EE1102 – Fundamentals of Electrical Engineering

4.0 Electric and Magnetic Fields

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Electric and Magnetic Fields (4 hrs)

Learning Outcome:

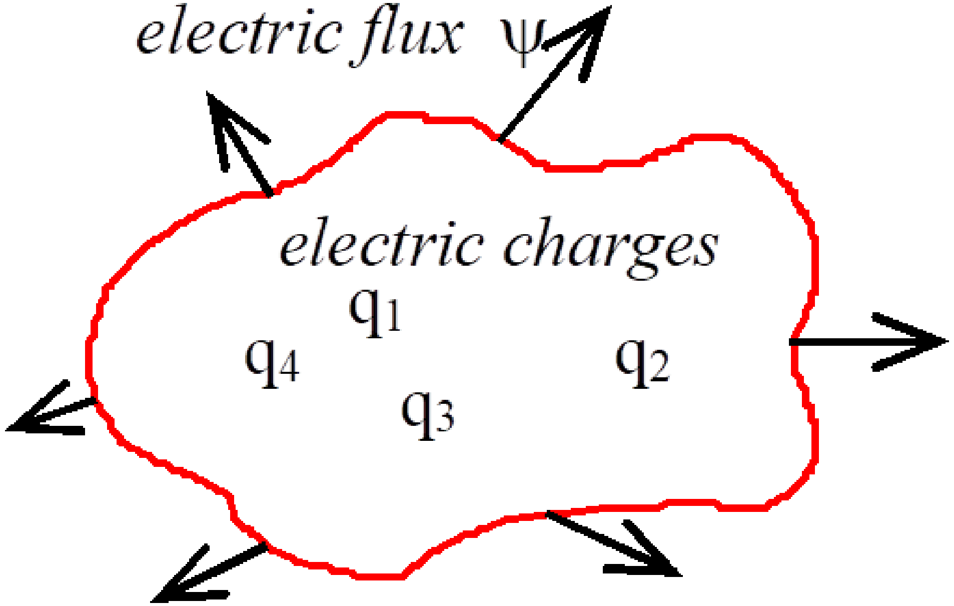
- Demonstrate knowledge of electricity and magnetism
- Determine force and energy in electric and magnetic fields
- Solve problems on capacitance, inductance and flux in a magnetic core

Content:

- Electrostatic Theory - Gauss’s Law, Coulomb’s Inverse Square Law
- Capacitance of dielectric – parallel plate and cylindrical
- Energy stored and force exerted in electric and magnetic fields
- Ohm's law in an electric field
- Electromagnetic Theory - Ampere's Law
- Calculation of flux in a magnetic circuit
- self and mutual inductance

Electrostatic Theory

Amber - Latin word Electrica from Greek word ἤλεκτρον (*ēlektron*)



Electrostatics is based on two laws

i. Gauss's Law

ii. Coulomb's Inverse Square Law

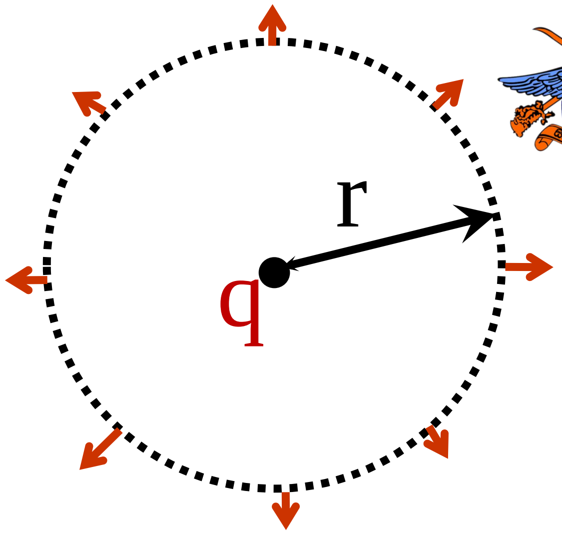
Gauss's Law

Total electric flux coming out of a closed surface is equal to the algebraic sum of the charge enclosed.

$$\psi = \Sigma q$$

[Unit of both electric flux ψ and charge q is coulomb (C)]

Electric flux density D



Amount of electric flux coming out per unit area normal to flux. [Unit: ***coulomb per meter²*** or ***C/m²***]

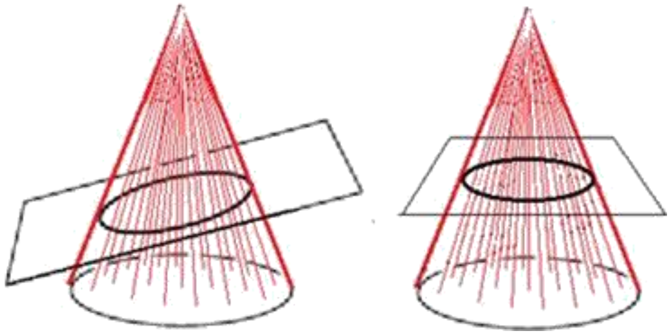
$$D = \psi / A$$

Consider imaginary sphere surrounding charge **q**.

