Chapter - 02

Infinite Series

Independent Study Material No. 01

Strategy for Testing the Series

1. If the series is of the form $\sum \frac{1}{n^p}$, it is a p-series, which we know to be convergent if $p > 1$ and divergent if $p \leq 1$.

2. If the series has the form $\sum ar^{n-1}$ or $\sum ar^n$, it is a geometric series, which converges if $|r| < 1$ and diverges if $|r| \geq 1$. Some preliminary algebraic manipulation may be required to bring the series into this form.

3. If the series has a form that is similar to a p-series or a geometric series, then one of the comparison tests should be considered.
   In particular, if a_n is a rational function or an algebraic function of n (involving roots of polynomials), then the series should be compared with a p-series.
   The comparison tests apply only to series with positive terms, but if $\sum a_n$ has some negative terms, then we can apply the Comparison Test to $\sum |a_n|$ and test for absolute convergence.

4. If you can see at a glance that $\lim_{n\to\infty} a_n \neq 0$, then the Test for Divergence should be used.

5. If the series is of the form $\sum (-1)^{n-1}\, b_n$ or $\sum (-1)^n\, b_n$, then the Alternating Series Test is an obvious possibility.

6. Series that involve factorials or other products (including a constant raised to the n^th power) are often conveniently tested using the Ratio Test. Bear in mind that $\left|\frac{a_{n+1}}{a_n}\right| \to 1$ as $n \to \infty$ for all p-series and therefore all rational or algebraic functions of n. Thus the Ratio Test should not be used for such series.

7. If a_n is of the form $(b_n)^n$, then the Root Test may be useful.

8. If $a_n = f(n)$, where $\int_1^\infty f(x)\, dx$ is easily evaluated, then the Integral Test is effective (assuming the hypotheses of this test are satisfied).

Illustrative Example 01

$$\sum_{n=1}^{\infty} \frac{n-1}{2n+1}$$

Answer:

Since $a_n \to \frac{1}{2}$ as $n \to \infty$, we should use the Test for Divergence.

Illustrative Example 02

$$\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3+4n^2+2}$$

Answer:

Since a_n is an algebraic function of n, we compare the given series with a p-series. The comparison series for the Limit Comparison Test is $\sum b_n$, where

$$b_n = \frac{\sqrt{n^3}}{3n^3} = \frac{n^{\frac{3}{2}}}{3n^3} = \frac{1}{3n^{\frac{3}{2}}}$$

Illustrative Example 03

$$\sum_{n=1}^{\infty} ne^{-n^2}$$

Answer:

Since the integral $\int_1^{\infty} xe^{-x^2}\, dx$ is easily evaluated, we use the Integral Test. The Ratio Test also works.

Illustrative Example 04

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+1}$$

Answer:

Since the series is alternating, we use the Alternating Series Test.

Illustrative Example 05

$$\sum_{k=1}^{\infty} \frac{2^k}{k!}$$

Answer:

Since the series involves $k!$, we use the Ratio Test.

Illustrative Example 06

$$\sum_{n=1}^{\infty} \frac{1}{2+3^n}$$

Answer:

Since the series is closely related to the geometric series $\sum \frac{1}{3^n}$, we use the Comparison Test.

Practice Problems

1. Use the Definition

   27–42 Determine whether the series is convergent or divergent. If it is convergent, find its sum.

   27. $\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \frac{1}{12} + \frac{1}{15} + \cdots$

   28. $\frac{1}{3} + \frac{2}{9} + \frac{1}{27} + \frac{2}{81} + \frac{1}{243} + \frac{2}{729} + \cdots$

   29. $\sum_{n=1}^{\infty} \frac{2+n}{1-2n}$

   30. $\sum_{k=1}^{\infty} \frac{k^2}{k^2-2k+5}$

   31. $\sum_{n=1}^{\infty} 3^{n+1}4^{-n}$

   32. $\sum_{n=1}^{\infty} [(-0.2)^n + (0.6)^{n-1}]$

   33. $\sum_{n=1}^{\infty} \frac{1}{4+e^{-n}}$

   34. $\sum_{n=1}^{\infty} \frac{2^n+4^n}{e^n}$

   35. $\sum_{k=1}^{\infty} (\sin 100)^k$

   36. $\sum_{n=1}^{\infty} \frac{1}{1+\left(\frac{2}{3}\right)^n}$

   37. $\sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$

   38. $\sum_{k=0}^{\infty} (\sqrt{2})^{-k}$

2. Integral Test

3–8 Use the Integral Test to determine whether the series is convergent or divergent.

