Tutorial (02)
Solutions

(01)

(a)
$$3x + 7y - 4z = -46$$
$$5w + 4x + 8y + z = 7$$
$$8w + 4y - 2z = 0$$
$$-w + 6x + 2z = 13$$

$\therefore$ Augmented Matrix :

$$\left(\begin{array}{cccc|c} 0 & 3 & 7 & -4 & -46 \\ 5 & 4 & 8 & 1 & 7 \\ 8 & 0 & 4 & -2 & 0 \\ -1 & 6 & 0 & 2 & 13 \end{array}\right)$$

$$\Big\downarrow\ R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cccc|c} 5 & 4 & 8 & 1 & 7 \\ 0 & 3 & 7 & -4 & -46 \\ 8 & 0 & 4 & -2 & 0 \\ -1 & 6 & 0 & 2 & 13 \end{array}\right)$$

$$\Big\downarrow\ R_1/5 \rightarrow R_1$$

$$\left(\begin{array}{cccc|c} 1 & 0.8 & 1.6 & 0.2 & 1.4 \\ 0 & 3 & 7 & -4 & -46 \\ 8 & 0 & 4 & -2 & 0 \\ -1 & 6 & 0 & 2 & 13 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_3 - 8R_1 \longrightarrow R_3 \\ R_4 + R_1 \longrightarrow R_4 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 0.8 & 1.6 & 0.2 & 1.4 \\ 0 & 3 & 7 & -4 & -46 \\ 0 & -6.4 & -8.8 & -3.6 & -11.2 \\ 0 & 6.8 & 1.6 & 2.2 & 14.4 \end{array}\right)$$

$$\Big\downarrow R_2/3 \longrightarrow R_2$$

$$\left(\begin{array}{cccc|c} 1 & 0.8 & 1.6 & 0.2 & 1.4 \\ 0 & 1 & \frac{7}{3} & -\frac{4}{3} & -\frac{46}{3} \\ 0 & -6.4 & -8.8 & -3.6 & -11.2 \\ 0 & 6.8 & 1.6 & 2.2 & 14.4 \end{array}\right)$$

$$\left\downarrow \begin{array}{l} R_1 - 0.8R_2 \to R_1 \\ R_3 + 6.4R_2 \to R_3 \\ R_4 - 6.8R_2 \to R_4 \end{array}\right.$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{4}{15} & \frac{19}{15} & \frac{41}{3} \\ 0 & 1 & \frac{7}{3} & -\frac{4}{3} & -\frac{46}{3} \\ 0 & 0 & \frac{92}{15} & -\frac{182}{15} & -\frac{328}{3} \\ 0 & 0 & -\frac{214}{15} & \frac{169}{15} & \frac{356}{3} \end{array}\right)$$

$$\Big\downarrow\ R_3 / \frac{92}{15} \to R_3$$

$$\left(\begin{array}{cccc|c} 1 & 0 & -\frac{4}{15} & \frac{19}{15} & \frac{41}{3} \\ 0 & 1 & \frac{7}{3} & -\frac{4}{3} & -\frac{46}{3} \\ 0 & 0 & 1 & -\frac{91}{46} & -\frac{410}{23} \\ 0 & 0 & -\frac{214}{15} & \frac{169}{15} & \frac{356}{3} \end{array}\right)$$

$$\left\downarrow \begin{array}{l} R_1 + \frac{4}{15}R_3 \to R_1 \\ R_2 - \frac{7}{3}R_3 \to R_2 \\ R_4 + \frac{214}{15}R_3 \to R_4 \end{array}\right.$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{17}{23} & \frac{205}{23} \\ 0 & 1 & 0 & \frac{151}{46} & \frac{604}{23} \\ 0 & 0 & 1 & -\frac{91}{46} & -\frac{410}{23} \\ 0 & 0 & 0 & \frac{-390}{23} & -\frac{3120}{23} \end{array}\right)$$

$$\Big\downarrow \; R_4 / -\frac{390}{23} \longrightarrow R_4$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & \frac{17}{23} & \frac{205}{23} \\ 0 & 1 & 0 & \frac{151}{46} & \frac{604}{23} \\ 0 & 0 & 1 & -\frac{91}{46} & -\frac{410}{23} \\ 0 & 0 & 0 & 1 & 8 \end{array}\right)$$

$$R_1 - \frac{17}{23} R_4 \longrightarrow R_1$$
$$R_2 - \frac{151}{46} R_4 \longrightarrow R_2$$
$$R_3 + \frac{91}{46} R_4 \longrightarrow R_3$$

