Chapter - 01

Differentiation and Integration

1.1 Concept of a Function

1.1.1 Introduction

Functions arise whenever one quantity depends on another. Consider the following four situations.

A. The area A of a circle depends on the radius r of the circle. The rule that connects r and A is given by the equation $A = \pi r^2$. With each positive number r there is associated one value of A, and we say that A is a function of r.

B. The human population of the world P depends on the time t. The table gives estimates of the world population $P(t)$ at time t, for certain years.

Year	Population (Millions)	Year	Population (Millions)	Year	Population (Millions)
1900	1650	1940	2300	1980	4450
1910	1750	1950	2560	1990	5280
1920	1860	1960	3040	2000	6080
1930	2070	1970	3710	2010	6870

For instance, $P(1950) \approx 2560,000,000$. But for each value of the time t there is a corresponding value of P, and we say that P is a function of t.

C. The cost C of mailing an envelope depends on its weight w. Although there is no simple formula that connects w and C, the post office has a rule for determining C when w is known.

D. The vertical acceleration a of the ground as measured by a seismograph during an earthquake is a function of the elapsed time t. Figure 1 shows a graph generated by seismic activity during the Northridge earthquake that shook Los Angeles in 1994. For a given value of t, the graph provides a corresponding value of a.

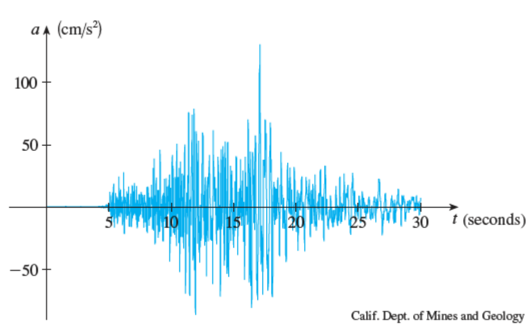


Figure 1 : Vertical ground acceleration during the Northridge earthquake

Each of these examples describes a rule whereby, given a number ($r, t, w,$ or t), another number ($A, P, C,$ or a) is assigned. In each case we say that the second number is a function of the first number.

1.1.2 Definition : Function

A function f is a rule that assigns to each element x in a set D exactly one element, called $f(x)$, in a set E.

- We usually consider functions for which the sets D and E are sets of real numbers. The set D is called the domain of the function.
- The number $f(x)$ is the value of f at x and is read " f of x".
- The range of f is the set of all possible values of $f(x)$ as x varies throughout the domain.
- A symbol that represents an arbitrary number in the domain of a function f is called an independent variable.
- A symbol that represents a number in the range of f is called a dependent variable.
- In Example A, for instance, r is the independent variable and A is the dependent variable.

It's helpful to think of a function as a machine (see Figure 2).

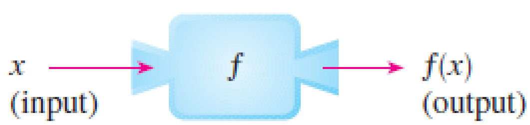


Figure 2 : Machine diagram for a function f

If x is in the domain of the function f, then when x enters the machine, it's accepted as an input and the machine produces an output $f(x)$ according to the rule of the function. Thus we can think of the domain as the set of all possible inputs and the range as the set of all possible outputs.
Another way to picture a function is by an arrow diagram as in Figure 3.

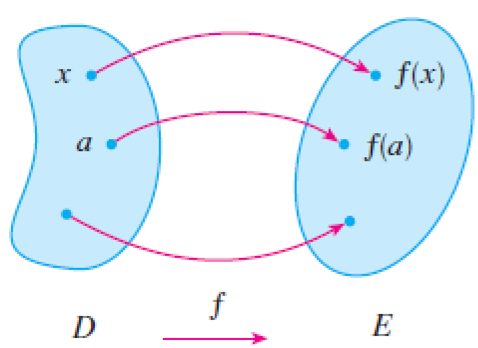


Figure 3 : Arrow diagram for f

Each arrow connects an element of D to an element of E. The arrow indicates that $f(x)$ is associated with x, $f(a)$ is associated with a, and so on.
The most common method for visualizing a function is its graph. If f is a function with domain D, then its graph is the set of ordered pairs

$$\{(x, f(x)) \mid x \in D\}.$$

(Notice that these are input-output pairs.) In other words, the graph of f consists of all points (x, y) in the coordinate plane such that $y = f(x)$ and x is in the domain of f.
The graph of a function f gives us a useful picture of the behavior of a function. Since the $y-$coordinate of any point (x, y) on the graph is $y = f(x)$, we can read the value of $f(x)$ from the graph as being the height of the graph above the point x (see Figure 4).

