ERROR RATE DUE TO NOISE
(Probability ERROR of Matched Filter)

- Consider a binary PCM system based on polar non return to zero signaling. In this form of signaling, symbol 1 & 0 are represented by positive and & negative rectangular pulses with amplitude A for equal duration.
- The channel noise is modelled as additive white Gaussian noise $w(t)$ of zero mean & power spectral density $N_0/2$.

In the signaling interval $0 \le t \le T_b$, the received signal is written as,

$$x = \begin{cases} +A + w, & \text{Symbol 1 was sent} \\ -A + w, & \text{Symbol 0 was sent} \end{cases} \quad \text{------} \rightarrow (1)$$

where T_b = bit duration, A = transmitted pulse amplitude

- Given the noisy signal $x(t)$, the receiver is required to make a decision in each signaling interval as to whether the transmitted symbol is a '1' or a "0".

Figure: Receiver for baseband transmission of binary-encoded PCM wave using Polar NRZ signaling

- If the output sample value is exactly equals to threshold value, the receiver may choose it as 1 (or) 0. Here there two possible kinds of errors to be considered.
  - Symbol 1 is chosen where a 0 was actually transmitted
  - Symbol 0 is chosen where a 1 was actually transmitted

Symbol '0' was sent

The received signal is expressed as

$$x(t) = -A + w \quad \text{for } 0 \le t \le T_b \quad \text{---} \rightarrow (2)$$

The output of matched filter is expressed as $Y = \int_0^{T_b} x(t)\, dt$

$$\Rightarrow y = -A + \frac{1}{T_b}\int_0^{T_b} w(t)\, dt \quad \text{--- (3)}$$

• The above equation (3) represents the sample value of random variable. It is characterized as

• The random variable Y is Gaussian distributed with a mean of $-A$.

• The variance of the random variable Y is

$$\sigma_Y^2 = E[(Y+A)^2] \Rightarrow \sigma_Y^2 = \frac{1}{T_b^2} E\left[\int_0^{T_b}\int_0^{T_b} w(t)\, w(u)\, dt\, du\right]$$

$$\Rightarrow \sigma_y^2 = \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} E[w(t)\, w(u)]\, dt\, du$$

$$\Rightarrow \sigma_Y^2 = \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} R_w(t, u)\, dt\, du \quad \text{----} \rightarrow (4)$$

where $R_w(t, u)$ is the auto correlation function of the white noise $w(t)$.

$$R_w(t, u) = \frac{N_0}{2}\, \delta(t-u) \quad \text{--} \rightarrow (5)$$

where $\delta(t-u)$ is the shifted delta function

(5) into (4)

$$\Rightarrow \sigma_Y^2 = \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} \frac{N_0}{2}\, \delta(t-u)\, dt\, du$$

~~(5) into (4)~~

$$\Rightarrow \sigma_Y^2 = \frac{1}{T_b^2}\frac{N_0}{2}\int_0^{T_b} 1\, du \int_0^{T_b} \delta(t-u)\, dt$$

$$\sigma_Y^2 = \frac{1}{T_b^2}\frac{N_0}{2}[u]_0^{T_b}\, [1]_{t=u}$$

$$\sigma_Y^2 = \frac{1}{T_b^2}\frac{N_0}{2}[T_b] \Rightarrow \sigma_Y^2 = \frac{N_0}{2T_b} \quad \text{---} \rightarrow (6)$$

