


ME 1102: Applied Mechanics

Equilibrium of Rigid Bodies

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Contents

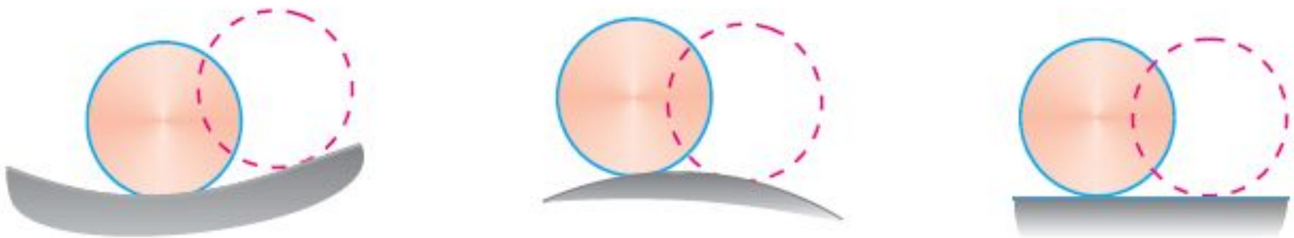
Introduction

- For a rigid body in static equilibrium, the external forces and moments are balanced and will impart no translational or rotational motion to the body.

- Principles of Equilibrium (Though there are many principles of equilibrium, yet the following three are important from the subject point of view):
  - **Two force principle**: As per this principle, if a body in equilibrium is acted upon by two forces, then they must be equal, opposite and collinear
  - **Three force principle**: If a body in equilibrium is acted upon by three forces, then the resultant of any two forces must be equal, opposite and collinear with the third force
  - **Four force principle**: If a body in equilibrium is acted upon by four forces, then the resultant of any two forces must be equal, opposite and collinear with the resultant of the other two forces

Introduction

- Types of Equilibrium



(a) Stable (b) Unstable (c) Neutral

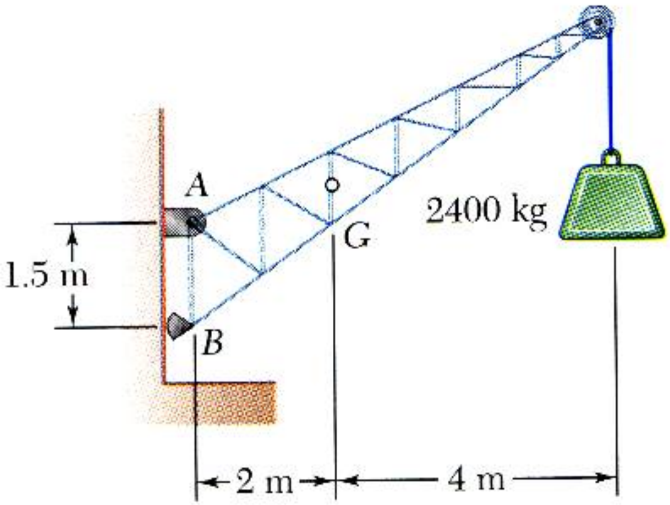
- The necessary and sufficient condition for the static equilibrium of a body are that the resultant force and couple from all external forces of the system equivalent to zero,

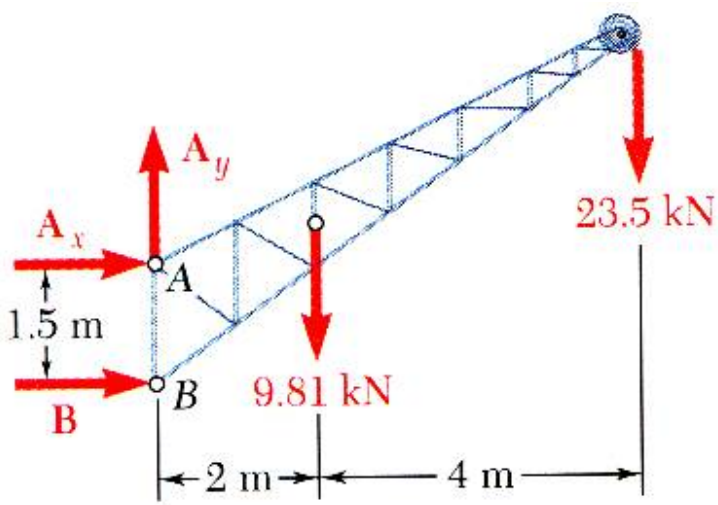
$$\sum \vec{F} = 0 \qquad \sum \vec{M}_O = \sum (\vec{r} \times \vec{F}) = 0$$

- Resolving each force and moment into its rectangular components leads to 6 scalar equations which also express the conditions for static equilibrium,