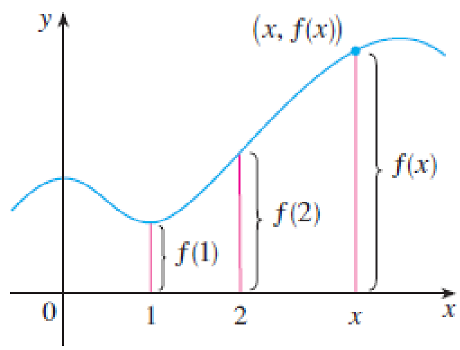


Figure 4

The graph of f also allows us to picture the domain of f on the $x-$axis and its range on the $y-$axis as in Figure 5.

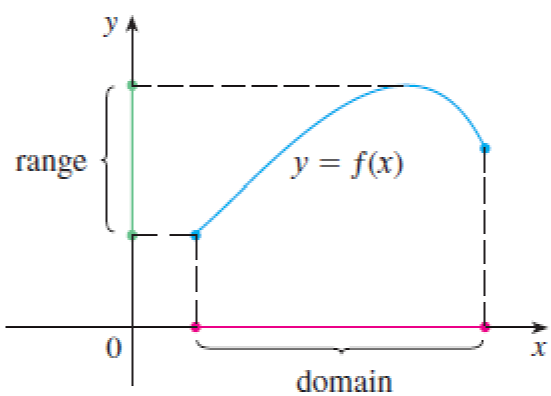
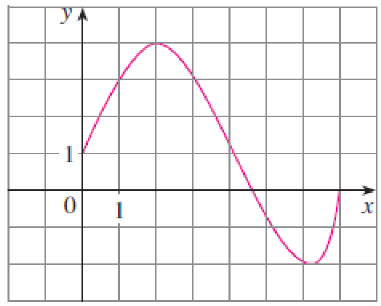


Figure 5

$\boxed{\text{Example 01}}$ The graph of a function f is shown in the figure.

(i) Find the values of $f(1)$ and $f(5)$.

(ii) What are the domain and range of f ?



$\boxed{\text{Example 02}}$ Sketch the graph and find the domain and range of each function.

(i) $f(x) = 2x - 1$

(ii) $g(x) = x^2$

$\boxed{\text{Example 03}}$ Find the domain of each function.

(i) $f(x) = \sqrt{x+2}$

(ii) $g(x) = \dfrac{1}{x^2 - x}$

1.1.3 Piecewise Defined Functions

The functions are defined by different formulas in different parts of their domains are called piecewise defined functions.

$\boxed{\text{Example 04}}$ A function f is defined by $f(x) = \begin{cases} 1 - x & \text{if } x \leq -1 \\ x^2 & \text{if } x > -1. \end{cases}$
Evaluate $f(-2)$, $f(-1)$, and $f(0)$ and sketch the graph.

$\boxed{\text{Example 05}}$ Sketch the graph of the absolute value function $f(x) = |x|$.

1.1.4 Symmetry of Functions

If a function f satisfies $f(-x) = f(x)$ for every number x in its domain, then f is called an even function.
For instance, the function $f(x) = x^2$ is even because

$$f(-x) = (-x)^2 = x^2 = f(x).$$

The geometric significance of an even function is that its graph is symmetric with respect to the $y-$axis (see Figure 6).

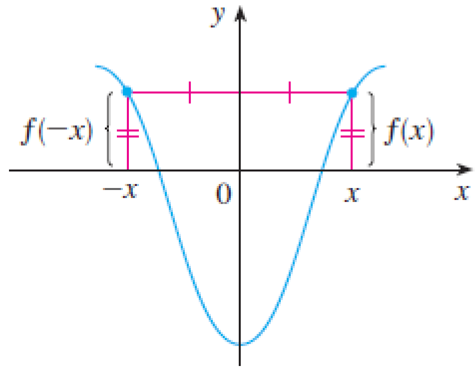


Figure 6 : An even function

This means that if we have plotted the graph of f for $x \geq 0$, we obtain the entire graph simply by reflecting this portion about the y-axis.
If f satisfies $f(-x) = -f(x)$ for every number x in its domain, then f is called an odd function.
For example, the function $f(x) = x^3$ is odd because

$$f(-x) = (-x)^3 = -x^3 = -f(x).$$

The graph of an odd function is symmetric about the origin (see Figure 7).

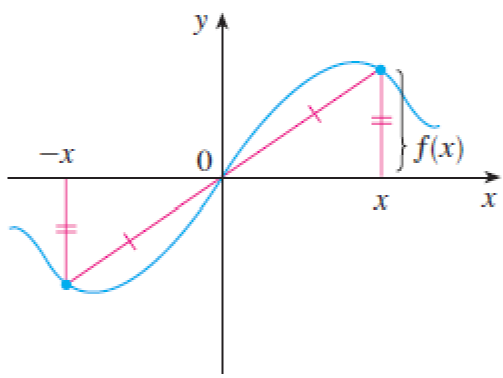


Figure 7 : An odd function

If we already have the graph of f for $x \geq 0$, we can obtain the entire graph by rotating this portion through 180^0 about the origin.

