Laplace Transform

- The Laplace transform is a mathematical operation that takes an equation from being a function of time, *t*, to being a function of the Laplace variable *s*.

<u>Definition:</u>

- For a function *f(t)* (*f(t)=0* for *t<0*):

$$\boxed{F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty f(t)e^{-st}dt}$$

  s is a complex variable

  F(s) is the Laplace transform of *f(t)*.

- We will never solve this integral but use the tables derived from it.

Common Laplace Functions

	$f(t)$	$F(s)$
Impulse	$\delta(t)$	1
Step	$1(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
	t^n	$\frac{n}{s^n}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Properties of the Laplace Transform (1)

1. Linearity

$$\begin{aligned} \mathcal{L}[a.f(t) + b.g(t)] &= a\mathcal{L}[f(t)] + b\mathcal{L}[g(t)] \\ &= aF(s) + bG(s) \end{aligned}$$

- Constants factor out and Laplace operation distributes over addition and subtraction.

Note: $\mathcal{L}[f(t).g(t)] \neq F(s).G(s)$

Properties of the Laplace Transform (2)

2. Integration

$$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s} + \boxed{\frac{[\int f(t)]_{t=0}}{s}}$$

Initial conditions

3. Differentiation

$$\mathcal{L}\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

$$\mathcal{L}\left[\frac{d^2f}{dt^2}\right] = s^2F(s) - s.f(0^-) - f^1(0^-)$$

Initial conditions

Properties of the Laplace Transform (4)

4. Multiplication by e^{-at}

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

- Important for damped response.

<u>Example:</u> $\mathcal{L}[e^{-at}\underbrace{\cos(\omega t)}_{f(t)}]$

From Laplace Table: $F(s) = \mathcal{L}[\cos(\omega t)] = \dfrac{s}{s^2+\omega^2}$

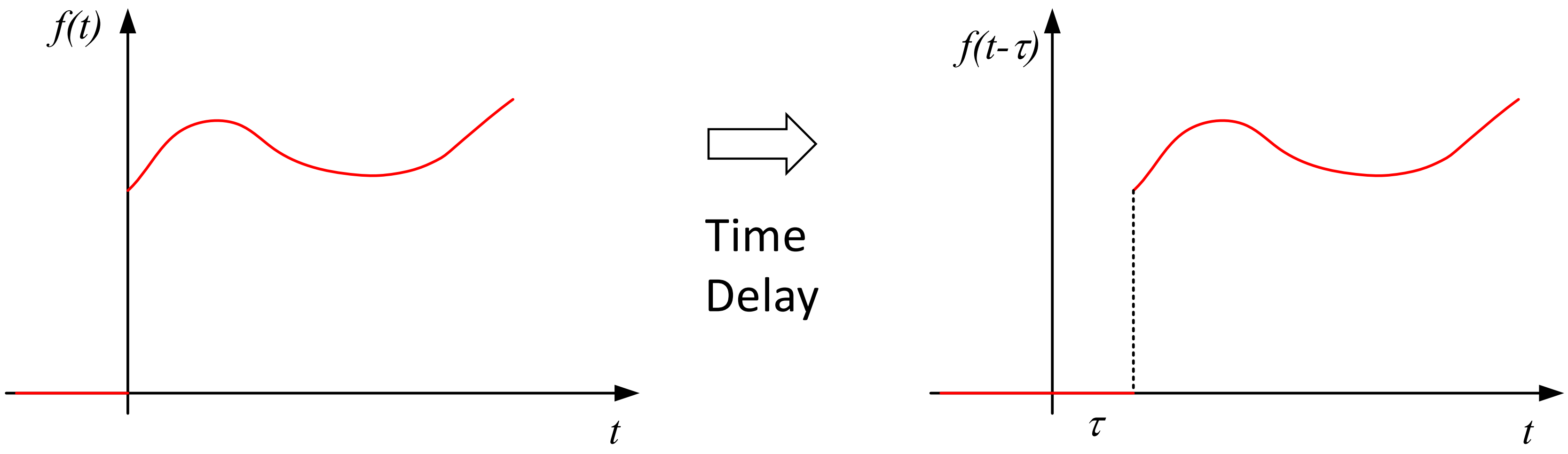
From Proposition above: $F(s+a) = \dfrac{s+a}{(s+a)^2+\omega^2}$

Properties of the Laplace Transform (5)

