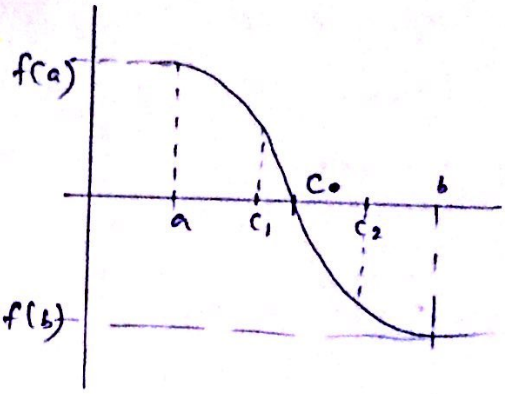
Numerical Methods

* Solutions of non - linear equations

① Bisection method



$f(a).f(b) < 0$

interval $\rightarrow (a, b)$

$c = \left(\frac{a+b}{2}\right)$

$\underline{c = c_1}$

If $f(a).f(c_1) > 0$

interval (c_1, b)

If $\underline{c = c_2}$

$f(a)\, f(c_2) < 0$

interval (a, c_2)

Ex :- Find a root of an equation $f(x) = x^3 - 4x - 9$ between the interval $(2, 3)$

$f(x) = x^3 - 4x - 9$

1st iteration

$f(2) = -9 \;,\; f(3) = 6$

$f(2) f(3) < 0$

$\therefore$ Root lies between $(2, 3)$

$x_0 = \frac{2+3}{2} = 2.5$

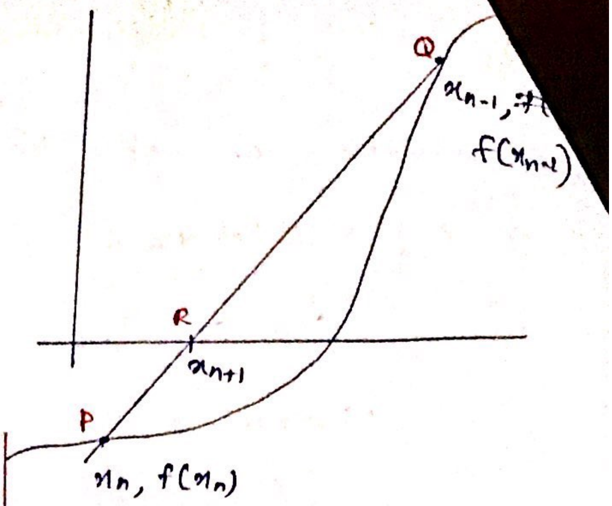
n	a	b	x_0	$f(x_0)$	$f(a)f(x_0)$
1	2	3	2.5	-3.37	(+)(+)
2	2.5	3	2.75	0.796	(−) (+)
3	2.5	2.75	2.625	-1.412	(−) ~~(+)~~(−)
4	2.625	2.75	2.6875	-0.339	(−) ~~(+)~~(−)
5	2.6875	2.75	2.7188	0.221	(−) (+)
6	2.6875	2.7188	2.7031	-0.0611	(−) (−)
7	2.7031	2.7188	2.7109	0.0794	(−) (+)
⋮					

$\therefore$ Approximate root of $x^3 - 4x - 9 = 0$ using bisection method after 7 iterations is, $x = 2.7109$ //

② Secant Method

Slope of PR = Slope of PQ

$$\frac{f(x_n) - 0}{(x_n - x_{n+1})} = \frac{f(x_n) - f(x_{n-1})}{(x_n - x_{n-1})}$$



$$\boxed{x_{n+1} = x_n - \frac{(x_n - x_{n-1})\, f(x_n)}{f(x_n) - f(x_{n-1})}}$$

Ex :- $f(x) = x^3 - 4x - 9$ interval $(2, 3)$

$f(x) = x^3 - 4x - 9$, $x_0 = 2$, $x_1 = 3$, $f(x_0) = -9$
$f(x_1) = 6$

1st Iteration (n = 1)

$$x_2 = x_1 - \frac{(x_1 - x_0)\, f(x_1)}{f(x_1) - f(x_0)}$$

$$x_2 = 3 - \frac{(3-2)\, 6}{(6 - (-9))}$$

$$\underline{x_2 = 2.6}$$

2nd Iteration (n = 2)

$$x_3 = x_2 - \frac{(x_2 - x_1)\, f(x_2)}{f(x_2) - f(x_1)}$$

$$= 2.6 - \frac{(2.6 - 3)(-1.824)}{-1.824 - 6}$$

$$\underline{x_3 = 2.6933}$$

3rd Iteration (n = 3)