3. $\sum_{n=1}^{\infty} n^{-3}$

4. $\sum_{n=1}^{\infty} n^{-0.3}$

5. $\sum_{n=1}^{\infty} \frac{2}{5n-1}$

6. $\sum_{n=1}^{\infty} \frac{1}{(3n-1)^4}$

7. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

8. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

3. Comparison Tests

3–32 Determine whether the series converges or diverges.

3. $\sum_{n=1}^{\infty} \frac{1}{n^3+8}$

4. $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}-1}$

5. $\sum_{n=1}^{\infty} \frac{n+1}{n\sqrt{n}}$

6. $\sum_{n=1}^{\infty} \frac{n-1}{n^3+1}$

7. $\sum_{n=1}^{\infty} \frac{9^n}{3+10^n}$

8. $\sum_{n=1}^{\infty} \frac{6^n}{5^n-1}$

9. $\sum_{k=1}^{\infty} \frac{\ln k}{k}$

10. $\sum_{k=1}^{\infty} \frac{k\sin^2 k}{1+k^3}$

11. $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k}}{\sqrt{k^3+4k+3}}$

12. $\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

13. $\sum_{n=1}^{\infty} \frac{1+\cos n}{e^n}$

14. $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{3n^4+1}}$

15. $\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$

16. $\sum_{n=1}^{\infty} \frac{1}{n^n}$

17. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$

18. $\sum_{n=1}^{\infty} \frac{2}{\sqrt{n}+2}$

19. $\sum_{n=1}^{\infty} \frac{n+1}{n^3+n}$

20. $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^4+n^2}$

21. $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{1+n}}{2+n}$

22. $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$

23. $\displaystyle\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$

24. $\displaystyle\sum_{n=1}^{\infty} \frac{n+3^n}{n+2^n}$

25. $\displaystyle\sum_{n=1}^{\infty} \frac{e^n+1}{ne^n+1}$

26. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

27. $\displaystyle\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^2 e^{-n}$

28. $\displaystyle\sum_{n=1}^{\infty} \frac{e^{1/n}}{n}$

29. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!}$

30. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

31. $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

32. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

4. Alternating Series Test

2–20 Test the series for convergence or divergence.

2. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \frac{2}{11} - \cdots$

3. $-\frac{2}{5} + \frac{4}{6} - \frac{6}{7} + \frac{8}{8} - \frac{10}{9} + \cdots$

4. $\frac{1}{\ln 3} - \frac{1}{\ln 4} + \frac{1}{\ln 5} - \frac{1}{\ln 6} + \frac{1}{\ln 7} - \cdots$

5. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3+5n}$

6. $\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

7. $\sum_{n=1}^{\infty} (-1)^n \frac{3n-1}{2n+1}$

8. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$

9. $\sum_{n=1}^{\infty} (-1)^n e^{-n}$

10. $\sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{2n+3}$

11. $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$

12. $\sum_{n=1}^{\infty} (-1)^{n+1} n e^{-n}$

13. $\sum_{n=1}^{\infty} (-1)^{n-1} e^{2/n}$

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \arctan n$

15. $\sum_{n=0}^{\infty} \frac{\sin\left(n+\frac{1}{2}\right)\pi}{1+\sqrt{n}}$

16. $\sum_{n=1}^{\infty} \frac{n \cos n\pi}{2^n}$

5. Ratio Test

7–24 Use the Ratio Test to determine whether the series is convergent or divergent.

7. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{5^n}$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^2}$

9. $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{2^n n^3}$

10. $\displaystyle\sum_{n=0}^{\infty} \frac{(-3)^n}{(2n+1)!}$

11. $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$

12. $\displaystyle\sum_{k=1}^{\infty} ke^{-k}$

13. $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

14. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{100^n}$

15. $\displaystyle\sum_{n=1}^{\infty} \frac{n\pi^n}{(-3)^{n-1}}$

16. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{10}}{(-10)^{n+1}}$

17. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\pi/3)}{n!}$

18. $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

19. $\displaystyle\sum_{n=1}^{\infty} \frac{n^{100}100^n}{n!}$

20. $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$

21. $$1 - \frac{2!}{1 \cdot 3} + \frac{3!}{1 \cdot 3 \cdot 5} - \frac{4!}{1 \cdot 3 \cdot 5 \cdot 7} + \cdots + (-1)^{n-1}\frac{n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} + \cdots$$

6. Root Test

25–30 Use the Root Test to determine whether the series is convergent or divergent.

25. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n^2+1}{2n^2+1}\right)^n$

26. $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^n}$

27. $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{(\ln n)^n}$

28. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{-2n}{n+1}\right)^{5n}$

29. $\displaystyle\sum_{n=1}^{\infty} \left(1 + \frac{1}{n}\right)^{n^2}$

30. $\displaystyle\sum_{n=0}^{\infty} (\arctan n)^n$

7. .

1–38 Test the series for convergence or divergence.

1. $\sum_{n=1}^{\infty} \frac{n^2 - 1}{n^3 + 1}$

2. $\sum_{n=1}^{\infty} \frac{n - 1}{n^3 + 1}$

3. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 1}{n^3 + 1}$

4. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2 - 1}{n^2 + 1}$

5. $\sum_{n=1}^{\infty} \frac{e^n}{n^2}$

6. $\sum_{n=1}^{\infty} \frac{n^{2n}}{(1 + n)^{3n}}$

7. $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln n}}$

8. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^4}{4^n}$

9. $\sum_{n=0}^{\infty} (-1)^n \frac{\pi^{2n}}{(2n)!}$

10. $\sum_{n=1}^{\infty} n^2 e^{-n^3}$

11. $\sum_{n=1}^{\infty} \left(\frac{1}{n^3} + \frac{1}{3^n}\right)$

12. $\sum_{k=1}^{\infty} \frac{1}{k\sqrt{k^2 + 1}}$

13. $\sum_{n=1}^{\infty} \frac{3^n n^2}{n!}$

14. $\sum_{n=1}^{\infty} \frac{\sin 2n}{1 + 2^n}$

15. $\sum_{k=1}^{\infty} \frac{2^{k-1} 3^{k+1}}{k^k}$

16. $\sum_{n=1}^{\infty} \frac{\sqrt{n^4 + 1}}{n^3 + n}$

17. $\sum_{n=1}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n - 1)}{2 \cdot 5 \cdot 8 \cdot \cdots \cdot (3n - 1)}$

18. $\sum_{n=2}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n} - 1}$

19. $\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{\sqrt{n}}$

20. $\sum_{k=1}^{\infty} \frac{\sqrt[3]{k} - 1}{k(\sqrt{k} + 1)}$

21. $\sum_{n=1}^{\infty} (-1)^n \cos(1/n^2)$

22. $\sum_{k=1}^{\infty} \frac{1}{2 + \sin k}$

23. $\sum_{n=1}^{\infty} \tan(1/n)$

24. $\sum_{n=1}^{\infty} n \sin(1/n)$

25. $\sum_{n=1}^{\infty} \frac{n!}{e^{n^2}}$

26. $\sum_{n=1}^{\infty} \frac{n^2 + 1}{5^n}$

27. $\sum_{k=1}^{\infty} \frac{k \ln k}{(k + 1)^3}$

28. $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$

29. $\sum_{n=1}^{\infty} \frac{(-1)^n}{\cosh n}$

30. $\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j + 5}$

31. $\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$

32. $\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

33. $\sum_{n=1}^{\infty} \left(\frac{n}{n + 1}\right)^{n^2}$

34. $\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$

35. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

36. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^{\ln n}}$

37. $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$

38. $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$