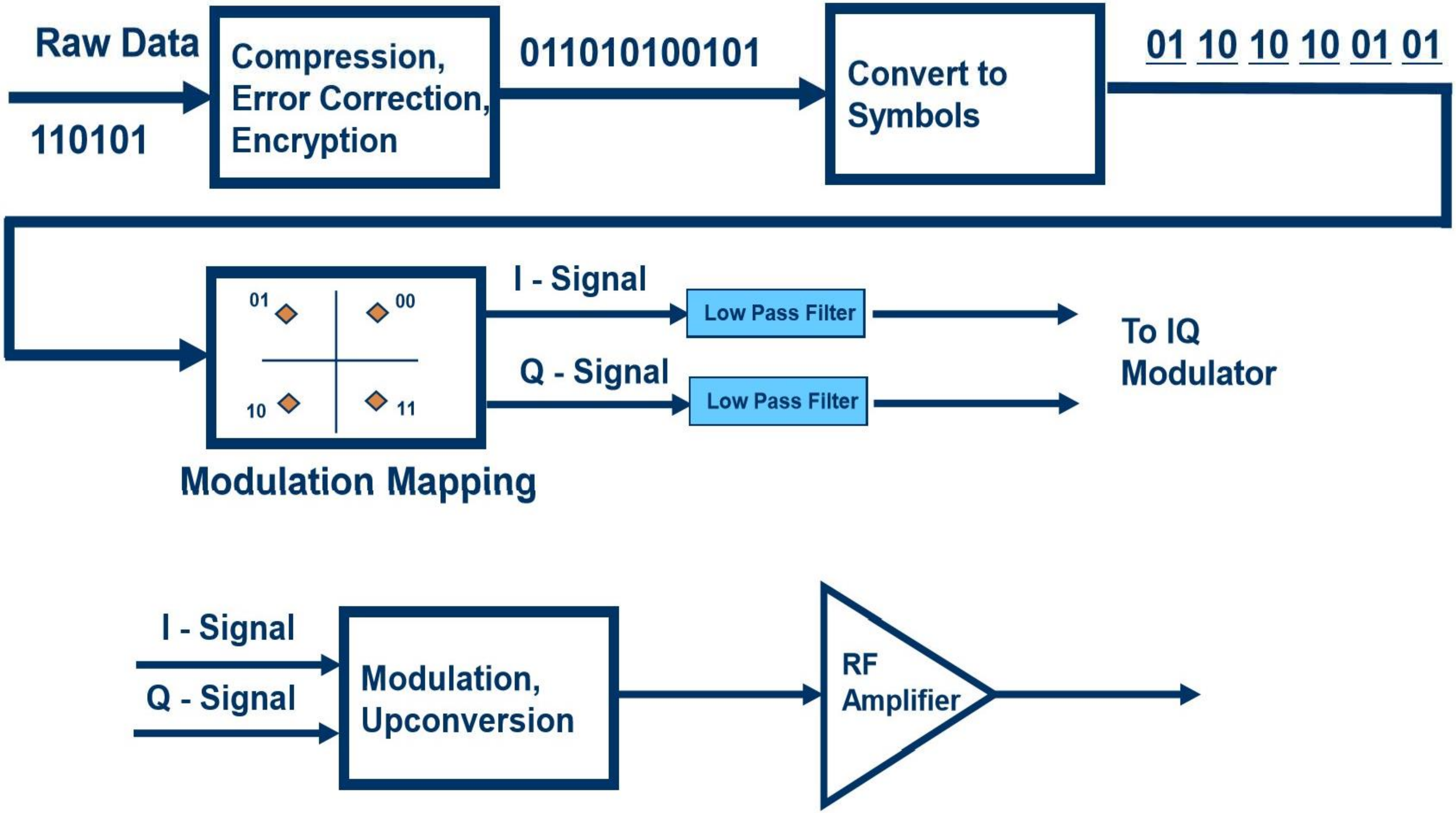
Communication Theory II

Lecture 5: Modulation on M-ary

M-ary Modulation

- M-ary (or multi-symbol) modulation schemes include
  - multi-phase
  - multi-amplitude
  - combined multi-phase/multi-amplitude
- They are commonly used in telephone, microwave, WiFi, cellular mobile phones and satellite communications to achieve higher spectrum efficiency.