$$x_4 = x_3 - \frac{(x_3 - x_2)\, f(x_3)}{f(x_3) - f(x_2)}$$

$$= 2.6933 - \frac{(2.6933 - 2.6)(-0.2372)}{-0.2372 - (-1.824)}$$

$$\underline{x_4 = 2.7072}$$

4th Iteration (n = 4)

$$x_5 = x_4 - \frac{(x_4 - x_3)\, f(x_4)}{f(x_4) - f(x_3)}$$

$$\underline{x_5 = 2.706}$$

$$f(x_5) = 2.706^3 - 4(2.706) - 9$$
$$= 0 \,//$$

∴ Approximate root of the equation $x^3 - 4x - 9 = 0$ after 4 iterations is

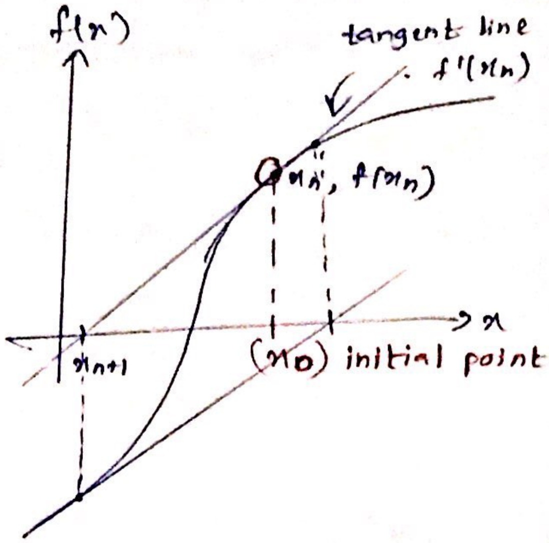
$$\underline{x = 2.706}$$

③ Newton - Raphson Method

Slope of the tangent line at $(x_n, f(x_n))$

$$f'(x_n) = \frac{f(x_n) - 0}{x_n - x_{n+1}}$$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$



Ex:- $\cancel{3}x^3 - 4x - 9$ $\quad [2,3]$

if we get $x_0 = 2$ $\quad f'(x) = 3x^2 - 4$

1st Iteration (n=0)

$$x_{0+1} = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 2 - \frac{-9}{8}$$

$$x_1 = 3.125$$

2nd Iteration (n=1)

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 3.125 - \frac{9.0176}{25.269}$$

$$x_2 = 2.7685$$

⋮

After 5 iterations $\longrightarrow$ $x = 2.7065$ //

* System of linear equations

① Gauss - Jacobi

initial approximation = (0, 0, 0)

Ex:- ~~27x + 6y − z = 85~~

$$4x_1 + x_2 + x_3 = 2$$
$$x_1 + 5x_2 + 2x_3 = -6$$
$$x_1 + 2x_2 + 3x_3 = -4$$

$\longrightarrow$

$$x_1 = \frac{(2 - x_2 - x_3)}{4}$$
$$x_2 = \frac{(-6 - 2x_3 - x_1)}{5}$$
$$x_3 = \frac{(-4 - x_1 - 2x_2)}{3}$$