Example 06 Determine whether each of the following functions is even, odd, or neither even nor odd.

(i) $f(x) = x^5 + x$

(ii) $g(x) = 1 - x^4$

(iii) $h(x) = 2x - x^2$

1.1.5 Increasing and Decreasing Functions

The graph shown in Figure 8 rises from A to B, falls from B to C, and rises again from C to D.

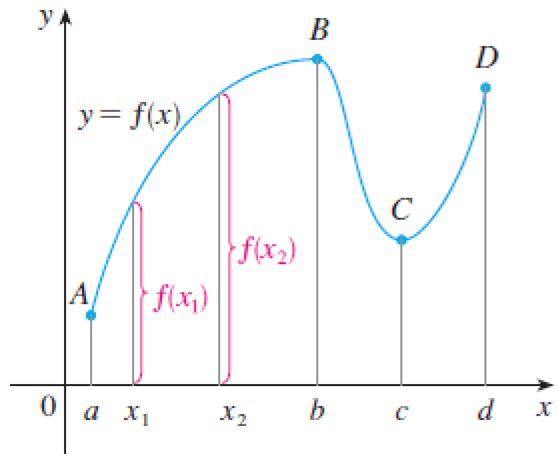


Figure 8

The function f is said to be increasing on the interval (a, b), decreasing on (b, c), and increasing again on (c, d).
Notice that if x_1 and x_2 are any two numbers between a and d with $x_1 < x_2$, then $f(x_1) < f(x_2)$. We use this as the defining property of an increasing function.

- A function f is called increasing on an interval I if

$$f(x_1) < f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I$$

- It is called decreasing on I if

$$f(x_1) > f(x_2) \qquad \text{whenever } x_1 < x_2 \text{ in } I.$$

1.2 Basic Functions

1.2.1 Linear Functions

If y is a **linear function** of x, then the graph of the function is a line. So we can use the slope-intercept form of the equation of a line to write a formula for the function as

$$y = f(x) = mx + c$$

where m is the slope of the line and c is the $y-$intercept.

Example 07

(i) As dry air moves upward, it expands and cools. If the ground temperature is 20^0C and the temperature at a height of 1 km is 10^0C, express the temperature T (in ^0C) as a function of the height h (in kilometers), assuming that a linear model is appropriate.

(ii) Draw the graph of the function in part (i). What does the slope represent?

(iii) What is the temperature at a height of 2.5 km?

1.2.2 Polynomial Functions

- A function P is called a polynomial if

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2x^2 + a_1x + a_0$$

  where n is a non-negative integer and the numbers $a_0, a_1, a_2, ..., a_n$ are constants called the coefficients of the polynomial.

- The domain of any polynomial is $\mathbb{R}$.
- If the leading coefficient $a_n \neq 0$, then the degree of the polynomial is n.
- A polynomial of degree 1 is of the form $P(x) = a_1x + a_0 = mx + c$ and so it is a linear function.
- A polynomial of degree 2 is of the form $P(x) = a_2x^2 + a_1x + a_0 = ax^2 + bx + c$ and is called a quadratic function.
- A polynomial of degree 3 is of the form $P(x) = a_3x^3 + a_2x^2 + a_1x + a_0 = ax^3 + bx^2 + cx + d$ and is called a cubic function.

1.2.3 Power Functions

A function of the form $f(x) = x^a$, where a is a constant, is called a power function.
We consider several cases.

Case I : $a = n$, where n is a positive integer
The graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4,$ and 5 are shown in Figure 9.

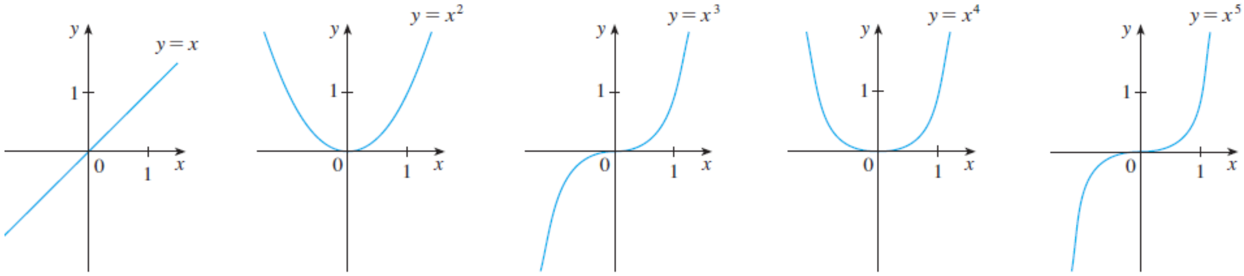


Figure 9 : Graphs of $f(x) = x^n$ for $n = 1, 2, 3, 4, 5$

The general shape of the graph of $f(x) = x^n$ depends on whether n is even or odd.
If n is even, then $f(x) = x^n$ is an even function and its graph is similar to the parabola $y = x^2$.
If n is odd, then $f(x) = x^n$ is an odd function and its graph is similar to that of $y = x^3$.

