$$A_{32} = (-1)^{3+2} |M_{32}|$$

$$= - \begin{vmatrix} 1 & 5 & 6 \\ 2 & 0 & 3 \\ 4 & 2 & 7 \end{vmatrix} = -8.$$

$$A_{24} = (-1)^{2+4} \begin{vmatrix} 1 & -3 & 3 \\ 1 & 5 & 9 \\ 4 & 0 & 2 \end{vmatrix} = -192$$

Def$^{\underline{n}}$ (Determinant)

Let A be an $n \times n$ matrix. Then the determinant of A is given by

$$\det(A) = |A|$$

$$= a_{11} A_{11} + a_{12} A_{12} + \cdots + a_{1n} A_{1n}$$

$$|A| = \sum_{k=1}^{n} a_{1k} A_{1k}.$$

Note

⊗ Above definition said as expanding by cofactors in the first row of A.

⊗ We can calculate determinant of A expanding by cofactors in any row or column.

Ex. Calculate $\begin{vmatrix} 3 & 5 & 2 \\ 4 & 2 & 3 \\ -1 & 2 & 4 \end{vmatrix}$

$$\begin{vmatrix} 3 & 5 & 2 \\ 4 & 2 & 3 \\ -1 & 2 & 4 \end{vmatrix} = 3\begin{vmatrix} 2 & 3 \\ 2 & 4 \end{vmatrix} - 5\begin{vmatrix} 4 & 3 \\ -1 & 4 \end{vmatrix} + 2\begin{vmatrix} 4 & 2 \\ -1 & 2 \end{vmatrix}$$

$$= 3(8-6) - 5(16+3) + 2(8+2) = -69$$

Ex. If $A = \begin{pmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 2 & 1 & 9 & 6 \\ 3 & 2 & 4 & 8 \end{pmatrix}_{4\times 4}$,

then calculate $|A|$

$$|A| = a_{11} A_{11} + a_{12} a_{12} + a_{13} A_{13} + a_{14} A_{14}$$

$$= 1\begin{vmatrix} -1 & 3 & 4 \\ 1 & 9 & 6 \\ 2 & 4 & 8 \end{vmatrix} - 3\begin{vmatrix} 0 & 3 & 4 \\ 2 & 9 & 6 \\ 3 & 4 & 8 \end{vmatrix}$$

$$+5\begin{vmatrix} 0 & -1 & 4 \\ 2 & 1 & 6 \\ 3 & 2 & 8 \end{vmatrix} - 2\begin{vmatrix} 0 & -1 & 3 \\ 2 & 1 & 9 \\ 3 & 2 & 9 \end{vmatrix}.$$

$$= 1(-92) - 3(-70) + 5(2) - 2(-16)$$

$$= 160 \,//$$

Properties of Determinands

Let A & B are square matrices.

① If one row or one column of a square matrix A is zero, then,

$$\det A = 0$$

Suppose the ith row of A contains all zeros. Then.

$$|A| = a_{i1}A_{i1} + a_{i2}A_{i2} + \cdots + a_{in}A_{in}.$$

$$= 0\,A_{i1} + 0\,A_{i2} + \cdots + 0\,A_{in}.$$

$$|A| = 0$$

(2) If the ith row or jth column of A is multiplied by a scalar k and let that new matrix be B. Then,

$$|B| = k|A|$$

Note :

$$|kA| = k^n |A|$$. If A is $n \times n$ matrix

(3) If the matrix B formed by interchanging two rows or columns of A, then

$$|B| = -|A|$$

(4) If A has two equal rows or columns, then

$$|A| = 0$$

<u>Proof</u>:

Suppose the ith & jth rows of A are equal. Let B be the matrix formed by interchanging these rows.

$$A = B \quad \text{--- (1)}$$

But; $|B| = -|A|$ (Property 3.)

$\therefore$ $|A| = |B|$ (from (1).)

$|A| = -|A|$

$2|A| = 0$

$|A| = 0$

(5) If the matrix B formed by adding a constant times one row (or column) of A to another row (or column) of A, then

$$|A| = |B|.$$

Note: If one row (column) of A is a scalar multiple of another row (column), then

$$|A| = 0$$

(6) $|A^T| = |A|$

(7) $|AB| = |A|\,|B|$

<u>Note</u>: $|A+B| \neq |A| + |B|$

<u>Ex</u>. Find $|A| = \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 2 & 1 & 9 & 6 \\ 3 & 2 & 4 & 8 \end{vmatrix}$

$\xrightarrow[r_4 \to r_4 + (-3)r_1]{r_3 \to r_3 + (-2)r_1} \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & -5 & -1 & 2 \\ 0 & -7 & -1 & 2 \end{vmatrix}$

