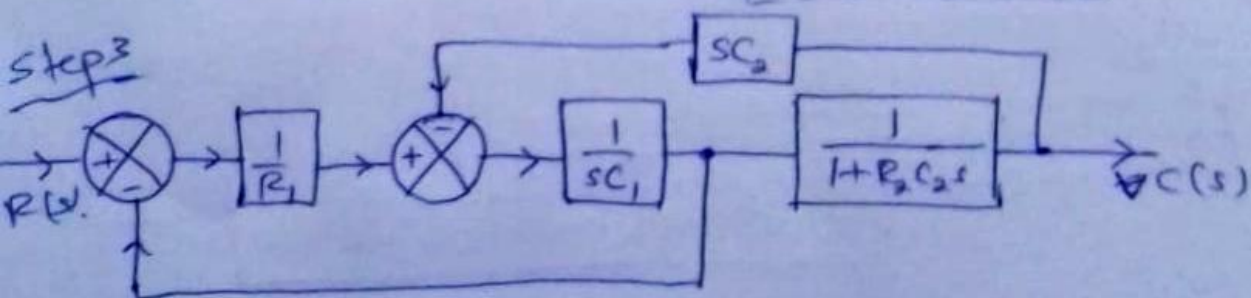
Step 1: shift the take off point beyond the block  $\frac{1}{sC_2}$ .



