Kinematic Singularities and Jacobians

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Introduction

- the position and orientation of the manipulator end-effecter were evaluated in relation to **joint displacements**.
- The joint displacements corresponding to a given end-effecter location were obtained by solving the **kinematic equation** for the manipulator
- In order to move the end-effecter in a **specified direction at a specified speed**, it is necessary to coordinate the motion of the individual joints.
- in order to coordinate joint motions, **we derive the differential relationship** between the joint displacements and the end-effecter location, and then solve for the individual joint motions.

Kinematic Singularities

- If we try to control a manipulator in Cartesian space, we can sometimes run into difficulties since the inverse mapping from Cartesian space to joint space can sometimes become a problem.
- These problem positions of the robot are referred to as **singularities**
- At a singularity, the **mobility** of a manipulator is **reduced**.
- Usually, arbitrary motion of the manipulator in a Cartesian direction is lost.
- This is referred to as "**Losing a DOF**".

Kinematic Singularities…

- Boundary Singularities (workspace singularities) are a common type of singularity.
- They are usually caused by a full extension of a joint, and asking the manipulator to move beyond where it can be positioned.
- Typically, this is trying to reach out of the workspace at the farthest extent of the workspace.
- Internal Singularities (joint space singularities).
- They are generally caused by an alignment of the robots axes in space.
- For example, if 2 axes become aligned in space, rotation of one can be canceled by counterrotation of the other, leaving the actual joint location indeterminate.

Jacobian Matrix

$$\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\\ \theta_6\end{bmatrix} \underset{\text{Inverse}}{\overset{\text{Forward}}{\rightleftarrows}} \begin{bmatrix}x\\ y\\ z\\ \phi\\ \theta\\ \varphi\end{bmatrix}$$

Kinematics

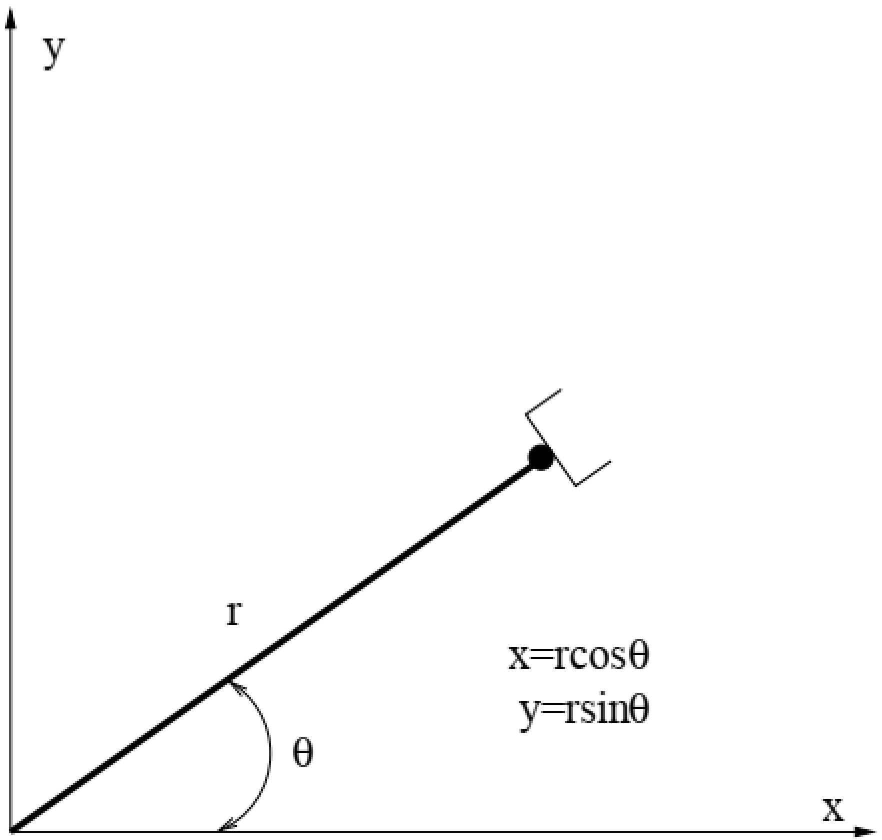
Joint Space Task Space

$$\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\\ \dot\theta_3\\ \dot\theta_4\\ \dot\theta_5\\ \dot\theta_6\end{bmatrix} \overset{\text{Jacobian Matrix}}{\Longleftrightarrow} \begin{bmatrix}\dot x\\ \dot y\\ \dot z\\ \omega_x\\ \omega_y\\ \omega_z\end{bmatrix}$$