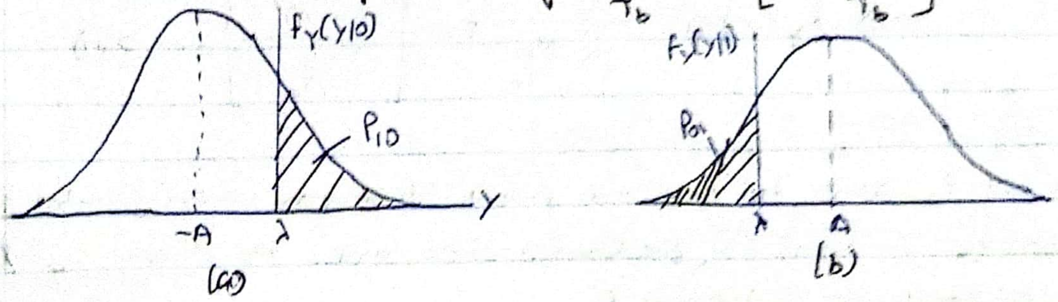
We know that, Gaussian density function is

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(x-m)^2}{2\sigma^2}}$$

Then the conditional probability density function of random variable Y where binary '0' is transmitted is

$$f_Y(y|0) = \frac{1}{\sqrt{2\pi \frac{N_0}{2T_b}}} e^{\frac{-(y+A)^2}{2\frac{N_0}{2T_b}}}$$

$$\Rightarrow f_Y(y|0) = \frac{1}{\sqrt{\pi \frac{N_0}{T_b}}} \exp\left[\frac{-(y+A)^2}{\frac{N_0}{T_b}}\right] \dashrightarrow (7)$$



- Fig: (a) Pdf of random variable y at matched filter output when '0' is transmitted.
- Fig: (b) Pdf of random variable y at matched filter output when '1' is transmitted.

Let P_{10} denote the conditional probability of error, when symbol 0 was sent. $P_{10} = P(y>\lambda \,|\, \text{symbol } 0 \text{ was sent})$

$$P_{10} = \int_{\lambda}^{\infty} f_Y(y|0)\,dy \Rightarrow P_{10} = \int_{\lambda}^{\infty} \frac{1}{\sqrt{\pi \frac{N_0}{T_b}}} \exp\left[\frac{-(y+A)^2}{\frac{N_0}{T_b}}\right] dy$$

$$\Rightarrow P_{10} = \frac{1}{\sqrt{\pi \frac{N_0}{T_b}}} \int_{\lambda}^{\infty} \exp\left[\frac{-(y+A)^2}{\frac{N_0}{T_b}}\right] dy \dashrightarrow (8)$$

Introduce the complementary error function for simplification of equation (8)

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-z^2}\,dz \dashrightarrow (9)$$

Simplifying the equation (8), let us take, $z = \frac{y+A}{\sqrt{\frac{N_0}{T_b}}}$

$$P_{10} = \frac{1}{\sqrt{\pi}} \int_{\frac{\lambda+A}{\sqrt{\frac{N_0}{T_b}}}}^{\infty} e^{-z^2}\,dz \Rightarrow P_{10} = \frac{1}{2}\operatorname{erfc}\left(\frac{\lambda+A}{\sqrt{\frac{N_0}{T_b}}}\right) \dashrightarrow (10)$$

Symbol '1' was sent

The conditional probability density function of random var 'y' where binary '1' is transmitted is

$$f_Y(y|1) = \frac{1}{\sqrt{2\pi \frac{N_0}{2T_b}}} e^{\frac{-(y-A)^2}{2\frac{N_0}{2T_b}}}$$

$$\Rightarrow f_Y(y|1) = \frac{1}{\sqrt{\pi \frac{N_0}{T_b}}} \exp\left[\frac{-(y-A)^2}{\frac{N_0}{T_b}}\right] \text{ - - - -> } (11)$$

- Let P_{01} denote the conditional probability errors when symbol 1 was transmitted. In figure (b), the shaded area under the curve $f_Y(y|1)$ over the limits $-\infty$ to λ gives P_{01}.

