Eigenvalues & Eigenvectors

If A is an $n \times n$ matrix, then a scalar λ is called an eigenvalue of A, if associated with it there is a non-zero vector $\underline{v}$, called an eigenvector, such that

$$A\underline{v} = \lambda \underline{v}$$

(*) To find the eigenvalues solve the characteristic equation.

$$|A - \lambda I| = 0$$

(*) To find the eigenvectors solve

$$(A - \lambda I)\underline{v} = \underline{0}.$$

(*) Note that, scalar λ can be real number or complex number.

<u>Ex</u>. Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}_{2\times 2}.$$

<u>Sol^{n}</u>.

First find eigenvalues using characteristic equation.

$$|A - \lambda I| = 0$$

$$\left| \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)(-3-\lambda) + 10 = 0$$

$$\lambda^2 - \lambda - 2 = 0$$

$$\lambda = -1, 2.$$

Next find corresponding eigenvectors for each eigen values.

For $\lambda = -1$

To find eigenvectors solve $(A - \lambda I)\underline{v} = \underline{0}$ for $\lambda = -1$

i.e. $\begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

when $\underline{v} = \begin{pmatrix} x \\ y \end{pmatrix}_{2\times 1}$.

$$\begin{pmatrix} 5x - 5y \\ 2x - 2y \end{pmatrix}_{2\times 1} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}_{2\times 1}$$

$$\Rightarrow \left.\begin{aligned} 5x - 5y &= 0 \\ 2x - 2y &= 0 \end{aligned}\right\} \quad x - y = 0$$

If we take t is any parameter and set $y = t$, then $x = y = t$.

Therefore corresponding eigen vectors for $\lambda = -1$,

$$E_\lambda = \underline{V} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} t \\ t \end{pmatrix} = t \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$E_\lambda = \left\{ t\begin{pmatrix}1\\1\end{pmatrix} \middle/ t \in \mathbb{R} \right\}$

eigen space for eigen value $\lambda = -1$.

For $\lambda = 2$

$(A - \lambda I)\underline{V} = 0$

$$\begin{pmatrix} 2 & -5 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow 2x - 5y = 0$

$\Rightarrow$ If we set $y = t$, then

$x = \frac{5y}{2} = \frac{5t}{2}$.

$\therefore$ Corresponding eigenvector for $\lambda = 2$

is

eigen space

$E_\lambda = \underline{V} = \begin{pmatrix} x \\ y \end{pmatrix} = \left\{ t\begin{pmatrix} 5/2 \\ 1 \end{pmatrix} \middle/ t \in \mathbb{R} \right\}$.

Note

We denote E_λ is to represent all the eigenvectors together with 0 vector correspondond to eigenvalue λ.

E_λ called as eigenspace.

Ex

Find eigenvalues & eigenvectors of $\bar{A} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}_{2\times 2}$

Sol^n.

For eigenvalues $|A-\lambda I| = 0$

$$\begin{vmatrix} 4-\lambda & 0 \\ 0 & 4-\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad \lambda = 4, 4 \quad$ (i.e. one - eigen value)

For $\lambda = 4$

$$(A-\lambda I)\underline{V} = \underline{0}$$

$$\begin{pmatrix} 4-4 & 0 \\ 0 & 4-4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ x & y can be any ~~vector~~ value.

$\Rightarrow$ Let s & t the any parameter. Then

$x = s$ & $y = t$ given eigen vectors.

∴ Corresponding eigen ~~vectors~~ space:

for $\lambda = 4$,

$$E_\lambda = \begin{pmatrix} x \\ y \end{pmatrix} = \left\{ \begin{pmatrix} s \\ t \end{pmatrix} \middle/ s,t \in \mathbb{R} \right\}$$

$$E_\lambda = \left\{ s\begin{pmatrix} 1 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix} \middle/ s,t \in \mathbb{R} \right\}.$$

Example 4.

Find the eigenvalues & Eigenspaces of the matrix

$$A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}_{3\times 3}$$

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Sol^{n}:

Let us find values of λ which satisfy the characteristic eq^{n} of the matrix A.

$$|A-\lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad 16+12\lambda-\lambda^3 = 0$$

$$\Rightarrow \quad \lambda = -2, 4.$$

E_λ for $\lambda=4$:

$$(A-\lambda I)\underline{v} = \underline{0} \; ; \text{ where } \vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Here we have three equations with three variables. Let us solve this linear system by GA Gauss-Elimination method.

$$(A-\lambda I) = \begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix}.$$

$$(A-\lambda I)\underline{v} = \underline{0}$$

$$\left(\begin{array}{ccc|c} -3 & -3 & 3 & 0 \\ 3 & -9 & 3 & 0 \\ 6 & -6 & 0 & 0 \end{array}\right) \cdot \xrightarrow{\text{Aftere elementrary row operations}}$$

$$\left(\begin{array}{ccc|c} 1 & 0 & -1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0. \end{array}\right).$$

$$\Rightarrow \quad x_1 - \frac{1}{2}x_3 = 0$$

$$x_2 - \frac{1}{2}x_3 = 0$$

If $x_3 = t$, then $x_1 = t/2$ and $x_2 = t/2$.

So eigen space corresponding to the eigenvalue $\lambda = 4$,

$$E_4 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} t/2 \\ t/2 \\ t \end{pmatrix} = \left\{ t \begin{pmatrix} 1/2 \\ 1/2 \\ 1 \end{pmatrix} \Big/ t \in \mathbb{R} \right\}$$

E_{-2} for $\lambda = -2$

$$A + 2I = \begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix}.$$

$(A + 2I)\underline{V} = \underline{0}$.

$$\left(\begin{array}{ccc|c} 3 & -3 & 3 & 0 \\ 3 & -3 & 3 & 0 \\ 6 & -6 & 6 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 1 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$x_1 - x_2 + x_3 = 0$

Then setting parameters s & t as $x_2 = s$, $x_3 = t$, we have $x_1 = s - t$

$$E_{-2} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s-t \\ s \\ t \end{pmatrix} = \begin{pmatrix} -t \\ 0 \\ t \end{pmatrix} + \begin{pmatrix} +s \\ s \\ 0 \end{pmatrix}.$$

$$E_{-2} = \left\{ t\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} +1 \\ 1 \\ 0 \end{pmatrix} \middle/ s, t \in \mathbb{R} \right\}$$

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

be a polynomial in x. If A

is an $n \times n$ matrix then $f(A)$

is defined as

$$f(A) = a_n A^n + a_{n-1} A^{n-1} + \dots + a_1 A + a_0 I$$

; where I is the identity matrix of

order n.