$\xrightarrow[r_4 \to r_4 + (-7)r_2]{r_3 \to r_3 + (-5)r_2} \begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & -16 & -18 \\ 0 & 0 & -32 & -26 \end{vmatrix}$

$$=(-16)\begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 9/8 \\ 0 & 0 & -32 & -26 \end{vmatrix} \quad \left(\because 2^{nd} \text{ propedng}\right)$$

$$r_4 \longrightarrow r_4 + (32) r_3 \Rightarrow (-16)\begin{vmatrix} 1 & 3 & 5 & 2 \\ 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 9/8 \\ 0 & 0 & 0 & 10 \end{vmatrix}$$

$$|A| = (-16)\left[1\begin{vmatrix} -1 & 3 & 4 \\ 0 & 1 & 9/8 \\ 0 & 0 & 10 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 & 2 \\ 0 & 1 & 9/8 \\ 0 & 0 & 10 \end{vmatrix} + 0\begin{vmatrix} 3 & 5 & 2 \\ -1 & 3 & 4 \\ 0 & 0 & 10 \end{vmatrix} - 0\begin{vmatrix} 3 & 5 & 2 \\ -1 & 3 & 4 \\ 0 & 1 & 9/8 \end{vmatrix}\right]$$

$$=(-16)\begin{vmatrix} -1 & 3 & 4 \\ 0 & 1 & 9/8 \\ 0 & 0 & 10 \end{vmatrix}$$

$$|A| = (-16)(-1)(1)(10)$$

$$|A| = 160$$

Ex. Random

$$\begin{vmatrix} bc & b+c & 1 \\ ca & a+c & 1 \\ ab & a+b & 1 \end{vmatrix}$$

$$\xrightarrow[r_3 \to r_3 + (-1)r_1]{r_2 \to r_2 + (-1)r_1} \begin{vmatrix} bc & b+c & 1 \\ c(a-b) & (a-b) & 0 \\ b(a-c) & a-c & 0 \end{vmatrix}$$

$$= (a-b)(a-c) \begin{vmatrix} bc & b+c & 1 \\ c & 1 & 0 \\ b & 1 & 0 \end{vmatrix}$$

$$= (a-b)(a-c)(c-b)$$

Theorem

If A is invertible, then

$|A| \neq 0$ and.

$$|A^{-1}| = \frac{1}{|A|}$$

The Adjoint of matrix A.

Let A be an $n \times n$ matrix.
Then the adjoint of A, denoted by adj A, is the transpose of the matrix obtained by replacing each element of A with its cofactor.

i.e. $\text{adj } A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & & A_{nn} \end{pmatrix}^T_{n \times n}$

<u>Ex</u>

If $A = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & -1 \\ 3 & 5 & 7 \end{pmatrix}$, find

~~adj~~ adj A.

$\text{adj } A = \begin{pmatrix} 12 & -3 & 3 \\ -13 & 5 & 2 \\ -7 & 2 & 2 \end{pmatrix}^T$

$\text{adj } A = \begin{pmatrix} 12 & -13 & -7 \\ -3 & 5 & 2 \\ 3 & 2 & 2 \end{pmatrix}$

Theorem

Let A be $n \times n$ matrix. Then

$$A\,(\operatorname{adj} A) = |A|\, I$$

Theorem

Let A be $n \times n$ matrix. Then A is invertible if and only if $|A| \neq 0$.

If $|A| \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{|A|}\,(\operatorname{adj} A)}$$

Suppose A is invertible.

$$\Rightarrow \quad |A| \neq 0$$

$$A\,(\operatorname{adj} A) = |A|\, I$$

$$\frac{1}{|A|} A\,(\operatorname{adj} A) = \frac{|A|}{|A|}\, I \quad (\text{Since } |A| \neq 0)$$

$$\frac{1}{|A|} A \cdot \operatorname{adj}(A) = I$$

$$\times A^{-1} \Rightarrow \frac{1}{|A|} \cdot \underbrace{A^{-1} \cdot A}(\operatorname{adj} A) = A^{-1} I$$

$$\frac{1}{|A|} I \cdot (\operatorname{adj}(A)) = A^{-1}$$

$$A^{-1} = \frac{1}{|A|} (\operatorname{adj} A)$$

Ex.

Let $A = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & -1 \\ 3 & 5 & 7 \end{pmatrix}$.

Determine whether A is invertible and calculate A^{-1} if it is.

Sol^{n}: det $A = 3 \neq 0$. Therefore by the previous theorem A is invertible. Also,

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$= \frac{1}{3}\begin{pmatrix} 12 & -13 & -7 \\ -3 & 5 & 2 \\ -3 & 2 & 2 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 4 & -13/3 & -7/3 \\ -1 & 5/3 & 2/3 \\ -1 & 2/3 & 2/3 \end{pmatrix}$$

Cramer's Rule

Consider the system of n linear equations in n-unknowns. Let the coefficent matrix be $A_{n \times n}$ and suppose that $\det A \neq 0$. Then the unique solution to the system $A\underline{x} = \underline{b}$ is given by

$$x_1 = \frac{|\Delta_1|}{|A|}, \quad x_2 = \frac{|\Delta_2|}{|A|}, \ldots$$

$$\ldots x_n = \frac{|\Delta_n|}{|A|}$$

where Δ_i is the matrix obtained by replacing the ith column of A by the column vector $\underline{b}$.

