
Chapter - 01

Differentiation and Integration

Independent Study Material No. 02

Calculating Limits Using Limit Laws

Suppose that c is a constant and the limits $\lim_{x \rightarrow a} f(x)$ and $\lim_{x \rightarrow a} g(x)$ exist. Then,

1. Sum Law:

$$\lim_{x \rightarrow a} [f(x) + g(x)] = \lim_{x \rightarrow a} f(x) + \lim_{x \rightarrow a} g(x)$$

2. Difference Law:

$$\lim_{x \rightarrow a} [f(x) - g(x)] = \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} g(x)$$

3. Constant Multiple Law:

$$\lim_{x \rightarrow a} [cf(x)] = c \lim_{x \rightarrow a} f(x)$$

4. Product Law:

$$\lim_{x \rightarrow a} [f(x)g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$$

5. Quotient Law:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$$

if $\lim_{x \rightarrow a} g(x) \neq 0$

6. Power Law:

$$\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$$

where n is a positive integer

7. $\lim_{x \rightarrow a} c = c$

8. $\lim_{x \rightarrow a} x = a$

9. $\lim_{x \rightarrow a} x^n = a^n$ where n is a positive integer

10. $\lim_{x \rightarrow a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer (If n is even, we assume that $a > 0$.)

11. Root Law:

$$\lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)}$$

where n is a positive integer (If n is even, we assume that $\lim_{x \rightarrow a} f(x) > 0$.)

Illustrative Example 01 Evaluate the following limits and justify each step.

$$(i) \lim_{x \rightarrow 5} (2x^2 - 3x + 4)$$

$$(ii) \lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x}$$

Answer:

(i)

$$\begin{aligned}\lim_{x \rightarrow 5} (2x^2 - 3x + 4) &= \lim_{x \rightarrow 5} (2x^2) - \lim_{x \rightarrow 5} (3x) + \lim_{x \rightarrow 5} (4) \text{ (by Law 2 and 1)} \\ &= 2 \lim_{x \rightarrow 5} (x^2) - 3 \lim_{x \rightarrow 5} (x) + \lim_{x \rightarrow 5} 4 \text{ (by Law 3)} \\ &= 2(5)^2 - 3(5) + 4 \text{ by 9,8, and 7} \\ &= 39\end{aligned}$$

(ii)

$$\begin{aligned}\lim_{x \rightarrow -2} \frac{x^3 + 2x^2 - 1}{5 - 3x} &= \frac{\lim_{x \rightarrow -2} (x^3 + 2x^2 - 1)}{\lim_{x \rightarrow -2} (5 - 3x)} \text{ (by Law 5)} \\ &= \frac{\lim_{x \rightarrow -2} x^3 + 2 \lim_{x \rightarrow -2} x^2 - \lim_{x \rightarrow -2} 1}{\lim_{x \rightarrow -2} 5 - 3 \lim_{x \rightarrow -2} x} \text{ (by Law 1,2, and 3)} \\ &= \frac{(-2)^3 + 2(-2)^2 - 1}{5 - 3(-2)} \text{ by 9,8, and 7} \\ &= \frac{-1}{11}\end{aligned}$$

Direct Substitution Property

If f is a polynomial or a rational function and a is in the domain of f , then

$$\lim_{x \rightarrow a} f(x) = f(a).$$

However, not all limits can be evaluated by direct substitution, as the following examples show.

Illustrative Example 02 Find $\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1}$.

Answer:

Let $f(x) = \frac{x^2 - 1}{x - 1}$.

We can't find the limit by substituting $x = 1$ because $f(1)$ isn't defined. Nor can we apply the Quotient Law, because the limit of the denominator is 0. Instead, we need to do some preliminary algebra. We factor the numerator as a difference of squares:

$$\frac{x^2 - 1}{x - 1} = \frac{(x - 1)(x + 1)}{(x - 1)}$$

The numerator and denominator have a common factor of $(x - 1)$. When we take the limit as x approaches 1, we have $x \neq 1$ and so $x - 1 \neq 0$. Therefore we can cancel the common factor and then compute the limit by direct substitution as follows:

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(x - 1)(x + 1)}{(x - 1)} \\ &= \lim_{x \rightarrow 1} (x + 1) \\ &= 1 + 1 \\ &= 2\end{aligned}$$

Illustrative Example 03 Find $\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h}$.

