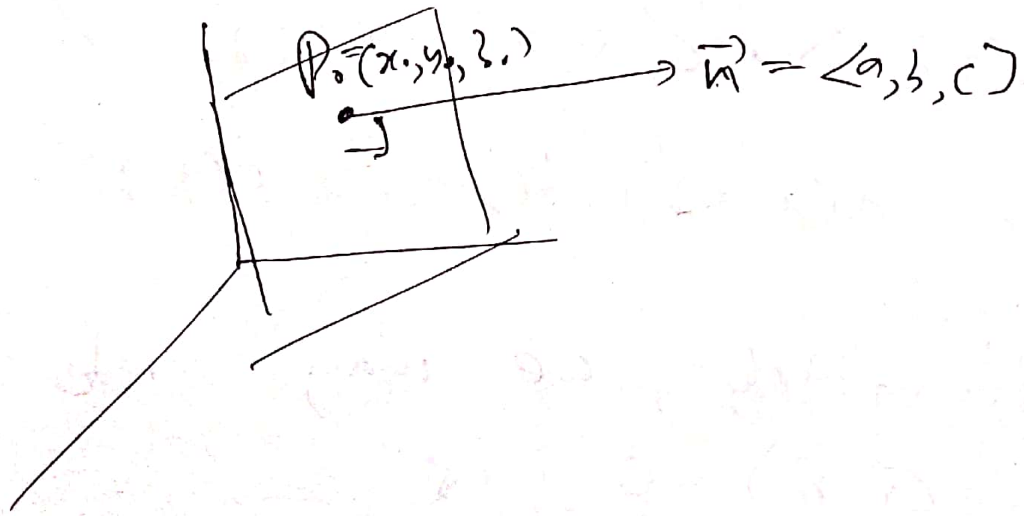
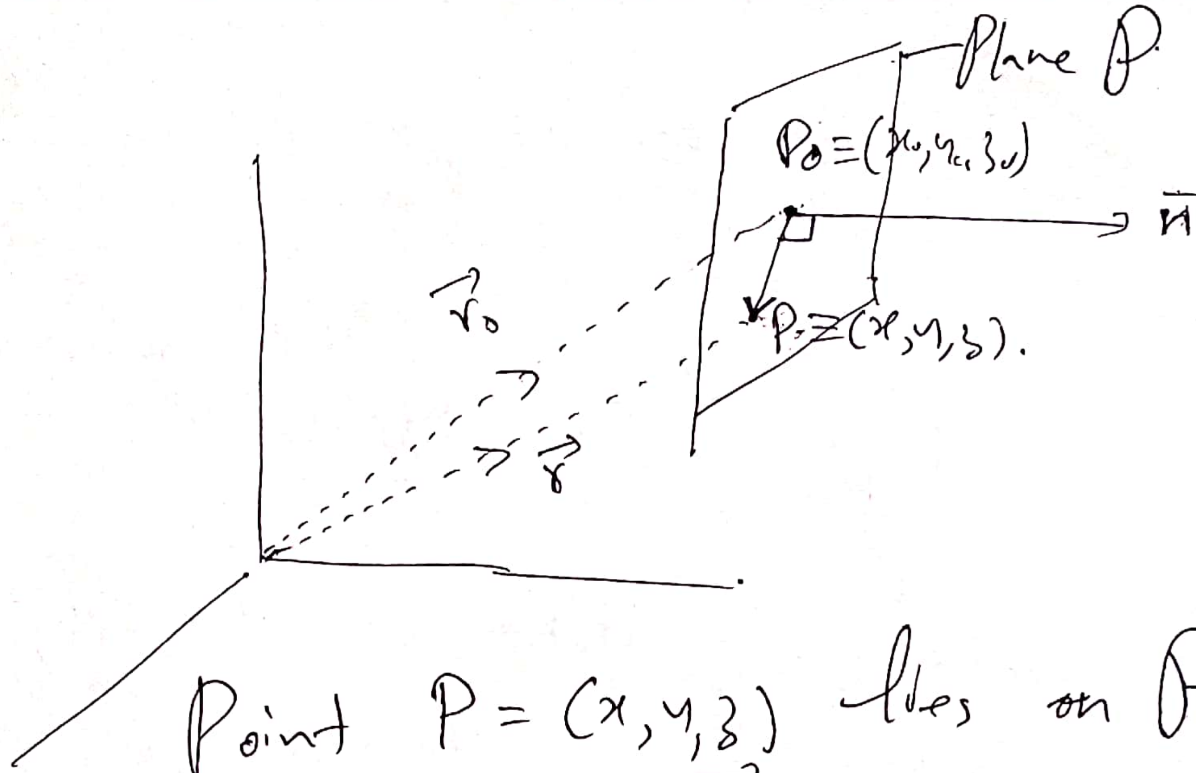
Plane in Three-space

