7: MAGNETOSTATICS

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CHAPTER OVERVIEW

7: Magnetostatics

Magnetostatics is the theory of the magnetic field in conditions in which its behavior is independent of electric fields, including

- The magnetic field associated with various spatial distributions of steady current
- The energy associated with the magnetic field
- Inductance, which is the ability of a structure to store energy in a magnetic field

The word ending “-statics” refers to the fact that these aspects of electromagnetic theory can be developed by assuming that the sources of the magnetic field are time-invariant; we might say that magnetostatics is the study of the magnetic field at DC. However, many aspects of magnetostatics are applicable at “AC” as well.

7.1: Comparison of Electrostatics and Magnetostatics

Students encountering magnetostatics for the first time have usually been exposed to electrostatics already. Electrostatics and magnetostatics exhibit many similarities. These are summarized in Table $7.1.1$. The elements of magnetostatics presented in this table are all formally introduced in other sections; the sole purpose of this table is to point out the similarities. The technical term for these similarities is *duality*. Duality also exists between voltage and current in electrical circuit theory.

Table $7.1.1$: A summary of the duality between electrostatics and magnetostatics

	electrostatics	**magnetostatics**
Sources	static charge	steady current, magnetizable material
Field intensity	$\mathbf{E}$ (V/m)	$\mathbf{H}$ (A/m)
Flux density	$\mathbf{D}$ (C/m^2)	$\mathbf{B}$ (Wb/m^2=T)
Material relations	$\mathbf{D}=\epsilon\mathbf{E}$	$\mathbf{B}=\mu\mathbf{H}$
	$\mathbf{J}=\sigma\mathbf{E}$	
Force on charge q	$\mathbf{F}=q\mathbf{E}$	$\mathbf{F}=q\mathbf{v}\times\mathbf{B}$
Maxwell's Eqs. (integral)	$\oint_S \mathbf{D}\cdot d\mathbf{s}=Q_{end}$	$\oint_S \mathbf{B}\cdot d\mathbf{s}=0$
	$\oint_C \mathbf{E}\cdot d\mathbf{l}=0$	$\oint_C \mathbf{H}\cdot d\mathbf{l}=I_{end}$
Maxwell's Eqs. (differential)	$\nabla\cdot\mathbf{D}=\rho_v$	$\nabla\cdot\mathbf{B}=0$
	$\nabla\times\mathbf{E}=0$	$\nabla\times\mathbf{H}=\mathbf{J}$
Boundary Conditions	$\hat{\mathbf{n}}\times[\mathbf{E}_1-\mathbf{E}_2]=0$	$\hat{\mathbf{n}}\times[\mathbf{H}_1-\mathbf{H}_2]=\mathbf{J}_s$
	$\hat{\mathbf{n}}\cdot[\mathbf{D}_1-\mathbf{D}_2]=\rho_s$	$\hat{\mathbf{n}}\cdot[\mathbf{B}_1-\mathbf{B}_2]=0$
Energy storage	Capacitance	Inductance
Energy density	$w_e=\frac{1}{2}\epsilon E^2$	$w_m=\frac{1}{2}\mu H^2$
Energy dissipation	Resistance	

7.2: Gauss' Law for Magnetic Fields - Integral Form

Gauss' Law for Magnetic Fields (GLM) is one of the four fundamental laws of classical electromagnetics, collectively known as *Maxwell's Equations*. Before diving in, the reader is strongly encouraged to review Section Section 2.5. In that section, GLM emerges from the "flux density" interpretation of the magnetic field. GLM is not identified in that section, but now we are ready for an explicit statement:

> *Gauss' Law for Magnetic Fields* (Equation 7.2.1) states that the flux of the magnetic field through a closed surface is zero.

This is expressed mathematically as follows:

$$\boxed{\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0} \tag{7.2.1}$$

where $\mathbf{B}$ is magnetic flux density and $\mathcal{S}$ is a closed surface with outward-pointing differential surface normal $d\mathbf{s}$. It may be useful to consider the units. $\mathbf{B}$ has units of Wb/m^2; therefore, integrating $\mathbf{B}$ over a surface gives a quantity with units of Wb, which is magnetic flux, as indicated above.

GLM can also be interpreted in terms of magnetic field lines. For the magnetic flux through a closed surface to be zero, every field line entering the volume enclosed by $\mathcal{S}$ must also exit this volume – field lines may not begin or end within the volume. The only way this can be true for every possible surface $\mathcal{S}$ is if magnetic field lines always form closed loops. This is in fact what we find in practice, as shown in Figure $7.2.1$.

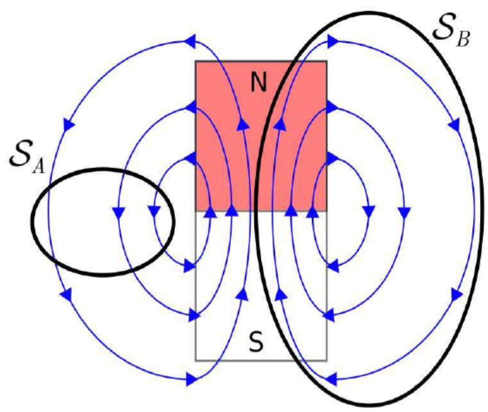

Figure $7.2.1$: Gauss' Law for Magnetostatics applied to a two-dimensional bar magnet. For the surface $\mathcal{S} = \mathcal{S}_A$, every field line entering $\mathcal{S}$ also leaves $\mathcal{S}$, so the flux through $\mathcal{S}$ is zero. For the surface $\mathcal{S} = \mathcal{S}_B$, every field line within $\mathcal{S}$ remains in $\mathcal{S}$, so the flux through $\mathcal{S}$ is again zero. Image used wtih permission (CC BY SA 4.0; K. Kikkeri)

Following this argument one step further, GLM implies there can be no particular particle or structure that can be the source of the magnetic field (because then that would be a start point for field lines). This is one way in which the magnetic field is very different from the electrostatic field, for which every field line begins at a charged particle. So, when we say that current (for example) is the source of the magnetic field, we mean only that the field coexists with current, and *not* that the magnetic field is somehow attached to the current. Summarizing, there is no "localizable" quantity, analogous to charge for electric fields, associated with magnetic fields. This is just another way in which magnetic fields are weird!

Summarizing

> Gauss' Law for Magnetic Fields requires that magnetic field lines form closed loops. Furthermore, there is no particle that can be identified as the source of the magnetic field.

7.3: Gauss' Law for Magnetism - Differential Form

The integral form of Gauss' Law states that the magnetic flux through a closed surface is zero. In mathematical form:

$$\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0 \tag{7.3.1}$$

where $\mathbf{B}$ is magnetic flux density and $\mathcal{S}$ is the enclosing surface. Just as Gauss's Law for electrostatics has both integral and differential forms, so too does Gauss' Law for Magnetic Fields. Here we are interested in the differential form for the same reason. Given a differential equation and the boundary conditions imposed by structure and materials, we may then solve for the magnetic field in very complicated scenarios.

The equation we seek may be obtained from Equation 7.3.1 using the Divergence Theorem, which in the present case may be written:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{B})\, dv = \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s}$$

Where $\mathcal{V}$ is the mathematical volume bounded by the closed surface $\mathcal{S}$. From Equation 7.3.1 we see that the right hand side of the equation is zero, leaving:

$$\int_{\mathcal{V}} (\nabla \cdot \mathbf{B})\, dv = 0$$

The above relationship must hold regardless of the specific location or shape of $\mathcal{V}$. The only way this is possible is if the integrand is everywhere equal to zero. We conclude:

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{7.3.2}$$

> The differential ("point") form of Gauss' Law for Magnetic Fields (Equation 7.3.2) states that the flux per unit volume of the magnetic field is always zero.

This is another way of saying that there is no point in space that can be considered to be the source of the magnetic field, for if it were, then the total flux through a bounding surface would be greater than zero. Said yet another way, the source of the magnetic field is not localizable.

7.4: Ampere's Circuital Law (Magnetostatics) - Integral Form

Ampere's circuital law (ACL) relates current to the magnetic field associated with the current. In the magnetostatic regime, the law is (see also Figure 7.4.1):

$$\boxed{\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl}} \tag{7.4.1}$$

That is, the integral of the magnetic field intensity $\mathbf{H}$ over a closed path $\mathcal{C}$ is equal to the current enclosed by that path, I_{encl}. Before proceeding to interpret this law, it is useful to see that it is dimensionally correct. That is, $\mathbf{H}$ (units of A/m) integrated over a distance (units of m) yields a quantity with units of current (i.e., A).

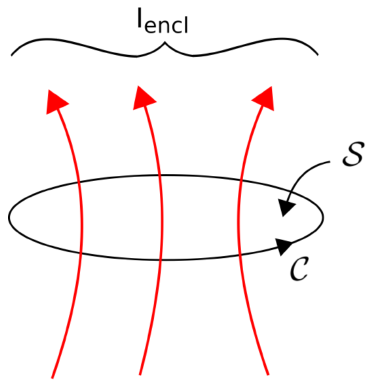

Figure 7.4.1: Reference directions for Ampere's Circuital Law (Equation 7.4.1). (CC BY SA 4.0; K. Kikkeri)

In general, I_{encl} may be either positive or negative. The direction corresponding to positive current flow must be correctly associated with $\mathcal{C}$. The relationship follows the *right-hand rule* of Stokes' Theorem summarized as follows. The direction of positive I_{encl} is the direction in which the fingers of the right hand intersect any surface $\mathcal{S}$ bordered by $\mathcal{C}$ when the thumb of the right hand points in the direction of integration. The connection to Stokes' Theorem is not a coincidence. See Section 7.9 for more about this.

Note that $\mathcal{S}$ can be *any* surface that is bounded by $\mathcal{C}$ – not just the taut surface implied in Figure 7.4.1.

> The integral form of Ampere's Circuital Law for magnetostatics (Equation 7.4.1) relates the magnetic field along a closed path to the total current flowing through any surface bounded by that path.

ACL plays a role in magnetostatics that is very similar to the role played by the integral form of Gauss' Law in electrostatics (Section 5.5). That is, ACL provides a means calculate the magnetic field given the source current. ACL also has a similar limitation. Generally, symmetry is required to simplify the problem sufficiently so that the integral equation may be solved. Fortunately, a number of important problems fall in this category. Examples include the problems addressed in Sections 7.5 (magnetic field of a line current), 7.6 (magnetic field inside a straight coil), 7.7 (magnetic field of a toroidal coil), 7.8 (magnetic field of a current sheet), and 7.11(boundary conditions on the magnetic field intensity). For problems in which the necessary symmetry is not available, the differential form of ACL may be required (Section 7.9).

