4: VECTOR ANALYSIS

Steven W. Ellingson
Virginia Polytechnic Institute and State University

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CHAPTER OVERVIEW

4: Vector Analysis

A *vector* is a mathematical object that has both a scalar part (i.e., a magnitude and possibly a phase), as well as a direction. Many physical quantities are best described as vectors. For example, the rate of movement through space can be described as *speed*; i.e., as a scalar having SI base units of m/s. However, this quantity is more completely described as *velocity*; i.e., as a vector whose scalar part is speed and direction indicates the direction of movement. Similarly, force is a vector whose scalar part indicates magnitude (SI base units of N), and direction indicates the direction in which the force is applied. Electric and magnetic fields are also best described as vectors.

4.1: Vector Arithmetic

In mathematical notation, a real-valued vector $\mathbf{A}$ is said to have a magnitude $A = |\mathbf{A}|$ and direction $\hat{\mathbf{a}}$ such that

$$\mathbf{A} = A\hat{\mathbf{a}}$$

where $\hat{\mathbf{a}}$ is a unit vector (i.e., a real-valued vector having magnitude equal to one) having the same direction as $\mathbf{A}$. If a vector is complex-valued, then A is similarly complex-valued.

Cartesian Coordinate System

Fundamentals of vector arithmetic are most easily grasped using the Cartesian coordinate system. This system is shown in Figure 4.1.1. Note carefully the relative orientation of the x, y, and z axes. This orientation is important. For example, there are two directions that are perpendicular to the $z = 0$ plane (in which the x- and y-axes lie), but the $+z$ axis is specified to be one of these in particular.

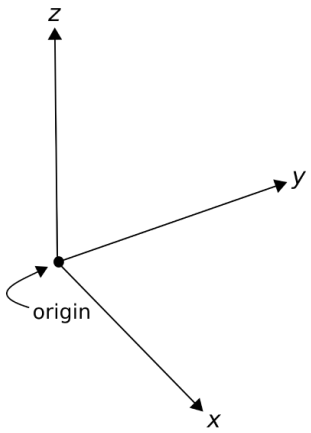

Figure 4.1.1: Cartesian coordinate system. (CC BY SA 4.0; K. Kikkeri).

Position-Fixed vs. Position-Free Vectors

It is often convenient to describe a position in space as a vector for which the magnitude is the distance from the origin of the coordinate system and for which the direction is measured from the origin toward the position of interest. This is shown in Figure 4.1.2. These *position vectors* are "position-fixed" in the sense that they are defined with respect to a single point in space, which in this case is the origin. Position vectors can also be defined as vectors that are defined with respect to some other point in space, in which case they are considered position-fixed to that position.

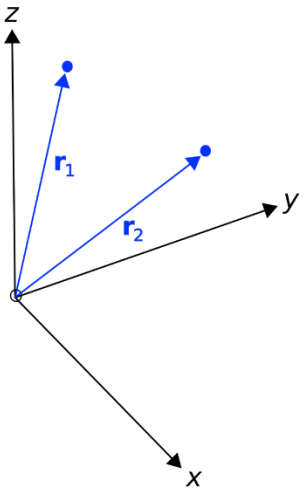

Figure 4.1.2: Position vectors. The vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are position-fixed and refer to particular locations. (CC BY SA 4.0; K. Kikkeri).

Position-free vectors, on the other hand, are not defined with respect to a particular point in space. An example is shown in Figure 4.1.2. Particles 1 m apart may both be traveling at 2 m/s in the same direction. In this case, the velocity of each particle can be described using the same vector, even though the particles are located at different points in space.

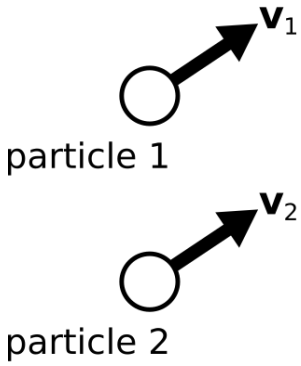

Figure 4.1.3: Two particles exhibiting the same velocity. In this case, the velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ are position-free and equal. (CC BY SA 4.0; K. Kikkeri).

Position-free vectors are said to be equal if they have the same magnitudes and directions. Position-fixed vectors, on the other hand, must also be referenced to the same position (e.g., the origin) to be considered equal.

Basis Vectors

Each coordinate system is defined in terms of three basis vectors which concisely describe all possible ways to traverse three-dimensional space. A basis vector is a position-free unit vector that is perpendicular to all other basis vectors for that coordinate system. The basis vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ of the Cartesian coordinate system are shown in Figure 4.1.4. In this notation, $\hat{\mathbf{x}}$ indicates the direction in which x increases most rapidly, $\hat{\mathbf{y}}$ indicates the direction in which y increases most rapidly, and $\hat{\mathbf{z}}$ indicates the direction in which z increases most rapidly. Alternatively, you might interpret $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$ as unit vectors that are parallel to the x-, y-, and z-axes and point in the direction in which values along each axis increase.

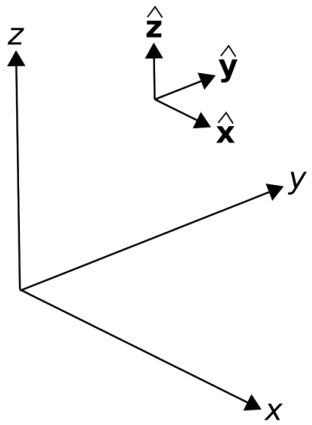

Figure 4.1.4: Basis vectors in the Cartesian coordinate system. (CC BY SA 4.0; K. Kikkeri).

Vectors in the Cartesian Coordinate System

In Cartesian coordinates, we may describe any vector $\mathbf{A}$ as follows:

$$\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$$

where A_x, A_y, and A_z are scalar quantities describing the components of $\mathbf{A}$ in each of the associated directions, as shown in Figure 4.1.5. This description makes it clear that the magnitude of $\mathbf{A}$ is:

$$|\mathbf{A}| = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

and therefore, we can calculate the associated unit vector as

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{\mathbf{A}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$
$$= \hat{\mathbf{x}} A_x \left(A_x^2 + A_y^2 + A_z^2\right)^{-1/2}$$
$$+ \hat{\mathbf{y}} A_y \left(A_x^2 + A_y^2 + A_z^2\right)^{-1/2}$$
$$+ \hat{\mathbf{z}} A_z \left(A_x^2 + A_y^2 + A_z^2\right)^{-1/2}$$

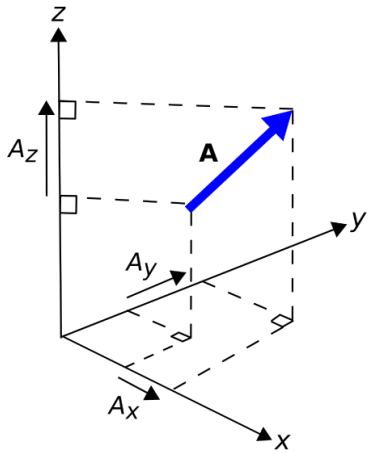

Figure 4.1.5: Components of a vector A in the Cartesian coordinate system. (CC BY SA 4.0; K. Kikkeri).

Vector Addition and Subtraction

It is common to add and subtract vectors. For example, vectors describing two forces $\mathbf{A}$ and $\mathbf{B}$ applied to the same point can be described as a single force vector $\mathbf{C}$ that is the sum of $\mathbf{A}$ and $\mathbf{B}$; i.e., $\mathbf{C} = \mathbf{A} + \mathbf{B}$. This addition is quite simple in the Cartesian coordinate system:

$$\begin{aligned}\mathbf{C} &= \mathbf{A} + \mathbf{B} \\ &= (\hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z) + (\hat{\mathbf{x}} B_x + \hat{\mathbf{y}} B_y + \hat{\mathbf{z}} B_z) \\ &= \hat{\mathbf{x}} (A_x + B_x) + \hat{\mathbf{y}} (A_y + B_y) + \hat{\mathbf{z}} (A_z + B_z)\end{aligned}$$

In other words, the $\hat{\mathbf{x}}$ component of $\mathbf{C}$ is the sum of the $\hat{\mathbf{x}}$ components of $\mathbf{A}$ and $\mathbf{B}$, and similarly for $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. From the above example, it is clear that vector addition is commutative; i.e.,

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A}$$

In other words, vectors may be added in any order. Vector subtraction is defined similarly:

$$\begin{aligned}\mathbf{D} &= \mathbf{A} - \mathbf{B} \\ &= (\hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z) - (\hat{\mathbf{x}} B_x + \hat{\mathbf{y}} B_y + \hat{\mathbf{z}} B_z) \\ &= \hat{\mathbf{x}} (A_x - B_x) + \hat{\mathbf{y}} (A_y - B_y) + \hat{\mathbf{z}} (A_z - B_z)\end{aligned}$$

In other words, the $\hat{\mathbf{x}}$ component of $\mathbf{D}$ is the difference of the $\hat{\mathbf{x}}$ components of $\mathbf{A}$ and $\mathbf{B}$, and similarly for $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$. Like scalar subtraction, vector subtraction is *not* commutative.

