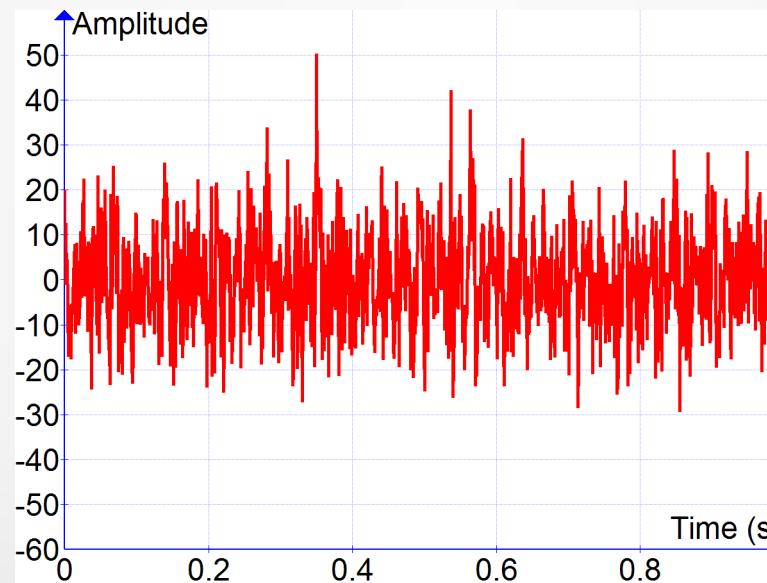


# Introduction

Random Signals & Processes  
Lecture 1  
Eng. (Mrs.) PN Karunananayake

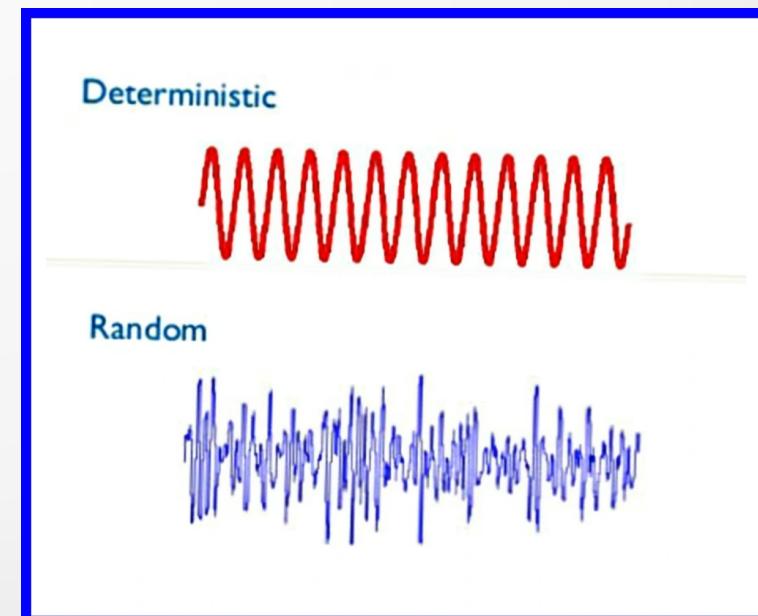
# What is a Signal?

- A signal is a physical quantity propagating through space and time that “contains” information about an event. It propagates from one physical location to another through some medium.



# Random Signal Vs Deterministic Signal

- Signals which can be **defined exactly by a mathematical formula** are known as **deterministic signals**.
- A signal is said to be non-deterministic if **there is uncertainty with respect to its value at some instant of time**.
- Non-deterministic signals are random in nature hence they are called random signals.