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

Free-Body Diagram

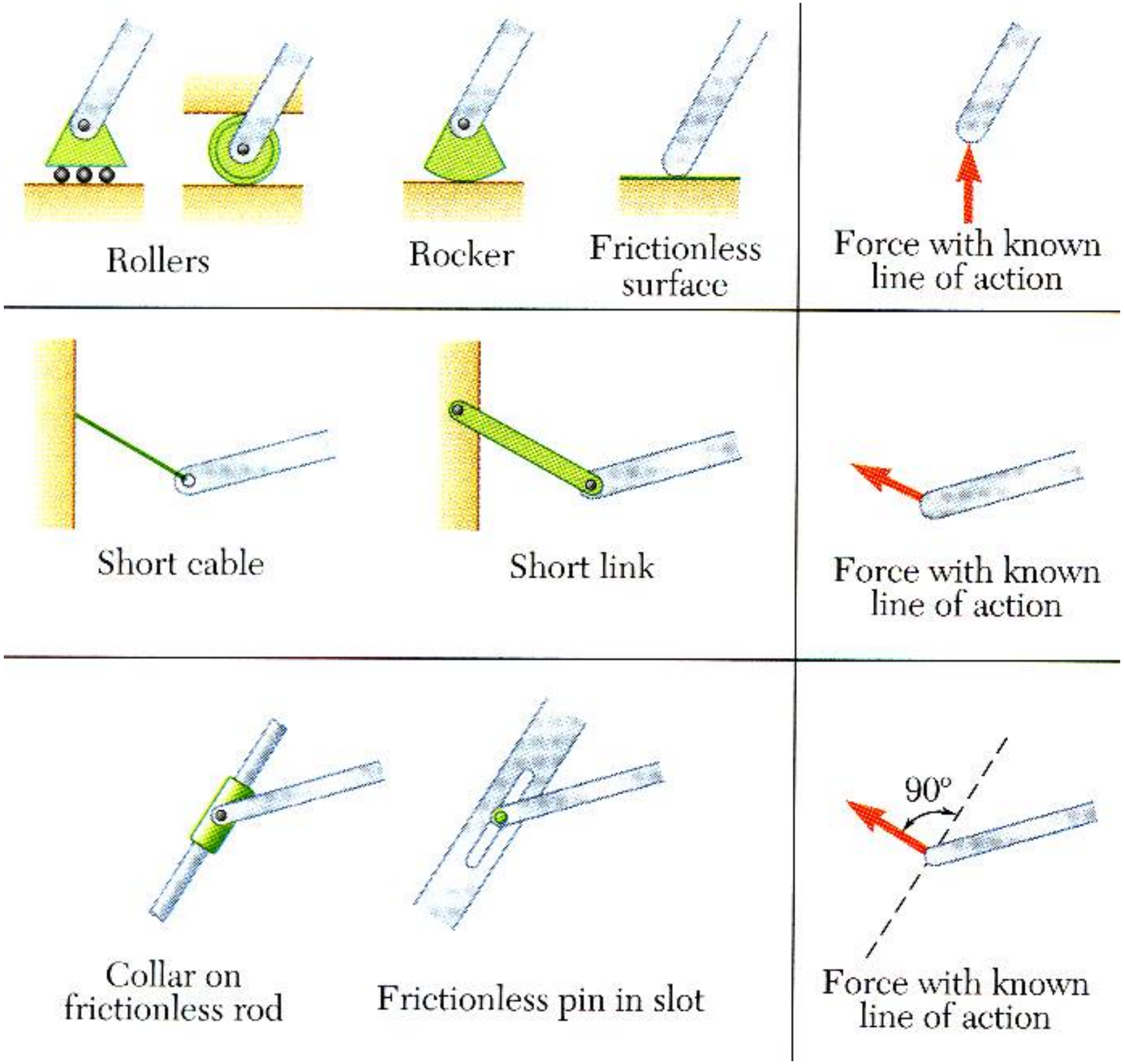




First step in the static equilibrium analysis of a rigid body is identification of all forces acting on the body with a *free-body* diagram.

- Select the extent of the free-body and detach it from the ground and all other bodies.
- Indicate point of application, magnitude, and direction of known external forces, including the rigid body weight.
- Indicate point of application and assumed direction of unknown applied forces. These usually consist of reactions through which the ground and other bodies oppose the possible motion of the rigid body.
- Include the dimensions necessary to compute the moments of the forces.

Reactions at Supports and Connections for a Two-Dimensional Structure

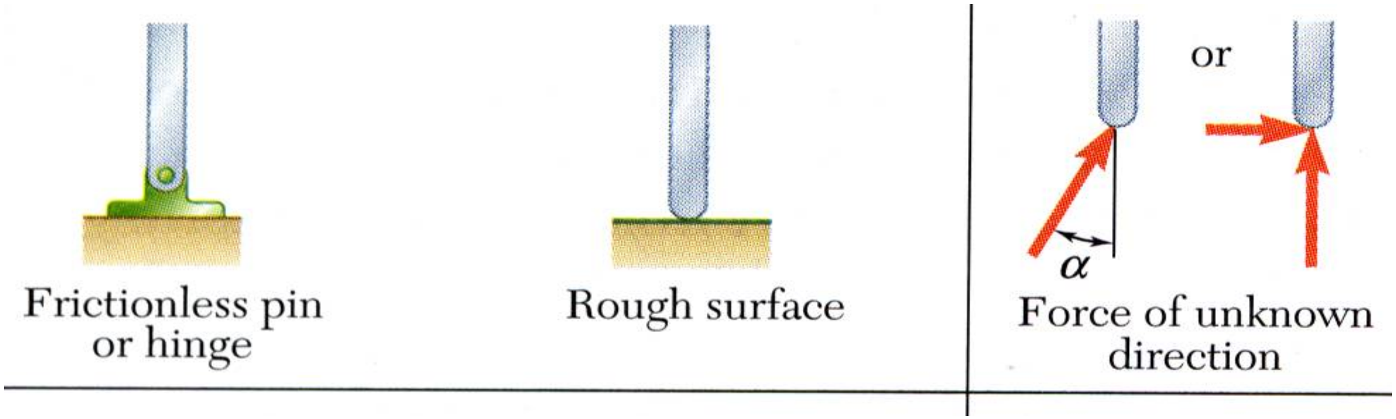


- Reactions equivalent to a force with known line of action.

Reactions at Supports and Connections for a Two-Dimensional Structure

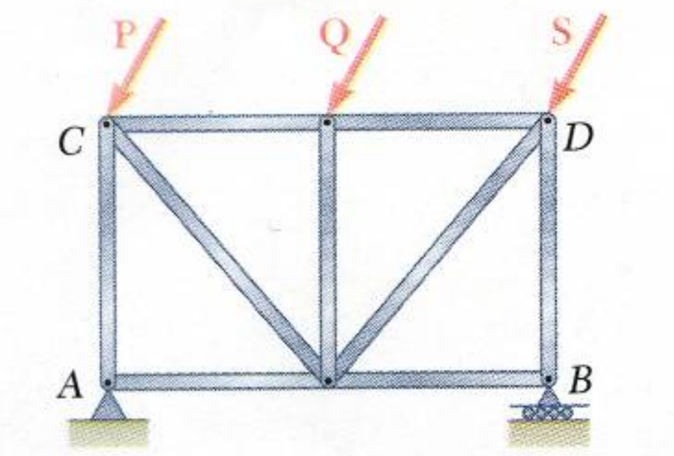
cont…

- Reactions equivalent to a force of unknown direction and magnitude.

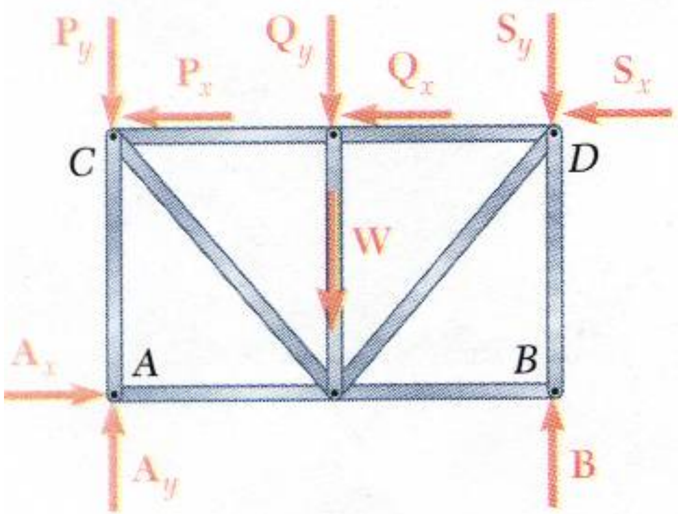


- Reactions equivalent to a force and a couple.