Case II : $a = \frac{1}{n}$, where n is a positive integer

The function $f(x) = x^{\frac{1}{n}} = \sqrt[n]{x}$ is a root function.
For $n = 2$ it is the square root function $f(x) = \sqrt{x}$, whose domain is $[0, \infty)$ and whose graph is the upper half of the parabola $x = y^2$. [See Figure 10(a).]
For other even values of n, the graph of $y = \sqrt[n]{x}$ is similar to that of $y = \sqrt{x}$.
For $n = 3$ we have the cube root function $f(x) = \sqrt[3]{x}$ whose domain is $\mathbb{R}$ and whose graph is shown in Figure 10(b).
The graph of $y = \sqrt[n]{x}$ for n odd $(n > 3)$ is similar to that of $f(x) = \sqrt[3]{x}$.

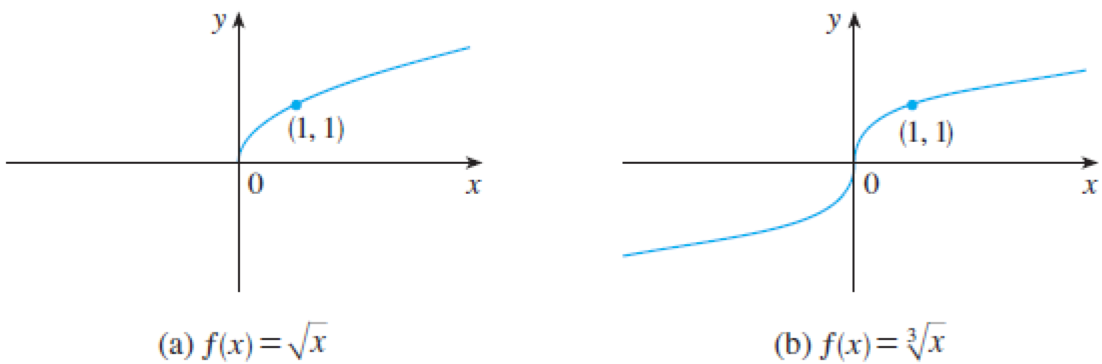


Figure 10 : Graphs of root functions

Case III : $a = -1$

The graph of the reciprocal function $f(x) = x^{-1} = \frac{1}{x}$ is shown in Figure 11. Its graph has the equation $y = \frac{1}{x}$, or $xy = 1$, and is a hyperbola with the coordinate axes as its asymptotes.

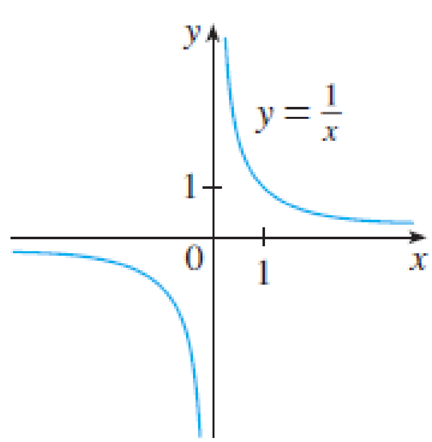


Figure 11 : The reciprocal function

1.2.4 Rational Functions

A rational function f is a ratio of two polynomials:

$$f(x) = \frac{P(x)}{Q(x)}$$

where P and Q are polynomials.
The domain consists of all values of x such that $Q(x) \neq 0$.

1.2.5 Algebraic Functions

A function f is called an algebraic function if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots) starting with polynomials.
Any rational function is automatically an algebraic function.

1.2.6 Trigonometric Functions

In calculus the convention is that radian measure is always used (except when otherwise indicated). For example, when we use the function $f(x) = \sin x$, it is understood that $\sin x$ means the sine of the angle whose radian measure is x.
The graphs of the sine and cosine functions are as shown in Figure 12.

