2.6 Ratio and Root Tests

2.6.1 Absolute Convergence

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called absolutely convergent if the series of absolute values $\sum_{n=1}^{\infty} |a_n|$ is convergent.

Example (14)

Show that the following series are absolutely convergent or not.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

2.6.2 Conditionally Convergence

Definition

A series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent if it is convergent but not absolutely convergent.

Theorem

If a series $\sum_{n=1}^{\infty} a_n$ *is absolutely convergent, then it is convergent.*

Example (15)

Determine whether the series

$$\sum_{n=1}^{\infty} \frac{\cos n}{n^2}$$

is convergent or divergent.

2.6.3 Ratio Test

1. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
2. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L > 1$ or $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = 1$ the Ratio Test is inconclusive; that is, no conclusion can be drawn about the convergence or divergence of $\sum_{n=1}^{\infty} a_n$.

Example (16)

Test the series $\sum_{n=1}^{\infty}(-1)^n \frac{n^3}{3^n}$ for absolute convergence.

Example (17)

Test the convergence of the series $\sum_{n=1}^{\infty} \frac{n^n}{n!}$.

2.6.4 Root Test

1. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$, then the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent (and therefore convergent).
2. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.
3. If $\lim_{n\to\infty} \sqrt[n]{|a_n|} = 1$ the Root Test is inconclusive.

Example (18)

Test the convergence of the series $\displaystyle\sum_{n=1}^{\infty}\left(\frac{2n+3}{3n+2}\right)^{n}$.

2.7 Power Series

2.7.1 Power Series

Definition

A power series is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots \tag{1}$$

where x is a variable and the c_n's are constants called the coefficients of the series.

- For each fixed x, the series (1) is a series of constants that we can test for convergence or divergence.
- A power series may converge for some values of x and diverge for other values of x.
- The sum of the series is a function

$$f(x) = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

whose domain is the set of all x for which the series converges.
- Notice that f resembles a polynomial.
- The only difference is that f has infinitely many terms.

- For instance, if we take $c_n = 1$ for all n, the power series becomes the geometric series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots + x^n + \dots$$

which converges when $-1 < x < 1$ and diverges when $|x| \geq 1$.

- If we put $x = \frac{1}{2}$, we get the convergent series

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$$

but if we put $x = 2$, we get the divergent series

$$\sum_{n=0}^{\infty} 2^n = 1 + 2 + 4 + 8 + 16 + \dots$$

- More generally, a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

  is called a power series in $(x-a)$ or a power series centered at a or a power series about a.

- Notice that when $x = a$, all of the terms are 0 for $n \geq 1$ and so the power series always converges when $x = a$.

Example (19)

1. For what values of x is the series
$$\sum_{n=0}^{\infty} n!x^n$$
convergent?

2. For what values of x does the series
$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n}$$
converge?

Theorem

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, *there are only three possibilities:*

(i) *The series converges only when* $x = a$.

(ii) *The series converges for all* x.

(iii) *There is a positive number* R *such that the series converges if* $|x-a| < R$ *and diverges if* $|x-a| > R$.

The number R *in case (iii) is called the* *radius of convergence* *of the power series.*

- By convention, the radius of convergence is $R = 0$ in case (i) and $R = \infty$ in case (ii).
- The interval of convergence of a power series is the interval that consists of all values of x for which the series converges.
- In case (i) the interval consists of just a single point a.
- In case (ii) the interval is $(-\infty, \infty)$.
- In case (iii) note that the inequality $|x - a| < R$ can be rewritten as $a - R < x < a + R$.
- When x is an *endpoint* of the interval, that is, $x = a \pm R$, anything can happen—the series might converge at one or both endpoints or it might diverge at both endpoints.
- Thus in case (iii) there are four possibilities for the interval of convergence:

$$(a-R, a+R) \quad (a-R, a+R] \quad [a-R, a+R) \quad [a-R, a+R]$$

Example (20)

Find the radius of convergence and interval of convergence of the series

1. $$\sum_{n=0}^{\infty} \frac{(-3)^n x^n}{\sqrt{n+1}}.$$
2. $$\sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}.$$

2.7.2 Representation of Functions as Power Series

- In this section we learn how to represent certain types of functions as sums of power series by manipulating geometric series or by differentiating or integrating such a series.
- We start with an equation that we have seen before:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + ... = \sum_{n=0}^{\infty} x^n \qquad |x| < 1$$

Example (21)

1. Express $\dfrac{1}{(1+x^2)}$ as the sum of a power series and find the interval of convergence.
2. Find a power series representation for $\dfrac{1}{x+2}$.
3. Find a power series representation for $\dfrac{x^3}{x+2}$.