1st Iteration

$x_1^{(1)} = \frac{1}{4}(2 - x_2 - x_3)$

$x_1^{(1)} = \frac{1}{4}(2 - 0 - 0)$

$x_1^{(1)} = \underline{0.5}$

$x_2^{(1)} = \frac{1}{5}(-6 - x_1 - 2x_3)$

$= \frac{1}{5}(-6 - 0 - 0)$

$x_2^{(1)} = \underline{-1.2}$

$x_3^{(1)} = \frac{1}{3}(-4 - x_1 - 2x_2)$

$= \frac{1}{3}(-4 - 0 - 0)$

$x_3^{1} = \underline{-1.33}$

2nd Iteration

$x_1^{(2)} = \frac{1}{4}(2 - x_2^{(1)} - x_3^{(1)})$

$= \frac{1}{4}(2 - (-1.2) - (-1.33))$

$x_1^{(2)} = \underline{1.133}$

$x_2^{(2)} = \frac{1}{5}(-6 - 0.5 - 2(-1.33))$

$x_2^{(2)} = \underline{-0.766}$

$x_3^{(2)} = \frac{1}{3}(-4 - 0.5 - 2(-1.2))$

$x_3^{(2)} = \underline{-0.7}$

② Gauss - Seidel

1st Iteration

$x_1^{(1)} = \frac{1}{4}(2 - x_2 - x_3)$

$= \frac{1}{4}(2 - 0 - 0)$

$= \underline{0.5}$

$x_2^{(1)} = \frac{1}{5}(-6 - x_1 - 2x_3)$

$= \frac{1}{5}[-6 - (0.5) - 2(0)]$

$= \underline{-1.3}$

$x_3^{(1)} = \frac{1}{3}[-4 - x_1 - 2x_2)$

$= \frac{1}{3}[-4 - 0.5 - 2(-1.3)]$

$= \underline{-0.633}$

2nd Iteration

$x_1^{(2)} = \frac{1}{4}[2 - (-1.3) - (-0.633)]$

$= \underline{0.983}$

$x_2^{(2)} = \frac{1}{5}(-6 - 0.983 - 2(-0.633))$

$= \underline{-1.143}$

$x_3^{(2)} = \frac{1}{3}[-4 - 0.983 - 2(-1.14)]$

$= \underline{-0.8981}$

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Lagrange Polynomial

$[x_0, f(x_0)]$ $[x_1, f(x_1)]$ $[x_2, f(x_2)]$
(x_0, y_0) (x_1, y_1) (x_2, y_2)

Ex:- Given that data $D = \{ (0,6), (1,0), (2,2) \}$ find the lagrange interpolation polynomial.

$$\boxed{P_n(x) = \sum_{k=0}^{n} l_k(x) f(x_k)} \quad ; \quad n = 2 \text{ (order)}, \; k = 0, 1, 2$$

* If we have n data sets, the order of polynomial is $(n-1)$, Therefore 3 data sets and order is 2.

Let us find l_0, l_1, l_2

$$l_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x-1)(x-2)}{(0-1)(0-2)} = \frac{x^2 - 3x + 2}{2}$$

$$l_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x-0)(x-2)}{(1-0)(1-2)} = \frac{x^2 - 2x}{(-1)}$$

$$l_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x-0)(x-1)}{(2-0)(2-1)} = \frac{x^2 - x}{2}$$

The interpolating polynomial

$$P_2(x) = \sum_{k=0}^{2} l_k(x) f(x_k)$$

$$= l_0(x) f(x_0) + l_1(x) f(x_1) + l_2(x) f(x_2)$$

$$= \frac{(x^2 - 3x + 2)}{2} \times 6 + \frac{(x^2 - 2x)}{(-1)} \times 0 + \frac{(x^2 - x)}{2} \times 2$$

$$= 3(x^2 - 3x + 2) + (x^2 - x)$$

$$P_2(x) = 4x^2 - 10x + 6 \; //$$

Numerical Integration

Trapezoidal method:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} f(x)\,dx$$

$$= \sum_{i=1}^{n} \left[\frac{h}{2} \left(f(x_{i-1}) + f(x_i) \right) \right]$$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2} \Big[\underset{\text{1st}}{f(x_0)} + \underset{\text{last}}{f(x_n)} + 2 \underbrace{\{ f(x_1) + f(x_2) + \dots f(x_{n-1}) \}}_{\text{rest of it.}} \Big]}$$

* $h = \left(\frac{b-a}{n} \right)$ ← sub intervals

Ex:- $I = \int_0^1 \frac{dx}{1+x}$ sub intervals $(2, 4)$

2 subintervals

$n = 2$: $h = \frac{1-0}{2} = 0.5$ ⟹

	x_0	x_1	x_2
x :	0,	0.5	1.0
$f(x)$:	1	0.666	0.5

$$\int_0^1 \frac{dx}{1+x} = \frac{h}{2} \left[f(x_0) + f(x_2) + 2 f(x_1) \right]$$

$$= \frac{0.5}{2} \left[1 + 0.5 + 2(0.666) \right]$$

$$= 0.705$$

4 subintervals

$n = 4$ $h = \frac{1-0}{4} = 0.25$ ⟹

	x_0	x_1	x_2	x_3	x_4
x	0	0.25	0.5	0.75	1
$f(x)$	1	0.8	0.66	0.571	0.5

$$\int_0^1 \frac{dx}{1+x} = \frac{0.25}{2} \left\{ f(x_0) + f(x_4) + 2 \left[f(x_1) + f(x_2) + f(x_3) \right] \right\}$$

$$= \frac{0.25}{2} \left(1 + 0.5 + 2(0.8 + 0.66 + 0.571) \right)$$

$$= 0.6952$$

Simpson's 1/3 Rule

$$\int_a^b f(x)\,dx = \frac{h}{3}\left\{ \underbrace{[f(x_0)+f(x_n)]}_{1^{st}\ \&\ \text{last}} + 2\underbrace{[f(x_2)+f(x_4)+\cdots]}_{\text{even}} + 4\underbrace{[f(x_3)+f(x_5)+\cdots]}_{\text{odd}} \right\}$$

$$h = \left(\frac{b-a}{n}\right)$$

Numerical Differentiation

i) Build an interpolating polynomial to approximate $f(x)$, then use the derivative of the interpolating polynomial as the approximation of the $f'(x_0)$.