Relative Positions and Distances

A common task in vector analysis is to describe the position of one point in space relative to a different point in space. Let us identify those two points using the position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$, as indicated in Figure 4.1.6. We may identify a third vector $\mathbf{r}_{12}$ as the position of $\mathbf{r}_2$ relative to $\mathbf{r}_1$:

$$\mathbf{r}_{12} \triangleq \mathbf{r}_2 - \mathbf{r}_1$$

Now $|\mathbf{r}_{12}|$ is the distance between these points, and $\mathbf{r}_{12} / |\mathbf{r}_{12}|$ is a unit vector indicating the direction to $\mathbf{r}_2$ from $\mathbf{r}_1$.

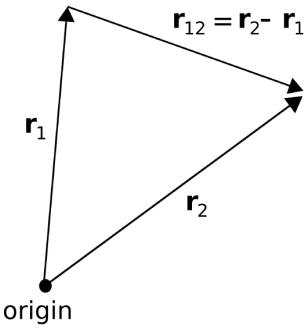

Figure 4.1.6: Relative position (distance and direction) between locations identified by their position vector. (CC BY SA 4.0; K. Kikkeri).

Example 4.1.1: Direction and distance between positions.

Consider two positions, identified using the position vectors $\mathbf{r}_1 = 2\hat{\mathbf{x}} + 3\hat{\mathbf{y}} + 1\hat{\mathbf{z}}$ and $\mathbf{r}_2 = 1\hat{\mathbf{x}} - 2\hat{\mathbf{y}} + 3\hat{\mathbf{z}}$, both expressed in units of meters. Find the direction vector that points from $\mathbf{r}_1$ to $\mathbf{r}_2$, the distance between these points, and the associated unit vector.

Solution

The vector that points from $\mathbf{r}_1$ to $\mathbf{r}_2$ is

$$\begin{aligned}\mathbf{R} &= \mathbf{r}_2 - \mathbf{r}_1 \\ &= (1-2)\hat{\mathbf{x}} + (-2-3)\hat{\mathbf{y}} + (3-1)\hat{\mathbf{z}} \\ &= -\hat{\mathbf{x}} - 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}}\end{aligned}$$

The distance between $\mathbf{r}_1$ and $\mathbf{r}_2$ is simply the magnitude of this vector:

$$|\mathbf{R}| = \sqrt{(-1)^2 + (-5)^2 + (2)^2} \cong 5.48 \text{ m}$$

The unit vector $\hat{\mathbf{R}}$ is simply $\mathbf{R}$ normalized to have unit magnitude:

$$\begin{aligned}\hat{\mathbf{R}} &= \mathbf{R}/|\mathbf{R}| \\ &\cong (-\hat{\mathbf{x}} - 5\hat{\mathbf{y}} + 2\hat{\mathbf{z}})/5.48 \\ &\cong -0.182\hat{\mathbf{x}} - 0.913\hat{\mathbf{y}} + 0.365\hat{\mathbf{z}}\end{aligned}$$

Multiplication of a Vector by a Scalar

Let's say a particular force is specified by a vector $\mathbf{F}$. What is the new vector if this force is doubled? The answer is simply $2\mathbf{F}$ – that is, twice the magnitude applied in the same direction. This is an example of scalar multiplication of a vector. Generalizing, the product of the scalar α and the vector $\mathbf{A}$ is simply $\alpha\mathbf{A}$.

Scalar ("Dot") Product of Vectors

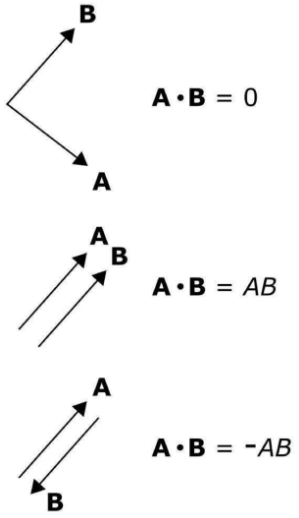

Figure 4.1.7: Special cases of the dot product. (CC BY SA 4.0; K. Kikkeri).

Another common task in vector analysis is to determine the similarity in the direction in which two vectors point. In particular, it is useful to have a metric which, when applied to the vectors $\mathbf{A} = \hat{\mathbf{a}}A$ and $\mathbf{B} = \hat{\mathbf{b}}B$, has the following properties (Figure 4.1.7):

- If $\mathbf{A}$ is perpendicular to $\mathbf{B}$, the result is zero.
- If $\mathbf{A}$ and $\mathbf{B}$ point in the same direction, the result is AB.
- If $\mathbf{A}$ and $\mathbf{B}$ point in opposite directions, the result is $-AB$.
- Results intermediate to these conditions depend on the angle ψ between $\mathbf{A}$ and $\mathbf{B}$, measured as if $\mathbf{A}$ and $\mathbf{B}$ were arranged "tail-to-tail" as shown in Figure 4.1.8.

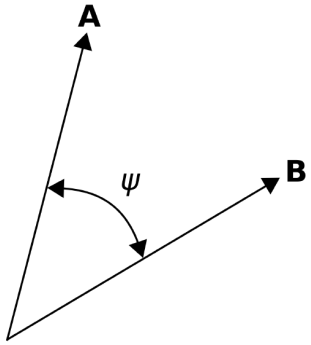

Figure 4.1.8: Calculation of the dot product. (CC BY SA 4.0; K. Kikkeri).

In vector analysis, this operator is known as the *scalar product* (not to be confused with scalar multiplication) or the *dot product*. The dot product is written $\mathbf{A} \cdot \mathbf{B}$ and is given in general by the expression:

$$\mathbf{A} \cdot \mathbf{B} = AB\cos\psi \tag{4.1.1}$$

Note that this expression yields the special cases previously identified, which are $\psi = \pi/2$, $\psi = 0$, and $\psi = \pi$, respectively. The dot product is commutative; i.e.,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

The dot product is also distributive; i.e.,

$$\mathbf{A} \cdot (\mathbf{B} + \mathbf{C}) = \mathbf{A} \cdot \mathbf{B} + \mathbf{A} \cdot \mathbf{C}$$

The dot product has some other useful properties. For example, note:

$$\begin{aligned}\mathbf{A}\cdot\mathbf{A} &= (\hat{\mathbf{x}}A_x+\hat{\mathbf{y}}A_y+\hat{\mathbf{z}}A_z)\cdot(\hat{\mathbf{x}}A_x+\hat{\mathbf{y}}A_y+\hat{\mathbf{z}}A_z)\\ &= \hat{\mathbf{x}}\cdot\hat{\mathbf{x}}A_x^2+\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}A_xA_y+\hat{\mathbf{x}}\cdot\hat{\mathbf{z}}A_xA_z\\ &+\hat{\mathbf{y}}\cdot\hat{\mathbf{x}}A_xA_y+\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}A_y^2+\hat{\mathbf{y}}\cdot\hat{\mathbf{z}}A_yA_z\\ &+\hat{\mathbf{z}}\cdot\hat{\mathbf{x}}A_xA_z+\hat{\mathbf{z}}\cdot\hat{\mathbf{y}}A_yA_z+\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}A_z^2\end{aligned}$$

which looks pretty bad until you realize that

$$\hat{\mathbf{x}}\cdot\hat{\mathbf{x}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{z}}\cdot\hat{\mathbf{z}}=1$$

and any other dot product of basis vectors is zero. Thus, the whole mess simplifies to:

$$\mathbf{A}\cdot\mathbf{A}=A_x^2+A_y^2+A_z^2$$

This is the square of the magnitude of $\mathbf{A}$, so we have discovered that

$$\mathbf{A}\cdot\mathbf{A}=|\mathbf{A}|^2=A^2$$

Applying the same principles to the dot product of potentially different vectors $\mathbf{A}$ and $\mathbf{B}$, we find:

$$\mathbf{A}\cdot\mathbf{B} = (\hat{\mathbf{x}}A_x+\hat{\mathbf{y}}A_y+\hat{\mathbf{z}}A_z)\cdot(\hat{\mathbf{x}}B_x+\hat{\mathbf{y}}B_y+\hat{\mathbf{z}}B_z) \tag{4.1.2}$$

$$= A_xB_x+A_yB_y+A_zB_z \tag{4.1.3}$$

This is a particularly easy way to calculate the dot product, since it eliminates the problem of determining the angle ψ. In fact, an easy way to calculate ψ given $\mathbf{A}$ and $\mathbf{B}$ is to first calculate the dot product using Equation 4.1.3 and then use the result to solve Equation 4.1.1 for ψ.

Example 4.1.2: Angle between two vectors

Consider the position vectors $\mathbf{C}=2\hat{\mathbf{x}}+3\hat{\mathbf{y}}+1\hat{\mathbf{z}}$ and $\mathbf{D}=3\hat{\mathbf{x}}-2\hat{\mathbf{y}}+2\hat{\mathbf{z}}$, both expressed in units of meters. Find the angle between these vectors.

Solution

From Equation 4.1.1

$$\mathbf{C}\cdot\mathbf{D}=CD\cos\psi$$

where $C=|\mathbf{C}|$, $D=|\mathbf{D}|$, and ψ is the angle we seek. From Equation 4.1.3:

$$\begin{aligned}\mathbf{C}\cdot\mathbf{D} &= C_xD_x+C_yD_y+C_zD_z\\ &= 2\cdot3+3\cdot(-2)+1\cdot2\text{m}^2\\ &= 2\text{ m}^2\end{aligned}$$

also

$$C=\sqrt{C_x^2+C_y^2+C_z^2}\cong3.742\text{ m}$$

$$D=\sqrt{D_x^2+D_y^2+D_z^2}\cong4.123\text{ m}$$

so

$$\cos\psi=\frac{\mathbf{C}\cdot\mathbf{D}}{CD}\cong0.130$$

Taking the inverse cosine, we find $\psi=82.6^\circ$.