2.7.3 Differentiation and Integration of Power Series

- The sum of a power series is a function $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$ whose domain is the interval of convergence of the series.
- We would like to be able to differentiate and integrate such functions, and the following theorem says that we can do so by differentiating or integrating each individual term in the series, just as we would for a polynomial.
- This is called term-by-term differentiation and integration.

Theorem

If the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ *has radius of convergence* $R > 0$, *then the function* f *defined by*

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + ... = \sum_{n=0}^{\infty} c_n(x-a)^n$$

is differentiable (and therefore continuous) on the interval $(a-R, a+R)$ *and*

(i) $$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + ... = \sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$$

(ii) $$\int f(x)\,dx = C + c_1\frac{(x-a)^2}{2} + ... = C + \sum_{n=0}^{\infty} c_n\frac{(x-a)^{n+1}}{n+1}$$

The radii of convergence of the power series in Equations (i) and (ii) are both R.

Equations (i) and (ii) in Theorem can be rewritten in the form

1. $$\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}[c_n(x-a)^n]$$
2. $$\int \left[\sum_{n=0}^{\infty} c_n(x-a)^n\right] dx = \sum_{n=0}^{\infty} \int [c_n(x-a)^n] \; dx$$

Example (22)

1. Express $\dfrac{1}{(1-x)^2}$ as a power series by differentiating $\dfrac{1}{1-x}$. What is the radius of convergence?
2. Find a power series representation for $\ln(1+x)$ and its radius of convergence.

2.8 Taylor series and Maclaurin Series

2.8.1 Introduction

- Suppose that f is any function that can be represented by a power series

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots \qquad |x-a| < R$$

- Let's try to determine what the coefficients c_n must be in terms of f.
- To begin, notice that if we put $x = a$ in above equation, then all terms after the first one are 0 and we get

$$f(a) = c_0$$

- We can differentiate the series in above equation term by term:

$$f'(x) = c_1+2c_2(x-a)+3c_3(x-a)^2+4c_4(x-a)^3+... \quad |x-a| < R$$

and substitution of $x = a$ gives

$$f'(a) = c_1$$

- Now we differentiate both sides of Equation and obtain

$$f''(x) = 2c_2 + 2\cdot 3c_3(x-a) + 3\cdot 4c_4(x-a)^2 + ... \quad |x-a| < R$$

Again we put $x = a$. The result is

$$f''(a) = 2c_2$$

- Let's apply the procedure one more time.
- Differentiation of the series in Equation gives

$$f'''(x) = 2\cdot 3c_3 + 2\cdot 3\cdot 4c_4(x-a) + 3\cdot 4\cdot 5c_5(x-a)^2 + \ldots \quad |x-a| < R$$

and substitution of $x = a$ gives

$$f'''(x) = 2 \cdot 3c_3 = 3!c_3$$

- If we continue to differentiate and substitute $x = a$, we obtain

$$f^{(n)}(a) = 2 \cdot 3 \cdot 4 \cdot \ldots \cdot nc_n = n!c_n$$

- Solving this equation for the n^{th} coefficient c_n, we get

$$c_n = \frac{f^{(n)}(a)}{n!}$$

Theorem

If f has a power series representation (expansion) at a, that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad |x-a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}.$$

- Substituting this formula for c_n back into the series, we see that if f has a power series expansion at a, then it must be of the following form.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \ldots$$

- The series is called the Taylor series of the function f at a (or about a or centered at a).

- For the special case $a = 0$ the Taylor series becomes

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 + \ldots$$

- This case arises frequently enough that it is given the special name Maclaurin series.

Example (23)

1. Find the Maclaurin series of the function $f(x) = e^x$ and its radius of convergence.
2. Find the Taylor series for $f(x) = e^x$ at $a = 2$.
3. Find the Maclaurin series for $\sin x$.
4. Find the Maclaurin series for $\cos x$.
5. Find the Maclaurin series for $x \cos x$.