5. Time shift

$$\mathcal{L}[f(t-a)1(t-a)] = e^{-as}F(s), a > 0$$

Important for analysing time delays



Example:

Find $\mathcal{L}[\underbrace{2te^{-3t} + 5]}_{\textit{0 for } t<0}$

$= 2.\mathcal{L}[te^{-3t}] + 5.\mathcal{L}[1(t)]$

$\Rightarrow \mathcal{L}[t] = \dfrac{1}{s^2}$ $\Rightarrow \mathcal{L}[te^{-3t}] = \dfrac{1}{(s+3)^2}$ (by property 4)

$= \boxed{\dfrac{2}{(s+3)^2} + \dfrac{5}{s}}$

Laplace/ Time Domain Relationship

- Initial Value Theorem:

This theorem is used to relate frequency domain expressions to the time domain behavior as time approaches zero

$$f(0^+) = \lim_{t \to 0} f(t) = \lim_{s \to \infty} sF(s) \quad \text{, if the lim exists}$$

- Final Value Theorem:

This theorem is used to relate frequency domain expressions to the time domain behavior as time approaches infinity.

$$f(\infty) = \lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

Example: Find the initial value of $f(t)$ where, $F(s) = \frac{s+3}{s(s^2+6s+13)}$

$$f(0^+) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} \cancel{s}.\frac{s+3}{\cancel{s}(s^2+6s+13)}$$

$$= \lim_{s\to\infty} \frac{s+3}{(s^2+6s+13)}$$

$$= \lim_{s\to\infty} \frac{1/s+3/s^2}{(1+6/s+13/s^2)}$$

$$= 0$$

<u>Example:</u> Find the final value of $f(t)$ where, $F(s) = \frac{s+3}{s(s^2+6s+13)}$

$$f(\infty) = \lim_{s\to 0} sF(s) \quad = \lim_{s\to 0} \cancel{s}.\frac{s+3}{\cancel{s}(s^2+6s+13)}$$

$$= \lim_{s\to 0} \frac{s+3}{(s^2+6s+13)}$$

$$= \lim_{s\to 0} \frac{0+3}{(0^2+6.0+13)}$$

$$= 3/13$$

Example

Find $\mathcal{L}^{-1}\left[\frac{3}{s+1}\right] = 3.\mathcal{L}^{-1}\left[\frac{1}{s+1}\right]$

$$= 3.e^{-t}, \mathrm{t} \geq 0$$

	$\boldsymbol{f(t)}$	$\boldsymbol{F(s)}$
Impulse	$\delta(t)$	1
Step	$1(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
	t^n	$\frac{n}{s^n}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Example

Find $\mathcal{L}^{-1}\left[\frac{5}{s^2+4s+13}\right]$

$$= \mathcal{L}^{-1}\left[\frac{5}{(s+2)^2+3^2}\right]$$

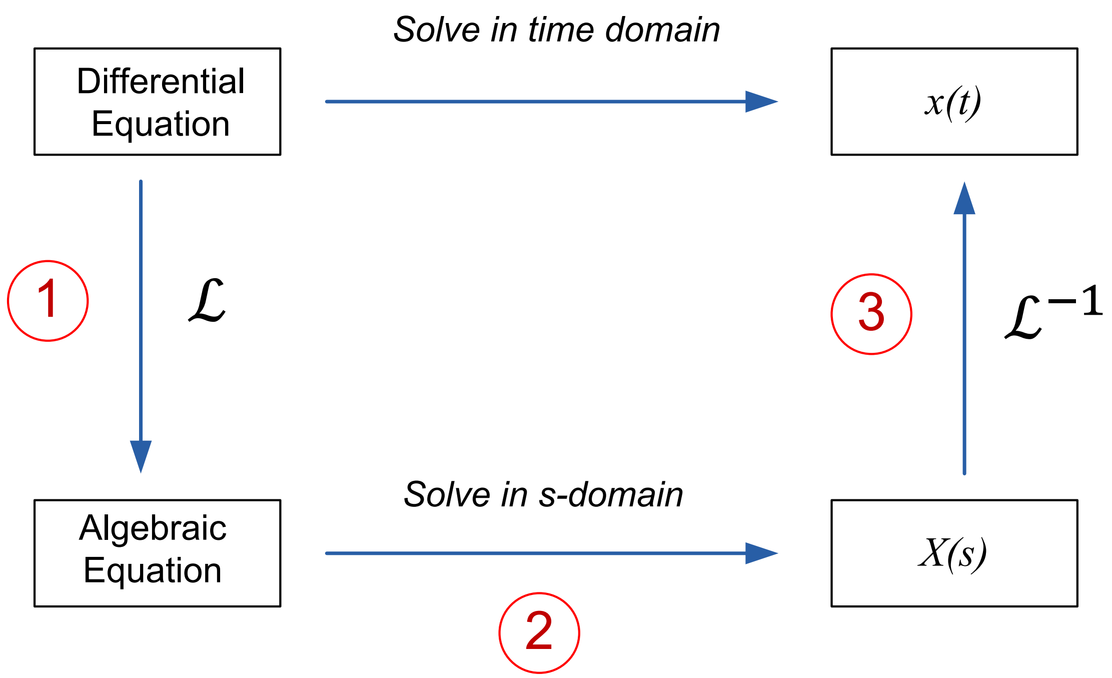
$$= \frac{5}{3}.\mathcal{L}^{-1}\left[\frac{3}{(s+2)^2+3^2}\right]$$

