Polar form of Complex number

Complex number $z = x + iy$

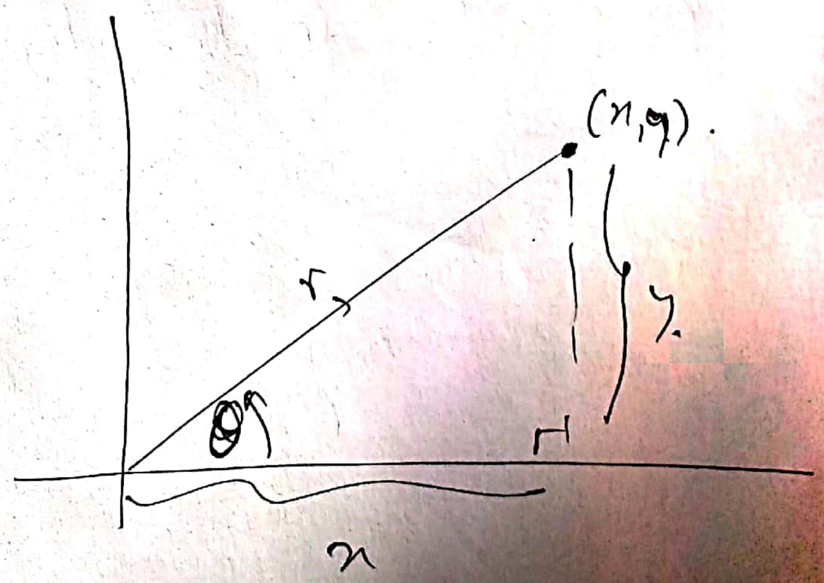
can be written as

$$z = r(\cos\theta + i\sin\theta);$$

where r, a positive real number, is called the modulus,

θ an angle such that $-\pi < \theta \leq \pi$,

θ is called the principle argument.



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$r = |z| = \sqrt{x^2+y^2}$$

* Note that θ, the argument, is not unique. The argument of z could also be $\theta \pm 2\pi$, $\theta \pm 4\pi$, ---, etc.

* To avoid duplication of θ, we usually quote θ in the range $-\pi < \theta \leq \pi$ and refer to it as the principle argument and denoted by $Arg(z)$

(*) If we not restrict to above range for argument then it is called as argument, denoted by $\arg(z)$.

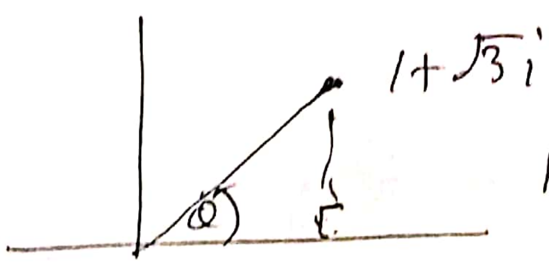
(*) The argument of $z=0$ cannot be defined.

