Complex Analysis

$$\mathbb{C} = \{x+iy \,/\, x,y \in \mathbb{R}, \; i^2 = -1\}$$

$Re(z) = x$ ← real part

$Im(z) = y$ ← imaginary part

- Complex numbers are not ordered.

  $z_1 < z_2$ ✗ $\qquad |z_1| < |z_2|$ ✓

- Absolute or modulus value

  $z = x+iy \qquad |z| = \sqrt{x^2+y^2}$

Properties

① for any $z \in \mathbb{C}$, $|z| \geq 0$

$|z| = 0$ if and only if $z = 0$

② $|z| = |-z|$

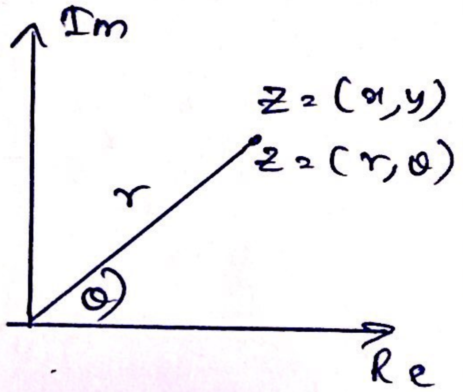
③ $|z_1 z_2| = |z_1||z_2|$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}$$

④ $|z_1 + z_2| \leq |z_1| + |z_2|$

$|z_1| - |z_2| \leq |z_1 - z_2|$

$$\boxed{|z|^2 = z\bar{z}}$$



$z = x+iy$

$z = re^{i\theta}$

- **Argument (angle)**

$z = re^{i\theta}$ θ is argument $\arg(z)$

The particular value of $\arg(z)$ which lies in the interval $[-\pi, \pi]$ is refers to as the principle value of the argument $\mathrm{Arg}(z)$

$$\boxed{\arg(i\cancel{z}) = \frac{\pi}{2} \pm 2k\pi}$$

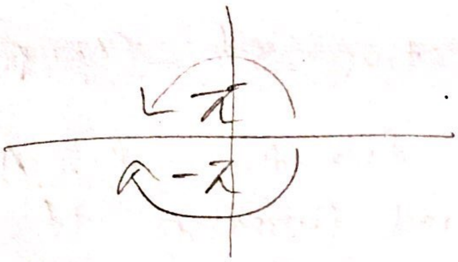
$$\boxed{\begin{array}{c}\mathrm{Arg}(z)\\ [-\pi, \pi)\end{array}}$$

$$\arg(z) = \mathrm{Arg}(z) + 2k\pi$$

Ex:- $z = 1+\sqrt{3}i$ $|z| = \sqrt{1^2+3} = \sqrt{4} = 2$

$$z = 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$$



$$\arg(1+\sqrt{3}i) = \frac{\pi}{2} + 2k\pi$$

$$-\pi < \mathrm{Arg}(z) < \pi \longrightarrow \mathrm{Arg}(1+\sqrt{3}i) = \frac{\pi}{3}$$

Properties

(1) $\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$

(2) $\arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2)$

(3) $\arg(\bar{z}) = -\arg(z)$

(4) $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}(z_1) + \mathrm{Arg}(z_2)$ Not true for all times.

(5) $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) \neq \mathrm{Arg}(z_1) - \mathrm{Arg}(z_2)$

Power

$z = x + iy = re^{i\theta}$

$z^n = (x+iy)^n = (re^{i\theta})^n$

$= r^n e^{in\theta}$

$$\boxed{z^n = r^n \left(\cos(n\theta) + i \sin(n\theta) \right)}$$

Root

$z = re^{i\theta}$ nth root,

$$\boxed{z_k = r^{1/n} e^{i\left(\frac{\theta + 2k\pi}{n}\right)}} \quad k = 0, 1, 2, \ldots (n-1)$$

Function of Complex variables

A function $f : A \longrightarrow B$ is said to be complex valued function if $A \subseteq \mathbb{C}$ & $B \subseteq \mathbb{C}$ and we write $w = f(z)$ for all $z \in A$

$$\boxed{f(z) = u(x,y) + i\, v(x,y)}$$

- **Continuity**

$\lim_{z \to z_0} f(z) = f(z_0) \longrightarrow$ continuous at point z_0.