$$= \frac{5}{3}.e^{-2t}.\sin 3t, t \geq 0$$

	$f(t)$	$F(s)$
Impulse	$\delta(t)$	1
Step	$1(t)$	$\frac{1}{s}$
Ramp	t	$\frac{1}{s^2}$
	t^n	$\frac{n}{s^n}$
Exponential	e^{-at}	$\frac{1}{s+a}$
Sine	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
Cosine	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Solving Linear Differential Equations Using Laplace Transform

- Laplace transform converts Linear Time Invariant (LTI) differential equations to an algebraic equation (AE).

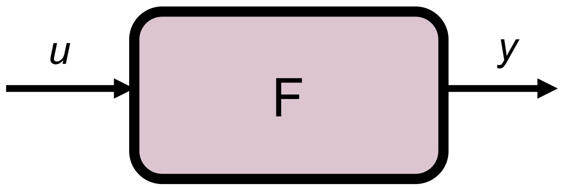


Systems

- In this unit, we are mainly concerned with systems which are linear and time-invariant, or which can be approximated by linear time-invariant (or LTI) systems.
- *Real systems* are almost all non-linear and time-varying, however many can be reasonably approximated by LTI equations (especially close to an equilibrium), and can be controlled using linear controllers.

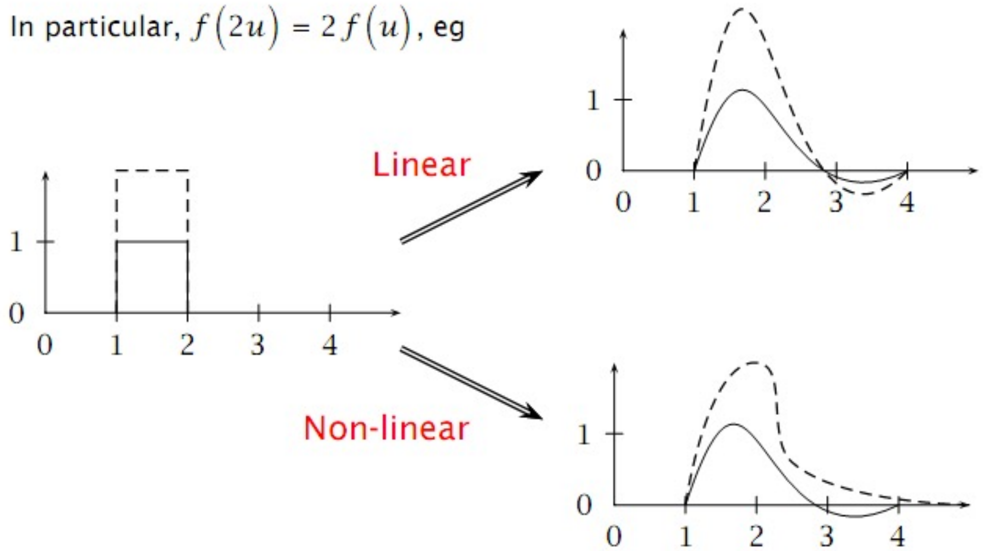
Linearity

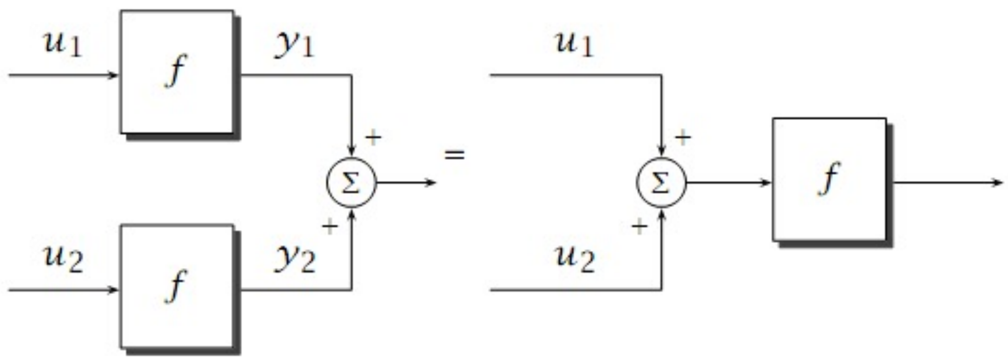
- A system is described as linear if it obeys the principles of *additivity* and *homogeneity*
  - Additive: adding inputs → adding outputs
