1.11 Sketch Graphs of Functions

1.11.1 Guidelines for Sketching a Curve

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand.

Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.)

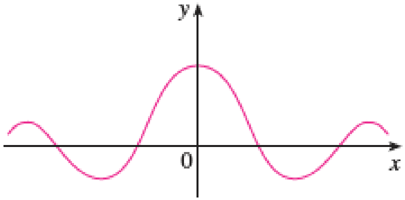
But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. **Domain :** It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

B. **Intercepts :** The $y-$intercept is $f(0)$ and this tells us where the curve intersects the $y-$axis.
To find the $x-$intercepts, we set $y = 0$ and solve for x.

C. Symmetry :

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an even function and the curve is symmetric about the $y-$axis. This means that our work is cut in half.
If we know what the curve looks like for $x \geq 0$, then we need only reflect about the $y-$axis to obtain the complete curve [see Figure 30(a)].

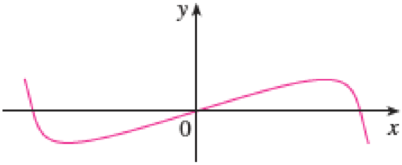


(a) Even function: reflectional symmetry

Figure 30(a)

C. **Symmetry :**

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an odd function and the curve is symmetric about the origin.
Again we can obtain the complete curve if we know what it looks like for $x \geq 0$. [Rotate 180° about the origin; see Figure 30(b).]



(b) Odd function: rotational symmetry

Figure 30(b)

C. Symmetry :

(ii) If $f(x+p) = f(x)$ for all x in D, where p is a positive constant, then f is called a periodic function and the smallest such number p is called the period.
If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 31).

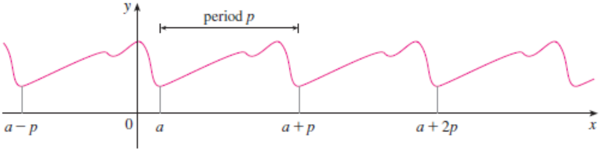


Figure 31 : Periodic function: translational symmetry

D. Asymptotes :

(i) ***Horizontal Asymptotes*** *:*

If either $\lim_{x \to \infty} f(x) = L$ or $\lim_{x \to -\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$.

If it turns out that $\lim_{x \to \infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but this fact is still useful information for sketching the curve.

D. Asymptotes :

(ii) ***Vertical Asymptotes*** *:*

The line $x = a$ is a vertical asymptote if at least one of the following statements is true:

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

(For rational functions you can locate the vertical asymptotes by equating the denominator to 0 after canceling any common factors. But for other functions this method does not apply.)
Furthermore, in sketching the curve it is very useful to know exactly which of the statements above is true.
If $f(a)$ is not defined but a is an endpoint of the domain of f, then you should compute $\lim_{x \to a^-} f(x)$ or $\lim_{x \to a^+} f(x)$, whether or not this limit is infinite.

E. **Intervals of Increase or Decrease :** Compute $f'(x)$ and find the intervals on which $f'(x)$ is positive (f is increasing) and the intervals on which $f'(x)$ is negative (f is decreasing).

F. **Local Maximum and Minimum Values :** Find the critical numbers of f [the numbers c where $f'(c) = 0$ or $f'(c)$ does not exist].

Then use the First Derivative Test.

If f' changes from positive to negative at a critical number c, then $f(c)$ is a local maximum.

If f' changes from negative to positive at c, then $f(c)$ is a local minimum.

Although it is usually prefer able to use the First Derivative Test, you can use the Second Derivative Test if $f'(c) = 0$ and $f''(c) \neq 0$.

Then $f''(c) > 0$ implies that $f(c)$ is a local minimum, whereas $f''(c) < 0$ implies that $f(c)$ is a local maximum.

G. Concavity and Points of Inflection : Compute $f''(x)$ and use the Concavity Test.

The curve is concave upward where $f''(x) > 0$ and concave downward where $f''(x) < 0$.

Inflection points occur where the direction of concavity changes.

H. **Sketch the Curve :** Using the information in items A–G, draw the graph.
Sketch the asymptotes as dashed lines.
Plot the intercepts, maximum and minimum points, and inflection points.
Then make the curve pass through these points, rising and falling according to E, with concavity according to G, and approaching the asymptotes.

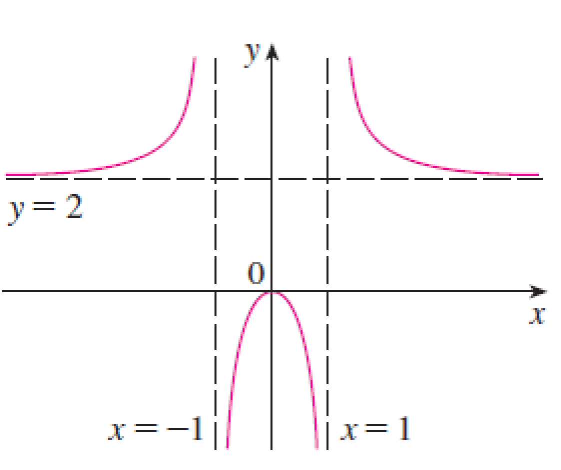
Example (17)

(i) Use the guidelines to sketch the curve $y = \dfrac{2x^2}{x^2 - 1}$.

(ii) Sketch the graph of $y = \dfrac{2x^2}{\sqrt{x + 1}}$.

(iii) Sketch the graph of $y = xe^x$.

Example (17 (i))



1.12 Rolle's Theorem

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Theorem

Let f be a function that satisfies the following three hypotheses:

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*
3. $f(a) = f(b)$

Then there is a number c in (a, b) such that $f'(c) = 0$.

Let's take a look at the graphs of some typical functions that satisfy the three hypotheses.

Figure 32 shows the graphs of four such functions.

In each case it appears that there is at least one point $(c, f(c))$ on the graph where the tangent is horizontal and therefore $f'(c) = 0$.

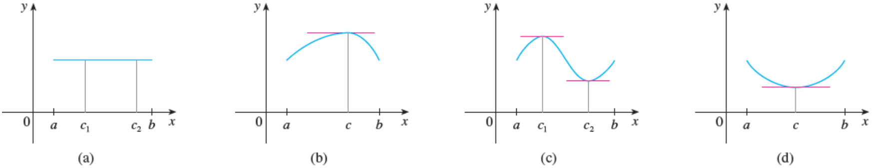

Figure 32

1.13 The Mean Value Theorem

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Theorem

Let f be a function that satisfies the following hypotheses:

1. *f is continuous on the closed interval $[a, b]$.*
2. *f is differentiable on the open interval (a, b).*

Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

Example (18)

Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all values of x. How large can $f(2)$ possibly be?