Electric flux $\boldsymbol{\psi = q}$

Cross-section taken perpendicular to direction of flux will go out equally in all directions through a surface area of $\boldsymbol{4\pi r^2}$ normal to this surface.

$\therefore$ electric flux density at radius ***r*** is $$D_r = \frac{q}{4\pi r^2}$$



Cross-section taken perpendicular to direction of flux

Coulomb's Inverse Square Law of Electrostatics

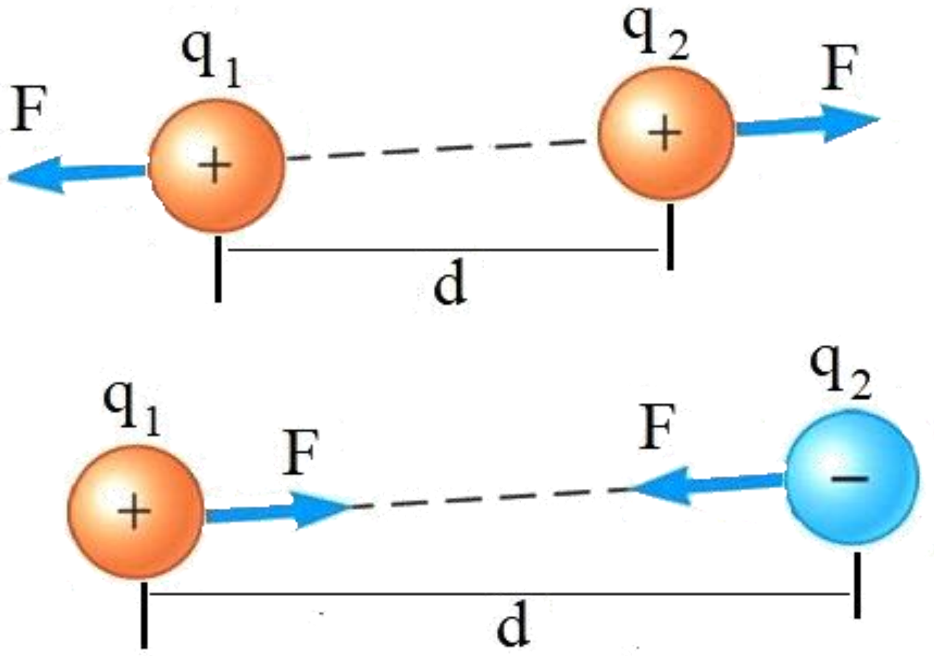
Force exerted by 2 point charges is proportional to

— product of charges

— inversely proportional to square of distance

$$F \propto \frac{q_1 . q_2}{d^2}$$

$$F = k.\frac{q_1 . q_2}{d^2}$$



The Electric field ξ at a point is defined as the force exerted on a unit positive charge placed at that point.

If field is produced by charge q_1, then force acting on a charge q2 = +1 will be at a distance r from the charge.

$$\xi_r = k.\frac{q_1 \times 1}{r^2}$$

Permittivity

Electric flux density $D_r \propto q$ and $\propto \frac{1}{r^2}$, Electric field $\xi_r \propto q$ and $\propto \frac{1}{r^2}$

thus $D_r \propto \xi_r$

Permittivity ε of medium is defined as the ratio of D to ξ. i.e. $D = \varepsilon\, \xi$.

[Unit of permittivity - farad per meter or F/m]

Constant k becomes $1/k = 4\pi\, \varepsilon$

and inverse square law can be stated as $$F = \frac{q_1 . q_2}{4\pi \varepsilon\ d^2}$$

Permittivity of free space is $\varepsilon_0 = 8.854 \times 10^{-12}$ F/m.

Usual to express permittivity as relative value by comparing with free space.

Relative Permittivity (also dielectric constant) ε_r of a material is defined as

$\varepsilon_r = \varepsilon / \varepsilon_0$ or $\varepsilon = \varepsilon_0\, \varepsilon_r$. [$\varepsilon_r$ is always greater than 1.]

[Relative permittivity has no dimension and is thus without a unit].

[In practice relative permittivity of air (1.00060) is taken as unity]

Electric potential difference

Electric potential difference between two points A and B is defined as

- work that must be done, against the forces of an electric field,
- in moving a unit positive charge from point A to point B.

➢ When a unit positive charge is moved an elemental distance ***dx*** in the direction of the force F,

- change in energy $\boldsymbol{dW = F \, . \, dx}$.
- change in potential = $\boldsymbol{dV}$

➢ force acting on a unit positive charge = ξ.

$\therefore \; \boldsymbol{\xi \, . \, dx \;=\; - \, dV}$

or
$$\xi = -\frac{dV}{dx}$$

[Unit for electric field is ***volt per meter*** or ***V/m***]

Capacitance of a dielectric

Capacitance of a dielectric (or insulating material) is the

– ability to store charge
– when placed between two electrodes
– when a potential difference has been applied.

