Cayley – Hamilton Theorem

Every square matrix satisfy it's own characteristic equation.

i.e. If the characteristic equation of A is $f(\lambda)$, then $f(A) = 0$

Ex:

If $A = \begin{pmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{pmatrix}_{3\times3}$ then

find A^{-1} using Cayley-Hamilton theorem.

Sol^n:

First let us find characteristic equation.

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$$|A-\lambda I| = 0$$

$$\begin{vmatrix} -\lambda & c & -b \\ -c & -\lambda & a \\ b & -a & -\lambda \end{vmatrix} = 0$$

$$-\lambda^3 - (a^2+b^2+c^2) = 0$$ → characteristic eqⁿ.

By Cayley - Hamilton theorem,

$$\therefore \quad -A^3 - (a^2+b^2+c^2)I = 0$$

$$A^3 + (a^2+b^2+c^2)I = 0$$

$$A^{-1}A^3 + (a^2+b^2+c^2)A^{-1}I = 0$$

$$A^2 + (a^2+b^2+c^2)A^{-1} = 0$$

$$A^{-1} = \left(\frac{-1}{a^2+b^2+c^2}\right)A^2$$; when $a^2+b^2+c^2 \neq 0$

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Roots of algebraic equation

The Quadratic Case

The simplest case of Vieta's states the following:

Theorem 1

Let r_1 & r_2 be the roots of the quadratic equation $ax^2+bx+c=0$. Then the two identities

$$\left.\begin{aligned} r_1+r_2 &= \frac{-b}{a} \\ r_1 r_2 &= \frac{c}{a} \end{aligned}\right\} \quad a \neq 0$$

both hold.

Note

This allows us to find the sum and the product of the roots of any quadratic polynomial

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without actually computing the roots themselves.

$$ax^2 + bx + c \equiv a(x - r_1)(x - r_2)$$

$$ax^2 + bx + c \equiv ax^2 - a x(r_1 + r_2) + ar_1 r_2.$$

Coefficient of x^2

$$a = a$$

Coefficient of x

$$b = -a(r_1 + r_2).$$

$$\boxed{r_1 + r_2 = \frac{-b}{a}.}$$

~~Coe~~ Constant coefficient

$$c = ar_1 r_2.$$

$$\boxed{r_1 r_2 = \frac{c}{a}}$$

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Ex

Suppose p & q are the roots of the equation t^2-7t+5. Find p^2+q^2.

Sol^n:

From Vieta's formulas we have

$$p+q = \frac{-b}{a} = \frac{7}{1} = 7.$$

$$pq = \frac{c}{a} = \frac{5}{1} = 5.$$

Therefore.

$$p^2+q^2 = (p+q)^2 - 2pq$$

$$= 7^2 - 2\times5 = 49-10$$

$$p^2+q^2 = 39$$ //.

Ex. Let m & n be the roots of the equation $2x^2+15x+16=0$. What is the value of $\frac{1}{m}+\frac{1}{n}$?

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Sol^{n}

From Vieta's formulas

$$m+n = \frac{-15}{2}$$

$$mn = \frac{16}{2} = 8.$$

Therefore,

$$\frac{1}{m}+\frac{1}{n} = \frac{m+n}{mn} = \frac{-15/2}{8} = \frac{-15}{16}$$ //

Cubic case

$$ax^3+bx^2+cx+d = a(x-r_1)(x-r_2)(x-r_3)$$

$$= ax^3 - a(r_1+r_2+r_3)x^2 \cancel{+(r_1r_2+r_2r_3+r_3r_1)}$$

$$+a(r_1r_2+r_2r_3+r_3r_1)x - r_1r_2r_3a.$$

Theorem 2

Let r_1, r_2, r_3 be the roots of the cubic equation $ax^3+bx^2+cx+d=0$.

Then we have,

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$$\left.\begin{aligned} r_1 + r_2 + r_3 &= \frac{-b}{a} \\ r_1 r_2 + r_2 r_3 + r_3 r_1 &= \frac{c}{a} \\ r_1 r_2 r_3 &= \frac{-d}{a} \end{aligned}\right\} \quad a \neq 0$$

Next consider polynomial ~~[illegible]~~ with order 4.

Let r_1, r_2, r_3 & r_4 be the roots of the quartic equation, ~~ax^4+bx^3+c~~

$$ax^4 + bx^3 + cx^2 + dx + e = 0, \quad a \neq 0.$$

Then we have.

$$r_1 + r_2 + r_3 + r_4 = -\frac{b}{a}$$

$$r_1 r_2 + r_1 r_3 + r_1 r_4 + r_2 r_3 + r_2 r_4 + r_3 r_4 = \frac{c}{a}$$

$$r_1 r_2 r_3 + r_1 r_2 r_4 + r_1 r_3 r_4 + r_2 r_3 r_4 = -\frac{d}{a}$$

$$r_1 r_2 r_3 r_4 = \frac{e}{a}.$$

Example

Suppose p, q and r are the roots of the polynomial $t^3 - 2t^2 + 3t - 4$.
Find $(p+1)(q+1)(r+1)$.