  - Homogeneous: scaling inputs → scaling outputs



$$F(a.u_1 + b.u_2) = a.F(u_1) + b.F(u_2)$$

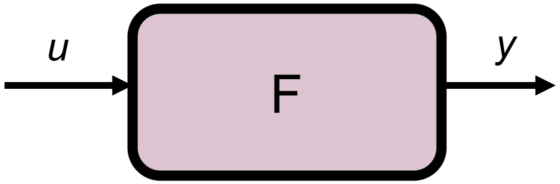
In particular, $f(2u) = 2f(u)$, eg



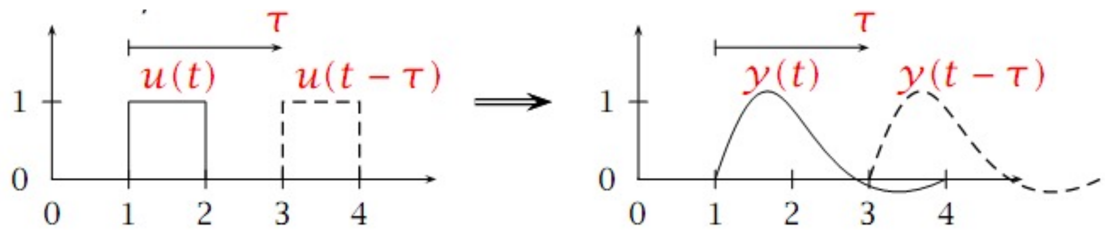


Time-invariance

- A system is described as time-invariant if a delayed input results in the same output delayed by the same time:

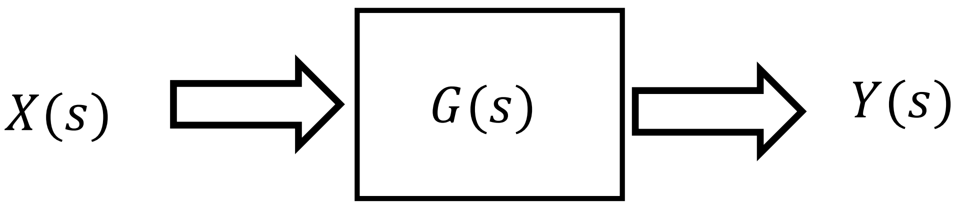


$$u(t) \rightarrow y(t) \Rightarrow u(t-\tau) \rightarrow y(t-\tau)$$



Transfer Functions

- A transfer function gives the relationship between the input and output. It is often expressed in s domain.
- A typical transfer function in given by:



$$G(s) = \frac{Y(s)}{X(s)}$$

- Usually the *G(s)* is proper. This means that:

$$\deg[X(s)] \geq \deg[Y(s)]$$

- Suppose we factorize the denominator, it would be given by:

$$G(s) = \frac{Y(s)}{(s - p_1)(s - p_2)\ldots(s - p_n)}$$

What is an Electromechanical System?

Electromechanical systems comprise both machinery and processes that are made up of both

electrical and mechanical components.

These systems can be as simple as a push button motor starting circuit or as complex as a complete industrial process.