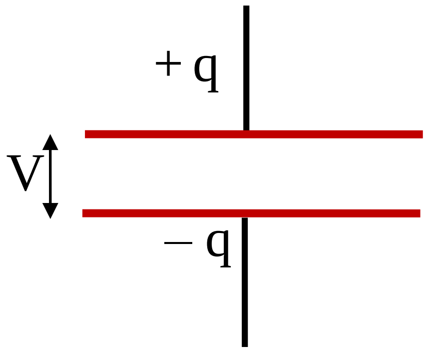
Capacitance is defined as ratio of

– amount of charge transferred
– to the applied potential difference.

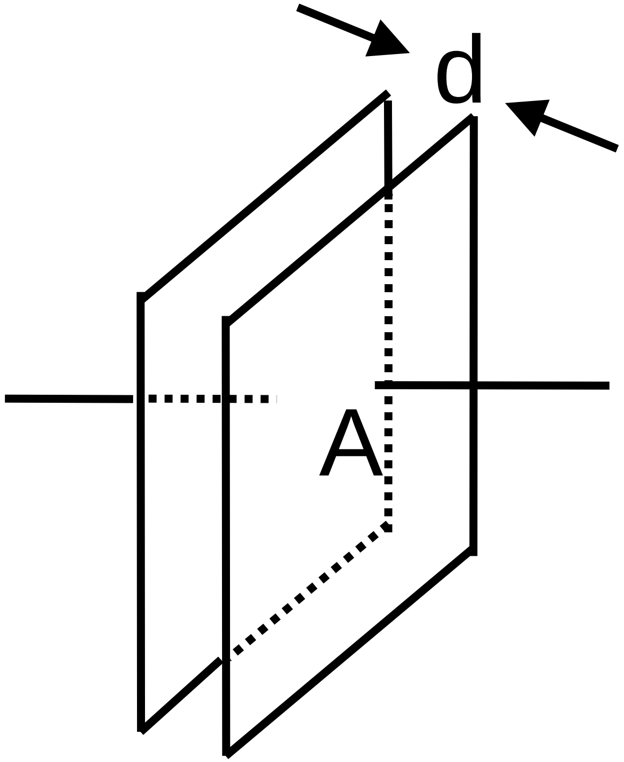
i.e. $\boldsymbol{C = q/V}$ or $\boldsymbol{q = C \,.\, V}$

[Unit of capacitance is the ***farad*** (***F***)]

Parallel plate capacitor



Simplest form of capacitor, where a dielectric (permittivity ε) of thickness **d** separates two parallel electrodes of cross-section area ***A***.



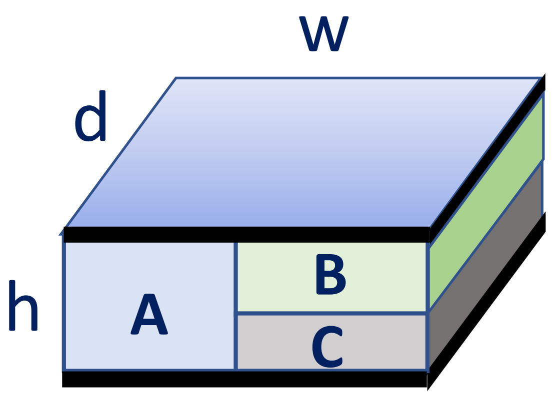
From Gauss's theorem, $\psi = q$

$\therefore$ $D = \psi/A = q/A,$ also $D = \varepsilon . \xi = q$

and for a uniform field, $\xi = dV/dx = V/d$

$$\therefore \quad D = \varepsilon \frac{V}{d} = \frac{q}{A}, \qquad C = \frac{q}{V} = \frac{\varepsilon A}{d}$$

Calculation of Capacitance



A parallel plate capacitor is built up of 3 materials,

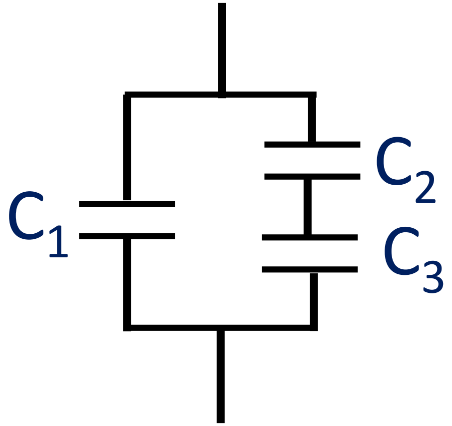
A (relative permittivity ε_{r1}, height h, width w_1 and depth d),

B (relative permittivity ε_{r2}, height h_2, width w_2 and depth d),

C (relative permittivity ε_{r3}, height h_3, width w_2 and depth d),

such that $h_2+h_3 = h$ and $w_1+w_2=w$.