Cross Product

The cross product is a form of vector multiplication that results in a vector that is perpendicular to both of the operands. The definition is as follows:

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB \sin \psi_{AB} \tag{4.1.4}$$

As shown in Figure 4.1.9, the unit vector $\hat{\mathbf{n}}$ is determined by the "right hand rule." Using your right hand, curl your fingers to traverse the angle ψ_{AB} beginning at $\mathbf{A}$ and ending at $\mathbf{B}$, and then $\hat{\mathbf{n}}$ points in the direction of your fully-extended thumb.

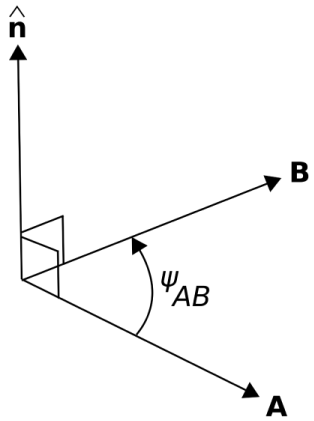

Figure 4.1.9: The cross product $\mathbf{A} \times \mathbf{B}$. (CC BY SA 4.0; K. Kikkeri).

It should be apparent that the cross product is not commutative but rather is *anticommutative*; that is,

$$\mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

You can confirm this for yourself using either Equation 4.1.4 or by applying the right-hand rule.

The cross product is distributive:

$$\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$$

There are two useful special cases of the cross product that are worth memorizing. The first is the cross product of a vector with itself, which is zero:

$$\mathbf{A} \times \mathbf{A} = 0$$

The second is the cross product of vectors that are perpendicular; i.e., for which $\psi_{AB} = \pi/2$. In this case:

$$\mathbf{A} \times \mathbf{B} = \hat{\mathbf{n}} AB$$

Using these principles, note:

$$\hat{\mathbf{x}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{z}} = 0$$

whereas

$$\hat{\mathbf{x}} \times \hat{\mathbf{y}} = \hat{\mathbf{z}}$$

$$\hat{\mathbf{y}} \times \hat{\mathbf{z}} = \hat{\mathbf{x}}$$

$$\hat{\mathbf{z}} \times \hat{\mathbf{x}} = \hat{\mathbf{y}}$$

A useful diagram that summarizes these relationships is shown in Figure 4.1.10.

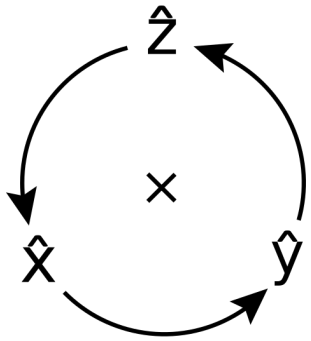

Figure 4.1.10: Cross products among basis vectors in the Cartesian system. The cross product of any two basis vectors is the third basis vector when the order of operands is counter-clockwise, as shown in the diagram, and is −1 times the third basis vector when the order of operands is clockwise with respect to the arrangement in the diagram. (CC BY SA 4.0; K. Kikkeri).

It is typically awkward to "manually" determine $\hat{\mathbf{n}}$ in Equation 4.1.4. However, in Cartesian coordinates the cross product may be calculated as:

$$\begin{aligned}\mathbf{A}\times\mathbf{B} = \quad & \hat{\mathbf{x}}\,(A_yB_z - A_zB_y) \\ & +\hat{\mathbf{y}}\,(A_zB_x - A_xB_z) \\ & +\hat{\mathbf{z}}\,(A_xB_y - A_yB_x)\end{aligned}$$

This may be easier to remember as a matrix determinant:

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Similar expressions are available for other coordinate systems.

Vector analysis routinely requires expressions involving both dot products and cross products in different combinations. Often, these expressions may be simplified, or otherwise made more convenient, using the vector identities listed in Appendix B3.

4.2: Cartesian Coordinates

The Cartesian coordinate system is introduced in Section 4.1. Concepts described in that section – i.e., the dot product and cross product – are described in terms of the Cartesian system. In this section, we identify some additional features of this system that are useful in subsequent work and also set the stage for alternative systems; namely the cylindrical and spherical coordinate systems.

Integration Over Length

Consider a vector field $\mathbf{A} = \hat{\mathbf{x}} A(\mathbf{r})$, where $\mathbf{r}$ is a position vector. What is the integral of $\mathbf{A}$ over some curve $\mathcal{C}$ through space? To answer this question, we first identify a differential-length segment of the curve. Note that this segment of the curve can be described as

$$d\mathbf{l} = \hat{\mathbf{x}} dx + \hat{\mathbf{y}} dy + \hat{\mathbf{z}} dz$$

The contribution to the integral for that segment of the curve is simply $\mathbf{A} \cdot d\mathbf{l}$. We integrate to obtain the result; i.e.,

$$\int_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}$$

For example, if $\mathbf{A} = \hat{\mathbf{x}} A_0$ (i.e., $A(\mathbf{r})$ is a constant) and if $\mathcal{C}$ is a straight line from $x = x_1$ and $x = x_2$ along some constant y and z, then $d\mathbf{l} = \hat{\mathbf{x}} dx$, $\mathbf{A} \cdot d\mathbf{l} = A_0 dx$, and subsequently the above integral is

$$\int_{x_1}^{x_2} A_0 dx = A_0 \left(x_2 - x_1 \right)$$

In particular, notice that if $A_0 = 1$, then this integral gives the length of $\mathcal{C}$. Although the formalism seems unnecessary in this simple example, it becomes very useful when integrating over paths that vary in more than one direction and with more complicated integrands.

Note that the Cartesian system was an appropriate choice for preceding example because this allowed two of the three basis directions (i.e., y and z) to be immediately eliminated from the calculation. Said differently, the preceding example is expressed with the minimum number of varying coordinates in the Cartesian system. Here's a counter-example. If $\mathcal{C}$ had been a circle in the $z = 0$ plane, then the problem would have required two basis directions to be considered – namely, both x and y. In this case, another system – namely, cylindrical coordinates (Section 4.3) – minimizes the number of varying coordinates (to just one, which is ϕ).

Integration Over Area

Now we ask the question, what is the integral of some vector field $\mathbf{A}$ over some surface $\mathcal{S}$? The answer is

$$\int_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{s} \tag{4.2.1}$$

We refer to $d\mathbf{s}$ as the differential surface element, which is a vector having magnitude equal to the differential area ds, and is normal (perpendicular) to each point on the surface. There are actually two such directions. We'll return to clear up the ambiguity in a moment. Now, as an example, if $\mathbf{A} = \hat{\mathbf{z}}$ and $\mathcal{S}$ is the surface bounded by $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$, then

$$d\mathbf{s} = \hat{\mathbf{z}} \, dx \, dy$$

since $dx \, dy$ is differential surface area in the $z = 0$ plane and $\hat{\mathbf{z}}$ is normal to the $z = 0$ plane. So $\mathbf{A} \cdot d\mathbf{s} = dx dy$, and subsequently the integral in Equation 4.2.1 becomes

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} dx \, dy = \left(x_2 - x_1 \right) \left(y_2 - y_1 \right)$$

Note that this has turned out to be a calculation of area.

Once again, we see the Cartesian system was an appropriate choice for this example because this choice minimizes the number of varying coordinates; in the above example, the surface of integration is described by a constant value of z with variable values of x and y. If the surface had instead been a cylinder or a sphere, not only would all three basis directions be variable, but also the surface normal would be variable, making the problem dramatically more complicated.

Now let's return to the issue of the direction of $d\mathbf{s}$. We chose $+\hat{\mathbf{z}}$, but why not choose $-\hat{\mathbf{z}}$ – also a normal to the surface – as this direction? To answer simply, the resulting area would be negative. "Negative area" is the expected ("positive") area, except with respect to the opposite-facing normal vector. In the present problem, the sign of the area is not important, but in some problems this sign becomes important. One example of a class of problems for which the sign of area is important is when the quantity of interest is a *flux*. If $\mathbf{A}$ were a flux density, then the integration over area that we just performed indicates the magnitude *and direction* of flux, and so the direction chosen for $d\mathbf{s}$ defines the direction of positive flux. Section 2.4 describes the electric field in terms of a flux (i.e., as electric flux density $\mathbf{D}$), in which case positive flux flows away from a positively-charged source.

Integration Over Volume

Another common task in vector analysis is integration of some quantity over a volume. Since the procedure is the same for scalar or vector quantities, we shall consider integration of a scalar quantity $A(\mathbf{r})$ for simplicity. First, we note that the contribution from a differential volume element

$$dv = dx\ dy\ dz$$

is $A(\mathbf{r})\ dv$, so the integral over the volume $\mathcal{V}$ is

$$\int_{\mathcal{V}} A(\mathbf{r})\ dv$$

For example, if $A(\mathbf{r}) = 1$ and $\mathcal{V}$ is a cube bounded by $x_1 \leq x \leq x_2$, $y_1 \leq y \leq y_2$, and $z_1 \leq z \leq z_2$, then the above integral becomes

$$\int_{x_1}^{x_2} \int_{y_1}^{y_2} \int_{z_1}^{z_2} dx\ dy\ dz = (x_2 - x_1)\,(y_2 - y_1)\,(z_2 - z_1)$$

i.e., this is a calculation of volume.

