MA 1203
Calculus

3 Credits

Chapter 04
Ordinary Differential Equations

4.1 Introduction

Definition : Differential Equation

Definition

A differential equation is an equation that contains an unknown function and one or more of its derivatives.

- Consider the differential equation,

$$\frac{dy}{dx} = 2$$

Here y is a dependent variable and x is the independent variable.

  ★ A differential equation which involve only one independent variable is called ordinary differential equation.

Now consider,

$$\frac{dy}{dx} = 2$$

By integrating w.r.t. x,

$$\int \frac{dy}{dx} \, dx = \int 2 \, dx$$

$$\int dy = \int 2 \, dx$$

$$y = 2x + C$$

where C is an arbitrary constant.

☆ By solving a differential equation, an algebraic equation can be obtained.

Order of a Differential Equation

Definition

The order of a differential equation is the order of the highest derivative that occurs in the equation.

Example (01)

Find the order of the following differential equations.

1 $\dfrac{dy}{dx} = 3$

2 $\dfrac{d^2y}{dx^2} = 100$

3 $\dfrac{d^3y}{dx^3} + \dfrac{dy}{dx} - 10 = 0$

Degree of a Differential Equation

Definition

The degree of the polynomial is the power of the highest order derivative after making the differential equation free from rational and fractional indices as far as the derivatives are concerned.

Example (02)

Find the order and degree of the following differential equations.

1. $\left(\dfrac{d^3y}{dx^3}\right)^2 + \left(\dfrac{d^2y}{dx^2}\right)^4 + \dfrac{dy}{dx} = 6$

2. $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 0$

3. $3\dfrac{d^2y}{dx^2} + \left(1 + \dfrac{dy}{dx}\right)^{\frac{3}{2}} + 9\left(\dfrac{d^2y}{dx^2}\right)^2 = 10$

Linear Differential Equations

- A differential equation is called linear if there are no multiplications among dependent variables and their derivatives.

- In other words, all coefficients are functions of independent variable.

- A differential equation do not satisfy the definition of linear equation is a nonlinear differential equation.

- ☆ A function f is called a **solution** of a differential equation if the equation is satisfied when $y = f(x)$ and its derivatives are substituted in to the equation.

4.2 Formation of Differential Equations

Leibnitz's Theorem of Differentiation

Theorem

If u and v are two differentiable functions, the nth derivative of uv with respect to x is

$$\frac{d^n(uv)}{dx^n} = \frac{d^n u}{dx^n}v + {}^n C_1 \frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + {}^n C_2 \frac{d^{n-2}u}{dx^{n-2}}\frac{d^2 v}{dx^2} + ... + u\frac{d^n v}{dx^n}$$

$$\frac{d^n(uv)}{dx^n} = \sum_{i=0}^{\infty} {}^n C_i \left(\frac{d^{n-i}u}{dx^{n-i}}\right)\left(\frac{d^i v}{dx^i}\right)$$

Example (03)

If $y = \tan^{-1} x$, show that

$$(1 + x^2)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = 0.$$

By differentiating this equation n times, show that

$$(1 + x^2)\frac{d^{n+2}y}{dx^{n+2}} + 2(n+1)x\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^ny}{dx^n} = 0.$$

Example (04)

If $y = (\cos^{-1} x)^2$, show that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$

Use Leibnitz's theorem to show that

$$(1 - x^2)\frac{d^{n+2}y}{dx^{n+2}} - (2n+1)x\frac{d^{n+1}y}{dx^{n+1}} - n^2\frac{d^ny}{dx^n} = 0.$$

Formation of Differential Equations by Elimination

- The problem of elimination gives an idea as to what kind of solution a differential equation may have.
- Here we eliminate arbitrary constants by formation of ordinary differential equations.

Example (05)

Consider the equation of simple harmonic motion,

$$x = A\cos(pt - \alpha).$$

(i) Eliminate the constants A and α.

(ii) Eliminate the constants A, α and p.

4.3 First Order First Degree Differential Equations

We will discuss the standard methods of solving the differential equation of following types:

1. Method of Separable Equations
2. Linear equations of first order
3. Exact Equation
4. Homogeneous Equation

Solving ODEs using method of separable equations

Definition

A separable equation is a first -order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y).$$

Strategy of Solution

$$\frac{dy}{dx} = g(x)h(y)$$

Step 01: Separate the Variables

$$\frac{dy}{h(y)} = g(x)\ dx$$

Step 02: Integrate both sides

$$\int \frac{dy}{h(y)} = \int g(x)\ dx$$

Example (06)

(i) Solve the differential equation

$$\frac{dy}{dx} = \frac{x^2}{y^2}.$$

(ii) Find the solution of this equation that satisfies the initial condition $y(0) = 2$.

Example (07)

Solve the differential equation

$$\frac{dy}{dx} = \frac{6x^2}{2y + \cos y}.$$

Example (08)

Solve the differential equation

$$y' = x^2 y.$$

Real World Examples

1 **Radioactive Decay**

$$\frac{dA}{dt} = -kA$$

At time $t = 0$, $A = A(0)$; which is the initial number of radioactive atoms.