Equilibrium of a Rigid Body in Two Dimensions



(a)



(b)

- For all forces and moments acting on a two-dimensional structure,

$$F_z = 0 \quad M_x = M_y = 0 \quad M_z = M_O$$

- Equations of equilibrium become

$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum M_A = 0$$

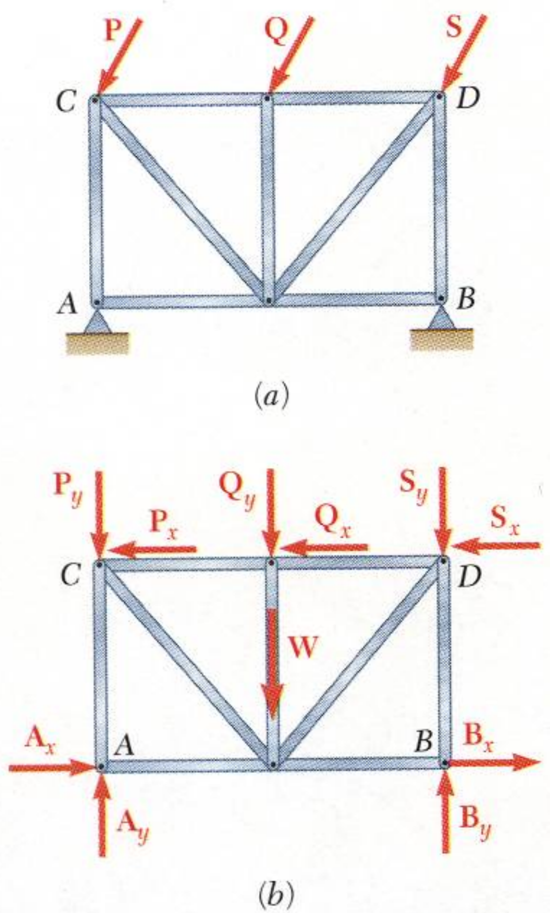
where A is any point in the plane of the structure.

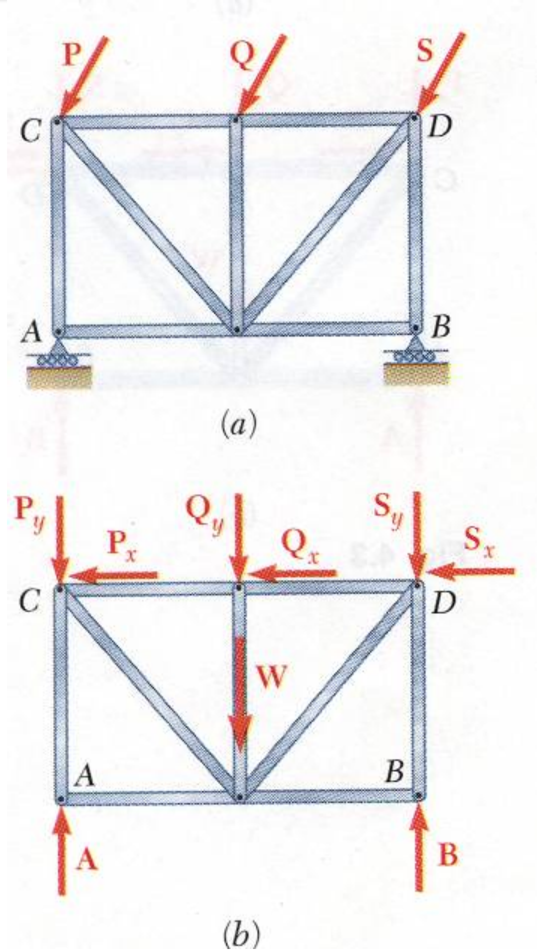
- The 3 equations can be solved for no more than 3 unknowns.

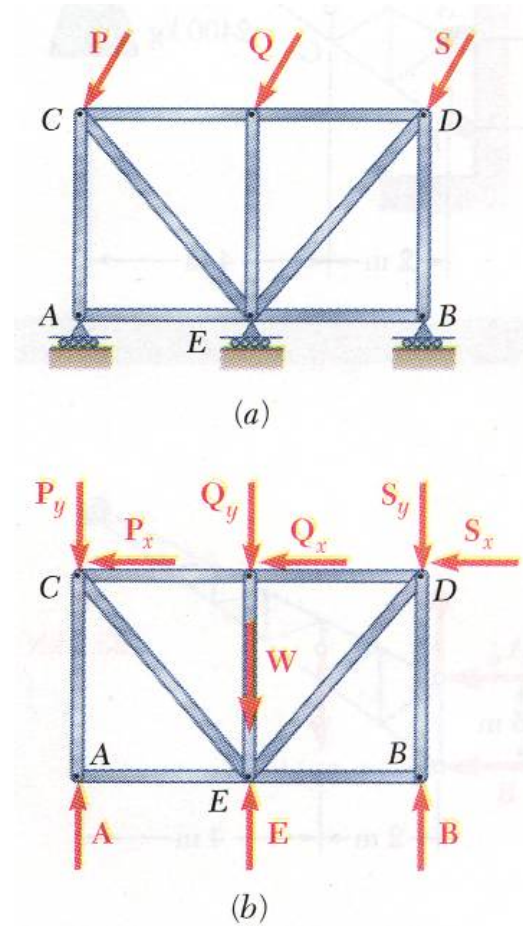
- The 3 equations can not be augmented with additional equations, but they can be replaced

$$\sum F_x = 0 \quad \sum M_A = 0 \quad \sum M_B = 0$$

Statically Indeterminate Reactions







- More unknowns than equations

- Fewer unknowns than equations, partially constrained

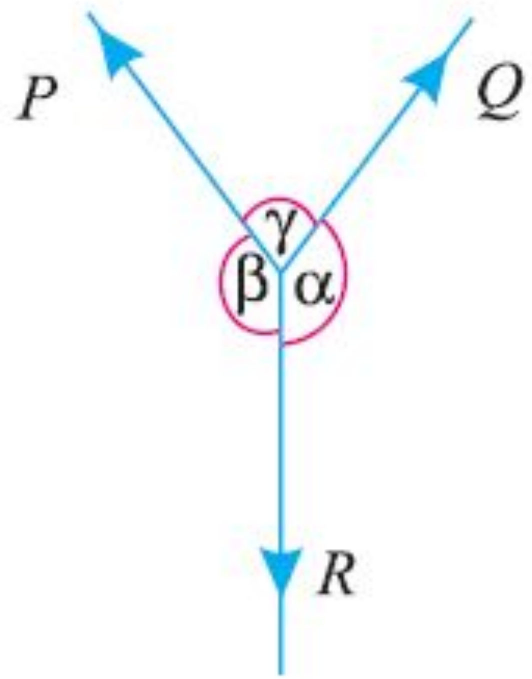
- Equal number of unknowns and equations but improperly constrained

Lami's Theorem

- *"If three coplanar forces acting at a point be in equilibrium, then each force is proportional to the sine of the angle between the other two"*

- Mathematically,

$$\frac{P}{\sin\alpha} = \frac{Q}{\sin\beta} = \frac{R}{\sin\gamma}$$



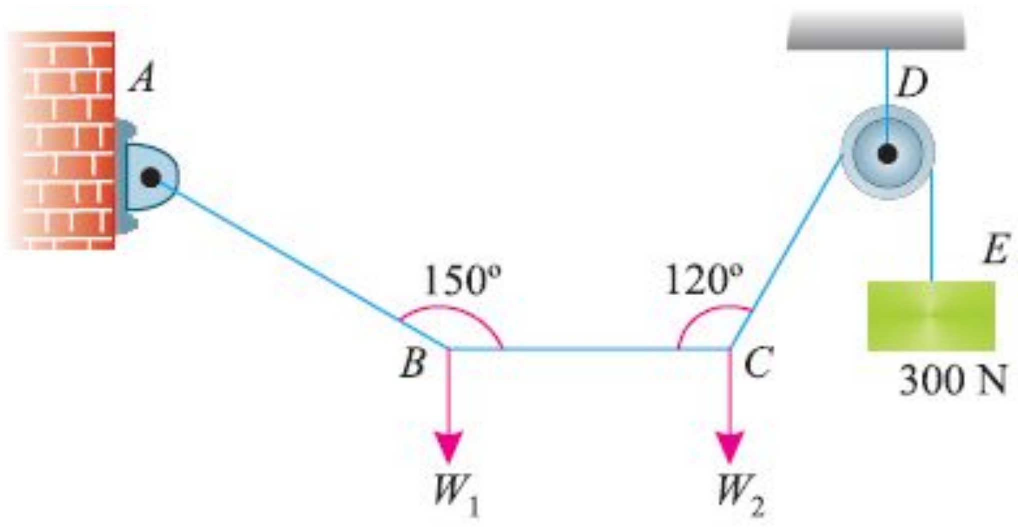
Problems: 4.1

- A light string ABCDE whose extremity A is fixed, has weight W_1 and W_2 attached to it at B and C. It passes round a small smooth peg at D carrying a weight of 300 N at the free end E as shown in the Figure.

