Homogeneous Equation

Definition

A homogeneous equation of the first order and first degree is one which can be written in the form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \longrightarrow (\star)$$

- To test whether a function of x and y can be written in the form of the right hand side of $(\star)$, it is convenient to put $\frac{y}{x} = v$ or $y = vx$.
- If the result is of the form $f(v)$, that is if all the x's cancel, the test is satisfied.
- By putting $v = yx$ to a homogeneous equation the left hand side becomes
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$
- Therefore, by putting $y = vx$, we get
$$v + x\frac{dv}{dx} = f(v).$$

Example (16)

Solve the differential equation:

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2x^2}.$$

Equations Reducible to the Homogeneous Form

Example (17)

Solve the differential equation:

$$\frac{dy}{dx} = \frac{y - x + 1}{y + x + 5}.$$

Equations Reducible to the Homogeneous Form

Example (18)

Solve the differential equation:

$$\frac{dy}{dx} = \frac{y - x + 1}{y - x + 5}.$$

Additional Problems

1. $$\frac{dy}{dx} = (x + y + 1)^2$$
2. $$\frac{dy}{dx} = \tan^2(x + y)$$
3. $$\frac{dy}{dx} = 2 + \sqrt{y - 2x + 3}$$
4. $$\frac{dy}{dx} = 1 + e^{y-x+5}$$
5. $$\frac{dy}{dx} = \frac{4x + 6y + 5}{3y + 2x + 4}$$

4.4 Second Order Differential Equations with Constant Coefficients

Second Order Linear Differential Equations

Definition

A second order linear differential equation has the form

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = G(x)$$

where P, Q, R and G are continuous functions.

- When $G(x) = 0$, we call the differential equation is **homogeneous** and when $G(x) \neq 0$, we call the differential equation is **non-homogeneous**.
- The form of a second order linear homogeneous differential equation is

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$$

where P, Q and R are continuous functions.

Initial Value Problem

Definition

An **initial-value problem** for the second-order linear homogeneous or non-homogeneous differential equation consists of finding a solution y of the differential equation that also satisfies initial conditions of the form $y(x_0) = y_0$ and $y'(x_0) = y_1$, where y_0 and y_1 are given constants.

Second Order Differential Equations with Constant Coefficients

Definition

A second order linear differential equation with constant coefficients has the form

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = g(x)$$

or

$$ay'' + by, + cy = g(x)$$

, where a, b, c are constants and $g(x)$ is a continuous function.

Solving Second Order Linear Homogeneous ODE with Constant Coefficients

Consider the second order linear homogeneous differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \longrightarrow (\star)$$

where a, b and c are constants.

The solution of equation $(\star)$ suggests that

$$y = Ae^{mx}$$

where A and m are constants.

Then,

$$y' = mAe^{mx}$$

and

$$y'' = m^2Ae^{mx}$$

Substituting in $(\star)$,

$$a[m^2 Ae^{mx}] + b[mAe^{mx}] + c[Ae^{mx}] = 0$$

$$[am^2 + bm + c]\ Ae^{mx} = 0$$

Since $Ae^{mx} \neq 0$;

$$am^2 + bm + c = 0$$

Thus, if m is a root of $am^2 + bm + c = 0$, then $y = Ae^{mx}$ is a solution of equation $(\star)$, whatever the value of A.

Let the roots of equation $am^2 + bm + c = 0$ be α and β.

Then, if α and β are unequal, we have two solutions of equation $(\star)$, namely $y = Ae^{\alpha x}$ and $y = Be^{\beta x}$.

Now, if we substitute $y = Ae^{\alpha x} + Be^{\beta x}$ in equation $(\star)$, we shall get

$$Ae^{\alpha x}(a\alpha^2 + b\alpha + c) + Be^{\beta x}(a\beta^2 + b\beta + c) = 0$$

which is true as α and β are the roots of equation $am^2 + bm + c = 0$.

Thus the sum of two solutions gives the third solution. As this third solution contains two arbitrary constants, we shall regard it as the general solution.

Equation $am^2 + bm + c = 0$ is known as the **auxiliary equation**.

Example (19)

1. Solve $2\dfrac{d^2y}{dx^2} + 5\dfrac{dy}{dx} + 2y = 0.$
2. Solve $\dfrac{d^2y}{dx^2} - 9y = 0.$

Example (20)

Solve the IVP:

$$\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 24y = 0 \ ; \ y(0) = 0 \ ; \ y'(0) = -7$$

Modification when the auxiliary equation has imaginary or complex roots

When the auxiliary equation $am^2 + bm + c = 0$ has roots of the form $p + iq$, $p - iq$ where $i = \sqrt{-1}$, it is best to modify the solution

$$y = Ae^{(p+iq)x} + Be^{(p-iq)x}$$

so as to present it without imaginary quantities.

Since $e^{iqx} = \cos qx + i \sin qx$ and $e^{-iqx} = \cos qx - i \sin qx$,

$$y = e^{px}\left[A(\cos qx + i \sin qx) + B(\cos qx - i \sin qx)\right]$$

$$y = e^{px}\left[E \cos qx + F \sin qx\right]$$

where $E = A + B$ and $F = i(A - B)$, are arbitrary constants.

Here, note that F is not necessarily an imaginary number.