The Cartesian system was an appropriate choice for this example because $\mathcal{V}$ is a cube, which is easy to describe in Cartesian coordinates and relatively difficult to describe in any other coordinate system.

4.3: Cylindrical Coordinates

Cartesian coordinates (Section 4.1) are not convenient in certain cases. One of these is when the problem has cylindrical symmetry. For example, in the Cartesian coordinate system, the cross-section of a cylinder concentric with the z-axis requires two coordinates to describe: x and y. However, this cross section can be described using a single parameter – namely the radius – which is ρ in the cylindrical coordinate system. This results in a dramatic simplification of the mathematics in some applications.

The cylindrical system is defined with respect to the Cartesian system in Figure 4.3.1. In lieu of x and y, the cylindrical system uses ρ, the distance measured from the closest point on the z axis, and ϕ, the angle measured in a plane of constant z, beginning at the $+x$ axis ($\phi = 0$) with ϕ increasing toward the $+y$ direction.

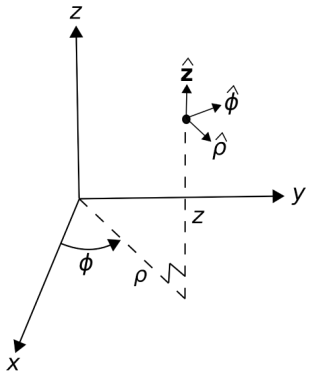

Figure 4.3.1: Cylindrical coordinate system and associated basis vectors. (CC BY SA 4.0; K. Kikkeri).

The basis vectors in the cylindrical system are $\hat{\rho}$, $\hat{\phi}$, and $\hat{\mathbf{z}}$. As in the Cartesian system, the dot product of like basis vectors is equal to one, and the dot product of unlike basis vectors is equal to zero. The cross products of basis vectors are as follows:

$$\hat{\rho} \times \hat{\phi} = \hat{\mathbf{z}}$$

$$\hat{\phi} \times \hat{\mathbf{z}} = \hat{\rho}$$

$$\hat{\mathbf{z}} \times \hat{\rho} = \hat{\phi}$$

A useful diagram that summarizes these relationships is shown in Figure 4.3.2.

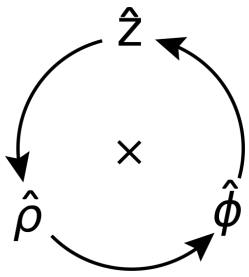

Figure 4.3.2: Cross products among basis vectors in the cylindrical system. (See Figure 4.1.10 for instructions on the use of this diagram.) (CC BY SA 4.0; K. Kikkeri).

The cylindrical system is usually less useful than the Cartesian system for identifying absolute and relative positions. This is because the basis directions depend on position. For example, $\hat{\rho}$ is directed radially outward from the $\hat{\mathbf{z}}$ axis, so $\hat{\rho} = \hat{\mathbf{x}}$ for locations along the x-axis but $\hat{\rho} = \hat{\mathbf{y}}$ for locations along the y axis. Similarly, the direction $\hat{\phi}$ varies as a function of position. To overcome this awkwardness, it is common to set up a problem in cylindrical coordinates in order to exploit cylindrical symmetry, but at some point to convert to Cartesian coordinates. Here are the conversions:

$$x = \rho \cos \phi$$

$$y = \rho \sin \phi$$

and z is identical in both systems. The conversion from Cartesian to cylindrical is as follows:

$$\rho = \sqrt{x^2 + y^2}$$

$$\phi = \arctan(y, x)$$

where arctan is the *four-quadrant* inverse tangent function; i.e., $\arctan(y/x)$ in the first quadrant ($x > 0$, $y > 0$), but possibly requiring an adjustment for the other quadrants because the signs of both x and y are individually significant.

Similarly, it is often necessary to represent basis vectors of the cylindrical system in terms of Cartesian basis vectors and vice-versa. Conversion of basis vectors is straightforward using dot products to determine the components of the basis vectors in the new system. For example, $\hat{\mathbf{x}}$ in terms of the basis vectors of the cylindrical system is

$$\hat{\mathbf{x}} = \hat{\rho}\,(\hat{\rho}\cdot\hat{\mathbf{x}}) + \hat{\phi}\left(\hat{\phi}\cdot\hat{\mathbf{x}}\right) + \hat{\mathbf{z}}\,(\hat{\mathbf{z}}\cdot\hat{\mathbf{x}})$$

The last term is of course zero since $\hat{\mathbf{z}}\cdot\hat{\mathbf{x}} = 0$. Calculation of the remaining terms requires dot products between basis vectors in the two systems, which are summarized in Table 4.3.1. Using this table, we find

$$\hat{\mathbf{x}} = \hat{\rho}\cos\phi - \hat{\phi}\sin\phi$$

$$\hat{\mathbf{y}} = \hat{\rho}\sin\phi + \hat{\phi}\cos\phi$$

and of course $\hat{\mathbf{z}}$ requires no conversion. Going from Cartesian to cylindrical, we find

$$\hat{\rho} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$$

$$\hat{\phi} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi$$

Table 4.3.1: Dot products between basis vectors in the cylindrical and Cartesian coordinate systems.

$\cdot$	$\hat{\boldsymbol{\rho}}$	$\hat{\boldsymbol{\phi}}$	$\hat{\mathbf{z}}$
$\hat{\mathbf{x}}$	$\cos\phi$	$-\sin\phi$	0
$\hat{\mathbf{y}}$	$\sin\phi$	$\cos\phi$	0
$\hat{\mathbf{z}}$	0	0	1

Integration Over Length

A differential-length segment of a curve in the cylindrical system is described in general as

$$d\mathbf{l} = \hat{\rho}d\rho + \hat{\phi}\rho d\phi + \hat{\mathbf{z}}\,dz$$

Note that the contribution of the ϕ coordinate to differential length is $\rho d\phi$, not simply $d\phi$. This is because ϕ is an angle, not a distance. To see why the associated distance is $\rho d\phi$, consider the following. The circumference of a circle of radius ρ is $2\pi\rho$. If only a fraction of the circumference is traversed, the associated arclength is the circumference scaled by $\phi/2\pi$, where ϕ is the angle formed by the traversed circumference. Therefore, the distance is $2\pi\rho\cdot\phi/2\pi = \rho\phi$, and the differential distance is $\rho d\phi$.

As always, the integral of a vector field $\mathbf{A}(\mathbf{r})$ over a curve $\mathcal{C}$ is

$$\int_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{l}$$

To demonstrate the cylindrical system, let us calculate the integral of $\mathbf{A}(\mathbf{r}) = \hat{\phi}$ when $\mathcal{C}$ is a circle of radius ρ_0 in the $z = 0$ plane, as shown in Figure 4.3.3. In this example, $d\mathbf{l} = \hat{\phi}\,\rho_0\,d\phi$ since $\rho = \rho_0$ and $z = 0$ are both constant along $\mathcal{C}$. Subsequently, $\mathbf{A}\cdot d\mathbf{l} = \rho_0 d\phi$ and the above integral is

$$\int_0^{2\pi} \rho_0\,d\phi = 2\pi\rho_0$$

i.e., this is a calculation of circumference.

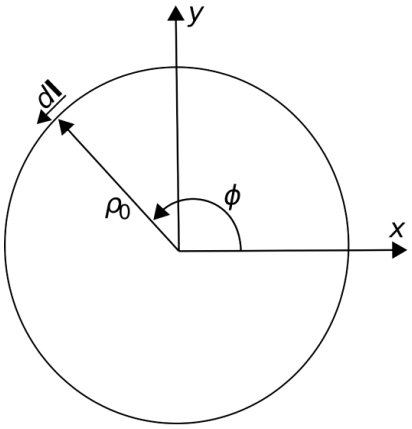

Figure 4.3.3: Example in cylindrical coordinates: The circumference of a circle. (CC BY SA 4.0; K. Kikkeri).

Note that the cylindrical system is an appropriate choice for the preceding example because the problem can be expressed with the minimum number of varying coordinates in the cylindrical system. If we had attempted this problem in the Cartesian system, we would find that both x and y vary over $\mathcal{C}$, and in a relatively complex way.

Integration Over Area

Now we ask the question, what is the integral of some vector field $\mathbf{A}$ over a circular surface $\mathcal{S}$ in the $z = 0$ plane having radius ρ_0? This is shown in Figure Figure 4.3.4. The differential surface vector in this case is

$$d\mathbf{s} = \hat{\mathbf{z}}\ (d\rho)\,(\rho d\phi) = \hat{\mathbf{z}}\,\rho\,d\rho\,d\phi \tag{4.3.1}$$

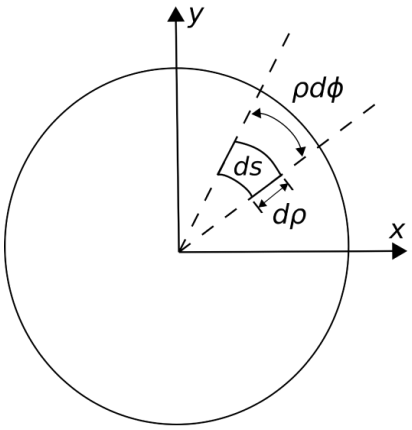

Figure 4.3.4: Example in cylindrical coordinates: The area of a circle. (CC BY SA 4.0; K. Kikkeri).