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 8 \end{array}\right)$$

$$\therefore \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ -2 \\ 8 \end{pmatrix}.$$

(02)

(a) Let x, y, z represent the number of small, medium, and large T-shirts, respectively.

$$x + y + z = 30$$

$$4x + 5y + 6z = 154$$

$$4x - 6z = -40$$

∴ the corresponding Augmented Matrix :

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 4 & 5 & 6 & 154 \\ 4 & 0 & -6 & -40 \end{array}\right)$$

then using Gauss - Jordan elimination, we can obtain the following row reduced-echelon form.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 12 \end{array}\right)$$

∴ $x = 8$; $y = 10$; $z = 12$

(02) (b) (ii)

$$x - y + 2z = 4$$
$$-x + 3y + z = -6$$
$$x + y + 5z = 3$$

Augmented Matrix :

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ -1 & 3 & 1 & -6 \\ 1 & 1 & 5 & 3 \end{array}\right)$$

$$\Big\downarrow \; R_1 + R_2 \longrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 2 & 3 & -2 \\ 1 & 1 & 5 & 3 \end{array}\right)$$

$$\Big\downarrow \; -R_1 + R_3 \longrightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 2 & 3 & -2 \\ 0 & 2 & 3 & -1 \end{array}\right) \xrightarrow{-R_2 + R_3 \longrightarrow R_3}$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

$\therefore$ the last row represents

$$0 . x + 0 . y + 0 . z = 1$$

$$\therefore \quad 0 = 1$$

Contradiction

$\therefore$ the system is inconsistent and has no solution.

(02)

(c)

(i) $x + 2y + 3z = 10$

$x + y + z = 7$

$3x + 2y + z = 18$

$\therefore$ the augmented matrix :

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 1 & 1 & 1 & 7 \\ 3 & 2 & 1 & 18 \end{array}\right)$$

$$\Big\downarrow -R_1 + R_2 \rightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -1 & -2 & -3 \\ 3 & 2 & 1 & 18 \end{array}\right)$$

$$\Big\downarrow -3R_1 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & -1 & -2 & -3 \\ 0 & -4 & -8 & -12 \end{array}\right) \xrightarrow{-R_2 \rightarrow R_2}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 10 \\ 0 & 1 & 2 & 3 \\ 0 & -4 & -8 & -12 \end{array}\right)$$

$$\Big\downarrow (-2)R_2 + R_1 \rightarrow R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & -4 & -8 & -12 \end{array}\right)$$

$$\Big\downarrow 4R_2 + R_3 \rightarrow R_3$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ the last row corresponds to the identity $0 = 0$ which is true for every value of $x, y,$ and z.

from the first two equations,

$$x - z = 4$$

$$y + 2z = 3$$

$\therefore \quad x = z+4$

$\quad\quad y = 3-2z$

$\therefore$ the solutions of the system

$$(x, y, z) = (z+4,\ 3-2z,\ z)$$

$\therefore$ the solution set :

$$\{(z+4,\ 3-2z,\ z) \mid z \in \mathbb{R}\}.$$

(c) (ii)

$$x + y + z = 1$$
$$2x - 2y + 6z = 10$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -2 & 6 & 10 \end{array}\right)$$

$$\Big\downarrow -2R_1 + R_2 \longrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -4 & 4 & 8 \end{array}\right)$$

$$\Big\downarrow \left(-\tfrac{1}{4}\right) R_2 \longrightarrow R_2$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & -2 \end{array}\right)$$

$$\Big\downarrow (-1)R_2 + R_1 \longrightarrow R_1$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \end{array}\right)$$

$x + 2z = 3 \qquad \therefore \quad x = 3 - 2z$

$y - z = -2 \qquad \qquad y = z - 2$

$(x, y, z) = (3 - 2z, z - 2, z).$

Homogeneous System of Linear Equations

a homogeneous system of linear equations is as follows:

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = 0$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = 0$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = 0$$

where a_{ij} ; $i = 1, 2, \dots m$; $j = 1, 2, \dots, n$ are constants.

$\therefore$ the augmented matrix

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right)$$

$$(A \mid b)$$

The above system is always satisfied by $x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0$

This solution is called the trivial solution.

*∴ Any system of homogeneous linear equations is always consistent.

Case I :-

if $|A| \neq 0$ then the homogeneous system of linear equations has a unique solution and hence the only solution to the system is the trivial solution.

Case II :-

If $|A| = 0$ then the homogeneous system has a non-trivial solution.