M-ary Modulation Block Diagram



In-phase and quadrature components

It is possible to create an arbitrarily phase-shifted sine wave, by mixing together **two sine waves that are 90° out of phase in different proportions**

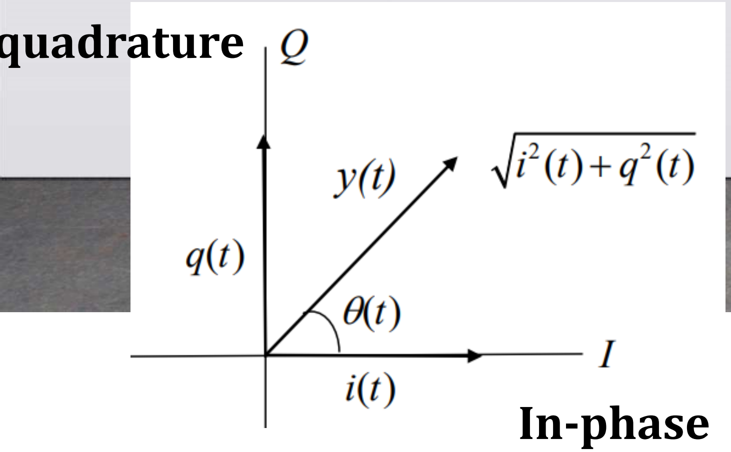
Complex baseband signal as S(t)=i(t) + jq(t)

In-phase quadrature

Transmit signal $y(t)=i(t)\cos(2\pi f_c t)+q(t)\cos(2\pi f_c t+\pi/2)$

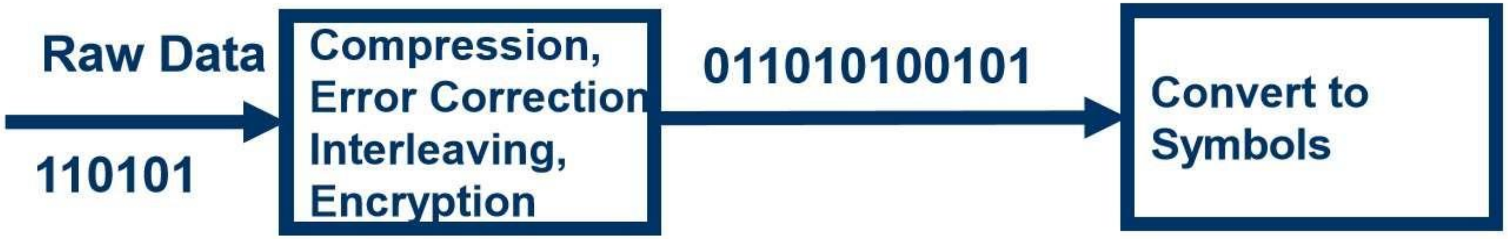
$$y(t)=i(t)\cos(2\pi f_c t)-q(t)\sin(2\pi f_c t)=\mathrm{Re}(y(t)e^{j2\pi f_c t})$$

$$y(t)=\sqrt{i^2(t)+q^2(t)}\cos\left(2\pi f_c t+\theta(t)\right)$$



$$\theta(t)=\tan^{-1}\{q(t)/\,i(t)\}$$

Raw Data Conversion



- Raw data comes from the user
  - Digitized voice, keystrokes, jpegs…
- Compression is employed for efficiency.
- Error correction is applied for transmission quality.
- Interleaving creates signal-dropout resistance.
- Encryption is applied for security.

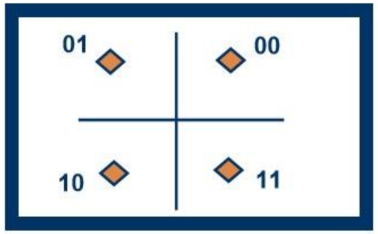
Data Bits to Symbols

- Symbols are represented by the possible states of digital modulation.
- Higher order modulation allows more bits per symbol.
- What in the world does that mean?
  - Mapping symbols to I and Q.

$$\text{Symbol rate} = \frac{\text{bit rate}}{\text{number of bits transmitted with each symbol}}$$