Modelling Mechanical Systems

<u>Approach</u>

1. Choose coordinates and orientation.
2. Draw free body diagrams of each inertia.
   - Note Assumptions made.
3. General equations of motion using Newton's 2nd Law and Euler's 2nd Law

$$\boxed{\sum F = ma} \qquad \boxed{\sum T = J\ddot{\theta}}$$

4 Double check

One Dimensional Translational Mechanical Elements

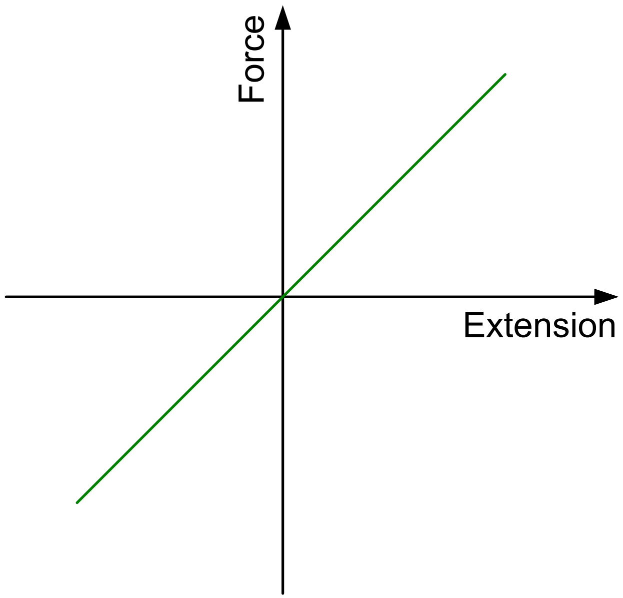
- Spring Elements
- Damper Elements
- Mass or inertia elements

Constitutive Relation:

Functional relationship between the relative displacement of the two end points and the force is transmitted through the elements.

Translational Springs

- Force is generated to resist deflection.
- Spring elements store potential energy.
- For linear translational Spring

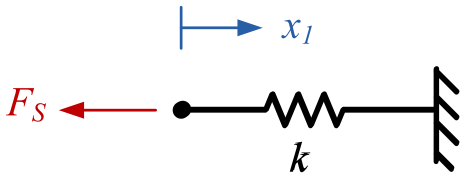


$$F \propto \Delta x$$

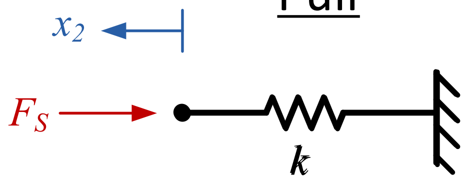
$$F = k.x$$

Free-fixed ends:

Push



Pull

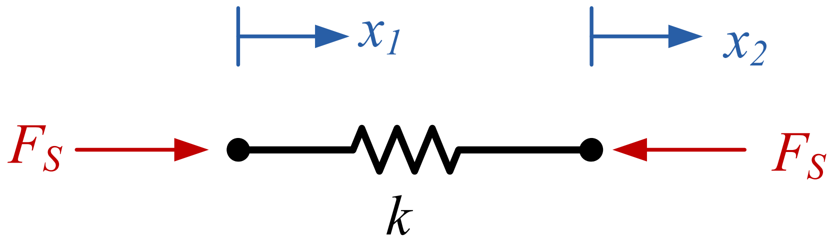


F_s is called spring force.

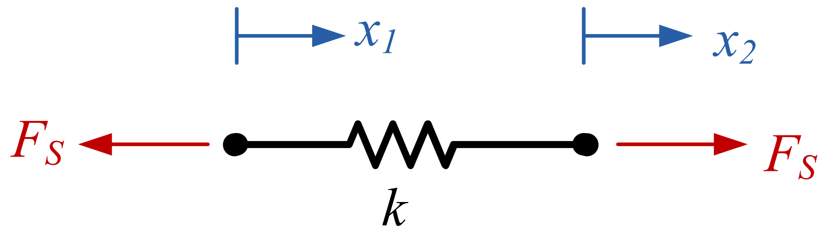
F_s is positive when directed opposite to the displacement.