Theorem

If $\lim_{n \to \infty} z_n = z_0$ and $f(z)$ is continuous, then,

$$\lim_{n \to \infty} f(z_n) = f(z_n)$$

Analyticity

Let $f(z)$ be a complex valued function defined in a neighbourhood at z_0. Then the derivative of $f(z)$ at z_0 is given by,

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$

$$\frac{df(z)}{dz} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$ (If limit exist)

The Cauchy- Riemann Equation (CR equation)

$$\boxed{f(z) = u(x,y) + i\,v(x,y)}$$

$$\left.\begin{aligned} u_x &= v_y \\ u_y &= -v_x \end{aligned}\right\} \text{CR - equations}$$

If $f(z) = u+iv$ is differentiable then,

$f'(z) = u_x + i v_x \rightarrow x$ direction

$f'(z) = u_y - i v_y \rightarrow y$ direction

Note

- f is differentiable at $z_0 \rightarrow$ CR- holds at z_0.
- CR do not hold at z_0 $\rightarrow$ f is not differentiable
  $\rightarrow$ f is not analytic

Entire Function

(differentiable everywhere)

If $f(z)$ is analytic on the whole complex plane; then it is said to be entire function.

Ex:- all polynomial function of z are entire.

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Theorem

Let $f(z) = u(x,y) + iV(x,y)$ be defined in some open set G. If the 1st partial derivatives are continuous and satisfies the CR-equations at all points of G, the $f(z)$ is analytic in G.

$$\text{[CR holds]} + \begin{bmatrix} \text{1st partial derivatives} \\ \text{are continuous} \\ u_x, u_y, V_x, V_y \end{bmatrix} \Longrightarrow f \text{ is analytic}$$

Harmonic Function

A real valued function $\phi(x,y)$ is said to be harmonic in a domain D if all it's second partial derivatives are continuous in D, and if at each point of D, ϕ satisfies $\nabla^2\phi = 0$

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \longrightarrow \nabla^2\phi = 0$$

Harmonic Conjugate

Suppose V is harmonic in domain D. A harmonic fun^n^ V in D is said to be a harmonic conjugate of u if $f(z) = u + iV$ is analytic in D.

Note

- V is harmonic conjugate of u

$$f(z) = u + iV$$

- u is harmonic conjugate of V

$$f(z) = V + iu$$

- The exponential functions (e^z)

$$e^z = e^{x+iy}$$
$$= e^x . e^{iy}$$
$$\boxed{e^z = e^x(\cos y + i\sin y)}$$

* $e^{z_1} . e^{z_2} = e^{z_1+z_2}$
* $\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$
* $\frac{d(e^z)}{dz} = e^z$
* $e^0 = 1$
* $|e^z| = e^x$

* $\arg(e^z) = y + 2k\pi$; $k \in \mathbb{C}$
* $e^z = 1$ if and only if $z = 2k\pi i$
* $e^{z_1} = e^{z_2}$ iff $z_1 = z_2 + 2k\pi i$

$\sin(z) =$	$\frac{e^{iz} - e^{-iz}}{2i}$
$\cos(z) =$	$\frac{e^{iz} + e^{-iz}}{2}$

$e^t = x$
$\log_e x = t$
$\ln x = t$

- The Logorithamic function $[\log_e z \ (\ln z)]$

$$\boxed{\log z = \underbrace{\ln|z|}_{\text{real}} + i\underbrace{(\theta + 2\pi k)}_{\text{real}}}$$

$K \in \mathbb{R}$

$\theta \longrightarrow \arg z$

$\log z \in \mathbb{C}$

when $k = 0$

$\text{Log}(z) = \log|z| + i\theta$

If $e^w = z$

$w = \log|z| + i \text{ Arg } z + 2k\pi i$

$e^w = e^{\log|z|} + e^{i(\text{Arg } z + 2k\pi)}$

$e^w = |z| . e^{i \text{Arg } z} \longleftarrow re^{i\theta}$ form.

