Time Domain Analysis

Introduction: Any system containing, energy storing element like inductors, Capacitors, etc. posses certain energy.

If the energy state of the system is disturbed then it takes a certain time to change from one state to another state.

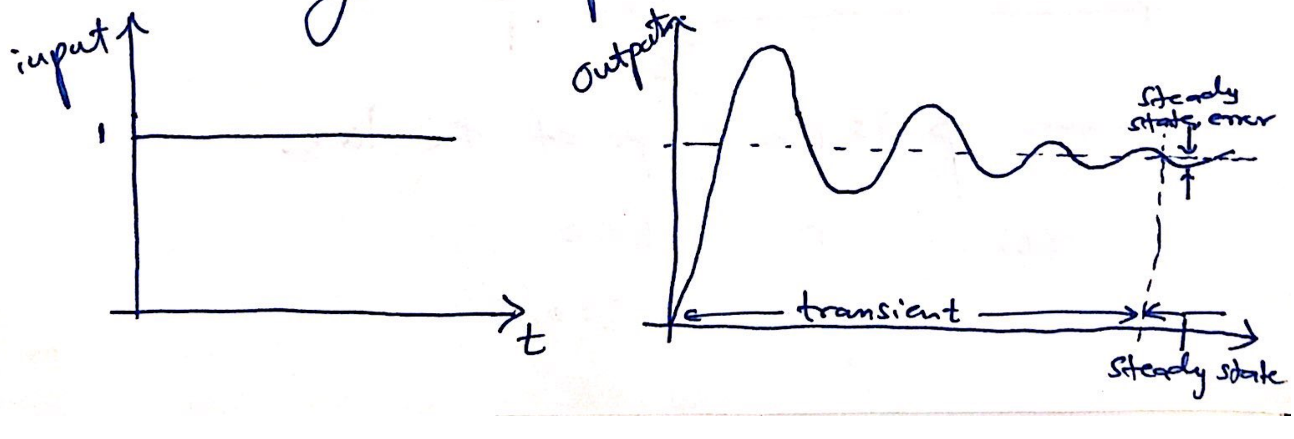
This time required to change from one state to another state is known as transient time.

The values of current or voltage or may be a force, displacement or angular displacement or etc. is called transient response.

Therefore, the time response of a Control System can be divided into two parts,

(a) transient response.

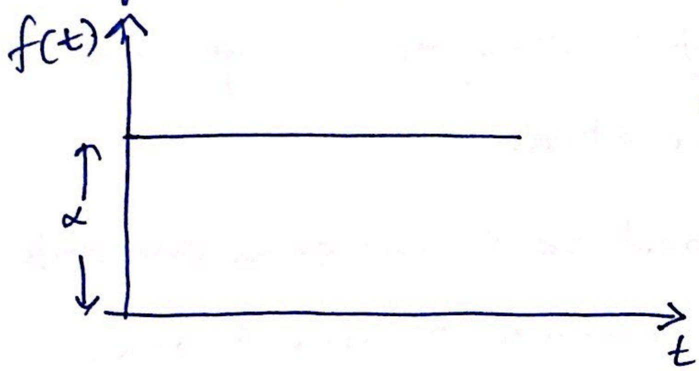
(b) Steady state response.



Test input signals for transient analysis.

For the analysis of time response of a Control system, the following signals can be used.

Step function.