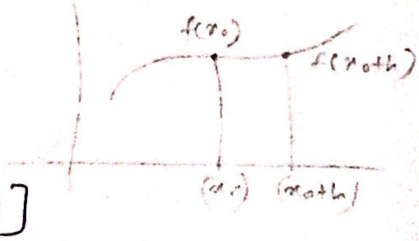
Two points difference formula

$$\boxed{f'(x_0) \approx \frac{f(x_0+h) - f(x_0)}{h}}$$

derivative approximation

when $h > 0$; forward difference

when $h < 0$; backward difference



Three points difference formular

* $f'(x_0) \approx \frac{1}{2h}\left[-3f(x_0) + 4f(x_0+h) + f(x_0+2h)\right]$
* $f'(x_0) \approx \frac{1}{2h}\left[-f(x_0-h) + f(x_0+h)\right]$
* $f'(x_0) \approx \frac{1}{2h}\left[f(x_0+2h) - 4f(x_0+h) + 3f(x_0)\right]$

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Numerical Solution for ODE

$y' = \underbrace{\cos t + t^2 + 1}_{f(t,y)}$.

① Euler's method

Let ODE & initial condition as

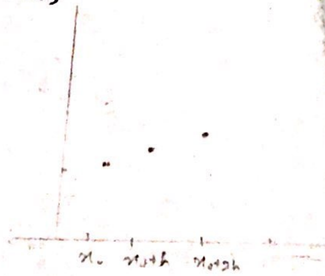
$$y' = f(t, y) \quad \text{for} \quad t_0 < t < b$$
$$y(t_0) = y_0$$

we restrict our consideration for which the function $f(x,y)$ is sufficiently smooth that the ODE has a unique solution satisfying the initial condition $y(t_0) = y_0$. Then formula for Euler's method,

$$\boxed{y_{n+1} = y_n + \left(h\, f(t_n, y_n)\right)}$$

; where h is step size.

This method is not practically accurate except for very small step size.



Example :

Consider ODE with initial condition,

$$y' = t + y \quad ; \; y(0) = 1 \longrightarrow y(t_0) = 1$$

Apply Euler's method on a interval $0 < t < 0.1$ (find the y values) with six steps , $h = 0.02$

$\therefore f(t, y) = y + t$

$y_{n+1} = y_n + h f(t_n, y_n)$

$y_{n+1} = y_n + h\,(t + y)$

n	t_n	y_n	y'_n ($y' = t+y$)	$h y'_n$ ($0.02 \times y'$)	y_{n+1}
0	0	1 (y_{n+1})	1	0.02	1.02
1	0.02	1.02	1.04	0.0208	1.0408
2	0.04	1.0408	1.0808	0.0216	1.0624
3	0.06	1.0624	1.1224	0.0224	1.0848
4	0.08				
5	0.1				

x	$f(x)$
0	
0.02	
0.04	

* In general the range of step size over which a method convergers will depend on the details of the system being solved and also on the type of solutions method used.

Runge - Kutta method

Consider ODE with initial condition $y' = f(t, y)$; $y(t_0) = y_0$
Commonly use a 4th order Runge - kutta method , which leads to the following equations.

$$k_1 = h f(t_i, y_i)$$

$$k_2 = h f\left(t_i + h/2 ,\ y_i + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(t_i + h/2 ,\ y_i + \frac{k_2}{2}\right)$$

$$k_4 = h f(t_{i+1},\ y_i + k_3)$$

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \quad ; i = 0, 1, 2, \dots$$

Example Let ODE $y' = \overbrace{y - t^2 + 1}^{f(t,y)}$, $0 \le t \le 2$ $y(0) = 0.5$
Apply 4th order range - kutta method with $h = 0.2$ & $N = 10$

$h = \frac{2}{10}$ ← max t, ← steps

$$k_1 = h f(t_i, y_i) = 0.2(y - t^2 + 1)$$

n	t	y	k_1	k_2	k_3	k_4	$y(t_{n})$
0	0	0.5	0.3 0.3	0.328	0.3308	0.35816	0.5
1	0.2						
2	0.4						
3	0.6						
⋮	⋮						
10	2						