Stability

Concept of stability: is very important to analyse and design the system.

A system is said to be stable if its response cannot be made to increase indefinitely, by the application of a bounded input excitation.

If the output approaches towards infinite value for sufficiently large time, the system is said to be unstable.

A linear time invariant (LTI) system is stable if

1. the system is excited by a bounded input, the output is bounded (BIBO stability criteria)
2. In absence of the input, the output ~~towar~~ tends towards zero.

Consider a system whose closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{1000}{s^2+40s+300} = \frac{1000}{(s+10)(s+30)}$$

If $R(s) = \frac{1}{s}$.

$$C(s) = \frac{1000}{s(s+10)(s+30)} = \cancel{33.3}$$

$$= \frac{3.33}{s} - \frac{5}{(s+10)} + \frac{1.67}{(s+30)}$$

$$c(t) = 3.33 - 5e^{-10t} + 1.67e^{-30t}$$

as $t \to \infty$, the output will be steady state.

such systems are known as absolutely stable systems.

Now consider,

$$\frac{C(s)}{R(s)} = \frac{3}{(s-1)(s+2)}.$$

Here one closed loop pole is located in right half ~~plane~~ of s-plane.

$R(s) = \frac{1}{s}$,

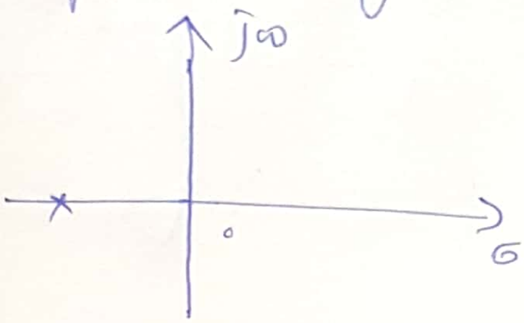
$$C(s) = \frac{3}{s(s-1)(s+2)} = \frac{-1.5}{s} + \frac{1}{(s-1)} + \frac{0.5}{(s+2)}$$

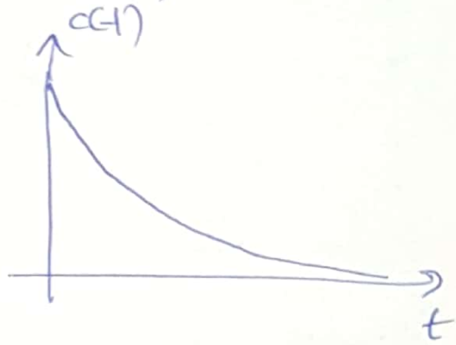
$$c(t) = -1.5 + e^{t} + 0.5e^{-2t}.$$

So as $t \to \infty$, the transient goes to increase in amplitude.

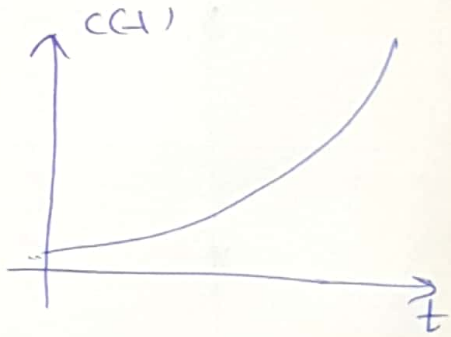
Effect of location of poles on stability