  If in the equilibrium position, BC is horizontal and AB and CD make 150° and 120° with BC, find

  (i) Tensions in the portion AB, BC and CD of the string

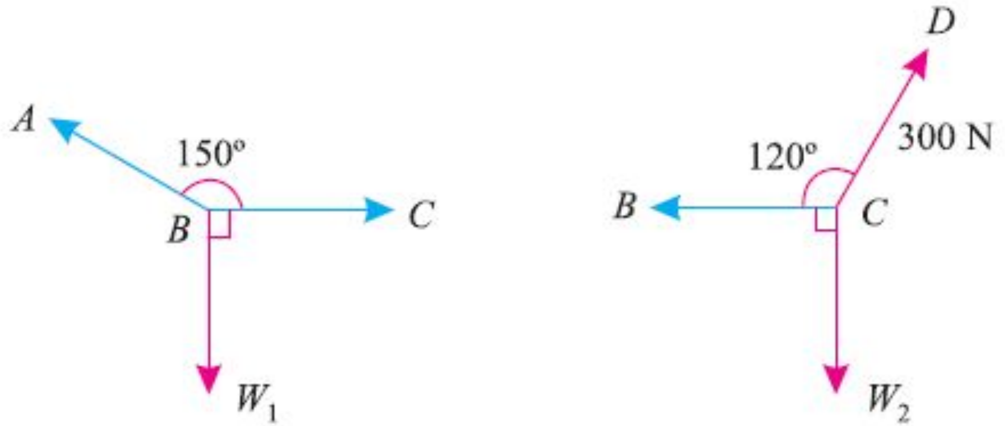
  (ii) Magnitudes of W1 and W2



Solution: 4.1

Solution. Given : Weight at E = 300 N

For the sake of convenience, let us split up the string $ABCD$ into two parts. The system of forces at joints B and C is shown in Fig. 5.8. (a) and (b).



(a) Joint B (b) Joint C

Fig. 5.8.

(i) *Tensions is the portion AB, BC and CD of the string*

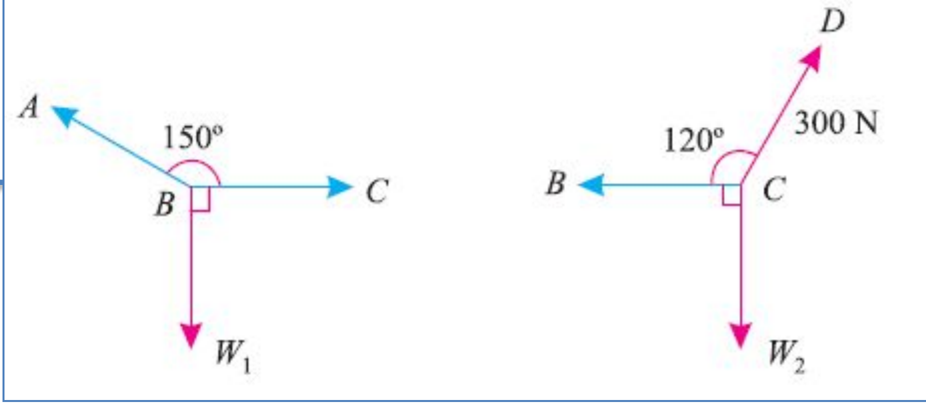
Let T_{AB} = Tension in the portion AB, and

T_{BC} = Tension in the portion BC,

We know that tension in the portion CD of the string.

$$T_{CD} = T_{DE} = 300 \text{ N} \quad \textbf{Ans.}$$

Solution: 4.1 cont…



Applying Lami's equation at C,

$$\frac{T_{BC}}{\sin 150^\circ} = \frac{W_2}{\sin 120^\circ} = \frac{300}{\sin 90^\circ}$$

$$\frac{T_{BC}}{\sin 30^\circ} = \frac{W_2}{\sin 60^\circ} = \frac{300}{1} \qquad \text{...}[\because \sin(180^\circ - \theta) = \sin\theta]$$

$\therefore$ $\quad T_{BC} = 300 \sin 30^\circ = 300 \times 0.5 = 150 \text{ N}$ **Ans.**

and $\quad W_2 = 300 \sin 60^\circ = 300 \times 0.866 = 259.8 \text{ N}$

Again applying Lami's equation at *B*,

$$\frac{T_{AB}}{\sin 90^\circ} = \frac{W_1}{\sin 150^\circ} = \frac{T_{BC}}{\sin 120^\circ}$$

$$\frac{T_{AB}}{1} = \frac{W_1}{\sin 30^\circ} = \frac{150}{\sin 60^\circ} \qquad \text{...}[\because \sin(180^\circ - \theta) = \sin\theta]$$

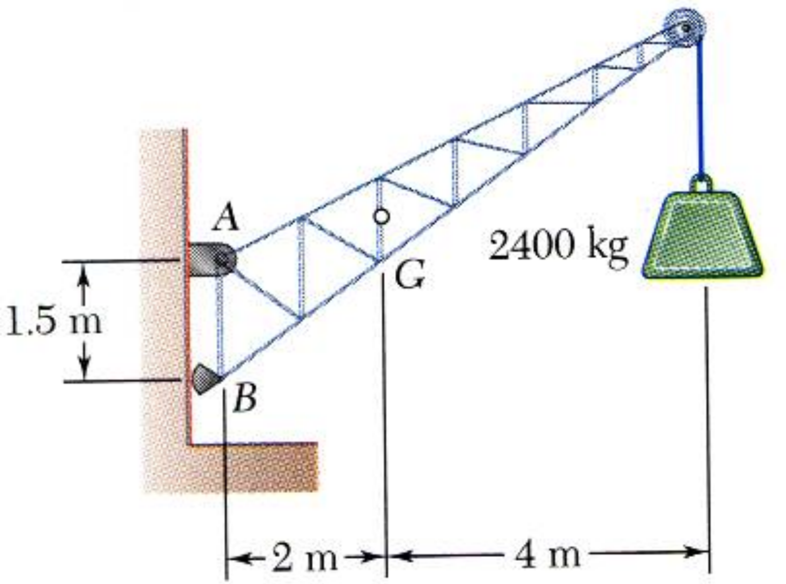
$\therefore$ $\quad T_{AB} = \dfrac{150}{\sin 60^\circ} = \dfrac{150}{0.866} = 173.2 \text{ N}$ **Ans.**

and $\quad W_1 = \dfrac{150 \sin 30^\circ}{\sin 60^\circ} = \dfrac{150 \times 0.5}{0.866} = 86.6 \text{ N}$