Finally, be aware that the form of ACL addressed here applies to magnetostatics only. In the presence of a *time-varying* electric field, the right side of ACL includes an additional term known as the *displacement current* (Section 8.9).

7.5: Magnetic Field of an Infinitely-Long Straight Current-Bearing Wire

In this section, we use the magnetostatic form of Ampere's Circuital Law (ACL) to determine the magnetic field due to a steady current I (units of A) in an infinitely-long straight wire. The problem is illustrated in Figure 7.5.1. The wire is an electrically-conducting circular cylinder of radius a. Since the wire is a cylinder, the problem is easiest to work in cylindrical coordinates with the wire aligned along the z axis.

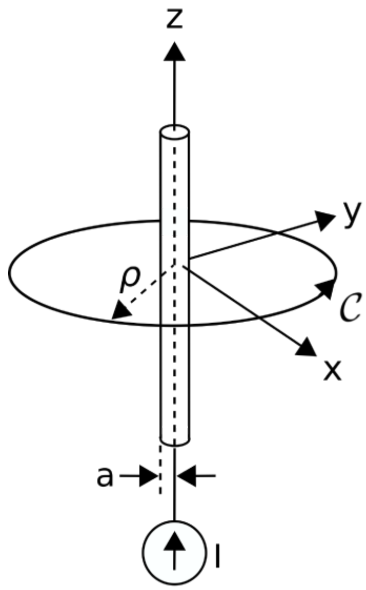

Figure 7.5.1: Determination of the magnetic field due to steady current in an infinitely-long straight wire. (CC BY SA 4.0; K. Kikkeri).

Here's the relevant form of ACL:

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl} \tag{7.5.1}$$

where I_{encl} is the current enclosed by the closed path $\mathcal{C}$. ACL works for *any* closed path, so to exploit the symmetry of the cylindrical coordinate system we choose a circular path of radius ρ in the $z=0$ plane, centered at the origin. With this choice we have

$$I_{encl} = I \text{ for } \rho \geq a$$

For $\rho < a$, we see that $I_{encl} < I$. a steady (DC) current will be distributed uniformly throughout the wire (Section 6.4). Since the current is uniformly distributed over the cross section, I_{encl} is less than the total current I by the same factor that the area enclosed by $\mathcal{C}$ is less than πa^2, the cross-sectional area of the wire. The area enclosed by $\mathcal{C}$ is simply $\pi\rho^2$, so we have

$$I_{encl} = I\frac{\pi\rho^2}{\pi a^2} = I\frac{\rho^2}{a^2} \text{ for } \rho < a$$

For the choice of $\mathcal{C}$ made above, Equation 7.5.1 becomes

$$\int_{\phi=0}^{2\pi} \mathbf{H} \cdot \left(\hat{\phi}\, \rho\, d\phi\right) = I_{encl} \tag{7.5.2}$$

Note that we have chosen to integrate in the $+\phi$ direction. Therefore, the right-hand rule specifies that positive I_{encl} corresponds to current flowing in the $+z$ direction, which is consistent with the direction indicated in Figure 7.5.1. (Here's an excellent exercise to test your understanding. Change the direction of the path of integration and confirm that you get the same result obtained at the end of this section. Changing the direction of integration should not change the magnetic field associated with the current!)

The simplest way to solve for $\mathbf{H}$ from Equation 7.5.2 is to use a symmetry argument, which proceeds as follows:

- Since the distribution of current is uniform and infinite in the z-dimension, $\mathbf{H}$ can't depend on z, and so $\mathbf{H} \cdot \hat{\mathbf{z}}$ must be zero everywhere.

- The problem is identical after any amount of rotation in ϕ; therefore, the magnitude of $\mathbf{H}$ cannot depend on ϕ. This is a form of radial symmetry. Since we determined above that $\mathbf{H}$ can't depend on z either, it must be that the magnitude of $\mathbf{H}$ can depend only on ρ.
- The radial symmetry of the problem also requires that $\mathbf{H}\cdot\hat{\rho}$ be equal to zero. If this were not the case, then the field would not be radially symmetric. Since we determined above that $\mathbf{H}\cdot\hat{\mathbf{z}}$ is also zero, $\mathbf{H}$ must be entirely $\pm\hat{\phi}$-directed.

From the above considerations, the most general form of the magnetic field intensity can be written $\mathbf{H}=\hat{\phi}H(\rho)$. Substituting this into Equation 7.5.2, we obtain

$$\begin{aligned} I_{encl} &= \int_{\phi=0}^{2\pi}[\hat{\phi}H(\rho)]\cdot(\hat{\phi}\rho d\phi) \\ &= \rho H(\rho)\int_{\phi=0}^{2\pi} d\rho \\ &= 2\pi\rho H(\rho) \end{aligned}$$

Therefore, $H(\rho)=I_{encl}/2\pi\rho$. Reassociating the known direction, we obtain:

$$\mathbf{H}=\hat{\phi}\frac{I_{encl}}{2\pi\rho}$$

Therefore, the field outside of the wire is:

$$\boxed{\mathbf{H}=\hat{\phi}\frac{I}{2\pi\rho} \text{ for } \rho\geq a} \tag{7.5.3}$$

whereas the field inside the wire is:

$$\boxed{\mathbf{H}=\hat{\phi}\frac{I\rho}{2\pi a^2} \text{ for } \rho< a} \tag{7.5.4}$$

(By the way, this is a good time for a units check.)

Note that as ρ increases from zero to a (i.e., inside the wire), the magnetic field is proportional to ρ and therefore increases. However, as ρ continues to increase beyond a (i.e., outside the wire), the magnetic field is proportional to ρ^{-1} and therefore decreases.

If desired, the associated magnetic flux density can be obtained using $\mathbf{B}=\mu\mathbf{H}$.

Summarizing

> The magnetic field due to current in an infinite straight wire is given by Equations [m0119_eACLLCe] (outside the wire) and [m0119_eACLLCi] (inside the wire). The magnetic field is $+\hat{\phi}$-directed for current flowing in the $+z$ direction, so the magnetic field lines form concentric circles perpendicular to and centered on the wire.

Finally, we point out another "right-hand rule" that emerges from this solution, shown in Figure $7.5.2$ and summarized below:

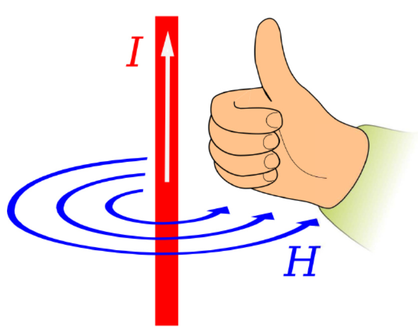

Figure $7.5.2$: Right-hand rule for the relationship between the direction of current and the direction of the magnetic field. (CC BY SA 4.0 (modified); Jfmelero).

> The magnetic field due to current in an infinite straight wire points in the direction of the curled fingers of the right hand when the thumb of the right hand is aligned in the direction of current flow.

This simple rule turns out to be handy in quickly determining the relationship between the directions of the magnetic field and current flow in many other problems, and so is well worth committing to memory.

7.6: Magnetic Field Inside a Straight Coil

In this section, we use the magnetostatic integral form of Ampere's Circuital Law (ACL) to determine the magnetic field inside a straight coil of the type shown in Figure $7.6.1$ in response to a steady (i.e., DC) current. The result has a number of applications, including the analysis and design of inductors, solenoids (coils that are used as magnets, typically as part of an actuator), and as a building block and source of insight for more complex problems.

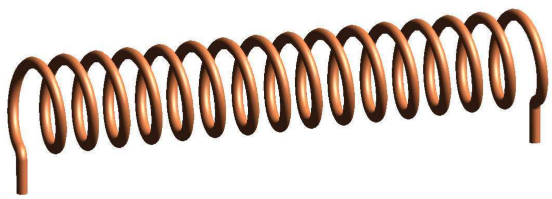

Figure $7.6.1$: A straight coil. (public domain; Zureks).

The present problem is illustrated in Figure $7.6.2$. The coil is circular with radius a and length l, and consists of N turns ("windings") of wire wound with uniform winding density. Since the coil forms a cylinder, the problem is easiest to work in cylindrical coordinates with the axis of the coil aligned along the z axis.

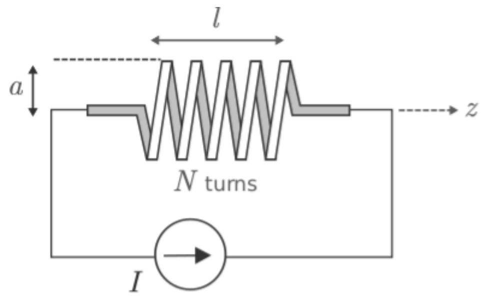

Figure $7.6.2$: Determination of the magnetic field due to DC current in a coil.

To begin, let's take stock of what we already know about the answer, which is actually quite a bit. The magnetic field deep inside the coil is generally aligned with axis of the coil as shown in Figure $7.6.3$. This can be explained using the result for the magnetic field due to a straight line current (Section 7.5), in which we found that the magnetic field follows a "right-hand rule." The magnetic field points in the direction of the fingers of the right hand when the thumb of the right hand is aligned in the direction of current flow. The wire comprising the coil is obviously *not* straight, but we can consider just one short segment of one turn and then sum the results for all such segments. When we consider this for a single turn of the coil, the situation is as shown in Figure $7.6.4$. Summing the results for many loops, we see that the direction of the magnetic field inside the coil must be generally in the $+\hat{\mathbf{z}}$ direction when the current I is applied as shown in Figures $7.6.2$ and $7.6.3$. However, there is one caveat. The windings must be sufficiently closely-spaced that the magnetic field lines can only pass through the openings at the end of the coil and do not take any "shortcuts" between individual windings.