The quantities in parentheses of Equation 4.3.1 are the radial and angular dimensions, respectively. The direction of $d\mathbf{s}$ indicates the direction of positive flux – see the discussion in Section 4.2 for an explanation. In general, the integral over a surface is

$$\int_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{s}$$

To demonstrate, let's consider $\mathbf{A} = \hat{\mathbf{z}}$; in this case $\mathbf{A} \cdot d\mathbf{s} = \rho\,d\rho\,d\phi$ and the integral becomes

$$\int_0^{\rho_0}\int_0^{2\pi} \rho\,d\rho\,d\phi = \left(\int_0^{\rho_0} \rho\,d\rho\right)\left(\int_0^{2\pi} d\phi\right) \tag{4.3.2}$$

$$= \left(\frac{1}{2}\rho_0^2\right)(2\pi) \tag{4.3.3}$$

$$= \pi\rho_0^2 \tag{4.3.4}$$

which we recognize as the area of the circle, as expected. The corresponding calculation in the Cartesian system is quite difficult in comparison.

Whereas the previous example considered a planar surface, we might consider instead a curved surface. Here we go. What is the integral of a vector field $\mathbf{A} = \hat{\rho}$ over a cylindrical surface $\mathcal{S}$ concentric with the z axis having radius ρ_0 and extending from $z = z_1$ to $z = z_2$? This is shown in Figure 4.3.5.

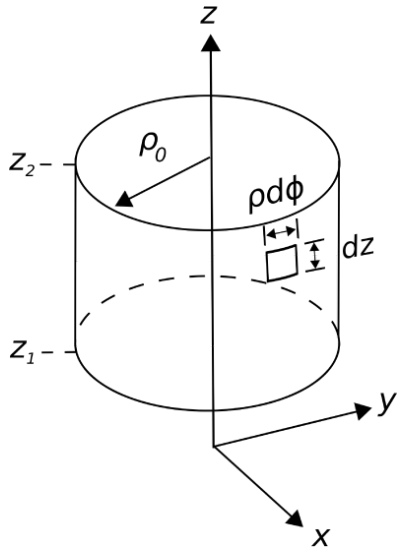

Figure 4.3.5: Example in cylindrical coordinates: The area of the curved surface of a cylinder. (CC BY SA 4.0; K. Kikkeri).

The differential surface vector in this case is

$$d\mathbf{s} = \hat{\rho}\,(\rho_0 d\phi)\,(dz) = \hat{\rho}\,\rho_0\; d\phi\; dz$$

The integral is

$$\int_{\mathcal{S}} \mathbf{A}\cdot d\mathbf{s} = \int_0^{2\pi}\int_{z_1}^{z_2} \rho_0\; d\phi dz \tag{4.3.5}$$

$$= \rho_0 \left(\int_0^{2\pi} d\phi\right)\left(\int_{z_1}^{z_2} dz\right) \tag{4.3.6}$$

$$= 2\pi\rho_0\,(z_2 - z_1) \tag{4.3.7}$$

which is the area of $\mathcal{S}$, as expected. Once again, the corresponding calculation in the Cartesian system is quite difficult in comparison.

Integration Over Volume

The differential volume element in the cylindrical system is

$$dv \;=\; d\rho\;\;(\rho d\phi)\;\;dz \;=\; \rho\; d\rho\; d\phi\; dz$$

For example, if $A(\mathbf{r}) = 1$ and the volume $\mathcal{V}$ is a cylinder bounded by $\rho \le \rho_0$ and $z_1 \le z \le z_2$, then

$$\int_{\mathcal{V}} A(\mathbf{r})\; dv = \int_0^{\rho_0}\int_0^{2\pi}\int_{z_1}^{z_2} \rho\; d\rho\; d\phi\; dz \tag{4.3.8}$$

$$= \left(\int_0^{\rho_0} \rho\; d\rho\right)\left(\int_0^{2\pi} d\phi\right)\left(\int_{z_1}^{z_2} dz\right) \tag{4.3.9}$$

$$= \pi\rho_0^2\,(z_2 - z_1) \tag{4.3.10}$$

i.e., area times length, which is volume.

Once again, the procedure above is clearly more complicated than is necessary if we are interested only in computing volume. However, if the integrand is not constant-valued then we are no longer simply computing volume. In this case, the formalism is appropriate and possibly necessary.

4.4: Spherical Coordinates

The spherical coordinate system is defined with respect to the Cartesian system in Figure 4.4.1. The spherical system uses r, the distance measured from the origin; θ, the angle measured from the $+z$ axis toward the $z = 0$ plane; and ϕ, the angle measured in a plane of constant z, identical to ϕ in the cylindrical system.

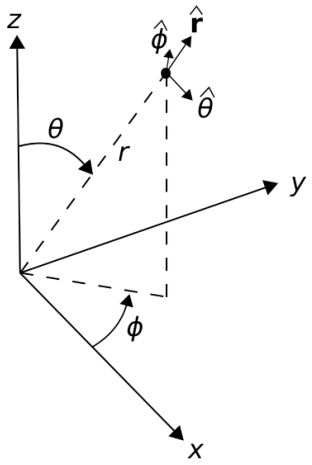

Figure 4.4.1: Spherical coordinate system and associated basis vectors. (CC BY SA 4.0; K. Kikkeri).

Spherical coordinates are preferred over Cartesian and cylindrical coordinates when the geometry of the problem exhibits spherical symmetry. For example, in the Cartesian coordinate system, the surface of a sphere concentric with the origin requires all three coordinates (x, y, and z) to describe. However, this surface can be described using a single constant parameter – the radius r – in the spherical coordinate system. This leads to a dramatic simplification in the mathematics in certain applications.

The basis vectors in the spherical system are $\hat{\mathbf{r}}$, $\hat{\theta}$, and $\hat{\phi}$. As always, the dot product of like basis vectors is equal to one, and the dot product of unlike basis vectors is equal to zero. For the cross-products, we find:

$$\hat{\mathbf{r}} \times \hat{\theta} = \hat{\phi}$$

$$\hat{\theta} \times \hat{\phi} = \hat{\mathbf{r}}$$

$$\hat{\phi} \times \hat{\mathbf{r}} = \hat{\theta}$$

A useful diagram that summarizes these relationships is shown in Figure 4.4.2.

Like the cylindrical system, the spherical system is often less useful than the Cartesian system for identifying absolute and relative positions. The reason is the same: Basis directions in the spherical system depend on position. For example, $\hat{\mathbf{r}}$ is directed radially outward from the origin, so $\hat{\mathbf{r}} = \hat{\mathbf{x}}$ for locations along the x-axis but $\hat{\mathbf{r}} = \hat{\mathbf{y}}$ for locations along the y axis and $\hat{\mathbf{r}} = \hat{\mathbf{z}}$ for locations along the z axis. Similarly, the directions of $\hat{\theta}$ and $\hat{\phi}$ vary as a function of position. To overcome this awkwardness, it is common to begin a problem in spherical coordinates, and then to convert to Cartesian coordinates at some later point in the analysis. Here are the conversions:

$$x = r\cos\phi\sin\theta \tag{4.4.1}$$

$$y = r\sin\phi\sin\theta \tag{4.4.2}$$

$$z = r\cos\theta \tag{4.4.3}$$

The conversion from Cartesian to spherical coordinates is as follows:

$$r = \sqrt{x^2 + y^2 + z^2} \tag{4.4.4}$$

$$\theta = \arccos(z/r) \tag{4.4.5}$$

$$\phi = \arctan(y, x) \tag{4.4.6}$$

where $\arctan$ is the four-quadrant inverse tangent function.

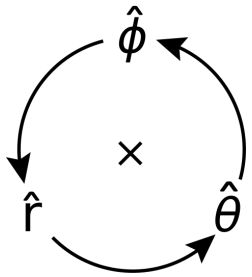

Figure 4.4.2 Cross products among basis vectors in the spherical system. (See Figure 4.1.10 for instructions on the use of this diagram.) (CC BY SA 4.0; K. Kikkeri).

Dot products between basis vectors in the spherical and Cartesian systems are summarized in Table 4.4.1. This information can be used to convert between basis vectors in the spherical and Cartesian systems, in the same manner described in Section 4.3; e.g.

$$\hat{\mathbf{x}} = \hat{\mathbf{r}}\,(\hat{\mathbf{r}}\cdot\hat{\mathbf{x}}) + \hat{\theta}\,\left(\hat{\theta}\cdot\hat{\mathbf{x}}\right) + \hat{\phi}\,\left(\hat{\phi}\cdot\hat{\mathbf{x}}\right)$$

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\,(\hat{\mathbf{x}}\cdot\hat{\mathbf{r}}) + \hat{\mathbf{y}}\,(\hat{\mathbf{y}}\cdot\hat{\mathbf{r}}) + \hat{\mathbf{z}}\,(\hat{\mathbf{z}}\cdot\hat{\mathbf{r}})$$

and so on.

Table 4.4.1: Dot products between basis vectors in the spherical and Cartesian coordinate systems.

·	$\hat{\mathbf{r}}$	$\hat{\theta}$	$\hat{\phi}$
$\hat{\mathbf{x}}$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
$\hat{\mathbf{y}}$	$\sin\theta\sin\phi$	$\cos\theta\sin\phi$	$\cos\phi$
$\hat{\mathbf{z}}$	$\cos\theta$	$-\sin\theta$	0

Example 4.4.1: Cartesian to spherical conversion

A vector field $\mathbf{G} = \hat{\mathbf{x}}\,xz/y$. Develop an expression for $\mathbf{G}$ in spherical coordinates.

