Mathematics - MA 1103

Tutorial 03 - Solutions

(01) (d) $(AB)^T = B^T A^T$

Proof:-

Let $A = [a_{ij}]_{m \times n}$

$B = [b_{ij}]_{n \times p}$

$AB = [c_{ij}]_{m \times p}$ where

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

$\therefore (AB)^T = [d_{ij}]_{p \times m}$ where

$$d_{ij} = c_{ji}$$
$$= \sum_{k=1}^{n} a_{jk} b_{ki}$$

$A^T = [e_{ij}]_{n \times m}$; $e_{ij} = a_{ji}$

$B^T = [f_{ij}]_{p \times n}$; $f_{ij} = b_{ji}$

$$B^T A^T = [g_{ij}]_{p\times m} \quad \text{where}$$

$$g_{ij} = \sum_{k=1}^{n} f_{ik}\, e_{kj}$$

$$\therefore \quad g_{ij} = \sum_{k=1}^{n} b_{ki}\, a_{jk}$$

$$\therefore \quad g_{ij} = \sum_{k=1}^{n} a_{jk}\, b_{ki}$$

$$\therefore \quad B^T A^T = [g_{ij}]_{p\times m}$$

$$(AB)^T = [d_{ij}]_{p\times m}$$

where

$$g_{ij} = \sum_{k=1}^{n} a_{jk}\, b_{ki} = d_{ij}$$

$$\therefore \quad (AB)^T = B^T A^T \; \blacksquare$$

(02)

(a)

(i) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 5 \\ 3 & 5 & 8 \end{pmatrix}$ (03)

Augmenting with I:

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 5 & 0 & 1 & 0 \\ 3 & 5 & 8 & 0 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -1 & -1 & -3 & 0 & 1 \end{array}\right)$$

$$\Big\downarrow R_3 \to R_3 + R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -2 & -5 & 1 & 1 \end{array}\right)$$

$$\Big\downarrow R_3 \to -\frac{1}{2} R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_1 \to R_1 - 3R_3 \\ R_2 \to R_2 + R_3 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 2 & 0 & -13/2 & 3/2 & 3/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right)$$

$$\Big\downarrow R_1 \to R_1 - 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -15/2 & 1/2 & 5/2 \\ 0 & 1 & 0 & 1/2 & 1/2 & -1/2 \\ 0 & 0 & 1 & 5/2 & -1/2 & -1/2 \end{array}\right)$$

$$\therefore A^{-1} = \frac{1}{2}\begin{pmatrix} -15 & 1 & 5 \\ 1 & 1 & -1 \\ 5 & -1 & -1 \end{pmatrix}$$

(02) (a)

(iii) (05)

$$A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$$

$$|A| = 3(-3+4) + 3(2-0) - 4(-2-0)$$

$\therefore |A| = 1$

let C be the matrix of cofactors.

$$\therefore \quad C = \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \quad \text{where}$$

$A_1 = \begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = 1$; $A_2 = -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} = -2$

$A_3 = \begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} = -2$; $B_1 = -\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} = -1$

$B_2 = \begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} = 3$; $B_3 = -\begin{vmatrix} 3 & -3 \\ 0 & -1 \end{vmatrix}$

$= 3$

$$c_1 = \begin{vmatrix} -3 & 4 \\ -3 & 4 \end{vmatrix} = 0 \quad ; \quad c_2 = -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} = -4$$

$$c_3 = \begin{vmatrix} 3 & -3 \\ 2 & -3 \end{vmatrix} = -3$$

$$\therefore \quad C = \begin{pmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{pmatrix}$$

$$\therefore \quad Adj.\ A = C^T$$

$$= \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|} . Adj.\ A$$

$$\therefore A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$$

(02)

(b) let $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

$$\therefore \quad A^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{pmatrix}$$

$$(2A)^{-1} = \frac{1}{2} A^{-1}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -3 & 3 & -3 \end{pmatrix}$$

$$(A^T)^{-1} = (A^{-1})^T$$

$$= \begin{pmatrix} 1 & -2 & -2 \\ -1 & 3 & 3 \\ 0 & -4 & -3 \end{pmatrix}$$

(03) (i)

(a)

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 3 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$|A| = 0(-1)^{1+1}\begin{vmatrix} 0 & 2 & 0 \\ 0 & 1 & 3 \\ 2 & 1 & 0 \end{vmatrix} + 1(-1)^{1+2}\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 1 & 1 & 0 \end{vmatrix}$$

$$+ 2(-1)^{1+3}\begin{vmatrix} 1 & 0 & 0 \\ 2 & 0 & 3 \\ 1 & 2 & 0 \end{vmatrix} + 3(-1)^{1+4}\begin{vmatrix} 1 & 0 & 2 \\ 2 & 0 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$= 0 - 3 + 2(-6) - 3 \times 6$$

$$= -33$$

(04)

(c) $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix} = 0$

$$\Big\downarrow\ C_1 \rightarrow C_1 + C_2 + C_3$$

$$\begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+c & a \\ x+a+b+c & a & x+b \end{vmatrix} = 0$$

$$\therefore \quad (x+a+b+c) \begin{vmatrix} 1 & b & c \\ 1 & x+c & a \\ 1 & a & x+b \end{vmatrix} = 0$$

$$\Big\downarrow\ \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}$$

$$(x+a+b+c) \begin{vmatrix} 1 & b & c \\ 0 & x-b+c & a-c \\ 0 & a-b & x+b-c \end{vmatrix} = 0$$

by expansion along the first column;

$$(x+a+b+c)\left[(x-b+c)(x+b-c)-(a-c)(a-b)\right]=0$$

$$(x+a+b+c)\left[x^2-(b-c)^2-(a^2-ab-ac+bc)\right]=0$$

$$(x+a+b+c)(x^2-b^2-c^2+2bc-a^2+ab+ac-bc)=0$$

$$(x+a+b+c)(x^2-a^2-b^2-c^2+ab+bc+ca)=0$$

$\therefore$ $x+a+b+c=0$ or

$$x^2-a^2-b^2-c^2+ab+bc+ca=0$$

$\therefore$ $x=-(a+b+c)$ is a root of the given equation.