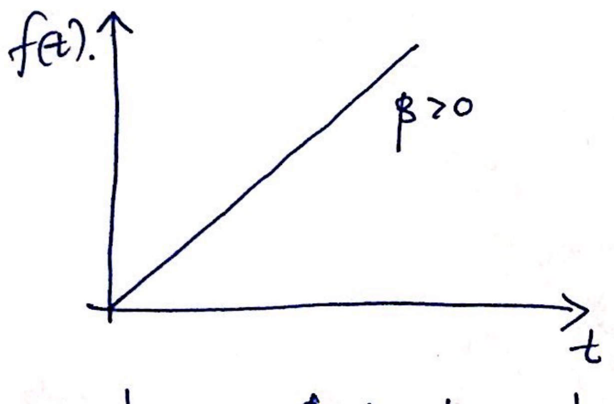


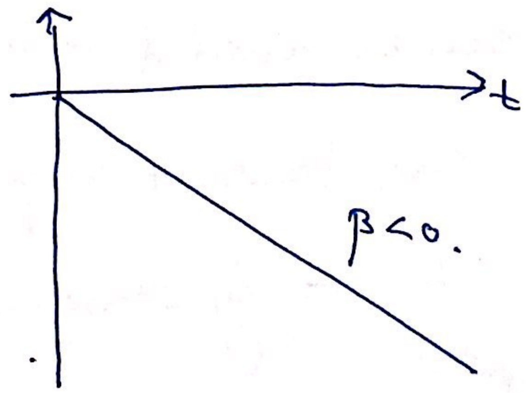
$$f(t) = \begin{cases} \alpha & t \geq 0 \\ 0 & t < 0. \end{cases}$$

Laplace transform of $f(t)$ is, α/s.

$$L\{f(t)\} = \frac{\alpha}{s}.$$

Ramp function.



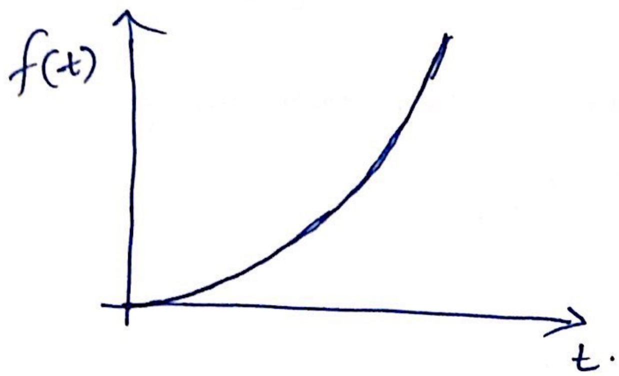


where β is the slope of the line.

$$f(t) = \begin{cases} 0 & t < 0 \\ \beta t & t \geq 0. \end{cases}$$

$$L\{f(t)\} = \beta/s^2$$

Parabolic functions

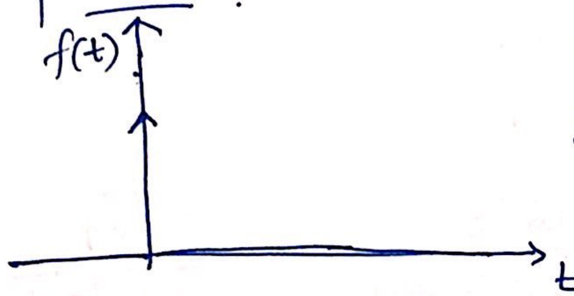


$$f(t) = \frac{\gamma t^2}{2} \quad t \geq 0$$
$$\quad\quad 0 \quad t < 0.$$

where γ is a constant.

So, $L\{f(t)\} = \gamma/s^3$

Impulse function.



The pulse for which the duration of the pulse approaches zero, the amplitude approaches infinity, but the area of the pulse is unity.

A unit impulse function is defined as,

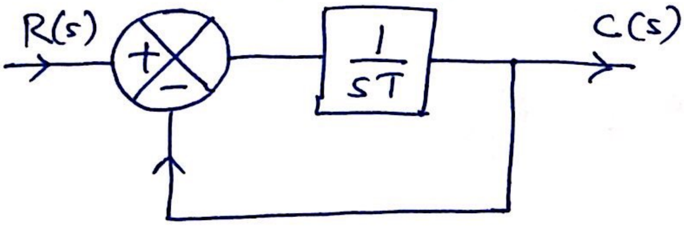
$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

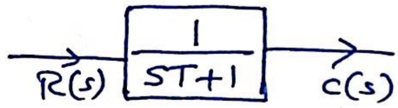
such that $\int_{-\infty}^{+\infty} \delta(t)\,dt = 1.$

$$L\{\delta(t)\} = 1.$$

Time Response of a First order System.

Consider a first order system with unity feedback as shown below.