Consider a plane P passing through a point $P_0 = (x_0, y_0, z_0)$. Although infinitely many planes pass through P_0, we can identify the particular plane P by specifying a vector $\vec{n} = \langle a, b, c \rangle$ normal to P.



Let $P = (x, y, z)$ be any point in the plane and position vectors are $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$ & $\vec{r} = \langle x, y, z \rangle$.



Point $P = (x, y, z)$ lies on P if and only if $\overrightarrow{P_0P}$ is perpendicular to $\vec{n}$.

i.e. $\vec{n} \cdot \overrightarrow{P_0P} = 0$

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$

$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

$\Rightarrow \quad a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

Alternatively, we may write

$\vec{n} \cdot (\vec{r} - \vec{r}_0) = 0$ as,

$$\boxed{\begin{gathered} \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r}_0 \quad \text{or} \\ \langle a, b, c \rangle \cdot \langle x, y, z \rangle = \langle a, b, c \rangle \cdot \langle x_0, y_0, z_0 \rangle \\ \Rightarrow \quad ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{= d}. \end{gathered}}$$

Equation of a Plane

An equation of the plane through $P_0 = (x_0, y_0, z_0)$ with non-zero normal vector $\vec{n} = \langle a, b, c \rangle$ can be written in the following three ways.

$$\boxed{\begin{aligned} &\vec{n} \cdot \langle x, y, z \rangle = d. \\ &a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \\ &ax + by + cz = d \end{aligned}}$$

where $d = \vec{n} \cdot \langle x_0, y_0, z_0 \rangle$

$d = ax_0 + by_0 + cz_0$.

Ex Find an equation of the plane through $P_0 = (3, 1, 0)$ with normal vector $\vec{n} = \langle 3, 2, -5 \rangle$.

Sol^n.

$\langle 3, 2, -5 \rangle \cdot \langle x, y, z \rangle = \langle 3, 2, -5 \rangle \cdot \langle 3, 1, 0 \rangle$

$3x + 2y - 5z = 11$

Ex. Find an equation of the plane P through $P_0 = (1, 2, 0)$ with normal vector $\vec{n} = \langle 0, 0, 3 \rangle$.

Sol^n.

Note that P_0 lies on the xy-plane and $\vec{n}$ points along the z-axis, orothogonal to the xy-plane.

$\vec{n} \cdot \langle x, y, z \rangle = d = \vec{n} \cdot \langle 1, 2, 0 \rangle$.

$\langle 0, 0, 3 \rangle \cdot \langle x, y, z \rangle = \langle 0, 0, 3 \rangle \cdot \langle 1, 2, 0 \rangle$.

$3z = 0$

$z = 0$

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Linear Algebra

Ex. 3 kg of Oranges and 2kg of Apples cost ~~340/20~~ 340 ruppees. 1 kg of Oranges and 4 kg of Apples cost 280 rupees. Find the cost of 1 kg of oranges & 1 kg of Apple?

Simultaneous Linear Equations

A system of simultaneous linear equations is a set of equations of the form,

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_m = b_m$$

The coefficients a_{ij} ($i=1, \ldots, m$ and $j=1, \ldots n$) and the quantities b_i ($i=1, 2, \ldots, m$) are known constant.

The x_j ($j=1, 2, \ldots, n$) are variables.

Note: A solution for the above system is a set of values for variables, that makes all the equations true.

Ex: Solve following system.

$$\cancel{x+y} \quad x - y = 7$$
$$x + y = 5.$$

$\begin{cases} x = 6 \\ y = -1 \end{cases} \longrightarrow$ system has a unique solution.