(*ii*) *Magnitudes of* W_1 *and* W_2

From the above calculations, we find that the magnitudes of W_1 and W_2 are 86.6 N and 259.8 N respectively. **Ans.**

Problems: 4.2



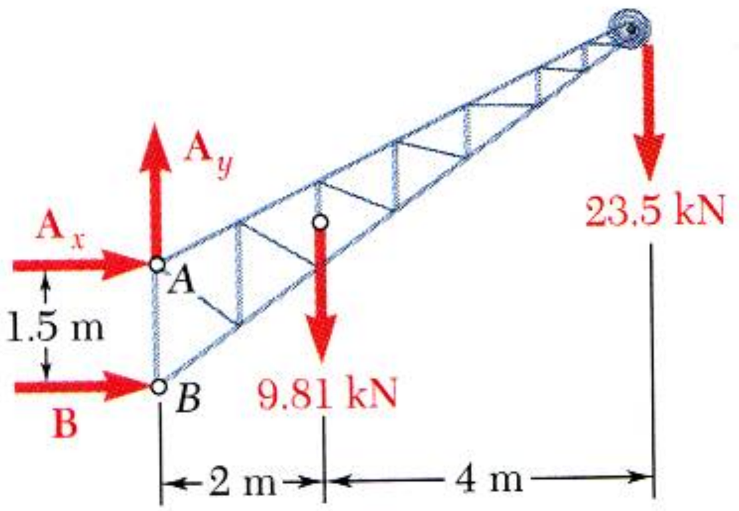
A fixed crane has a mass of 1000 kg and is used to lift a 2400 kg crate. It is held in place by a pin at *A* and a rocker at *B*. The center of gravity of the crane is located at *G*.

Determine the components of the reactions at *A* and *B*.

SOLUTION:

- Create a free-body diagram for the crane.
- Determine *B* by solving the equation for the sum of the moments of all forces about *A*. Note there will be no contribution from the unknown reactions at *A*.
- Determine the reactions at A by solving the equations for the sum of all horizontal force components and all vertical force components.
- Check the values obtained for the reactions by verifying that the sum of the moments about *B* of all forces is zero.

Solution: 4.2



- Create the free-body diagram.

- Determine B by solving the equation for the sum of the moments of all forces about A.

$$\sum M_A = 0: \quad +B(1.5\text{m}) - 9.81\,\text{kN}(2\text{m}) - 23.5\,\text{kN}(6\text{m}) = 0$$

$$\boxed{B = +107.1\,\text{kN}}$$

- Determine the reactions at A by solving the equations for the sum of all horizontal forces and all vertical forces.

$$\sum F_x = 0: \quad A_x + B = 0$$

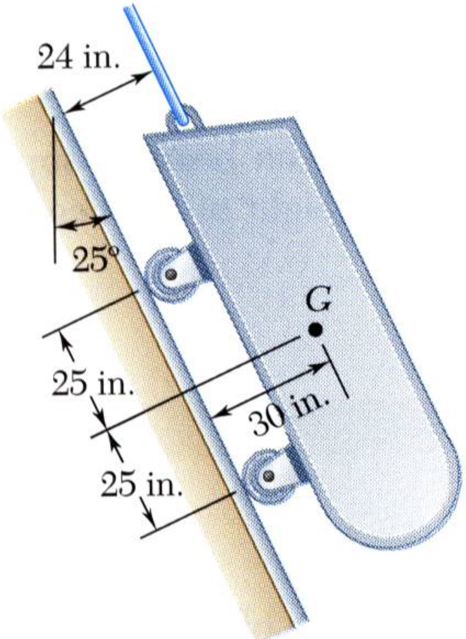
$$\boxed{A_x = -107.1\,\text{kN}}$$

$$\sum F_y = 0: \quad A_y - 9.81\,\text{kN} - 23.5\,\text{kN} = 0$$

$$\boxed{A_y = +33.3\,\text{kN}}$$

- Check the values obtained.

Problems: 4.3



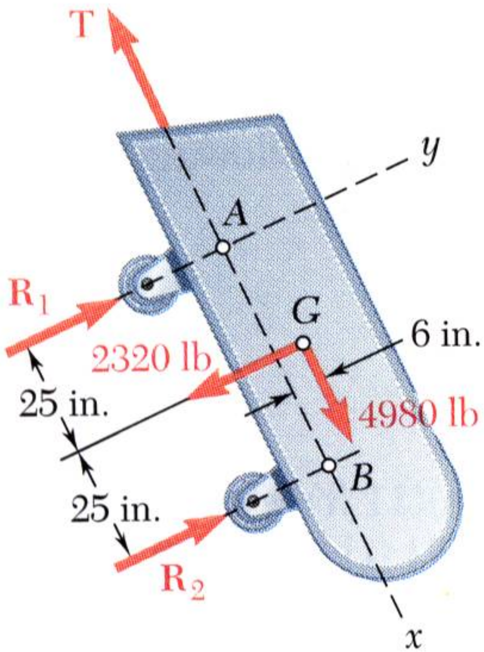
A loading car is at rest on an inclined smooth track. The gross weight of the car and its load is 5500 lb, and it is applied at *G*. The cart is held in position by the cable.

Determine the tension in the cable and the reaction at each pair of wheels.

SOLUTION:

- Create a free-body diagram for the car with the coordinate system aligned with the track.
- Determine the reactions at the wheels by solving equations for the sum of moments about points above each axle.
- Determine the cable tension by solving the equation for the sum of force components parallel to the track.
- Check the values obtained by verifying that the sum of force components perpendicular to the track are zero.

Solution: 4.3



- Create a free-body diagram

$$W_x = +(5500\text{ lb})\cos 25^\circ$$
$$= +4980\text{ lb}$$

$$W_y = -(5500\text{ lb})\sin 25^\circ$$
$$= -2320\text{ lb}$$

- Determine the reactions at the wheels.

$$\sum M_A = 0: \quad -(2320\text{ lb})25\text{in.} - (4980\text{ lb})6\text{in.}$$
$$+ R_2(50\text{in.}) = 0$$

$$\boxed{R_2 = 1758\text{ lb}}$$

$$\sum M_B = 0: \quad +(2320\text{ lb})25\text{in.} - (4980\text{ lb})6\text{in.}$$
$$- R_1(50\text{in.}) = 0$$