(a) poles on negative real axis,

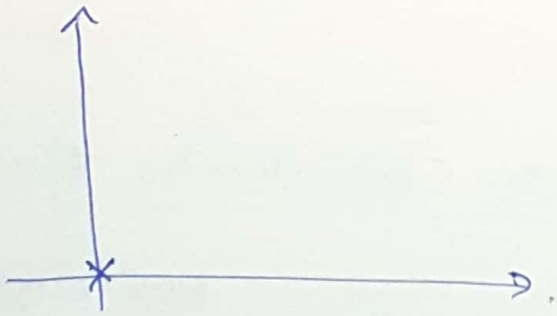


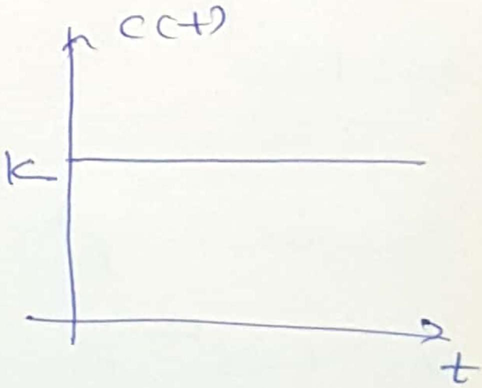


(b) pole on positive real axis,

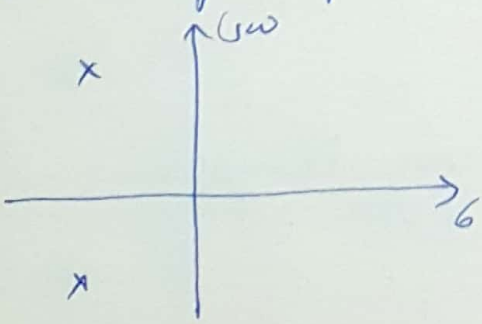


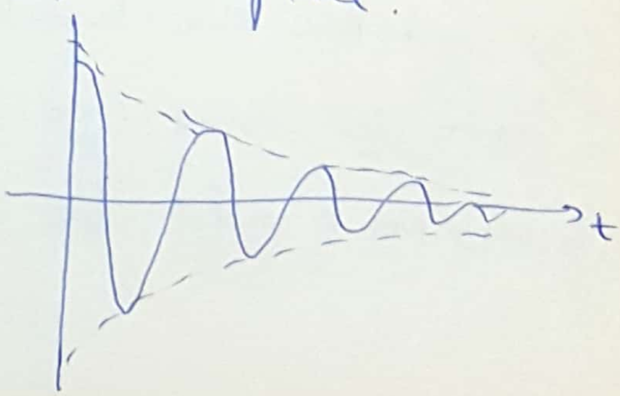
(c). pole at origin.



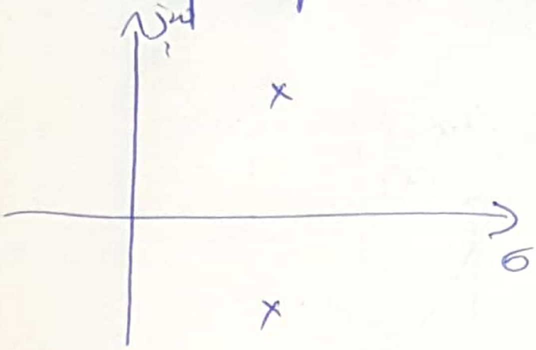


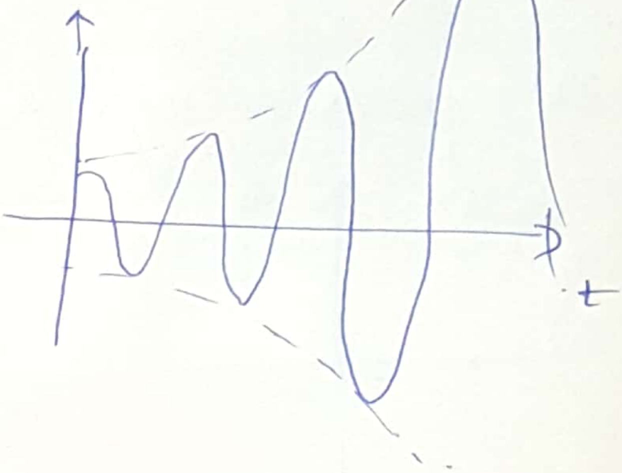
(d) Complex pole in left half of s-plane.