(03)

(a)

(i)

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 3 & 4 & 4 & 0 \\ 7 & 10 & 12 & 0 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - 3R_1 \\ R_3 \to R_3 - 7R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -2 & -5 & 0 \\ 0 & -4 & -9 & 0 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 /(-1) \\ R_3 \to R_3 /(-1) \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 4 & 9 & 0 \end{array}\right)$$

$$\Big\downarrow R_3 \to R_3 - 2R_2$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right) \xrightarrow{R_3 \to R_3 /(-1)}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 2 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right)$$

$\therefore x = y = z = 0$

$\therefore$ the system has a unique solution.

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 4 \\ 7 & 10 & 12 \end{vmatrix}$$

$$= -2 \neq 0$$

$\therefore$ the system has only the trivial solution.

(04)

(ii) $x + 3y - 2z = 0$

$2x - y + 4z = 0$

$x - 11y + 14z = 0$

$$\left(\begin{array}{ccc|c} 1 & +3 & -2 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & -11 & 14 & 0 \end{array}\right)$$

$$\Big\downarrow R_2 \to R_2 - 2R_1$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & -7 & 8 & 0 \\ 0 & -14 & 16 & 0 \end{array}\right)$$

$$\Big\downarrow R_2 \to R_2/(-7)$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -2 & 0 \\ 0 & 1 & -\frac{8}{7} & 0 \\ 0 & -14 & 16 & 0 \end{array}\right)$$

$$\Big\downarrow R_1 \to R_1 - 3R_2$$

$$\left(\begin{array}{ccc|c} 1 & 0 & \frac{10}{7} & 0 \\ 0 & 1 & -\frac{8}{7} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ the system has an infinite number of solutions.

$$x + \frac{10}{7}z = 0 \Rightarrow x = -\frac{10z}{7}$$

$$y - \frac{8}{7}z = 0 \Rightarrow y = \frac{8z}{7}$$

$\therefore$ the solution set:

$$\left\{ (x, y, z) \;\middle|\; x = \frac{-10z}{7} ; y = \frac{8z}{7} ; z \in \mathbb{R} \right\}.$$

$$= \left\{ \left(\frac{-10z}{7}, \frac{8z}{7}, z \right) \;\middle|\; z \in \mathbb{R} \right\}$$

(05)

$$kx + y + z = 1$$
$$x + ky + z = 1$$
$$x + y + kz = 1$$

$\therefore \quad AX = b$ where

$$A = \begin{pmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{pmatrix} ; \quad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} ; \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(i) For the system of linear equations to have a unique solution,

$$|A| \neq 0$$

$$\therefore \quad \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix} \neq 0$$

$$k(k^2-1) - 1(k-1) + 1(1-k) \neq 0$$
$$k(k-1)(k+1) - 2(k-1) \neq 0$$
$$(k-1)(k^2+k-2) \neq 0$$
$$(k-1)^2(k+2) \neq 0$$

$\therefore \quad k \neq 1$ or $k \neq -2$

$\therefore$ $|A| = 0$ when $k = 1$ or $k = -2$

$\therefore$ When $k=1$ or $k=-2$ the system of linear equations are inconsistent. or (dependent with) infinitely many Solutions.

(iii)
When $k=1$, the augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$\therefore$ When $k=1$; the system has infinitely many Solutions, and the Solution set:

$$\left\{ (x, y, z) \;\middle|\; x+y+z = 1 \; ; \; x, y, z \in \mathbb{R} \right\}$$

(iii)
When $k=-2$; the augmented matrix:

$$\left(\begin{array}{ccc|c} -2 & 1 & 1 & 1 \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{array}\right)$$

$$\Big\downarrow R_1 \to R_1/(-2)$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & -2 & 1 & 1 \\ 1 & 1 & -2 & 1 \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_2 \longrightarrow R_2 - R_1 \\ R_3 \longrightarrow R_3 - R_1 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \end{array}\right)$$

$$\Big| R_2 \to R_2 /$$

$$\left(\begin{array}{ccc|c} 1 & -\frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -1 & -1 \\ 0 & \frac{3}{2} & -\frac{3}{2} & \frac{3}{2} \end{array}\right)$$

$$\Big\downarrow \begin{array}{l} R_1 \to R_1 + \frac{1}{2} R_2 \\ R_3 \to R_3 - \frac{3}{2} R_2 \end{array}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 3 \end{array}\right)$$

$\therefore$ the last row yields the following

Contradiction: $0 = 3$

but $0 \neq 3$

$\therefore$ the system of linear equations is inconsistent or has no solutions when

$k = -2$