# What is a Random Process?



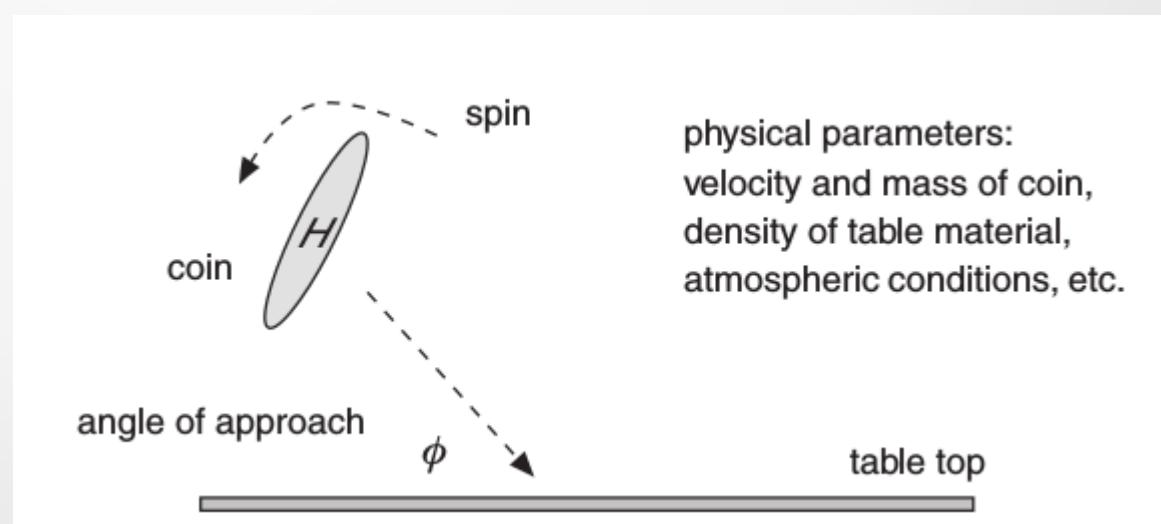
**Random process includes random variable(s) that evolves in time by some random mechanism** (the time variable can be replaced by a space variable, or some other variable, in application). Or a collection of random variables.

The variable can have a discrete set of values at a given time, or a continuum of values may be available.

**Random Process is also known as also called a stochastic process**

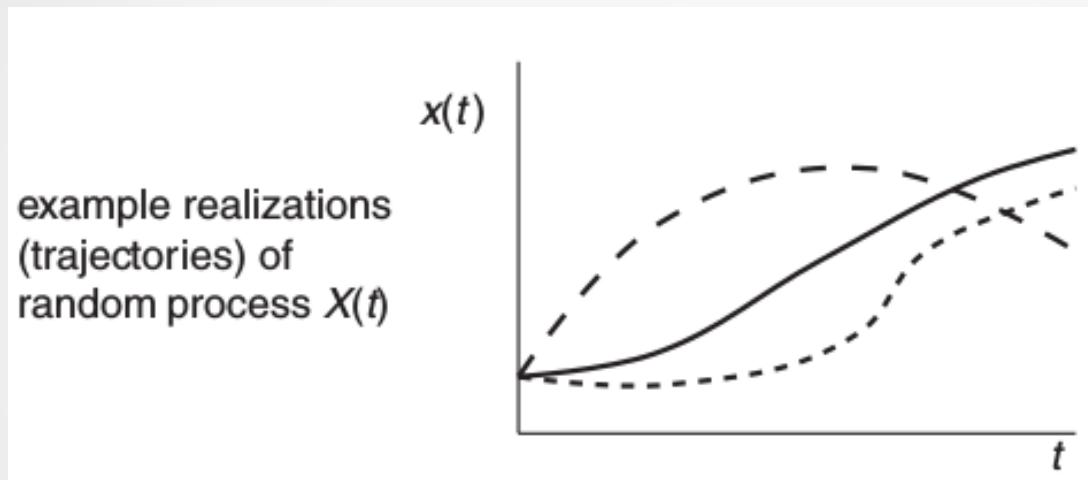
# What is randomness

- A lack of complete information about a physical process.  
Example: Tossing a coin [H,T]
- Required information to predict exact outcome:
  1. Velocity of the coin just before impact
  2. The angle of approach  $\phi$  with respect to the table top
  3. The mass of the coin
  4. Atmospheric conditions (temperature, humidity, and so on), and other relevant physical parameters



# Modelling the Process

- Ex. Tornado in the x-direction as a function of time



Note : A **random process is denoted by  $X(t)$  (uppercase letter)** and a realization of the process by  **$x(t)$  (lowercase letter)**.

The **outcome at a specific time  $t_o$  is also random**: it is a random variable  $X(t_o)$ , and a particular outcome at that time is  $x(t_o)$ .

A **random process is a collection of random variables that are indexed by time**, for one realization  $x(t)$  and for one outcome  $x(t_o)$  at  $t = t_o$  .

# Example Related to Communication

- Task: Recover the sender's message (voice, data, video)
  1. The signal
  2. The characteristic of a radio channel
  3. The interfering signals
  4. Thermal noise introduced at the antenna and the front-end RF amplifier.

# Axioms of probability

## Sample space

- Mathematical abstraction of the collection of all possible experimental outcomes

## Event

- An event is a set of sample points. Usually events are denoted by capital letters.