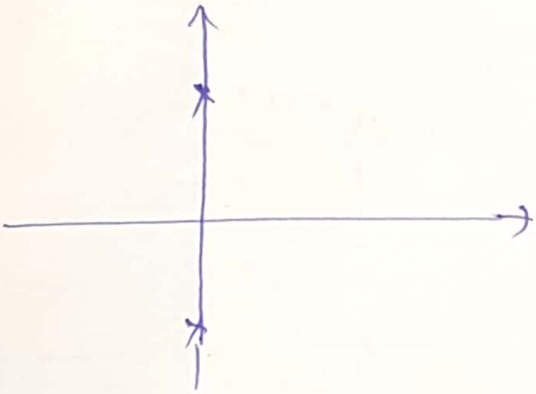


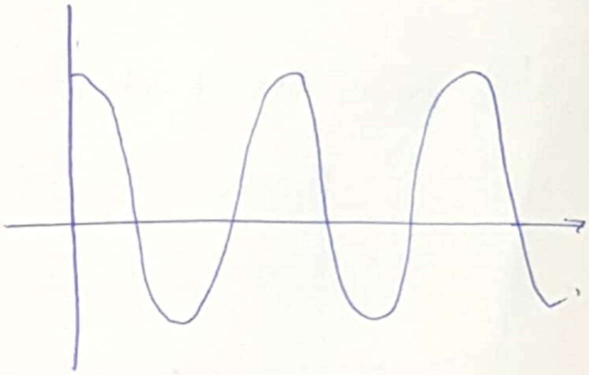
(e) complex poles in the right half of s-plane.





(f) poles on $j\omega$ axis.





Necessary but not sufficient Condition for stability.

Consider a system with characteristic equation.

$a_0 s^n + a_1 s^{n-1} + \dots + a_n = 0$, $\quad 1 + G(s)H(s) = 0$.

(a) All the coefficients of the equation should have the same sign.

(b) There should be no missing terms.

If above two conditions are not satisfied, the system will be unstable. But if all the coefficients have the same sign and there is no missing term we have no guarantee that the system will be stable. For stability we use Routh-Hurwitz Criterion.

The Routh Hurwitz Criterion.

Consider the following characteristic ~~eq~~ polynomial,

$a_0 s^n + a_1 s^{n-1} + \cdots + a_n = 0$.

where $a_0, a_1, a_2, \ldots, a_n$ are all same sign & none is zero.

Step 1

Row 1	a_0	a_2	a_4	...
Row 2	a_1	a_3	a_5	...

Step 2: From this we have ~~two rows~~ third row,

a_0	a_2	a_4	...
a_1	a_3	a_5	...
b_1	b_2	b_3	...

$$b_1 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}$$

$$b_2 = -\frac{1}{a_1}\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}.$$

step3:

$$\begin{array}{cccc} a_0 & a_2 & a_4 & \cdots \\ a_1 & a_3 & a_5 & \cdots \\ b_1 & b_2 & b_3 & \cdots \\ c_1 & c_2 & c_3 & \cdots \end{array}$$

$$c_1 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \qquad c_2 = -\frac{1}{b_1}\begin{vmatrix} a_1 & a_5 \\ b_1 & b_5 \end{vmatrix}.$$

Statement of Routh Hurwitz

The system is stable if and only if all the elements in the 1st column have the same algebraic sign. If all elements are not of the same sign then the number of sign changes of the elements in first column equals the number of roots of the characteristic equation in the right half of the s-plane.

ex: Check the stability of the system given by,

$$s^4 + 2s^3 + 6s^2 + 4s + 1 = 0.$$

s^4	2	1	2
s^3	2	3	
s^2	-2 b_1		
s^1	5 c_1		
s^0	2 d_1		

$$b_1 = \frac{-1}{2}\begin{vmatrix} 2 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= -2$$

$$c_1 = -\frac{1}{(-2)}\begin{vmatrix} 2 & 3 \\ -2 & 2 \end{vmatrix}$$

$$= 5$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 2 & 2 \\ 2 & 0 \end{vmatrix} = 2.$$

$$d_1 = -\frac{1}{5}\begin{vmatrix} -2 & 2 \\ 5 & 0 \end{vmatrix} = 2.$$

There are two sign changes in the first column (from 2 to -2 & from -2 to 5), hence there are two roots in the right half of s-plane.