Ex: The value of $\arg(i)$ are

$\pi/2,\ \pi/2 \pm 2\pi,\ \pi/2 \pm 4\pi, \ldots$

i.e. $\arg(i) = \frac{\pi}{2} + 2k\pi \ ;\ k \in \mathbb{Z}$

Ex: Let $z = 1+\sqrt{3}\,i$



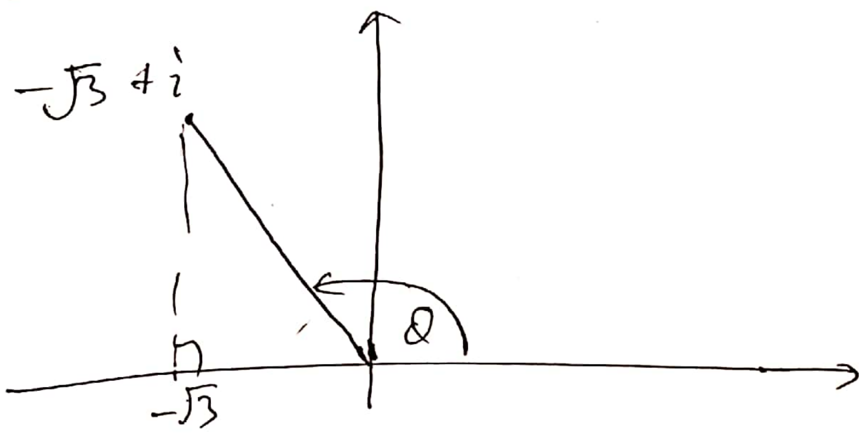
$\boxed{\theta = \pi/3}$

$\therefore\ \arg(1+\sqrt{3}\,i) = \pi/3 + 2k\pi\ ,\ k \in \mathbb{Z}.$

Note: $\mathrm{Arg}(1+\sqrt{3}i) = \pi/3$,

Since $-\pi < \mathrm{Arg}(z) \leq \pi$)

Ex:
Express $z = -\sqrt{3} + i$ in the form $r(\cos\theta + i\sin\theta)$, where $-\pi < \theta \leq \pi$.



$r = \sqrt{(-\sqrt{3})^2 + 1^2} = 2$

$\theta = \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \pi - \pi/6 = 5\pi/6$

$\therefore$ Therefore,

$z = 2\left(\cos 5\pi/6 + i\sin 5\pi/6\right)$

Euler's Formula

Euler's formula is

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

* We can write the cosine & sine in terms of the exponential.

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

* $z = re^{i\theta}$ representation we called as exponential form.

After some trigonometric calculations we can get following exponential form.

Let $z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$

then,

$$z_1 z_2 = r_1 r_2 \, e^{i(\theta_1+\theta_2)}$$

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) e^{i(\theta_1-\theta_2)} \quad ; \; r_2 \neq 0$$

Absolute Value / Modulus

Let $z = x+iy$ be any Complex number. Then its absolute value, denoted by $|z|$, defined as

$$|z| = \sqrt{x^2+y^2} = r.$$

Some important properties of complex numbers

① $|z_1 z_2| = |z_1||z_2|$

② $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $|z_2| \neq 0$

③ $|z_1 + z_2| \leq |z_1| + |z_2|$.

Also for any complex numbers z_1 & z_2 other than zero following properties hold.

④ $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

⑤ $\arg\left(\dfrac{z_1}{z_2}\right) = \arg. z_1 - \arg z_2$

(6) $\arg(\bar{z}) = -\arg(z)$.

[Complex conjugate

If $z = x+iy$, its complex conjugate ~~...~~ denoted by $\bar{z}$, defined by $\bar{z} = x - iy$.]

[If $z = re^{i\theta}$ then
$\bar{z} = re^{-i\theta}$.]

(7) $\mathrm{Arg}(z_1 z_2) \neq \mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2$

(8) $\mathrm{Arg}\left(\frac{z_1}{z_2}\right) \neq \mathrm{Arg}\, z_1 - \mathrm{Arg}\, z_2$.

Ey

① Express $z = \sqrt{2}\left(\cos\frac{\pi}{10} + i\sin\frac{\pi}{10}\right)$ in the exponential form.

i.e. $re^{i\theta}$ form ; where $-\pi < \theta \leq \pi$

Ans. $z = \sqrt{2}\,(\cos \pi/10 + i \sin \pi/10)$.

So $r = \sqrt{2}$, & $\theta = \pi/10$.

$\therefore \quad z = \sqrt{2}\, e^{\frac{i\pi}{10}}$

② Express $z = \sqrt{2}\, e^{\frac{3\pi}{4} i}$ in the form $x + iy$.

Ans. $z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$.

$= \sqrt{2}\left(\frac{-1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)$.

$z = -1 + i$

(3) Express $z = 2e^{\left(\frac{23}{5}\pi i\right)}$ in the form $r(\cos\theta + i\sin\theta)$; where $-\pi < \theta \leq \pi$.