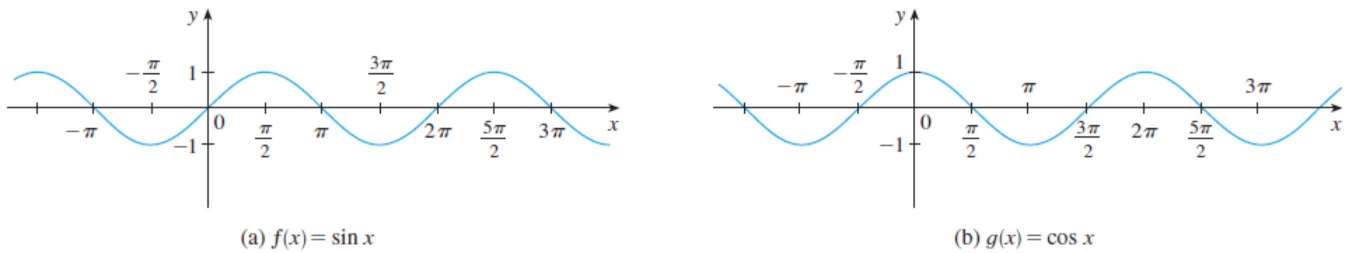


Figure 12

Notice that for both the sine and cosine functions the domain is $\mathbb{R}$ and the range is the closed interval $[-1, 1]$.
Thus, for all values of x, we have

$$-1 \leq \sin x \leq 1 \qquad -1 \leq \cos x \leq 1$$

The zeros of the sine function occur at the integer multiples of π; that is,

$$\sin x = 0 \qquad \text{when } x = n\pi \quad n \text{ is an integer}$$

- An important property of the sine and cosine functions is that they are periodic functions and have period 2π.
- This means that, for all values of x,

$$\sin(x+2\pi) = \sin x \qquad \cos(x+2\pi) = \cos x$$

1.3 Logarithmic and Exponential Functions

1.3.1 Exponential Functions

An exponential function is a function of the form $f(x) = b^x$, where b is a positive constant.

- If $x = n$, a positive integer, then
$$b^n = \underbrace{b \cdot b \cdot \ldots \cdot b}_{n \text{ factors}}.$$
- If $x = 0$, then $b^0 = 1$.
- If $x = -n$, where n is a positive integer, then $b^{-n} = \dfrac{1}{b^n}$.
- If x is a rational number, $x = \frac{p}{q}$, where p, q are integers and $q > 0$, then $b^x = b^{\frac{p}{q}} = \sqrt[q]{b^p} = (\sqrt[q]{b})^p$.

The graphs of members of the family of functions $y = b^x$ are shown in Figure 13 for various values of the base b.

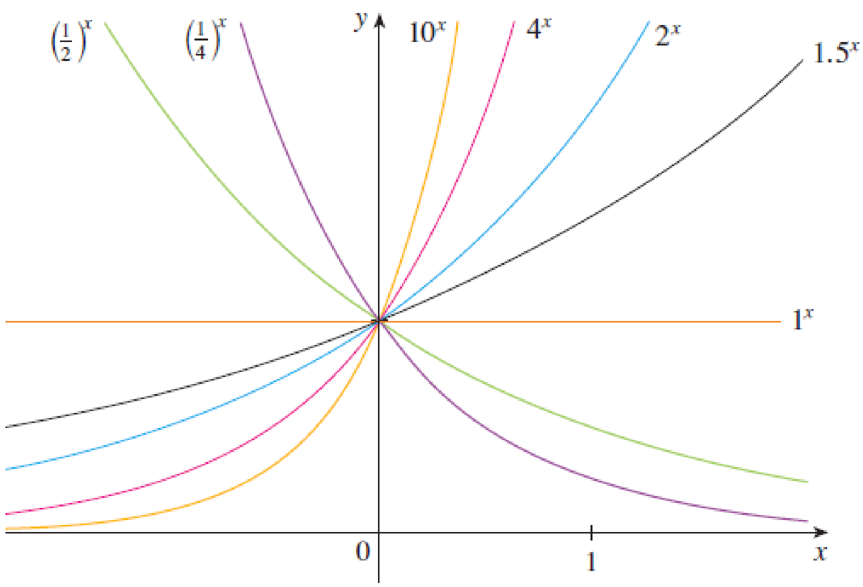


Figure 13

Notice that all of these graphs pass through the same point $(0, 1)$ because $b^0 = 1$ for $b \neq 0$.
Notice also that as the base b gets larger, the exponential function grows more rapidly (for $x > 0$).

Example 08

The half-life of Strontium-90, ^{90}Sr, is 25 years. This means that half of any given quantity of ^{90}Sr will disintegrate in 25 years.

(i) If a sample of ^{90}Sr has a mass of 24 mg, find an expression for the mass $m(t)$ that remains after t years.

(ii) Find the mass remaining after 40 years, correct to the nearest milligram.