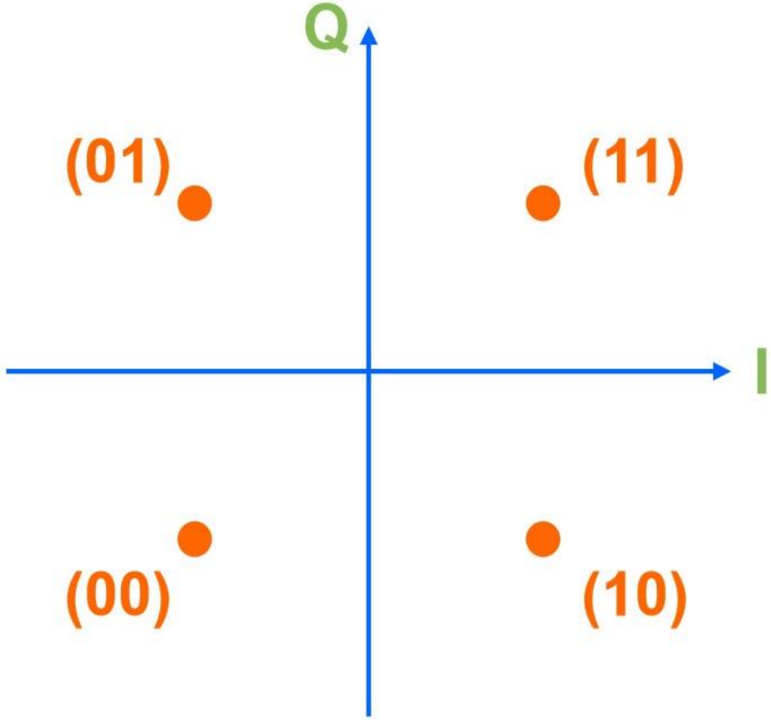
IQ Mapping

- **What is Mapping:**
  - **Translate a Symbol to a point in the IQ space**

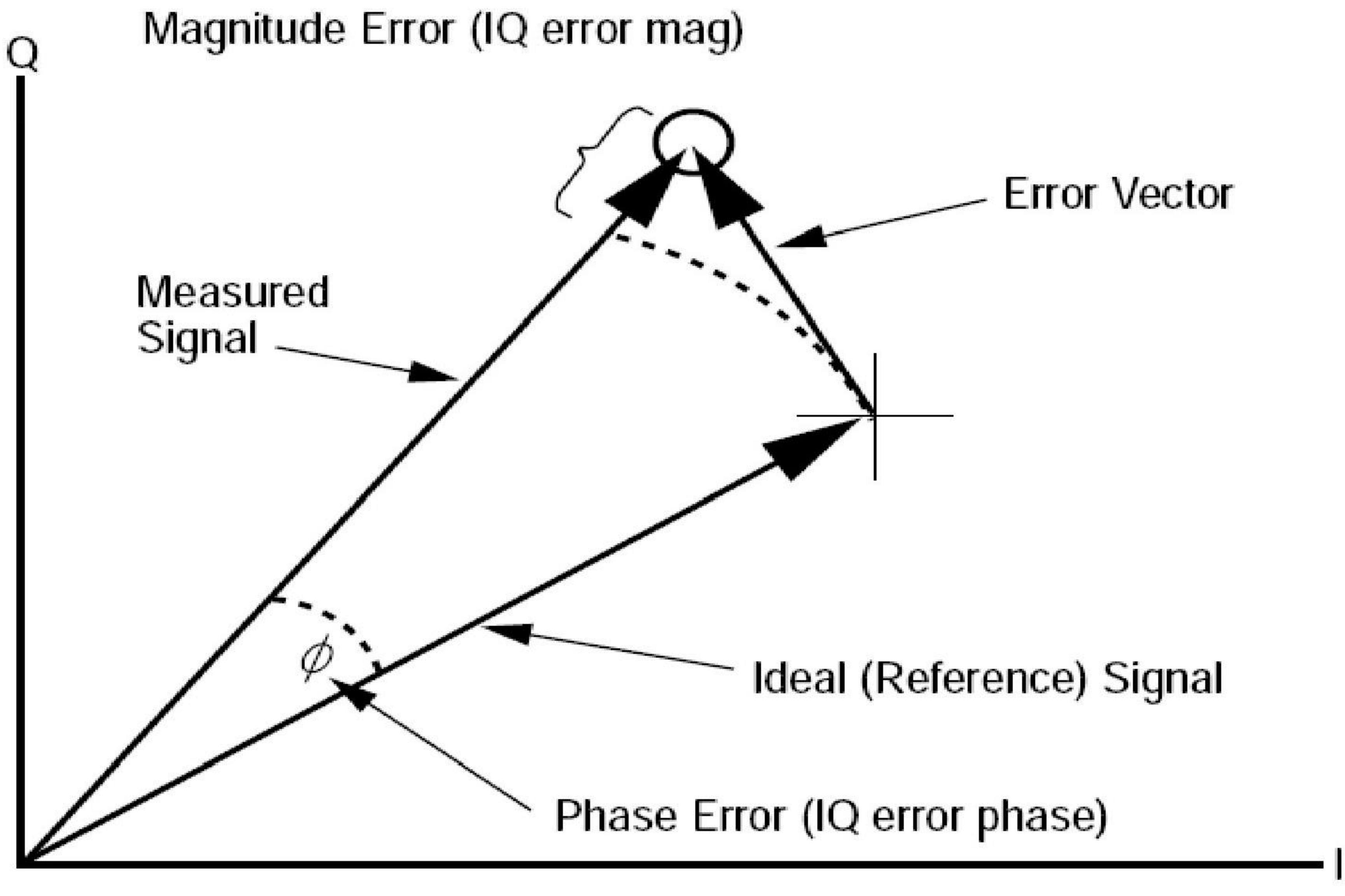


Modulation Mapping

- **Example**



Error Vector Magnitude (EVM)



