

3.0 Alternating Current Theory

The advantage of the alternating waveform for electric power is that it can be stepped up or stepped down in potential easily for transmission and utilisation. Alternating waveforms can be of many shapes. The one that is used with electric power is the sinusoidal waveform. This has an equation of the form

$$v(t) = V_m \sin(\omega t + \phi)$$

The **peak value** of the waveform is the maximum value of the waveform and for the a.c. waveform it is V_m .

$$\text{mean value} = \frac{1}{T} \int_{T_o}^{T_o+T} v(t) \cdot dt$$

The **mean value** of the sinusoidal a.c. waveform is 0 since positive and negative areas cancel. (It can also be shown by integration). If $T_o = \phi/\omega$

$$\text{average value} = \frac{1}{T} \int_{T_o}^{T_o+T} v_{rect}(t) \cdot dt$$

The **average value** of the a.c. waveform is defined as the average value of the rectified waveform and can be shown to be equal to $\frac{2}{\pi} V_m$

$$\text{rms value} = \sqrt{\frac{1}{T} \int_{T_o}^{T_o+T} v^2(t) \cdot dt}$$

The **effective value** or **r.m.s.** value of a waveform is defined as above so that the power in a resistor is given correctly. That is

$$\frac{V^2}{R} \cdot T = \int_{T_o}^{T_o+T} \frac{v^2(t)}{R} \cdot dt, \text{ or } V_{\text{effective}}^2 = \frac{1}{T} \int_{T_o}^{T_o+T} v^2(t) \cdot dt$$

The effective value or rms value of the waveform is thus the *square root of the mean of the squared waveform* and can be shown to be equal to $\frac{1}{\sqrt{2}} V_m$ for the sinusoidal a.c. waveform.

Unless otherwise specified, the rms value is the value that is always specified for ac waveforms, whether it be a voltage or a current. For example, **230 V** in the mains supply is an **rms value** of the voltage. Similarly when we talk about a **5 A**, **13 A** or **15A** socket outlet (plug point), we are again talking about the **rms value** of the rated current of the socket outlet.

For a given waveform, such as the *sinusoid*, the *peak value*, *average value* and the *rms value* are dependant on each other. The **peak factor** and the **form factor** are the two factors that are most commonly defined.

$$\text{Form Factor} = \frac{\text{rms value}}{\text{average value}} \quad \text{and for a sinusoidal waveform, Form Factor} = \frac{V_m}{\sqrt{2} / \frac{2V_m}{\pi}} = \frac{\pi}{2\sqrt{2}} = 1.1107 \approx 1.111$$

The form factor is useful such as when the average value has been measured using a rectifier type moving coil meter and the rms value is required to be found. [Note: You will be studying about these meters later]

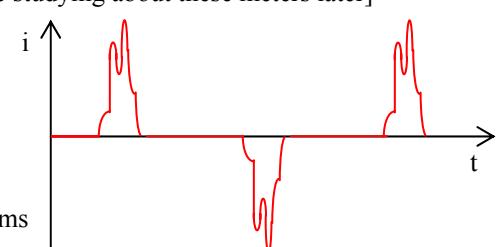
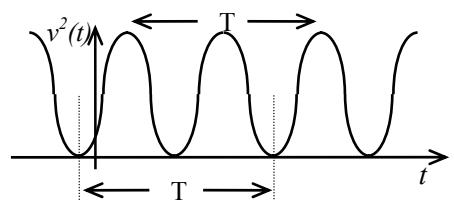
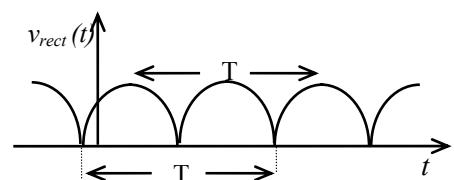
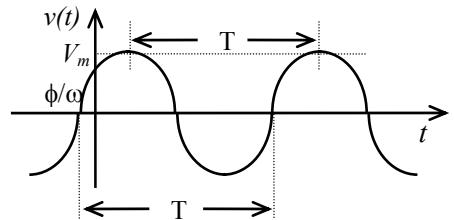
$$\text{Peak Factor} = \frac{\text{peak value}}{\text{rms value}}$$

$$\text{Peak Factor} = \frac{V_m}{\sqrt{2} / \frac{2V_m}{\pi}} = \sqrt{2} = 1.4142$$

The peak factor is useful when defining highly distorted waveforms such as the current waveform of compact fluorescent lamps.

Some advantages of the sinusoidal waveform for electrical power applications

- a. Sinusoidally varying voltages are easily generated by rotating machines
- b. Differentiation or integration of a sinusoidal waveform produces a sinusoidal waveform of the same frequency, differing only in magnitude and phase angle. Thus when a sinusoidal current is passed through (or a sinusoidal voltage applied across) a resistor, inductor or a capacitor a sinusoidal voltage waveform (or current waveform) of the same frequency, differing only in magnitude and phase angle, is obtained.



If $i(t) = I_m \sin(\omega t + \phi)$,

for a resistor, $v(t) = R.i(t) = R.I_m \sin(\omega t + \phi) = V_m \sin(\omega t + \phi)$ – magnitude changed by R but no phase shift

for an inductor,

$$v(t) = L \cdot \frac{di}{dt} = L \cdot \frac{d}{dt}(I_m \sin(\omega t + \phi)) = L \cdot \omega \cdot I_m \cos(\omega t + \phi) = L \cdot \omega \cdot I_m \sin(\omega t + \phi + \pi/2)$$

– magnitude changed by $L\omega$ and phase angle changed by $\pi/2$

for a capacitor,

$$v(t) = \frac{1}{C} \cdot \int i \cdot dt = \frac{1}{C} \int I_m \sin(\omega t + \phi) \cdot dt = \frac{-1}{C \cdot \omega} \cdot I_m \cos(\omega t + \phi) = \frac{1}{C \cdot \omega} \cdot I_m \sin(\omega t + \phi - \pi/2)$$

– magnitude changed by $1/C\omega$ and phase angle changed by $-\pi/2$

c. Sinusoidal waveforms have the property of remaining unaltered in shape when other sinusoids having the same frequency but different in magnitude and phase are added to them.

$$\begin{aligned} A \sin(\omega t + \alpha) + B \sin(\omega t + \beta) &= A \sin \omega t \cos \alpha + A \cos \omega t \sin \alpha + B \sin \omega t \cos \beta + B \cos \omega t \sin \beta \\ &= (A \cos \alpha + B \cos \beta) \sin \omega t + (A \sin \alpha + B \sin \beta) \cos \omega t \\ &= C \sin(\omega t + \phi), \text{ where } C \text{ and } \phi \text{ are constants obtained from trigonometry.} \end{aligned}$$

d. Periodic, but non-sinusoidal waveforms can be broken up to its fundamental and harmonics.

