Mathematics - MA 1103
Tutorial 01 - Vectors - Solutions.
Intake 38 - 03rd Batch

(01) $A(2,4,1)$ $\quad \therefore \overrightarrow{OA} = \langle 2,4,1 \rangle = \underline{a}$

$B(3,2,-1)$ $\quad \therefore \overrightarrow{OB} = \langle 3,2,-1 \rangle = \underline{b}$

$$\overrightarrow{OC} = 2.\overrightarrow{OB}$$

(a) (i) $\overrightarrow{OC} = 2.\langle 3, 2, -1 \rangle$

$\therefore \underline{c} = \langle 6, 4, -2 \rangle$

(ii) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \langle 3, 2, -1 \rangle - \langle 2, 4, 1 \rangle$

$= \langle 1, -2, -2 \rangle$

(b) (i) $\overrightarrow{AC} = \underline{c} - \underline{a}$

$= \langle 6, 4, -2 \rangle - \langle 2, 4, 1 \rangle$

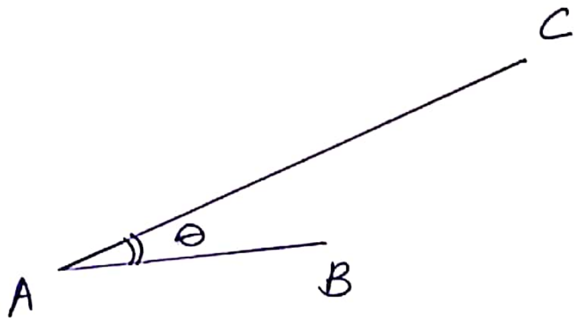
$= \langle 6-2, 4-4, -2-1 \rangle$

$= \langle 4, 0, -3 \rangle$

$\| \overrightarrow{AC} \|^2 = 4^2 + 0^2 + (-3)^2 = 25$

$\therefore AC = 5$

(ii)



$$\overrightarrow{AB} \cdot \overrightarrow{AC} = \| \overrightarrow{AB} \| \cdot \| \overrightarrow{AC} \| \cos \theta$$

$$\langle 1, -2, -2 \rangle \cdot \langle 4, 0, -3 \rangle = \sqrt{1^2 + (-2)^2 + (-2)^2} \cdot \sqrt{4^2 + 0^2 + (-3)^2} \cos \theta$$

$$4 + 6 = 3 \cdot 5 \cdot \cos \theta$$

$$10 = 15 \cos \theta$$

$$\frac{2}{3} = \cos \theta$$

$$\therefore \theta = \cos^{-1}\left(\frac{2}{3}\right)$$

$$= 48.189 \simeq 48^\circ$$

(iii) $P(\alpha, \beta, \gamma)$ $\quad \therefore \underline{p} = \langle \alpha, \beta, \gamma \rangle$

$$\overrightarrow{BP} = \underline{p} - \underline{b}$$

$$= \langle \alpha, \beta, \gamma \rangle - \langle 3, 2, -1 \rangle$$

$$= \langle \alpha - 3, \beta - 2, \gamma + 1 \rangle$$

since $BP \perp AC$

$$\overrightarrow{BP} \cdot \overrightarrow{AC} = 0$$

$$\therefore \langle \alpha - 3, \beta - 2, \gamma + 1 \rangle \cdot \langle 4, 0, -3 \rangle = 0$$

$$4(\alpha - 3) - 3(\gamma + 1) = 0$$

$$4\alpha - 12 - 3\gamma - 3 = 0$$

$$\therefore 4\alpha - 3\gamma = 15$$

(02) $A \equiv (3, -2, 4)$ $\therefore$ $\underline{a} = \langle 3, -2, 4 \rangle$

$B \equiv (5, 4, 0)$ $\underline{b} = \langle 5, 4, 0 \rangle$

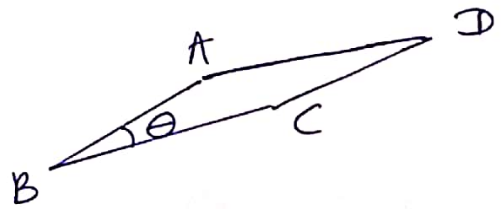
$C \equiv (11, 6, -4)$ $\underline{c} = \langle 11, 6, -4 \rangle$