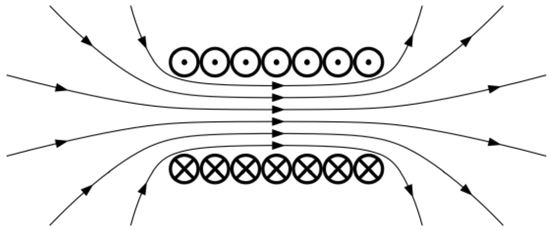

Figure $7.6.3$: Magnetic field lines inside a straight coil with closely-spaced windings. (Dotted circles represent current flowing up/out from the page; crossed circles represent current flowing down/into the page.) (© CC BY SA 3.0; Geek3)

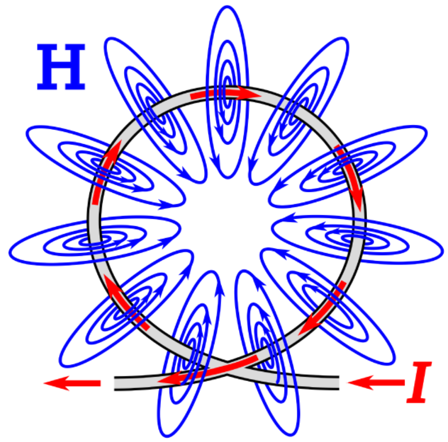

Figure 7.6.4: Magnetic field due to a single loop (© CC0 1.0 (modified); Chetvorno)

Figure $7.6.3$ also indicates that the magnetic field lines near the ends of the coil diverge from the axis of the coil. This is understandable since magnetic field lines form closed loops. The relatively complex structure of the magnetic fields near the ends of the coil (the "fringing field") and outside of the coil make them relatively difficult to analyze. Therefore, here we shall restrict our attention to the magnetic field deep inside the coil. This restriction turns out to be of little consequence as long as $l \gg a$.

Also, it is apparent from the radial symmetry of the coil that the magnitude of the magnetic field cannot depend on ϕ. Putting these findings together, we find that the most general form for the magnetic field intensity deep inside the coil is $\mathbf{H} \approx \hat{\mathbf{z}} H(\rho)$. That is, the direction of $\mathbf{H}$ is $\pm\hat{\mathbf{z}}$ and the magnitude of $\mathbf{H}$ depends, at most, on ρ. In fact, we will soon find with the assistance of ACL that the magnitude of $\mathbf{H}$ doesn't depend on ρ either.

Here's the relevant form of ACL:

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl} \tag{7.6.1}$$

where I_{encl} is the current enclosed by the closed path $\mathcal{C}$. ACL works for *any* closed path that encloses the current of interest. Also, for simplicity, we prefer a path that lies on a constant-coordinate surface. The selected path is shown in Figure $7.6.5$. The benefits of this particular path will soon become apparent. However, note for now that this particular choice is consistent with the right-hand rule relating the direction of $\mathcal{C}$ to the direction of positive I. That is, when I is positive, the current in the turns of the coil pass through the surface bounded by $\mathcal{C}$ in the same direction as the fingers of the right hand when the thumb is aligned in the indicated direction of $\mathcal{C}$.

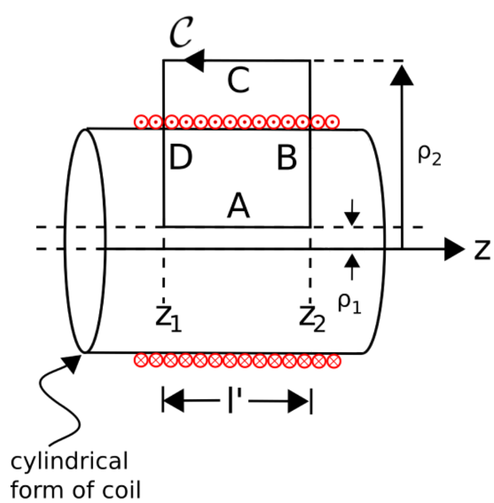

Figure 7.6.5: Selected path of integration. (© CC BY SA 4.0: K. Kikkeri)

Let's define N to be the number of windings in the coil. Then, the winding density of the coil is N/l (turns/m). Let the path length in the z direction be l', as indicated in Figure $7.6.5$. Then the enclosed current is

$$I_{encl} = \frac{N}{l} l' I$$

That is, the number of turns per unit length times length gives number of turns, and this quantity times the current through the wire is the total amount of current crossing the surface bounded by $\mathcal{C}$.

For the choice of $\mathcal{C}$ made above, and taking our approximation for the form of $\mathbf{H}$ as exact, Equation 7.6.1 becomes

$$\oint_{\mathcal{C}} [\hat{\mathbf{z}} H(\rho)] \cdot d\mathbf{l} = \frac{N}{l} l' I \tag{7.6.2}$$

The integral consists of segments A, B, C, and D, as shown in Figure 7.6.5. Let us consider the result for each of these segments individually:

- The integral over segments B and D is zero because $d\mathbf{l} = \hat{\rho} d\rho$ for these segments, and so $\mathbf{H} \cdot d\mathbf{l} = 0$ for these segments.
- It is also possible to make the contribution from Segment C go to zero simply by letting $\rho_2 \to \infty$. The argument is as follows. The magnitude of $\mathbf{H}$ outside the coil must decrease with distance from the coil, so for ρ sufficiently large, $\mathbf{H}(\rho)$ becomes negligible. If that's the case, then the integral over Segment C also becomes negligible.

With $\rho_2 \to \infty$, only Segment A contributes significantly to the integral over $\mathcal{C}$ and Equation 7.6.2 becomes:

$$\begin{aligned} \frac{N}{l} l' I &= \int_{z_1}^{z_2} [\hat{\mathbf{z}} H(\rho_1)] \cdot (\hat{\mathbf{z}} dz) \\ &= H(\rho_1) \int_{z_1}^{z_2} dz \\ &= H(\rho_1) [z_2 - z_1] \end{aligned}$$

Note $z_2 - z_1$ is simply l'. Also, we have found that the result is independent of ρ_1, as anticipated earlier. Summarizing:

$$\boxed{\mathbf{H} \approx \hat{\mathbf{z}} \frac{NI}{l} \text{ inside coil}} \tag{7.6.3}$$

Let's take a moment to consider the implications of this remarkably simple result.

- Note that it is dimensionally correct; that is, current divided by length gives units of A/m, which are the units of $\mathbf{H}$.
- We have found that the magnetic field is simply winding density (N/l) times current. To increase the magnetic field, you can either use more turns per unit length or increase the current.
- We have found that the magnetic field is uniform inside the coil; that is, the magnetic field along the axis is equal to the magnetic field close to the cylinder wall formed by the coil. However, this does not apply close to ends of the coil, since we have neglected the fringing field.

These findings have useful applications in more complicated and practical problems, so it is worthwhile taking note of these now. Summarizing:

> The magnetic field deep inside an ideal straight coil (Equation 7.6.3) is uniform and proportional to winding density and current.

7.7: Magnetic Field of a Toroidal Coil

A toroid is a cylinder in which the ends are joined to form a closed loop. An example of a toroidal coil is shown in Figure 7.7.1. Toroidal coils are commonly used to form inductors and transformers. The principal advantage of toroidal coils over straight coils in these applications is *magnetic field containment* – as we shall see in this section, the magnetic field outside of a toroidal coil can be made negligibly small. This reduces concern about interactions between this field and other fields and structures in the vicinity.

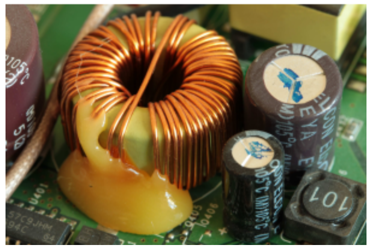

Figure 7.7.1: A toroidal coil used as a large-value inductor in the power supply of a wireless router. (© CC0 1.0; Slick)

In this section, we use the magnetostatic form of Ampere's Circuital Law (ACL) to determine the magnetic field due to a steady (DC) current flowing through a toroidal coil. The problem is illustrated in Figure 7.7.2. The toroid is circular with inner and outer radii a and b, respectively. The coil consists of N windings (turns) of wire wound with uniform winding density. This problem is easiest to work in cylindrical coordinates with the toroid centered on the origin in $z = 0$ plane.

To begin, let's take stock of what we already know about the answer, which is actually quite a bit. First, a review of Section 7.6 ("Magnetic Field Inside a Straight Coil") is recommended. There it is shown that the magnetic field deep inside a straight coil is aligned with axis of the coil. This can be explained using the result for the magnetic field due to a straight line current (Section 7.5), in which we found that the magnetic field follows a "right-hand rule" – The magnetic field points in the direction of the curled fingers of the right hand when the thumb of the right hand is aligned in the direction of current flow. The wire comprising the coil is obviously *not* straight, but we can consider just one short segment of one turn and then sum the results for all such segments. When we do this, we see that the direction of the magnetic field inside the coil must be in the $+\hat{\phi}$ direction when the current I is applied as shown in Figure 7.7.2. Also, because the problem is identical after any amount of rotation around the z axis, the magnitude of the magnetic field cannot depend on ϕ. Putting these findings together, we find that the most general form for the magnetic field intensity inside or outside the coil is $\mathbf{H} = \hat{\phi} H(\rho, z)$.

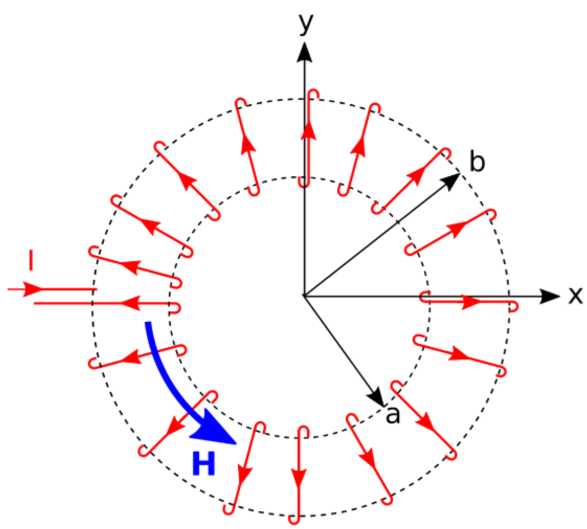

Figure 7.7.2: Geometry of a toroidal coil.

Here's the relevant form of ACL:

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl} \tag{7.7.1}$$

where I_{encl} is the current enclosed by the closed path $\mathcal{C}$. ACL works for *any* closed path, but we need one that encloses some current so as to obtain a relationship between I and $\mathbf{H}$. Also, for simplicity, we prefer a path that lies on a constant-coordinate surface. The selected path is a circle of radius ρ centered on the origin in the $z = z_0$ plane, as shown in Figure 7.7.3. We further require $\mathcal{C}$ to lie entirely inside the coil, which ensures that the enclosed current includes the current of all the windings as they pass through the hole at the center of the toroid. We choose the direction of integration to be in the $+\phi$ direction, which is consistent with the indicated direction of positive I. That is, when I is positive, the current in the windings of the coil pass through the surface bounded by $\mathcal{C}$ in the same direction as the curled fingers of the right hand when the thumb is aligned in the indicated direction of $\mathcal{C}$.

