EE 1102 – Fundamentals of Electrical Engineering

Step Response of RL and RC Circuits

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Learning outcomes

After successful completion of this module, you should be able to

1. Describe the properties of unit steps and unit impulse

2. Solve simple circuits for step response.

2

Outline Syllabus

Overview of Electrical Engineering (2 hrs)

Role of Electrical Engineer, Introduction to Generation, Transmission, Distribution and Utilisation.

SI Units (2 hrs)

Basic and supplementary units, Derived units, Symbols

Basic DC circuit analysis (4 hrs)

Circuit elements, Circuit laws, Circuit solutions with DC – Step Response

Network Theorems (4 hrs)

Ohm's Law and Kirchhoff's Laws, Other network theorems.

Alternating Current theory (8 hrs)

Sinusoidal waveform, phasor and complex representation, Impedance and Admittance, Power and Energy, Power factor. Solution of simple R, L, and C circuit problems by phasor and complex variables.

Electrostatic and Electromagnetic theory (4hrs)

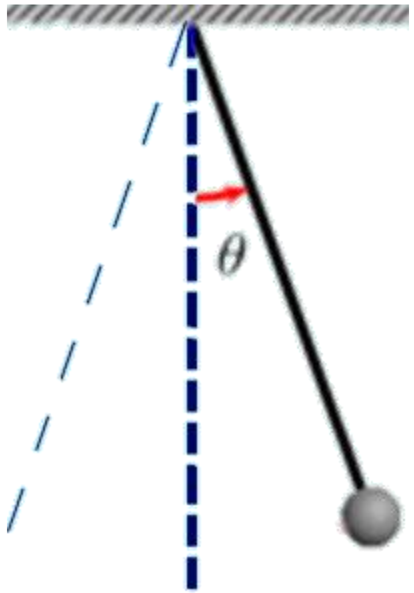
Basic Laws, Calculation of field and force

Electrical Installations (4 hrs)

Fuses, miniature circuit breakers, earth leakage circuit breakers, residual current circuit breakers, earthing, electric shock. Wiring regulations, basic domestic installations.

Natural Behaviour of R-L-C Circuits

- Does not depend on the external forcing functions
- Depends on internal properties of the system.



Give a pendulum an initial swing and let go

- behaviour depends on natural frequency
- if we push at some other rate, behaviour would also depend on forcing frequency.

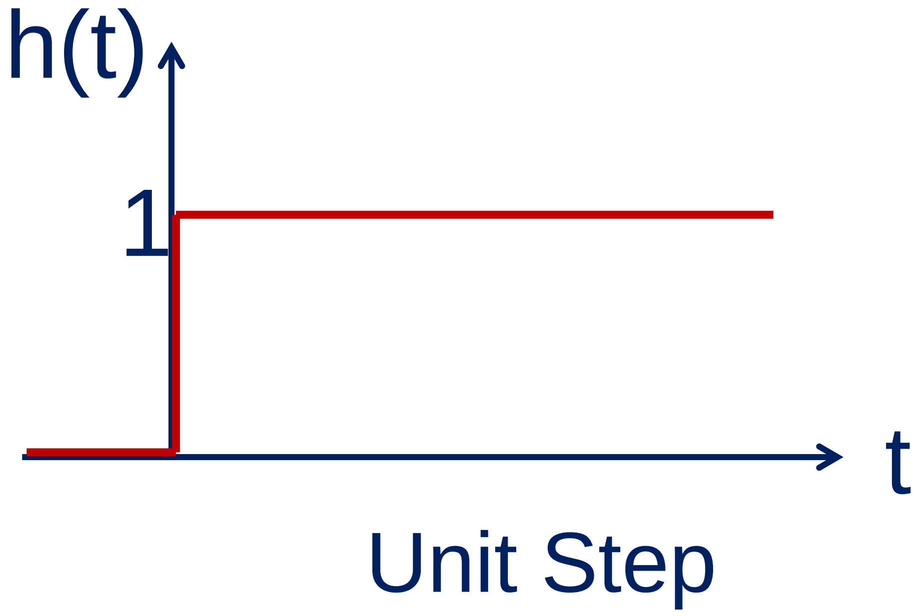
To determine the natural behaviour

- forcing function must not have its own specific frequency –step function, impulse function.

Unit Step Function H(t)

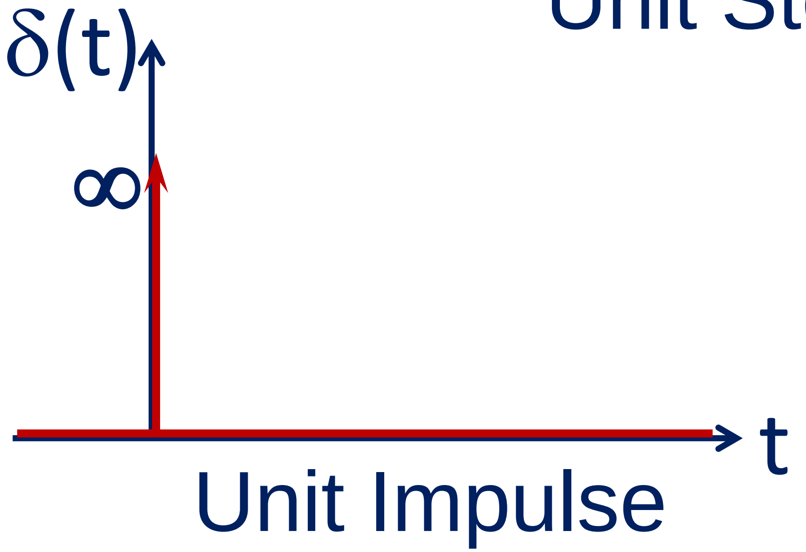
- similar to a staircase step

$$h(t) = 0, \qquad t < 0$$
$$h(t) = 1, \qquad t > 0$$



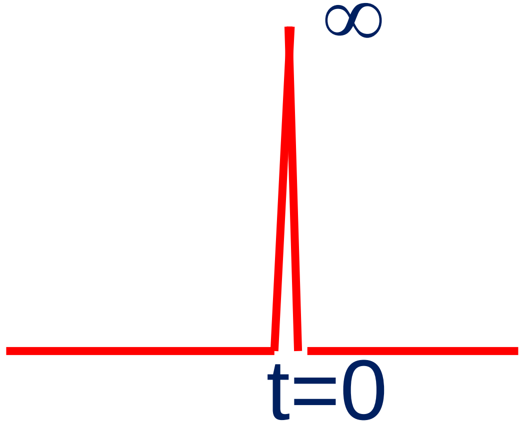
Unit Impulse Function δ(t)

$$\delta(t) = 0 \text{ ,} t < 0$$
$$\delta(t) = \infty \text{ ,} t = 0$$
$$\delta(t) = 0 \text{ ,} t > 0$$
$$\int_{-\infty}^{\infty} \delta(t) \cdot dt = 1$$