Free-free Ends:

Pull ($x_2 > x_1$)



Push ($x_2 < x_1$)



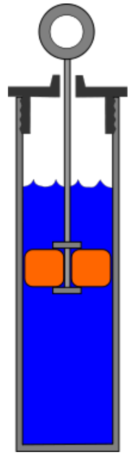
Computing Forces:

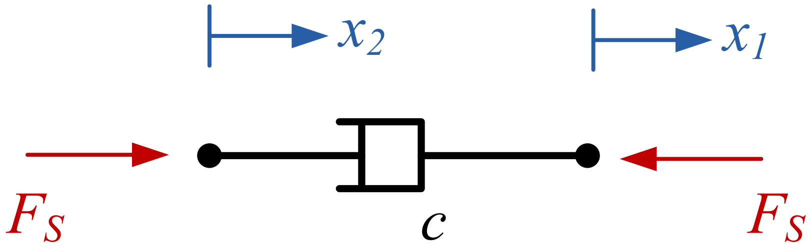
$$F_S = k(x_2 - x_1)$$

$$F_S = k(x_1 - x_2)$$

Translational Damper

- Force is generated to resist motion.
- Examples: dashpot, friction, wind drag.
- Damper dissipates energy (different from Spring and Mass)





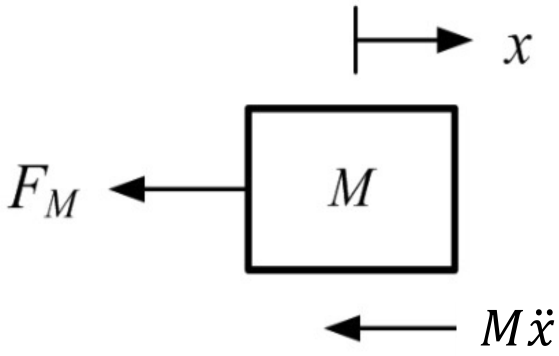
Constitutive Relation,

$$F_D = c(\dot{x}_1 - \dot{x}_2)$$

c - damping coefficient

Power dissipation = Force x Velocity

Mass Element



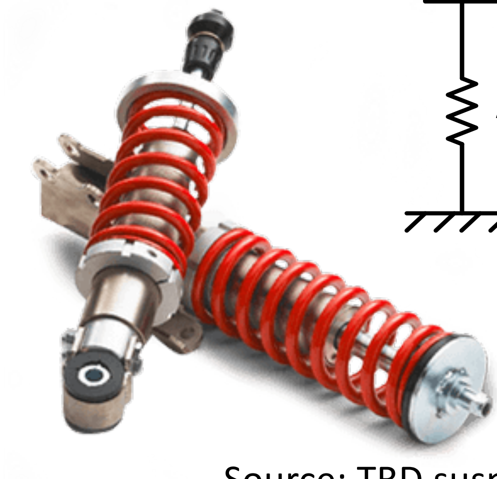
Force balance,

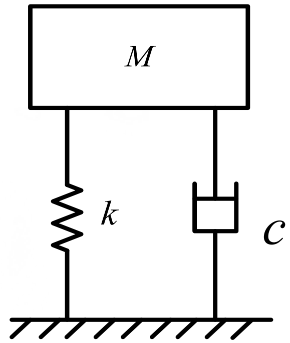
$$F_M - M\ddot{x} = 0$$

$$F_M = M\ddot{x}$$

Example of a System with a Mass, Spring and Damper

Source: Honda Motors

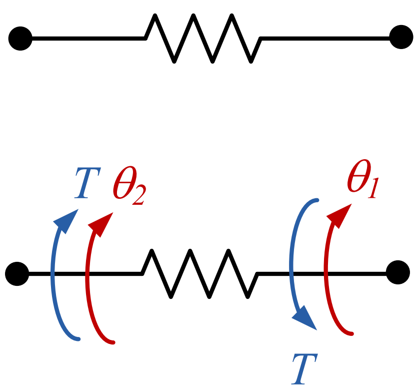




Source: TRD suspension

Torsional Systems

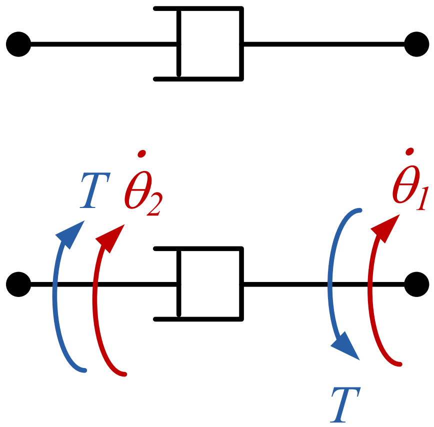
Torsional Spring



$$T = k(\theta_1 - \theta_2)$$

k torsional stiffness

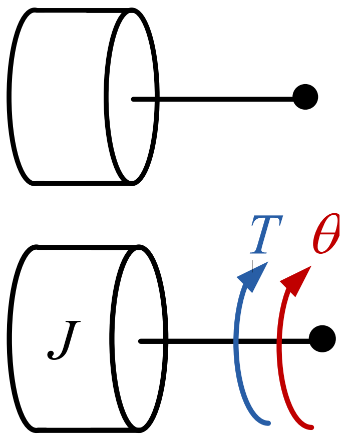