Here, $G(s) = \frac{1}{sT}$

$H(s) = 1$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)} = \frac{1/sT}{1+\frac{1}{sT}}$$

$$= \frac{1}{sT+1}$$

$$\therefore \quad \frac{C(s)}{R(s)} = \frac{1}{1+sT}.$$

Response of the first order System w.r.t unit step input.

$$R(s) = \frac{1}{s} \quad \text{or} \quad r(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\therefore \quad C(s) = \frac{1}{s(sT+1)}.$$

taking partial fractions,

$$C(s) = \frac{1}{s} - \frac{T}{1+sT}$$

$$= \frac{1}{s} - \frac{1}{s+\frac{1}{T}}$$

$$\therefore \quad c(t) = L^{-1}C(s) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+\frac{1}{T}}\right).$$

$$= 1 - e^{-t/T}$$

when $t = T$,

$$c(T) = 1 - e^{-1}$$

$$= 0.632 \quad \text{or} \quad 63.2\%.$$

where 'T' is known as the time Constant.

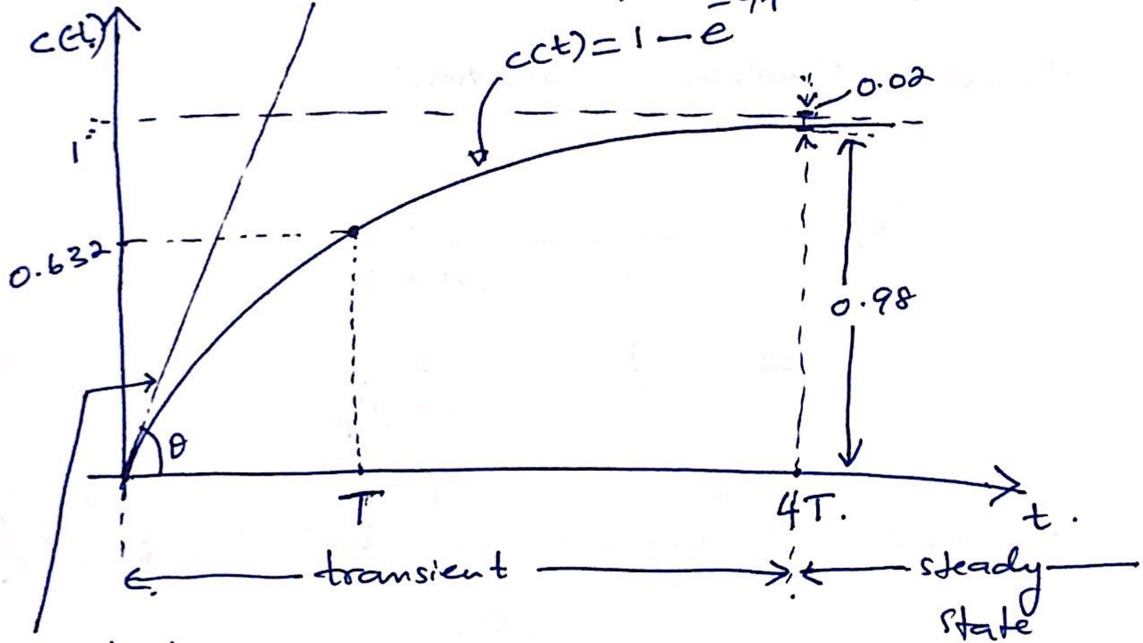
It is defined as the time required for the signal to attain 63.2% of final or steady state value.

Also time constant indicates how fast the system reaches the final value.

smaller time constant $\Longrightarrow$ faster in system response

large time constant $\Longrightarrow$ slow in " " .