Example (21)

Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 9y = 0.$

When the roots of the auxiliary equation are equal

Suppose auxiliary equation has equal roots $\alpha = \beta$.

Then, the solution is

$$y = Ae^{\alpha x} + Bxe^{\alpha x}$$

$$y = (A + Bx)e^{\alpha x}$$

Example (22)

Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0.$

Additional Examples

1. $2y" - 5y' - 3y = 0$
2. $y" - 10y' + 25y = 0$
3. $y" + 4y' + 7y = 0$
4. $y" + 3y' - 10y = 0,\ y(0) = 4,\ y'(0) = -2$
5. $3y" - 2y' - 8y = 0,\ y(0) = -6,\ y'(0) = -18$
6. $4y" - 5y' = 0,\ y(-2) = 0,\ y'(-2) = 7$
7. $y" - 8y' + 17y = 0,\ y(0) = -4,\ y'(0) = -1$

Second Order Linear Non-homogeneous Differential Equations with Constant Coefficients

Theorem

The general solution of the second order non-homogeneous differential equation

$$ay'' + by' + cy = g(x)$$

where a, b, c are constants, can be written as

$$y = Complementary\ Function + Particular\ Integral$$

$$y = CF + PI$$

- The complementary function is the solution of the equation

$$ay" + by' + cy = 0$$

- If $y = u$ is a particular integral of

$$ay" + by' + cy = g(x)$$

,
so that

$$a\frac{d^2u}{dx^2} + b\frac{du}{dx} + cu = g(x)$$

Solution by D-Operator

In calculus, differentiation is often denoted by the capital letter D, which represents $\dfrac{d}{dx}$ and called D-operator.

$$\frac{dy}{dx} = Dy$$

Higher-order derivatives can be expressed in terms of D,

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = D(Dy) = D^2y$$

Similarly,

$$\frac{d^ny}{dx^n} = D^ny$$

- Any linear differential equation can be expressed in terms of the D-operator,

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

$$aD^2y + bDy + cy = 0$$

$$(aD^2 + bD + c)y = 0$$

- In generally, a higher order homogeneous ODE,

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + ... + a_1 y' + a_0 y = 0$$

can be expressed in terms of the D-operator,

$$(a_n D^n + a_{n-1}D^{n-1} + ... + a_1 D + a_0)y = 0$$

Properties of D-Operator

1. $D(u+v) = Du + Dv$
2. $D^m D^n u = D^{m+n} u \; ; \; m, n \in \mathbb{Z}^+$
3. $D(cu) = cDu \; ; \; c$ is a constant

Theorem

Let $F(D) = a_n D^n + a_{n-1} D^{n-1} + ... + a_1 D + a_0$*, where* a_i*'s are constants,* $\forall\ i = 1, 2, ..., n$ *and* $n \in \mathbb{Z}^+$*.*
Then,

1. $F(D)e^{ax} = e^{ax} F(a)$
2. $F(D)\left\{e^{ax} V(x)\right\} = e^{ax} F(D + a) V(x)$
3. $F(D^2)\cos ax = F(-a^2)\cos ax$

Symbolical methods of finding the Particular Integral when $g(x) = e^{ax}$

We use the notation

$$PI = \frac{1}{F(D)}\, g(x)$$

to denote a particular integral of the equation $F(D)y = g(x)$.

If $f(x) = e^{ax}$, then

$$F(D)e^{ax} = e^{ax}F(a)$$

(i) If $F(a) \neq 0$,

$$PI = \frac{1}{F(a)}\, e^{ax}$$

(ii) If $F(a) = 0$, then $(D - a)$ be a factor of $F(D)$.
Suppose that,

$$F(D) = (D-a)^p \phi(D)$$

where $\phi(a) \neq 0$

$$PI = \frac{1}{F(D)}\, e^{ax} = \frac{1}{(D-a)^p \phi(D)}\, e^{ax} = \frac{1}{(D-a)^p} \left\{ \frac{e^{ax} \cdot 1}{\phi(a)} \right\}$$

Since $F(D)\,\{e^{ax} V(x)\} = e^{ax} F(D+a) V(x)$ with $V = 1$,

$$PI = \frac{e^{ax}}{\phi(a)} \cdot \frac{1}{D^p} \cdot 1 = \frac{e^{ax}}{\phi(a)} \cdot \frac{x^p}{p!}$$

- ☆ $\dfrac{1}{D}$ is the inverse operator of D, that is the operator that integrates with respect to x.
- ☆ $\dfrac{1}{D^p} \longrightarrow$ integrate p times

Example (23)

Find the general solution of the differential equation:

$$(D + 3)^2 y = 50e^{2x}$$

Example (24)

Find the general solution of the differential equation:

$$(D-2)^2 y = 50e^{2x}$$

Particular integral when $g(x) = \cos ax$

$$F(D^2)\cos ax = F(-a^2)\cos ax$$

This suggests that we may obtain the PI by writing $-a^2$ for D^2 whenever it occurs.

Example (25)

Find the general solution of the differential equation:

$$(D^2 + 3D + 2)y = \cos 2x$$

Example (26)

Find the general solution of the differential equation:

$$(D^3 + 6D^2 + 11D + 6)y = 2\sin 3x$$