Jaconian Matrix: Relationship between joint space velocity with task space velocity

Manipulator Jacobians

- Joint 1 is a revolute joint and joint 2 is a prismatic joint
- Let us find the rate of change of x and y
- their velocities, using the chain rule to differentiate x and y with respect to time t



2-DOF Polar, planar manipulator. The endpoint has coordinates (rCosθ ; rSinθ).

$$\frac{dx}{dt} = \frac{\partial(r\ cos\theta)}{\partial r}\frac{dr}{dt} + \frac{\partial(r\ cos\theta)}{\partial\theta}\frac{d\theta}{dt} \quad ==> \quad \dot{x} = cos\theta\dot{r} - r sin\theta\dot{\theta}$$

$$\frac{dy}{dt} = \frac{\partial(r\ sin\theta)}{\partial r}\frac{dr}{dt} + \frac{\partial(r\ sin\theta)}{\partial\theta}\frac{d\theta}{dt} \quad ==> \quad \dot{y} = sin\theta\dot{r} + r\ cos\theta\dot{\theta}$$

J

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial r} \end{bmatrix} \begin{bmatrix} \frac{d\theta}{dt} \\ \frac{dr}{dt} \end{bmatrix} \Longrightarrow \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} -r\ sin\ \theta & cos\ \theta \\ r\ cos\ \theta & sin\ \theta \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix}$$

Chain rule $\quad z = F(x, y) \quad dz = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy \quad \frac{dz}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}$

Manipulator Jacobians…

- The matrix which relates changes in joint parameter velocities to Cartesian velocities is called the **Jacobian Matrix**.
- The Jacobian for this manipulator is:

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = J \begin{bmatrix} \dot{\theta} \\ \dot{r} \end{bmatrix} \quad where \quad J = \begin{bmatrix} -r\,sin\,\theta & cos\,\theta \\ r\,cos\,\theta & sin\,\theta \end{bmatrix}$$

Manipulator Jacobians…

- **If we specify the Cartesian velocities, we can find the joint parameter velocities with the inverse Jacobian**.
- The inverse Jacobian is:

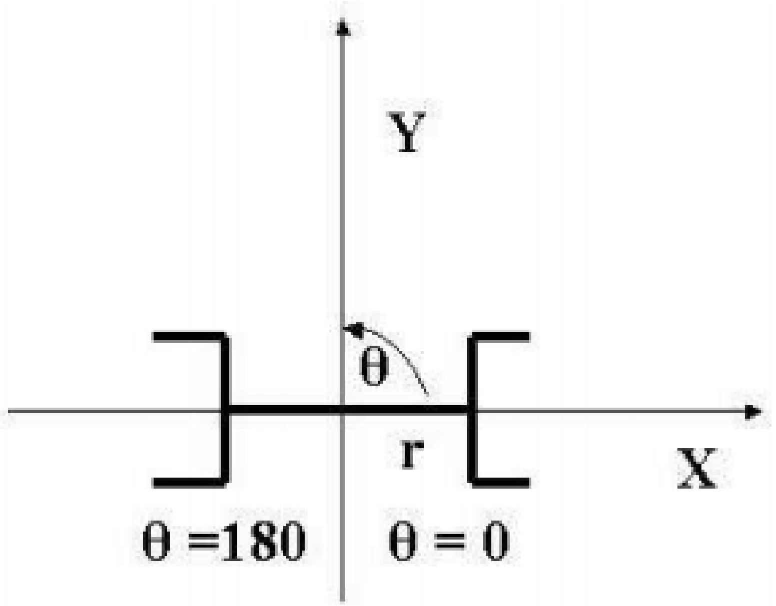
$$[J^{-1}]\begin{bmatrix}\dot{x}\\ \dot{y}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\\ \dot{r}\end{bmatrix} \quad where \quad J^{-1} = \begin{bmatrix}\frac{-\sin\theta}{r} & \frac{\cos\theta}{r}\\ \cos\theta & \sin\theta\end{bmatrix}$$

