Mathematics - MA 1103
Intake 38 - 03rd Batch
Tutorial 07 - Solutions.

(01) (a)

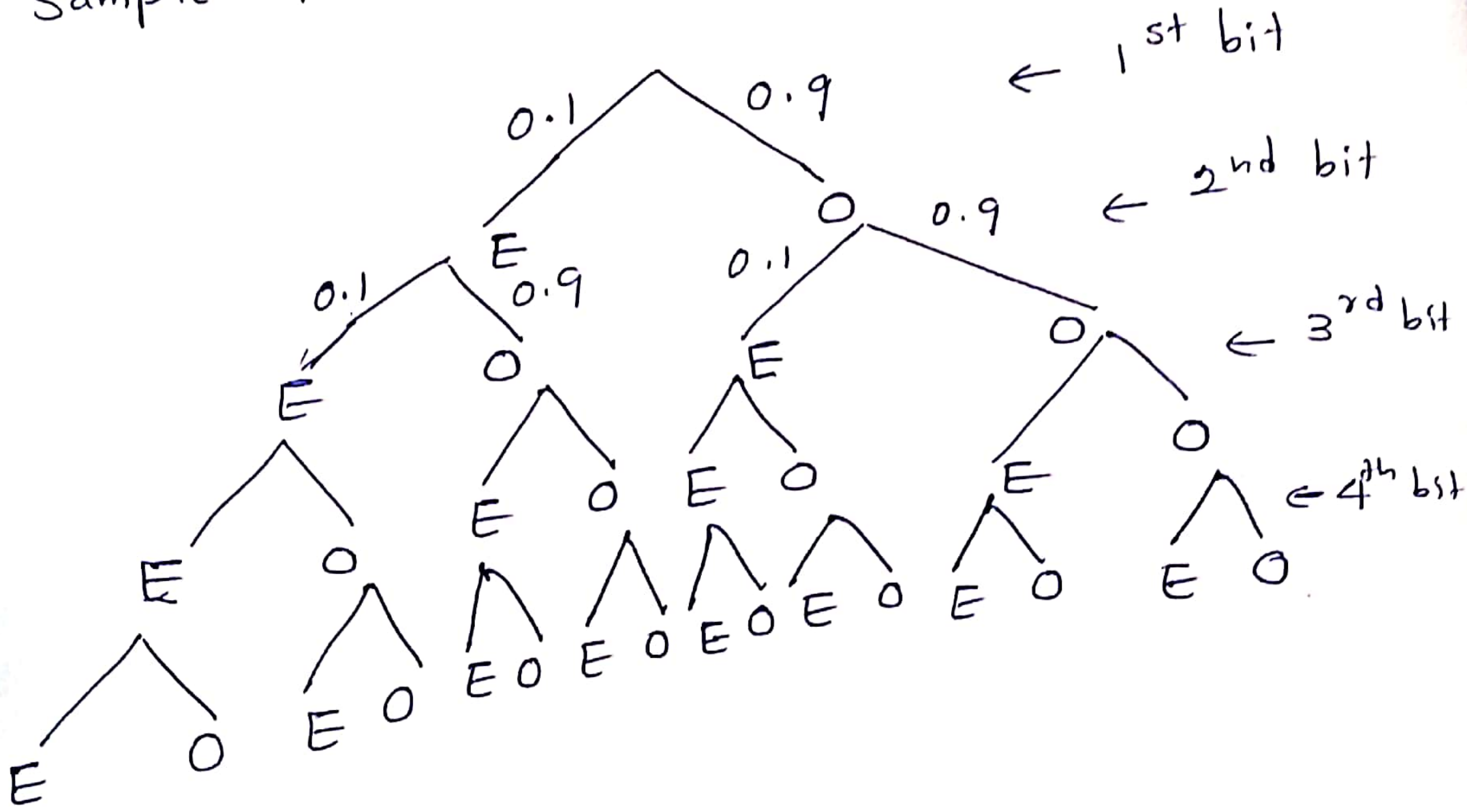
Let

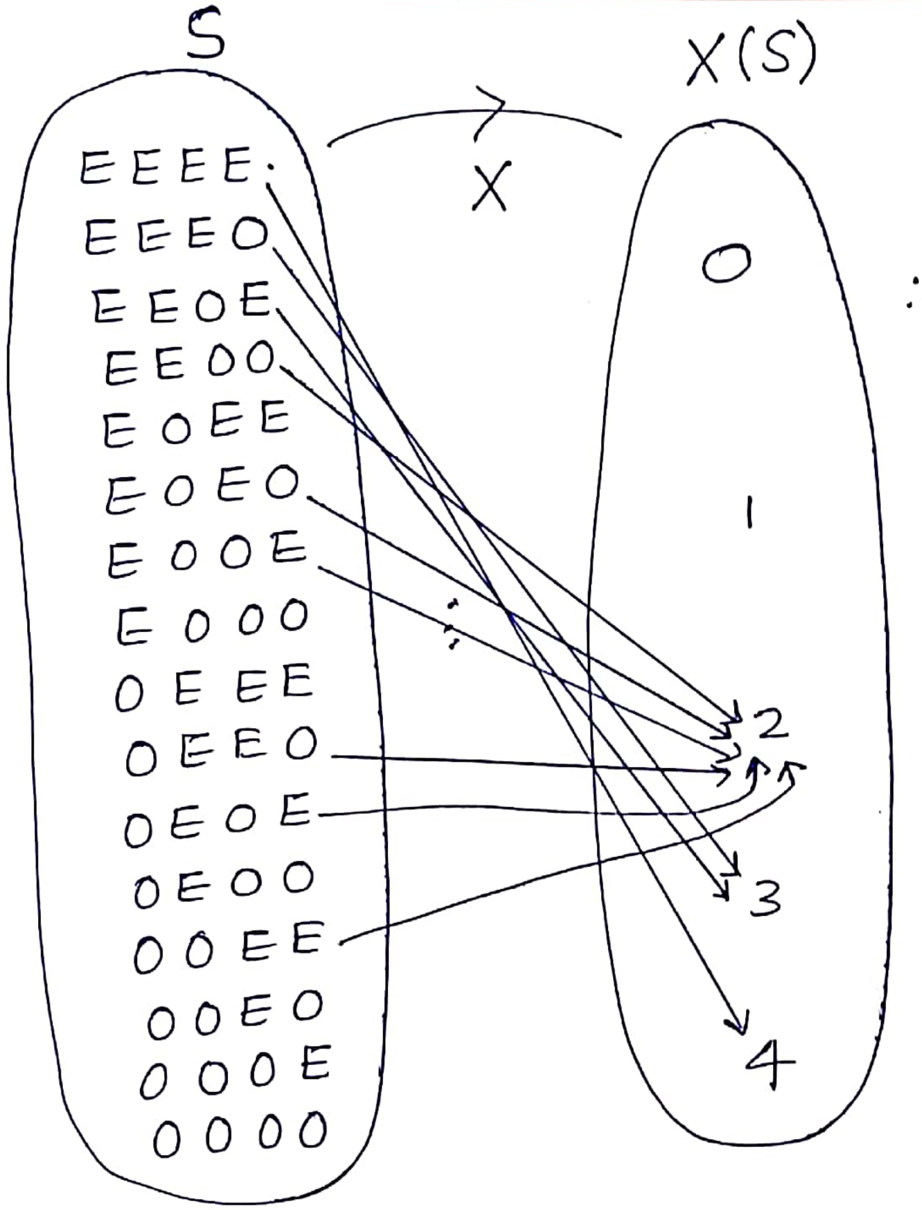
E denote a bit in error

O denote a bit received without error

X - # of bits in error in the next 4 bits transmitted.

Sample space:





$\therefore X(S) = \{0, 1, 2, 3, 4\}$

$\therefore \quad X(E_2) = 2$ where

$E_2 = \{EEOO,\ EOEO,\ EOOE,\ OEEO,$

$OEOE,\ OOEO\}$

since the trials are independent,

$P(\{EEOO\}) = P(E) \cdot P(E) \cdot P(O) \cdot P(O)$

$= (0 \cdot 1)^2 (0 \cdot 9)^2$

similarly, $P(\{EOEO\}) = (0 \cdot 1)^2 (0 \cdot 9)^2$

$\vdots$

$P(\{OOEO\}) = (0 \cdot 1)^2 (0 \cdot 9)^2$

$$\therefore \; P(X=2) = P(E_2)$$
$$= 6 \times (0.1)^2 (0.9)^2$$
$$= 0.0081$$

$\therefore$ generally,

$$P(X=x) = \left\{ \begin{array}{l} \text{\# of outcomes} \\ \text{that result in} \\ x \text{ errors.} \end{array} \right\} \times (0.1)^x \times (0.9)^{4-x}$$