Torsional Damper



$$T = C.(\dot{\theta}_1 - \dot{\theta}_2)$$

C torsional dashpot constant

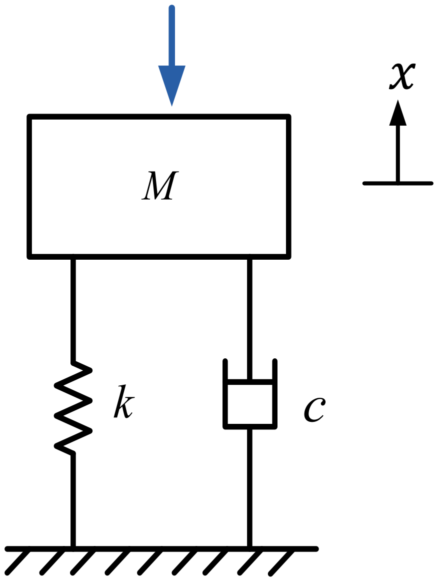
Inertia Element

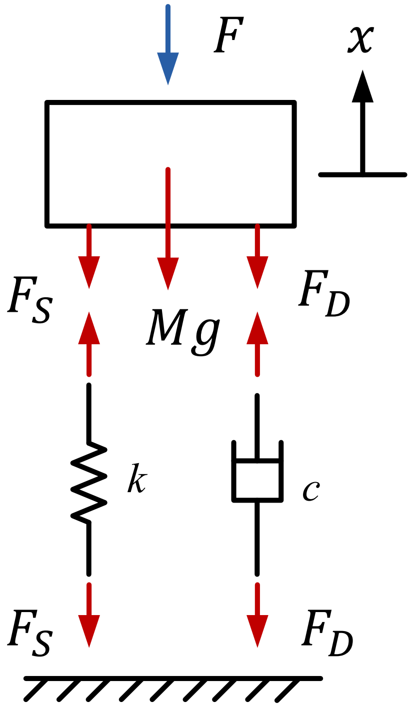


$$T = J\ddot{\theta}$$

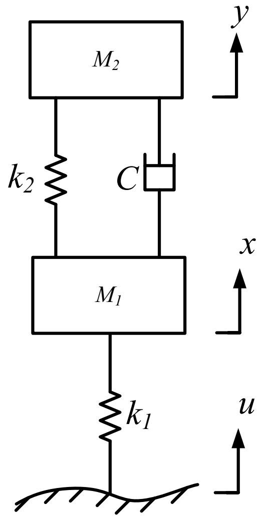
J mass moment of inertia

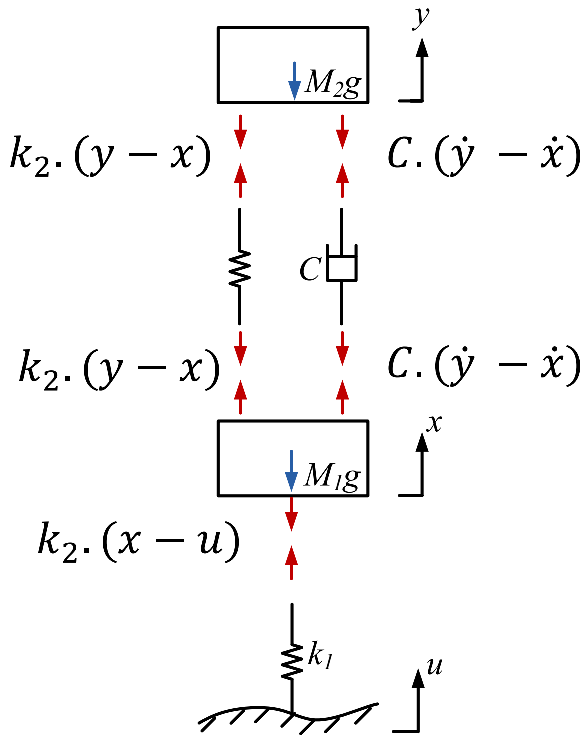
Example 1



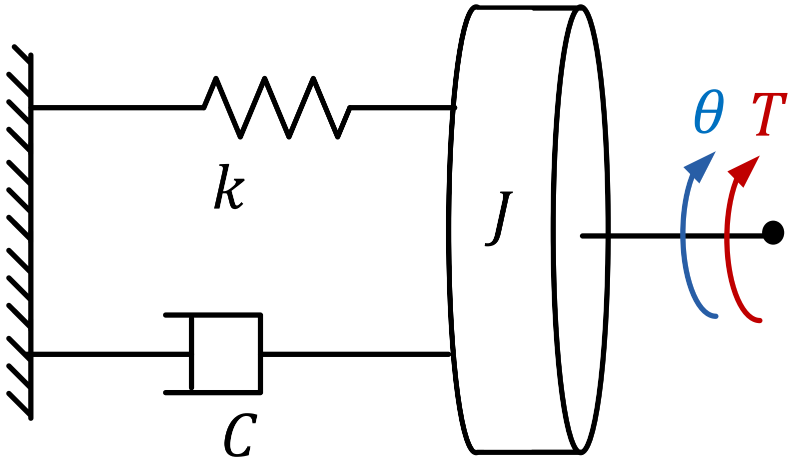


The Quarter Car Model (2)





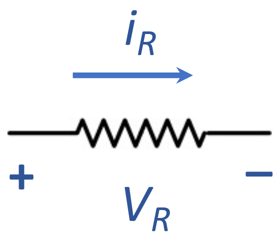
Example 3



Circuit Analysis In Laplace Domain

In s-domain,

Resistors

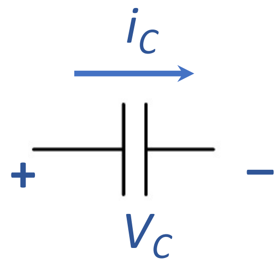


$$V = IR$$

$$V(s) = RI(s)$$

$$I(s) = \frac{1}{R}V(s)$$

Capacitors

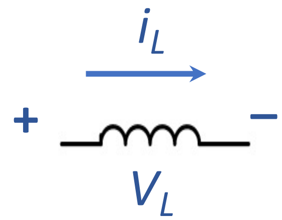


$$I = C\frac{dV}{dt}$$

$$V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}$$

$$I(s) = sCV(s) - Cv(0^-)$$

Inductors



$$V = L\frac{dI}{dt}$$

$$V(s) = sLI(s) - Li(0^-)$$

$$I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}$$

Analyzing Electrical Systems - Method of Generalized Impedances (1)