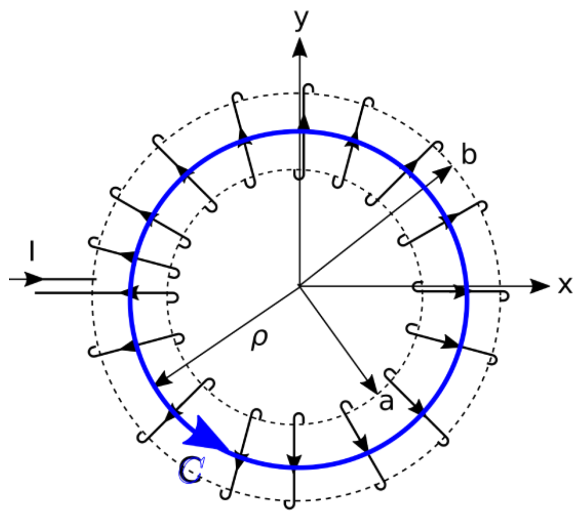

Figure 7.7.3: Selected path of integration.

In terms of the variables we have defined, the enclosed current is simply

$$I_{encl} = NI$$

Equation 7.7.1 becomes

$$\oint_{\mathcal{C}} \left[\hat{\phi} H(\rho, z_0)\right] \cdot d\mathbf{l} = NI \tag{7.7.2}$$

Now evaluating the integral:

$$\begin{aligned} NI &= \int_0^{2\pi} \left[\hat{\phi} H\left(\rho, z_0\right)\right] \cdot \left(\hat{\phi} \rho d\phi\right) \\ &= \rho H\left(\rho, z_0\right) \int_0^{2\pi} d\phi \\ &= 2\pi\rho H\left(\rho, z_0\right) \end{aligned}$$

It is now clear that the result is independent of z_0. Summarizing:

$$\boxed{\mathbf{H} = \hat{\phi} \frac{NI}{2\pi\rho} \text{ inside coil}} \tag{7.7.3}$$

Let's take a moment to consider the implications of this result.

- Note that it is dimensionally correct; that is, current divided by the circumference of $\mathcal{C}$ ($2\pi\rho$) gives units of A/m, which are the units of $\mathbf{H}$.
- We have found that the magnetic field is proportional to winding density (i.e., number of windings divided by circumference) times current. To increase the magnetic field you can either use more windings or increase the current.
- Remarkably, we have found that the magnitude of the magnetic field inside the coil depends only on ρ; i.e., the distance from the central (here, z) axis. It is independent of z.

Summarizing:

The magnetic field inside a toroidal coil (Equation 7.7.3) depends only on distance from the central axis and is proportional to winding density and current.

Now let us consider what happens outside the coil. For this, we consider any path of integration ($\mathcal{C}$) that lies completely outside the coil. Note that any such path encloses no current and therefore $I_{encl} = 0$ for any such path. In this case we have:

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = 0$$

There are two ways this could be true. Either $\mathbf{H}$ could be zero everywhere along the path, or $\mathbf{H}$ could be non-zero along the path in such a way that the integral windings out to be zero. The radial symmetry of the problem rules out the second possibility – if $\mathbf{H}$ is radially symmetric and $\mathcal{C}$ is radially symmetric, then the sign of $\mathbf{H} \cdot d\mathbf{l}$ should not change over $\mathcal{C}$. Therefore:

The magnetic field everywhere outside an ideal toroidal coil is zero.

Note the caveat signaled by the use of the adjective "ideal." In a practical toroidal coil, we expect there will be some leakage of magnetic flux between the windings. In practice, this leakage can be made negligibly small by using a sufficiently high winding density and winding the wire on material on a toroidal form (a "core") having sufficiently large permeability. The use of a high-permeability core, as shown in Figure 7.7.1, will dramatically improve the already pretty-good containment. In fact, the use of such a core allows the spacing between windings to become quite large before leakage becomes significant.

One final observation about toroidal coils is that at no point in the derivation of the magnetic field did we need to consider the cross-sectional shape of the coil; we merely needed to know whether $\mathcal{C}$ was inside or outside the coil. Therefore:

The magnetic field inside an ideal toroidal coil does not depend on the cross-sectional shape of the coil.

7.8: Magnetic Field of an Infinite Current Sheet

We now consider the magnetic field due to an infinite sheet of current, shown in Figure $7.8.1$. The solution to this problem is useful as a building block and source of insight in more complex problems, as well as being a useful approximation to some practical problems involving current sheets of finite extent including, for example, microstrip transmission line and ground plane currents in printed circuit boards.

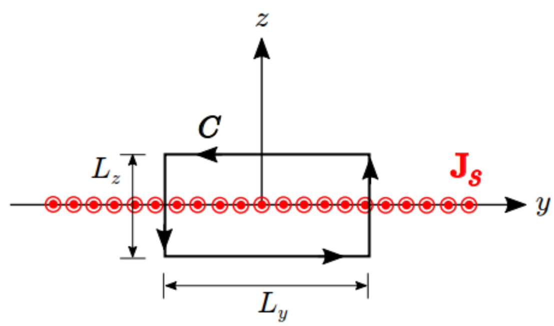

Figure 7.8.1: Analysis of the magnetic field due to an infinite thin sheet of current.

The current sheet in Figure $7.8.1$ lies in the $z=0$ plane and the current density is $\mathbf{J}_s=\hat{\mathbf{x}}J_s$ (units of A/m); i.e., the current is uniformly distributed such that the total current crossing any segment of width Δy along the y direction is $J_s\Delta y$.

To begin, let's take stock of what we already know about the answer, which is actually quite a bit. For example, imagine the current sheet as a continuum of thin strips parallel to the x axis and very thin in the y dimension. Each of these strips individually behaves like a straight line current $I=J_s\Delta y$ (units of A). The magnetic field due to each of these strips is determined by a "right-hand rule" – the magnetic field points in the direction of the curled fingers of the right hand when the thumb of the right hand is aligned in the direction of current flow. (Section 7.5). It is apparent from this much that $\mathbf{H}$ can have no $\hat{\mathbf{y}}$ component, since the field of each individual strip has no $\hat{\mathbf{y}}$ component. When the magnetic field due to each strip is added to that of all the other strips, the $\hat{\mathbf{z}}$ component of the sum field must be zero due to symmetry. It is also clear from symmetry considerations that the magnitude of $\mathbf{H}$ cannot depend on x or y. Summarizing, we have determined that the most general form for $\mathbf{H}$ is $\hat{\mathbf{y}}H(z)$, and furthermore, the sign of $H(z)$ must be positive for $z<0$ and negative for $z>0$.

It's possible to solve this problem by actually summing over the continuum of thin current strips as imagined above.[1] However, it's far easier to use Ampere's Circuital Law (ACL; Section 7.4). Here's the relevant form of ACL:

$$\oint_{\mathcal{C}}\mathbf{H}\cdot d\mathbf{l}=I_{encl} \tag{7.8.1}$$

where I_{encl} is the current enclosed by a closed path $\mathcal{C}$. ACL works for *any* closed path, but we need one that encloses some current so as to obtain a relationship between $\mathbf{J}_s$ and $\mathbf{H}$. Also, for simplicity, we prefer a path that lies on a constant-coordinate surface. A convenient path in this problem is a rectangle lying in the $x=0$ plane and centered on the origin, as shown in Figure $7.8.1$. We choose the direction of integration to be counter-clockwise from the perspective shown in Figure $7.8.1$, which is consistent with the indicated direction of positive J_s according to the applicable right-hand rule from Stokes' Theorem. That is, when J_s is positive (current flowing in the $+\hat{\mathbf{x}}$ direction), the current passes through the surface bounded by $\mathcal{C}$ in the same direction as the curled fingers of the right hand when the thumb is aligned in the indicated direction of $\mathcal{C}$.

Let us define L_y to be the width of the rectangular path of integration in the y dimension and L_z to be the width in the z dimension. In terms of the variables we have defined, the enclosed current is simply

$$I_{encl}=J_sL_y$$

Equation 7.8.1 becomes

$$\oint_{\mathcal{C}}[\hat{\mathbf{y}}H(z)]\cdot d\mathbf{l}=J_sL_y \tag{7.8.2}$$

Note that $\mathbf{H}\cdot d\mathbf{l}=0$ for the vertical sides of the path, since $\mathbf{H}$ is $\hat{\mathbf{y}}$-directed and $d\mathbf{l}=\hat{\mathbf{z}}dz$ on those sides. Therefore, only the horizontal sides contribute to the integral and we have:

$$\int_{-L_{y/2}}^{+L_w/2} \left[\hat{\mathbf{y}} H\left(-\frac{L_z}{2}\right)\right] \cdot (\hat{\mathbf{y}} dy)$$
$$+\int_{+L_v/2}^{-L_y/2} \left[\hat{\mathbf{y}} H\left(+\frac{L_z}{2}\right)\right] \cdot (\hat{\mathbf{y}} dy) = J_s L_y$$

Now evaluating the integrals:

$$H\left(-\frac{L_z}{2}\right) L_y - H\left(+\frac{L_z}{2}\right) L_y = J_s L_y$$

Note that all factors of L_y cancel in the above equation. Furthermore, $H(-L_z/2) = -H(+L_z/2)$ due to (1) symmetry between the upper and lower half-spaces and (2) the change in sign between these half-spaces, noted earlier. We use this to eliminate $H(+L_z/2)$ and solve for $H(-L_z/2)$ as follows:

$$2H(-L_z/2) = J_s$$

yielding

$$H(-L_z/2) = +\frac{J_s}{2}$$

and therefore

$$H(+L_z/2) = -\frac{J_s}{2}$$

Furthermore, note that $\mathbf{H}$ is independent of L_z; for example, the result we just found indicates the same value of $H(+L_z/2)$ regardless of the value of L_z. Therefore, $\mathbf{H}$ is uniform throughout all space, except for the change of sign corresponding for the field above vs. below the sheet.

Summarizing

$$\boxed{\mathbf{H} = \pm\hat{\mathbf{y}}\frac{J_s}{2} \text{ for } z \lessgtr 0} \tag{7.8.3}$$

> The magnetic field intensity due to an infinite sheet of current (Equation 7.8.3) is spatially uniform except for a change of sign corresponding for the field above vs. below the sheet.

1. In fact, this is pretty good thing to try, if for no other reason than to see how much simpler it is to use ACL instead.↩

7.9: Ampere's Law (Magnetostatics) - Differential Form

The integral form of Amperes' Circuital Law (ACL) for magnetostatics relates the magnetic field along a closed path to the total current flowing through any surface bounded by that path. In mathematical form:

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl}$$

where $\mathbf{H}$ is magnetic field intensity, $\mathcal{C}$ is the closed curve, and I_{encl} is the total current flowing through any surface bounded by $\mathcal{C}$. In this section, we derive the differential form of this equation. In some applications, this differential equation, combined with boundary conditions associated with discontinuities in structure and materials, can be used to solve for the magnetic field in arbitrarily complicated scenarios. A more direct reason for seeking out this differential equation is that we gain a little more insight into the relationship between current and the magnetic field, disclosed at the end of this section.