Solution

Simply substitute expressions in terms of spherical coordinates for expressions in terms of Cartesian coordinates. Use Table 4.4.1 and Equations 4.4.1- 4.4.3. Making these substitutions and applying a bit of mathematical clean-up afterward, one obtains

$$\mathbf{G} = \left(\hat{\mathbf{r}}\sin\theta\cot\phi + \hat{\phi}\cos\theta\cot\phi - \hat{\phi}\right)\cdot r\cos\theta\cos\phi$$

Integration Over Length

A differential-length segment of a curve in the spherical system is

$$d\mathbf{l} = \hat{\mathbf{r}}\,dr + \hat{\theta}\,r\,d\theta + \hat{\phi}\,r\sin\theta\,d\phi$$

Note that θ is an angle, as opposed to a distance. The associated distance is $r\,d\theta$ in the θ direction. Note also that in the ϕ direction, distance is $r\,d\phi$ in the $z = 0$ plane and less by the factor $\sin\theta$ for $z <> 0$.

As always, the integral of a vector field $\mathbf{A}(\mathbf{r})$ over a curve $\mathcal{C}$ is

$$\int_{\mathcal{C}} \mathbf{A}\cdot d\mathbf{l}$$

To demonstrate line integration in the spherical system, imagine a sphere of radius a centered at the origin with "poles" at $z = +a$ and $z = -a$. Let us calculate the integral of $\mathbf{A}(\mathbf{r}) = \hat{\theta}$, where $\mathcal{C}$ is the arc drawn directly from pole to pole along the surface of the

sphere, as shown in Figure 4.4.3. In this example, $d\mathbf{l} = \hat{\theta}\ a\ d\theta$ since $r = a$ and ϕ (which could be any value) are both constant along $\mathcal{C}$. Subsequently, $\mathbf{A} \cdot d\mathbf{l} = a\ d\theta$ and the above integral is

$$\int_0^{\pi} a\ d\theta = \pi a$$

i.e., half the circumference of the sphere, as expected.

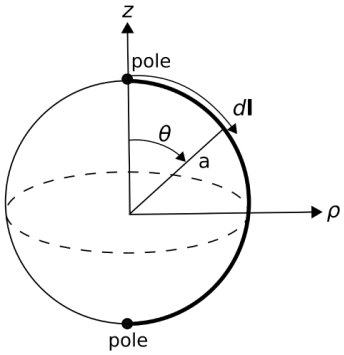

Figure 4.4.3: Example in spherical coordinates: Poleto-pole distance on a sphere. (CC BY SA 4.0; K. Kikkeri).

Note that the spherical system is an appropriate choice for this example because the problem can be expressed with the minimum number of varying coordinates in the spherical system. If we had attempted this problem in the Cartesian system, we would find that both z and either x or y (or all three) vary over $\mathcal{C}$ and in a relatively complex way.

Integration Over Area

Now we ask the question, what is the integral of some vector field $\mathbf{A}$ over the surface $\mathcal{S}$ of a sphere of radius a centered on the origin? This is shown in Figure 4.4.4. The differential surface vector in this case is

$$d\mathbf{s} = \hat{\mathbf{r}}\ \ (r\ d\theta)\,(r \sin\theta\ d\phi) = \hat{\mathbf{r}}\ r^2\ \sin\theta\ d\theta\ d\phi$$

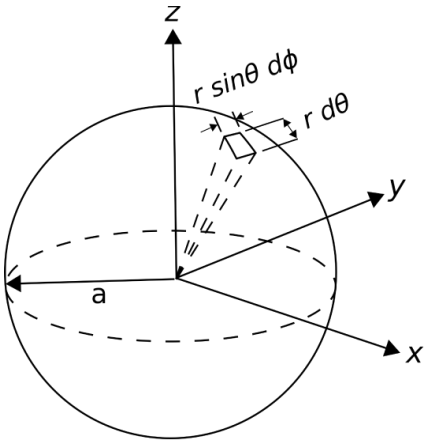

Figure 4.4.4: Example in spherical coordinates: The area of a sphere. (CC BY SA 4.0; K. Kikkeri).

As always, the direction is normal to the surface and in the direction associated with positive flux. The quantities in parentheses are the distances associated with varying θ and ϕ, respectively. In general, the integral over a surface is

$$\int_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{s}$$

In this case, let's consider $\mathbf{A} = \hat{\mathbf{r}}$; in this case $\mathbf{A} \cdot d\mathbf{s} = a^2\ \sin\theta\ d\theta\ d\phi$ and the integral becomes

$$\int_0^{\pi}\int_0^{2\pi} a^2\,\sin\theta\,d\theta\,d\phi = a^2\left(\int_0^{\pi}\sin\theta d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$
$$= a^2\,(2)\,(2\pi)$$
$$= 4\pi a^2$$

which we recognize as the area of the sphere, as expected. The corresponding calculation in the Cartesian or cylindrical systems is quite difficult in comparison.

Integration Over Volume

The differential volume element in the spherical system is

$$dv \;=\; dr\,(rd\theta)\,(r\sin\theta d\phi) \;=\; r^2 dr\;\sin\theta\,d\theta\,d\phi$$

For example, if $A(\mathbf{r}) = 1$ and the volume $\mathcal{V}$ is a sphere of radius a centered on the origin, then

$$\int_{\mathcal{V}} A(\mathbf{r})\,dv = \int_0^a\int_0^{\pi}\int_0^{2\pi} r^2 dr\;\sin\theta\,d\theta\,d\phi$$
$$= \left(\int_0^a r^2 dr\right)\left(\int_0^{\pi}\sin\theta\,d\theta\right)\left(\int_0^{2\pi} d\phi\right)$$
$$= \left(\frac{1}{3}a^3\right)(2)\,(2\pi)$$
$$= \frac{4}{3}\pi a^3$$

which is the volume of a sphere.

4.5: Gradient

The gradient operator is an important and useful tool in electromagnetic theory. Here's the main idea:

> The *gradient* of a scalar field is a vector that points in the direction in which the field is most rapidly increasing, with the scalar part equal to the rate of change.

A particularly important application of the gradient is that it relates the electric field intensity $\mathbf{E}(\mathbf{r})$ to the electric potential field $V(\mathbf{r})$. This is apparent from a review of Section 2.2; see in particular, the battery-charged capacitor example. In that example, it is demonstrated that:

- The *direction* of $\mathbf{E}(\mathbf{r})$ is the direction in which $V(\mathbf{r})$ decreases most quickly, and
- The *scalar part* of $\mathbf{E}(\mathbf{r})$ is the rate of change of $V(\mathbf{r})$ in that direction. Note that this is also implied by the units, since $V(\mathbf{r})$ has units of V whereas $\mathbf{E}(\mathbf{r})$ has units of V/m.

The gradient is the mathematical operation that relates the vector field $\mathbf{E}(\mathbf{r})$ to the scalar field $V(\mathbf{r})$ and is indicated by the symbol "∇" as follows:

$$\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$$

or, with the understanding that we are interested in the gradient as a function of position $\mathbf{r}$, simply

$$\mathbf{E} = -\nabla V$$

At this point we should note that the gradient is a very general concept, and that we have merely identified one application of the gradient above. In electromagnetics there are many situations in which we seek the gradient ∇f of some scalar field $f(\mathbf{r})$. Furthermore, we find that other differential operators that are important in electromagnetics can be interpreted in terms of the gradient operator ∇. These include *divergence* (Section 4.6), *curl* (Section 4.8), and the *Laplacian* (Section 4.10).

In the Cartesian system:

$$\nabla f = \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \tag{4.5.1}$$

Example 4.5.1: Gradient of a ramp function.

Find the gradient of $f = ax$ (a "ramp" having slope a along the x direction).

Solution

Here, $\partial f/\partial x = a$ and $\partial f/\partial y = \partial f/\partial z = 0$. Therefore $\nabla f = \hat{\mathbf{x}}a$. Note that ∇f points in the direction in which f most rapidly increases, and has magnitude equal to the slope of f in that direction.

The gradient operator in the cylindrical and spherical systems is given in Appendix B2.

4.6: Divergence

In this section, we present the divergence operator, which provides a way to calculate the flux associated with a point in space. First, let us review the concept of flux.

The integral of a vector field over a surface is a scalar quantity known as *flux*. Specifically, the flux F of a vector field $\mathbf{A}(\mathbf{r})$ over a surface $\mathcal{S}$ is

$$\int_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{s} = F$$

Note that $\mathbf{A}$ could be fairly described as a *flux density*; i.e., a quantity having units equal to the units of F, but divided by area (i.e., m^2). Also worth noting is that the flux of a vector field that has unit magnitude and is normal to all points on $\mathcal{S}$ is simply the area of $\mathcal{S}$.

It is quite useful to identify some electromagnetic quantities as either fluxes or flux densities. Here are two important examples:

- The electric flux density $\mathbf{D}$, having units of C/m^2, is a description of the electric field as a flux density. (See Section 2.4 for more about electric flux density.) The integral of $\mathbf{D}$ over a closed surface yields the enclosed charge Q_{encl}, having units of C. This relationship is known as Gauss' Law:

$$\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s} = Q_{encl} \tag{4.6.1}$$

  (See Section 5.5 for more about Gauss' Law.)
- The magnetic flux density $\mathbf{B}$, having units of Wb/m^2, is a description of the magnetic field as a flux density. (See Section 2.5 for more about magnetic flux density.) The integral of $\mathbf{B}$ over a surface (open *or* closed) yields the magnetic flux Φ, having units of Wb:

$$\int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = \Phi$$

  This is important because, for example, the time rate of change of Φ is proportional to electric potential. (See Section 8.3 for more about this principle, called *Faraday's Law*.)