$e^w = z$

* $\log z = \log|z| + i\arg(z)$
* $\log z = \log|z| + i[\text{Arg}(z) + 2k\pi]$

$K = 0 \rightarrow \log z = \log|z| + i\text{Arg}(z)$

Note

1) $e^{\log z} = z$

2) $\log e^{z} = z + 2k\pi i$

3) $\log(z_1 z_2) = \log z_1 + \log z_2$

4) $\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$

5) $w = \log z$ is one to one function

6) $\frac{d}{dz}(\log z) = \frac{1}{z}$ for all $z \in D^* = \mathbb{C} \setminus \{x / x \leq 0\}$

* principle value of $\log z$

$$\operatorname{Log} z = \log|z| + i \operatorname{Arg} z \qquad (k=0)$$

$$\boxed{z^{\alpha} = e^{\alpha \log z}} \qquad \boxed{P.V.(z^{\alpha}) = e^{\alpha \operatorname{Log} z}}$$

Principle value.

Complex Integral

$$\boxed{L = \int_{\gamma} f(z)\,dz} \qquad \boxed{\int_a^b f(t)\,dt = F(b) - F(a)}$$

$$\boxed{\int_{\gamma} f(z)\,dz = \int_a^b f(z(t)) \cdot z'(t)\,dt}$$

$\gamma \to$ • line • circle • Semi circle.

(t-parametric)

where γ is $z(t)$ and $z'(t)$ is derivative

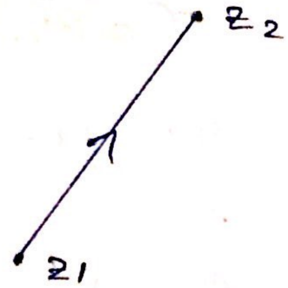
C - contour consisting of directed smooth curves $(\gamma_1, \gamma_2, \dots \gamma_n)$

$$\boxed{\int_C f(z)\,dz = \sum_{r=1}^{n} \int_{\gamma_n} f(z)\,dz} \qquad \boxed{\int_{-\gamma} f(z)\,dz = -\int_{\gamma} f(z)\,dz}$$

If C consists of a single point

$$\boxed{\int_C f(z)\,dz = 0}$$

Line segmet in parametrization



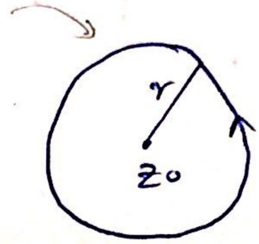
The straight line segment which join z_1 & z_2 points

$$\boxed{z(t) = z_1 + t(z_2 - z_1) \quad ; 0 \le t \le 1}$$

Set lower limit to get z_1 & upper limit to get z_2.

The circle path

$|z - z_0| = r$



centerpoint, radius

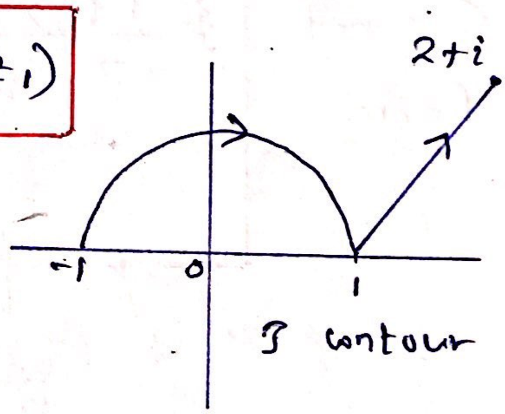
$$\boxed{z(t) = z_0 + re^{it} \quad ; 0 \le t \le 2\pi}$$

if it's a semi circle $(0 \le t \le \pi)$

Theorem

Suppose that the function $f(z)$ is con. in domain D and has an antiderivative $F(z)$ throughout D ; i.e. $\frac{d}{dz}F(z) = f(z)$ for each z in D. Then for any contour J lying in D, with initial point z_1 and and terminal point z_T we have,
(Then we don't need to have parametrize t)

$$\boxed{\int_J f(z)\,dz = F(z_T) - F(z_1)}$$



Ex:- $\int_J \cos z \, dz$

has the antiderivative $F(z) = \sin z$ for all z.