Answer:

If we define

$$F(h) = \frac{(3 + h)^2 - 9}{h},$$

then we can't compute $\lim_{h \rightarrow 0} F(h)$ by letting $h = 0$ since $F(0)$ is undefined. But if we simplify $F(h)$ algebraically, we find that

$$F(h) = \frac{(3 + h)^2 - 9}{h} = \frac{(9 + 6h + h^2) - 9}{h} = \frac{6h + h^2}{h} = 6 + h$$

(Recall that we consider only $h \neq 0$ when letting h approach 0.) Thus

$$\lim_{h \rightarrow 0} \frac{(3 + h)^2 - 9}{h} = \lim_{h \rightarrow 0} (6 + h) = 6$$

Illustrative Example 04 Find $\lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2}$.

Answer:

We can't apply the Quotient Law immediately, since the limit of the denominator is 0. Here the preliminary algebra consists of rationalizing the numerator:

$$\begin{aligned}
 \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} &= \lim_{t \rightarrow 0} \frac{\sqrt{t^2 + 9} - 3}{t^2} \cdot \frac{\sqrt{t^2 + 9} + 3}{\sqrt{t^2 + 9} + 3} \\
 &= \lim_{t \rightarrow 0} \frac{(t^2 + 9) - 9}{t^2(\sqrt{t^2 + 9} + 3)} \\
 &= \lim_{t \rightarrow 0} \frac{t^2}{t^2(\sqrt{t^2 + 9} + 3)} \\
 &= \lim_{t \rightarrow 0} \frac{1}{\sqrt{t^2 + 9} + 3} \\
 &= \frac{1}{\sqrt{\lim_{t \rightarrow 0}(t^2 + 9)} + 3} \\
 &= \frac{1}{3 + 3} \\
 &= \frac{1}{6}
 \end{aligned}$$

Practice Problems:

$$\lim_{x \rightarrow 5} \frac{x^2 - 6x + 5}{x - 5}$$

$$\lim_{x \rightarrow 5} \frac{x^2 - 5x + 6}{x - 5}$$

$$\lim_{t \rightarrow -3} \frac{t^2 - 9}{2t^2 + 7t + 3}$$

$$\lim_{h \rightarrow 0} \frac{(-5 + h)^2 - 25}{h}$$

$$\lim_{x \rightarrow -3} \frac{x^2 + 3x}{x^2 - x - 12}$$

$$\lim_{x \rightarrow 4} \frac{x^2 + 3x}{x^2 - x - 12}$$

$$\lim_{x \rightarrow -1} \frac{2x^2 + 3x + 1}{x^2 - 2x - 3}$$

$$\lim_{h \rightarrow 0} \frac{(2 + h)^3 - 8}{h}$$

$$\lim_{x \rightarrow -2} \frac{x + 2}{x^3 + 8}$$

$$\lim_{t \rightarrow 1} \frac{t^4 - 1}{t^3 - 1}$$

$$\lim_{h \rightarrow 0} \frac{\sqrt{9 + h} - 3}{h}$$

$$\lim_{u \rightarrow 2} \frac{\sqrt{4u + 1} - 3}{u - 2}$$

$$\lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x - 3}$$

$$\lim_{h \rightarrow 0} \frac{(3 + h)^{-1} - 3^{-1}}{h}$$

$$\lim_{t \rightarrow 0} \frac{\sqrt{1 + t} - \sqrt{1 - t}}{t}$$

$$\lim_{t \rightarrow 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$$

$$\lim_{x \rightarrow 16} \frac{4 - \sqrt{x}}{16x - x^2}$$

$$\lim_{x \rightarrow 2} \frac{x^2 - 4x + 4}{x^4 - 3x^2 - 4}$$

$$\lim_{t \rightarrow 0} \left(\frac{1}{t\sqrt{1 + t}} - \frac{1}{t} \right)$$

$$\lim_{x \rightarrow -4} \frac{\sqrt{x^2 + 9} - 5}{x + 4}$$

$$\lim_{h \rightarrow 0} \frac{(x + h)^3 - x^3}{h}$$

$$\lim_{h \rightarrow 0} \frac{\frac{1}{(x + h)^2} - \frac{1}{x^2}}{h}$$