Summarizing

> Flux is the scalar quantity obtained by integrating a vector field, interpreted in this case as a flux density, over a specified surface.

The concept of flux applies to a surface of finite size. However, what is frequently of interest is behavior at a single point, as opposed to the sum or average over a region of space. For example, returning to the idea of electric flux density ($\mathbf{D}$), perhaps we are not concerned about the total charge (units of C) enclosed by a surface, but rather the charge density (C/m^3) at a point. In this case, we could begin with Equation 4.6.1 and divide both sides of the equation by the volume V enclosed by $\mathcal{S}$:

$$\frac{\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s}}{V} = \frac{Q_{encl}}{V}$$

Now we let V shrink to zero, giving us an expression that must be true at whatever point we decide to converge upon. Taking the limit as $V \to 0$:

$$\lim_{V \to 0} \frac{\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{s}}{V} = \lim_{V \to 0} \frac{Q_{encl}}{V}$$

The quantity on the right hand side is by definition the *volume charge density* ρ_v (units of C/m^3) at the point at which we converge. The left hand side is the *divergence* of $\mathbf{D}$, sometimes abbreviated "div $\mathbf{D}$." Thus, the above equation can be written

$$\text{div}\, \mathbf{D} = \rho_v$$

Summarizing

> Divergence is the flux per unit volume through an infinitesimally-small closed surface surrounding a point.

We will typically not actually want to integrate and take a limit in order to calculate divergence. Fortunately, we do not have to. It turns out that this operation can be expressed as the dot product $\nabla \cdot \mathbf{D}$; where, for example,

$$\nabla \triangleq \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

in the Cartesian coordinate system. This is the *same* "∇" that appears in the definition of the gradient operator (Section 4.5) and is same operator that often arises when considering other differential operators. If we expand $\mathbf{D}$ in terms of its Cartesian components:

$$\mathbf{D} = \hat{\mathbf{x}}D_x + \hat{\mathbf{y}}D_y + \hat{\mathbf{z}}D_z$$

Then

$$\text{div}\,\mathbf{D} = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

This seems to make sense for two reasons. First, it is dimensionally correct. Taking the derivative of a quantity having units of C/m ^{2} with respect to distance yields a quantity having units of C/m^3. Second, it makes sense that flux from a point should be related to the sum of the rates of change of the flux density in each basis direction. Summarizing:

> The divergence of a vector field $\mathbf{A}$ is $\nabla \cdot \mathbf{A}$.

Example 4.6.1: Divergence of a uniform field

A field $\mathbf{A}$ that is constant with respect to position is said to be *uniform*. A completely general description of such a field is $\mathbf{A} = \hat{\mathbf{x}}A_x + \hat{\mathbf{y}}A_y + \hat{\mathbf{z}}A_z$ where A_x, A_y, and A_z are all constants. We see immediately that the divergence of such a field must be zero. That is, $\nabla \cdot \mathbf{A} = 0$ because each component of $\mathbf{A}$ is constant with respect to position. This also makes sense from the perspective of the "flux through an infinitesimally-small closed surface" interpretation of divergence. If the flux is uniform, the flux into the surface equals the flux out of the surface resulting in a net flux of zero.

Example 4.6.2: Divergence of a linearly-increasing field

Consider a field $\mathbf{A} = \hat{\mathbf{x}}A_0 x$ where A_0 is a constant. The divergence of $\mathbf{A}$ is $\nabla \cdot \mathbf{A} = A_0$. If we interpret $\mathbf{A}$ as a flux density, then we have found that the net flux per unit volume is simply the rate at which the flux density is increasing with distance.

To compute divergence in other coordinate systems, we merely need to know ∇ for those systems. In the cylindrical system:

$$\nabla = \hat{\rho}\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial \phi}\rho + \hat{\mathbf{z}}\frac{\partial}{\partial z}$$

and in the spherical system:

$$\nabla = \hat{\mathbf{r}}\frac{1}{r^2}\frac{\partial}{\partial r}r^2 + \hat{\theta}\frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\sin\theta + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \tag{4.6.2}$$

Alternatively, one may use the explicit expressions for divergence given in Appendix B2.

Example 4.6.3: Divergence of a radially-decreasing field

Consider a vector field that is directed radially outward from a point and which decreases linearly with distance; i.e., $\mathbf{A} = \hat{\mathbf{r}}A_0/r$ where A_0 is a constant. In this case, the divergence is most easily computed in the spherical coordinate system since partial derivatives in all but one direction (r) equal zero. Neglecting terms that include these zero-valued partial derivatives, we find:

$$\nabla \cdot \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\left[\frac{A_0}{r}\right]\right) = \frac{A_0}{r^2}$$

In other words, if we interpret $\mathbf{A}$ as a flux density, then the flux per unit volume is decreasing with as the square of distance from the origin.

It is useful to know that divergence, like ∇ itself, is a linear operator; that is, for any constant scalars a and b and vector fields $\mathbf{A}$ and $\mathbf{B}$:

$$\nabla \cdot (a\mathbf{A} + b\mathbf{B}) = a\nabla \cdot \mathbf{A} + b\nabla \cdot \mathbf{B}$$

4.7: Divergence Theorem

The *Divergence Theorem* relates an integral over a volume to an integral over the surface bounding that volume. This is useful in a number of situations that arise in electromagnetic analysis. In this section, we derive this theorem.

Consider a vector field $\mathbf{A}$ representing a flux density, such as the electric flux density $\mathbf{D}$ or magnetic flux density $\mathbf{B}$. The divergence of $\mathbf{A}$ is

$$\nabla \cdot \mathbf{A} = f \tag{4.7.1}$$

where is $f(\mathbf{r})$ is the flux per unit volume through an infinitesimally-small closed surface surrounding the point at $\mathbf{r}$. Since f is flux per unit volume, we can obtain flux for any larger contiguous volume $\mathcal{V}$ by integrating over $\mathcal{V}$; i.e.,

$$\text{flux through } \mathcal{V} = \int_{\mathcal{V}} f \, dv$$

In the Cartesian system, $\mathcal{V}$ can be interpreted as a three-dimensional grid of infinitesimally-small cubes having side lengths dx, dy, and dz, respectively. Note that the flux *out of* any face of one of these cubes is equal to the flux *into* the cube that is adjacent through that face. That is, the portion of the total flux that flows between cubes cancels when added together. In fact, the only fluxes which do not cancel in the integration over $\mathcal{V}$ are those corresponding to faces which lie on the bounding surface $\mathcal{S}$, since the integration stops there. Stating this mathematically:

$$\int_{\mathcal{V}} f \, dv = \oint_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{s} \tag{4.7.2}$$

Thus, we have converted a volume integral into a surface integral.

To obtain the Divergence Theorem, we return to Equation 4.7.1. Integrating both sides of that equation over $\mathcal{V}$, we obtain

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{A}) \, dv = \int_{\mathcal{V}} f \, dv$$

Now applying Equation 4.7.2 to the right hand side:

$$\boxed{\int_{\mathcal{V}} (\nabla \cdot \mathbf{A}) \, dv = \oint_{\mathcal{S}} \mathbf{A} \cdot d\mathbf{s}} \tag{4.7.3}$$

> The Divergence Theorem (Equation 4.7.3) states that the integral of the divergence of a vector field over a volume is equal to the flux of that field through the surface bounding that volume.

The principal utility of the Divergence Theorem is to convert problems that are defined in terms of quantities known throughout a volume into problems that are defined in terms of quantities known over the bounding surface and vice-versa.

4.8: Curl

Curl is an operation, which when applied to a vector field, quantifies the *circulation* of that field. The concept of circulation has several applications in electromagnetics. Two of these applications correspond to directly to Maxwell's Equations:

- The circulation of an electric field is proportional to the rate of change of the magnetic field. This is a statement of the *Maxwell-Faraday Equation* (Section 8.8), which includes as a special case *Kirchoff's Voltage Law for electrostatics* (Section 5.11).
- The circulation of a magnetic field is proportional to the source current and the rate of change of the electric field. This is a statement of *Ampere's Law* (Sections 7.9 and 8.9)

Thus, we are motivated to formally define circulation and then to figure out how to most conveniently apply the concept in mathematical analysis. The result is the *curl* operator. So, we begin with the concept of circulation:

> "Circulation" is the integral of a vector field over a closed path.

Specifically, the circulation of the vector field $\mathbf{A}(\mathbf{r})$ over the closed path $\mathcal{C}$ is

$$\oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}$$

The circulation of a uniform vector field is zero for any valid path. For example, the circulation of $\mathbf{A} = \hat{\mathbf{x}} A_0$ is zero because non-zero contributions at each point on $\mathcal{C}$ cancel out when summed over the closed path. On the other hand, the circulation of $\mathbf{A} = \hat{\phi} A_0$ over a circular path of constant ρ and z is a non-zero constant, since the non-zero contributions to the integral at each point on the curve are equal and accumulate when summed over the path.

Example 4.8.1: Circulation of the magnetic field intensity surrounding a line current

Consider a current I (units of A) flowing along the z axis in the $+z$ direction, as shown in Figure 4.8.1.