Ans.

$$z = 2e^{\left(\frac{23}{5}\pi i\right)}$$

So $r = 2$, $\theta = \frac{23\pi}{5}$

$$\theta = \frac{23\pi}{5} \longrightarrow \frac{13\pi}{5} \longrightarrow \frac{3\pi}{5}$$

(Continue to subtract/add 2π from θ unitl $-\pi < \theta \leq \pi$).

$$\therefore \quad z = \sqrt{2}\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$$

(4) Express $3\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right) \times 4\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

Ans.

$$= 3\times 4\left(\cos\left[\frac{5\pi}{12}+\frac{\pi}{12}\right] + i\sin\left[\frac{5\pi}{12}+\frac{\pi}{12}\right]\right).$$

polar & Exponential form of multiplication.

$$= 12\left(\cos\frac{\pi}{2} + i\sin\pi/2\right).$$

$$= 12\,(0 + i(1))$$

$$= 12i.$$

(5) Express $\dfrac{\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)}{2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}$

in the form $x+iy$.

Ans.

$$= \frac{\sqrt{2}}{2}\left(\cos\left[\frac{\pi}{12} - \frac{5\pi}{6}\right] + i\sin\left[\frac{\pi}{12} - \frac{5\pi}{6}\right]\right).$$

$$= \frac{\sqrt{2}}{2}\left(\cos\left(-\frac{3\pi}{4}\right) + i\sin\left(-\frac{3\pi}{4}\right)\right).$$

$$= \frac{\sqrt{2}}{2}\left(\frac{-1}{\sqrt{2}} + i\frac{-1}{\sqrt{2}}\right)$$

$$= \frac{-1}{2} - \frac{1}{2}i$$

Recall De' Movier's theorem

$$\boxed{\begin{aligned} z^n &= \left[r(\cos\theta + i\sin\theta)\right]^n \\ &= r^n(\cos n\theta + i\sin n\theta) \end{aligned}}$$

This theorem is true for any integer n. i.e. for any negative integers, theorem hold.

Scanned with CamScanner

Ex. Simplify

$$\frac{\left(\cos\frac{9\pi}{17} + i\sin\frac{9\pi}{17}\right)^5}{\left(\cos\frac{2\pi}{17} - i\sin\frac{2\pi}{17}\right)^3}$$

Ans

$$= \frac{\cos\left(\frac{45\pi}{17}\right) + i\sin\left(\frac{45\pi}{17}\right)}{\left[\cos\left(-\frac{2\pi}{17}\right) + i\sin\left(-\frac{2\pi}{17}\right)\right]^3}$$ ← De' Moivre's theorem

$$= \frac{\cos\left(\frac{45\pi}{17}\right) + i\sin\left(\frac{45\pi}{17}\right)}{\cos\left(\frac{-6\pi}{17}\right) + i\sin\left(\frac{-6\pi}{17}\right)}$$

$$= \cos\left(\frac{45\pi}{17} + \frac{6\pi}{17}\right) + i\sin\left(\frac{45\pi}{17} + \frac{6\pi}{17}\right)$$

$\left(\frac{z_1}{z_2}\text{, Apply result}\right)$

Scanned with CamScanner

$$= \cos\frac{51\pi}{17} + i\sin\frac{51\pi}{17}$$

$$= \cos\pi + i\sin\pi = -1$$

Ex.

Express $(1+\sqrt{3}\,i)^7$ in the form $x+iy$.

$$r = \sqrt{1^2+(\sqrt{3})^2} = 2$$

$$\theta = \tan^{-1}\left(\frac{\sqrt{3}}{1}\right) = \pi/3$$

So. $$(1+\sqrt{3}\,i)^7 = \left[2\cos(\pi/3 + i\sin\pi/3)\right]^7$$

$$= 2^7\left(\cos\frac{7\pi}{3} + i\sin\frac{7\pi}{3}\right).$$

$$= 128\left(\cos\pi/3 + i\sin\pi/3\right)$$

$$= 128\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right).$$

$$= 64 + 64\sqrt{3}\,i$$

nth roots of a Complex number

nth root of a Complex number $z = re^{i\theta}$ is given by,

$$z_k = r^{1/n} e^{\frac{i(\theta+2k\pi)}{n}}$$

where $k = 0, 1, 2, \ldots, (n-1)$

Ex (1)