(a) (i) $\overrightarrow{BA} = \underline{a} - \underline{b}$

$= \langle 3, -2, 4 \rangle - \langle 5, 4, 0 \rangle$

$= \langle -2, -6, 4 \rangle$

(ii)



$\overrightarrow{BC} = \underline{c} - \underline{b}$

$= \langle 11, 6, -4 \rangle - \langle 5, 4, 0 \rangle$

$= \langle 6, 2, -4 \rangle$

$\|\overrightarrow{BA}\|^2 = (-2)^2 + (-6)^2 + 4^2 = 56$

$\|\overrightarrow{BC}\|^2 = 6^2 + 2^2 + (-4)^2 = 56$

$\overrightarrow{BA} \cdot \overrightarrow{BC} = \|\overrightarrow{BA}\| \cdot \|\overrightarrow{BC}\| \cos\theta$

$\langle -2, -6, 4 \rangle \cdot \langle 6, 2, -4 \rangle = \sqrt{56} \cdot \sqrt{56} \cdot \cos\theta$

$-12 - 12 - 16 = 56 \cos\theta$

$\frac{-\cancel{40}\,5}{\cancel{56}\,7} = \cos\theta$

$\therefore \cos\theta = -\frac{5}{7}$

$\theta = \cos^{-1}\left(-\frac{5}{7}\right)$

(b) l

(i) $r(t) = <8, -3, 2> + t<1, 3, -2>$

(i) Suppose C lies on l.

$\therefore <11, 6, -4> = <8, -3, 2> + t<1, 3, -2>$

$\therefore <11, 6, -4> = <8+t, -3+3t, 2-2t>$

$\therefore t = 3$

$\therefore$ $8 + t = 11$

$-3 + 3t = 6$

$2 - 2t = -4$

} these equations are satisfied when $t = 3$.

$\therefore$ C lies on l

(ii) $\overrightarrow{AB} = <2, 6, -4> = 2<1, 3, -2>$

$\therefore$ both $\overrightarrow{AB}$ and l are parallel to the vector $<1, 3, -2>$.

$\therefore$ $\overrightarrow{AB}$ and l are also parallel.

(c) $\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD} = \overrightarrow{OC} + \overrightarrow{BA}$

$= <11, 6, -4> + <-2, -6, 4>$

$= <9, 0, 0>$

$\therefore D \equiv (9, 0, 0)$

(03) $A \equiv (2, 1, 3)$

$B \equiv (6, 5, 3)$

$C \equiv (6, 1, -1)$

$D \equiv (2, -3, -1)$

$l_1 : \quad r(t) = \langle 6, 1, -1 \rangle + t \langle 1, 1, 0 \rangle$

(a) (i) $\overrightarrow{AB} = \underline{b} - \underline{a}$

$= \langle 6, 5, 3 \rangle - \langle 2, 1, 3 \rangle$

$= \langle 4, 4, 0 \rangle$

(ii) $\overrightarrow{AB} = 4 \langle 1, 1, 0 \rangle$

$\therefore$ both $\overrightarrow{AB}$ and l_1 are parallel to the vector $\langle 1, 1, 0 \rangle$. and hence parallel to each other

(iii) suppose D lies on l_1.

$\therefore \langle 2, -3, -1 \rangle = \langle 6, 1, -1 \rangle + t \langle 1, 1, 0 \rangle$

$\langle 2, -3, -1 \rangle = \langle 6+t, 1+t, -1 \rangle$

$\therefore \quad 2 = 6 + t \qquad \therefore \quad t = -4$

$-3 = 1 + t \quad \leftarrow$ satisfied by $t = -4$

$-1 = -1$

$\therefore$ D lies on l_1

(b) (i) $\overrightarrow{DM} = \underline{m} - \underline{d}$

$= < 4, 1, 1 > - < 2, -3, -1 >$

$= < 2, 4, 2 >$

$\therefore$ the equation of l_2:

$r(t) = < 2, -3, -1 > + t < 2, 4, 2 >$

(ii) $\overrightarrow{AC} = \underline{c} - \underline{a}$

$= < 6, 1, -1 > - < 2, 1, 3 >$

$= < 4, 0, -4 >$

the angle between $\overrightarrow{AC}$ and l_2 is equal to the angle between $\overrightarrow{AC}$ and $< 2, 4, 2 >$

$\overrightarrow{AC} \cdot < 2, 4, 2 > = < 4, 0, -4 > \cdot < 2, 4, 2 >$

$= 8 + 0 - 8$

$= 0$

$\therefore$ $\overrightarrow{AC}$ is perpendicular to $< 2, 4, 2 >$ and hence perpendicular to l_2.

(04) (a)

$l_1: \quad x = -1 + 2t_1 \qquad y = 1 - 2t_1 \qquad z = 1 + 4t_1$

$l_2: \quad x = 1 - t_2 \qquad y = t_2 \qquad z = 3 - 2t_2$

$l_3: \quad x = 1 + 2t_3 \qquad y = -1 - t_3 \qquad z = 4 + 3t_3$

(i) Suppose l_2 and l_3 intersect.

$\therefore 1 - t_2 = 1 + 2t_3$ — ① $\qquad t_2 = -1 - t_3$ — ② $\qquad 3 - 2t_2 = 4 + 3t_3$ — ③

$\therefore t_2 = -2t_3 \qquad -2t_3 = -1 - t_3$

$$\left.\begin{aligned} 1 &= t_3 \\ \therefore t_2 &= -2 \end{aligned}\right\} \text{substitute in ③}$$

$$3 - 2t_2 = 4 + 3t_3$$

$$3 - 2(-2) = 4 + 3(1)$$

$$7 = 7$$

$\therefore$ Our assumption is correct and hence l_2 and l_3 are intersected by each other.

Let the acute angle between l_2 and l_3 be θ.

let $\underline{b}$ and $\underline{c}$ be the direction vectors of l_2 and l_3 respectively.

$\therefore \underline{b} = \langle -1, 1, -2 \rangle$

$\underline{c} = \langle 2, -1, 3 \rangle$

$\therefore |\underline{b} \cdot \underline{c}| = \|\underline{b}\| \cdot \|\underline{c}\| \cdot \cos\theta$

$$\left|(-1)\cdot 2 + 1(-1) + (-2)\cdot 3\right| = \sqrt{(-1)^2+1^2+(-2)^2}\cdot\sqrt{2^2+(-1)^2+3^2}\cdot\cos\theta$$

$$9 = \sqrt{6}\,\sqrt{14}\,\cos\theta$$

$$\therefore \cos\theta = \frac{9}{\sqrt{84}}$$

$$\theta = \cos^{-1}\left(\frac{9}{\sqrt{84}}\right)$$

(ii)

coplanar - lie on the same plane
(intersecting, parallel, coincident)

skew - not coplanar
(neither parallel nor intersecting)

let the direction vector of l_1 be $\underline{a}$.

$\therefore \underline{a} = \langle 2, -2, 4 \rangle$

suppose $\underline{a} = d \cdot \underline{c}$; $d \in \mathbb{R}$.

$2 = d2$
$1 = d$

$-2 = d(-1)$
$-2 = 1(-1)$
$-2 = -1$
false

$4 = d \cdot 3$
$4 = 1(3)$
$4 = 3$
false

$\therefore$ we can't find $d \in \mathbb{R}$ such that $\underline{a} = d \cdot \underline{c}$

$\therefore$ for all $d \in \mathbb{R}$; $\underline{a} \neq d\underline{c}$ $\therefore$ $\underline{a}$ and $\underline{c}$ are not parallel.

Similar to part (i) above, we can show that (10)
l_1 and l_3 are not intersecting.

$\therefore$ l_1 and l_3 are not coplanar.

(05) (a)

(i) $\overrightarrow{PQ} = \langle 1, 8, -5 \rangle$

$\overrightarrow{PR} = \langle 4, 1, -5 \rangle$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 8 & -5 \\ 4 & 1 & -5 \end{vmatrix} = \langle -35, -15, -31 \rangle$$

let $\vec{n} = \langle 35, 15, 31 \rangle$

$\therefore$ $\vec{n}$ is perpendicular to the plane which goes through the three points P, Q and R.

