GENERAL SIR JOHN KOTELAWALA DEFENCE UNIVERSITY

Faculty of Engineering
Department of Mathematics

BSc Engineering Degree
Semester 2 Examination - November 2022
(Intake 39 - All Engineering Streams)

MA 1203 - CALCULUS

Time : 3 hours | 25 November, 2022

INSTRUCTIONS TO CANDIDATES

- This paper contains 5 questions on 4 pages.
- Answer **all** questions.
- This is a closed book examination.
- This examination accounts for 70% of the module assessment. A total maximum mark obtainable is 100. The marks assigned for each question and parts thereof are indicated in square brackets.
- If you have any doubt as to the interpretation of the wordings of a question, make your own decision, but clearly state it on the script
- No penalty will be applied for incorrect answers.
- Assume reasonable values for any data not given in or provided with the question paper, clearly make such assumptions made in the script Marks will be lost if all necessary work is not clearly shown.
- All examinations are conducted under the rules and regulations of the KDU.

Question 1

(a) Evaluate the following limits:

(i) $\displaystyle\lim_{x \to 1} \frac{4 - \sqrt{x + 15}}{x^2 - 1}$. [15]

(ii) $\displaystyle\lim_{x \to 0} \frac{\tan x - x}{x^3}$. [25]

(b) Let a be a positive real number and the function $f(x)$ be defined by

$$f(x) = \begin{cases} x^2 - a^2 x & \text{if } x < 4, \\ x - 13a & \text{if } x \geq 4. \end{cases}$$

If $f(x)$ is continuous at $x = 4$, find the value(s) of a. [20]

(c) (i) State the Leibnitz's theorem of differentiation. [10]

(ii) If $y = 2\tan^{-1} x$, show that

$$(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$$

Use Leibnitz's theorem to show that

$$(1 + x^2)\frac{d^{n+2}y}{dx^{n+2}} + 2(n + 1)x\frac{d^{n+1}y}{dx^{n+1}} + n(n + 1)\frac{d^n y}{dx^n} = 0.$$

[30]

[100 Marks]

Question 2

(a) (i) Show that $\displaystyle\int_0^8 \frac{dx}{x^{\frac{4}{3}} + x^{\frac{7}{3}}} = 3\tan^{-1} 2$. [20]

(ii) Find $\displaystyle\frac{d}{dx}\int_1^{x^4} \sec t \; dt$. [20]

(b) Given that $f(x) = \dfrac{2x^2 - 3}{(x - 2)^2(x^2 + 1)}$ can be expressed in the form

$$f(x) = \frac{A}{(x - 2)} + \frac{B}{(x - 2)^2} + \frac{Cx + D}{(x^2 + 1)}, \text{ where } x \neq 2.$$

Find the value of A, B, C, and D.
Hence evaluate the integral

$$\int \frac{10x^2 - 15}{(x - 2)^2(x^2 + 1)}\, dx.$$

[30]

(c) The following Figure 01 shows a curve and straight line with equations $y = 5 + 7x - x^2$ and $y = x - 2$ respectively.

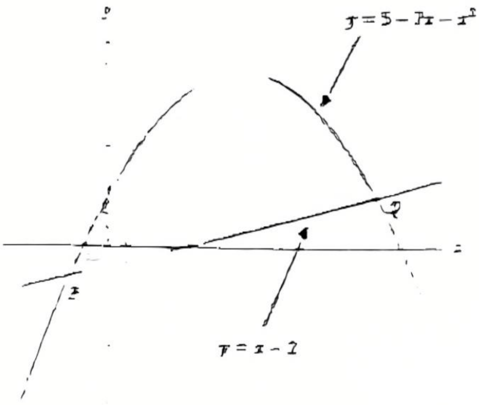

Figure 01

The curve and straight line meet at the points P and Q.

