GENERAL SIR JOHN KOTELAWALA DEFENCE UNIVERSITY

Faculty of Engineering
Department of Mathematics

BSc Engineering Degree
Semester 3 Examination - July 2022
(Intake 38 - All Engineering Streams)

MA 2103 - ADVANCED CALCULUS

Time : 3 hours — 25 July, 2022

INSTRUCTIONS TO CANDIDATES

- This paper contains 5 questions on 3 pages
- Answer all questions
- This is a closed book examination
- This examination accounts for 70% of the module assessment. A total maximum mark obtainable is 100. The marks assigned for each question and parts thereof are indicated in square brackets
- If you have any doubt as to the interpretation of the wordings of a question, make your own decision, but clearly state it on the script
- No penalty will be applied for incorrect answers
- Assume reasonable values for any data not given in or provided with the question paper, clearly make such assumptions made in the script Marks will be lost if all necessary work is not clearly shown.
- All examinations are conducted under the rules and regulations of the KDU.

Please go on to the next page. . .

Question 1

(a) Find the Fourier Sine Transform of $f(x) = e^{-x}$.
Hence show that

$$\int_0^\infty \frac{t \sin mt}{1+t^2}\, dt = \frac{\pi}{2} e^{-m}.$$

[40]

(b) Show that the Fourier Transform of $f(x) = \begin{cases} 1-x^2 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}$ is

$$2\sqrt{\frac{2}{\pi}}\left(\frac{\sin w - w\cos w}{w^3}\right).$$

Hence deduce that

$$\int_0^\infty \frac{\sin x - x\cos x}{x^3}\, dx = \frac{\pi}{4}.$$

[60]

[100 Marks]

Question 2

(a) (i) Show that the limit $\lim_{(x,y)\to(0,0)} \frac{xy\cos y}{3x^2+y^2}$ does not exist. [10]

(ii) Show that the function $f(x,y) = \sqrt{x} + \sqrt{1-x^2-y^2}$ is continuous at $\left(\frac{1}{2}, \frac{1}{2}\right)$. [10]

(b) (i) The pressure P (in kilo-pascals), volume V (in liters), and temperature T (in kelvins) of a mole of an ideal gas are related by the equation $PV = 8.31T$. The pressure of 1 mole of an ideal gas is increasing at a rate of $0.05kPa/s$ and the temperature is increasing at a rate of $0.15K/s$. Find the rate of change of the volume when the pressure is $20kPa$ and the temperature is $320K$. [20]

(ii) A surface S is defined by the Cartesian equation

$$z = x\sin(x+y).$$

Find an equation of the tangent plane on S at the point $(-1, 1, 0)$. [20]

(c) Consider the function

$$f(x,y) = x^4 + y^4 - 4xy + 1.$$

(i) Find all critical points of f. [20]

(ii) Determine the nature of the points in part (i). [20]

[100 Marks]

Question 3

(a) Evaluate the triple integral

$$\iiint_E 4x^2z^3\, dV,$$

where E is a solid region given by

$$E = \{(x, y, z) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1 - x,\ 0 \leq z \leq \sqrt{y}\}.$$

[20]

(b) Consider the vector field

$$\mathbf{F} = \sin y\ \mathbf{i} + (x\cos y + \cos z)\ \mathbf{j} - y\sin z\ \mathbf{k}.$$

(i) Show that $\mathbf{F}$ is conservative over space. [10]

(ii) Determine the potential function for $\mathbf{F}$. [20]

(iii) Using Part (ii), evaluate the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where C is the curve given by $\mathbf{r}(t) = \sin t\ \mathbf{i} + t\ \mathbf{j} + 2t\ \mathbf{k},\ 0 \leq t \leq \frac{\pi}{2}$. [20]

(c) Use the **divergence theorem** to calculate the outward flux of $\mathbf{F}$ across S if

$$\mathbf{F}(x, y, z) = 3xy^2\ \mathbf{i} + +xe^z\ \mathbf{j} + z^3\ \mathbf{k}$$

and S is the surface of the solid bounded by the cylinder $y^2 + z^2 = 1$ and the planes $x = 1$ and $x = 2$.

[30]

[100 Marks]

Question 4

The one dimensional heat equation for the temperature, $u(x, t)$, satisfies

$$\frac{\partial u}{\partial t} - 2\frac{\partial^2 u}{\partial x^2} = 0, \text{ for } 0 < x < 5,\ t \geq 0,$$

where t is the time and x is a spatial dimension.

(a) Using the method of separation of variables, show that the heat equation can be reduced to two ordinary differential equations (ODEs) of the form

$$X''(x) - CX(x) = 0$$

and

$$T'(t) - 2CT(t) = 0$$

where C is the separation constant.

[20]

(b) By solving the ODEs in part (a) for the $C = -\lambda^2$, determine the solution of the heat equation subject to the given Dirichlet boundary conditions $u(0,t) = u(5,t) = 0$ and the initial condition $u(x,0) = 3\sin 2\pi x$. [80]

[100 Marks]

Question 5

(a) It is given that $u = u(x,y)$ satisfies the partial differential equation

$$3\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} - 5\frac{\partial^2 u}{\partial y^2} = (y-x)e^x.$$

Find the general solution of the above partial differential equation. [45]

(b) The temperature $u(x,t)$ at any point of an infinite bar satisfies the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

for which $-\infty < x < \infty$ and $t > 0$.
The initial temperature along the length of the bar is given by

$$u(x,0) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| > 1 \end{cases}.$$

Using the **Fourier Transforms**, show that

$$u(x,t) = \frac{2}{\pi}\int_0^\infty e^{-w^2 t}\left(\frac{\sin w \cos wx}{w}\right) dw.$$

[55]

[100 Marks]

End of the Question Paper.