1.3.2 Natural Exponential Function

The function $f(x) = e^x$ is called as the natural exponential function.
The value of e, correct to five decimal places, is

$$e \approx 2.71828.$$

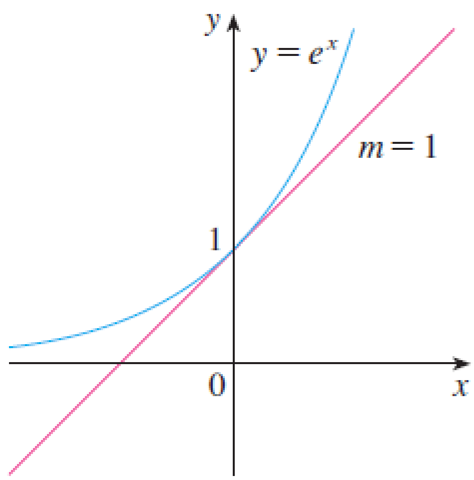


Figure 14 : The natural exponential function crosses the $y-$axis with a slope of 1.

1.3.3 Logarithmic Functions

The logarithmic functions $f(x) = \log_b x$, where the base b is a positive constant, are the inverse functions of the exponential functions.

$$\log_b x = y \iff b^y = x$$

- $\log_b(b^x) = x$ for every $x \in \mathbb{R}$.
- $b^{\log_b x} = x$ for every $x > 0$.

The logarithmic function $\log_b$ has domain $(0, \infty)$ and range $\mathbb{R}$.
Its graph is the reflection of the graph of $y = b^x$ about the line $y = x$.
Figure 15 shows the case when $b > 1$.

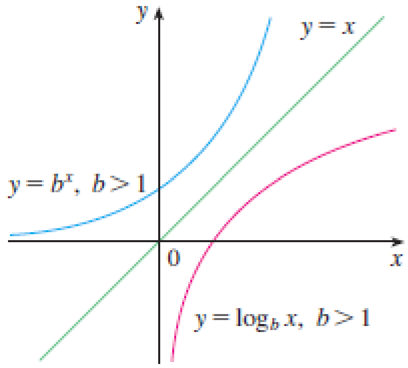


Figure 15

The Figure 16 shows the graphs of $y = \log_b x$ with various values of the base $b > 0$. Since $\log_b 1 = 0$, the graphs of all logarithmic functions pass through the point $(1, 0)$.

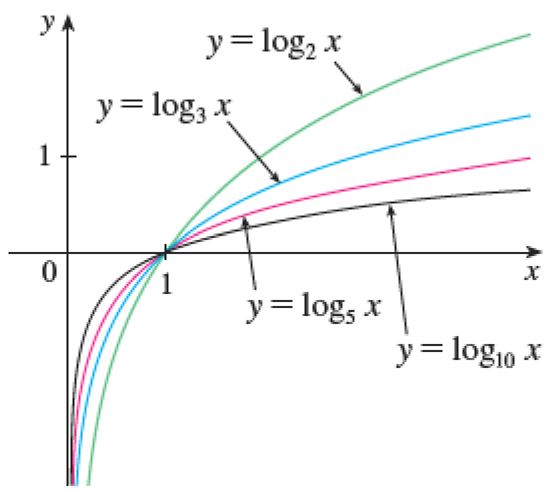


Figure 16

1.3.4 Natural Logarithmic Function

The logarithm with base e is called the natural logarithm and has a special notation:

$$\log_e x = \ln x.$$

$$\ln x = y \iff e^y = x$$

- If $x = 1$, then $\ln e = 1$.

The graphs of the exponential function $y = e^x$ and its inverse function, the natural logarithm function, are shown in Figure 17.
Because the curve $y = e^x$ crosses the $y-$axis with a slope of 1, it follows that the reflected curve $y = \ln x$ crosses the $x-$axis with a slope of 1.

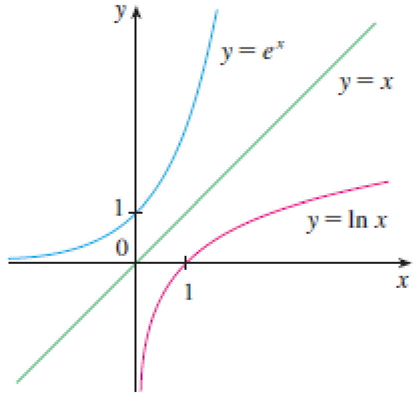


Figure 17 : The graph of $y = \ln x$ is the reflection of the graph of $y = e^x$ about the line $y = x$.