Ex

Using Cramer's rule solve the system.

$$2x_1 + 4x_2 + 6x_3 = 18$$
$$4x_1 + 5x_2 + 6x_3 = 24$$
$$3x_1 + x_2 - 2x_3 = 4$$

$$|A| = \begin{vmatrix} 2 & 4 & 6 \\ 4 & 5 & 6 \\ 3 & 1 & -2 \end{vmatrix} = 6$$

$\therefore |A| \neq 0 \implies$ system has a unique solution,

$$x_1 = \frac{|\Delta_1|}{|A|} = \frac{\begin{vmatrix} 18 & 4 & 6 \\ 24 & 5 & 6 \\ 4 & 1 & -2 \end{vmatrix}}{6} = \frac{24}{6} = 4.$$

$$x_2 = \frac{|\Delta_2|}{|A|} = \begin{vmatrix} 2 & 18 & 6 \\ 4 & 24 & 6 \\ 3 & 4 & -2 \end{vmatrix}$$

$$= \frac{-12}{6} = -2$$

$$x_3 = \frac{|\Delta_3|}{|A|} = \frac{\begin{vmatrix} 2 & 4 & 18 \\ 4 & 5 & 24 \\ 3 & 1 & 4 \end{vmatrix}}{|A|}$$

$$x_3 = \frac{18}{6} = 3$$

$$\left\{ \begin{array}{l} x_1 = 4 \\ x_2 = -2 \\ x_3 = 3 \end{array} \right\}$$

Ex. Prove that:

① $|AB| = |BA|$; where A & B are $n\times n$ matrices

② $|A^2| = |A|^2$

Proof.

① Let A & B are $n\times n$ matrices.

$|AB| = |A||B|$ —— ① ($\because$ 7th property)

$|BA| = |B||A|$ —— ②

Since $|A|$ & $|B|$ are scalars, $|A||B| = |B||A|$.

Therefore by ① & ②.

$|AB| = |BA|$

② Clearly

$|A^2| = |A\cdot A| = |A||A|$

$= |A|^2$

Ex Raedn

$$\begin{vmatrix} x^2 & x & 1 \\ 1 & x^2 & x \\ x & 1 & x^2 \end{vmatrix} \xrightarrow{r_1 \to r_1 + r_2 + r_3}$$

$$\begin{vmatrix} x^2+x+1 & x^2+x+1 & x^2+x+1 \\ 1 & x^2 & x \\ x & 1 & x^2 \end{vmatrix}$$

$$= (x^2+x+1) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x^2 & x \\ x & 1 & x^2 \end{vmatrix} \xrightarrow[C_3 \to C_3 - C_1]{C_2 \to C_2 - C_1}$$

$$= (x^2+x+1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & x^2-1 & x-1 \\ x & 1-x & x(x-1) \end{vmatrix}$$

$$= (x^2+x+1)(x-1) \begin{vmatrix} 1 & 0 & 0 \\ 1 & (x-1)(x+1) & 1 \\ x & (1-x) & x \end{vmatrix}$$

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$$= (x^2+x+1)(x-1)^2 \begin{vmatrix} 1 & 0 & 0 \\ 1 & (x+1) & 1 \\ x & -1 & x \end{vmatrix}$$

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$$= (x^2+x+1)(x-1)^2 \left[1 \cdot x(x+1) + 1 \right]$$

$$= (x^2+x+1)^2 (x-1)^2$$

Ex
Solve for x, $\begin{vmatrix} x-3 & 1 & -1 \\ -7 & x+5 & -1 \\ -6 & 6 & x+4 \end{vmatrix} = 0$

$$C_1 \longrightarrow C_1 + C_2 \begin{vmatrix} x-2 & 1 & -1 \\ x-2 & x+5 & -1 \\ 0 & 6 & x+4 \end{vmatrix}$$

$$= (x-2) \begin{vmatrix} 1 & 1 & -1 \\ 1 & x+5 & -1 \\ 0 & 6 & x+4 \end{vmatrix} \xrightarrow{r_2 \to r_2 - r_1}$$

$$(x-2)\begin{vmatrix} 1 & 1 & -1 \\ 0 & x+4 & 0 \\ 0 & 6 & x+4 \end{vmatrix}$$

$$=(x-2)(x+4)\begin{vmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 6 & x+4 \end{vmatrix}$$

$$=(x-2)(x+4)\left[(x+4)-0\right]$$

$$\Rightarrow (x-2)(x+4)^2 = 0$$

$$x = 2, -4, -4$$

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