- A singularity occurs when the joint velocity in joint space becomes infinite to maintain Cartesian velocity.
- A singularity occurs whenever the determinant of the **Jacobian is 0**
- The associated Jacobian matrix is said to be singular.

$$det\,(J) = 0$$

Manipulator Jacobians…

- In this case, det (J) = −r. The determinant is 0 when r = 0.
- Since r is the radius of the manipulator, the robot has a singularity when we try to move through the origin in Cartesian space.
- At this point, the joint space velocity of joint 1 becomes infinite to achieve any Cartesian velocity vector



Jacobian of 2-link Revolute-Revolute (RR) Manipulator

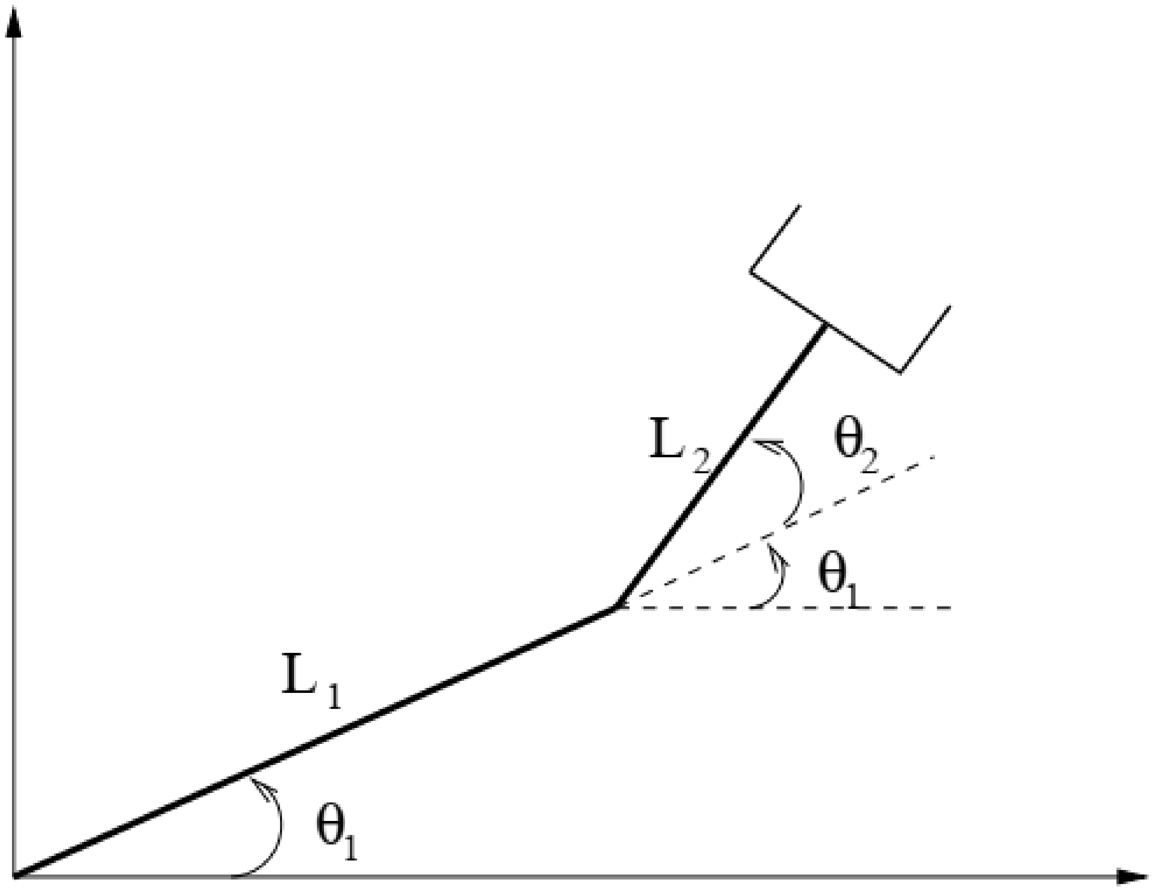


Figure 3: 2-Link RR Planar Manipulator

$$T_1^0 = \begin{bmatrix} C_1 & -S_1 & 0 & C_1L_1 \\ S_1 & C_1 & 0 & S_1L_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^1 = \begin{bmatrix} C_2 & -S_2 & 0 & C_2L_2 \\ S_2 & C_2 & 0 & S_2L_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad T_2^0 = \begin{bmatrix} C_{12} & -S_{12} & 0 & C_1L_1 + L_2C_{12} \\ S_{12} & C_{12} & 0 & S_1L_1 + L_2S_{12} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