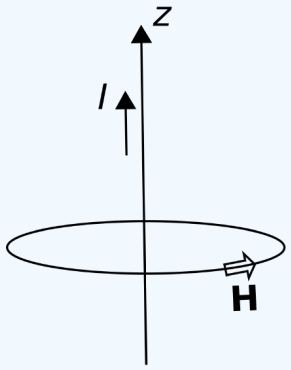

Figure 4.8.1: Magnetic field intensity due to a current flowing along the z axis. (CC BY SA 4.0; K. Klikkeri).

It is known that this current gives rise to a magnetic field intensity $\mathbf{H} = \hat{\phi} H_0 / \rho$, where H_0 is a constant having units of A since the units of $\mathbf{H}$ are A/m. (Feel free to consult Section 7.5 for the details; however, no additional information is needed to follow the example being presented here.) The circulation of $\mathbf{H}$ along any circular path of radius a in a plane of constant z is therefore

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = \int_{\phi=0}^{2\pi} \left(\hat{\phi} \frac{H_0}{a} \right) \cdot \left(\hat{\phi}\, a\, d\phi \right) = 2\pi H_0$$

Note that the circulation of $\mathbf{H}$ in this case has two remarkable features: (1) It is independent of the radius of the path of integration; and (2) it has units of A, which suggests a current. In fact, it turns out that the circulation of $\mathbf{H}$ in this case is *equal* to the enclosed source current I. Furthermore, it turns out that the circulation of $\mathbf{H}$ along *any* path enclosing the source current is equal to the source current! These findings are consequences of *Ampere's Law*, as noted above.

Curl is, in part, an answer to the question of what the circulation *at a point in space* is. In other words, what is the circulation as $\mathcal{C}$ shrinks to it's smallest possible size. The answer in one sense is zero, since the arclength of $\mathcal{C}$ is zero in this limit – there is nothing

to integrate over. However, if we ask instead what is the circulation *per unit area* in the limit, then the result should be the non-trivial value of interest. To express this mathematically, we constrain $\mathcal{C}$ to lie in a plane, and define $\mathcal{S}$ to be the open surface bounded by $\mathcal{C}$ in this case. Then, we define the scalar part of the curl of $\mathbf{A}$ to be:

$$\lim_{\Delta s \to 0} \frac{\oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}}{\Delta s}$$

where Δs is the area of $\mathcal{S}$, and (important!) we require $\mathcal{C}$ and $\mathcal{S}$ to lie in the plane that maximizes the above result.

Because $\mathcal{S}$ and it's boundary $\mathcal{C}$ lie in a plane, it is possible to assign a direction to the result. The chosen direction is the normal $\hat{\mathbf{n}}$ to the plane in which $\mathcal{C}$ and $\mathcal{S}$ lie. Because there are two normals at each point on a plane, we specify the one that satisfies the *right hand rule*. This rule, applied to the curl, states that the correct normal is the one which points through the plane in the same direction as the fingers of the right hand when the thumb of your right hand is aligned along $\mathcal{C}$ in the direction of integration. Why is this the correct orientation of $\hat{\mathbf{n}}$? See Section 4.9 for the answer to that question. For the purposes of this section, it suffices to consider this to be simply an arbitrary sign convention.

Now with the normal vector $\hat{\mathbf{n}}$ unambiguously defined, we can now formally define the curl operation as follows:

$$\text{curl } \mathbf{A} \triangleq \lim_{\Delta s \to 0} \frac{\hat{\mathbf{n}} \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}}{\Delta s} \tag{4.8.1}$$

where, once again, Δs is the area of $\mathcal{S}$, and we select $\mathcal{S}$ to lie in the plane that maximizes the magnitude of the above result. Summarizing:

> The curl operator quantifies the circulation of a vector field at a point.

> The magnitude of the curl of a vector field is the circulation, per unit area, at a point and such that the closed path of integration shrinks to enclose zero area while being constrained to lie in the plane that maximizes the magnitude of the result.

> The direction of the curl is determined by the right-hand rule applied to the associated path of integration.

Curl is a very important operator in electromagnetic analysis. However, the definition (Equation 4.8.1) is usually quite difficult to apply. Remarkably, however, it turns out that the curl operation can be defined in terms of the ∇ operator; that is, the same ∇ operator associated with the gradient, divergence, and Laplacian operators. Here is that definition:

$$\text{curl } \mathbf{A} \triangleq \nabla \times \mathbf{A}$$

For example: In Cartesian coordinates,

$$\nabla \triangleq \hat{\mathbf{x}} \frac{\partial}{\partial x} + \hat{\mathbf{y}} \frac{\partial}{\partial y} + \hat{\mathbf{z}} \frac{\partial}{\partial z}$$

and

$$\mathbf{A} = \hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z$$

so curl can be calculated as follows:

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

or, evaluating the determinant:

$$\begin{aligned}\nabla \times \mathbf{A} &= \hat{\mathbf{x}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) \\ &+ \hat{\mathbf{y}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) \\ &+ \hat{\mathbf{z}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\end{aligned}$$

Expressions for curl in each of the three major coordinate systems is provided in Appendix B2.

It is useful to know is that curl, like ∇ itself, is a linear operator; that is, for any constant scalars a and b and vector fields $\mathbf{A}$ and $\mathbf{B}$:

$$\nabla \times (a\mathbf{A} + b\mathbf{B}) = a\nabla \times \mathbf{A} + b\nabla \times \mathbf{B}$$

4.9: Stokes' Theorem

Stokes' Theorem relates an integral over an open surface to an integral over the curve bounding that surface. This relationship has a number of applications in electromagnetic theory. Here is the theorem:

$$\boxed{\int_{\mathcal{S}} (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_{\mathcal{C}} \mathbf{A} \cdot d\mathbf{l}}$$

where $\mathcal{S}$ is the open surface bounded by the closed path $\mathcal{C}$. The direction of the surface normal $d\mathbf{s} = \hat{\mathbf{n}} ds$ is related to the direction of integration along $\mathcal{C}$ by the *right-hand rule*, illustrated in Figure 4.9.1. In this case, the right-hand rule states that the correct normal is the one that points through the surface in the same direction as the fingers of the right hand when the thumb of your right hand is aligned along $\mathcal{C}$ in the direction of integration.

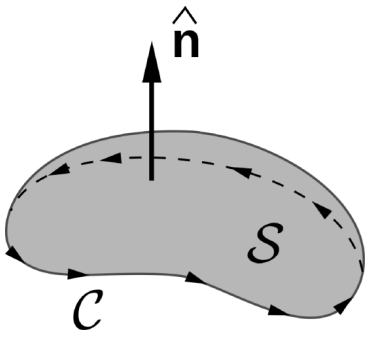

Figure 4.9.1: The relative orientations of the direction of integration $\mathcal{C}$ and surface normal $\vec{\mathbf{n}}$ in Stokes' Theorem. (CC BY SA 3.0; Cronholm144).

Stokes' Theorem is a purely mathematical result and not a principle of electromagnetics *per se*. The relevance of the theorem to electromagnetic theory is primarily as a tool in the associated mathematical analysis. Usually the theorem is employed to transform a problem expressed in terms of an integration over a surface into an integration over a closed path or vice-versa. For more information on the theorem and its derivation, see "Additional Reading" at the end of this section.

4.10: The Laplacian Operator

The Laplacian $\nabla^2 f$ of a field $f(\mathbf{r})$ is the divergence of the gradient of that field:

$$\nabla^2 f \triangleq \nabla \cdot (\nabla f) \tag{4.10.1}$$

Note that the Laplacian is essentially a definition of the second derivative with respect to the three spatial dimensions. For example, in Cartesian coordinates,

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \tag{4.10.2}$$

as can be readily verified by applying the definitions of gradient and divergence in Cartesian coordinates to Equation 4.10.1.

The Laplacian relates the electric potential (i.e., V, units of V) to electric charge density (i.e., ρ_v, units of C/m^3). This relationship is known as *Poisson's Equation*:

$$\nabla^2 V = -\frac{\rho_v}{\epsilon}$$

where ϵ is the permittivity of the medium. The fact that V is related to ρ_v in this way should not be surprising, since electric field intensity ($\mathbf{E}$, units of V/m) is proportional to the derivative of V with respect to distance (via the gradient) and ρ_v is proportional to the derivative of $\mathbf{E}$ with respect to distance (via the divergence).

The Laplacian operator can also be applied to *vector* fields; for example, Equation 4.10.2 is valid even if the scalar field "f" is replaced with a vector field. In the Cartesian coordinate system, the Laplacian of the vector field $\mathbf{A} = \hat{\mathbf{x}} A_x + \hat{\mathbf{y}} A_y + \hat{\mathbf{z}} A_z$ is

$$\nabla^2 \mathbf{A} = \hat{\mathbf{x}} \nabla^2 A_x + \hat{\mathbf{y}} \nabla^2 A_y + \hat{\mathbf{z}} \nabla^2 A_z$$

An important application of the Laplacian operator of vector fields is the *wave equation*; e.g., the wave equation for $\mathbf{E}$ in a lossless and source-free region is

$$\nabla^2 \mathbf{E} + \beta^2 \mathbf{E} = 0$$

where β is the **phase propagation constant**.

It is sometimes useful to know that the Laplacian of a vector field can be expressed in terms of the gradient, divergence, and curl as follows:

$$\nabla^2 \mathbf{A} = \nabla (\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A})$$

The Laplacian operator in the cylindrical and spherical coordinate systems is given in Appendix B2.