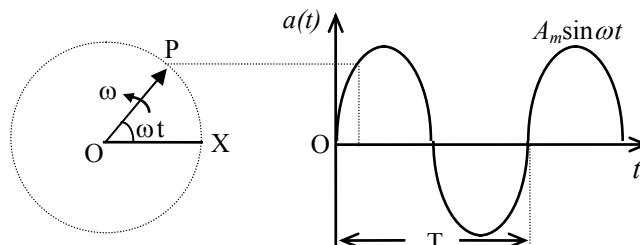
e. Sinusoidal waveforms can be represented by the projections of a rotating phasor.

3.1 Phasor Representation of Sinusoids

You may be aware that $\sin \theta$ can be written in terms of exponentials and complex numbers.

$$\text{i.e. } e^{j\theta} = \cos \theta + j \sin \theta$$

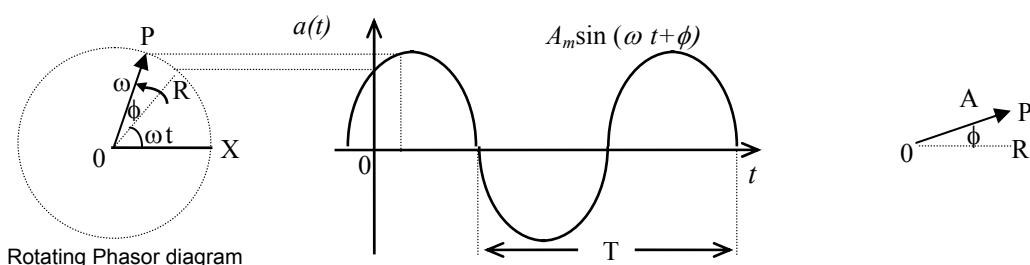
$$\text{or } e^{j\omega t} = \cos \omega t + j \sin \omega t$$



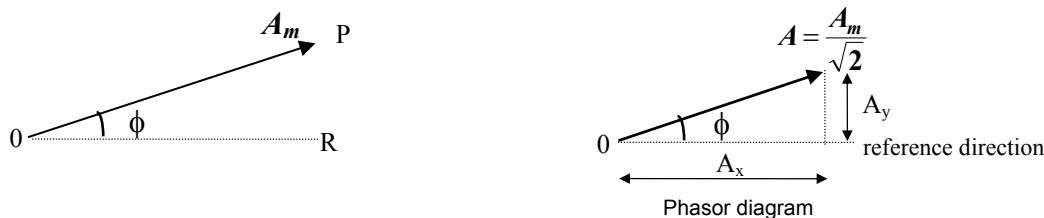
Consider a line OP of length A_m which is in the horizontal direction OX at time $t=0$.

If OP rotates at an angular velocity ω , then in time t its position would correspond to an angle of ωt .

The projection of this rotating phasor OP (a phasor is somewhat similar to a vector, except that it does not have a physical direction in space but a phase angle) on the y-axis would correspond to $OP \sin \omega t$ or $A_m \sin \omega t$ and on the x-axis would correspond to $A_m \cos \omega t$. Thus the sinusoidal waveform can be thought of being the projection on a particular direction of the complex exponential $e^{j\omega t}$.



If we consider more than one phasor, and each phasor rotates at the same angular frequency, then there is no relative motion between the phasors. Thus if we fix the reference phasor OR in a particular reference direction (without showing its rotation), then all others phasors moving at the same angular frequency would also be fixed at a relative position. Usually this reference direction is chosen as horizontal on the diagram for convenience.



It is also usual to draw the **Phasor diagram** using the **rms value** A of the sinusoidal waveform, rather than with the **peak value** A_m . This is shown on an enlarged diagram. Thus unless otherwise specified it is the rms value that is drawn on a phasor diagram.

It should be noted that the values on the phasor diagram are no longer time variables. The phasor \mathbf{A} is characterised by its magnitude $|A|$ and its phase angle ϕ . These are also the polar co-ordinates of the phasor and is commonly written as $|A|\angle\phi$. The phasor \mathbf{A} can also be characterised by its cartesian co-ordinates \mathbf{A}_x and \mathbf{A}_y and usually written using complex numbers as $\mathbf{A} = \mathbf{A}_x + j \mathbf{A}_y$.

Note: In electrical engineering, the letter j is always used for the complex operator $\sqrt{-1}$ because the letter i is regularly used for electric current.

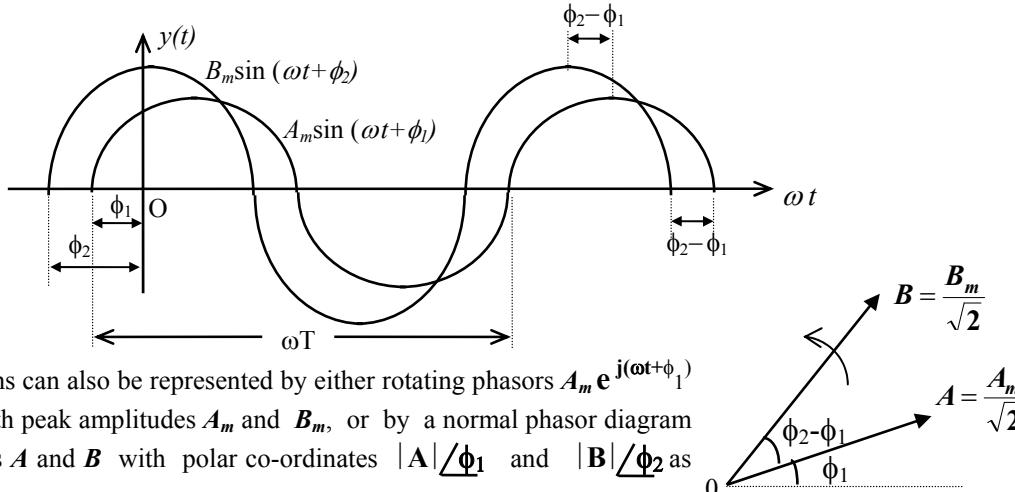
$$\text{It is worth noting that } |A| = \sqrt{A_x^2 + A_y^2} \text{ and that } \tan \phi = \frac{A_y}{A_x} \text{ or } \phi = \tan^{-1}\left(\frac{A_y}{A_x}\right)$$

$$\text{Also, } \mathbf{A}_x = |A| \cos \phi, \quad \mathbf{A}_y = |A| \sin \phi \quad \text{and} \quad |A| e^{j\phi} = |A| \cos \phi + j |A| \sin \phi = \mathbf{A}_x + j \mathbf{A}_y$$

Note: If the period of a sinusoidal waveform is T , then the corresponding angle would be ωT . Also, the period of a waveform corresponds to 1 complete cycle or 2π radians or 360° . $\therefore \omega T = 2\pi$

3.2 Phase difference

Consider the two waveforms $A_m \sin(\omega t + \phi_1)$ and $B_m \sin(\omega t + \phi_2)$ as shown in the figure. It can be seen that they have different amplitudes and different phase angles with respect to a common reference.