Determine the capacitance of the arrangement.



$$C_1 = \frac{d.w_1\varepsilon_0\varepsilon_{r1}}{h},\quad C_2 = \frac{d.w_2\varepsilon_0\varepsilon_{r2}}{h_2},\quad C_3 = \frac{d.w_2\varepsilon_0\varepsilon_{r3}}{h_3},$$

Therefore total C = C_1 parallel (C_2 series C_3) = $C_1 + \dfrac{C_2C_3}{C_2+C_3}$

$$= \frac{d.w_1\varepsilon_0\varepsilon_{r1}}{h} + \frac{d.w_2\varepsilon_0\varepsilon_{r2}\varepsilon_{r3}}{\varepsilon_{r2}h_3+\varepsilon_{r3}h_2}$$

with $h_2+h_3 = h$ and $w_1+w_2=w$

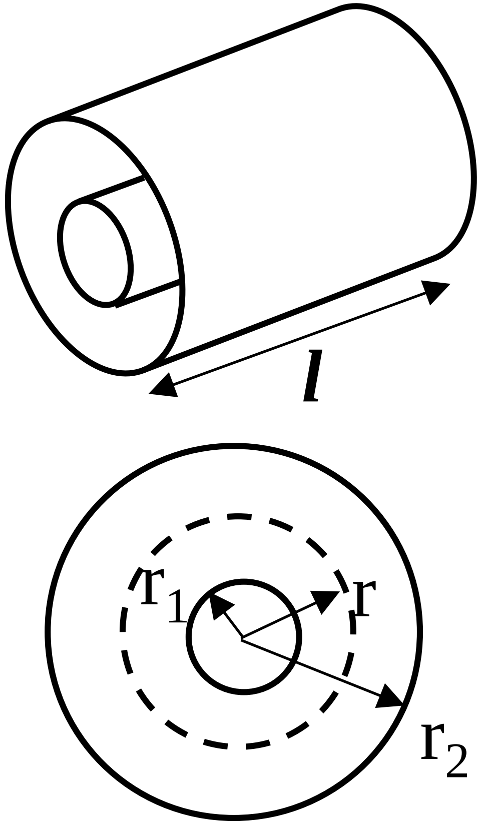
Cylindrical plate capacitor

From Gauss's theorem, $\psi = q$

$D_r = \psi / A = q/2\pi r\, l$, also $D = \varepsilon\, \xi = q/A$

and $\xi = - dV/dr$

$$-\frac{dV}{dr} = \frac{q}{2\pi\,\varepsilon\, r l} \qquad \int_{V_1}^{V_2} dV = \int_{r_1}^{r_2} \frac{q}{2\pi\varepsilon\ l}\frac{dr}{r}$$



Integration gives

$$-V_1 + V_2 = \frac{q}{2\pi\,\varepsilon\, l} \ln\left[\frac{r_2}{r_1}\right]$$

$$C = \frac{2\pi\,\varepsilon\, l}{\ln\left[r_2 / r_1\right]}$$

Energy stored in an electric field

For a parallel plate capacitor of cross-section ***A*** and spacing ***d***

Energy stored in a capacitor = $\int \boldsymbol{v \cdot i \cdot dt} = \int \boldsymbol{v \cdot dq} = \int \boldsymbol{v \cdot C \cdot dv} = \tfrac{1}{2}\boldsymbol{Cv^2}$

Energy stored/unit volume = $\int \frac{v \cdot dq}{\text{volume}} = \int \frac{v \cdot dq}{d.A} = \int \xi \cdot dD = \int \frac{D}{\varepsilon} \cdot dD$

$= \tfrac{1}{2}\frac{\boldsymbol{D}^2}{\varepsilon} = \tfrac{1}{2}\boldsymbol{D} \cdot \xi$ J/m^3

Force exerted in an electric field

Move electrodes of parallel plate capacitor to change spacing by *dx*

Change in energy stored = ½ *D.*ξ ×(change in volume) = ½ *D.*ξ *.A. dx*

Change in energy stored = work done = *F . dx*

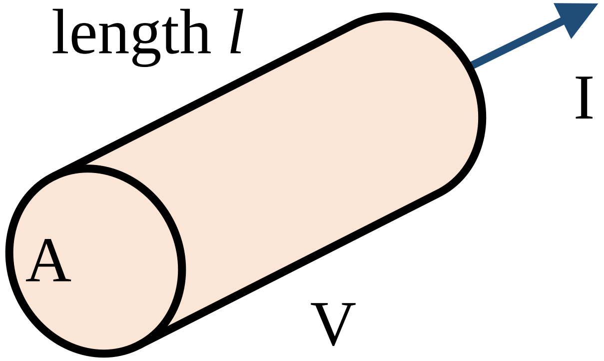
$\therefore$ *F . dx* = ½ *D.*ξ *A. dx* or *F* = ½ *D.*ξ*.A*