$\vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{P}$

$\therefore$ $\langle 35, 15, 31 \rangle \cdot \langle x, y, z \rangle = \langle 35, 15, 31 \rangle \langle 1, -1, 4 \rangle$

$\therefore$ $35x + 15y + 31z = 144$

(b) (ii)

$$l: \vec{r(t)} = \langle 1, -2, -1 \rangle + t \underbrace{\langle 4, 5, 6 \rangle}_{a}$$

$$p: \quad x + 2y + 3z = 5$$

$$\underbrace{\langle 1, 2, 3 \rangle}_{\vec{u}} \cdot \langle x, y, z \rangle = 5$$

$$\underline{a} \cdot \underline{u} = \langle 4, 5, 6 \rangle \cdot \langle 1, 2, 3 \rangle = 32$$

$$\therefore \underline{a} \cdot \underline{u} \neq 0$$

$\therefore$ $\underline{a}$ and $\underline{u}$ are not perpendicular to each other and hence l and p are not parallel. i.e. l and p intersect with each other. Let the point of intersection of l and p be $P_0(x_0, y_0, z_0)$.

P_0 on $l \Rightarrow$

$$\langle x_0, y_0, z_0 \rangle = \langle 1+4t, -2+5t, -1+6t \rangle$$

P_0 on $p \Rightarrow$

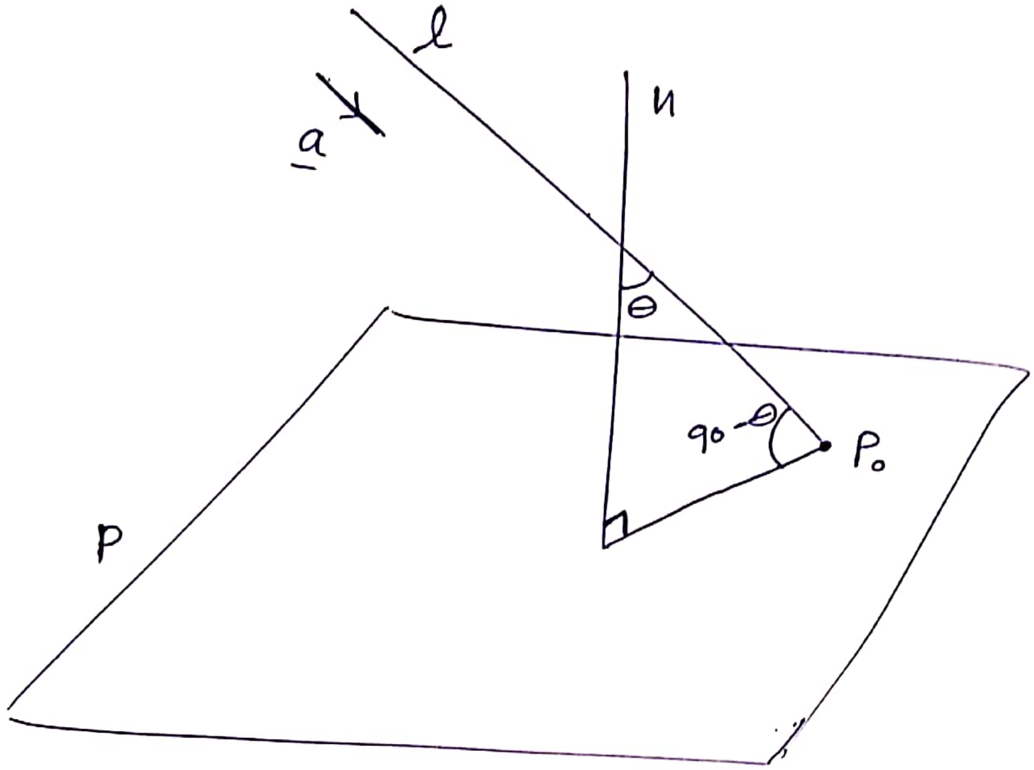
$$x_0 + 2y_0 + 3z_0 = 5$$

$$\therefore 1 + 4t + 2(-2+5t) + 3(-1+6t) = 5$$

$$t = \frac{11}{32}$$

$$\therefore \langle x_0, y_0, z_0 \rangle = \left\langle 1 + 4\left(\frac{11}{32}\right), -2 + 5\left(\frac{11}{32}\right), -1 + 6\left(\frac{11}{32}\right) \right\rangle$$