$$\therefore \; P(X=x) = b(x; n, p) = \binom{n}{x} p^x q^{n-x} \; ;$$

$$q = 1-p \; .$$
$$x = 0, 1, 2, \ldots, n$$

$\therefore$ when $x = 2$,

$$P(X=2) = b(2; 4, 0.1) = \binom{4}{2} (0.1)^2 (0.9)^2$$
$$= 0.0081$$

(b) $\mu_X = E(X) = np$

$$\sigma_X^2 = np(1-p)$$

$n = 4$; $p = 0.1$

$$\therefore \; \mu_X = E(X) = 4(0.1) = 0.4$$
$$\sigma_X^2 = 4(0.1)(0.9) = 0.36$$

(02) $P = 0.1$

(a)

Let X = # of samples that contain the pollutant in the next 18 samples analyzed.

$\therefore$ X is a binomial random variable which follows the binomial distribution.

$$\therefore P(X=x) = b(x; n, P)$$

$\therefore$ $n = 18$, $x = 2$, $P = 0.1$

$$\therefore P(X=2) = b(2; 18, 0.1)$$
$$= \binom{18}{2}(0.1)^2(0.9)^{16}$$
$$= 0.284$$

(b) $$P(X \geq 4) = \sum_{x=4}^{18} b(x; 18, 0.1)$$

However,

$$P(X \geq 4) = 1 - P(X \leq 3)$$
$$= 1 - \sum_{x=0}^{3} b(x; 18, 0.1)$$
$$= 1 - \{0.150 + 0.3 + 0.284 + 0.168\}$$
$$= 0.098$$

(c) $P(3 \leq X < 7)$

$$= \sum_{x=3}^{6} b(x; 18, 0.1)$$

$$= \sum_{x=3}^{6} \binom{18}{x} (0.1)^{x} (0.9)^{18-x}$$

$$= \binom{18}{3}(0.1)^{3}(0.9)^{15} + \binom{18}{4}(0.1)^{4}(0.9)^{14} + \binom{18}{5}(0.1)^{5}(0.9)^{13} + \binom{18}{6}(0.1)^{6}(0.9)^{12}$$

$$= 0.168 + 0.070 + 0.022 + 0.005$$

$$= 0.265$$

(04)

(06)

(a) Let X denote the number of samples analyzed until a large particle is detected. Then X is a geometric random variable with $p = 0.01$ and hence follows the following distribution.

$$P(X=x) = (1-p)^{x-1} p \quad ; \quad x = 1, 2, 3, \ldots$$
$$= (0.99)^{x-1}(0.01)$$

$$\therefore \; P(X=125) = (0.99)^{124}(0.01)$$
$$= 0.0029$$

(05) (a)

$$P(X=x) = p(x;\lambda) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ;$$

$$x = 0, 1, 2, \ldots$$

where $\lambda = E(X) = V(X)$

(i) $\lambda = 2.3$ = mean per mm

Let X denote the # of flaws in 1 mm of wire.

$$\therefore \; P(X=2) = p(2;\lambda) = \frac{e^{-2.3}\, 2.3^{2}}{2!} = 0.265$$

(ii) Let X denote the # of flaws in 5 mm wire. $\therefore E(X) = 5\lambda$

$$\therefore P(X=10) = p(10, 5\lambda) = \frac{e^{-2.3\times5}(2.3\times5)^{10}}{10!}$$

$$= \frac{e^{-11.5}\, 11.5^{10}}{10!}$$

$$= 0.113$$

(iii) Let X denote the # of flaws in 2 mm wire. $\therefore E(X) = 2\lambda = 4.6$

$$P(X \geq 1) = 1 - P(X<1)$$

$$= 1 - P(X=0)$$

$$= 1 - P(0; 4.6)$$

$$= 1 - \frac{e^{-4.6} \times 4.6^{0}}{0!}$$

$$= 1 - e^{-4.6}$$

$$= 0.9899$$

(06)