Some properties of Unit Impulse δ(t)

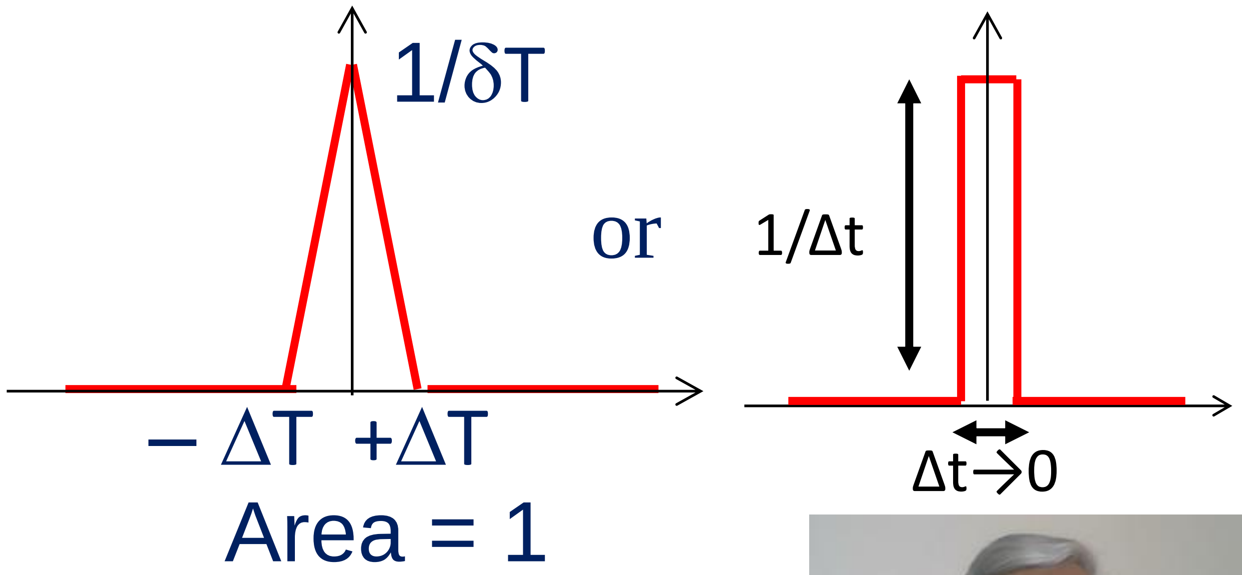
Area under curve is unity. $$\int_{-\infty}^{\infty} \delta(t)\cdot dt = 1$$



This gives $$\int_{0^-}^{0^+} \delta(t)\cdot dt = 1$$

Also $$\int_{-\infty}^{\infty} f(t)\cdot\delta(t)\cdot dt = f(0)$$

and $$\int_{-\infty}^{\infty} f(t-\tau)\cdot\delta(t)\cdot dt = f(\tau)$$



Transient Analysis

Solution of ordinary differential equation

- Particular integral → steady state solution
- Complementary function → transient solution

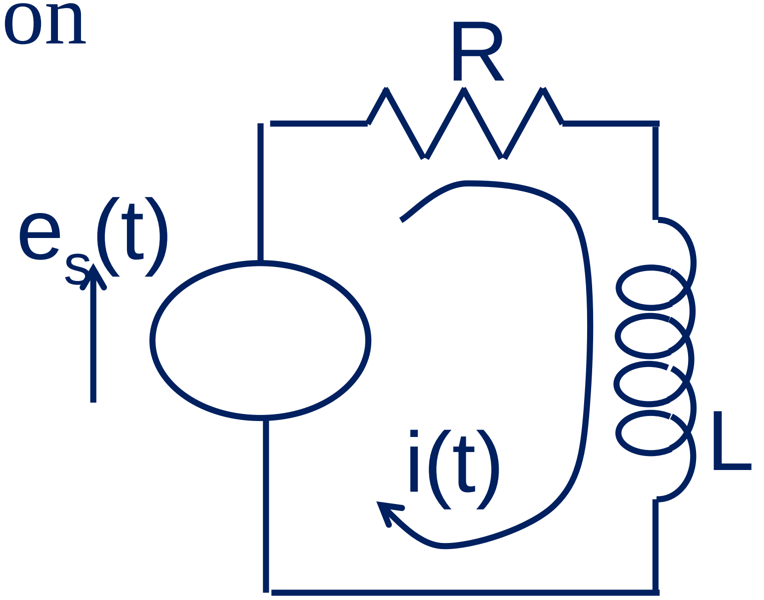
Complementary function → transient solution