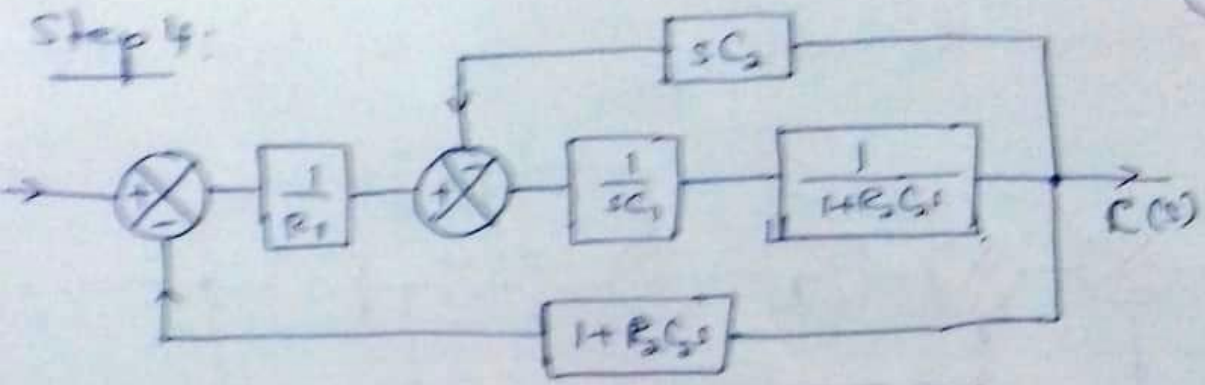
Step 2: Cascade two blocks



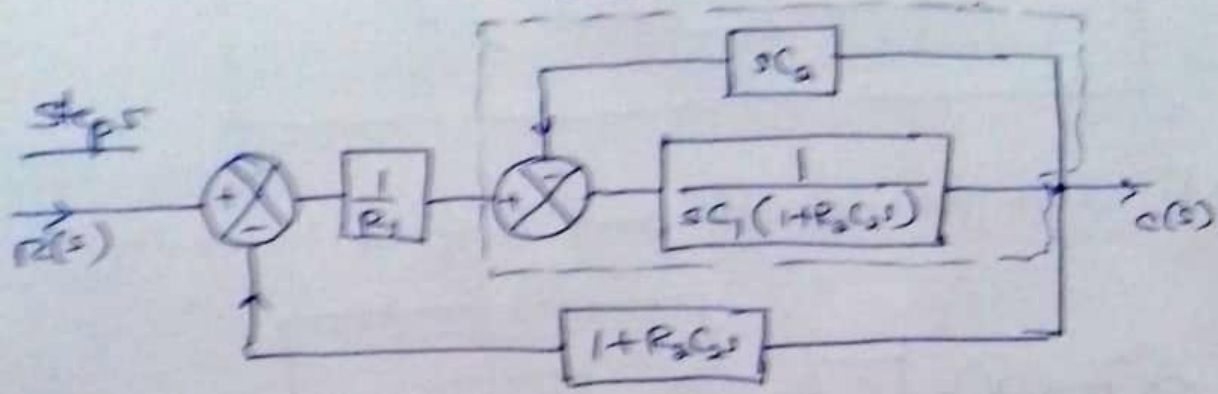
Step 3



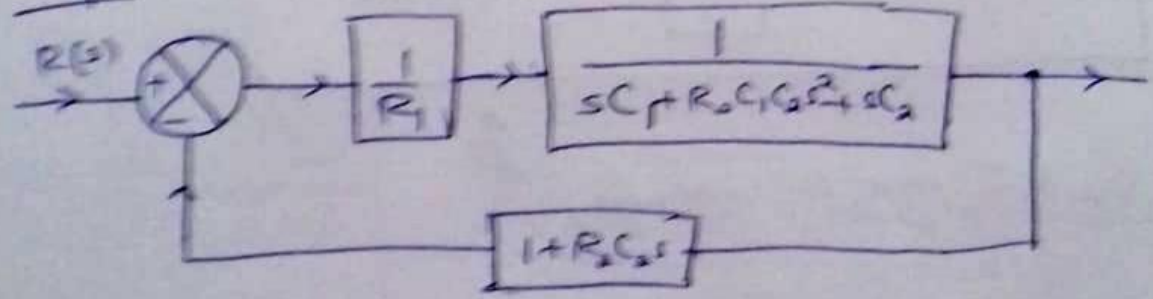
Step 4:



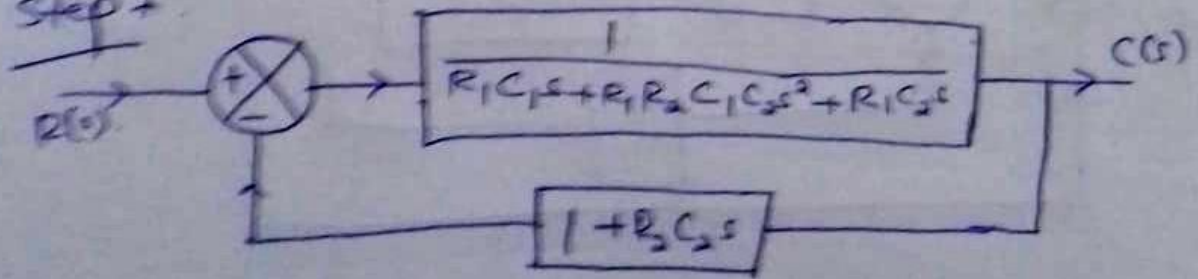
Step 5:



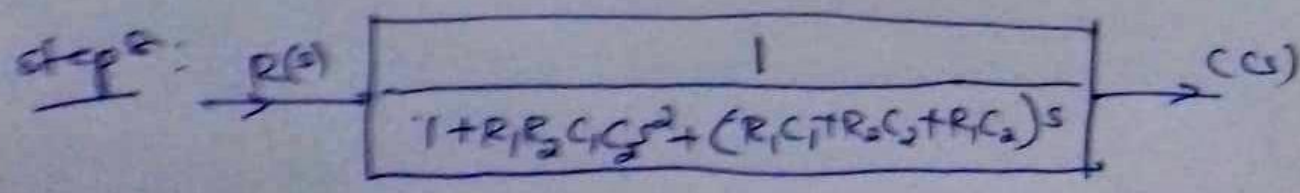
Step 6:



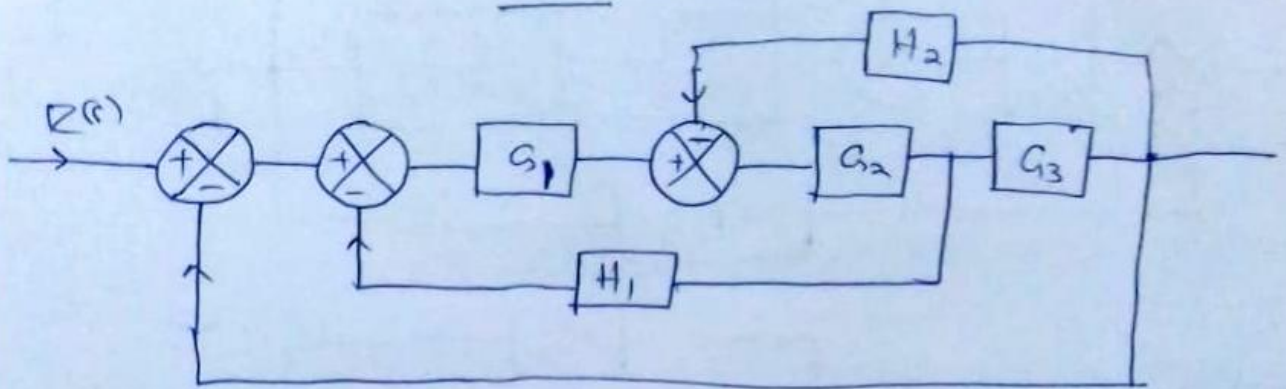
Step 7:



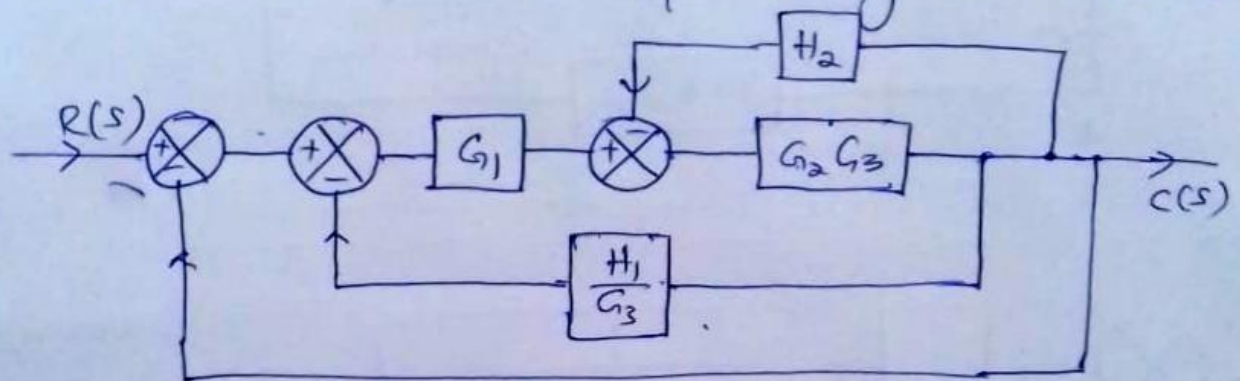
Step 8:



Determine the ratio  $\frac{C(s)}{R(s)}$  for the system shown (8)

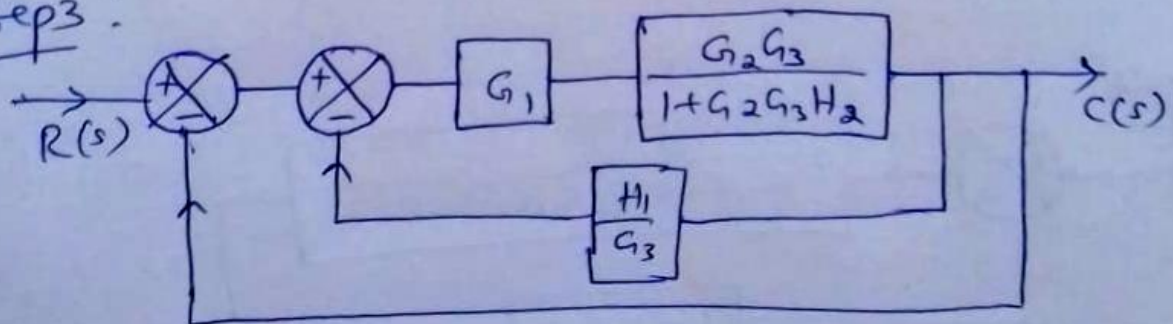


Step 1: Shift the takeoff point beyond block  $G_3$ .

