2.9 Fourier Series

2.9.1 Introduction

- Fourier series are infinite series that represent periodic functions in terms of cosines and sines.
- A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x, except possibly at some points, and if there is some positive number p, called a period of $f(x)$, such that

$$f(x+p) = f(x) \qquad \text{for all } x.$$

- The graph of a periodic function has the characteristic that it can be obtained by periodic repetition of its graph in any interval of length p.
- The smallest positive period is often called the fundamental period.

- The series to be obtained will be a trigonometric series, that is, a series of the form

$$a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \dots$$

$$= a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

- $a_0, a_1, b_1, a_2, b_2, \dots$ are constants, called the coefficients of the series.
- We see that each term has the period 2π.
- Hence if the coefficients are such that the series converges, its sum will be a function of period 2π.

2.9.2 Definition of Fourier Series

Definition

Suppose that $f(x)$ is a given function of period 2π and is such that it can be represented by a series $a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$, that is, the series converges and, moreover, has the sum $f(x)$. Then, using the equality sign, we write

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

and call the Fourier series of $f(x)$.

2.9.3 Coefficients of Fourier Series

Definition

The coefficients of $f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ are the so-called Fourier coefficients of $f(x)$, given by the Euler formulas

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\ dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx\ ;\ \ n = 1, 2, ...$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx\ ;\ \ n = 1, 2, ...$$

2.9.4 Fourier Series for Functions of Period 2π

Example (24)

Periodic Rectangular Wave

Find the Fourier coefficients of the following periodic function of period 2π and obtain the Fourier series.

$$f(x) = \begin{cases} -k & \text{if } -\pi < x < 0 \\ k & \text{if } 0 < x < \pi. \end{cases}$$

Example (25)

Find the Fourier series of the given function $f(x)$, which is assumed to have the period 2π . Show the details of your work.

1. $f(x) = \begin{cases} x & \text{if } -\pi < x < 0 \\ \pi - x & \text{if } 0 < x < \pi. \end{cases}$
2. $f(x) = x^2$; $-\pi < 0 < \pi$

2.9.5 Orthogonality of the Trigonometric System

Theorem

The trigonometric system is orthogonal on the interval $-\pi \leq x \leq \pi$ *(hence also on* $0 \leq x \leq 2\pi$ *or any other interval of length*2π *because of periodicity); that is, the integral of the product of any two functions bellow over that interval is 0, so that for any integers* n *and* m*,*

1. $\displaystyle\int_{-\pi}^{\pi} \cos nx \cos mx = 0 \qquad (n \neq m)$
2. $\displaystyle\int_{-\pi}^{\pi} \sin nx \sin mx = 0 \qquad (n \neq m)$
3. $\displaystyle\int_{-\pi}^{\pi} \sin nx \cos mx = 0 \qquad (n \neq m \;\; or \;\; n = m)$

2.9.5.1 Application of Theorem to the Fourier Series

Consider the Fourier series expansion of the function $f(x)$,

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \longrightarrow (\star)$$

Integrating on both sides of $(\star)$ from $-\pi$ to π , we get

$$\int_{-\pi}^{\pi} f(x)\ dx = \int_{-\pi}^{\pi} \left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\ dx$$

We now assume that term-wise integration is allowed. Then we obtain

$$\int_{-\pi}^{\pi} f(x)\,dx = a_0 \int_{-\pi}^{\pi} dx + \sum_{n=1}^{\infty} \left(a_n \int_{-\pi}^{\pi} \cos nx \ dx + b_n \int_{-\pi}^{\pi} \sin nx \ dx \right)$$

The first term on the right equals $2\pi a_0$.
Integration shows that all the other integrals are 0.
Hence division by gives

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx.$$

Multiplying on both sides of $(\star)$ by $\cos mx$ with any fixed positive integer m and integrating from $-\pi$ to π , we have

$$\int_{-\pi}^{\pi} f(x)\cos mx\, dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)\right]\cos mx\, dx$$

We now integrate term by term. Then on the right we obtain an integral of $a_0 \cos mx$ which is 0; an integral of $a_n \cos nx \cos mx$, which is $a_m\pi$ for $n = m$ and 0 for $n \neq m$ by theorem; and an integral of $b_n \sin nx \cos mx$, which is 0 for all n and m by theorem. Hence the right side of the above integral equals $a_m\pi$.
Division by π gives

$$a_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\ dx$$

(with m instead of n).

Multiplying ($\star$) on both sides by with any fixed positive integer m and integrating from $-\pi$ to π, we get

$$\int_{-\pi}^{\pi} f(x)\sin mx\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)\right]\sin mx\,dx$$

Integrating term by term, we obtain on the right an integral of $a_0 \sin mx$, which is 0; an integral of $a_n \cos nx \sin mx$, which is 0 by theorem; and an integral of $b_n \sin nx \sin mx$, which is $b_m\pi$ if $n = m$ and 0 if $n \neq m$, by theorem. This implies π gives

$$b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\sin nx\ dx$$

(with n denoted by m).
This completes the proof of the Euler formulas for the Fourier coefficients.