Force exerted on unit area in an electric field = *F/A* = ½ *D.*ξ N/m^2

Ohm's Law in an electric field

For cylindrical volume, for a current flow I

Current density in conducting medium $\boldsymbol{J = I/A}$



for a uniform field in the medium

$\xi = -dV/dx = V/l$

Also, since $R = \rho\, l/A$ and $V = R\,.\, I$,

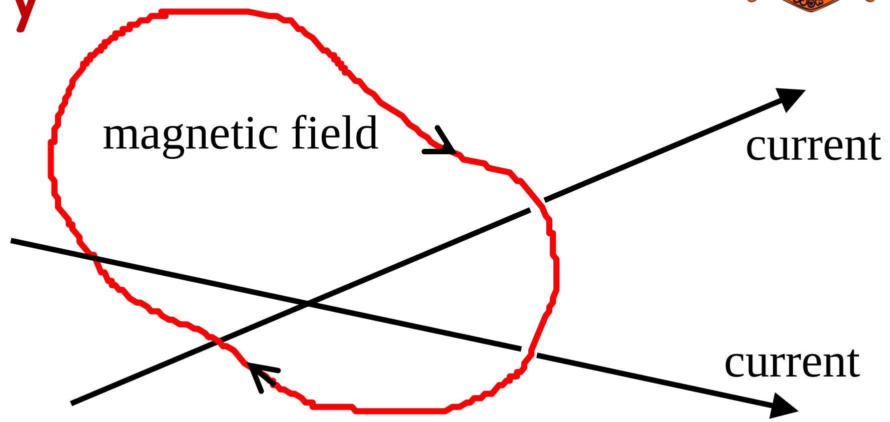
$$\xi \cdot \boldsymbol{l} = \frac{\rho\, \boldsymbol{l}}{\boldsymbol{A}} \cdot \boldsymbol{J} \cdot \boldsymbol{A}$$

This gives Ohm's law for an electric field as $\boldsymbol{\xi = \rho\,.\,J}$

or more commonly $\boldsymbol{J = \sigma\,.\,\xi}$

where conductivity $\sigma = 1/\rho$

Electromagnetic Theory



Basic theorem is **Ampere's Law**

Ampere's Law states that the line integral of the magnetic field ***H*** taken around a closed path is equal to the total current enclosed by the path.

$$\oint \boldsymbol{H} \cdot d\boldsymbol{l} = \sum \boldsymbol{I}$$

For a uniform field, ***H*** is a constant and $\boldsymbol{H} \,.\, \boldsymbol{l} = \Sigma \boldsymbol{I}$

or if ***H*** is constant over sections, then $\Sigma \boldsymbol{H} \,.\, \boldsymbol{l} = \Sigma \boldsymbol{I}$

[Unit of magnetic field is ***ampere per meter (A/m)***]

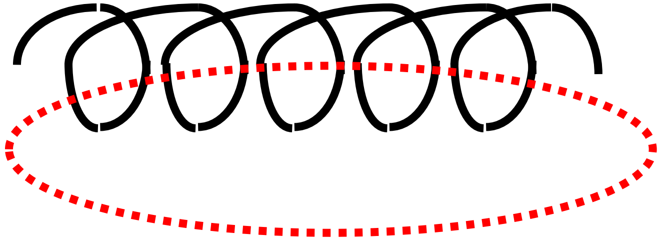
Magnetomotive force (mmf)

Magnetomotive force is flux producing ability of electric current in a magnetic circuit. [Similar to electromotive force in an electric circuit].

[Unit of magnetomotive force is ***ampere (A)***]

Note: Although some use *ampere-turns*, it is wrong as *turns* is not a unit

$$\text{mmf } \boldsymbol{\Im = \Sigma I}$$



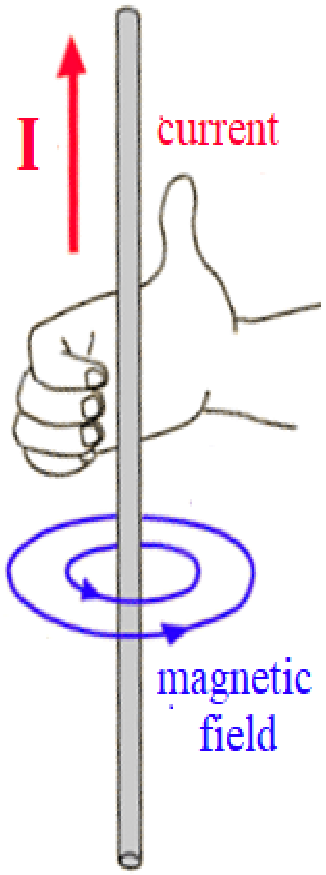
Flux path

Consider a coil having $\boldsymbol{N}$ turns

It will link the flux path with each turn, so that total current linking with the flux would be $\boldsymbol{\Sigma I = N.I}$

Thus from Ampere's Law, the **mmf** produced by a coil of $\boldsymbol{N}$ turns would be $\boldsymbol{N\,I}$, and $\boldsymbol{N\,I = H\,l}$.