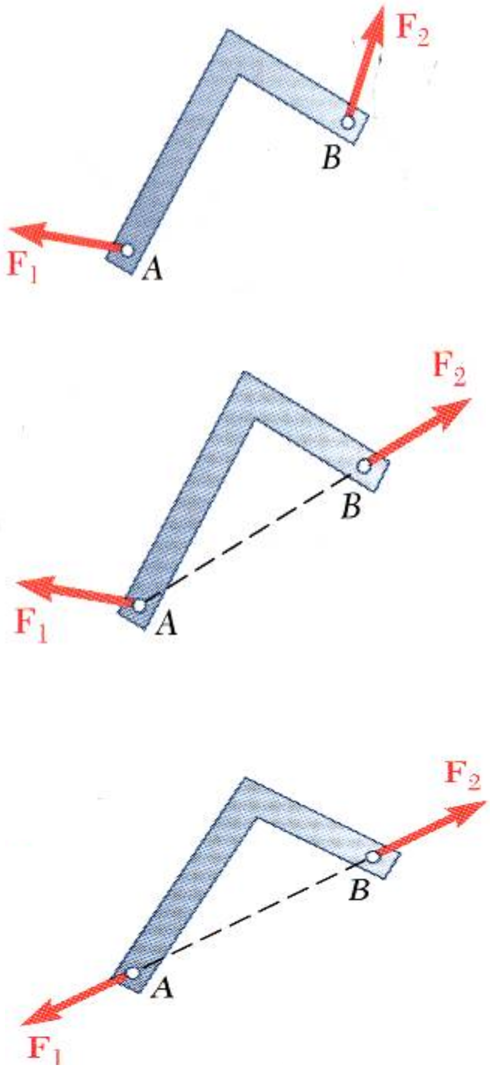
$$\boxed{R_1 = 562\text{ lb}}$$

- Determine the cable tension.

$$\sum F_x = 0: \quad +4980\text{ lb} - \mathrm{T} = 0$$

$$\boxed{T = +4980\text{ lb}}$$

Equilibrium of a Two-Force Body



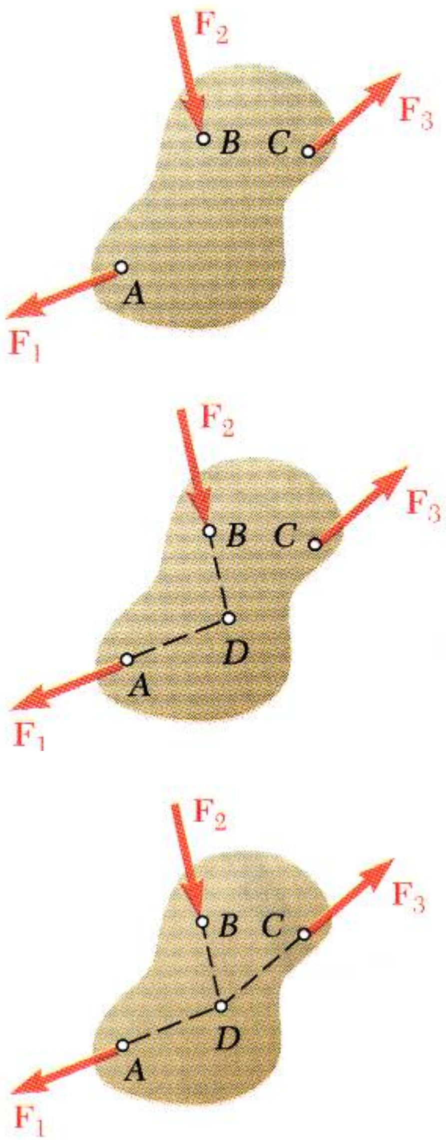
- Consider a plate subjected to two forces $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$

- For static equilibrium, the sum of moments about *A* must be zero. The moment of $\boldsymbol{F_2}$ must be zero. It follows that the line of action of $\boldsymbol{F_2}$ must pass through *A*.

- Similarly, the line of action of $\boldsymbol{F_1}$ must pass through *B* for the sum of moments about *B* to be zero.

- Requiring that the sum of forces in any direction be zero leads to the conclusion that $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ must have equal magnitude but opposite sense.

Equilibrium of a Three-Force Body



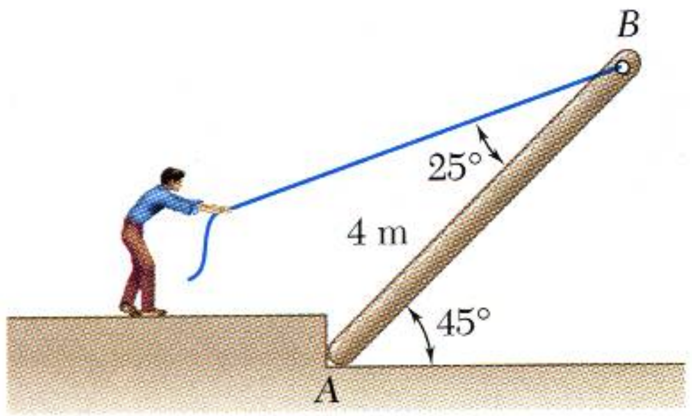
- Consider a rigid body subjected to forces acting at only 3 points.

- Assuming that their lines of action intersect, the moment of $\boldsymbol{F_1}$ and $\boldsymbol{F_2}$ about the point of intersection represented by D is zero.

- Since the rigid body is in equilibrium, the sum of the moments of $\boldsymbol{F_1}$, $\boldsymbol{F_2}$, and $\boldsymbol{F_3}$ about any axis must be zero. It follows that the moment of $\boldsymbol{F_3}$ about D must be zero as well and that the line of action of $\boldsymbol{F_3}$ must pass through D.

- The lines of action of the three forces must be concurrent or parallel.

Problems: 4.4



A man raises a 10 kg joist, of length 4 m, by pulling on a rope.

Find the tension in the rope and the reaction at A.

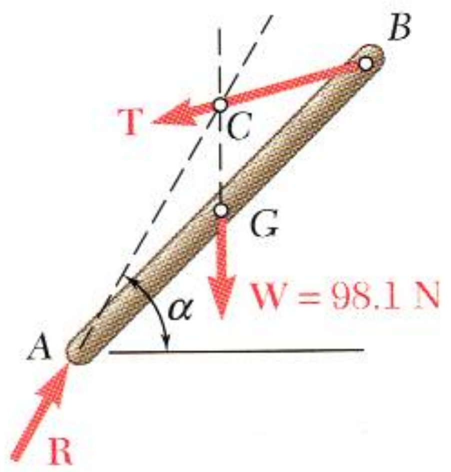
SOLUTION:

- Create a free-body diagram of the joist. Note that the joist is a 3 force body acted upon by the rope, its weight, and the reaction at A.

- The three forces must be concurrent for static equilibrium. Therefore, the reaction $\boldsymbol{R}$ must pass through the intersection of the lines of action of the weight and rope forces. Determine the direction of the reaction force $\boldsymbol{R}$.

- Utilize a force triangle to determine the magnitude of the reaction force $\boldsymbol{R}$.

Solution: 4.4



- Create a free-body diagram of the joist.

- Determine the direction of the reaction force ***R***.