2.9.6 Fourier Series for Functions of Any Period $p = 2L$

- Consider a function $f(x)$ of period $p = 2L$.
- Then we can introduce a new variable v such that $f(x)$, as a function of v, has period 2π.
- If we set

$$x = \frac{p}{2\pi} v$$

So that,

$$v = \frac{2\pi}{p} x = \frac{\pi}{L} x$$

then $v = \pm\pi$ corresponds to $x = \pm L$.

This means that f, as a function of v, has period 2π and, therefore, a Fourier series of the form

$$f(x) = f\left(\frac{L}{\pi}v\right) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nv + b_n \sin nv)$$

with coefficients obtained in the last section,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\ dv$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\cos nv\ dv$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{L}{\pi}v\right)\sin nv\ dv$$

We could use these formulas directly, but the change to x simplifies calculations.

Since

$$v = \frac{\pi}{L}x, \qquad \text{we have} \qquad dv = \frac{\pi}{L}\,dx$$

and we integrate over x from $-L$ to L.
Consequently, we obtain for a function $f(x)$ of period $2L$ the Fourier series

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L}x + b_n \sin \frac{n\pi}{L}x \right) \longrightarrow (\star\star)$$

The Fourier coefficients of $f(x)$ given by the Euler formulas,

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\ dx \qquad n = 1, 2, \ldots$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\ dx \qquad n = 1, 2, \ldots$$

We continue to call $(\star\star)$ with any coefficients a trigonometric series.
And we can integrate from 0 to $2L$ or over any other interval of length $p = 2L$.

Example (26)

Find the Fourier series of the function

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < -1 \\ k & \text{if } -1 < x < 1 \\ 0 & \text{if } 1 < x < 2. \end{cases}$$

Example (27)

Find the Fourier series of the function

$$f(x) = \begin{cases} -k & \text{if } -2 < x < 0 \\ k & \text{if } 0 < x < 2. \end{cases}$$

Example (28)

Show that the Fourier series expansion of the function

$$f(x) = \begin{cases} 0 & \text{if } -2 < x < 0 \\ 2 - x & \text{if } 0 < x < 2 \end{cases}$$

is

$$\frac{1}{2} + \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{2}x\right) + \sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}x\right).$$

2.9.10 Even and Odd Functions

- If $f(x)$ is an even function, its Fourier series reduces to a Fourier cosine series,

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x$$

with coefficients (note: integration from 0 to L only!)

$$a_0 = \frac{1}{L} \int_0^L f(x)\ dx$$

$$a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\ dx \qquad n = 1, 2, \ldots$$

- If $f(x)$ is an odd function, that is, its Fourier series reduces to a Fourier sine series,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

with coefficient

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx \qquad n = 1, 2, \ldots$$

Note:

1. For an even function $g(x)$,

$$\int_{-L}^{L} g(x)\ dx = 2\int_{0}^{L} g(x)\ dx$$

2. For an odd function $h(x)$,

$$\int_{-L}^{L} h(x)\ dx = 0$$

Summary:

Even Function of Period 2π

If f is even and $L = \pi$, then

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$$

with coefficients

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx \qquad n = 1, 2, \ldots$$

Summary:

Odd Function of Period 2π

If f is odd and $L = \pi$, then

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

with coefficient

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \ dx \qquad n = 1, 2, \ldots$$

2.9.11 Sum and Scalar Multiple of Fourier Series

Theorem

1. *The Fourier coefficients of a sum* $f_1 + f_2$ *are the sums of the corresponding Fourier coefficients of* f_1 *and* f_2.
2. *The Fourier coefficients of* cf *are* c *times the corresponding Fourier coefficients of* f.

Example (29)

Find the Fourier series of the function

$$f(x) = x + \pi \;\; \text{if} \; -\pi < x < \pi$$

and

$$f(x + 2\pi) = f(x).$$

2.9.12 Half-Range Expansions

- So far, we define a Fourier series expansion of a function $f(x)$ that was defined on an interval $-L < x < L$.
- However, in many instance we will need to expand a function in a Fourier series when the function is defined only for $0 < x < L$.
- This can be done in three different ways.
- The series produced is then called by a half-range Fourier series.

1. Reflect the graph of the function about the y axis onto $-L < x < 0$
   This is called Half-range Cosine Expansion of $f(x)$.
2. Reflect the graph of the function through the origin onto $-L < x < 0$
   This is called Half-range Sine Expansion of $f(x)$.
3. Define f on the $-L < x < 0$ by $f(x) = f(x + L)$

2.9.12.1 Fourier Series of Half-range Function

- A function can be expanded using half of its range from : 0 to L or, $-L$ to 0 or L to $2L$.
- That is the range of integration is L.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

$$a_0 = \frac{1}{L/2} \int_0^L f(x)\ dx$$

$$a_n = \frac{1}{L/2} \int_0^L f(x) \cos \frac{n\pi x}{L}\ dx$$

$$b_n = \frac{1}{L/2} \int_0^L f(x) \sin \frac{n\pi x}{L}\ dx$$

Example (30)

Expand $f(x) = x^2\ ,\ 0 < x < 1$.

(i) in a half-range cosine series

(ii) in a half-range sine series

(iii) in a Fourier series.