## Probability measure

- An assignment of real numbers to the events defined on sample space.

# Sigma Field

- An algebra for events called a sigma field is defined,  
A collection  $\mathcal{F}$  of subsets of  $\Omega$  is called a  $\sigma$ -field, if it satisfies the following properties:
  1.  $\Phi \in \mathcal{F}$
  2. if  $A \in \mathcal{F}$  then  $A^c \in \mathcal{F}$
  3. if  $A_1, A_2, A_3, \dots \in \mathcal{F}$ , then

$$\bigcup_{m=1}^{\infty} A_m \in \mathcal{F}.$$

# Examples

1. The smallest  $\sigma$  -field associated with  $\Omega$  is  $F = \{\emptyset, \Omega\}$ .
2. If  $A$  is a subset of  $\Omega$ , then  $F = \{\emptyset, A, A_c, \Omega\}$  is a  $\sigma$  -field.
3. Consider a sample space defined by  $= \{a, b, c, d\}$ . A set  $C = \{\{a\}, \{b\}\}$  is a subset of  $\Omega$ , but it is not a field.

The complement of a simple event  $\{a\}$  is  $\{a\}_c = \{b, c, d\}$ .

Similarly,  $\{b\}_c = \{a, c, d\}$ .

$$\{a\} \cup \{b\} = \{a, b\} \text{ and } (\{a\} \cup \{b\})_c = \{c, d\}.$$

Then collecting all these subsets of  $\Omega$ , we find that

- $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \Omega\}$  is the smallest  $\sigma$  -field containing all the elements of  $C$ .

# Probability measure and probability space

- A probability measure  $P$  defined on  $(\Omega, \mathcal{F})$  is a function that maps any element of  $\mathcal{F}$  into  $[0, 1]$  such that,

(a)  $P[\emptyset] = 0, P[\Omega] = 1$ ;

(b) if  $A_1, A_2, \dots \in \mathcal{F}$  and  $A_m \cap A_n = \emptyset$  ( $m \neq n$ ), then

$$P \left[ \bigcup_{m=1}^{\infty} A_m \right] = \sum_{m=1}^{\infty} P[A_m].$$

*The triple  $(\Omega, \mathcal{F}, P)$  is called a **probability space**.*

# Example

- Experiment of tossing two coins with possibly biased coins.
- The sample space of tossing the first coin (coin “1”) is denoted  $\Omega_1 = \{h, t\}$ . Its  $\sigma$  -field is  $F_1 = \{\emptyset, \{h\}, \{t\}, \Omega_1\}$ .
- $P_1[\emptyset] = 0$ ,  $P_1[\{h\}] = p_1$  ,  $P_1[\{t\}] = 1 - p_1$  , and  $P_1[1] = 1$  where  $p_1$  is a fixed real number in the interval  $[0, 1]$
- The sample space of tossing the second coin (coin “2”) is denoted  $\Omega_2 = \{h, t\}$ . Its  $\sigma$  -field is  $F_2 = \{\emptyset, \{h\}, \{t\}, \Omega_2\}$
- $P_2[\emptyset] = 0$ ,  $P_2[\{h\}] = p_2$  ,  $P_2[\{t\}] = 1 - p_2$  , and  $P_2[1] = 1$  where  $p_2$  is a fixed real number in the interval  $[0, 1]$

## Example

- The sample space of the experiment of tossing the two coins is the Cartesian product of the two sample spaces defined above.

$$\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}.$$

# Bernoulli trials and Bernoulli's theorem

- **Repeated independent trials** are called Bernoulli trials if there are **only two possible outcomes** for each trial and **their probabilities remain the same throughout the trials.**
- The sample space of each individual trial is :

$$\Omega = \{s, f\}$$

- probability of the simple event  $\{s\}$  denoted by  $p$

$$P [\{s\}] = p, \quad 0 \leq p \leq 1,$$

probability of the event  $\{f\}$  is given by

$$P [\{f\}] = 1 - p \triangleq q,$$

- The sample space for an experiment consisting of two independent Bernoulli trials,

$$\Omega^2 = \Omega \times \Omega = \{(ss), (sf), (fs), (ff)\}.$$

**Bernoulli distribution for two trials is given by {p<sup>2</sup> , pq, qp, q<sup>2</sup>}**

Sample space for n Bernoulli trials is the n<sup>th</sup>-fold Cartesian product of  $\Omega$ :

$$\Omega^n = \Omega \times \Omega \times \cdots \times \Omega = \{(ss \dots s), (ss \dots f), \dots, (ff \dots s), (ff \dots f)\}.$$

the probability of the outcome ssf . . . fsf is given by:

$$P[\{ssf \dots fsf\}] = ppq \dots qpq = p^k q^{n-k}$$

where k is the number of successes and n – k is that of failures in a given outcome of n Bernoulli trials