The equation we seek may be obtained using Stokes' Theorem, which in the present case may be written:

$$\int_{\mathcal{S}} (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} \tag{7.9.1}$$

where $\mathcal{S}$ is any surface bounded by $\mathcal{C}$, and $d\mathbf{s}$ is the differential surface area combined with the unit vector in the direction determined by the right-hand rule from Stokes' Theorem. ACL tells us that the right side of the above equation is simply I_{encl}. We may express I_{encl} as the integral of the volume current density $\mathbf{J}$ (units of A/m^2; Section 6.2) as follows:

$$I_{encl} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{s}$$

so we may rewrite Equation 7.9.1 as follows:

$$\int_{\mathcal{S}} (\nabla \times \mathbf{H}) \cdot d\mathbf{s} = \int_{\mathcal{S}} \mathbf{J} \cdot d\mathbf{s}$$

The above relationship must hold regardless of the specific location or shape of $\mathcal{S}$. The only way this is possible for all possible surfaces in all applicable scenarios is if the integrands are equal. Thus, we obtain the desired expression:

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{7.9.2}$$

That is, the curl of the magnetic field intensity at a point is equal to the volume current density at that point. Recalling the properties of the curl operator – in particular, that curl involves derivatives with respect to direction – we conclude:

> The differential form of Ampere's Circuital Law for magnetostatics (Equation 7.9.2) indicates that the volume current density at any point in space is proportional to the spatial rate of change of the magnetic field and is perpendicular to the magnetic field at that point.

7.10: Boundary Conditions on the Magnetic Flux Density (B)

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At an interface between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. Continuities and discontinuities in fields can be described mathematically by *boundary conditions* and used to constrain solutions for fields away from these interfaces.

In this section, we derive the boundary condition on the magnetic flux density $\mathbf{B}$ at a smooth interface between two material regions, as shown in Figure $7.10.1$[1] The desired boundary condition may be obtained from Gauss' Law for Magnetic Fields (GLM):

$$\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = 0$$

where $\mathcal{S}$ is any closed surface. Let $\mathcal{S}$ take the form of cylinder centered at a point on the interface, and for which the flat ends are parallel to the surface and perpendicular to $\hat{\mathbf{n}}$, as shown in Figure $7.10.1$ Let the radius of this cylinder be a, and let the length of the cylinder be $2h$.

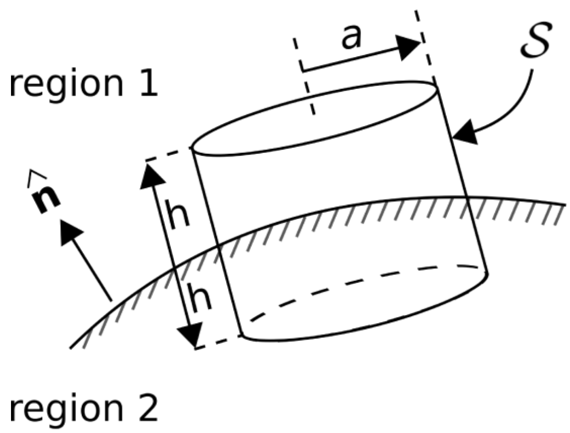

Figure 7.10.1: Determination of the boundary condition on B at the interface between material regions (© CC BY SA 4.0; K. Kikkeri)

From GLM, we have

$$\begin{aligned} \oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s} = & \int_{top} \mathbf{B} \cdot d\mathbf{s} \\ & + \int_{side} \mathbf{B} \cdot d\mathbf{s} \\ & + \int_{bottom} \mathbf{B} \cdot d\mathbf{s} \qquad = 0 \end{aligned}$$

Now let us reduce h and a together while (1) maintaining a constant ratio $h/a \ll 1$ and (2) keeping $\mathcal{S}$ centered on the interface. Because $h \ll a$, the area of the side can be made negligible relative to the area of the top and bottom. Then, as $h \to 0$, we are left with

$$\int_{top} \mathbf{B} \cdot d\mathbf{s} + \int_{bottom} \mathbf{B} \cdot d\mathbf{s} \to 0$$

As the area of the top and bottom sides become infinitesimal, the variation in $\mathbf{B}$ over these areas becomes negligible. Now we have simply:

$$\mathbf{B}_1 \cdot \hat{\mathbf{n}} \Delta A + \mathbf{B}_2 \cdot (-\hat{\mathbf{n}}) \, \Delta A \to 0$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are the magnetic flux densities at the interface but in regions 1 and 2, respectively, and ΔA is the area of the top and bottom sides. Note that the orientation of $\hat{\mathbf{n}}$ is important – we have assumed $\hat{\mathbf{n}}$ points into region 1, and we must now stick with this choice. Thus, we obtain

$$\boxed{\hat{\mathbf{n}} \cdot (\mathbf{B}_1 - \mathbf{B}_2) = 0} \tag{7.10.1}$$

where, as noted above, $\hat{\mathbf{n}}$ points into region 1.

Summarizing

> The normal (perpendicular) component of $\mathbf{B}$ across the boundary between two material regions is continuous.

It is worth noting what this means for the magnetic field intensity $\mathbf{H}$. Since $\mathbf{B} = \mu\mathbf{H}$, it must be that

> The normal (perpendicular) component of $\mathbf{H}$ across the boundary between two material regions is discontinuous if the permeabilities are unequal.

---

1. It may be helpful to note the similarity (duality, in fact) between this derivation and the derivation of the associated boundary condition on $\mathbf{D}$ presented in Section 5.18.↩

7.11: Boundary Conditions on the Magnetic Field Intensity (H)

In homogeneous media, electromagnetic quantities vary smoothly and continuously. At a boundary between dissimilar media, however, it is possible for electromagnetic quantities to be discontinuous. Continuities and discontinuities in fields can be described mathematically by *boundary conditions* and used to constrain solutions for fields away from these boundaries. In this section, we derive boundary conditions on the magnetic field intensity $\mathbf{H}$.

To begin, consider a region consisting of only two media that meet at a smooth boundary as shown in Figure $7.11.1$. The desired boundary condition can be obtained directly from Ampere's Circuital Law (ACL):

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl} \tag{7.11.1}$$

where $\mathcal{C}$ is any closed path and I_{encl} is the current that flows through the surface bounded by that path in the direction specified by the "right-hand rule" of Stokes' theorem.

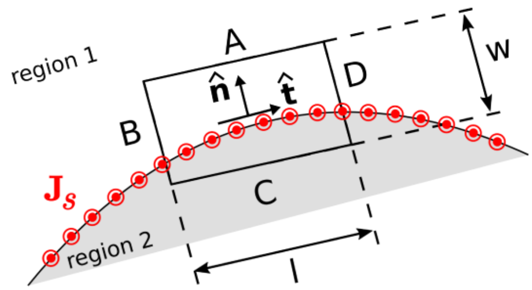

Figure 7.11.1: Determining the boundary condition on H at the smooth boundary between two material regions.

Let $\mathcal{C}$ take the form of a rectangle centered on a point on the boundary as shown in Figure $7.11.1$, perpendicular to the direction of current flow at that location. Let the sides A, B, C, and D be perpendicular and parallel to the boundary. Let the length of the parallel sides be l, and let the length of the perpendicular sides be w. Now we apply ACL. We must integrate in a counter-clockwise direction in order to be consistent with the indicated reference direction for $\mathbf{J}_s$. Thus:

$$\begin{aligned} \oint \mathbf{H} \cdot d\mathbf{l} = & \int_A \mathbf{H} \cdot d\mathbf{l} \\ & + \int_B \mathbf{H} \cdot d\mathbf{l} \\ & + \int_C \mathbf{H} \cdot d\mathbf{l} \\ & + \int_D \mathbf{H} \cdot d\mathbf{l} \qquad = I_{encl} \end{aligned}$$

Now we let w and l become vanishingly small while (1) maintaining the ratio l/w and (2) keeping $\mathcal{C}$ centered on the boundary. In this process, the contributions from the B and D segments become equal in magnitude but opposite in sign; i.e.,

$$\int_B \mathbf{H} \cdot d\mathbf{l} + \int_D \mathbf{H} \cdot d\mathbf{l} \to 0$$

This leaves

$$\int_A \mathbf{H} \cdot d\mathbf{l} + \int_C \mathbf{H} \cdot d\mathbf{l} \to I_{encl}$$

Let us define the unit vector $\hat{\mathbf{t}}$ ("tangent") as shown in Figure $7.11.1$. Now we have simply:

$$\mathbf{H}_1 \cdot \left(-\hat{\mathbf{t}}\Delta l\right) + \mathbf{H}_2 \cdot \left(+\hat{\mathbf{t}}\Delta l\right) = I_{encl} \tag{7.11.2}$$

where $\mathbf{H}_1$ and $\mathbf{H}_2$ are the fields evaluated on the two sides of the boundary, and $\Delta l \to 0$ is the length of sides A and C. As always, I_{encl} (units of A) may be interpreted as the flux of the current density $\mathbf{J}_s$ (units of A/m) flowing past a line on the surface having length Δl (units of m) perpendicular to $\hat{\mathbf{t}} \times \hat{\mathbf{n}}$, where $\hat{\mathbf{n}}$ is the normal to the surface, pointing into Region 1. Stated mathematically:

$$I_{encl} \rightarrow \mathbf{J}_s \cdot \left(\Delta l \, \hat{\mathbf{t}} \times \hat{\mathbf{n}}\right)$$

Before proceeding, note this is true regardless of the particular direction we selected for $\hat{\mathbf{t}}$; it is only necessary that $\hat{\mathbf{t}}$ be tangent to the boundary. Thus, $\hat{\mathbf{t}} \times \hat{\mathbf{n}}$ need not necessarily be in the same direction as $\mathbf{J}_s$. Now Equation 7.11.2 can be written:

$$\mathbf{H}_2 \cdot \hat{\mathbf{t}} \Delta l - \mathbf{H}_1 \cdot \hat{\mathbf{t}} \Delta l = \mathbf{J}_s \cdot \left(\hat{\mathbf{t}} \times \hat{\mathbf{n}}\right) \Delta l$$

Eliminating the common factor of Δl and arranging terms on the left:

$$\left(\mathbf{H}_2 - \mathbf{H}_1\right) \cdot \hat{\mathbf{t}} = \mathbf{J}_s \cdot \left(\hat{\mathbf{t}} \times \hat{\mathbf{n}}\right) \tag{7.11.3}$$

The right side may be transformed using a vector identity ($\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \mathbf{B} \cdot (\mathbf{C} \times \mathbf{A}) = \mathbf{C} \cdot (\mathbf{A} \times \mathbf{B})$ of Section B3) to obtain:

$$\left(\mathbf{H}_2 - \mathbf{H}_1\right) \cdot \hat{\mathbf{t}} = \hat{\mathbf{t}} \cdot \left(\hat{\mathbf{n}} \times \mathbf{J}_s\right) \tag{7.11.4}$$

Equation 7.11.4 is the boundary condition we seek. We have found that the component of $\mathbf{H}_2 - \mathbf{H}_1$ (the difference between the magnetic field intensities at the boundary) in any direction tangent to the boundary is equal to the component of the current density flowing in the perpendicular direction $\hat{\mathbf{n}} \times \mathbf{J}_s$. Said differently:

> A discontinuity in the tangential component of the magnetic field intensity at the boundary must be supported by surface current flowing in a direction perpendicular to this component of the field.