These two waveforms can also be represented by either rotating phasors $A_m e^{j(\omega t + \phi_1)}$ and $B_m e^{j(\omega t + \phi_2)}$ with peak amplitudes A_m and B_m , or by a normal phasor diagram with complex values \mathbf{A} and \mathbf{B} with polar co-ordinates $|A|\angle\phi_1$ and $|B|\angle\phi_2$ as shown .

Any particular value (such as positive peak, or zero) of $a(t)$ is seen to occur at a time T after the corresponding value of $b(t)$. i.e. the positive peak A_m occurs after an angle $(\phi_2 - \phi_1)$ after the positive peak B_m . Similarly the zero of $a(t)$ occurs after an angle $(\phi_2 - \phi_1)$ after the corresponding zero of $b(t)$. In such a case we say that the waveform $\mathbf{b}(t)$ **leads** the waveform $\mathbf{a}(t)$ by a phase angle of $(\phi_2 - \phi_1)$. Similarly we could say that the waveform $\mathbf{a}(t)$ **lags** the waveform $\mathbf{b}(t)$ by a phase angle of $(\phi_2 - \phi_1)$. [Note: Only the angle less than 180° is used to specify whether a waveform leads or lags another waveform].

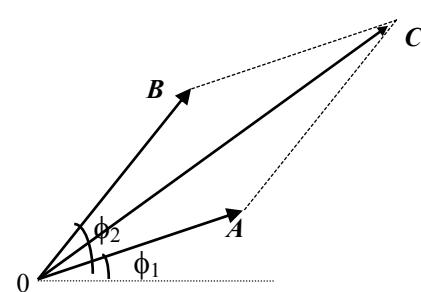
We could also define, **lead** and **lag** by simply referring to the phasor diagram. Since angles are always measured anticlockwise (convention), we can see from the phasor diagram, that \mathbf{B} **leads** \mathbf{A} by an angle of $(\phi_2 - \phi_1)$ anticlockwise or that \mathbf{A} **lags** \mathbf{B} by an angle $(\phi_2 - \phi_1)$.

Addition and subtraction of phasors can be done using the same parallelogram and triangle laws as for vectors, generally using complex numbers. Thus the addition of phasor \mathbf{A} and phasor \mathbf{B} would be

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (A \cos \phi_1 + j A \sin \phi_1) + (B \cos \phi_2 + j B \sin \phi_2) \\ &= (A \cos \phi_1 + B \cos \phi_2) + j (A \sin \phi_1 + B \sin \phi_2) \\ &= C_x + j C_y = /C\angle\phi_c = \mathbf{C} \end{aligned}$$

$$\text{where } |C| = \sqrt{C_x^2 + C_y^2} = \sqrt{(A \cos \phi_1 + B \cos \phi_2)^2 + (A \sin \phi_1 + B \sin \phi_2)^2}$$

$$\text{and } \phi_c = \tan^{-1}\left(\frac{C_y}{C_x}\right) = \tan^{-1}\left(\frac{(A \sin \phi_1 + B \sin \phi_2)}{(A \cos \phi_1 + B \cos \phi_2)}\right)$$



Example

Find the addition and the subtraction of the two complex numbers given by $10\angle 30^\circ$ and $25\angle 48^\circ$.

$$\text{Addition} = 10\angle 30^\circ + 25\angle 48^\circ = 10(0.8660 + j 0.5000) + 25(0.6691 + j 0.7431)$$

$$= (8.660 + 16.728) + j (5.000 + 18.577) = 25.388 + j 23.577 = 34.647\angle 42.9^\circ$$

$$\text{Subtraction} = 10\angle 30^\circ - 25\angle 48^\circ = (8.660 - 16.728) + j (5.000 - 18.577) = -8.068 - j 13.577 = 15.793\angle 239.3^\circ$$

Multiplication and division of phasors is most easily done using the polar form of complex numbers.

Thus the multiplication of phasor \mathbf{A} and phasor \mathbf{B} would be

$$\mathbf{A} * \mathbf{B} = |A|/\phi_1 * |B|/\phi_2 = |A| e^{j\phi_1} * |B| e^{j\phi_2} = |A| * |B| e^{j(\phi_1 + \phi_2)} = |C|/\phi_c$$

where $|C| = |A| * |B|$ and $\phi_c = \phi_1 + \phi_2$

In a similar way, it can be easily seen that for division

$$|C| = |A| // |B| \text{ and } \phi_c = \phi_1 - \phi_2$$

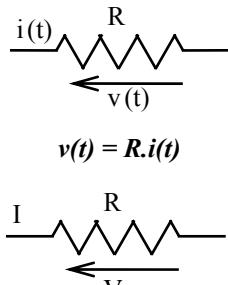
Thus, whenever we need to do addition and subtraction, we use the cartesian form of complex numbers, whereas for multiplication or division we use the polar form.