Let us consider the output of the system.



tangent to the curve at $t=0$. $[\tan\theta = 1/T]$.

$$\text{let } c(t) = 1 - e^{-t/T}$$

$$\frac{d}{dt} c(t) = \frac{1}{T} e^{-t/T}.$$

$$\text{when } t=0, \quad \left.\frac{d}{dt} c(t)\right|_{t=0} = \frac{1}{T}.$$

$$c(t) = 1 - e^{-t/T}$$

When $t = T$,

$$c(T) = 1 - e^{-1} = 0.632$$

when $t = 2T$,

$$c(2T) = 1 - e^{-2} = 0.864.$$

when $t = 4T$,

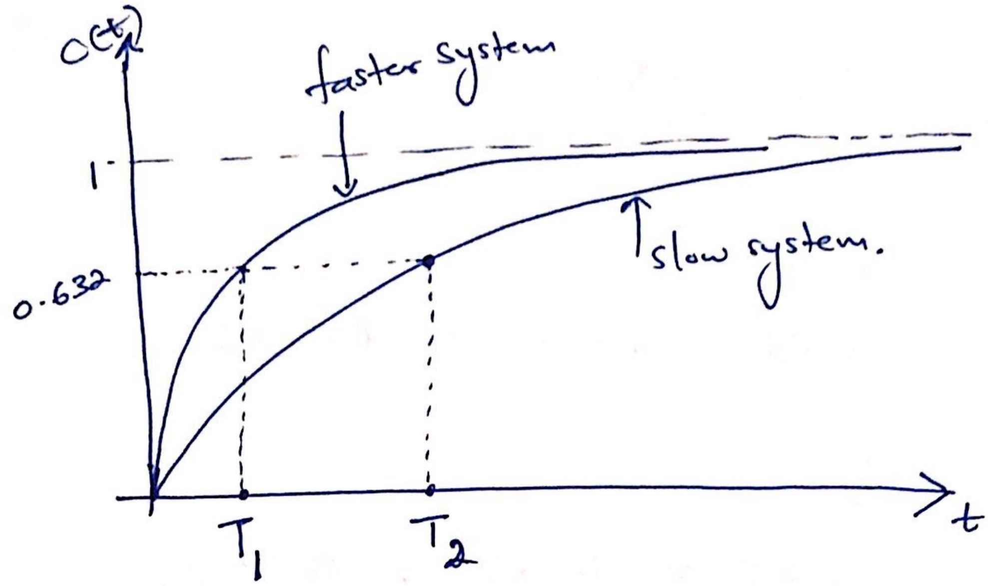
$$c(4T) = 1 - e^{-4} = 0.98.$$

Theoretically, we say that the actual output has reached the desired output, when $t \to \infty$

$$\lim_{t \to \infty} c(t) = \lim_{t \to \infty} (1 - e^{-t/T}) = 1$$

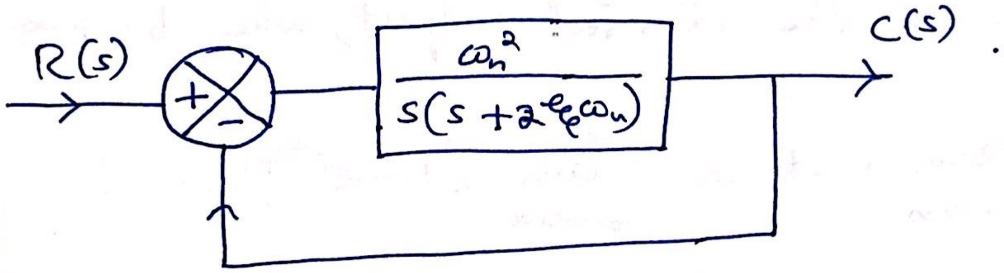
In practice, when actual output reached within 2% of the desired output, it is said that steady state has reached.

Here, $4T$ is known as settling time (t_s).



Time response of Second Order System.

The block diagram of a second order system is shown below.



Here, $G(s) = \dfrac{\omega_n^2}{s(s+2\xi\omega_n)}$, $H(s) = 1$.

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)H(s)}$$

$$= \frac{\omega_n^2/s(s+2\xi\omega_n)}{1+\dfrac{\omega_n^2}{s(s+2\xi\omega_n)}\times 1}$$

$$= \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Time Response of Second order System, with <u>unit step input</u>

when $r(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \Rightarrow L\{r(t)\} = R(s)$

$$R(s) = 1/s.$$

$$C(s) = G(s)\,R(s).$$

$$= \frac{1}{s} \times \frac{\omega_n^2}{(s^2 + 2\xi\omega_n s + \omega_n^2)}$$

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left[\left(\omega_n\sqrt{1-\xi^2}\right)t + \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}\right]$$

where, ω_n : natural frequency of oscillations.