Series R-L circuit

With step excitation $e_s(t) = E.H(t)$

governing behaviour is

$$L\frac{di}{dt} + R \cdot i = e_s(t) = E \cdot H(t)$$



Series R-L circuit

Particular integral → ***E/R***

Complementary function → $\boldsymbol{L\,p\,i + R\,i = 0}$

Solution of $L\dfrac{di}{dt} + R \cdot i = e_s(t) = E \cdot H(t)$

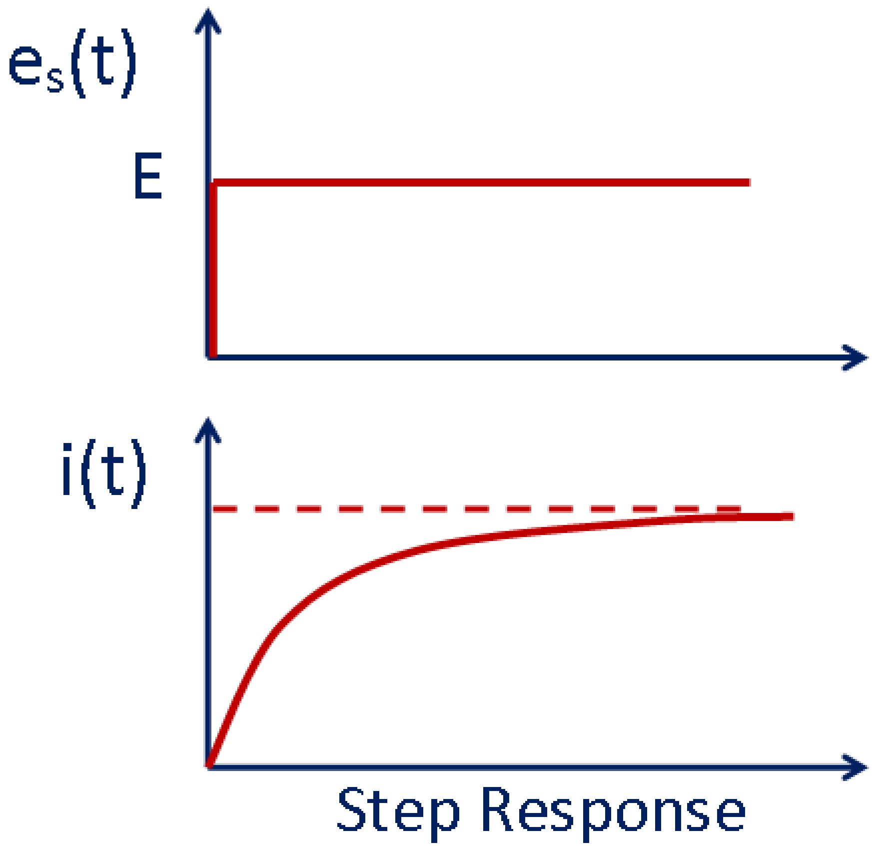
has the form $i(t) = A.e^{-\frac{R}{L}t} + \dfrac{E}{R}$

A is obtained from initial conditions

At t = 0, i = 0

$$\therefore A = -\frac{E}{R}$$

$$\therefore i(t) = \frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)$$



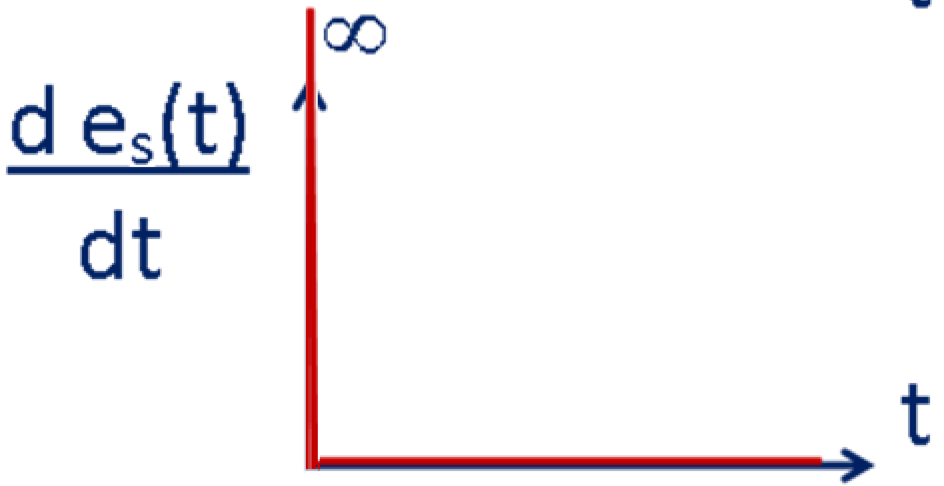
Relationship of Unit Step with Unit Impulse

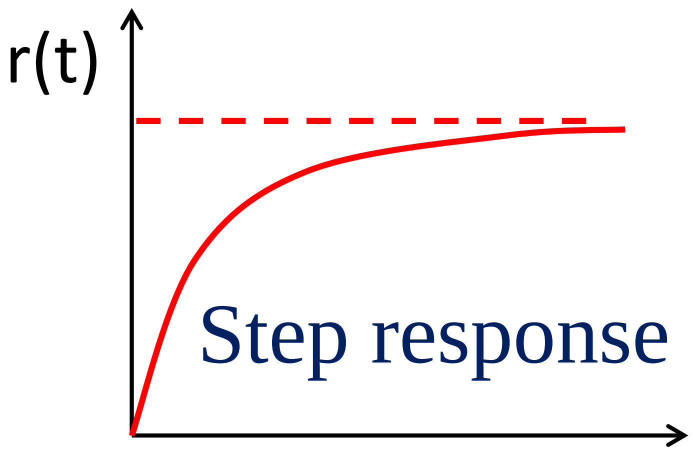
Unit impulse δ (t)

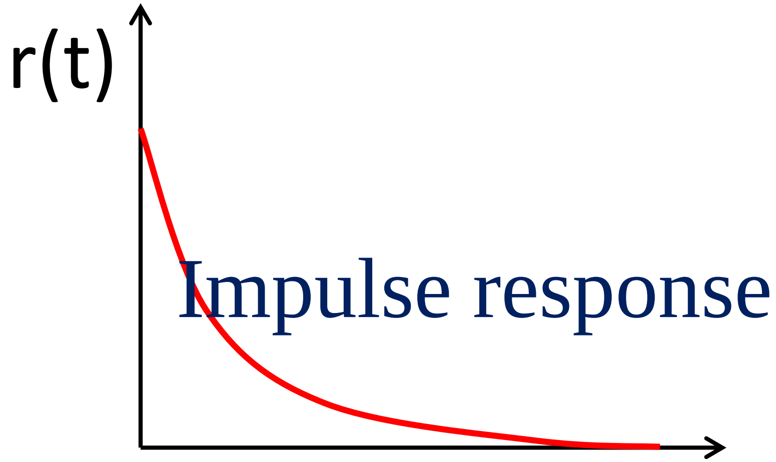
→ derivative of unit step H(t)

Response to unit impulse

→ derivative of response to unit step







RL Circuit – Impulse Response

With impulse excitation e(t) = E.δ(t)

Complementary function is unchanged.