Example

Find the multiplication and the division of the two complex numbers given by $10\angle 30^\circ$ and $25\angle 48^\circ$.

$$\text{Multiplication} = 10\angle 30^\circ * 25\angle 48^\circ = 250\angle 78^\circ$$

$$\text{Division} = 10\angle 30^\circ \div 25\angle 48^\circ = 0.4\angle -18^\circ$$

3.3 Currents and voltages in simple circuit elements**3.3.1 Resistor**

for a sinusoid, consider $i(t) = \text{Real part of } [I_m e^{j(\omega t + \theta)}]$ or $I_m \cos(\omega t + \theta)$

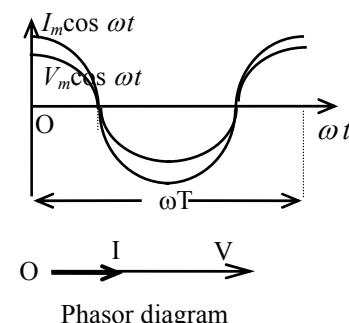
$$\therefore v(t) = \text{Real} [R \cdot I_m e^{j(\omega t + \theta)}] = \text{Real} [V_m e^{j(\omega t + \theta)}]$$

$$\text{or } v(t) = R \cdot I_m \cos(\omega t + \theta) = V_m \cos(\omega t + \theta)$$

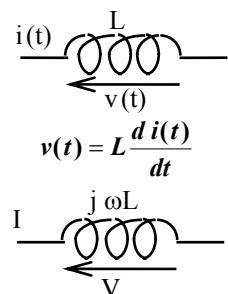
$$\therefore V_m = R \cdot I_m \text{ and } V_m/\sqrt{2} = R \cdot I_m/\sqrt{2}$$

$$\text{i.e. } V = R \cdot I$$

Note: V and I are rms values of the voltage and current and no additional phase angle change has occurred in the resistor .



Note also that the power dissipated in the resistor R is equal to $R \cdot I^2 = V \cdot I$

3.3.2 Inductor

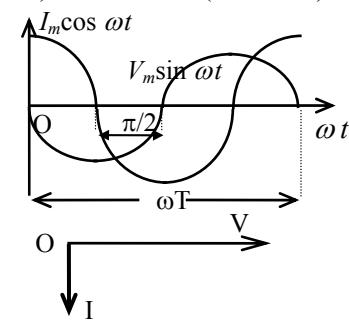
for a sinusoid, consider $i(t) = \text{Real part of } [I_m e^{j(\omega t + \theta)}]$ or $I_m \cos(\omega t + \theta)$

$$\therefore v(t) = \text{Real} [L \cdot \frac{d}{dt} I_m e^{j(\omega t + \theta)}] = \text{Real} [L \cdot j\omega I_m e^{j(\omega t + \theta)}] = \text{Real} [j V_m e^{j(\omega t + \theta)}]$$

$$\text{or } v(t) = L \cdot \frac{d}{dt} I_m \cos(\omega t + \theta) = -L \cdot \omega \cdot I_m \sin(\omega t + \theta) = L \cdot \omega \cdot I_m \cos(\omega t + \theta + \pi/2)$$

$$= V_m \cos(\omega t + \theta + \pi/2)$$

$$\therefore V_m = \omega L \cdot I_m \text{ and } V_m/\sqrt{2} = \omega L \cdot I_m/\sqrt{2}$$



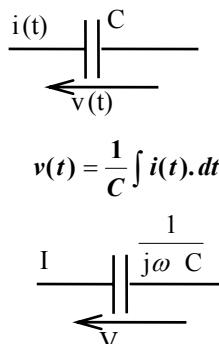
It can be seen that the **rms** magnitude of voltage is related to the **rms** magnitude of current by the multiplying factor ωL . It also seen that the voltage waveform leads the current waveform by 90° or $\pi/2$ radians or that the current waveform lags the voltage waveform by 90° for an inductor .

Thus it is usual to write the relationship as $V = j\omega L \cdot I$ or $V = \omega L \cdot I \angle 90^\circ$

The impedance Z of the inductance may thus be defined as $j\omega L$, and $V = Z \cdot I$ corresponds to the generalised form of Ohm's Law.

Remember also that the power dissipation in a pure inductor is zero, as energy is only stored and as there is no resistive part in it, but that the product $V \cdot I$ is not zero.

3.3.3 Capacitor



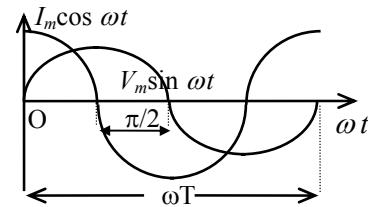
for a sinusoid, consider $i(t) = \text{Real part of } [I_m e^{j(\omega t + \theta)}]$ or $I_m \cos(\omega t + \theta)$

$$v(t) = \text{Real} \left[\frac{1}{C} \int I_m e^{j(\omega t + \theta)} dt \right] = \text{Real} \left[\frac{1}{C j \omega} \cdot I_m e^{j(\omega t + \theta)} \right] = \text{Real} \left[\frac{1}{j} V_m e^{j(\omega t + \theta)} \right]$$

or $v(t) = \frac{1}{C} \int I_m \cos(\omega t + \theta) dt = \frac{1}{C \omega} \cdot I_m \sin(\omega t + \theta) = \frac{1}{C \omega} \cdot I_m \cos(\omega t + \theta - \pi/2)$

$$v(t) = \frac{1}{C} \int i(t) dt$$

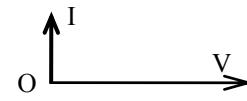
$$\therefore V_m = \frac{1}{\omega C} \cdot I_m \quad \text{and} \quad V_m/\sqrt{2} = \frac{1}{\omega C} \cdot I_m/\sqrt{2}$$



It can be seen that the **rms** magnitude of voltage is related to the **rms** magnitude of current by the multiplying factor $\frac{1}{\omega C}$.

It also seen that the voltage waveform lags the current waveform by 90° or $\pi/2$ radians or that the current waveform leads the voltage waveform by 90° for a capacitor.

Thus it is usual to write the relationship as $V = \frac{1}{j\omega C} \cdot I$ or $V = \frac{1}{\omega C} \angle 90^\circ$



Phasor diagram

The impedance Z of the inductance may thus be defined as $\frac{1}{j\omega C}$, and $V = Z \cdot I$ corresponds to the generalised form of Ohm's Law.

Remember also that the power dissipation in a pure capacitor is zero, as energy is only stored and as there is no resistive part in it, but that the product $V \cdot I$ is not zero.

3.4 Impedance and Admittance in an a.c. circuit

The **impedance** Z of an a.c. circuit is a complex quantity. It defines the relation between the complex rms voltage and the complex rms current. **Admittance** Y is the inverse of the impedance Z .

$$V = Z \cdot I, \quad I = Y \cdot V \quad \text{where} \quad Z = R + jX, \quad \text{and} \quad Y = G + jB$$

It is usual to express Z in cartesian form in terms of R and X , and Y in terms of G and B .

The real part of the impedance Z is resistive and is usually denoted by a resistance R , while the imaginary part of the impedance Z is called a *reactance* and is usually denoted by a reactance X .

It can be seen that the pure inductor and the pure capacitor has a reactance only and not a resistive part, while a pure resistor has only resistance and not a reactive part.