ω_d : damped frequency defined as,

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

ξ : damping coefficient.

For the second order system,

$$G(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}.$$

$s^2 + 2\xi\omega_n s + \omega_n^2 = 0$ is called the characteristic equation.

Example: Measurement Conducted on a servomechanism show the system response to be

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}$$

when subjected to the unit step input.

(a) obtain the expression for the closed loop transfer function.

(b) Determine the natural frequency and damping coefficient of the system.

Answer:

$$c(t) = 1 + 0.2e^{-60t} - 1.2e^{-10t}.$$

$$C(s) = \frac{1}{s} + \frac{0.2}{(s+60)} - \frac{1.2}{(s+10)}.$$

$$= \frac{600}{s(s+60)(s+10)}$$

(a) transfer function,

$$\frac{C(s)}{R(s)} = \frac{\frac{600}{s(s+60)(s+10)}}{1/s} = \frac{600}{(s+10)(s+60)}$$

(b) Characteristic equation;

$$s^2 + 70s + 600 = 0$$

Comparing this with $s^2 + 2\xi\omega_n s + \omega_n^2 = 0$,

we get,

$$\omega_n^2 = 600.$$

$$\omega_n = 24.49 \text{ rad } s^{-1}$$

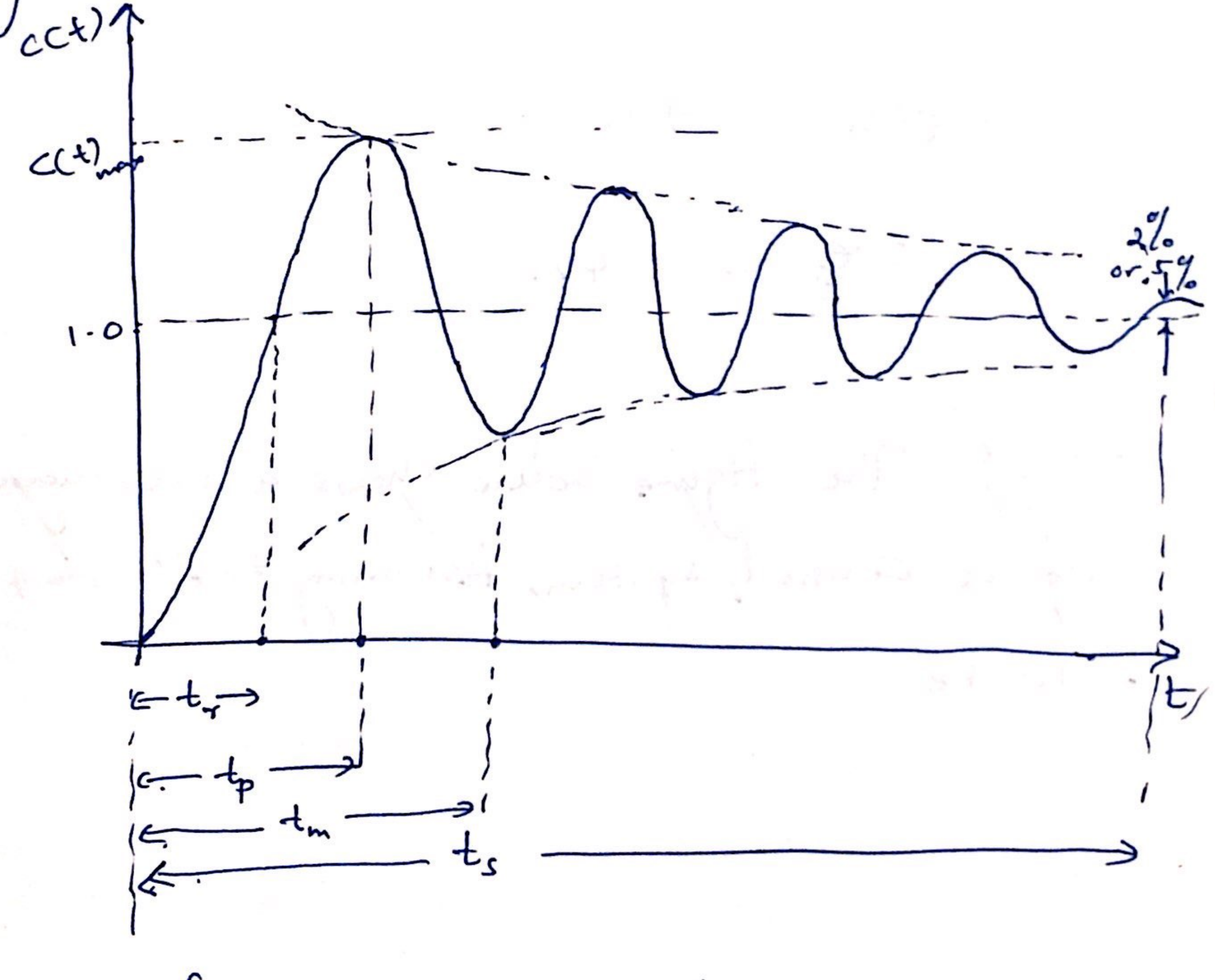
$$2\xi\omega_n = 70.$$

$$\therefore \xi = 1.42.$$

example: The figure below shows a block diagram of a control system. Assuming time constant T to be

Transient Response Specifications of Second order System

The performance of a control system are expressed in terms of transient response to a unit step input, because it is easy to generate.



The following are common transient response characteristics,

1. Rise time (t_r)
2. Peak time (t_p)
3. Maximum overshoot (M_p)
4. Settling time (t_s)
5. Steady state error (e_{ss})

1. Rise time (t_r): time required for the response to rise from ~~10% to 90%~~ 100% of the final value.