(b) (i) The density function for the uniformly distributed random variable X is

$$f(x) = \begin{cases} \frac{1}{4} & \text{if } 0 \leq x \leq 4 \\ 0, & \text{elsewhere.} \end{cases}$$

(ii) $$P(X \geq 3) = \int_3^4 f(x)\,dx$$

$$= \int_3^4 \frac{1}{4}\,dx = \frac{1}{4}$$

(07) The p.d.f. of T is as follows:

$$E(T) = 5 = \frac{1}{d} \Rightarrow d = \frac{1}{5}$$

$$\therefore \quad f(t) = \begin{cases} \frac{1}{5} e^{-\frac{t}{5}} & \text{if } t \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

the probability that a given component is still functioning after 8 years is given by:

$$P(T > 8) = \int_8^\infty f(t)\, dt$$

$$= \int_8^\infty \frac{1}{5} e^{-t/5}\, dt$$

$$= \frac{1}{5} \left[e^{-t/5} \cdot \left(-\frac{1}{5}\right) \right]_8^\infty$$

$$= \left[e^{-t/5} \right]_\infty^8$$

$$= e^{-8/5} - \underbrace{\lim_{t \to \infty} \frac{1}{e^{t/5}}}_{=0}$$

$$\therefore P(T>8) = e^{-8/5} \simeq 0.2$$

(10)

Let X be the # of components functioning after 8 years.

$$P(X \geq 2) = \sum_{x=2}^{5} b(x; 5, 0.2)$$

or

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - P(X=1)$$

$$= 1 - b(1; 5, 0.2)$$

$$= 1 - \binom{5}{1}(0.2)^1(0.8)^4$$

$$= 1 - 0.7373$$

$$= 0.2627$$

(08)

(a) $P(A|D) = \frac{P(D \cap A)}{P(D)} = \frac{0.78}{0.83}$

$= 0.94$

(b) $P(D|A) = \frac{P(D \cap A)}{P(A)} = \frac{0.78}{0.82}$

$= 0.95$

(09)
(a)

Let A be the event that the first fuse is defective.

B be the event that the second fuse is defective.

We want to find $P(A \cap B)$.

$\therefore \quad P(A \cap B) = P(A) \cdot P(B|A)$

$= \left(\frac{5}{20}\right) \cdot \left(\frac{4}{19}\right)$

$= \frac{1}{19}$

(c) Let A be the event that the fire engine is available

Let B be the event that the ambulance is available.

$\therefore$ We want to find $P(A \cap B)$.

Since A and B are independent events,

$$P(A \cap B) = P(A) \cdot P(B)$$
$$= (0.98)(0.92)$$
$$= 0.9016.$$

(10) Let

A be the event that the product is defective

B_1 be the event that the product made by machine B_1.

B_2 " " " " " " " " " B_2

B_3 " " " " " " " " " B_3

From the Rule of Elimination (Theorem of Total probability)

$$P(A) = P(B_1)\,P(A|B_1) + P(B_2)\,P(A|B_2) + P(B_3) \cdot P(A|B_3)$$

$$= (0.3)(0.02) + (0.45)(0.03) + (0.25)(0.02)$$

$$= 0.006 + 0.0135 + 0.005$$

$$= 0.0245$$

(11) $P(P_1) = 0.3$

$P(P_2) = 0.2$

$P(P_3) = 0.5$

We need to find $P(P_j | D)$; $j = 1, 2, 3$

From Bayes' rule,

$$P(P_1|D) = \frac{P(P_1)\,P(D|P_1)}{P(P_1)\cdot P(D|P_1) + P(P_2)\,P(D|P_2) + P(P_3)\cdot P(D|P_3)}$$

$$= \frac{(0.30)(0.01)}{(0.3)(0.01) + (0.2)(0.03) + (0.5)(0.02)}$$

$$= \frac{0.003}{0.019} = 0.158$$

Similarly,

$$P(P_2|D) = \frac{(0.03)(0.20)}{0.019} = 0.316$$

$$P(P_3|D) = \frac{(0.02)(0.50)}{0.019} = 0.526$$

$\therefore$ $P(P_3|D)$ is the largest, a defective for a random product is most likely the result of plan 3. ∎