Thus $Z = R + j0$ for a resistor, $Z = 0 + j\omega L$ for an inductor, and $Z = \frac{1}{j\omega C} = 0 - j\frac{1}{\omega C}$ for a capacitor.

The real part of the admittance Y is a *conductance* and is usually denoted by G , while the imaginary part of the admittance Y is called a *susceptance* and is denoted by B .

Relationships exist between the components of Z and the components of Y as follows.

$$G + jB = Y = \frac{1}{Z} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2} \quad \text{so that} \quad G = \frac{R}{R^2 + X^2}, \quad \text{and} \quad B = \frac{-X}{R^2 + X^2}$$

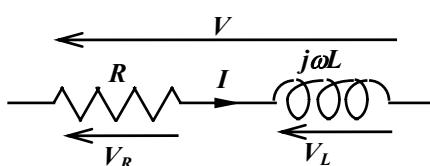
The reverse process can also be similarly done if necessary.

However, it must be remembered that in a complex circuit, G does not correspond to the inverse of the resistance R but its effective value is influenced by X as well as seen above.

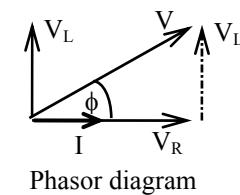
3.5 Simple Series Circuits

In the case of single elements R , L and C we found that the angle difference between the voltage and the current was either zero, or $\pm 90^\circ$. This situation changes when there are more than one component in a circuit.

3.5.1 R-L series circuit



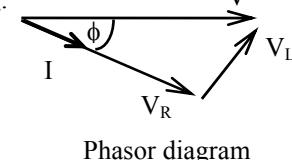
In the series R-L circuit,
considering current I as reference
 $V_R = R \cdot I$, $V_L = j\omega L \cdot I$, and $V = V_R + V_L$
 $\therefore V = (R + j\omega L) \cdot I$ so that
the total series impedance is $Z = R + j\omega L$



Phasor diagram

The above phasor diagram has been drawn with \mathbf{I} as reference. [i.e. \mathbf{I} is drawn along the x-axis direction]. The current was selected as reference in this example, because it is common to both the resistance and the inductance and makes the drawing of the circuit diagram easier. In this diagram, the voltage across the resistor \mathbf{V}_R is in phase with the current, whereas the voltage across the inductor \mathbf{V}_L is 90° leading the current. The total voltage \mathbf{V} is then obtained by the phasor addition (similar to vector addition) of \mathbf{V}_R and \mathbf{V}_L .

If the total voltage was taken as the reference, the diagram would just rotate as shown. In this diagram, the current is seen to be lagging the voltage by the same angle that in the earlier diagram the voltage was seen to be leading the current. \mathbf{V}_L has been drawn from the end of \mathbf{V}_R rather than from the origin for ease of obtaining the resultant \mathbf{V} from the triangular law.

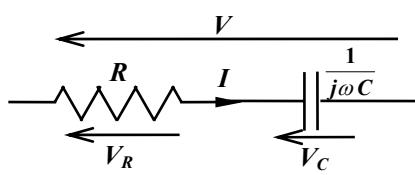


Phasor diagram

In an R-L circuit, the current lags the voltage by an angle less than 90° and the circuit is said to be inductive.

Note that the power dissipation can only occur in the resistance in the circuit and is equal to $\mathbf{R} \cdot \mathbf{I}^2$ and that this is not equal to product $\mathbf{V} \cdot \mathbf{I}$ for the circuit.

3.5.2 R-C series circuit

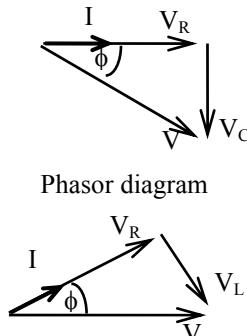


$$\therefore \mathbf{V} = (\mathbf{R} + \frac{1}{j\omega C}) \cdot \mathbf{I} \text{ so that the total series impedance is } Z = \mathbf{R} + \frac{1}{j\omega C}$$

In the series R-C circuit,

$$\mathbf{V}_R = \mathbf{R} \cdot \mathbf{I}, \quad \mathbf{V}_C = \frac{1}{j\omega C} \cdot \mathbf{I} = -j \frac{1}{\omega C} \cdot \mathbf{I}$$

$$\text{and } \mathbf{V} = \mathbf{V}_R + \mathbf{V}_C$$



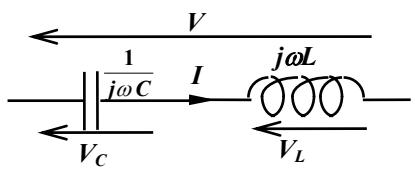
Phasor diagram

The phasor diagrams have been drawn first with current \mathbf{I} as reference and then with voltage \mathbf{V} as reference.

In an R-C circuit, the current leads the voltage by an angle less than 90° and the circuit is said to be capacitive.

Note that the power dissipation can only occur in the resistance in the circuit and is equal to $\mathbf{R} \cdot \mathbf{I}^2$ and that this is not equal to product $\mathbf{V} \cdot \mathbf{I}$ for the circuit.

3.5.3 L-C series circuit

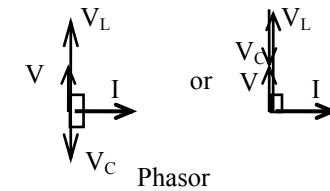


In the series L-C circuit,

$$\mathbf{V}_L = j\omega L \cdot \mathbf{I}, \quad \mathbf{V}_C = \frac{1}{j\omega C} \cdot \mathbf{I} = -j \frac{1}{\omega C} \cdot \mathbf{I}$$

$$\text{and } \mathbf{V} = \mathbf{V}_L + \mathbf{V}_C$$

$$\therefore \mathbf{V} = (j\omega L + \frac{1}{j\omega C}) \cdot \mathbf{I} \text{ so that the total series impedance is } Z = j\omega L + \frac{1}{j\omega C} = j\omega L - j \frac{1}{\omega C}$$



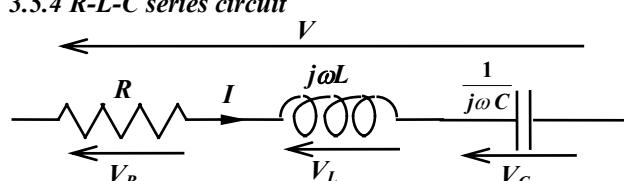
Phasor

It is seen that the total impedance is purely reactive, and that all the voltages in the circuit are inphase but perpendicular to the current. The resultant voltage corresponds to the algebraic difference of the two voltages \mathbf{V}_L and \mathbf{V}_C and the direction could be either up or down depending on which voltage is more.