2. Peak time (t_p): time required for the response to reach the first peak of the time response.

3. Maximum overshoot (M_p): normalised difference between peak of the time response and steady output.

4. Settling time (t_s): time required for the response to reach and stay with in the specified range (2% or 5%) of its final value.

5. Steady state error (e_{ss}): difference between actual output and desired output as $t \to \infty$.

$$\boxed{\begin{array}{l}\text{Expressions for rise time } (t_r) \\ t_r = \dfrac{\pi - \tan^{-1}\left(\dfrac{\sqrt{1-\xi^2}}{\xi}\right)}{\omega_n\sqrt{1-\xi^2}}\end{array}}$$

Expression for peak time (t_p).

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}.$$

First minimum t_m

$$t_m = \frac{2\pi}{\omega_n\sqrt{1-\xi^2}}$$

Maximum overshoot M_p.

$$M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100\%.$$

Settling time t_s,

$$t_s = \frac{4}{\xi\omega_n}$$

Example: when a second order system is subjected to a unit step input, the values of $\xi = 0.5$ and $\omega_n = 6\ rad\ s^{-1}$. Determine the rise time, peak time, settling time and peak overshoot.

Rise time

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}}{\omega_n\sqrt{1-\xi^2}}$$

$\xi = 0.5$

$\omega_n = 6 \text{ rad s}^{-1}$

$$t_r = \frac{\pi - \tan^{-1}\frac{\sqrt{1-(0.5)^2}}{0.5}}{6\sqrt{1-(0.5)^2}} = 0.403 \text{ s}$$

Peak time

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

$$= \frac{\pi}{6\sqrt{1-(0.5)^2}} = 0.605 \text{ s}$$

Settling time

$$t_s = \frac{4}{\omega_n \xi} = \frac{4}{0.5 \times 6} = 1.33 \text{ s}$$

Maximum overshoot

$$M_p = e^{\frac{-\pi\xi}{\sqrt{1-\xi^2}}} \times 100\%$$

$$= e^{\frac{-\pi \times 0.5}{\sqrt{1-0.5^2}}} \times 100$$

$$= 16.3\%$$