$$\int_J \cos z \, dz = \Big[\sin z\Big]_{-1}^{2+i}$$

$$= \sin(2+i) - \sin(-1)$$

Cauchy's 1st Integral Theorem

If $f(z)$ be a single valued and analytic fu^n through out a simply connected region IR, then the line integral of $f(z)$ around any closed path C which lies entirely inside R is zero.

$$\boxed{\oint_C f(z)\, dz = 0}$$

* If there are no singular points inside the close path.

(Cauchy's Integral Formula)

Cauchy's 2nd Integral Theorem

If $f(z)$ be a single valued fu^n and analytic through out a simply connected region R and if C be an contour interior to R and enclosed z_0, z_0 being an isolated singular point,

* one singular point.

$$\boxed{\oint_C \frac{f(z)}{z - z_0}\, dz = 2\pi i\, f(z_0)}$$

General Form

$$\boxed{\oint_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(z_0) = \frac{2\pi i}{n!}\left(\frac{d^n f(z)}{dz^n}\right)_{z=z_0}}$$

Series Representation

* $\left(\frac{1}{1-z}\right) = 1+z+z^2+z^3+\dots = \sum_{n=0}^{\infty} z^n$ if $|z|<1$

* $\left(\frac{1}{1+z}\right) = 1-z+z^2-z^3+\dots = \sum_{n=0}^{\infty} (-z)^n$ if $|z|<1$

Taylor's series (Analytic and no singular point)

$$\boxed{f(z) = \sum_{n=0}^{\infty} (z-z_0)^n \frac{f^{(n)}(z_0)}{n!}}$$

Note: When $z_0=0$ $f(z) = \sum_{n=0}^{\infty} A_n z^n$; $A_n = \frac{f^{(n)}(0)}{n!}$ is known as Maclaurin Series.

Laurent series (Ananalytic and there are singular point)

$$\boxed{f(z) = \underbrace{\sum_{n=0}^{\infty} A_n (z-z_0)^n}_{\text{Analytic part}} + \underbrace{\sum_{n=1}^{\infty} \frac{B_{-n}}{(z-z_0)^n}}_{\text{principle part}}}$$

$\frac{1}{1-z}$	$= \sum_{n=0}^{\infty} z^n$	$\lvert z\rvert<1$
$\frac{1}{1+z}$	$= \sum_{n=0}^{\infty} (-z)^n$	$\lvert z\rvert<1$

Taylor expansion of e^z

$$\boxed{e^z = \sum_{j=0}^{\infty} \left(\frac{z^j}{j!}\right)}$$

Residue Theorem

$f(z) = \frac{2z+1}{z(z-1)}$ ← order of z is 1

$$\boxed{Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)}$$

when pole of order is 1

$$\boxed{Res(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} (z - z_0)^m f(z) \right]_{z = z_0}}$$

when pole of order is m.

$$\frac{1}{z^m (z-1)}$$

Ex:- $f(z) = \frac{4z^2+1}{z^2 (z-9)^3}$

- $z = 0$ is pole of order 2
- $z = 9$ is pole of order 3

$Res(f, 0)$ $(m = 2)$

$$= \lim_{z \to 0} \frac{1}{(2-1)!} \left[\frac{d^{2-1}}{dz^{2-1}} (z-0)^2 f(z) \right]_{z=0}$$

$$= \lim_{z \to 0} \left\{ \frac{d}{dz} \frac{\cancel{z^2} \times (4z^2+1)}{\cancel{z^2} (z-9)^3} \right\} \text{ at } z=0$$