When $\omega L = \frac{1}{\omega C}$ the total impedance of the circuit becomes zero, so that the circuit current for a given supply voltage would become very large (only limited by the internal impedance of the source). This condition is known as series resonance. In an L-C circuit, the current either lags or leads the voltage by an angle equal to 90° and the resultant circuit is either purely inductive or capacitive.

Note that no power dissipation can occur in the circuit and but that the product $\mathbf{V} \cdot \mathbf{I}$ for the circuit is non zero.

3.5.4 R-L-C series circuit



$$\therefore \mathbf{V} = (\mathbf{R} + j\omega L + \frac{1}{j\omega C}) \cdot \mathbf{I} \text{ so that the total series impedance is } Z = \mathbf{R} + j\omega L + \frac{1}{j\omega C} = \mathbf{R} + j(\omega L - \frac{1}{\omega C})$$

In the series R-L-C circuit,

$$\mathbf{V}_R = \mathbf{R} \cdot \mathbf{I}, \quad \mathbf{V}_L = j\omega L \cdot \mathbf{I}, \quad \mathbf{V}_C = \frac{1}{j\omega C} \cdot \mathbf{I} = -j \frac{1}{\omega C} \cdot \mathbf{I}$$

$$\text{and } \mathbf{V} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C$$

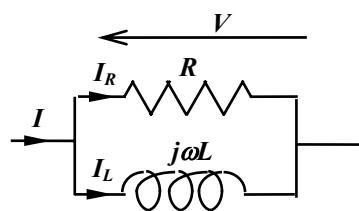
$|Z| = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$ has a minimum value at $\omega L = \frac{1}{\omega C}$. This is the series resonance condition.

In an R-L-C circuit, the current can either lag or lead the voltage, and the phase angle difference between the current and the voltage can vary between -90° and 90° and the resultant circuit is either inductive or capacitive.

Note that the power dissipation can only occur in the resistance in the circuit and is equal to $R \cdot I^2$ and that this is not equal to product $V \cdot I$ for the circuit.

3.6 Simple Parallel Circuits

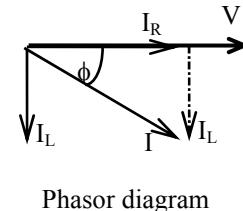
3.6.1 R-L parallel Circuit



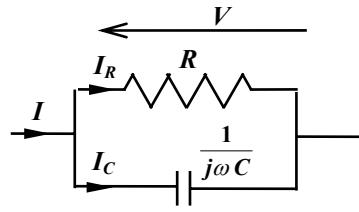
In the parallel R-L circuit,
considering \mathbf{V} as reference
 $\mathbf{V} = R \cdot \mathbf{I}_R$, $\mathbf{V} = j\omega L \cdot \mathbf{I}_L$, and $\mathbf{I} = \mathbf{I}_R + \mathbf{I}_L$

$$\therefore \mathbf{I} = \frac{\mathbf{V}}{R} + \frac{\mathbf{V}}{j\omega L}$$

$$\therefore \text{total shunt admittance} = \frac{1}{R} + \frac{1}{j\omega L}$$



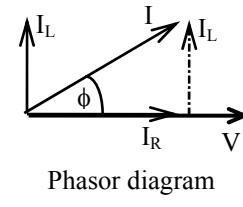
3.6.2 R-C parallel Circuit



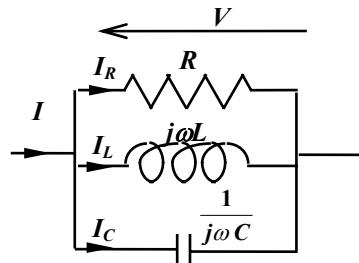
In the parallel R-C circuit,
considering \mathbf{V} as reference
 $\mathbf{V} = R \cdot \mathbf{I}_R$, $\mathbf{I}_C = j\omega C \cdot \mathbf{V}$, and $\mathbf{I} = \mathbf{I}_R + \mathbf{I}_C$

$$\therefore \mathbf{I} = \frac{\mathbf{V}}{R} + \mathbf{V} \cdot j\omega C$$

$$\therefore \text{total shunt admittance} = \frac{1}{R} + j\omega C$$



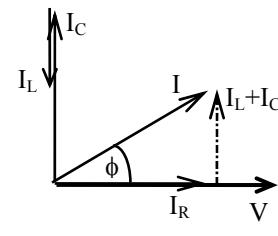
3.6.3 R-L-C parallel Circuit



In the parallel R-L-C circuit,
considering \mathbf{V} as reference
 $\mathbf{V} = R \cdot \mathbf{I}_R$, $\mathbf{V} = j\omega L \cdot \mathbf{I}_L$, $\mathbf{I}_C = j\omega C \cdot \mathbf{V}$
and $\mathbf{I} = \mathbf{I}_R + \mathbf{I}_L + \mathbf{I}_C$

$$\therefore \mathbf{I} = \frac{\mathbf{V}}{R} + \frac{\mathbf{V}}{j\omega L} + \mathbf{V} \cdot j\omega C$$

$$\therefore \text{total shunt admittance} = \frac{1}{R} + \frac{1}{j\omega L} + j\omega C$$



As in the case of the series circuit, shunt resonance will occur when $\frac{1}{\omega L} = \omega C$ giving a minimum value of shunt admittance.

Note that even in the case of a parallel circuit, power loss can only occur in the resistive elements and that the product $V \cdot I$ is not usually equal to the power loss.

3.7 Power and Power Factor

It was noted that in an a.c. circuit, power loss occurs only in resistive parts of the circuit and in general the power loss is not equal to the product $V \cdot I$ and that purely inductive parts and purely capacitive parts of a circuit did not have any power loss. To account for this apparent discrepancy, we define the product $V \cdot I$ as the **apparent power S** of the circuit. Apparent power has the unit volt-ampere (VA) and not the watt (W), and watt (W) is used only for the **active power P** of the circuit (which we earlier called the **power**)

$$\text{apparent power } S = V \cdot I$$

Since a difference exists between the **apparent power** and the **active power** we define a new term called the **reactive power Q** for the reactance X.

The instantaneous value of power $p(t)$ is the product of the instantaneous value of voltage $v(t)$ and the instantaneous value of current $i(t)$.