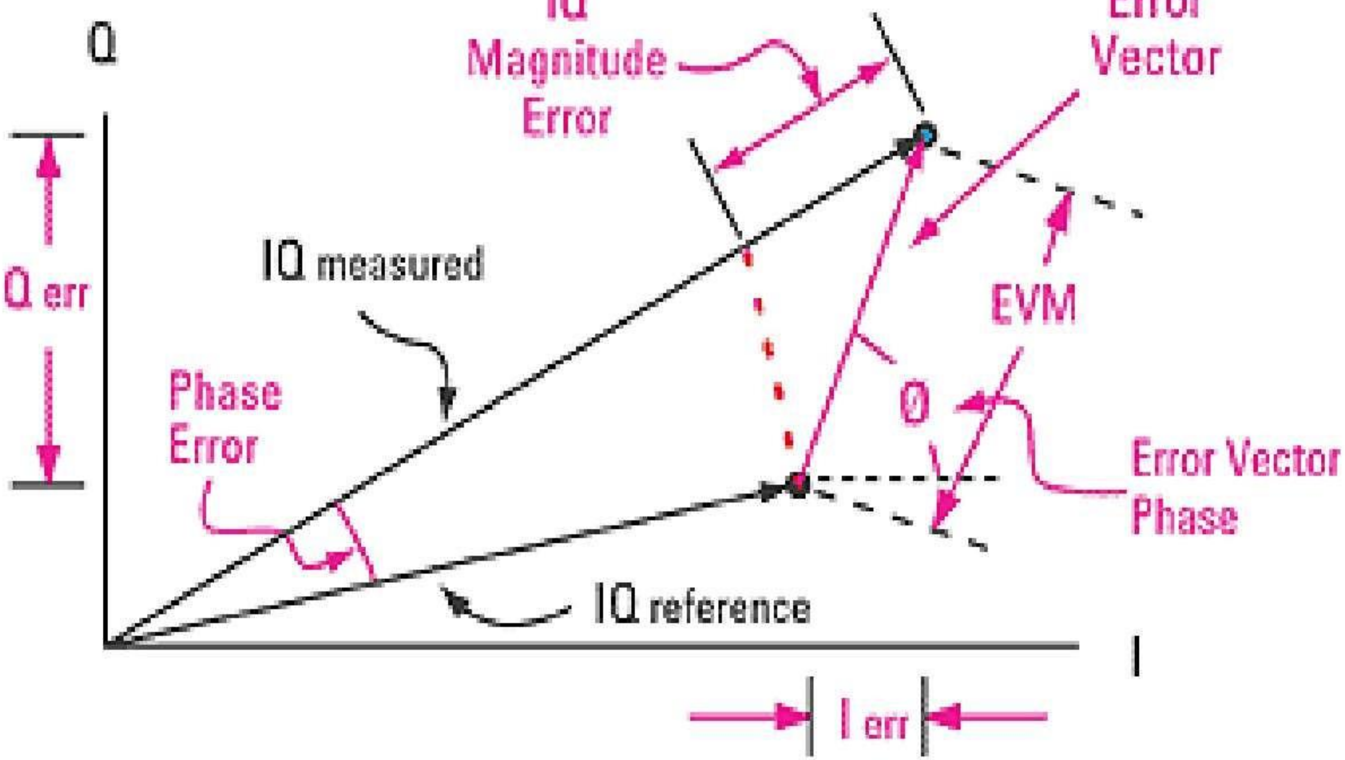
EVM Calculation

$$EVM\,[n] = \sqrt{Ierr[n]^2 + Qerr[n]^2}$$

where [n] = measurement at the symbol time

I err = I Ref - I Meas

Q err = Q Ref - Q Meas



M-ary PSK

- In M-ary PSK, the phase of the carrier takes on one of M possible values,

$$\theta_i = 2(i-1)\pi / M, \quad \text{where} \quad i = 1, 2, \ldots, M$$

- During each signaling interval of duration T, one of the following M possible signals is sent.

$$s_i(t) = \sqrt{\frac{2E_s}{T}} \cos\left(2\pi f_c t + \frac{2\pi}{M}(i-1)\right), \quad \text{where} \quad i = 1, 2, \ldots, M$$

- E_s is the energy per symbol.
- The carrier frequency $f_c = n_c / T$ for some fixed integer n_c

M-ary PSK

- Each $s_i(t)$ may be expanded in terms of two basis functions $\phi_1(t)$ and $\phi_2(t)$, defined as