$$= \left\langle \frac{76}{32}, \frac{-9}{32}, \frac{34}{32} \right\rangle$$



$$\underline{a} \cdot \underline{n} = \|\underline{a}\| \cdot \|\underline{n}\| \cdot \cos\theta$$

$$32 = \sqrt{16+25+36} \cdot \sqrt{1+4+9} \quad \cos\theta$$

$$32 = \sqrt{77}\ \sqrt{14} \quad \cos\theta$$

$$\frac{32}{\sqrt{1078}} = \cos\theta \qquad \therefore \theta = \cos^{-1}\left(\frac{32}{\sqrt{1078}}\right)$$

$\therefore$ angle between l and P =

$$\frac{\pi}{2} - \cos^{-1}\left(\frac{32}{\sqrt{1078}}\right)$$

(06) (a) (ii)

$P_1: \quad 3x - y + 2z - 4 = 0$

$P_2: \quad -2x + y - 4z + 3 = 0$

let $z = t$

$$\therefore \quad x = 1 + 2t$$
$$y = -1 + 8t$$
$$z = t$$

let l be the line of intersection of the two planes.

$\therefore l: \quad x = 1 + 2t \; ; \; y = -1 + 8t \; ; \; z = t$

The acute angle between two planes is the same as the angle between the two normal vectors.

$\therefore \vec{n_1} = \langle 3, -1, 2 \rangle$

$\vec{n_2} = \langle -2, 1, -4 \rangle$

$\|\vec{n_1}\| = \sqrt{9+1+4} = \sqrt{14}$

$\|\vec{n_2}\| = \sqrt{4+1+16} = \sqrt{21}$

$$\therefore \left| \vec{n_1} \cdot \vec{n_2} \right| = \|\vec{n_1}\| \cdot \|\vec{n_2}\| \cdot \cos\theta$$

$$|-6-1-6| = \sqrt{14} \cdot \sqrt{21} \cdot \cos\theta$$

$$\frac{13}{\sqrt{294}} = \cos\theta$$

$$\therefore \theta = \cos^{-1}\left(\frac{3}{\sqrt{294}}\right)$$

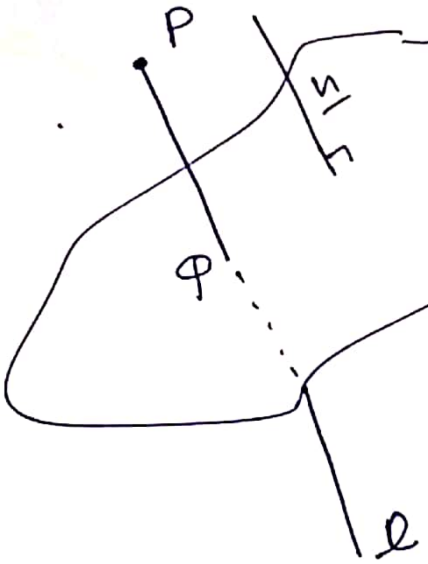
(b)

(ii) Let $Q(x_0, y_0, z_0)$ be the foot of the perpendicular from P to the plane. (14)

$P(3, 5, 2)$

$3x - 2y + z = 4$

$$\underbrace{\langle 3, -2, 1 \rangle}_{\vec{n}} \cdot \langle x, y, z \rangle = 4$$



$\therefore\ l: \ \vec{r(t)} = \langle 3, 5, 2 \rangle + t \langle 3, -2, 1 \rangle$