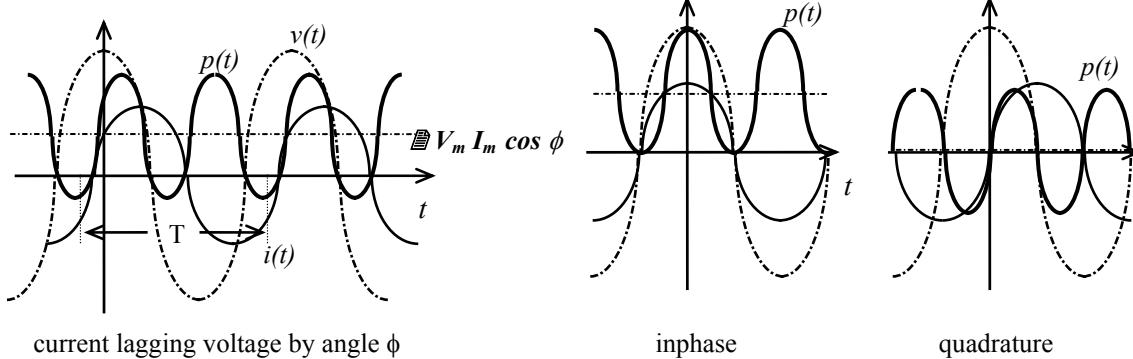
i.e. $p(t) = v(t) \cdot i(t)$

If $v(t) = V_m \cos \omega t$ and $i(t) = I_m \cos(\omega t - \phi)$,

where the voltage has been taken as reference and the current lags the voltage by a phase angle ϕ

then $p(t) = V_m \cos \omega t \cdot I_m \cos(\omega t - \phi) = V_m I_m \cdot \frac{1}{2} [2 \cos \omega t \cdot \cos(\omega t - \phi)]$

$$= \frac{1}{2} V_m I_m [\cos(2\omega t - \phi) + \cos \phi]$$



It can be seen that the waveform of power $p(t)$ has a sinusoidally varying component and a constant component.

Thus the average value of power (active power) P would be given by the constant value $\frac{1}{2} V_m I_m \cos \phi$.

$$\text{active power } P = \frac{1}{2} V_m I_m \cos \phi = \frac{V_m}{\sqrt{2}} \cdot \frac{I_m}{\sqrt{2}} \cdot \cos \phi = V \cdot I \cos \phi$$

The term $\cos \phi$ is defined as the **power factor**, and is the **ratio** of the **active power** to the **apparent power**.

Note that for a **resistor**, $\phi = 0^\circ$ so that $P = V \cdot I$

and that for an **inductor**, $\phi = 90^\circ$ lagging (i.e. current is lagging the voltage by 90°) so that $P = 0$

and that for an **capacitor**, $\phi = 90^\circ$ leading (i.e. current is leading the voltage by 90°) so that $P = 0$

For combinations of **resistor**, **inductor** and **capacitor**, P lies between $V \cdot I$ and 0

For an inductor or capacitor, $V \cdot I$ exists although $P = 0$. For these elements the product $V \cdot I$ is defined as the **reactive power** Q . This occurs when the **voltage** and the **current** are **quadrature** (90° out of phase). Thus reactive power is defined as the product of voltage and current components which are quadrature. This gives

$$\text{reactive power } Q = V \cdot I \sin \phi$$

Unlike in the case of inphase, where the same direction means positive, when quantities are in quadrature there is no natural positive direction. It is usual to define inductive reactive power when the current is lagging the voltage and capacitive reactive power when the current is leading the voltage. It is worth noting that inductive reactive power and capacitive reactive power have opposite signs.

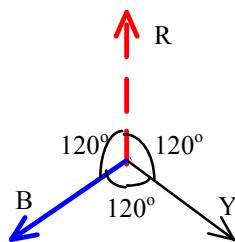
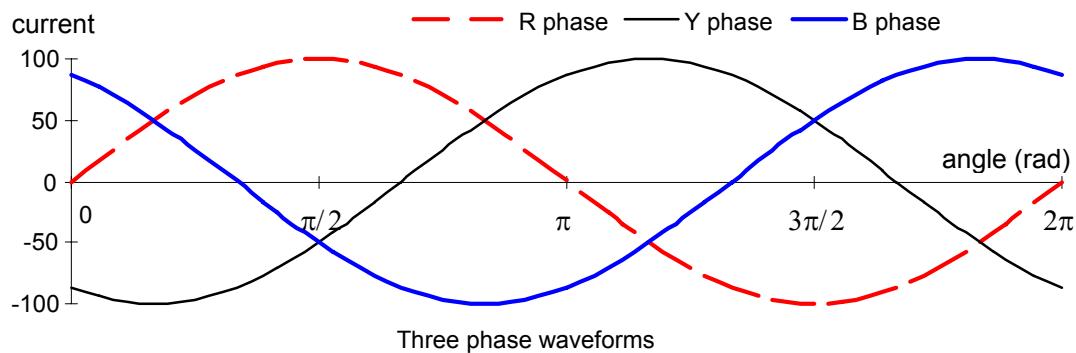
Although reactive power does not consume any energy, it reduces the power factor below unity. When the power factor is below unity, for the same power transfer P the current required becomes larger and the power losses in the circuit becomes still larger ($\text{power loss} \propto I^2$). This is why supply authorities encourage the industries to improve their power factors to be close to unity.

3.8 Three Phase Power

As you already know, to transmit power using single phase ac, we need two wires. Of course you may have an earth wire for protection, but this wire does not usually carry any current. You may also have seen that distribution lines usually have 4 wires. What are these 4 wires? It is the three phase wires and the neutral wire.

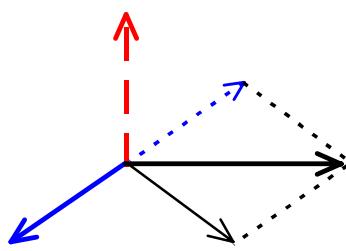
3.8.1 Balanced Three Phase

Now why do you need three phase wires? This is in order to make our transmission efficient. You are already aware that if we have three equal forces at angles of 120° to each other, if we use the triangular law we get an equilateral triangle in which the resultant force is zero. A similar thing happens in three phase except that we have phasors instead of vectors. That is, we have 3 voltages (or currents) which are equal in magnitude but differing in phase angle by 120° from each other.



If the R phase is taken as reference and drawn vertical (reference need not always be drawn horizontally), then the three phases would have the phasor diagram shown.