$\therefore$ $x^2-a^2-b^2-c^2+ab+bc+ca=0$

$$\therefore \quad x=\pm\sqrt{a^2+b^2+c^2-ab-bc-ca}$$

(05) (b)

Let the three numbers be x, y, and z.

$$x + y + z = 6$$

$$x + 2z = 7$$

$$3x + y + z = 12$$

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 12 \end{pmatrix}$$

$$A\,X = b$$

$$|A| = D = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix} = 4$$

$$D_1 = \begin{vmatrix} 6 & 1 & 1 \\ 7 & 0 & 2 \\ 12 & 1 & 1 \end{vmatrix} = 12$$

$$D_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 7 & 2 \\ 3 & 12 & 1 \end{vmatrix} = 4$$

$$D_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 0 & 7 \\ 3 & 1 & 12 \end{vmatrix} = 8$$

$$\therefore \quad x = \frac{D_1}{D} = \frac{12}{4} = 3$$

$$y = \frac{D_2}{D} = \frac{4}{4} = 1$$

$$z = \frac{D_3}{D} = \frac{8}{4} = 2$$

$\therefore$ the three numbers are

3, 1, 2

(06) (a) (ii)

let $A = \begin{pmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix}$

the Characteristic equation :

$$|A - d I| = 0$$

$$\therefore \begin{vmatrix} 6-d & -2 & 2 \\ -2 & 3-d & -1 \\ 2 & -1 & 3-d \end{vmatrix} = 0$$

$$(6-d)\{(3-d)^2 - 1\} + 2\{-2(3-d) + 2\}$$
$$+ 2\{2 - 2(3-d)\} = 0$$

$$(6-d)(9 - 6d + d^2 - 1) + 2(-6 + 2d + 2)$$
$$+ 2(2 - 6 + 2d) = 0$$

$$\therefore -d^3 + 12d^2 - 36d + 32 = 0$$

by trial, $d = 2$ is a root of the above equation.

$\therefore \ (\lambda-2)(\lambda^2-10\lambda+16)=0$

$\therefore \ (\lambda-2)(\lambda-2)(\lambda-8)=0$

$\therefore \ \lambda = 2, 2, 8$ are the eigen values.

For the eigen vectors;

$(A-\lambda I)\underline{V}=\underline{0}$ where $\underline{V}=\begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\underline{0}=\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

when $\lambda = 2$

$(A-2I)\underline{V}=\underline{0}$

$$\begin{pmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix}\underline{V}=\underline{0}$$

Using Gauss-Jordan Elimination,

$$\left(\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore \quad 2x - y + z = 0$$

$$y = 2x + z$$

$$\therefore \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 2x+z \\ z \end{pmatrix}$$

$$= \begin{pmatrix} 1 \cdot x + 0 \cdot z \\ 2 \cdot x + 1 \cdot z \\ 0 \cdot x + 1 \cdot z \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} z$$

$$\therefore E_2 = \left\{ \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} z \;\middle|\; x, z \in \mathbb{R} \right\}.$$

Similarly, E_8 can also be found.

(06) (b) (i)

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{pmatrix}$$

For the eigen values of A,

$$|A - dI| = 0$$

$$\begin{vmatrix} 1-d & 2 & -3 \\ 0 & 3-d & 2 \\ 0 & 0 & -2-d \end{vmatrix} = 0$$

$$(1-d)(3-d)(-2-d) = 0$$

$$\therefore \quad d = 1, 3, -2$$

let $E(A)$ denotes the eigen value of A.

$d = 1$

$E(A^3) = 1^3 = 1$

$E(A^2) = 1^2 = 1$

$d = 3$

$E(A^3) = 3^3 = 27$

$E(A^2) = 3^2 = 9$

$d = -2$

$E(A^3) = (-2)^3 = -8$

$E(A^2) = (-2)^2 = 4$

$E(I) = 1, 1, 1$

let $X = 3A^3 + 5A^2 - 6A + 2I$

$E(X) = 3(d^3) + 5(d^2) - 6(d) + 2(1)$

when $d = 1$;

$$E_1(X) = 3(1^3) + 5(1^2) - 6(1) + 2$$
$$= 3 + 5 - 6 + 2$$
$$= 4$$

When $d = 3$;

$$E_3(X) = 3(3^3) + 5(3^2) - 6(3) + 2$$
$$= 3 \times (27) + 5(9) - 16$$
$$= 110$$

when $d = -2$

$$E_{-2}(X) = 3(-2)^3 + 5(-2)^2 - 6(-2) + 2$$
$$= 3 \times (-8) + 20 + 12 + 2$$
$$= 10$$

$\therefore$ the eigen values of X :

4, 110, 10