Field produced by a long straight conductor



If a circular path of radius r is considered around a conductor carrying a current I,

then the field H_r along this path would be constant by symmetry.

For a single conductor, N = 1

$\therefore$ by Ampere's Law, $1.I = H_r.2\pi r$

$$H_r = \frac{I}{2\pi r}$$ at radial distance r from conductor.

Field produced inside a toroid

A toroid is wound uniformly with ***N*** turns.

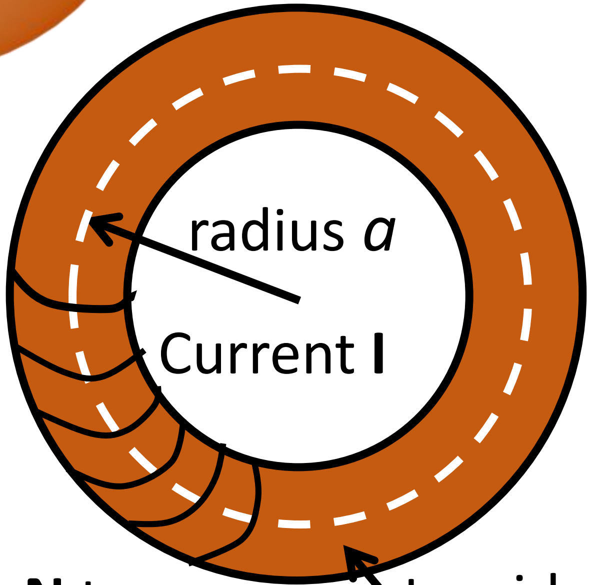
mean radius of magnetic path = ***a***,

then magnetic path length = ***2π a***,

total mmf produced = ***N I***.

From Ampere's Law

magnetic field is $\boldsymbol{H} = \dfrac{\boldsymbol{NI}}{2\pi\, \boldsymbol{a}}$ inside the toroid.



[variation of magnetic field inside the cross section of the toroid is usually not considered and is assumed uniform].

Magnetic flux density

Magnetic field ***H*** gives rise to a magnetic flux ϕ,

[Unit of magnetic flux - ***weber (Wb)***]

For a given area ***A*,** flux ϕ has a magnetic flux density $\boldsymbol{B} = \phi/A$

[Unit of magnetic flux density - ***tesla (T)***]

Relationship between ***B*** and ***H*** is given by permeability μ.

$$\mathbf{B} = \mu \, \mathrm{H}, \quad \text{where } \mu = \mu_o \, \mu_r,$$

μ_r = relative permeability

$\mu_o = 4\pi \times 10^{-7}$ H/m is permeability of free space

[permeability of air taken equal to that of free space, as relative permeability = 1.00000037]

[Unit of permeability is ***henry per meter (H/m)***].

Reluctance of a magnetic path

Reluctance ***S*** is offered to flow of magnetic flux when mmf is applied.

Thus $\boldsymbol{mmf = Reluctance \times flux}$ or $\mathfrak{I} = \boldsymbol{S} . \phi$

For a uniform field, $\mathfrak{I} = N\,I = H.l,\quad and \quad \phi = B.\,A = \mu\, H.\,A$

$\therefore\ H.l = S\,.\,\mu\, H.\,A \quad \rightarrow$ magnetic reluctance $\boldsymbol{S = \dfrac{l}{\mu\, A}}$

where ***l*** = length and ***A*** = cross-section

[Unit of magnetic reluctance is ***henry***$^{-1}$ (***H***$^{-1}$)]

Magnetic Permeance Λ is the inverse of the magnetic reluctance.

Thus $\Lambda = \dfrac{\mathbf{1}}{\boldsymbol{S}} = \dfrac{\boldsymbol{\mu\, A}}{\boldsymbol{l}}$

[Unit of magnetic permeance is ***henry (H)***]