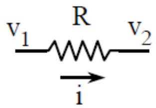
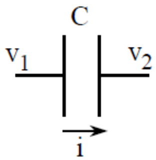
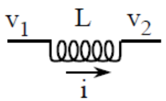
- The impedance description of an electrical device is governed by the relation:

$$V(s) = Z(s)I(s)$$

- Where $Z(s)$ is called the (s-domain) generalized impedance of the device.
- Note that the impedance description is only valid for electrical circuits with zero initial conditions.

Analyzing Electrical Systems - Method of Generalized Impedances (2)

- The generalized impedance for a resistor, capacitor and inductor can be summarized as follows:

Element	Schematic	Temporal Domain	Laplace Domain	Generalized Impedance
Resistance: **R**		$v_1 - v_2 = Ri$	$V_1(s) - V_2(s) = RI(s)$	$Z_R = R$
Capacitance: **C**		$\begin{cases} i = C\dfrac{d}{dt}(v_1 - v_2) \\ v_1 - v_2 = \dfrac{1}{C}\int i(t)\,dt \end{cases}$	$V_1(s) - V_2(s) = \dfrac{1}{Cs}I(s)$	$Z_C = \dfrac{1}{Cs}$
Inductance: **L**		$v_1 - v_2 = L\dfrac{di}{dt}$	$V_1(s) - V_2(s) = sLI(s)$	$Z_L = Ls$