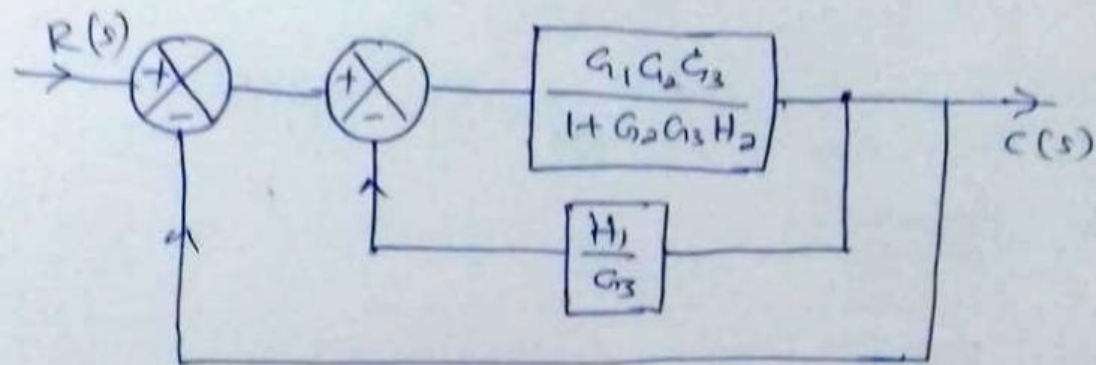
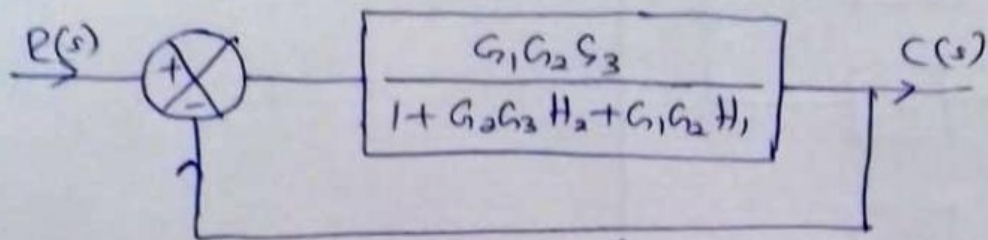
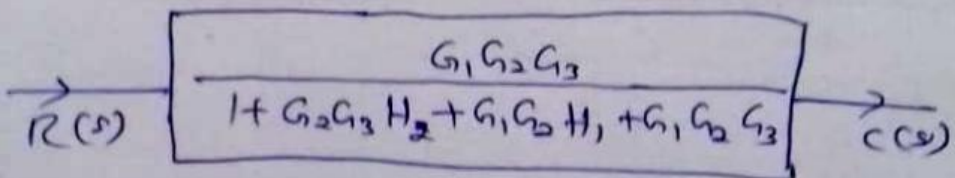


Step 2:  $G_2$  &  $G_3$  are in cascade,

Step 3:



(9)

Step 4Step 5Step 6:

$$\therefore \frac{C(s)}{R(s)} = \frac{G_1 G_2 G_3}{1 + G_2 G_3 H_2 + G_1 G_2 H_1 + G_1 G_2 G_3}$$

Determine the transfer functions,

$\frac{C_1}{R_1}$ ,  $\frac{C_2}{R_2}$ ,  $\frac{C_1}{R_2}$  and  $\frac{C_2}{R_1}$  from the block

diagram shown.