Particular integral is now different and equal to 0.

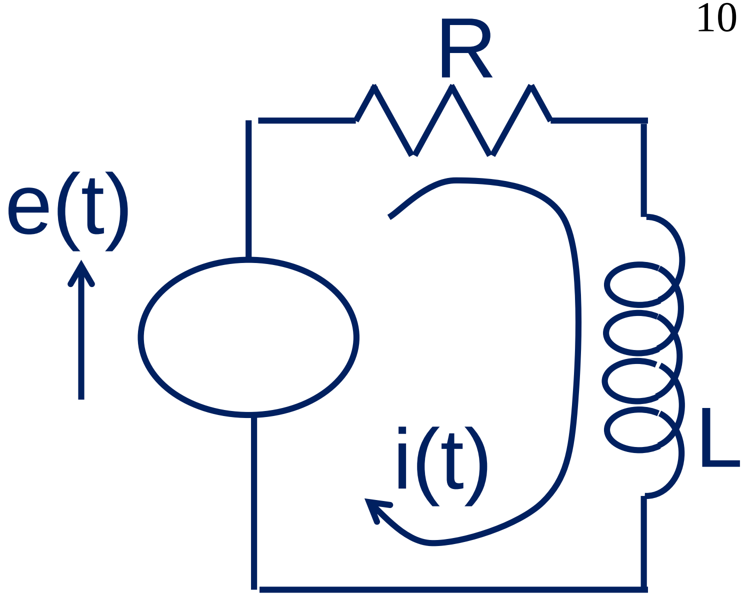
$$i(t) = A.e^{-\frac{R}{L}t} + 0$$

New ***A*** is obtained from new initial conditions.

$$i(t) = \frac{E}{L} e^{-\frac{R}{L}t}$$

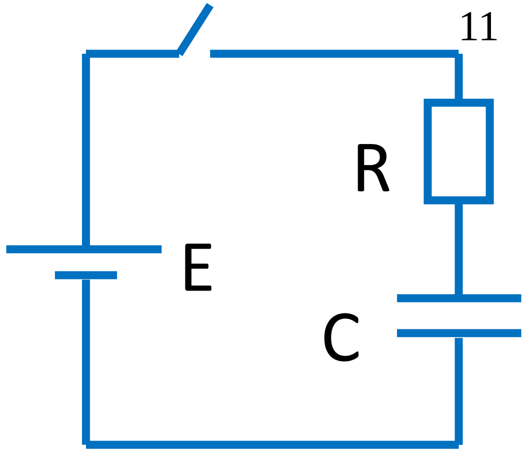
Or from derivative of unit step response as

$$i(t) = \frac{d}{dt}\left[\frac{E}{R}\left(1 - e^{-\frac{R}{L}t}\right)\right] = \frac{E}{L} e^{-\frac{R}{L}t}$$



Series R-L circuit

Simple R C circuit supplied from a step voltage



○ switch closed onto a battery ≡ step voltage source

Differential equation governing behaviour of circuit

$$e(t) = R\,.\,i(t) + \frac{1}{C}\int i(t)\,dt = R.i(t) + \frac{1}{C.p}\,i(t)$$

For complementary function, $R + \frac{1}{C\,p} = 0$, giving p = - 1/RC

i.e. complementary function = A $e^{-\,t/RC}$

Also, a particular solution i(t) = 0

solution can be written as $i(t) = 0 + A\cdot e^{-\frac{1}{RC}t}$ where *A* is a constant.

A can be obtained from the initial conditions

at t=0, $v_C(t) = 0$. [since voltage across capacitor cannot change suddenly]

$\therefore$ $v_R(t) = E$ and i(t) = E/R, $\frac{E}{R} = A\cdot e^{-\frac{1}{RC}\times 0}$ $\qquad i(t) = \frac{E}{R}e^{-\frac{1}{RC}t}$

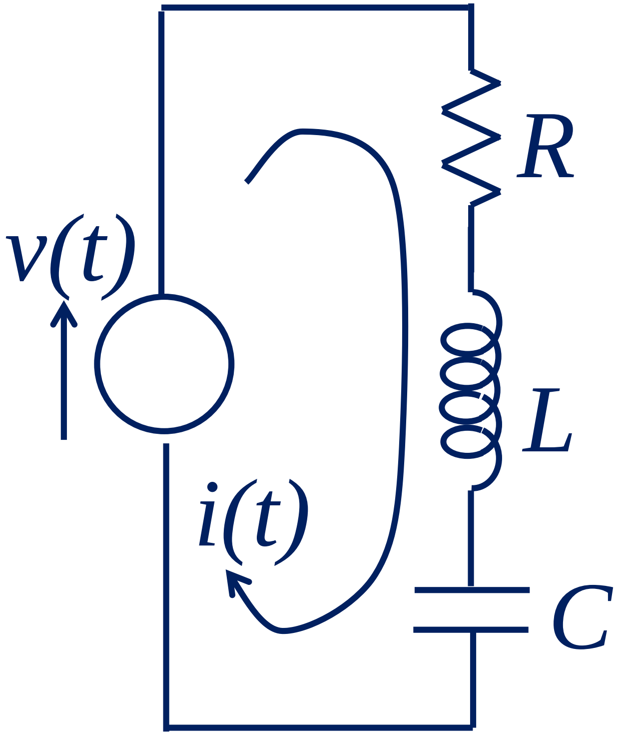
voltage and current can be sketched as earlier