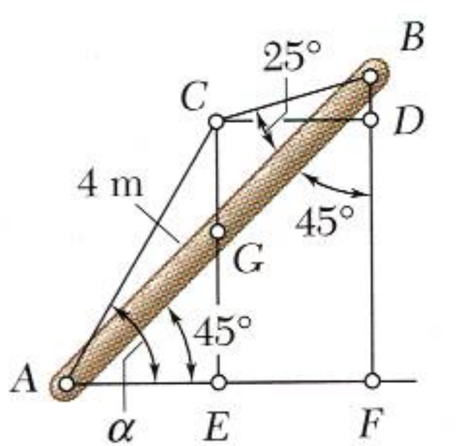
$$AF = AB\cos 45 = (4\,\mathrm{m})\cos 45 = 2.828\,\mathrm{m}$$

$$CD = AE = \tfrac{1}{2}AF = 1.414\,\mathrm{m}$$

$$BD = CD\cot(45+25) = (1.414\,\mathrm{m})\tan 20 = 0.515\,\mathrm{m}$$

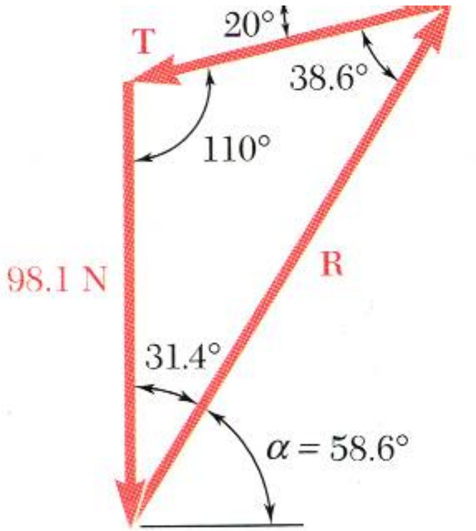
$$CE = BF - BD = (2.828 - 0.515)\,\mathrm{m} = 2.313\,\mathrm{m}$$

$$\tan\alpha = \frac{CE}{AE} = \frac{2.313}{1.414} = 1.636$$



$$\boxed{\alpha = 58.6^{\circ}}$$

Solution: 4.4 cont…



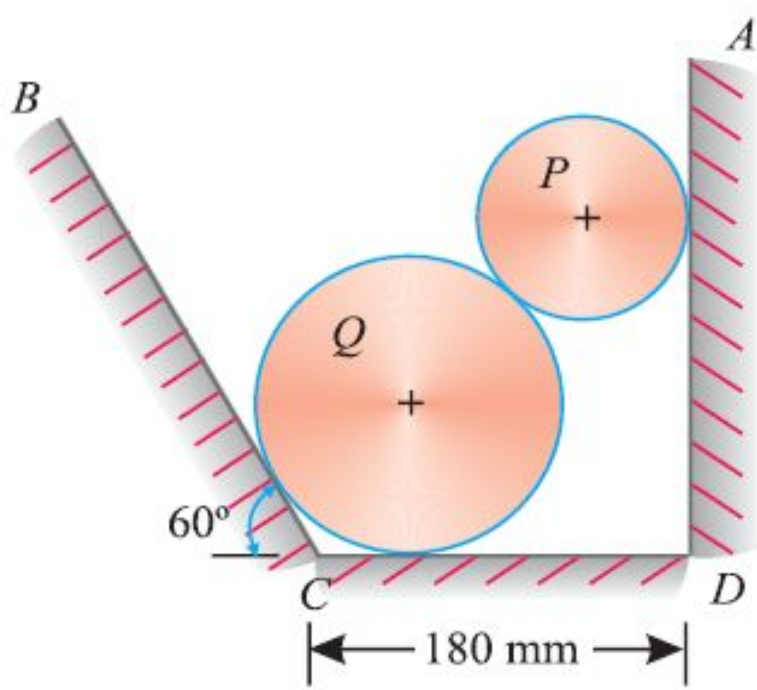
- Determine the magnitude of the reaction force ***R***.

$$\frac{T}{\sin 31.4^\circ} = \frac{R}{\sin 110^\circ} = \frac{98.1\,\text{N}}{\sin 38.6^\circ}$$

$$\boxed{\begin{aligned} T &= 81.9\,\text{N} \\ R &= 147.8\,\text{N} \end{aligned}}$$

Problems: 4.5

- Two cylinders P and Q rest in a smooth channel as shown in Figure. The cylinder P has diameter of 100 mm and weighs 200 N, whereas the cylinder Q has diameter of 180 mm and weighs 500 N.
  If the bottom width of the box is 180 mm, with one side vertical and the other inclined at 60◦, determine the pressures at all the four points of contact.



Equilibrium of a Rigid Body in Three Dimensions

- Six scalar equations are required to express the conditions for the equilibrium of a rigid body in the general three dimensional case.

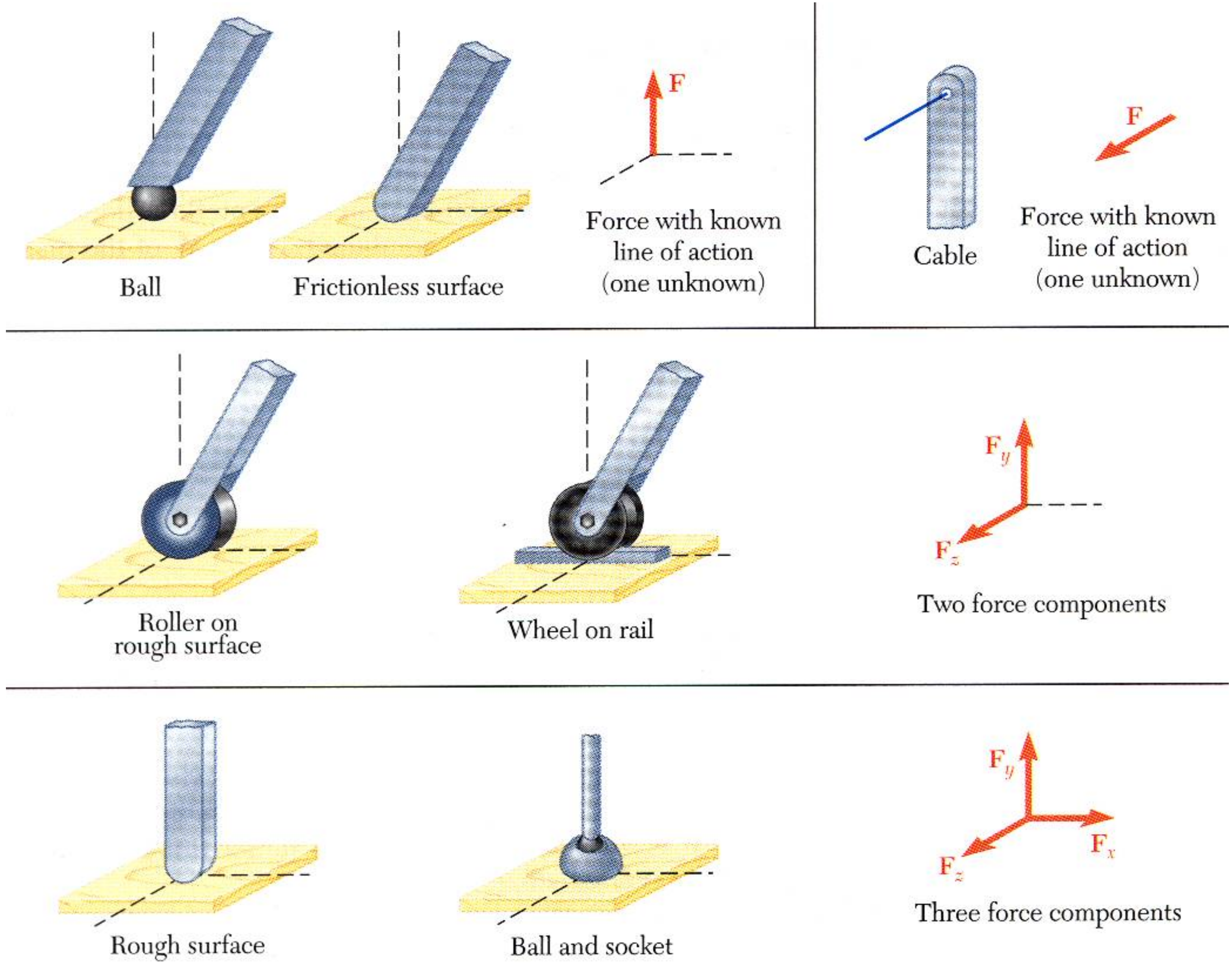
$$\sum F_x = 0 \quad \sum F_y = 0 \quad \sum F_z = 0$$
$$\sum M_x = 0 \quad \sum M_y = 0 \quad \sum M_z = 0$$