$$P_{01} = P(Y<\lambda \mid \text{symbol 1 was sent})$$

$$P_{01} = \int_{-\infty}^{\lambda} f_Y(y|1)\, dy \Rightarrow P_{01} = \int_{-\infty}^{\lambda} \frac{1}{\sqrt{\pi \frac{N_0}{T_b}}} \exp\left[\frac{-(y-A)^2}{\frac{N_0}{T_b}}\right] dy$$

$$\Rightarrow P_{01} = \frac{1}{\sqrt{\pi \frac{N_0}{T_b}}} \int_{-\infty}^{\lambda} \exp\left[\frac{-(y-A)^2}{\frac{N_0}{T_b}}\right] dy \text{ - - -> } (12)$$

For the simplification of eq. (12), we take $z = \frac{A-y}{\sqrt{\frac{N_0}{T_b}}}$

Interchanging the limits,

$$P_{01} = \frac{1}{\sqrt{\pi}} \int_{\frac{A-\lambda}{\sqrt{\frac{N_0}{T_b}}}}^{\infty} \exp[-z^2]\, dy$$

Using error function expression the above equation is written as

$$\Rightarrow P_{01} = \frac{1}{2} \operatorname{erfc}\left(\frac{A-\lambda}{\sqrt{\frac{N_0}{T_b}}}\right) \text{ - - -> } (13)$$

Therefore the average probability of symbol error, P_e is

$$P_e = P_0 P_{10} + P_1 P_{01}$$

$$P_e = \frac{P_0}{2} \operatorname{erfc}\left(\frac{A+\lambda}{\sqrt{\frac{N_0}{T_b}}}\right) + \frac{P_1}{2} \operatorname{erfc}\left(\frac{A-\lambda}{\sqrt{\frac{N_0}{T_b}}}\right)$$

If symbol 0 & 1 are equiprobable, then $P_0 = P_1 = 1/2$

Take the optimum value of threshold is equal to zero. i.e, $\lambda_{opt} = 0$

$$P_e = \frac{1/2}{2} \operatorname{erfc}\left(\frac{A+0}{\sqrt{\frac{N_0}{T_b}}}\right) + \frac{1/2}{2} \operatorname{erfc}\left(\frac{A-0}{\sqrt{\frac{N_0}{T_b}}}\right)$$

$$P_e = \frac{1}{4} \operatorname{erfc}\left(\frac{A}{\sqrt{\frac{N_0}{T_b}}}\right) + \frac{1}{4} \operatorname{erfc}\left[\frac{A}{\sqrt{\frac{N_0}{T_b}}}\right]$$

$$\Rightarrow P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{A}{\sqrt{N_0/T_b}}\right] \dashrightarrow (14)$$

The energy of transmitted signal, E_b is $E_b = A^2 T_b$

$$\Rightarrow A = \sqrt{\frac{E_b}{T_b}}$$

$A \to (14)$

$$P_e = \frac{1}{2} \operatorname{erfc}\left[\frac{\sqrt{E_b/T_b}}{\sqrt{N_0/T_b}}\right]$$

$$\Rightarrow P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Where, E_b is transmitted signal energy per bit and N_0 is the noise spectral density.

Nyquist's Criterion for distortionless Baseband binary Transmission

- we know that, $y(t_i) = a_i + \sum_{k=-\infty}^{\infty} a_k \, p[(i-k)T_b] + n(t_i)$

- **Time domain criterion**: From the above equation, we know that the second term (summation) must be zero to eliminate effect of ISI. This is possible if the received pulse $p(t)$ is controlled such that,

$$p[(i-k)T_b] = \begin{cases} 1 & \text{for } i = k \\ 0 & \text{for } i \neq k \end{cases} \quad \text{---} (1)$$

- If $p(t)$ satisfies the above condition, then we get a signal which is free from ISI. i.e, $y(t_i) = a_i$ for all i

Frequency domain criterion: Let $p(nT_b)$ represent the impulses at which $p(t)$ is sampled for decision. These samples are taken at the rate of T_b.