1.4 Hyperbolic Functions

The hyperbolic cosine function can be defined as

$$f(x) = \cosh x = \frac{e^x + e^{-x}}{2}$$

and the hyperbolic sine function can be defined as

$$f(x) = \sinh x = \frac{e^x - e^{-x}}{2}.$$

- The function $\cosh x$ is an even function and $\sinh x$ is an odd function.
- $\cosh x + \sinh x = e^x$.
- For all $x > 0$, $\cosh x > 0$.
- $\sinh x = 0 \iff e^x - e^{-x} = 0 \iff x = 0$.

The other hyperbolic functions are

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\operatorname{sech} x = \frac{1}{\cosh x}$$

$$\operatorname{csch} x = \frac{1}{\sinh x}$$

The domain of coth and csch is $x \neq 0$ while the domain of the other hyperbolic functions is all real numbers.

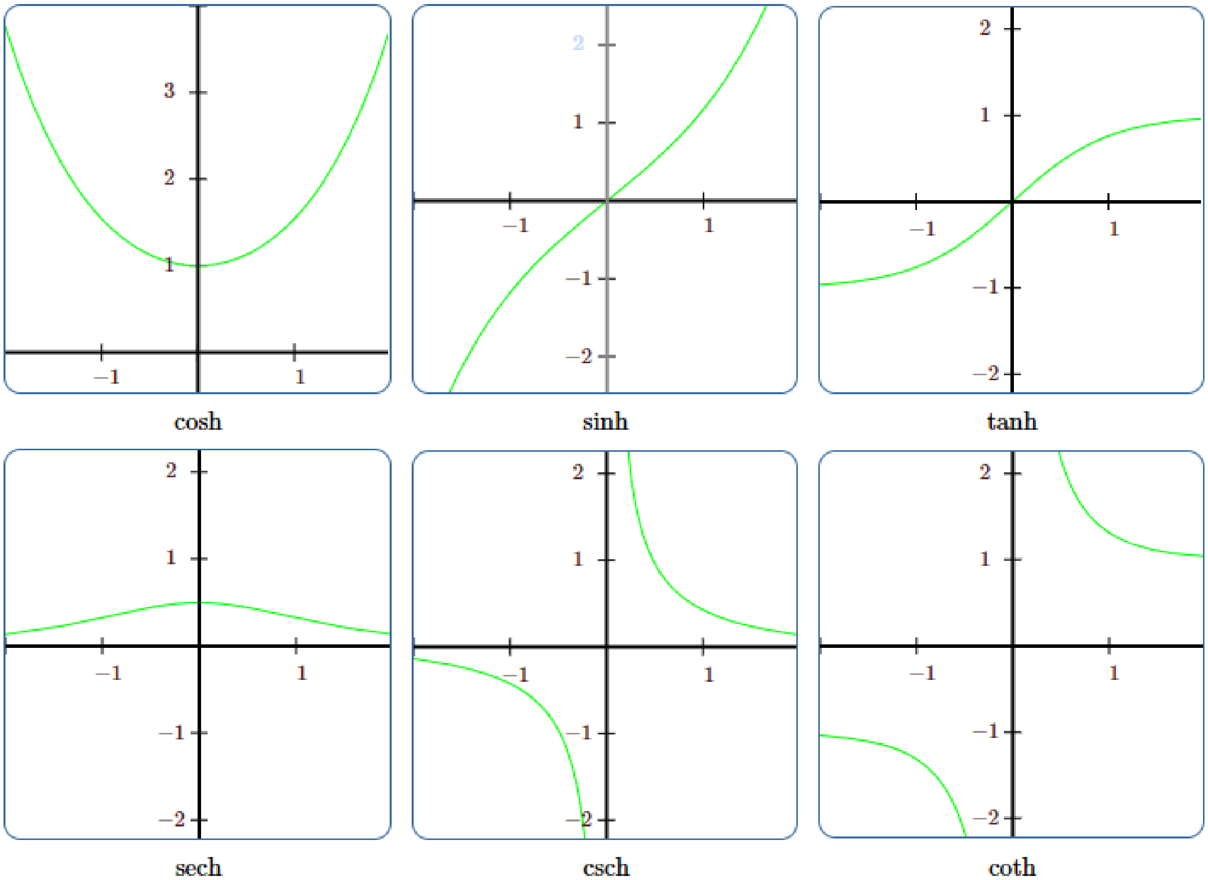


Figure 18

1.4.1 Series Representation of Hyperbolic Functions

Consider the series expansion of e^x,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ...$$

then,

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - ...$$

If we subtract, we get;

$$e^x - e^{-x} = 2x + \frac{2x^3}{3!} + \frac{2x^5}{5!} + ...$$

Dividing by 2;

$$\frac{e^x - e^{-x}}{2} = \sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...$$

If we add the two series and dividing by 2, we get;

$$\frac{e^x + e^{-x}}{2} = \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...$$

1.5 Periodic Functions

A function $f(x)$ is called a periodic function if $f(x)$ is defined for all real x, and if there is some positive number p such that $f(x+p) = f(x)$, for all x.
The positive number p is called a period of the function $f(x)$.
The graph of a periodic function has the characteristic that it can be obtained by periodic repetition of its graph in any interval of length p.