we will find J such that

$$\begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = J \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \end{bmatrix} \quad ; \qquad J = \begin{bmatrix} \frac{\partial X}{\partial \Theta_1} & \frac{\partial X}{\partial \Theta_2} \\ \frac{\partial Y}{\partial \Theta_1} & \frac{\partial Y}{\partial \Theta_2} \end{bmatrix}$$

$$X = C_1L_1 + L_2C_{12}\ ;\ \frac{\partial X}{\partial t} = \frac{\partial(C_1L_1 + L_2C_{12})}{\partial\Theta_1}\frac{\partial\Theta_1}{\partial t} + \frac{\partial(C_1L_l + L_2C_{12})}{\partial\Theta_2}\frac{\partial\Theta_2}{\partial t}$$

$$\dot{X} = (-S_1L_1 - L_2S_{12})\dot{\Theta}_1 - L_2S_{12}\dot{\Theta}_2$$

$$Y = S_1L_1 + L_2S_{12}\ ;\ \frac{\partial Y}{\partial t} = \frac{\partial(S_1L_1 + L_2S_{12})}{\partial\Theta_1}\frac{\partial\Theta_1}{\partial t} + \frac{\partial(S_1L_1 + L_2S_{12})}{\partial\Theta_2}\frac{\partial\Theta_2}{\partial t}$$

$$\dot{Y} = (C_1L_1 + L_2C_{12})\dot{\Theta}_1 + L_2C_{12}\dot{\Theta}_2$$

$$and \quad \begin{bmatrix} \dot{X} \\ \dot{Y} \end{bmatrix} = \begin{bmatrix} -S_1L_1 - L_2S_{12} & -L_2S_{12} \\ C_1L_1 + L_2C_{12} & L_2C_{12} \end{bmatrix} \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \end{bmatrix}$$

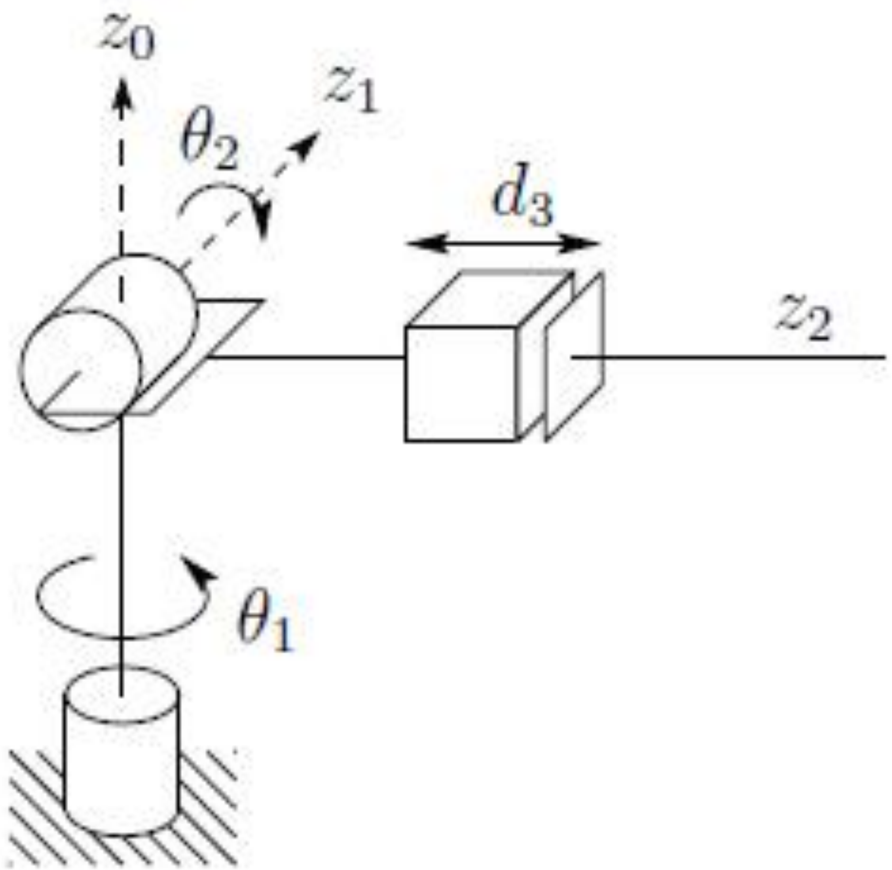
Finding Singularities of the 2-Link Manipulator