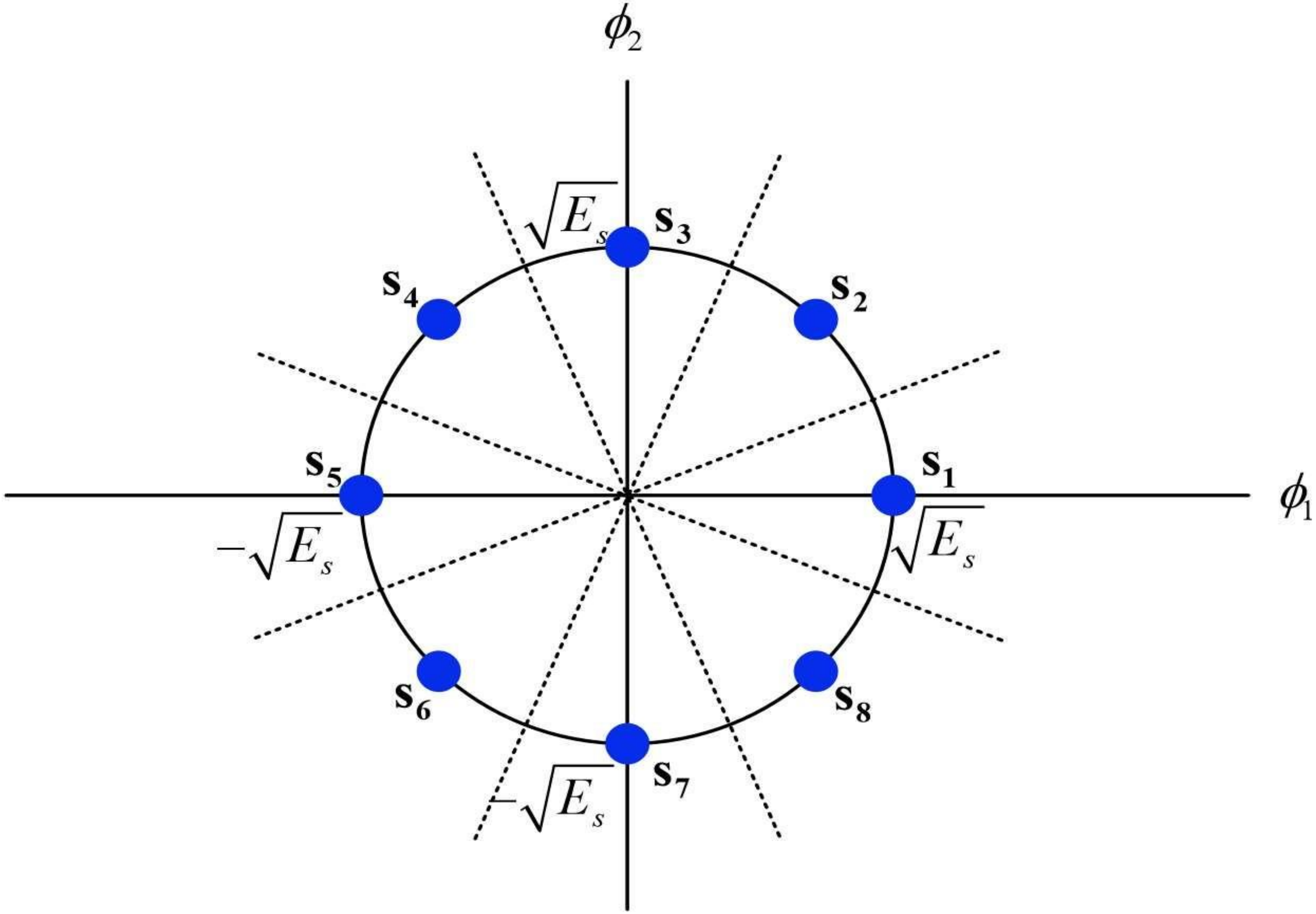
$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t\right), \qquad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin\left(2\pi f_c t\right), \qquad 0 \le t \le T$$

- Both basis functions have unit energy.

- The signal constellation of *M*-ary PSK is therefore two-dimensional. The *M* message points are equally spaced on a circle of radius $\sqrt{E_s}$ and center at the origin.

M-ary PSK with M = 8



M-ary PSK

- The average probability of symbol error for M-ary PSK is

$$p(e)=\sum_{m=1}^{M} p(e\,|\,m)\,p(\mathbf{s}_{\mathbf{m}})$$

  - $p(\mathbf{s}_{\mathbf{m}})$ is the probability that symbol $\mathbf{s}_{\mathbf{m}}$ was sent.
  - $p(e\,|\,m)$ is the probability of decision error when symbol $\mathbf{s}_{\mathbf{m}}$ was sent