RLC Circuit Analysis

Governing differential equation can be found from

$$v_R + v_L + v_C = v(t)$$

$$Ri(t) + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{\tau=t} i(\tau)\,d\tau = v(t)$$



Series R-L-C circuit

When source is a constant voltage,
differentiating and dividing by L gives

$$\frac{d^2 i(t)}{dt^2} + \frac{R}{L}\frac{di(t)}{dt} + \frac{1}{LC}i(t) = 0$$

Natural behavior of RLC circuits

Can be expressed in the more generally form:

$$\frac{d^2 i(t)}{dt^2} + 2\alpha \frac{di(t)}{dt} + {\omega_0}^2 i(t) = 0$$

α and ω_0 are both in units of angular frequency and defined as;

$$\alpha = \frac{R}{2L} \text{ and } \omega_0 = \frac{1}{\sqrt{LC}}$$

A useful parameter is the *damping factor* ζ

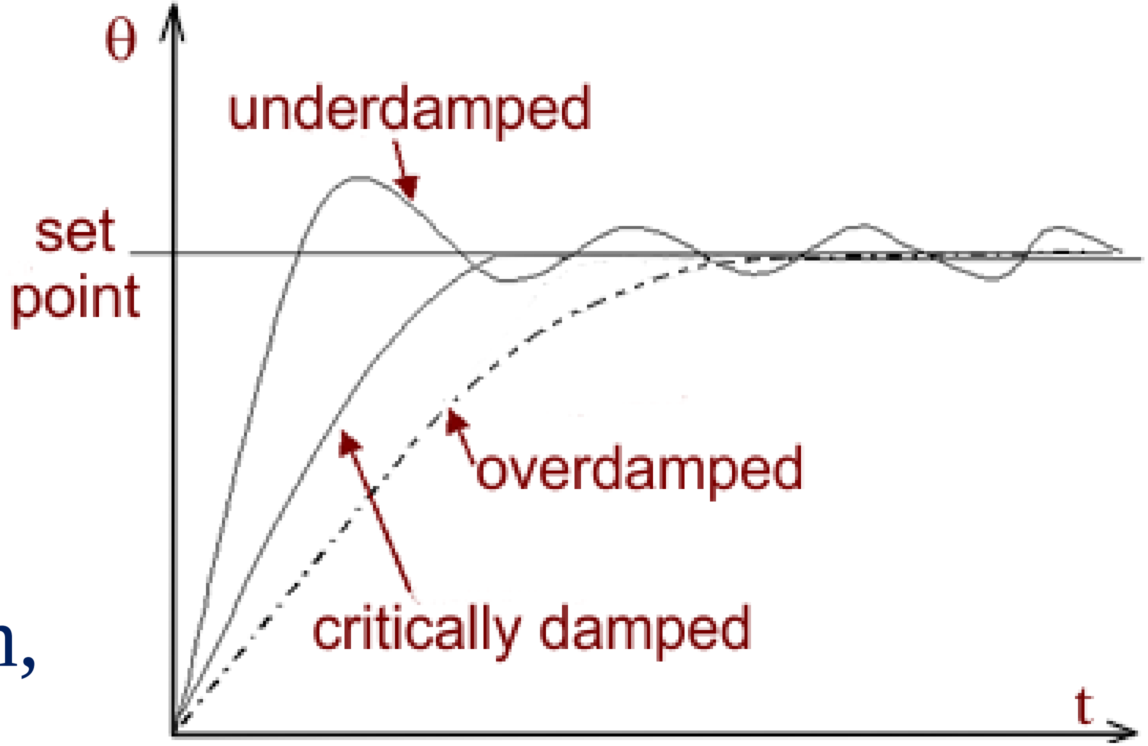
which is defined as the ratio of these two, $\zeta = \dfrac{\alpha}{\omega_0}$

In the case of the series RLC circuit, the damping factor

is given by, $\zeta = \dfrac{R}{2}\sqrt{\dfrac{C}{L}}$

Transient response

Differential equation for the circuit solves in three different ways dependi on the value of ζ.



underdamped $\zeta < 1$,

overdamped $\zeta > 1$,

and critically damped $\zeta = 1$.

Characteristic equation has the form,

$$s^2 + 2\alpha s + {\omega_0}^2 = 0$$

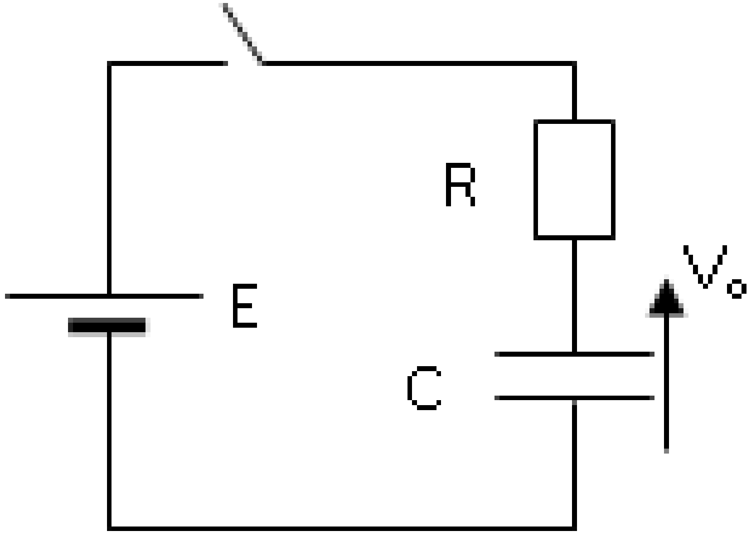
Roots of the equation in *s* are,

$$s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2} \quad \text{and} \quad s_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2}$$

general solution of is of the form $i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$

Where coefficients A_1 and A_2 are determined by the boundary conditions

R C circuit supplied from a step voltage source, but with C initially charged to V_o



Differential equation governing behaviour of circuit

$$e(t) = R\,.\,i(t) + \frac{1}{C}\int i(t)\,dt + V_o = R.i + \frac{1}{C.p}i + V_o$$

Complementary function = A $e^{-t/RC}$ as before.