An important consequence is that:

> If there is no surface current, then the tangential component of the magnetic field intensity is continuous across the boundary.

It is possible to obtain a mathematical form of the boundary condition that is more concise and often more useful than Equation 7.11.4. This form may be obtained as follows. First, we note that the dot product with respect to $\hat{\mathbf{t}}$ on both sides of Equation 7.11.4 means simply "any component that is tangent to the boundary." We need merely to make sure we are comparing the same tangential component on each side of the equation. For example $\hat{\mathbf{n}} \times \left(\mathbf{H}_2 - \mathbf{H}_1\right)$ is tangential to the boundary, since $\hat{\mathbf{n}}$ is perpendicular to the boundary and therefore any cross product involving $\hat{\mathbf{n}}$ will be perpendicular to $\hat{\mathbf{n}}$. The corresponding component of the current density is $\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{J}_s\right)$, so Equation 7.11.4 may be equivalently written as follows:

$$\hat{\mathbf{n}} \times \left(\mathbf{H}_2 - \mathbf{H}_1\right) = \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{J}_s \tag{7.11.5}$$

Applying a vector identity ($\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ of Section B3) to the right side of Equation 7.11.5 we obtain:

$$\begin{aligned} \hat{\mathbf{n}} \times \hat{\mathbf{n}} \times \mathbf{J}_s &= \hat{\mathbf{n}} \left(\hat{\mathbf{n}} \cdot \mathbf{J}_s\right) - \mathbf{J}_s \left(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}\right) \\ &= \hat{\mathbf{n}}\,(0) - \mathbf{J}_s\,(1) \\ &= -\mathbf{J}_s \end{aligned}$$

Therefore:

$$\hat{\mathbf{n}} \times \left(\mathbf{H}_2 - \mathbf{H}_1\right) = -\mathbf{J}_s$$

The minus sign on the right can be eliminated by swapping $\mathbf{H}_2$ and $\mathbf{H}_1$ on the left, yielding

$$\boxed{\hat{\mathbf{n}} \times \left(\mathbf{H}_1 - \mathbf{H}_2\right) = \mathbf{J}_s}$$

This is the form in which the boundary condition is most commonly expressed.

It is worth noting what this means for the magnetic field intensity $\mathbf{B}$. Since $\mathbf{B} = \mu \mathbf{H}$:

In the absence of surface current, the tangential component of $\mathbf{B}$ across the boundary between two material regions is discontinuous if the permeabilities are unequal.

7.12: Inductance

Current creates a magnetic field, which subsequently exerts force on other current-bearing structures. For example, the current in each winding of a coil exerts a force on every other winding of the coil. If the windings are fixed in place, then this force is unable to do work (i.e., move the windings), so instead the coil stores potential energy. This potential energy can be released by turning off the external source. When this happens, charge continues to flow, but is now propelled by the magnetic force. The potential energy that was stored in the coil is converted to kinetic energy and subsequently used to redistribute the charge until no current flows. At this point, the inductor has expended its stored energy. To restore energy, the external source must be turned back on, restoring the flow of charge and thereby restoring the magnetic field.

Now recall that the magnetic field is essentially defined in terms of the force associated with this potential energy; i.e., $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ where q is the charge of a particle comprising the current, $\mathbf{v}$ is the velocity of the particle, and $\mathbf{B}$ is magnetic flux density. So, rather than thinking of the potential energy of the system as being associated with the magnetic force applied to current, it is equally valid to think of the potential energy as being stored in the magnetic field associated with the current distribution. The energy stored in the magnetic field depends on the geometry of the current-bearing structure and the permeability of the intervening material because the magnetic field depends on these parameters.

The relationship between current applied to a structure and the energy stored in the associated magnetic field is what we mean by *inductance*. We may fairly summarize this insight as follows:

> Inductance is the ability of a structure to store energy in a magnetic field.

> The inductance of a structure depends on the geometry of its current-bearing structures and the permeability of the intervening medium.

Note that inductance does *not* depend on current, which we view as either a stimulus or response from this point of view. The corresponding response or stimulus, respectively, is the magnetic flux associated with this current. This leads to the following definition:

$$L = \frac{\Phi}{I} \text{ (single linkage)} \tag{7.12.1}$$

where Φ (units of Wb) is magnetic flux, I (units of A) is the current responsible for this flux, and L (units of H) is the associated inductance. (The "single linkage" caveat will be explained below.) In other words, a device with high inductance generates a large magnetic flux in response to a given current, and therefore stores more energy for a given current than a device with lower inductance.

To use Equation 7.12.1 we must carefully define what we mean by "magnetic flux" in this case. Generally, magnetic flux is magnetic flux density (again, $\mathbf{B}$, units of Wb/m^2) integrated over a specified surface $\mathcal{S}$, so

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s}$$

where $d\mathbf{s}$ is the differential surface area vector, with direction normal to $\mathcal{S}$. However, this leaves unanswered the following questions: Which $\mathcal{S}$, and which of the two possible normal directions of $d\mathbf{s}$? For a meaningful answer, $\mathcal{S}$ must uniquely associate the magnetic flux to the associated current. Such an association exists if we require the current to form a closed loop. This is shown in Figure 7.12.1. Here $\mathcal{C}$ is the closed loop along which the current flows, $\mathcal{S}$ is a surface bounded by $\mathcal{C}$, and the direction of $d\mathbf{s}$ is defined according to the right-hand rule of Stokes' Theorem. Note that $\mathcal{C}$ can be a closed loop of *any* shape; i.e., not just circular, and not restricted to lying in a plane. Further note that $\mathcal{S}$ used in the calculation of Φ can be *any* surface bounded by $\mathcal{C}$. This is because magnetic field lines form closed loops such that any one magnetic field line intersects any open surface bounded by $\mathcal{C}$ exactly once. Such an intersection is sometimes called a "linkage." So there we have it – we require the current I to form a closed loop, we measure the magnetic flux through this loop using the sign convention of the right-hand rule, and the ratio is the inductance.

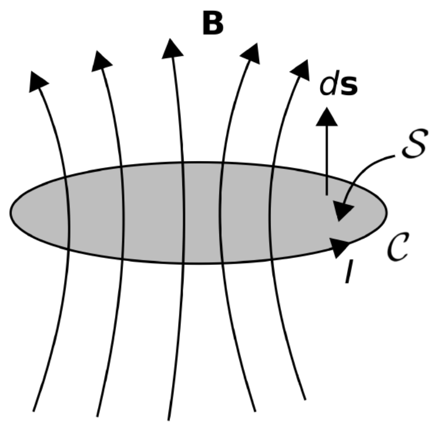

Figure 7.12.1: Association between a closed loop of current and the associated magnetic flux. (© CC BY SA 4.0; K. Kikkeri)

Many structures consist of multiple such loops – the coil is of course one of these. In a coil, each winding carries the same current, and the magnetic fields of the windings add to create a magnetic field, which grows in proportion to the winding density (Section 7.6). The magnetic flux density inside a coil is proportional to the number of windings, N, so the flux Φ in Equation 7.12.1 should properly be indicated as $N\Phi$. Another way to look at this is that we are counting the number of times the same current is able to generate a unique set of magnetic field lines that intersect $\mathcal{S}$.

Summarizing, our complete definition for inductance is

$$\boxed{L = \frac{N\Phi}{I} \text{ (identical linkages)}} \tag{7.12.2}$$

> An engineering definition of inductance is Equation 7.12.2, with the magnetic flux defined to be that associated with a single closed loop of current with sign convention as indicated in Figure 7.12.1, and N defined to be the number of times the same current I is able to create that flux.

What happens if the loops have different shapes? For example, what if the coil is not a cylinder, but rather cone-shaped? (Yes, there is such a thing – see "Additional Reading" at the end of this section.) In this case, one needs a better way to determine the factor $N\Phi$ since the flux associated with each loop of current will be different. However, this is beyond the scope of this section.

An *inductor* is a device that is designed to exhibit a specified inductance. We can now make the connection to the concept of the inductor as it appears in elementary circuit theory. First, we rewrite Equation 7.12.2 as follows:

$$I = \frac{N\Phi}{L}$$

Taking the derivative of both sides of this equation with respect to time, we obtain:

$$\frac{d}{dt} I = \frac{N}{L} \frac{d}{dt} \Phi \tag{7.12.3}$$

Now we need to reach beyond the realm of magnetostatics for just a moment. Section 8.3 ("Faraday's Law") shows that the change in Φ associated with a change in current results in the creation of an electrical potential equal to $-Nd\Phi/dt$ realized over the loop $\mathcal{C}$. In other words, the terminal voltage V is $+Nd\Phi/dt$, with the change of sign intended to keep the result consistent with the sign convention relating current and voltage in passive devices. Therefore, $d\Phi/dt$ in Equation 7.12.3 is equal to V/N. Making the substitution we find:

$$V = L \frac{d}{dt} I \tag{7.12.4}$$

This is the expected relationship from elementary circuit theory.

Another circuit theory concept related to inductance is *mutual inductance*. Whereas inductance relates changes in current to instantaneous voltage in the same device (Equation 7.12.4), mutual inductance relates changes in current in one device to instantaneous voltage in a *different* device. This can occur when the two devices are coupled ("linked") by the same magnetic field.

For example, *transformers* (Section 8.5) typically consist of separate coils that are linked by the same magnetic field lines. The voltage across one coil may be computed as the time-derivative of current on the other coil times the mutual inductance.