- Fourier spectrum of these impulses is given as

$$P_\delta(f) = f_b \sum_{n=-\infty}^{\infty} P(f - mf_b) \quad \text{---} (2)$$

- where, f_b is bit rate in bits per second (b/s)

$P_\delta(f)$ Fourier transform of infinite periodic sequence of delta functions of period T_b.

The time domain representation of pulse signal $P_\delta(f)$ is

$$P_\delta(t) = \sum_{n=-\infty}^{\infty} p(nT_b)\, \delta(t - nT_b) \quad \text{---} (3)$$

- The Fourier transform of $P_\delta(f)$ is

$$P_\delta(f) = \int_{-\infty}^{\infty} P_\delta(t)\, e^{-j2\pi f t}\, dt$$

$$\Rightarrow P_\delta(f) = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} p(nT_b)\, \delta(t - nT_b)\, e^{-j2\pi f t}\, dt$$

Let the integer, $n = i - k$

$$P_\delta(f) = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} p((i-k)T_b)\, \delta(t-(i-k)T_b)\, e^{-j2\pi f t}\, dt$$

• Now let us apply the condition of equation (1) to above equation

$$P_\delta(f) = \begin{cases} \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} p(0)\, \delta(t)\, e^{-j2\pi f t}\, dt & \text{for } i = k \\ \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} 0\, \delta(t)\, e^{-j2\pi f t}\, dt & \text{for } i \neq k \end{cases}$$

Therefore,

$$P_\delta(f) = \int_{-\infty}^{\infty} p(0)\, \delta(t)\, e^{-j2\pi f t}\, dt \quad \text{for } i = k$$

$$P_\delta(f) = p(0) \int_{-\infty}^{\infty} \delta(t)\, e^{-j2\pi f t}\, dt$$

Using shifting property of delta function,

$$P_\delta(f) = p(0)\left[e^{-j2\pi f t}\right] \text{ at, } t = 0 \text{ for } i = k$$

$$\Rightarrow P_\delta(f) = p(0) \Rightarrow P_\delta(f) = 1$$

Hence eq(2) becomes,

$$1 = f_b \sum_{n=-\infty}^{\infty} P(f - n f_b) \Rightarrow \sum_{n=-\infty}^{\infty} P(f - n f_b) = \frac{1}{f_b}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} P(f - n f_b) = T_b \quad \text{--> (4)}$$

~~In frequency function,~~

Above equation is called Nyquist Pulse shaping criterion for baseband transmission.

Raised Cosine Spectrum

(Ideal Nyquist Channel

The simplest way of satisfying

$$\sum_{n=-\infty}^{\infty} P(f-nf_b) = T_b \quad (\text{or}) \quad \sum_{n=-\infty}^{\infty} P(f-nR_b) = T_b \rightarrow ①$$

• For $n=0$, the L.H.S corresponds to $P(f)$ and it represents a frequency function with the narrowest band which statisfies equation ①

• The range of frequencies for $P(f)$ will extend from $-W$ to W ($-B_0$ to B_0) where W or B_0 corresponds to half the bit rate.

Hence, $W = \frac{f_b}{2}$ (or) $\frac{R_b}{2}$.

• This equation is to specify the frequency function $P(f)$ to be in the form of a rectangular function, it is shown as

$$P(f) = \begin{cases} \frac{1}{2W} & ; -W < f < W \\ 0 & ; \quad |f| > W \end{cases}$$

$$P(f) = \frac{1}{2W} \, rect\left(\frac{f}{2W}\right) \quad — ①$$

The overall system bandwidth 'W' is defined by,

$$W = \frac{R_b}{2} = \frac{1}{2T_b} \quad — ②$$

The signal that produces zero ISI can be obtained by taking the IFT of $P(f)$

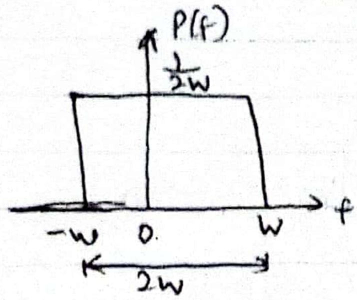
This means that we have,

$$P(t) = F^{-1}[P(f)] \Longrightarrow P(t) = F^{-1}\left[\frac{1}{2W} \, rect\left(\frac{f}{2W}\right)\right]$$

