MA 1203

Calculus

3 Credits

Chapter 02

Infinite Series

2.1 Difference Between Sequences and Series

2.1.1 Sequences

- A sequence can be thought of as a list of numbers written in a definite order:

$$a_1, a_2, a_3, ..., a_n, ...$$

- The number a_1 is called the first term, a_2 is the second term, and in general a_n is the n^{th} term.
- In an infinite sequences, each term an will have a successor a_{n+1}.
- For every positive integer n there is a corresponding number a_n and so a sequence can be defined as a function whose domain is the set of positive integers.
- But we usually write a_n instead of the function notation $f(n)$ for the value of the function at the number n.

2.1.2 Notation

The sequence

$$\{a_1, a_2, a_3, ...\}$$

is also denoted by

$$\{a_n\}$$

or

$$\{a_n\}_{n=1}^{\infty}$$

- Some sequences can be defined by giving a formula for the n^{th} term.
- In the following example we give three descriptions of the sequence:

Example

$$\left\{\frac{n}{n+1}\right\}_{n=1}^{\infty} \longleftarrow \text{by using the preceding notation}$$

$$a_n = \frac{n}{n+1} \longleftarrow \text{by using the defining formula}$$

$$\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right\} \longleftarrow \text{by writing terms of the sequence}$$

Example (01)

Obtain the defining formula and write the terms of the following sequences.

(i) $\left\{ \dfrac{(-1)^n(n+1)}{3^n} \right\}_{n=1}^{\infty}$

(ii) $\left\{ \sqrt{n-3} \right\}_{n=3}^{\infty}$

(iii) $\left\{ \cos\left(\dfrac{n\pi}{6} \right) \right\}_{n=0}^{\infty}$

Example (02)

Find a formula for the general term a_n of the sequence

$$\left\{ \frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \dots \right\}$$

assuming that the pattern of the first few terms continues.

- Some sequences that do not have a simple defining equation.

Examples

(a) The sequence $\{p_n\}$, where p_n is the population of the world as of January 1 in the year n.

(b) If we let a_n be the digit in the n^{th} decimal place of the number e, then

$$\{a_n\} = \{7, 1, 8, 2, 8, 1, 8, 2, 8, 4, 5, ...\}$$

is a well-defined sequence.

(c) The Fibonacci sequence $\{f_n\}$ is defined recursively by the conditions

$$f_1 = 1 \; ; \; f_2 = 1 \; ; \; f_n = f_{n-1} + f_{n-2} \; ; \; n \geq 3$$

Each term is the sum of the two preceding terms.
The first few terms are $\{1, 1, 2, 3, 5, 8, 13, 21, ...\}$.

Definition

A sequence $\{a_n\}$ has the limit L and we write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty$$

if we can make the terms a_n as close to L as we like by taking n sufficiently large.

If $\lim\limits_{n\to\infty} a_n$ exists, we say the sequence converges (or is convergent).

Otherwise, we say the sequence diverges (or is divergent).

2.1.3 Series

- An infinite series (or just a series) is an expression of the form

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots$$

which can obtain by adding the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$.

- It is denoted by the symbol

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum a_n.$$

2.2 Convergent and Divergent Series

2.2.1 Partial Sums

$$\begin{aligned}
s_1 &= a_1 \\
s_2 &= a_1 + a_2 \\
s_3 &= a_1 + a_2 + a_3 \\
&\vdots \\
s_n &= a_1 + a_2 + a_3 + \ldots + a_n
\end{aligned}$$

- These partial sums form a new sequence $\{s_n\}$, which may or may not have a limit.
- If $\lim\limits_{n\to\infty} s_n = s$ exists (as a finite number), then, we call it the sum of the infinite series $\sum a_n$.

Definition (2.2.2)

Given a series $\sum_{n=1}^{\infty} a_n$, let s_n denote its n^{th} partial sum:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + \ldots + a_n.$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$ exists as a real number, then the series $\sum_{n=1}^{\infty} a_n$ is called convergent and we write

$$a_1 + a_2 + a_3 + \ldots + a_n + \ldots = s \text{ or } \sum_{n=1}^{\infty} a_n = s$$

The number s is called the sum of the series. If the sequence $\{s_n\}$ is divergent, then the series is called divergent.