Let us conclude this section by taking a moment to dispel a common misconception about inductance. The misconception pertains to the following question. If the current does not form a closed loop, what is the inductance? For example, engineers sometimes refer to the inductance of a pin or lead of an electronic component. A pin or lead is not a closed loop, so the formal definition of inductance given above – ratio of magnetic flux to current – does not apply. The broader definition of inductance – the ability to store energy in a magnetic field – does apply, but this is *not* what is meant by "pin inductance" or "lead inductance." What is actually meant is the imaginary part of the impedance of the pin or lead – i.e., the reactance – expressed as an equivalent inductance. In other words, the reactance of an inductive device is positive, so any device that also exhibits a positive reactance can be viewed from a circuit theory perspective as an equivalent inductance. This is not referring to the storage of energy in a magnetic field; it merely means that the device can be modeled as an inductor in a circuit diagram. In the case of "pin inductance," the culprit is not actually inductance, but rather skin effect (see "Additional References" at the end of this section). Summarizing:

> Inductance implies positive reactance, but positive reactance does not imply the physical mechanism of inductance.

7.13: Inductance of a Straight Coil

In this section, we determine the inductance of a straight coil, as shown in Figure $7.13.1$. The coil is circular with radius a and length l and consists of N windings of wire wound with uniform winding density. Also, we assume the winding density N/l is large enough that magnetic field lines cannot enter or exit between windings but rather must traverse the entire length of the coil. Since the coil forms a cylinder, the problem is easiest to work in cylindrical coordinates with the axis of the coil aligned along the z axis.

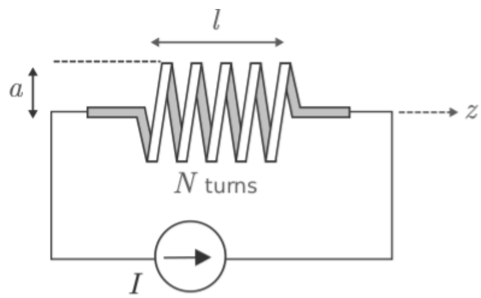

Figure 7.13.1: Determination of the inductance of a straight coil.

Inductance L in this case is given by (Section 7.12)

$$L = \frac{N\Phi}{I} \tag{7.13.1}$$

where I is current and Φ is the magnetic flux associated with one winding of the coil. Magnetic flux in this case is given by

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s}$$

where $\mathbf{B}$ is the magnetic flux density (units of T = Wb/m^2), $\mathcal{S}$ is the surface bounded by a single current loop, and $d\mathbf{s}$ points in the direction determined by the right hand rule with respect to the direction of positive current flow.

First, let's determine the magnetic field. The magnetic flux density deep inside the coil is (Section 7.6):

$$\mathbf{B} \approx \hat{\mathbf{z}} \frac{\mu N I}{l} \tag{7.13.2}$$

Is it reasonable to use this approximation here? Since inductance pertains to energy storage, the question is really what fraction of the energy is stored in a field that is well-described by this approximation, as opposed to energy stored in the "fringing field" close to the ends of the coil. If we make l sufficiently large relative to a, then presumably energy storage in the fringing field will be negligible in comparison. Since the alternative leads to a much more complicated problem, we shall assume that Equation 7.13.2 is valid for the interior of the coil.

Next, we determine Φ. In this case, a natural choice for $\mathcal{S}$ is the interior cross-section of the coil in a plane perpendicular to the axis. The direction of $d\mathbf{s}$ must be $+\hat{\mathbf{z}}$ since this is the direction in which the fingers of the right hand point when the current flows in the direction indicated in Figure $7.13.1$. Thus, we have

$$\begin{aligned} \Phi &\approx \int_{\mathcal{S}} \left(\hat{\mathbf{z}} \frac{\mu N I}{l}\right) \cdot (\mathbf{z}\, ds) \\ &= \frac{\mu N I}{l} \int_{\mathcal{S}} ds \\ &= \frac{\mu N I}{l} A \end{aligned}$$

where A is the cross-sectional area of the coil.

Finally from Equation 7.13.1 we obtain

$$\boxed{L \approx \frac{\mu N^2 A}{l} \quad (l \gg a)} \tag{7.13.3}$$

Note that this is dimensionally correct; that is, permeability (units of H/m) times area (units of m^2) divided by length (units of m) gives units of H, as expected. Also, it is worth noting that inductance is proportional to permeability and cross-sectional area, and inversely proportional to length. Interestingly the inductance is proportional to N^2 as opposed to N; this is because field strength increases with N, and independently there are N flux linkages. Finally, we note that the inductance does not depend on the *shape* of the coil cross-section, but only on the *area* of the cross-section. Summarizing:

The inductance of a long straight coil is given approximately by Equation $7.13.3$.

Again, this result is approximate because it neglects the non-uniform fringing field near the ends of the coil and the possibility that magnetic field lines escape between windings due to inadequate winding density. Nevertheless, this result facilitates useful engineering analysis and design.

7.14: Inductance of a Coaxial Structure

Let us now determine the inductance of coaxial structure, shown in Figure 7.14.1. The inductance of this structure is of interest for a number of reasons – in particular, for determining the characteristic impedance of coaxial transmission line, as addressed in Section 3.10.

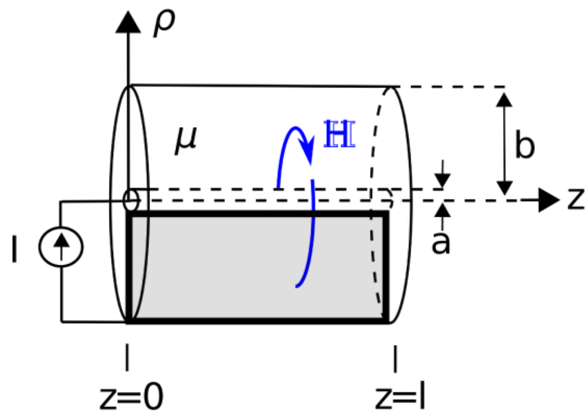

Figure 7.14.1: Determining the inductance of coaxial line.

For our present purpose, we may model the structure as shown in Figure 7.14.1. This model consists of two concentric perfectly-conducting cylinders of radii a and b, separated by a homogeneous material having permeability μ. To facilitate analysis, let us place the $+z$ axis along the common axis of the concentric cylinders, so that the cylinders may be described as the surfaces $\rho = a$ and $\rho = b$.

Below we shall find the inductance by assuming a current I on the inner conductor and integrating over the resulting magnetic field to obtain the magnetic flux Φ between the conductors. Then, inductance can be determined as the ratio of the response flux to the source current.

Before we get started, note the derivation we are about to do is similar to the derivation of the *capacitance* of a coaxial structure, addressed in Section 5.24. The reader may benefit from a review of that section before attempting this derivation.

The first step is to find the magnetic field inside the structure. This is relatively simple if we may neglect fringing fields, since then the internal field may be assumed to be constant with respect to z. This analysis will also apply to the case where the length l pertains to one short section of a much longer structure; in this case we will obtain the inductance *per length* as opposed to the total inductance for the structure. Note that the latter is exactly what we need for the transmission line lumped-element equivalent circuit model (Section 3.4).

To determine the inductance, we invoke the definition:

$$L \triangleq \frac{\Phi}{I} \tag{7.14.1}$$

A current I flowing in the $+z$ direction on the inner conductor gives rise to a magnetic field inside the coaxial structure. The magnetic field intensity for this scenario was determined in Section 7.5 where we found

$$\mathbf{H} = \hat{\phi}\frac{I}{2\pi\rho} \ , \ a \leq \rho \leq b$$

The reader should note that in that section we were considering merely a line of current; not a coaxial structure. So, on what basis do we claim the field for inside the coaxial structure is the same? This is a consequence of Ampere's Law (Section 7.14):

$$\oint_{\mathcal{C}} \mathbf{H} \cdot d\mathbf{l} = I_{encl}$$

If in this new problem we specify the same circular path $\mathcal{C}$ with radius greater than a and less than b, then the enclosed current is simply I. The presence of the outer conductor does not change the radial symmetry of the problem, and nothing else remains that can change the outcome. This is worth noting for future reference:

> The magnetic field inside a coaxial structure comprised of concentric conductors bearing current I is identical to the magnetic field of the line current I in free space.

We're going to need magnetic flux density ($\mathbf{B}$) as opposed to $\mathbf{H}$ in order to get the magnetic flux. This is simple since they are related by the permeability of the medium; i.e., $\mathbf{B} = \mu\mathbf{H}$. Thus:

$$\mathbf{B} = \hat{\phi}\frac{\mu I}{2\pi\rho} \ , \ \ a \leq \rho \leq b$$

Next, we get Φ by integrating over the magnetic flux density

$$\Phi = \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{s}$$

where $\mathcal{S}$ is any open surface through which all magnetic field lines within the structure must pass. Since this can be *any* such surface, we may as well choose the simplest one. The simplest such surface is a plane of constant ϕ, since such a plane is a constant-coordinate surface and perpendicular to the magnetic field lines. This surface is shown as the shaded area in Figure 7.14.1. Using this surface we find:

$$\begin{aligned} \Phi &= \int_{\rho=a}^{b} \int_{z=0}^{l} \left(\hat{\phi}\frac{\mu I}{2\pi\rho}\right) \cdot (\hat{\phi} d\rho dz) \\ &= \frac{\mu I}{2\pi}\left(\int_{z=0}^{l} dz\right)\left(\int_{\rho=a}^{b} \frac{d\rho}{\rho}\right) \\ &= \frac{\mu I l}{2\pi} \ln\left(\frac{b}{a}\right) \end{aligned}$$

Wrapping up:

$$L \triangleq \frac{\Phi}{I} = \frac{(\mu I l/2\pi)\ln(b/a)}{I}$$

Note that factors of I in the numerator and denominator cancel out, leaving:

$$\boxed{L = \frac{\mu l}{2\pi} \ln\left(\frac{b}{a}\right)}$$

Note that this is dimensionally correct, having units of H. Also note that this is expression depends only on materials (through μ) and geometry (through l, a, and b). Notably, it does *not* depend on current, which would imply non-linear behavior.

To make the connection back to lumped-element transmission line model parameters (Sections 3.4 and 3.10), we simply divide by l to get the per-unit length parameter:

$$\boxed{L' = \frac{\mu}{2\pi} \ln\left(\frac{b}{a}\right)} \tag{7.14.2}$$

which has the expected units of H/m.

Example 7.14.1: Inductance of RG-59 coaxial cable

RG-59 coaxial cable consists of an inner conductor having radius 0.292 mm, an outer conductor having radius 1.855 mm, and polyethylene (a non-magnetic dielectric) spacing material. Estimate the inductance per length of RG-59.

Solution

From the problem statement, $a = 0.292$ mm, $b = 1.855$ mm, and $\mu \cong \mu_0$ since the spacing material is non-magnetic. Using Equation 7.14.2, we find $L' \cong 370$ nH/m.