- These equations can be solved for no more than 6 unknowns which generally represent reactions at supports or connections.

- The scalar equations are conveniently obtained by applying the vector forms of the conditions for equilibrium,

$$\sum \vec{F} = 0 \qquad \sum \vec{M}_O = \sum (\vec{r} \times \vec{F}) = 0$$

Reactions at Supports and Connections for a Three-Dimensional Structure



Reactions at Supports and Connections for a Three-Dimensional Structure
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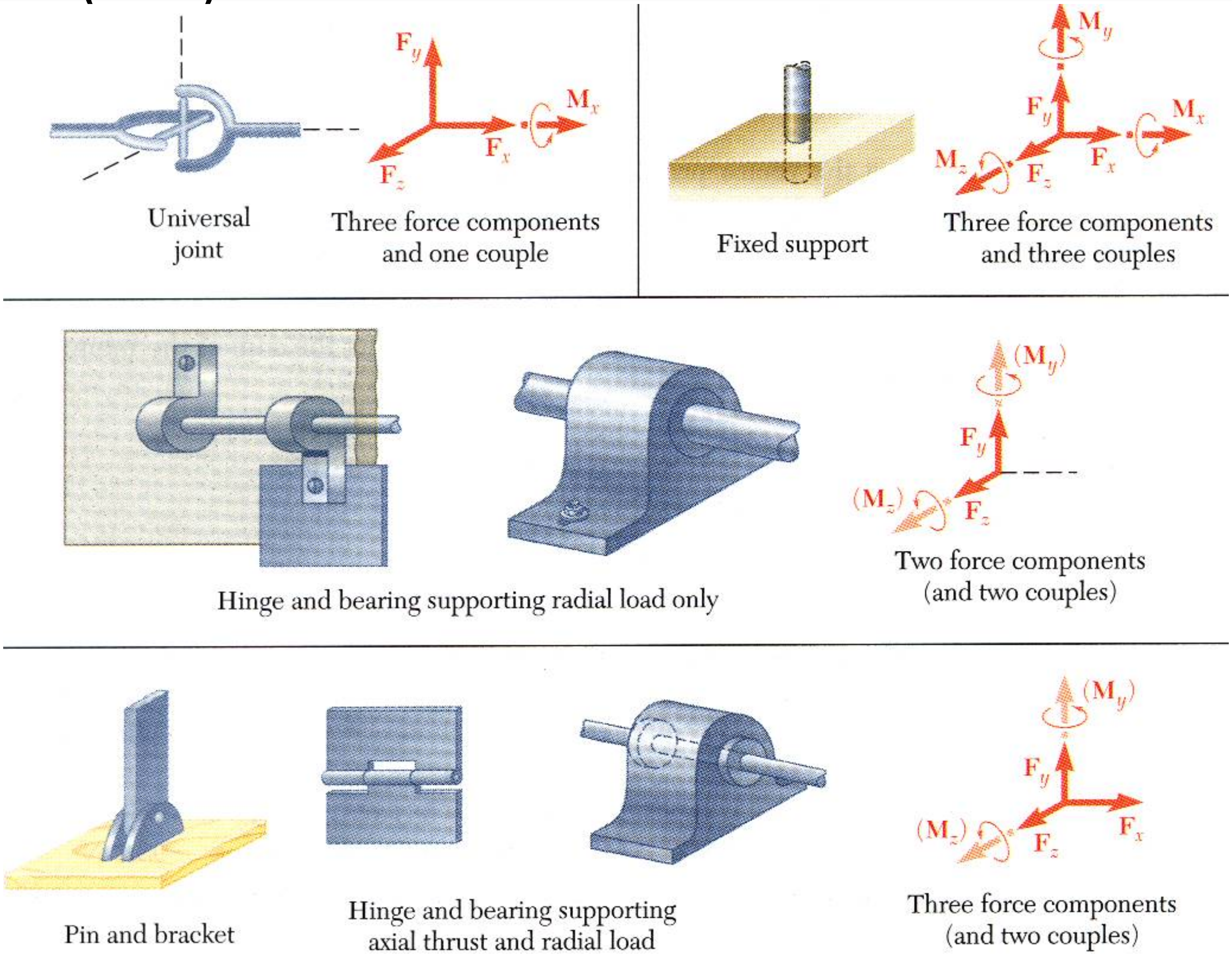
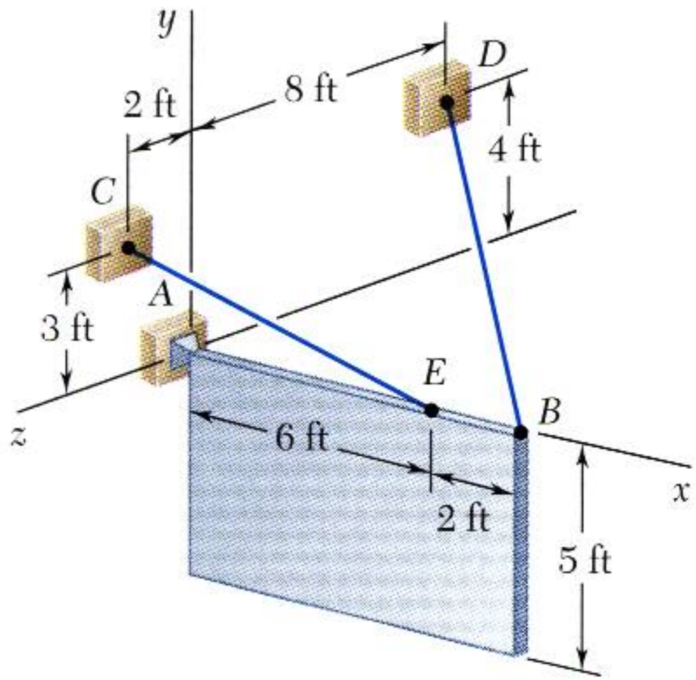


Problems: 4.6



SOLUTION:

- Create a free-body diagram for the sign.
- Apply the conditions for static equilibrium to develop equations for the unknown reactions.

A sign of uniform density weighs 270 lb and is supported by a ball-and-socket joint at *A* and by two cables.

Determine the tension in each cable and the reaction at *A*.