Determination of Flux in a Magnetic Circuit

Each outer limb – A_o, L_o, μ

Middle limb – A_m, L_m, μ

Air gap – A_a, L_a, μ_o

Assuming uniform field, Reluctance S = L/ μA

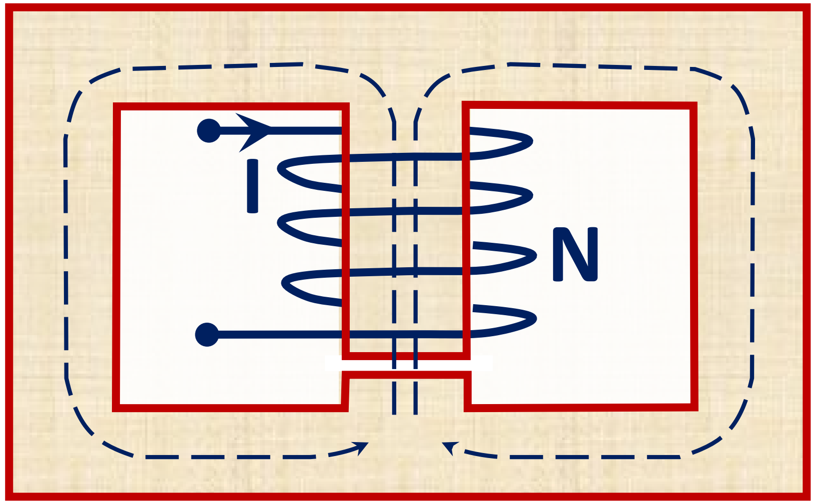
Thus $\Im$, S_o, S_m, and S_a are calculated as

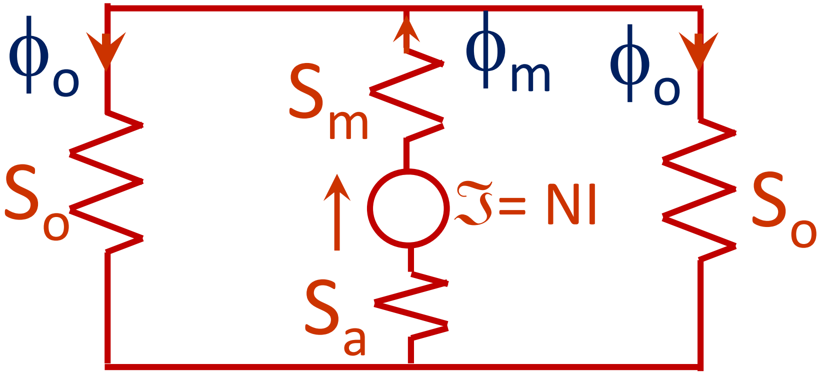
$\Im = N \, . \, I$

$S_o = L_o/\mu_o\mu_r A_o$

$S_m = L_m/\mu_o\mu_r A_m$ and

$S_a = L_a/\mu_o A_a$



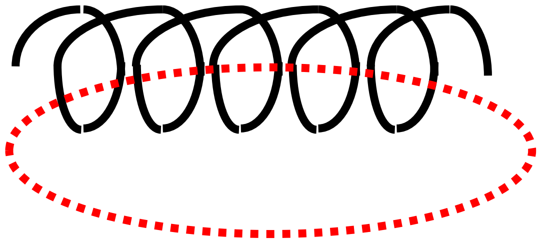


Magnetic equivalent circuit can be drawn as shown.

Due to symmetry, flux $\phi_m = 2\phi_o$ and calculated as in resistive circuit.

$$\emptyset_m = \frac{N.I}{S_m + S_a + \frac{1}{2}S_o} \quad \text{and} \quad \emptyset_o = \frac{N.I}{2S_m + 2S_a + S_o}$$

Self Inductance



While reluctance is a property of magnetic circuit, the corresponding electrical quantity is inductance.

Induced emf $$e = N\frac{d\phi}{dt} = L\frac{d\,i}{dt}\ ,\qquad N\,\phi = L\,i,\qquad L = \frac{N\phi}{i}$$

Self inductance ***L*** of a winding is the flux linkage produced in the same winding due to unit current flowing through it.

For ***N*** turn coil, if magnetic flux is ϕ, flux linkage with coil is $N.\phi$.