Ex. Solve following system

$$\left.\begin{array}{r} x - y = 7 \\ 2x - 2y = 14 \end{array}\right\} \Rightarrow x - y = 7$$

$\begin{cases} x = 7 \\ y = 0 \end{cases}, \begin{cases} x = 0 \\ y = -7 \end{cases}, \begin{cases} x = 8 \\ y = 1 \end{cases} \longrightarrow$

system has an infinite number of solution

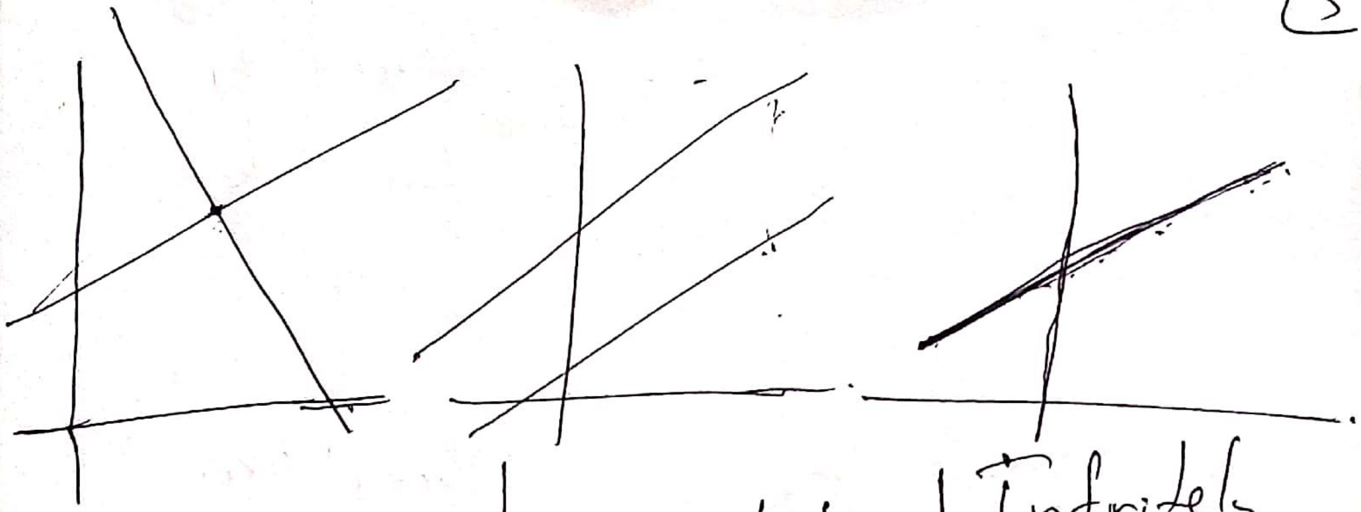
Ex Solve following system.

$$\left.\begin{aligned} x-y &= 7 \\ 2x-2y &= 13 \end{aligned}\right\}. \qquad \left\{\begin{aligned} x-y &= 7 \\ x-y &= 13/2 \end{aligned}\right.$$

$\rightarrow$ system has no solution

Note

- A set of simultaneous equations is consistent if it has at least one solution.
- A system with no solution is said to be ~~ka~~ inconsistent.



unique solution | no solution | Infinitely many solution

Gauss – Jordan Elimination

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$\vdots$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m.$$

Using matrix notation, we can write system as the <u>augmented</u> matrix.

$$\left(\begin{array}{cccccc|c} a_{11} & a_{12} & a_{13} & \cdots & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} & \cdots & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & \cdots & a_{mn} & b_m \end{array}\right)$$

Following are the elementrary row operations applied to the augumented matrix.

Elementrary Row Operations

① Interchanging two rows.

② Multiply (or divide) one row by a non-zero number

③ Add a multiple of one row to another row.