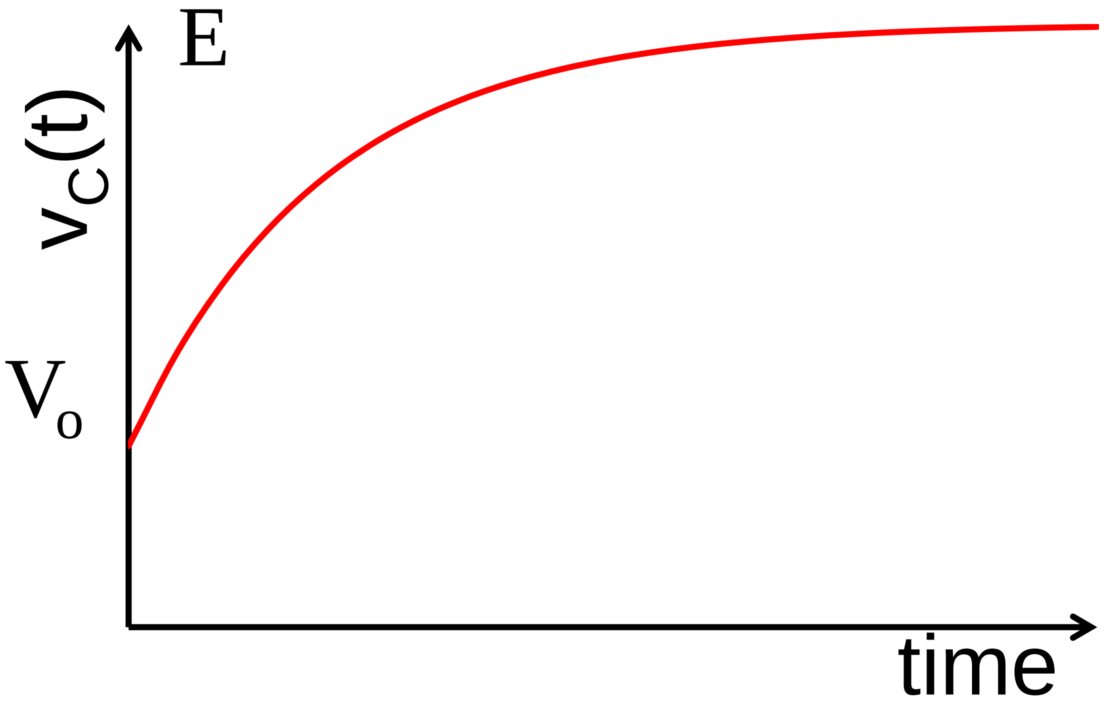
Particular solution would again be zero,
but capacitor is initially charged. So that at t=0, $v_C(t) = V_o$

$\therefore\ v_R(t) = E - V_o$ and $i(t) = (E - V_o)/R$

$$\therefore \frac{E - V_o}{R} = A \cdot e^{-\frac{1}{RC}\times 0} \quad \text{giving} \quad A = \frac{E - V_o}{R} \quad \therefore i(t) = \frac{E - V_o}{R}\cdot e^{-\frac{1}{RC}t}$$

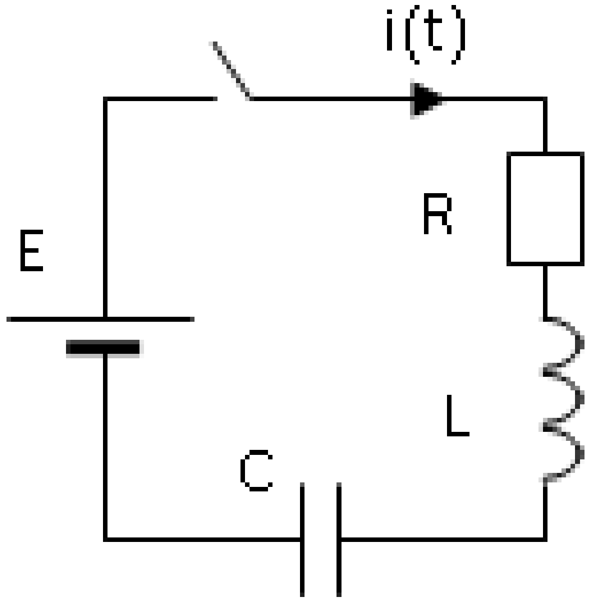
Using Ohm's law, the voltage across capacitor is now given by

$$v_C(t) = E - \left(E - V_o\right)\cdot e^{-\frac{1}{RC}t}$$



R L C circuit supplied from a step voltage source

$$e(t) = R.i(t) + L\frac{d\,i(t)}{d\,t} + \frac{1}{C}\int i(t)\,dt = R.i(t) + L.p.i(t) + \frac{1}{C.p}i(t)$$



where differential operator d/dt has been replaced by the operator *p*

Differentiation on either side of equation gives

$$p.e(t) = R.p.i(t) + L.p^2.i(t) + \frac{1}{C}i(t)$$

The complementary function is the part without the forcing function *e(t)*

i.e. $R.p + L.p^2 + 1/C = 0$

Solution of this equation can be real or complex dependant on the values of components.

(a) When solution has two distinct real roots

R = 480 Ω, L = 0.4 H, C = 2.5 μF, E = 120 V

complementary function is $0.4\ p^2 + 480\ p + 4\times10^5 = 0$

i.e. $p^2 + 1200\ p + 10^6 = 0$ giving $p = -600 \pm \sqrt{600^2 - 10^6} = -600 \pm j800$

Particular solution is i(t) = 0 at t = ∞

giving the solution $.e^{-600t}.e^{j800t} + B.e^{-600t}.e^{-j\,800t}$, or $C.e^{-600t}.\cos(800t+\theta)$

Using initial conditions [2 required due to 2 energy storing elements]

i(t) = 0 and $v_C(t)=0$ at t=0

$0 = C.e^{-600\times0}.\cos(800\times0+\theta)$ gives $\theta = \pm 90^o$

as C cannot be zero [trivial solution]