(i) Find the coordinates of the points P and Q. [10]

(ii) Find the exact area enclosed by the curve and the straight line. [20]

[100 Marks]

Question 3

(a) Solve the initial value problem

$$x + 3y^2\sqrt{x^2 - 1}\,\frac{dy}{dx} = 0, \quad y(0) = 1$$

by separation of variables. [20]

(b) Solve the differential equation

$$\frac{1}{x}\frac{dy}{dx} - \frac{2}{x^2}y = x\cos x$$

for $x > 0$ by the method of integrating factor. [20]

(c) The motion of the particle along $x-$ axis describes by the following differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{-3x} + 4\cos(x).$$

a) Show that $y = \frac{1}{25}(e^{-3x} - 4\sin x - 3\cos x)$ is a particular integral of the given differential equation. [30]

b) Find the general solution of the differential equation subject to the initial condition $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$. [30]

[100 Marks]

Question 4

(a) Determine whether the given series is convergent or divergent. Show your justifications and state the name of the convergence test you use.

(i) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ [15]

(ii) $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2+4n+3}$ [10]

(iii) $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{n\ln(n)}$ [10]

(iv) $\displaystyle\sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$ [15]

(b) Let $f(x)$ be the periodic function defined as:

$$f(x) = \begin{cases} 0 & \text{if } -3 < x < 0, \\ 3-x & \text{if } 0 < x < 3. \end{cases}$$

(i) Sketch the graph of $f(x)$ in the interval $-6 < x < 9$. [5]

(ii) Show that the Fourier series for $f(x)$ is

$$\frac{3}{4} + \frac{6}{\pi^2}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos\left(\frac{(2n+1)\pi}{3}x\right) + \frac{3}{\pi}\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{3}x\right).$$

[45]

[100 Marks]

Question 5

(a) Find the Laplace transform of the following functions:

(i) $f(t) = \cos(5t)\sin(3t)$ [15]

(ii) $f(t) = t^4 + \cos^2(2t)\, e^{-2t}$ [15]

(b) Find the inverse Laplace transform of each of the following functions:

(i) $F(s) = \dfrac{s^2+4s-15}{(s-1)(s^2+9)}$ [15]

(ii) $F(s) = \dfrac{4e^{-3s}}{s^2+2s+5}$ [15]

(c) The function $f(t)$ is defined as:

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < 2, \\ \sin(t-2) & \text{if } t < 2. \end{cases}$$

(i) Express the $f(t)$ in the terms of the unit step function, and hence find its Laplace transform. [10]

(ii) Hence solve the following initial value problem:

$$\frac{d^2y}{dt^2} + y = f(t), \ y(0) = 0, \ y'(0) = 0.$$

[30]

[100 Marks]

End of the Question Paper.

Table of Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
$t^n, \ n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
$t^p, \ p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$
$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
$\sin(at) - at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$
$\sin(at) + at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
$\cos(at) - at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$
$\cos(at) + at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
$\sin(at+b)$	$\frac{s\sin(b) + a\cos(b)}{s^2+a^2}$
$\cos(at+b)$	$\frac{s\cos(b) - a\sin(b)}{s^2+a^2}$
$\sinh(at)$	$\frac{a}{s^2-a^2}$
$\cosh(at)$	$\frac{s}{s^2-a^2}$

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
$e^{at}\sin(bt)$	$\dfrac{b}{(s-a)^2+b^2}$
$e^{at}\cos(bt)$	$\dfrac{s-a}{(s-a)^2+b^2}$
$e^{at}\sinh(bt)$	$\dfrac{b}{(s-a)^2-b^2}$
$e^{at}\cosh(bt)$	$\dfrac{s-a}{(s-a)^2-b^2}$
$t^n e^{at},\ n = 1, 2, 3, \ldots$	$\dfrac{n!}{(s-a)^{n+1}}$
$u(t-a)$	$\dfrac{e^{-as}}{s}$
$u(t-a)f(t-a)$	$e^{-as}F(s)$
$e^{at}f(t)$	$F(s-a)$
$\dfrac{1}{t}f(t)$	$\displaystyle\int_s^\infty F(u)\,du$
$\displaystyle\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0)$