2 **Logistic Equation**

$$\frac{dP}{dt} = p(1 - p)$$

At time $t = 0$, $p = p(0) =$ the initial population.

3 **Newton's Law of Cooling**

$$\frac{dT}{dt} = -k(T - T_m)$$

where $T(0) = T_0$ and T_m is the surrounding temperature.

Example (09)

Solve the differential equation

$$\frac{dy}{dx} + 2xy^2 = 0.$$

Additional Problems

1 $\dfrac{dy}{dx} = 3x^2 y^2$

2 $\dfrac{dy}{dx} = x\sqrt{y}$

3 $xyy' = x^2 + 1$

4 $y' + xe^y = 0$

5 $(e^y - 1)y' = 2 + \cos x$

6 $\dfrac{dy}{dx} = xe^y \; ; \; y(0) = 0$

7 $\dfrac{dy}{dx} = \dfrac{x \sin x}{y} \; ; \; y(0) = -1$

8 $\dfrac{du}{dt} = \dfrac{2t + \sec^2 t}{2u} \; ; \; u(0) = -5$

First Order Linear Differential Equations

Definition

A first order differential equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x),$$

where $P(x)$ and $Q(x)$ are continuous functions is said to be a **linear first order differential equation**.

- If $Q(x) = 0$, the differential equation is said to be *homogeneous*.
- If $Q(x) \neq 0$, the differential equation is said to be *non-homogeneous*.

Strategy of Solving Linear Equations

Step 01: Put the equation in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Step 02: Multiply both sides by the integrating factor.

$$\text{Integrating Factor} = e^{\int P(x)}$$

$$e^{\int P(x)}\frac{dy}{dx} + e^{\int P(x)}P(x)y = e^{\int P(x)}Q(x)$$

$$\frac{d}{dx}\left[e^{\int P(x)} \cdot y\right] = e^{\int P(x)} \cdot Q(x)$$

The differential equation becomes,

$$\frac{d}{dx}[IF \cdot y] = IF \cdot Q(x)$$

Step 03: Integrate both sides,

$$e^{\int P(x)} \cdot y = \int e^{\int P(x)} \cdot Q(x) + C$$

Step 04: Final Solution

$$y = e^{-\int P(x)} \left[\int e^{\int P(x)} \cdot Q(x) + C \right]$$

Example (10)

Solve the differential equation

$$\frac{dy}{dx} + 3x^2y = 6x^2.$$

Example (11)

Find the solution of the IVP:

$$x^2 y' + xy = 1$$

where $x > 0$, $y(1) = 2$.

Example (12)

Solve the differential equation

$$y' + 2xy = 1.$$

Additional Problems

1. $y' + y = 1$
2. $y' - y = e^x$
3. $y' = x - y$
4. $y' + xe^y = 0$
5. $4x^3y + x^4y' = \sin^3 x$
6. $xy' + y = 2\sqrt{x}$
7. $x^2y' + 2xy = \ln x$; $y(1) = 2$
8. $t^3\dfrac{dy}{dt} + 3t^2y = \cos t$; $y(\pi) = 0$

Exact Differential Equation

Definition

Necessary and Sufficient Condition:
A differential expression of the form

$$M(x, y)dx + N(x, y)dy = 0$$

is said to be an **exact differential equation** if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Solution for Exact Equation

$$dF = M(x, y)dx + N(x, y)dy = 0$$

Then, an implicit general solution of the differential equation is

$$F(x, y) = C$$

As

$$dF = M(x, y)dx + N(x, y)dy = 0$$

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy$$

Hence,

$$\frac{\partial F}{\partial x} = M(x, y) \qquad \frac{\partial F}{\partial y} = N(x, y).$$

Method of Solution

To solve an exact equation, first solve the equations:

$$\frac{\partial f(x,y)}{\partial x} = M(x,y) \qquad \frac{\partial f(x,y)}{\partial y} = N(x,y)$$

for $f(x,y)$.

The solution to the exact equation is then given by,

$$f(x,y) = C$$

, where C represents an arbitrary constant.

Example (13)

Is

$$\frac{dA}{dt} = -kA$$

, where k is a positive constant, exact?

Example (14)

Is

$$\frac{dy}{dx} = \frac{1 - 2x}{3y + 7}$$

exact? If yes, obtain the solutions.

Example (15)

Solve the differential equation

$$\cos(x + y)dx + (3y^2 + 2y + \cos(x + y))dy = 0.$$

Additional Problems

1. $2xy\,dx + x^2\,dy = 0$
2. $x^3\,dx + y^3\,dy = 0$
3. $\sin x \cos y \, dx + \cos x \sin y \, dy = 0$
4. $(x^2 + y^2)dx - 2xy\,dy = 0$
5. $3(y+1)dx = 2x\,dy$