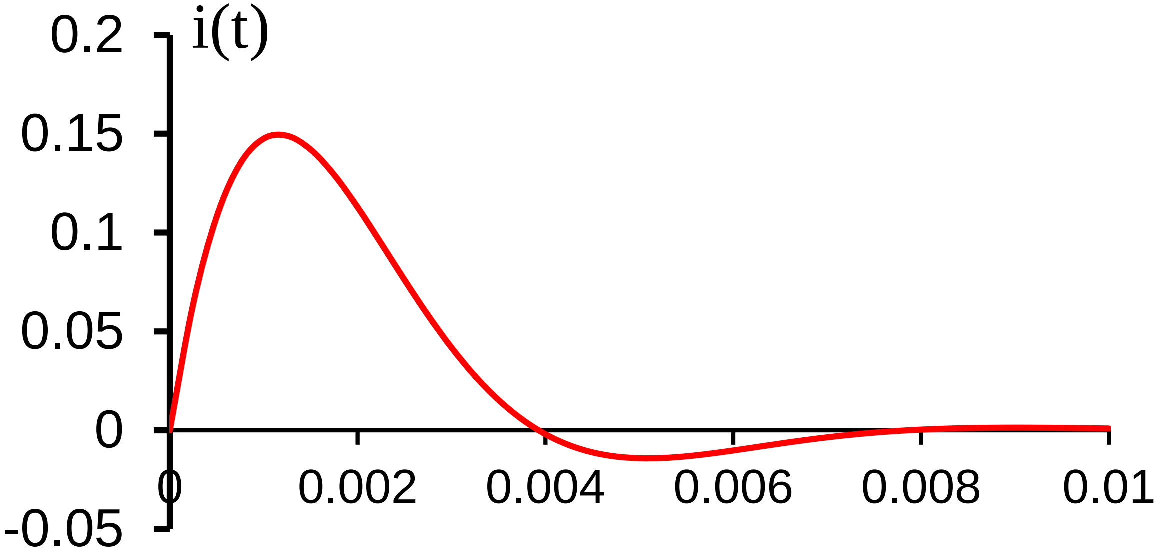
giving the solution $i(t) = C.\ e^{-600t}.\sin 800t$

Also since i(0)=0, $v_R(0) = 0$. $\therefore$ $v_L(0) = 120 = 0.4 \times \frac{di}{dt}$

giving $\frac{di}{dt} = 300$ at t=0

$$\frac{di}{dt} = C.\left[-600 \times e^{-600\times 0}.\sin 800 \times 0 + e^{-600\times 0}.800 \times \cos 800 \times 0\right] = 300$$

giving C = 300/800 = 0.375 $\therefore$ $i(t) = 0.375\, e^{-600t}.\sin 800t$ A

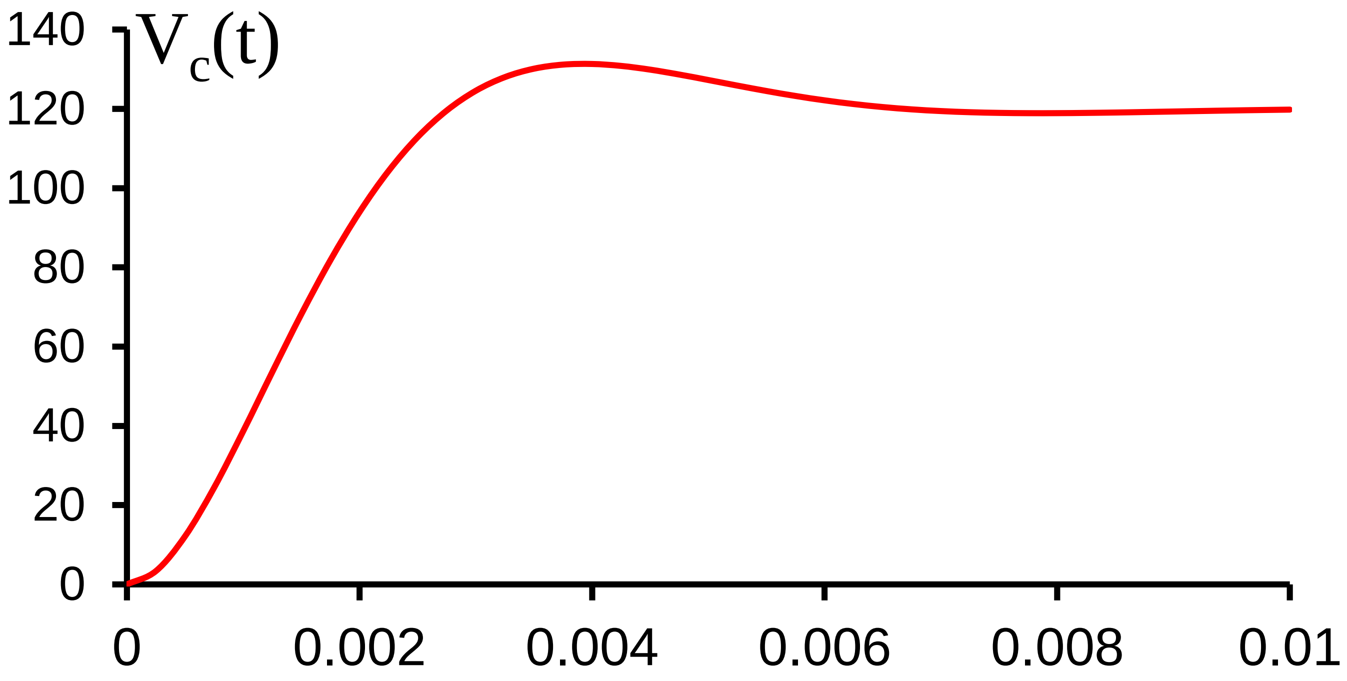


Using Ohm's law, voltages across the elements are

$v_R(t) = 480\ i(t) = 180\ e^{-600t}.\sin 800t \quad V$

$v_L(t) = 0.4\ p.i(t) = -\ 90\ e^{-600t}.\sin 800t + 120\ e^{-600t}.\cos 800t\ V$