- When each symbol is sent with the same probability,

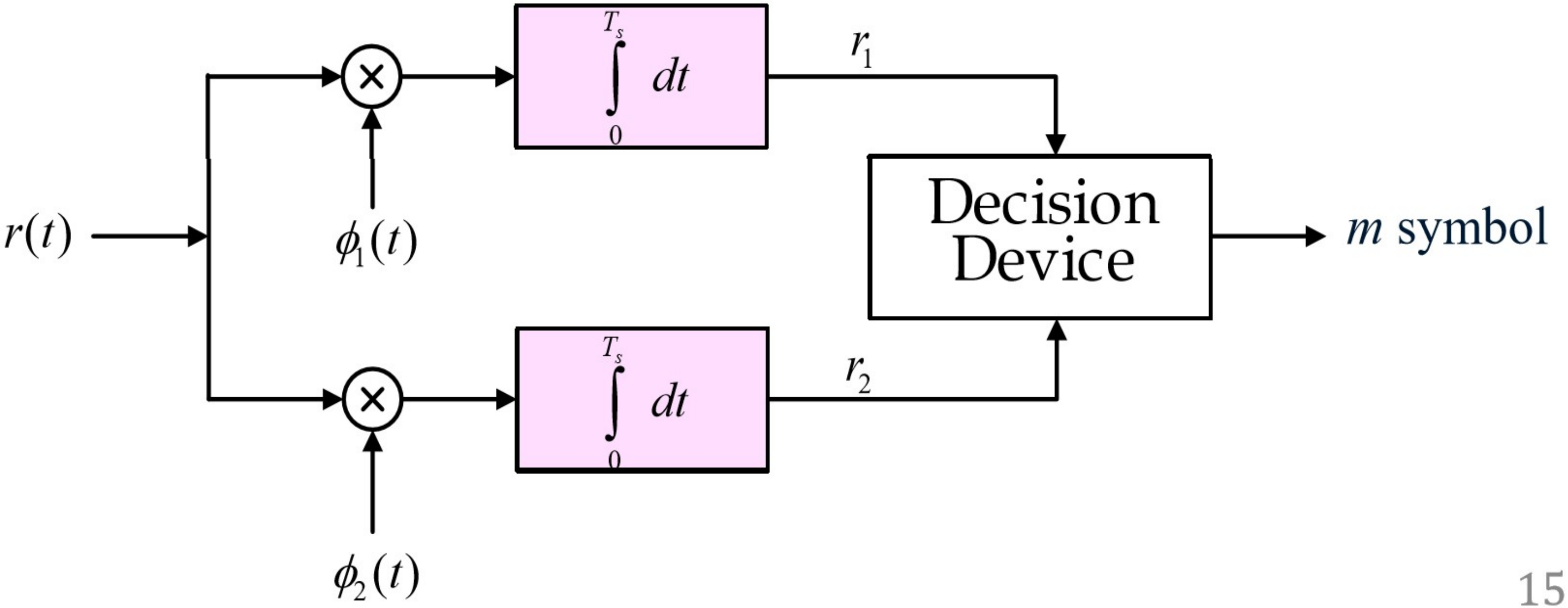
$$p(e)=\frac{1}{M}\sum_{m=1}^{M} p(e\,|\,m)$$

M-ary PSK

- It is difficult to compute $p(e\,|\,m)$ directly, we obtain it through

$$p(e\,|\,m) = 1 - p(c\,|\,m)$$

- $p(c\,|\,m)$ is the probability of "correct decision" when symbol $\mathbf{s}_m$ was sent.



M-ary PSK with M = 8

- Assume that $\mathbf{s}_1$ was sent, the receiver produces

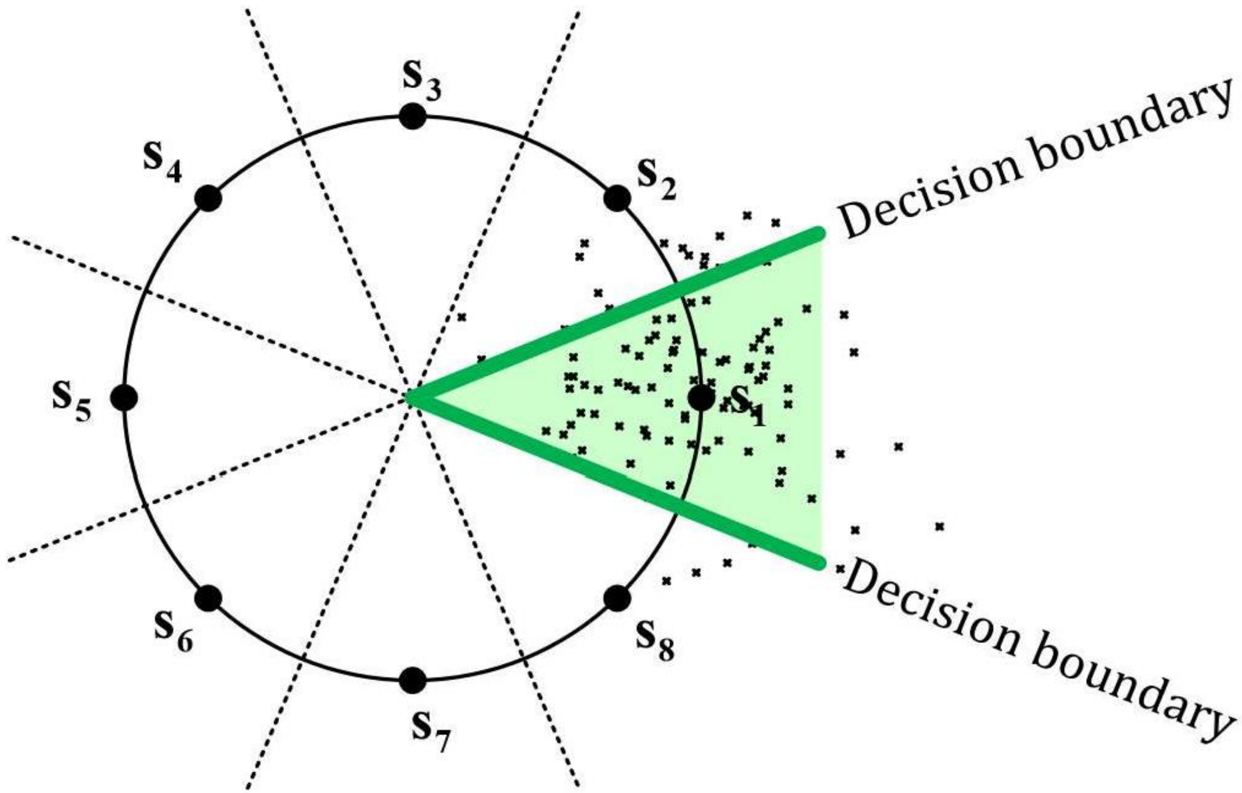
$$r_1 = \sqrt{E_s} + N_1 \qquad r_2 = N_2$$