Ex Solve following system.

$$2x + 4y + 6z = 18$$
$$4x + 5y + 6z = 24$$
$$3x + y - 2z = 4.$$

$$\left(\begin{array}{ccc|c} 2 & 4 & 6 & 18 \\ 4 & 5 & 6 & 24 \\ 3 & 1 & -2 & 4 \end{array}\right) \xrightarrow{r_1 \to \frac{1}{2} r_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 4 & 5 & 6 & 24 \\ 3 & 1 & -2 & 4 \end{array}\right)$$

$r_2 \to r_2 + (-4) r_1$
$r_3 \to r_3 + (-3) r_1$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & 1 & 2 & 4 \\ 0 & -5 & -11 & -23 \end{array}\right) \xleftarrow{r_2 \to \frac{r_2}{(-3)}} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -3 & -6 & -12 \\ 0 & -5 & -11 & -23 \end{array}\right)$$

$r_1 \to r_1 - 2 r_2$
$r_3 \to r_3 - 3 r_1$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -1 & -3 \end{array}\right) \xrightarrow{r_3 \to -r_3} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 3. \end{array}\right)$$

$r_1 \to r_1 + r_3$
$r_2 \to r_2 - 2 r_3$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right).$$

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$$\left\{\begin{array}{l} x=4 \\ y=-2 \\ z=3 \end{array}\right\}$$ //

Gauss- Jordan elimination method

means:

$$\left(\begin{array}{c|c} & \end{array}\right) \xrightarrow{\text{Convert}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & \square \\ 0 & 1 & 0 & \square \\ 0 & 0 & 1 & \square \end{array}\right)$$

augumented matrix

reduced row echelon form.

Ex. Solve the system

$$2x_2 + 3x_3 = 4$$

$$2x_1 - 6x_2 + 7x_3 = 15$$

$$x_1 - 2x_2 + 5x_3 = 10$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 3 & 4 \\ 2 & -6 & 7 & 15 \\ 1 & -2 & 5 & 10 \end{array}\right) \xrightarrow{r_1 \leftrightarrow r_3} \left(\begin{array}{ccc|c} 1 & -2 & 5 & 10 \\ 2 & -6 & 7 & 15 \\ 0 & 2 & 3 & 4 \end{array}\right)$$

$r_2 \to r_2 - 2r_1$

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$$\left(\begin{array}{ccc|c} 1 & -2 & 5 & 10 \\ 0 & -2 & -3 & -5 \\ 0 & 2 & 3 & 4 \end{array}\right) \xrightarrow{r_2 \to \frac{r_2}{(-2)}} \left(\begin{array}{ccc|c} 1 & -2 & 5 & 10 \\ 0 & 1 & 3/2 & 5/2 \\ 0 & 2 & 3 & 4 \end{array}\right)$$

$r_1 \to r_1 + 2r_2$

$r_3 \to r_3 - 2r_2$

$$\left(\begin{array}{ccc|c} 1 & 0 & 8 & 15 \\ 0 & 1 & 3/2 & 5/2 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

Last row means, ~~$0 \neq 1$~~

$0 = 1$, #.

$\therefore$ Thus system has no solution.

System is inconsistent.

Matrix

Def^{n} (Matrix).

A matrix is a rectangular array of numbers.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & & a_{2n} \\ \vdots & \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & & a_{mn} \end{pmatrix}_{m \times n}$$

Usually matrices denoted by capidial letters.

component.

order (or size)
m = no. of rows
n = no. of column

Sometimes write the matrix A as,

$$A = (a_{ij})_{m \times n} \quad ; \quad i = 1, \ldots, m \; ; \; j = 1, \ldots, n.$$

Note:

* If A is a $m \times n$ matrix with $m = n$, ~~then~~ A is called a square matrix

* An $m \times n$ matrix with all components equals to zero is called the $m \times n$ zero matrix.

Def^n. (Equality of Matrices).

Two matrices A & B are equal if

* they have same size and
* corresponding components are equal.

i.e. $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$.

Then $A = B$ if

$$a_{ij} = b_{ij} \text{ for any } i \text{ and } j$$

Matrix Operation

Addition of Matrices

Let $A = (a_{ij})$ & $B = (b_{ij})$ be two $m \times n$ matrices. (i.e. order must be equal). Then the sum of A & B is the $m \times n$ matrix $A+B$ given by.

$$A+B = (a_{ij}+b_{ij})_{m \times n}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

⊘ adding the corresponding components of A & B.

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