# Binomial coefficient

- If the order in which the successes occur does not matter, then the number of sample points belonging to this event is equal to the number of combinations of  $n$  things taken  $k$  at a time.
- This number is referred to as the binomial coefficient and denoted as:

$$\binom{n}{k} \triangleq \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k \times (k-1) \cdots 2 \times 1}.$$

# Binomial Distribution

- The set of probabilities  $B(k; n, p)$  is called the binomial distribution.

$$B(k; n, p) \triangleq \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

# Joint probability and conditional probability

- Example:

A joint (or compound) experiment that consists of one experiment having possible outcomes  $A_m$  ( $m = 1, 2, \dots, M$ ) and another having the possible outcomes  $B_n$  ( $n = 1, 2, \dots, N$ ) can be considered as a single experiment having the set of possible outcomes  $(A_m, B_n)$ .

- **Probabilities relating to such a combined experiment are known as joint (or compound) probabilities.**
- The joint probability of events A and B is often written as  $P[A, B]$  instead of  $P[A \cap B]$ .

$$0 \leq P[A, B] \leq 1.$$

# Conditional Probability

- The conditional probability that event B occurs given that event A occurs is defined as:

$$P[B | A] \triangleq \frac{P[A, B]}{P[A]},$$

# Bayes' theorem

- A set of events  $A_1, A_2, \dots, A_n$  is called a partition of the sample space if they are a set of mutually exclusive and exhaustive events in ; i.e.,  $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$  and  $A_i \cap A_j = \emptyset$  for  $i \neq j$ .
- Then we obtain for any event B:

$$\bigcup_{j=1}^n \{B \cap A_j\} = B,$$

and then

$$\sum_{j=1}^n P[B, A_j] = \sum_{j=1}^n P[A_j]P[B | A_j] = P[B].$$

# Bayes' theorem

- Let  $B$  be an event in a sample space and  $A_1, A_2, \dots, A_n$  be a partition of . Then it can be shown that

$$P[A_j | B] = \frac{P[A_j] P[B | A_j]}{P[B]} = \frac{P[A_j] P[B | A_j]}{\sum_{i=1}^n P[B | A_i] P[A_i]}.$$

# Example

- Consider some disease and its medical diagnosis test. The following statistics are known about this disease and its medical test.
  - For a person with this disease, the test yields a positive result 99% of the time and a negative result 1%.
  - For a person without this disease, the test yields a negative result 99% of the time and a positive result 1%.
  - Suppose that 1% of the population is infected by this disease and 99% of the population is not.
- Suppose that you have taken this test and, unfortunately, the test result is positive. What is the chance that you are indeed infected by this disease?

# Answer

- Let A represent a person's condition with respect to this disease and B represent their test result:
- $A = \text{"Not infected by the disease"}$
- $A_c = \text{"Infected by the disease"}$
- $B = \text{"Negative test result"}$
- $B_c = \text{"Positive test result."}$

$$P[A] = p, \quad P[A^c] = 1 - p;$$

$$P[B|A] = \alpha, \quad P[B^c|A] = 1 - \alpha;$$

$$P[B^c|A^c] = \beta, \quad P[B|A^c] = 1 - \beta.$$

$$P[A, B] = P[A]P[B|A] = p\alpha;$$

$$P[A, B^c] = p(1 - \alpha);$$

$$P[A^c, B] = (1 - p)(1 - \beta);$$

$$P[A^c, B^c] = (1 - p)\beta.$$

$$P[A^c|B^c] = \frac{P[A^c, B^c]}{P[A, B^c] + P[A^c, B^c]} = \frac{(1 - p)\beta}{p(1 - \alpha) + (1 - p)\beta}.$$

If we substitute  $p = \alpha = \beta = 0.99$ , then

$$P[A^c|B^c] = \frac{0.01 \times 0.99}{0.99 \times 0.01 + 0.01 \times 0.99} = \frac{0.0099}{0.0198} = 0.5.$$

That is, the probability that you have this disease is 50%.

Thank you!