Sol:

$(p+1)(q+1)(r+1) = pqr + (pq + qr + rp) + (p+q+r) + 1$ (1)

Vieta's formul.

$p+q+r = \frac{2}{1} = 2.$

$pq + qr + rp = \frac{3}{1} = 3.$

$pqr = \frac{4}{1} = 4$

$\therefore$ From (1),

$(p+1)(q+1)(r+1) = 4 + 3 + 2 + 1$

$= 10$

Ex.

The roots r_1, r_2 and r_3 of $x^3-2x^2-11x+a$ satisfy $r_1+2r_2+3r_3=0$. Find all possible values of a.

Sol:

From Vieta's formul.

$$r_1+r_2+r_3 = 2 \quad \text{———} \quad (1)$$

$$r_1r_2+r_2r_3+r_3r_1 = -11 \quad \text{———} \quad (2)$$

$$r_1+2r_2+3r_3 = 0 \quad \text{———} \quad (3)$$

$$(3)-(1) \Rightarrow r_2+2r_3 = -2$$

$$\therefore \boxed{r_2 = -2-2r_3}$$

$$(1) \Rightarrow r_1+(-2-2r_3)+r_3 = 2.$$

$$r_1 - 2r_3 + r_3 = 4$$

$$r_1 - r_3 = 4$$

$$\boxed{r_1 = r_3+4}$$

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② $\Rightarrow$

$$(-2-2r_3)(r_3+4)+(-2-2r_3)r_3+r_3(r_3+4) = -11$$

$$\therefore \quad 3r_3^2+8r_3-3=0$$

$$(3r_3-1)(r_3+3)=0$$

$$\boxed{r_3=\frac{1}{3},\ -3}$$

If $r_3 = 1/3$, $r_2 = -2-2r_3 = -2-2\cdot\frac{1}{3}$

$r_2 = -8/3$.

, $r_1 = r_3+4 = \frac{1}{3}+4 = \frac{13}{3}$.

Vieta's formula,

$$r_1 r_2 r_3 = -a$$

$$a = -r_1 r_2 r_3$$

$$a = -\frac{1}{3}\cdot\frac{-8}{3}\cdot\frac{1}{3} = \boxed{\frac{104}{27}}$$

If $r_3 = -3$, $r_2 = 4$, $r_1 = 1$.

$\therefore$ So $a = -r_1 r_2 r_3$

$$= -1\cdot 4\cdot(-3)$$

$$a = \boxed{12}$$

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Complex Numbers

Definition

An expression of form $z = a + ib$ is called a complex number where $a, b \in \mathbb{R}$ and $i^2 = -1$.

Ex $3 - i4 = 3 - 4i$ (commutative)

Set of Complex number is denoted by $\mathbb{C}$ and is given by the set

$$\mathbb{C} = \{x + iy \mid x, y \in \mathbb{R} \text{ and } i^2 = -1\}$$

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Note.

Let $z_1, z_2 \in \mathbb{C}$. Then $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$.

(i) $z_1 = z_2$ if $x_1 = x_2$ & $y_1 = y_2$.

(ii) x is called the real part of $z = x + iy$ & y is called an imaginary part of z and we denoted by.

$$x = Re(z) \in \mathbb{R}$$

$$y = Im(z) \in \mathbb{R}.$$

(iii) Addition of Complex numbers z_1 & z_2 is defined by

$$z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2)$$

$$= (x_1 + x_2) + i(y_1 + y_2) \in \mathbb{C}$$

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(iv) Multiplication of z_1 & z_2 is defined by.

$$z_1 \times z_2 = (x_1 + iy_1) \times (x_2 + iy_2)$$
$$= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1) \in \mathbb{C}$$

Properties of Complex numbers

① For any $z_1, z_2 \in \mathbb{C}$

(i) $z_1 + z_2 = z_2 + z_1$ (Commutative law)

(ii) $z_1 \times z_2 = z_2 \times z_1$

② For any $z_1, z_2, z_3 \in \mathbb{C}$

(i) $(z_1 + z_2) + z_2 = z_1 + (z_2 + z_3)$ (Associative law)

$(z_1 \times z_2) \times z_3 = z_1 \times (z_2 \times z_3)$

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(3) For any $z_1, z_2, z_3 \in \mathbb{C}$.

$$z_1 \times (z_2 + z_3) = (z_1 \times z_2) + (z_1 \times z_3)$$

(distributive law).

(4) There exists $0 \in \mathbb{C}$ for any $z \in \mathbb{C}$ such that.

$$z + 0 = z = 0 + z$$

(5) There ~~exists~~ exists $1 \in \mathbb{C}$ for any $z \in \mathbb{C}$ such that.

$$z \times 1 = 1 \times z = z.$$

(6) For any $z \in \mathbb{C}$ there exists $w \in \mathbb{C}$ such that.

$$z + w = w + z = 0$$

(i.e. $w = -z$).

additive inverse.

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(7) For any $z \in \mathbb{C}\setminus\{0\}$, there exists $w \in \mathbb{C}\setminus\{0\}$ such that,

$$z \times w = 1 = w \times z$$

multiplication ~~identity~~ inverse.

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