- If we invert the Jacobian, we get: $J^{-1}\dot{X} = \dot{\Theta}$
- The inverse is undefined whenever det (J)=0. So, by solving det (J)=0, we can find singularities in the robot workspace.

$$J = \begin{bmatrix} -S_1L_1 - L_2S_{12} & -L_2S_{12} \\ C_1L_1 + L_2C_{12} & L_2C_{12} \end{bmatrix}$$

$$\begin{aligned} det\,(J) &= (-S_1L_1 - L_2S_{12})(L_2C_{12}) + (L_2S_{12})(C_1L_1 + L_2C_{12}) \\ &= -S_1L_1L_2C_{12} - L_2^2S_{12}C_{12} + C_1L_1L_2S_{12} + L_2^2S_{12}C_{12} \\ &= L_1L_2(S_{12}C_1 - C_{12}S_1) = L_1L_2(S(\Theta_1 + \Theta_2 - \Theta_1)) \\ & \quad det\,(J) \quad = \quad L_1L_2S\Theta_2 \end{aligned}$$

RRP robot(Stanford arm)



$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} C_1S_2d_3 - S_1d_2 \\ S_1S_2d_3 + C_1d_2 \\ C_2d_3 + d_1 \end{bmatrix} ; \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = J \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{d}_3 \end{bmatrix} \; ; \begin{bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{bmatrix} = \begin{bmatrix} \frac{\partial X}{\partial \Theta_1} & \frac{\partial X}{\partial \Theta_2} & \frac{\partial X}{\partial d_3} \\ \frac{\partial Y}{\partial \Theta_1} & \frac{\partial Y}{\partial \Theta_2} & \frac{\partial Y}{\partial d_3} \\ \frac{\partial Z}{\partial \Theta_1} & \frac{\partial Z}{\partial \Theta_2} & \frac{\partial Z}{\partial d_3} \end{bmatrix} \begin{bmatrix} \dot{\Theta}_1 \\ \dot{\Theta}_2 \\ \dot{d}_3 \end{bmatrix}$$

$$J = \begin{bmatrix} -S_1S_2d_3 - C_1d_2 & C_1C_2d_3 & C_1S_2 \\ C_1S_2d_3 - S_1d_2 & S_1C_2d_3 & S_1S_2 \\ 0 & -S_2d_3 & C_2 \end{bmatrix}$$

Singularities: when det (J)=0

$$\begin{aligned} det\,(J) &= S_2d_3(S_1S_2(-S_1S_2d_3 - C_1d_2) - C_1S_2(C_1S_2d_3 - S_1d_2)) \\ &\quad +C_2((-S_1S_2d_3 - C_1d_2)(S_1C_2d_3) - (C_1S_2d_3 - S_1d_2)(C_1C_2d_3)) \\ &= S_2d_3[-S_1^2S_2^2d_3 - S_1S_2C_1d_2 - C_1^2S_1^2d_3 + C_1S_1S_2d_2] \\ &\quad +C_2[-S_1^2S_2C_2d_3^2 - C_1S_1C_2d_2d_3 - C_1^2S_2C_2d_3^2 + C_1S_1C_2d_2d_3] \\ &= S_2d_3[-d_3S_2^2[C_1^2 + S_1^2] + C_2[-S_2C_2d_3^2[S_1^2 + C_1^2]] \\ &= S_2d_3[-d_3S_2^2] - C_2^2S_2d_3^2 = -S_2^2d_3^2 - C_2^2S_2d_3^2 = -S_2d_3^2[S_2^2 + C_2^2] \\ det\,(J) &= -S_2d_3^2 \end{aligned}$$