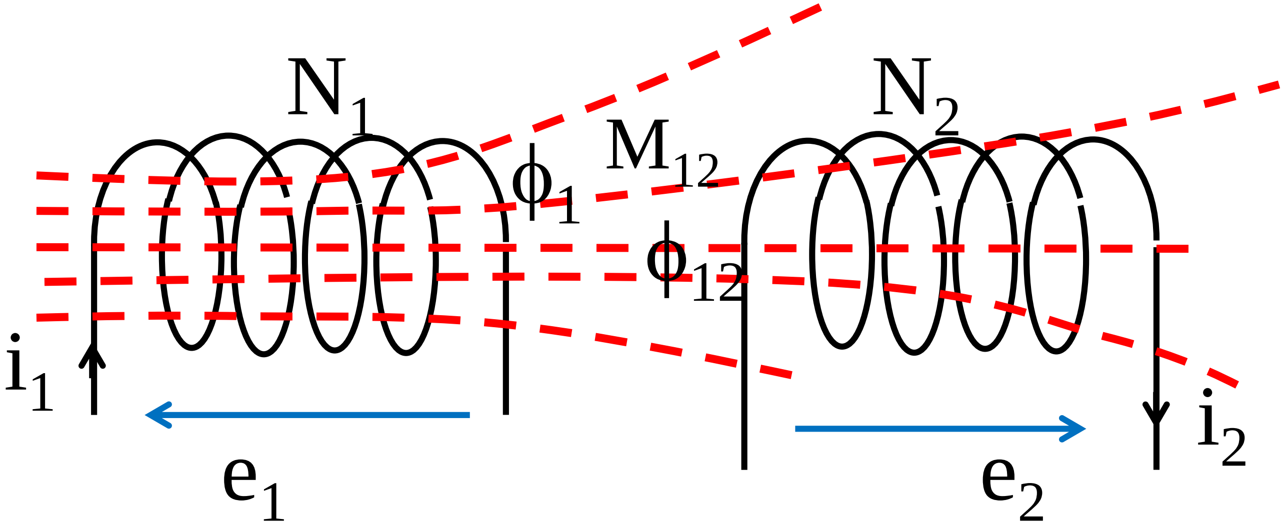
also since $N\,I = S\,\phi$,

$$L = \frac{N^2}{S} = \frac{N^2 \mu\, A}{l}$$

Thus inductance of a coil of ***N*** turns can be determined from the dimensions of the magnetic circuit.

Mutual Flux

Mutual coupling takes place when 2 coils are in the vicinity of each other's magnetic circuit



- Coil 1 produces a flux ϕ_1, part of which ϕ_{12} links with coil 2.
- when current in coil 1 varies, induced emf e_2 occurs in coil 2.

$$e_2 = N_2 \frac{d\phi_{12}}{dt} = M_{12} \frac{d\,i_1}{dt}$$

Where M_{12} is the Mutual Inductance between 1 and 2.

Mutual Inductance

Since $N_2 \frac{d\phi_{12}}{dt} = M_{12} \frac{di_1}{dt}$, $\quad N_2 \phi_{12} = M_{12}\, i_1 \quad \rightarrow \quad \boldsymbol{M_{12} = \frac{N_2 \phi_{12}}{i_1}}$

Mutual inductance $\boldsymbol{M}_{12}$, of coil 2 due to a current in coil 1, is the flux linkage in the coil 2 due to unit current flowing in coil 1.

also since $\boldsymbol{N_1\, I_1 = S\, \phi_l}$, and a fraction $\boldsymbol{k_{12}}$ of the primary flux would link with the secondary, $\phi_{12} = \boldsymbol{k_{12}} \cdot \phi_l$

$$\boldsymbol{M_{12} = \frac{k_{12} N_1 N_2}{S} = \frac{k_{12} N_1 N_2 \mu A}{l}}$$

$\boldsymbol{k_{12}}$ is known as the coefficient of coupling between the coils.

$\boldsymbol{k_{12} = k_{21}}$ so that $\boldsymbol{M_{12} = M_{21}}$.

For good coupling, $\boldsymbol{k_{12}}$ is very nearly equal to unity.

Energy stored in a magnetic field

Energy stored in an inductor

$$= \int v \cdot i \cdot dt = \int L \frac{di}{dt} \cdot i \cdot dt = \int L \cdot i \cdot di = \tfrac{1}{2} L i^2$$

Energy stored in a unit volume in magnetic field

$$= \int \frac{N \cdot \frac{d\phi}{dt} \cdot i \cdot dt}{\text{volume}} = \int \frac{N \cdot i \cdot d\phi}{A . l} = \int \frac{N \cdot i}{l} \cdot d \frac{\phi}{A} = \int H \cdot dB$$

$$= \int \frac{B}{\mu} \cdot dB = \tfrac{1}{2} \frac{B^2}{\mu} = \tfrac{1}{2} B \cdot H$$

J/m^3

Force exerted in a magnetic field

For a spacing change of $\boldsymbol{dx}$ by moving the electromagnet

change in energy stored = $\frac{1}{2}\, \boldsymbol{B}.\boldsymbol{H} \times$(change in volume)

$$= \frac{1}{2}\, \boldsymbol{B}.\boldsymbol{H} \times \boldsymbol{A}.\, \boldsymbol{dx}$$

Also, change in energy stored = work done = $\boldsymbol{F}\, .\, \boldsymbol{dx}$,

$$\therefore \quad \boldsymbol{F}\, .\, \boldsymbol{dx} = \frac{1}{2}\, \boldsymbol{B}.\boldsymbol{H}\, .\, \boldsymbol{A}.\, \boldsymbol{dx} \qquad \text{or} \qquad \boldsymbol{F} = \frac{1}{2}\, \boldsymbol{B}.\boldsymbol{H}\boldsymbol{A}$$

i.e. Force on unit area in electric field = $\boldsymbol{F}/\boldsymbol{A} = \frac{1}{2}\boldsymbol{B}.\boldsymbol{H}$ N/m^2

Recap of Electric and Magnetic Fields

Electrostatic Theory - Gauss’s Law, Coulomb’s Inverse Square Law

Capacitance of dielectric

Ohm's law in an electric field

Electromagnetic Theory - Ampere's Law

Energy stored and force exerted in electric and magnetic fields

mmf, flux density, reluctance

Calculation of flux in a magnetic circuit

Self and mutual inductance

END OF PRESENTATION