- The sum of a series is the limit of the sequence of partial sums.
- So when we write $\sum_{n=1}^{\infty} a_n = s$, we mean that by adding sufficiently many terms of the series we can get as close as we like to the number s.
- Notice that

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{i=1}^{n} a_i$$

Example (03)

Let the sum of the first n terms of the series $\sum_{n=1}^{\infty} a_n$ is

$$a_1 + a_2 + ... + a_n = \frac{2n}{3n+5}.$$

Find the sum of the series.

2.2.3 Geometric Series

The geometric series

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \qquad |r| < 1.$$

If $|r| \geq 1$, the geometric series is divergent.

Example (04)

(a) Find the sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \ldots$$

(b) Is the series

$$\sum_{n=1}^{\infty} 2^{2n} 3^{1-n}$$

convergent or divergent?

Example (05)

Show that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$

is convergent, and find its sum.

Theorem

If the series $\sum_{n=1}^{\infty} a_n$ *is convergent, then* $\lim_{n\to\infty} a_n = 0$.

Proof.

Let $s_n = a_1 + a_2 + ... + a_n$.
Then $a_n = s_n - s_{n-1}$.
Since $\sum a_n$ is convergent, the sequence $\{s_n\}$ is convergent.
Let $\lim_{n\to\infty} s_n = s$.
Since $n-1 \to \infty$ as $n \to \infty$, we also have $\lim_{n\to\infty} s_{n-1} = s$.
Therefore

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (s_n - s_{n-1}) = \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} = s - s = 0$$

$\square$

2.2.4 Test for Divergence

☆ If $\lim_{n\to\infty} a_n$ does not exist or if $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example (06)

Show that the series

$$\sum_{n=1}^{\infty} \frac{n^2}{5n^2 + 3}$$

diverges.

Theorem

If $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *are convergent series, then so are the series* $\sum_{n=1}^{\infty} ca_n$ *(where c is a constant),* $\sum_{n=1}^{\infty}(a_n + b_n)$*, and* $\sum_{n=1}^{\infty}(a_n - b_n)$*, and*

(i) $\sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum_{n=1}^{\infty}(a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum_{n=1}^{\infty}(a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n.$

Example (07)

Find the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right).$$

2.3 Integral Test

2.3.1 The Integral Test

☆ Suppose $f(x)$ is a function such that

$$f(n) = a_n$$

for $n = 1, 2, \dots.$

Further suppose that:

1. $f(x) > 0$ ($f(x)$ is positive)
2. $f(x)$ is a decreasing function.

Then, either

$$\int_1^{\infty} f(x)\,dx \text{ and } \sum_{n=1}^{\infty} a_n$$

both converge, or diverge.

Example (08)

Test the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 1}$$

for convergence or divergence.

Example (09)

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\ln n}{n}$$

converges or diverges.

2.3.2 p-Series Test

- The p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

is convergent if $p > 1$ and divergent if $p \leq 1$.

Example (10)

Test the following series for convergence or divergence.

(i) $$\sum_{n=1}^{\infty} \frac{1}{n^3}$$

(ii) $$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{3}}}$$

2.4 Comparison Tests

2.4.1 The Comparison Test

☆ Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

1. If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n, then $\sum a_n$ is also convergent.

2. If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n, then $\sum a_n$ is also divergent.

Example (11)

(i) Determine whether the series

$$\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$$

converges or diverges.

(ii) Test the series

$$\sum_{k=1}^{\infty} \frac{\ln k}{k}$$

for convergence or divergence.

2.4.2 The Limit Comparison Test

☆ Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

Example (12)

(i) Test the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$$

for convergence or divergence.

(ii) Test the series

$$\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$$

for convergence or divergence.

2.5 Alternating Series

2.5.1 Alternating Series

Definition

An alternating series is a series whose terms are alternately positive and negative.

- Generally, the n^{th} term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n$$

or

$$a_n = (-1)^n b_n$$

where b_n is a positive number.

2.5.2 Alternating Series Test

☆ If the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n,\ b_n > 0$ satisfies

(i) $b_{n+1} \leq b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Example (13)

Test the following series for convergence or divergence.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

(b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{3n}{4n-1}$

(c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$