$=$

Cauchy's Residue Theorem

$$\boxed{\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} Re(f, z_k)}$$

Trigonometric Integral over $[0, 2\pi]$

$$\boxed{I = \int_0^{2\pi} u(\cos\theta, \sin\theta)\, d\theta}$$

* Let $z = e^{i\theta}$

Cauchy's principle value of the integral of f over $(-\infty, \infty)$

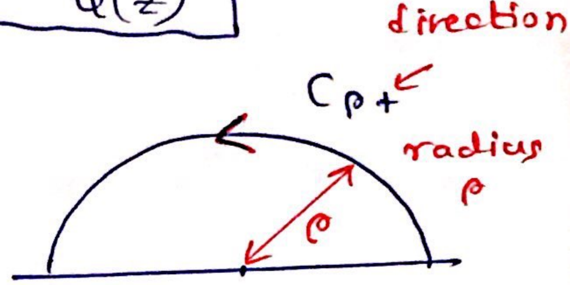
$$\boxed{P.V. \int_{-\infty}^{\infty} f(z)\, dz = \lim_{\rho \to \infty} \int_{-\rho}^{\rho} f(z)\, dz}$$

Lemma ①

If $f(z) = \dfrac{P(z)}{Q(z)}$

$\deg Q \geq 2 + \deg P$

$$\boxed{\lim_{\rho \to \infty} \int_{C_{\rho+}} f(z)\,dz = 0}$$



$C_{\rho+}$ = half circle of radius ρ with center 0 in the upper half plane.

Jordan Lemma ②

$\deg Q \geq 1 + \deg P$; $m > 0$

$$\boxed{\lim_{\rho \to \infty} \int_{C_{\rho+}} e^{imz} \times \frac{P(z)}{Q(z)}\,dz = 0}$$

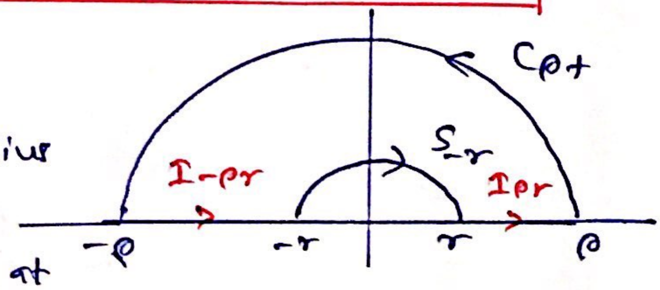
Lemma ③

If f has simple pole at $z = c$ (singular point at center)

$S_r : z = c + re^{i\theta}$; $\theta_1 \leq \theta \leq \theta_2$

$$\boxed{\lim_{r \to 0^+} \int_{S_r} f(z)\,dz = i(\theta_2 - \theta_1)\,\mathrm{Res}\,(f, c)}$$

define the paths ;



$C_{\rho+}$ = upper half circle with radius ρ at origin zero.

S_{-r} = Circular arc with simple pole at origin, center origin and radius r.
$(0 < \theta < \pi)$

$I_{-\rho r}$ = line segment of x axis from $(-\rho$ to $-r)$

$I_{\rho r}$ = line segment of x axis from $(\rho$ to $r)$

Residue Theory

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

1 Method → Laurent expansion

Residue is the coefficient of $\frac{1}{(z-z_0)}$ ~~at~~ in the Laurent expansion for $f(z)$ at z_0 point.

$$f(z) = \frac{e^z}{z} = \frac{1}{z}\left\{1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots\right\} \quad (\text{; at } z_0 = 0)$$

$$= \frac{1}{z} + 1 + \frac{z}{2!} + \frac{z^2}{3!} + \cdots$$

$\frac{1}{(z-z_0)}$ coefficient of 1st order pole.

$Res(f, 0) = 1$

2 Method

$$\boxed{Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)}$$

3 Method

Let $f(z) = \frac{P(z)}{Q(z)}$ where the fun $P(z)$ & $Q(z)$ are both analytic at z_0, and $Q(z)$ has simple zero at z_0 (one singular point)

$$\boxed{Res(f, z_0) = \frac{P(z_0)}{Q'(z_0)}}$$

4 Method

If f has pole of order m at z_0 then,

$$\boxed{Res(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)!} \left[\frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z)\right)\right]_{z = z_0}}$$

Cauchy's Residue Theorem

$$\boxed{\oint_C f(z)\, dz = 2\pi i \sum_{k=1}^{n} Res(f, z_k)}$$