$\therefore\ \langle x_0, y_0, z_0 \rangle = \langle 3 + 3t_0, 5 - 2t_0, 2 + t_0 \rangle$

since Q lies on the plane

$3(3 + 3t_0) - 2(5 - 2t_0) + 2 + t_0 = 4$

$\therefore\ t_0 = \frac{3}{14}$

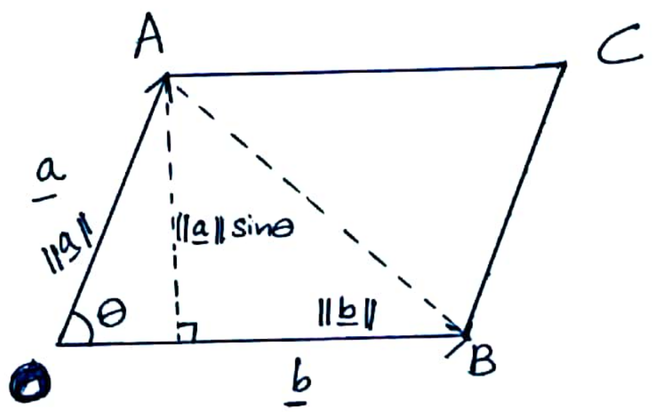
$$\therefore \langle x_0, y_0, z_0 \rangle = \langle 3 + 3\left(\frac{3}{14}\right), 5 - 2\left(\frac{3}{14}\right), 2 + \frac{3}{14} \rangle \quad (15)$$

$$\therefore \overrightarrow{PQ} = \frac{3}{14}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$$

$$\|\overrightarrow{PQ}\| = \frac{3}{14}\sqrt{14}$$

$$= 0.802.$$

Area



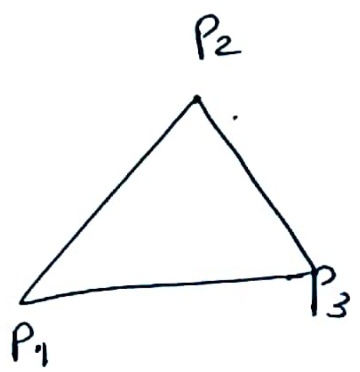
$\therefore$ Area of the Δ OAB $= \frac{1}{2} \cdot \|\underline{b}\| \cdot \|\underline{a}\| \sin\theta$

$= \frac{1}{2} \|\underline{a} \times \underline{b}\|$

Area of the parallelogram OACB

$= 2 \times$ Area of the Δ OAB

$= \|\underline{a} \times \underline{b}\|$



(08) (a)

(i) $\overrightarrow{P_1P_2} = \langle 1, 2, 3 \rangle$

$\overrightarrow{P_1P_3} = \langle 1, -3, -5 \rangle$

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_2P_3} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 1 & -3 & -5 \end{vmatrix}$$

$= \langle -1, 8, -5 \rangle$

$\therefore$ Area $= \frac{1}{2} \| \langle -1, 8, -5 \rangle \| = \frac{3}{2}\sqrt{10}$

Special Products

The Scalar triple product

$$\underline{a} \cdot (\underline{b} \times \underline{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Vector Triple Product

$$\underline{a} \times (\underline{b} \times \underline{c}) = (\underline{a} \cdot \underline{c})\,\underline{b} - (\underline{a} \cdot \underline{b})\,\underline{c}$$

Volume of a Parallelepiped

Suppose the vectors $\underline{a}$, $\underline{b}$, and $\underline{c}$ are not coplanar. Then,

$$\left.\begin{array}{l}\text{Volume of the} \\ \text{Parallelepiped}\end{array}\right\} = |\underline{a} \cdot (\underline{b} \times \underline{c})|$$

(08) (c)

(i) $\underline{a} = \langle 7, 1, 9 \rangle$

$\underline{b} = \langle 1, 2, 3 \rangle$

$\underline{c} = \langle 3, 0, 6 \rangle$

$$\text{Volume} = |\underline{a} \cdot (\underline{b} \times \underline{c})|$$

$$= \begin{vmatrix} 7 & 1 & 9 \\ 1 & 2 & 3 \\ 3 & 0 & 6 \end{vmatrix}$$

$$= 7(12-0) - 1(6-9) + 9(0-6)$$

$$= 84 + 3 - 54$$

$$= 33$$