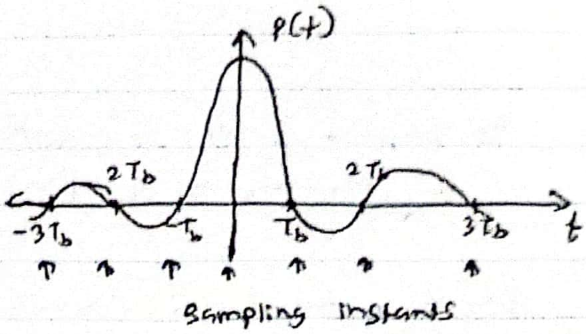
$$\Longrightarrow P(t) = sinc(2Wt) \quad or \quad P(t) = \frac{\sin(2\pi Wt)}{2\pi Wt} \quad — ③$$

The ideal base band data transmission system described by equation ① in frequency domain and equation ③ in time domain is called the

Ideal Nyquist Channel



(a) Graphical representation of P(f)



(b) Time-domain representation

Advantages of using the Sinc Pulse:

- Bandwidth requirement (of the channel) is reduced
- ISI is reduced to zero.

Possible Difficulties: This ideal Nyquist Channel have two Practical difficulties.

- The P(f) is plot from -W to W & zero elsewhere.

This is physically ~~no~~ unrealizable because of the abrupt transitions at the edges $\pm W$

Raised cosine channels (or) Raised cosine spectrum

we know that,

$$\sum_{n=-\infty}^{\infty} P(f-nf_b) = T_b \text{ (or) } \sum_{n=-\infty}^{\infty} P(f-nR_b) = T_b = \frac{1}{R_b} = \frac{1}{f_b}$$

$$\dots + P(f+3R_b) + P(f+2R_b) + P(f+R_b) + P(f) + P(f-R_b) + P(f-2R_b) + P(f-3R_b) \dots = \frac{1}{R_b} = T_b$$

- But, $W = \frac{R_b}{2} \implies R_b = 2W$.

$$\dots + P(f+6W) + P(f+4W) + P(f+2W) + P(f) + P(f-2W) + P(f-4W) + P(f-6W) + \dots = \frac{1}{2W}$$

$$P(f+2W) + P(f) + P(f-2W) = \frac{1}{2W}; \quad -W \leq f \leq W \text{ — ①}$$

- Raised cosine spectrum consists of a flat portion and a roll off portion. The raised cosine spectrum is expressed mathematically as

$$P(f) = \begin{cases} \frac{1}{2W}; \text{ (flat portion)} & 0 \leq |f| \leq f_1 \\ \frac{1}{4W}\left\{1 - \sin\left[\frac{\pi(|f| - W)}{2W - 2f_1}\right]\right\}; & f_1 \leq |f| \leq 2W - f_1 \text{ — ②} \\ 0; & |f| \geq 2W - f_1 \end{cases}$$

- The frequency parameter f_1 & bandwidth W are related by

$$\alpha = 1 - \frac{f_1}{W} \text{ — ③}$$

- The parameter α is called the roll off factor, it indicates excess Bandwidth over the ideal solution W.
- The transmission Bandwidth B_T, is defined by

$$B_T = 2W - f_1 \implies B_T = W(1+\alpha) \text{ — ④}$$

• The normalized frequency response of raised cos. function is obtained by multiplying P(f) by 2W and is plotted in figure (a), for different values of α. The corresponding time response p(t) is shown in figure (b).