N_1 and N_2 are two independent Gaussian random variables with zeros mean and variance $\sigma^2 = \frac{N_0}{2}$.

$$f_{r_1,r_2}(r_1,r_2) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{\left(r_1-\sqrt{E_s}\right)^2}{2\sigma^2}\right]\frac{1}{\sqrt{2\pi}\sigma}\exp\left[-\frac{\left(r_2\right)^2}{2\sigma^2}\right]$$

$$= \frac{1}{2\pi\sigma^2}\exp\left[-\frac{\left(r_1^2-2r_1\sqrt{E_s}+E_s+r_2^2\right)}{2\sigma^2}\right]$$

M-ary PSK with M = 8



M-ary PSK

- Since the decision is entirely based on phase, we transform (r_1, r_2) into (R, Θ).

$$R = \sqrt{r_1^2 + r_2^2} \qquad \Theta = \tan^{-1}(\frac{r_2}{r_1})$$

$$f_{r_1,r_2}(r_1,r_2) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{\left(r_1^2 - 2r_1\sqrt{E_s} + E_s + r_2^2\right)}{2\sigma^2}\right]$$

$$f_{R,\Theta}(r,\theta) = \frac{r}{2\pi\sigma^2}\exp\left[-\frac{r^2 + E_s - 2\sqrt{E_s}\,r\cos(\theta)}{2\sigma^2}\right]$$

M-ary PSK

- Since the decision is entirely based on phase, we focus only on Θ .

$$
\begin{aligned}
f_{\Theta}(\theta) &= \int_0^{\infty} f_{R,\Theta}(r,\theta)dr \\
&= \int_0^{\infty} \frac{r}{2\pi\sigma^2} \exp\left[-\frac{r^2 + E_s - 2\sqrt{E_s}\,r\cos(\theta)}{2\sigma^2}\right]dr \\
&= \int_0^{\infty} \frac{r}{2\pi\sigma^2} \exp\left[-\frac{r^2 - 2\sqrt{E_s}\,r\cos(\theta) + E_s\cos^2(\theta) - E_s\cos^2(\theta) + E_s}{2\sigma^2}\right]dr \\
&= \exp\left[-\frac{-E_s\cos^2(\theta) + E_s}{2\sigma^2}\right]\int_0^{\infty} \frac{r}{2\pi\sigma^2} \exp\left[-\frac{(r - \sqrt{E_s}\cos(\theta))^2}{2\sigma^2}\right]dr \\
&= \exp\left[-\frac{E_s}{2\sigma^2}\sin^2(\theta)\right]\int_0^{\infty} \frac{r}{2\pi\sigma^2} \exp\left[-\frac{(r - \sqrt{E_s}\cos(\theta))^2}{2\sigma^2}\right]dr
\end{aligned}
$$

M-ary PSK

$$\int_0^\infty \frac{r}{2\pi\sigma^2}\exp\left[-\frac{(r-\sqrt{E_s}\cos(\theta))^2}{2\sigma^2}\right]dr = \int_0^\infty \frac{(r-\sqrt{E_s}\cos(\theta))}{2\pi\sigma^2}\exp\left[-\frac{(r-\sqrt{E_s}\cos(\theta))^2}{2\sigma^2}\right]dr$$

$$+\int_0^\infty \frac{\sqrt{E_s}\cos(\theta)}{2\pi\sigma^2}\exp\left[-\frac{(r-\sqrt{E_s}\cos(\theta))^2}{2\sigma^2}\right]dr$$

$$= -\frac{1}{2\pi}\exp\left(-\frac{(r-\sqrt{E_s}\cos(\theta))^2}{2\pi\sigma^2}\right)\Bigg|_0^\infty + \frac{\sqrt{E_s}\cos(\theta)}{\sqrt{2\pi\sigma^2}}\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(r-\sqrt{E_s}\cos(\theta))^2}{2\sigma^2}\right]dr$$

$$= -\frac{1}{2\pi}\exp\left(-\frac{E_s\cos^2(\theta)}{2\pi\sigma^2}\right) + \frac{\sqrt{E_s}\cos(\theta)}{\sqrt{2\pi\sigma^2}}\int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left[-\frac{(r-\sqrt{E_s}\cos(\theta))^2}{2\sigma^2}\right]dr$$