$v_C(t) = 120 - 90\ e^{-600t}.\sin 800t - 120\ e^{-600t}.\cos 800t \quad V$



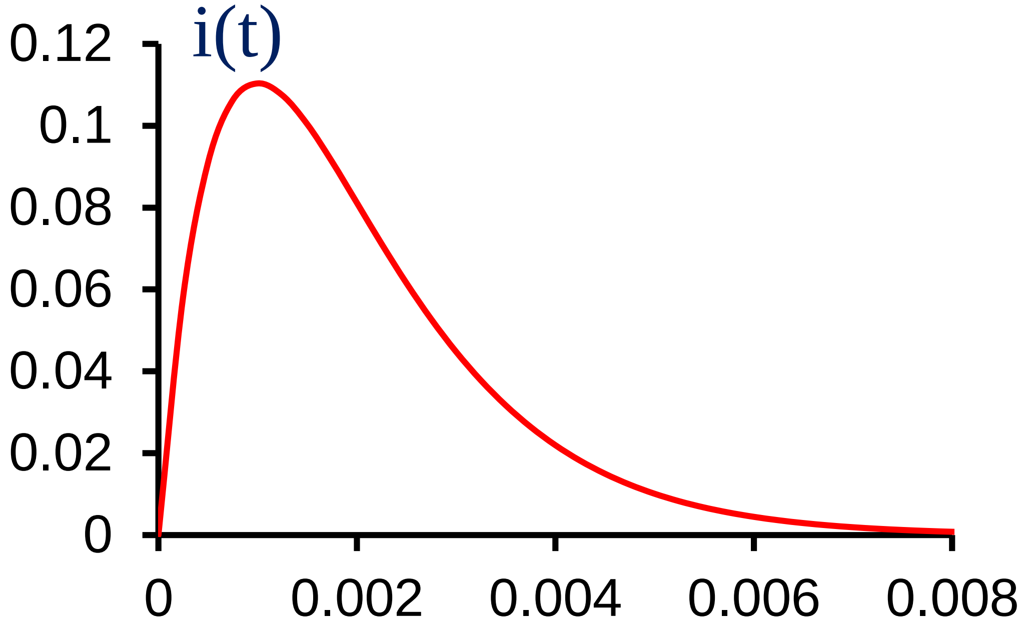
(b) When solution has two equal real roots

R = 800 W, L = 0.4 H, C = 2.5 mF, E = 120 V

complementary function is $0.4\,p^2 + 800\,p + 4\times10^5 = 0$

i.e. $p^2 + 2000\,p + 10^6 = 0$ giving $(p + 1000)^2 = 0$

or $p = -1000$ (repeated roots)



In this case the solution is of the form

$i(t) = A\,t\,e^{-1000t} + 0$

At t=0, i(t) = 0, [automatically satisfied]

and di(t)/dt = 300

$$\frac{di}{dt} = A\,e^{-1000\times0} - 1000A\times0.e^{-1000\times0} = 300$$

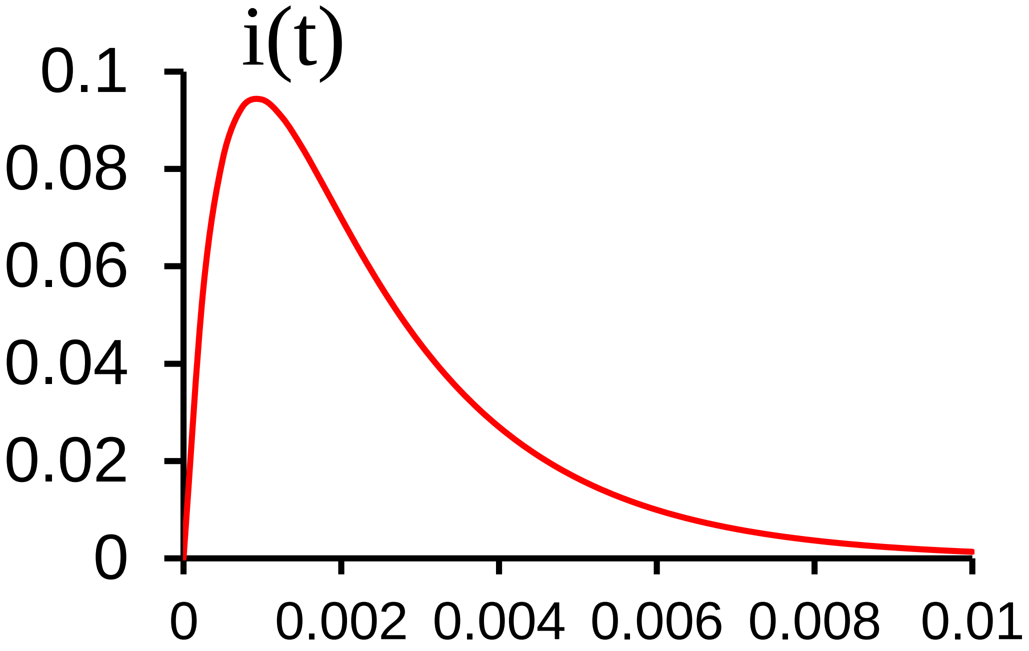
giving A = 300 $\quad\therefore\ i(t) = 300\,t\,e^{-1000t}$ A

(c) When solution has two complex roots

$R = 1000\ \Omega$, $L = 0.4$ H, $C = 2.5\ \mu F$, $E = 120$ V

complementary function is $0.4\ p^2 + 1000\ p + 4\times10^5 = 0$

i.e. $p^2 + 2500\ p + 10^6 = 0$ giving



$(p+500)(p+2000) = 0$ or $p = -500$ or -2000

The solution is of the form

$i(t) = A\ e^{-500t} + B\ e^{-2000t}$

At $t = 0$, $i(t) = 0 = A + B$

At $t = 0$, $v_C(t) = 0$, gives $di(t)/dt = 300$.

i.e. $-500A - 2000B = 300$, gives $A = 0.2 = -B$

$\therefore\ i(t) = 0.2\ (e^{-500t} - e^{-2000t})$ A

END OF PRESENTATION