7.15: Magnetic Energy

Consider a structure exhibiting inductance; i.e., one that is able to store energy in a magnetic field in response to an applied current. This structure could be a coil, or it could be one of a variety of inductive structures that are not explicitly intended to be an inductor; for example, a coaxial transmission line. When current is applied, the current-bearing elements of the structure exert forces on each other. Since these elements are not normally free to move, we may interpret this force as potential energy stored in the magnetic field associated with the current (Section 7.12).

We now want to know how much energy is stored in this field. The answer to this question has relevance in several engineering applications. One issue is that any system that includes inductance is using some fraction of the energy delivered by the power supply to energize this inductance. In many electronic systems – in power systems in particular – inductors are periodically energized and de-energized at a regular rate. Since power is energy per unit time, this consumes power. Therefore, energy storage in inductors contributes to the power consumption of electrical systems.

The stored energy is most easily determined using circuit theory concepts. First, we note that the electrical potential difference $v(t)$ (units of V) across an inductor is related to the current $i(t)$ (units of A) through the inductor as follows (Section 7.12):

$$v(t) = L\frac{d}{dt}i(t)$$

where L (units of H) is the inductance. The instantaneous power associated with the device is

$$p(t) = v(t)i(t)$$

Energy (units of J) is power (units of J/s) integrated over time. Let W_m be the energy stored in the inductor. At some time t_0 in the past, $i(t_0) = 0$ and $W_m = 0$. As current is applied, W_m increases monotonically. At the present time t, $i(t) = I$. Thus, the present value of the magnetic energy is:

$$W_m = \int_{t_0}^{t_0+t} p(\tau)d\tau$$

Now evaluating this integral using the relationships established above:

$$\begin{aligned} W_m &= \int_{t_0}^{t+t_0} v(\tau)i(\tau)d\tau \\ &= \int_{t_0}^{t+t_0} \left[L\frac{d}{d\tau}i(\tau)\right] i(\tau)d\tau \\ &= L\int_{t_0}^{t+t_0} \left[\frac{d}{d\tau}i(\tau)\right] i(\tau)d\tau \end{aligned}$$

Changing the variable of integration from τ (and $d\tau$) to i (and di) we have

$$\begin{aligned} W_m &= L\int_{t_0}^{t+t_0} \frac{di}{d\tau} i d\tau \\ &= L\int_0^I i di \end{aligned}$$

Evaluating the integral we obtain the desired expression

$$\boxed{W_m = \frac{1}{2}LI^2} \tag{7.15.1}$$

> The energy stored in an inductor in response to a steady current I is Equation 7.15.1. This energy increases in proportion to inductance and in proportion to the square of current.

The long straight coil (Section 7.13) is representative of a large number of practical applications, so it is useful to interpret the above findings in terms of this structure in particular. For this structure we found

$$L = \frac{\mu N^2 A}{l}$$

where μ is the permeability, N is the number of windings, A is cross-sectional area, and l is length. The magnetic field intensity inside this structure is related to I by (Section 7.6):

$$H = \frac{NI}{l}$$

Substituting these expressions into Equation 7.15.1, we obtain

$$\begin{aligned} W_m &= \frac{1}{2}\left[\frac{\mu N^2 A}{l}\right]\left[\frac{Hl}{N}\right]^2 \\ &= \frac{1}{2}\mu H^2 Al \end{aligned}$$

Recall that the magnetic field inside a long coil is approximately uniform. Therefore, the density of energy stored inside the coil is approximately uniform. Noting that the product Al is the volume inside the coil, we find that this energy density is W_m/Al; thus:

$$w_m = \frac{1}{2}\mu H^2 \tag{7.15.2}$$

which has the expected units of energy per unit volume (J/m^3).

The above expression provides an alternative method to compute the total magnetostatic energy in *any* structure. Within a mathematical volume $\mathcal{V}$, the total magnetostatic energy is simply the integral of the energy density over $\mathcal{V}$; i.e.,

$$W_m = \int_{\mathcal{V}} w_m \, dv$$

This works even if the magnetic field and the permeability vary with position. Substituting Equation 7.15.2 we obtain:

$$\boxed{W_m = \frac{1}{2}\int_{\mathcal{V}} \mu H^2 dv} \tag{7.15.3}$$

Summarizing:

> The energy stored by the magnetic field present within any defined volume is given by Equation 7.15.3.

It's worth noting that this energy increases with the permeability of the medium, which makes sense since inductance is proportional to permeability.

Finally, we reiterate that although we arrived at this result using the example of the long straight coil, Equations 7.15.2 and 7.15.3 are completely general.

7.16: Magnetic Materials

As noted in Section 2.5, magnetic fields arise in the presence of moving charge (i.e., current) and in the presence of certain materials. In this section, we address these "magnetic materials."

A magnetic material may be defined as a substance that exhibits permeability μ (Section 2.6) that is significantly different from the permeability of free space μ_0. Since the magnetic flux density $\mathbf{B}$ is related to the magnetic field intensity $\mathbf{H}$ via $\mathbf{B} = \mu\mathbf{H}$, magnetic materials may exhibit magnetic flux density in response to a given magnetic field intensity that is significantly greater than that of other materials. Magnetic materials are also said to be "magnetizable," meaning that the application of a magnetic field causes the material itself to become a source of the magnetic field.

Magnetic media are typically metals, semiconductors, or heterogeneous media containing such materials. An example is ferrite, which consists of iron particles suspended in a ceramic. Magnetic media are commonly classified according to the physical mechanism responsible for their magnetizability. These mechanisms include *paramagnetism*, *diamagnetism*, and *ferromagnetism*. All three of these mechanisms involve quantum mechanical processes operating at the atomic and subatomic level, and are not well-explained by classical physics. These processes are beyond the scope of this book (but information is available via "Additional References" at the end of this section). However, it is possible to identify some readily-observable differences between these categories of magnetic media.

Paramagnetic and diamagnetic materials exhibit permeability that is only very slightly different than μ_0 and typically by much less than 0.01%. These materials exhibit very weak and temporary magnetization. The principal distinction between paramagnetic and diamagnetic media is in the persistence and orientation of induced magnetic fields. Paramagnetic materials – including aluminum, magnesium, and platinum – exhibit a very weak persistent magnetic field, and the magnetic field induced in the material is aligned in the same direction as the impressed (external) magnetic field. Diamagnetic materials – including copper, gold, and silicon – do not exhibit a persistent magnetic field, and the magnetic field induced in the material is (counter to intuition!) aligned in the *opposite* direction as the impressed magnetic field. The magnetization of paramagnetic and diamagnetic media is typically so weak that it is not often a consideration in engineering analysis and design.

> Paramagnetic and diamagnetic media exhibit permeability only very slightly different than that of free space, with little or no magnetization.

Ferromagnetic materials, on the other hand, exhibit permeability that can be many orders of magnitude greater than μ_0. (See Appendix A.2 for some example values.) These materials can be readily and indefinitely magnetized, thus, permanent magnets are typically comprised of ferromagnetic materials. Commonly-encountered ferromagnetic materials include iron, nickel, and cobalt.

Ferromagnetic materials are significantly non-linear (see definition in Section 2.8), exhibiting *saturation* and *hysteresis*. This is illustrated in Figure 7.16.1. In this plot, the origin represents a ferromagnetic material that is unmagnetized and in a region free of an external magnetic field. The external magnetic field is quantified in terms of $\mathbf{H}$, plotted along the horizontal axis. As the external field is increased, so to is $\mathbf{B}$ in the material, according to the relationship $\mathbf{B} = \mu\mathbf{H}$. Right away we see the material is non-linear, since the slope of the curve – and hence μ – is not constant.

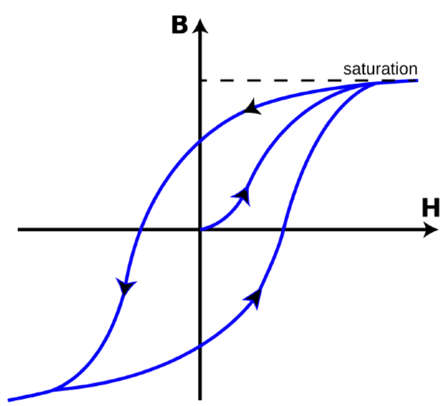

Figure 7.16.1: Non-linearity in a ferromagnetic material manifesting as saturation and hysteresis. ((modified) CC BY SA 3.0; Ndthe)

Once the external magnetizing field $\mathbf{H}$ exceeds a certain value, the response field $\mathbf{B}$ no longer significantly increases. This is saturation. Once saturated, further increases in the external field result do not significantly increase the magnetization of the material, so there is no significant increase in $\mathbf{B}$.

From this state of saturation, let us now reduce the external field. We find that the rate of decrease in $\mathbf{B}$ with respect to $\mathbf{H}$ is significantly less than the rate that $\mathbf{B}$ originally increased with respect to $\mathbf{H}$. In fact, $\mathbf{B}$ is still greater than zero even when $\mathbf{H}$ has been reduced to zero. At this point, the magnetization of the material is obvious, and a device comprised of this material could be used as a magnet.

If we now apply an external field in the reverse direction, we find that we are eventually able to zero and then redirect the response field. As we continue to decrease $\mathbf{H}$ (that is, increase the magnitude in the reverse direction), we once again reach saturation.

The same behavior is observed when we once again increase $\mathbf{H}$. The material is eventually demagnetized, remagnetized in the opposite direction and then saturated in that direction. At this point, it is apparent that a return to the start condition ($\mathbf{H}=\mathbf{B}=0$; i.e., demagnetized when there is no external field) is not possible.

Hysteresis is the name that we apply to this particular form of non-linear behavior. Hysteresis has important implications in engineering applications. First, as identified above, it is an important consideration in the analysis and design of magnets. In applications where a ferromagnetic material is being used because high permeability is desired – e.g., in inductors (Section 7.12) and transformers (Section 8.5) – hysteresis complicates the design and imposes limits on the performance of the device.

Hysteresis may also be exploited as a form of memory. This is apparent from Figure $7.16.1$. If $\mathbf{B}>0$, then recent values of $\mathbf{H}$ must have been relatively large and positive. Similarly, If $\mathbf{B}<0$, then recent values of $\mathbf{H}$ must have been relatively large and negative. Furthermore, the most recent sign of $\mathbf{H}$ can be inferred even if the present value of $\mathbf{H}$ is zero. In this sense, the material "remembers" the past history of its magnetization and thereby exhibits memory. This is the enabling principle for a number of digital data storage devices, including hard drives (see "Additional Reading" at the end of this section). Summarizing:

> Ferromagnetic media exhibit permeability μ that is orders of magnitude greater than that of free space and are readily magnetizable. These materials are also nonlinear in μ, which manifests as saturation and hysteresis.