$$1^{1/3} = [1(\cos 0 + i \sin 0)]^{1/3}$$

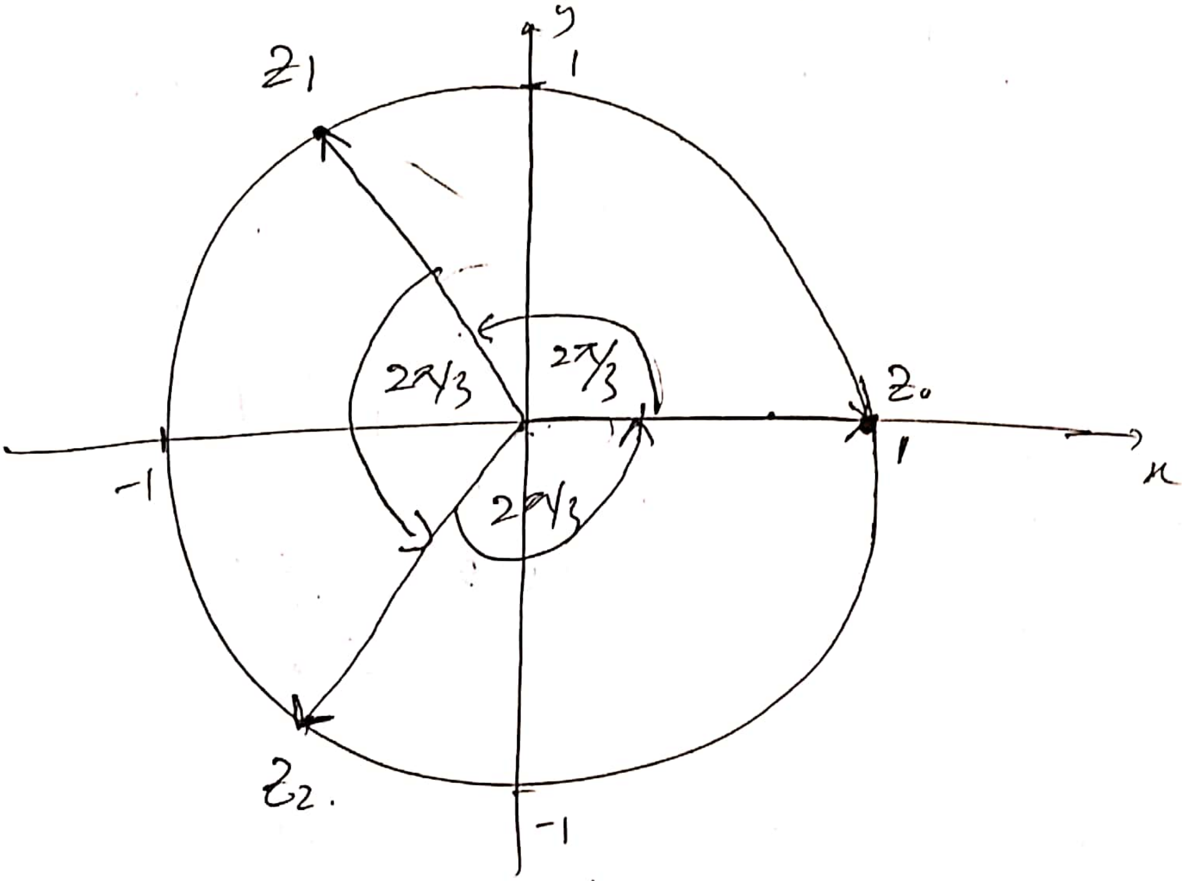
$$z_k = 1^{1/3} e^{\frac{i(0+2k\pi)}{3}} \; ; \; k = 0, 1, 2$$

when

$k=0 \Rightarrow z_0 = \cos 0 + i \sin 0 = 1$

$k=1 \Rightarrow z_1 = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) = \frac{-1}{2} + \frac{i\sqrt{3}}{2}$

$k=2 \Rightarrow z_2 = \cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) = \frac{-1}{2} - \frac{i\sqrt{3}}{2}.$



the Argand diagram.