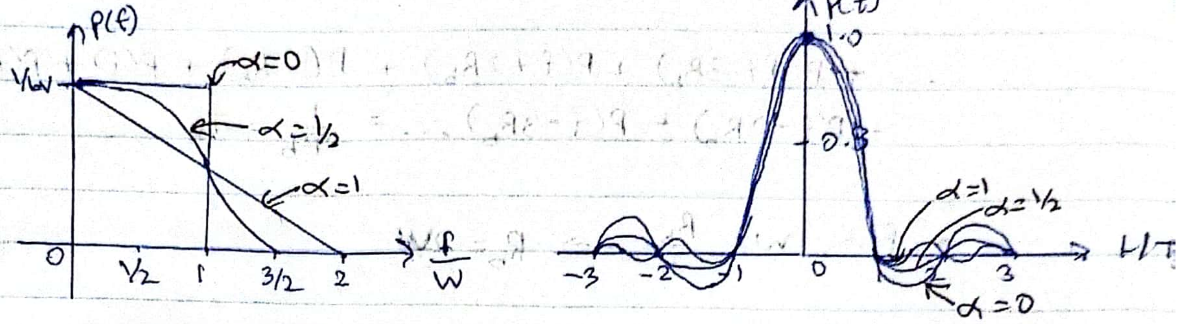


Fig. Responses for different roll-off factors, α

• a) For $\alpha = 0.5$ and 2, the characteristics of P(f) changes gradually with respect to frequency. Hence, it is easier to realize this characteristics practically. The time response p(t) is the ~~F.T.~~ I.F.T. of P(f) i.e. $p(t) = \text{sinc}(2Wt)\left[\frac{\cos(2\pi\alpha Wt)}{1-16\alpha^2W^2t^2}\right]$ —(5)

• (b) The time response has a sinc shape and all the sinc function pass through zero at $t = \pm T_b, \pm 2T_b, \ldots$

Effects of Gaussian Noise:-

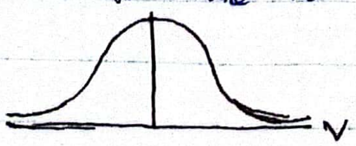
Noise → causes error

• error rate → depend on signal & noise amplitude & characteristics of noise

• Noise of large amplitude spike will introduce error in the detected signal

It is assumed to be gaussian

eg. ~~Gaussian~~ Thermal noise generated to a resistor



$V\uparrow \Rightarrow PDF\downarrow$

$M=0$

$\max = V=0$

CamScanner

Error Function

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-z^2}\,dy \qquad 0 \le \operatorname{erf} \le 1$$

$$\frac{2}{\sqrt{\pi}}\, f(y)$$

$$e^{x} = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$\frac{2}{\sqrt{\pi}} \int_0^y 1 + \frac{-z^2}{1} + \frac{(-z^2)^2}{2!} + \frac{(-z^2)^3}{3!} + \cdots dz$$

$$= \frac{2}{\sqrt{\pi}} \left[y - \frac{z^3}{3\cdot 1} + \frac{z^5}{5\cdot 2!} - \frac{z^7}{7\cdot 3!} + \cdots \cancel{dz} \right]_0^y$$

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \left[y - \frac{y^3}{3\cdot 1} + \frac{y^5}{5\cdot 2!} - \frac{y^7}{7\cdot 3!} + \cdots \right]$$

$$\operatorname{erf}(\infty) = \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-z^2}\,dz \qquad \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$$

$$= \frac{2}{\sqrt{\pi}} \times \frac{\sqrt{\pi}}{2} = 1$$

$$\operatorname{erf}(\infty) = 1$$

$$y = 0 \Rightarrow \operatorname{erf}(y) = 0$$

Symmetry Property: $\operatorname{erf}(y) = -\operatorname{erf}(-y)$

Complementary error function :-

$$\operatorname{erfc}(y) = 1 - \operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_y^\infty e^{(-z^2)}\,dz$$

$y \uparrow \Rightarrow \operatorname{erfc}(y) \downarrow$

decreasing function of y