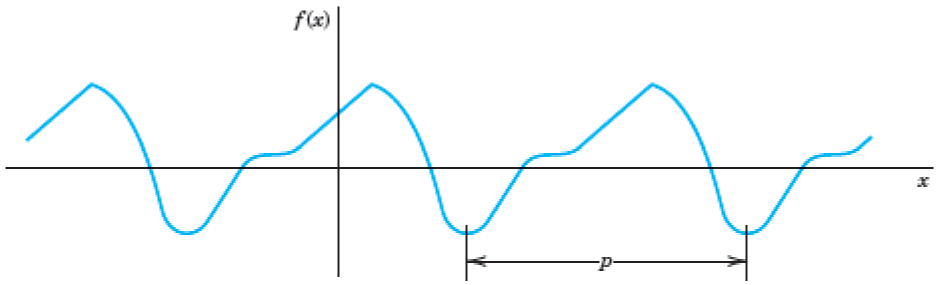


Figure 19 : Periodic function of period p

■ If a function $f(x)$ has period p, it also has the period $2p$ because

$$f(x+2p) = f([x+p]+p) = f(x+p) = f(x).$$

Thus for any integer $n = 1, 2, 3, ...$

$$f(x+np) = f(x) \quad \text{for all } x.$$

■ If $f(x)$ and $g(x)$ have period p, then $af(x) + bg(x)$ with any constants a and b has the period p.

1.6 Inverse of Functions

1.6.1 One-to-One Functions

A function f is called a one-to-one function if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \;\text{ whenever } x_1 \neq x_2.$$

If a horizontal line intersects the graph of f in more than one point, then we see from Figure 20 that there are numbers x_1 and x_2 such that $f(x_1) = f(x_2)$. This means that f is not one-to-one.

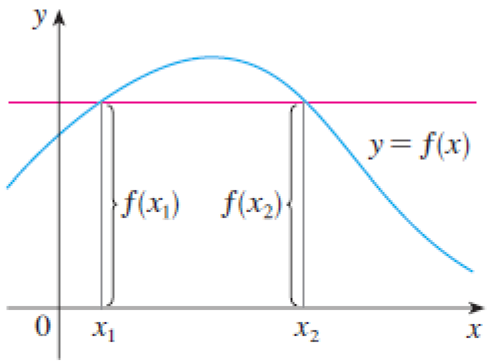


Figure 20

1.6.2 Definition : Inverse Function

Let f be a one-to-one function with domain A and range B. Then its inverse function f^{-1} has domain B and range A and is defined by

$$f^{-1}(y) = x \iff f(x) = y$$

for any y in B.

■ The definition says that if f maps x into y, then f^{-1} maps y back into x. (If f were not one-to-one, then f^{-1} would not be uniquely defined.)

The arrow diagram in Figure 21 indicates that f^{-1} reverses the effect of f.

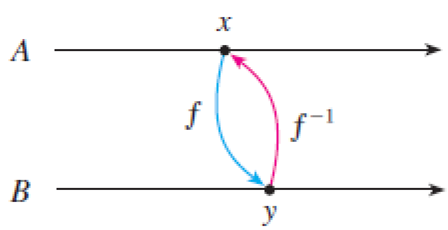


Figure 21

- ⋆ Domain of f^{-1} = Range of f
- ⋆ Range of f^{-1} = Domain of f

■ Do not mistake the -1 in f^{-1} for an exponent. Thus

$$f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}.$$

1.6.3 Method to Find the Inverse Function of a One-to-One Function f

Step 01 : Write $y = f(x)$.

Step 02 : Solve this equation for x in terms of y (if possible).

Step 03 : To express f^{-1} as a function of x, interchange x and y. The resulting equation is $y = f^{-1}(x)$.

Example 09

Find the inverse function of $f(x) = x^3 + 2$.

■ The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

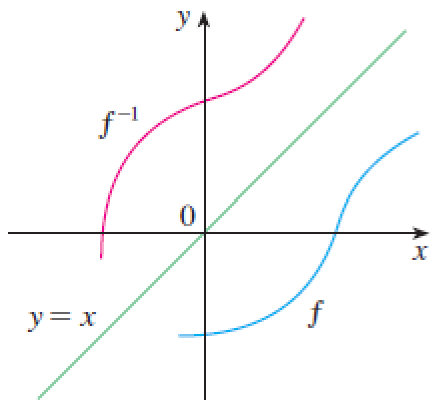


Figure 22