$\therefore$ The system is unstable.

example:

$$2s^4 + 5s^3 + 5s^2 + 2s + 1 = 0$$

s^4	2	5	1
s^3	5	2	
s^2	4.2.		
s^1	0.809		
s^0	1		

From the routh table:
No. of sign changes in first column $= 0$.

$\therefore$ No. of poles on the right hand side of s-plane $= 0$.

$\therefore$ System is stable.

Example: Investigate the stability,

$$s^5+s^4+2s^3+2s^2+3s+5=0$$

s^5	1	2	3
s^4	1	2	5
s^3	0		
s^2			
s^1			
s^0			

Now multiply the characteristic eq by $(s+1)$

$$(s+1)(s^5+s^4+2s^3+2s^2+3s+5)=0$$

$$\Rightarrow s^6+2s^5+3s^4+4s^3+5s^2+8s+5=0$$

s^6	1	3	5	5
s^5	2	4	8	
s^4	1	1	5	
s^3	2	-2		
s^2	2	5		
s^1	-7			
s^0	1			

No. of sign changes in first column = 2

No. of roots in the right half of s plane = 2

$\therefore$ System is not stable.

Corrected.

example: $s^4+2s^3+6s^2+4s+1=0.$

$$\begin{array}{c|ccc} s^4 & 1 & 6 & 1 \\ s^3 & 2 & 4 & \\ \hline s^2 & 4 & 1 & \\ s^1 & 3.5 & & \\ s^0 & 1 & & \end{array}$$

1st column

$$b_1 = -\frac{1}{2}\begin{vmatrix} 1 & 6 \\ 2 & 4 \end{vmatrix} = 4$$

$$b_2 = -\frac{1}{2}\begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix} = 1$$

$$c_1 = -\frac{1}{4}\begin{vmatrix} 2 & 4 \\ 4 & 1 \end{vmatrix} = 3.5$$

$$d_1 = -\frac{1}{3.5}\begin{vmatrix} 4 & 1 \\ 3.5 & 0 \end{vmatrix} = 1$$

Since all the coefficients of the 1st column are of the same sign (positive), the given equation has no roots with positive real parts.

Hence, the system is stable.

Example: The open loop transfer function of unity feedback system is $\frac{K}{s(1+0.4s)(1+0.25s)}$

Find the restriction of K, so that the closed loop system is absolutely stable.

Solution

$$G(s) = \frac{K}{s(1+0.4s)(1+0.25s)}$$

$$H(s) = 1.$$

characteristic equation,

$$1 + G(s)H(s) = 0.$$

$$1 + \frac{K}{s(1+0.4s)(1+0.25s)} \times 1 = 0.$$

$$s(1+0.4s)(1+0.25s) + K = 0.$$

$$s^3 + 6.5s^2 + 10s + 10K = 0.$$

s^3	1	10
s^2	6.5	10K.
s^1	$\frac{65-10K}{6.5}$	
s^0	10K	

For absolute stability, there should be no sign change in the first column, i.e. no root of the characteristic equation should lie in the right half of s-plane.

$\therefore$ $K > 0$, $65 - 10K > 0$

$\therefore$ $0 < K < 6.5$

<u>example</u>: check whether all the roots of the equation $s^3 + 7s^2 + 25s + 39 = 0$ have real parts more negative than -1.

<u>Solution</u>:

s^3	1	25
s^2	7	39
s^1	19.42	
s^0	39.	

Since there is no sign change in the 1st column, it means all the roots are in the left half of s-plane.

put $s = z - 1$ in the characteristic equation

$$(z-1)^3 + 7(z-1)^2 + 25(z-1) + 39 = 0.$$

$$\Rightarrow z^3 + 4z^2 + 14z + 20 = 0.$$

z^3	1	14
z^2	4	20
z	9	
z^0	20	

Since no sign change in the first column, the roots of the characteristic equation lie in the left of z-plane.

~~It means~~

It means all the roots of the original equation in s-domain lie to the left of ~~z~~ $s=-1$