If $V_R(t) = V_m \cos \omega t$, then $V_Y(t) = V_m \cos (\omega t - 2\pi/3)$ and $V_B(t) = V_m \cos (\omega t + 2\pi/3)$. It can easily be seen that the addition of these three waveforms at any instant is zero. This is not only true for balanced three phase voltages, but for balanced three phase currents. Thus if we have a balanced three phase system of currents, then their addition would become zero, and no neutral wire would be required. However in practice, three phase currents are never perfectly balanced and the neutral wire would carry the unbalance.



If V_R , V_Y , and V_B are the r.m.s. voltages of the three phases with respect to the neutral, then they are normally called the **phase voltages**. The voltage between any two phase wires (or lines) is called the **line voltage**.

As can be seen from the diagram, the magnitude of the line voltage is $\sqrt{3}$ times the phase voltage for a balanced system. In a three phase system, it is always the r.m.s. line voltage that is specified, unless otherwise specified. For example in the domestic supply, in single phase we say that the voltage is 230 V and in three phase we say that the voltage is 400 V ($\sim 230 \times \sqrt{3}$).

If V_p is the phase voltage magnitude, $V_R = V_p \angle 0^\circ$, $V_Y = V_p \angle -2\pi/3$, $V_B = V_p \angle 2\pi/3$.

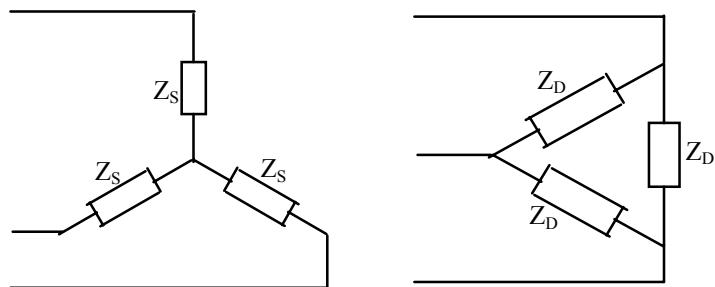
The Line voltage magnitude will be $\sqrt{3} \times V_p = V_L$ and the line voltages will be $V_{RY} = V_L \angle \pi/6$,

$V_{YB} = V_L \angle (-2\pi/3 + \pi/6) = V_L \angle -\pi/2$, $V_{BR} = V_L \angle (2\pi/3 + \pi/6) = V_L \angle 5\pi/6$.

Balanced loads may be connected either in star or delta as shown.

A balanced star connected load can be converted to an equivalent delta and the other way round as well. In such a case it can be shown that

$$Z_D = 3 Z_S$$



It can also be shown that the total three phase power in a star connected load, or delta connected load can be expressed in terms of the line quantities as

$$P = \sqrt{3} V_L I_L \cos \phi$$

where $\cos \phi$ is the power factor of the load, V_L is the line voltage and I_L is the line current.

The reactive power in a three phase circuit may be similarly defined as

$$Q = \sqrt{3} V_L I_L \sin \phi$$

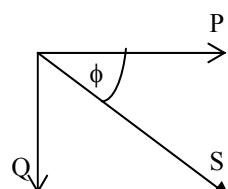
and the apparent power in a three phase circuit may be defined as

$$S = \sqrt{3} V_L I_L$$

Note: In both the single phase circuit as well as the three phase circuit,

$$S^2 = P^2 + Q^2 \quad \text{and} \quad \tan \phi = Q/P$$

Since the magnitudes of the three phases are equal (for both voltage and current) in a balanced three phase system, it is sufficient to calculate the quantities for one of the phases and obtain the others by symmetry.



Example

A 3 phase, 12 kW balanced load at a lagging power factor of 0.8 is supplied from a 3 phase, 400V, 50 Hz supply. Determine the line current in magnitude and phase relative to the supply voltage, the apparent power and the reactive power drawn.

$$P = 12 \text{ kW}, V_L = 400\text{V}, \cos \phi = 0.8 \text{ lag}, P = \sqrt{3} V_L I_L \cos \phi, \text{ i.e. } 12000 = \sqrt{3} \times 400 \times I_L \times 0.8$$

$$\therefore I_L = 21.65 \text{ A}, \phi = \cos^{-1}(0.8) = 36.87^\circ. \text{ i.e. } I_L = 21.65 \angle -36.87^\circ \text{ A.}$$

$$\text{Apparent power} = \sqrt{3} V_L I_L = \sqrt{3} \times 400 \times 21.65 = 15000 = 15 \text{ kVA.}$$

$$\text{Reactive power} = \sqrt{3} V_L I_L \sin \phi = \sqrt{3} \times 400 \times 21.65 \times \sin(36.87^\circ) = 9000 = 9 \text{ k var}$$

3.8.1 Unbalanced three phase systems

Unbalanced three phase systems would consist of either star connected or delta connected sources which need not be balanced and an unbalanced star connected or delta connected load. Such systems cannot be solved using one phase of the system, but could be solved using the normal network theorems.

3.9 Power factor correction

Since most practical loads are inductive in nature (as they have coils rather than capacitors), the power factor can be improved by using capacitors across the load. They may be either connected in star or delta.

A pure capacitor affects only the reactive power and not the active power.

Example

In the above example a delta connected bank of capacitors are used to improve the power factor to 0.95 lag. What should be the value of the required capacitors.

$$\text{New power factor} = 0.95 \text{ lag. } \therefore \text{new power factor angle} = \cos^{-1}(0.95) = 18.19^\circ$$

The active power of the load is unchanged by the capacitor bank, $\therefore P = 12000 \text{ W}$

$$\therefore \text{new reactive power} = P \tan \phi = 12000 \times \tan(18.19^\circ) = 3944 = 3.944 \text{ k var}$$

$$\therefore \text{reactive power supplied from capacitor bank} = 9000 - 3944 = 5056 = 5.056 \text{ k var.}$$

Each of the 3 capacitors in the delta connected bank must supply $= 5.056/3 = 1.685 \text{ k var.}$

Since they are connected in delta, each would get a voltage of 400 V.

$$\therefore V_L^2 C \omega = 400^2 \times C \times 2 \pi \times 50 = 1685$$

$$\therefore C = 33.5 \times 10^{-6} = 33.5 \mu\text{F.}$$

If the capacitors had been connected in star, then the reactive power would still have been the same, but the voltage across each would have been $400/\sqrt{3}$ and the value of each capacitor would be

$$V_p^2 C' \omega = (400/\sqrt{3})^2 \times C' \times 2 \pi \times 50 = 1685$$

$$\therefore C' = 100.6 \times 10^{-6} = 100.6 \mu\text{F.}$$