- Assume $\frac{E_s}{\sigma^2} >> 1$ and $|\Theta| < \frac{\pi}{2}$, we obtain

$$\int_0^\infty \frac{r}{2\pi\sigma^2}\exp\left[-\frac{(r-\sqrt{E_s}\cos(\theta))^2}{2\sigma^2}\right]dr \approx 0 + \frac{\sqrt{E_s}\cos(\theta)}{\sqrt{2\pi\sigma^2}}\cdot 1$$

M-ary PSK with M = 8

- Probability density function of Θ is .

$$f_{\Theta}(\theta) \approx \sqrt{\frac{E_s}{2\pi\sigma^2}} \cos\theta \exp\left[-\frac{E_s}{2\sigma^2}\sin^2\theta\right]$$

- The decision is correct, if $-\frac{\pi}{M} \leq \Theta \leq \frac{\pi}{M}$. Therefore,

$$p(c\,|\,1) \approx \int_{-\pi/M}^{\pi/M} \sqrt{\frac{E_s}{2\pi\sigma^2}} \cos\theta \exp\left[-\frac{E_s}{2\sigma^2}\sin^2\theta\right] d\theta$$

- Changing variables $z = \sqrt{\frac{E_s}{\sigma^2}} \sin\theta$ and $dz = \sqrt{\frac{E_s}{\sigma^2}} \cos\theta d\theta$

$$p(c\,|\,1) \approx \int_{-\sqrt{E_s/\sigma^2}\sin(\pi/M)}^{\sqrt{E_s/\sigma^2}\sin(\pi/M)} \frac{1}{\sqrt{2\pi}} \exp\left[-z^2/2\right] dz$$

M-ary PSK

$$p(c\,|\,1) \approx \int_{-\sqrt{E_s/\sigma^2}\sin(\pi/M)}^{\sqrt{E_s/\sigma^2}\sin(\pi/M)} \frac{1}{\sqrt{2\pi}} \exp\left[-z^2/2\right]dz$$

$$= \int_{-\sqrt{2E_s/N_0}\sin(\pi/M)}^{0} \frac{1}{\sqrt{2\pi}} \exp\left[-z^2/2\right]dz + \int_{0}^{\sqrt{2E_s/N_0}\sin(\pi/M)} \frac{1}{\sqrt{2\pi}} \exp\left[-z^2/2\right]dz$$

$$= 1 - Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right) - Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

$$= 1 - 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

where $$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^{\infty} \exp\left(-\frac{u^2}{2}\right) du, \qquad x \geq 0$$

In words, *the Q-function equals the area under the positive tail of the zero-mean, unit-variance Gaussian density function.*

M-ary PSK

$$p(e|1) = 1 - p(c|1) = 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

Since $p(e|m) = 1 - p(c|m)$ for m = 1, 2,…, M,

the average probability of **symbol error** for M-ary PSK is

$$p(e) = 2Q\left(\sqrt{\frac{2E_s}{N_0}}\sin\frac{\pi}{M}\right) = \mathrm{erfc}\left(\sqrt{\frac{E_s}{N_0}}\sin\frac{\pi}{M}\right)$$

where $\mathrm{erfc}(x) = \dfrac{2}{\sqrt{\pi}}\displaystyle\int_x^{\infty} \exp(-t^2)\,dt = 2Q(\sqrt{2}x)$

$$E_s = kE_b = (\log_2 M)E_b$$

M-ary PSK

The average probability of **bit error** for M-ary PSK is approximated as

$$p_b = \frac{1}{\log_2 M} p(e)$$

$$p_b = BER = \frac{1}{\log_2 M} 2Q\left(\sqrt{\frac{2\log_2 M \cdot E_b}{N_0}} \sin\frac{\pi}{M}\right)$$

$$p_b = BER = \frac{1}{\log_2 M} erfc\left(\sqrt{\frac{\log_2 M \cdot E_b}{N_0}} \sin\frac{\pi}{M}\right)$$

M-ary ASK

- In *M*-ary ASK, during each signaling interval of duration *T*, one of the following *M* possible signals is sent

$$s_i(t) = A_i \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t\right), \quad \text{where} \quad i = 1, 2, \ldots, M$$

- The carrier frequency $f_c = n_c / T$ for some fixed integer n_c

M-ary PSK

- Each $s_i(t)$ may be expanded in terms of one basis functions $\phi_1(t)$, defined as

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t\right), \qquad 0 \le t \le T$$

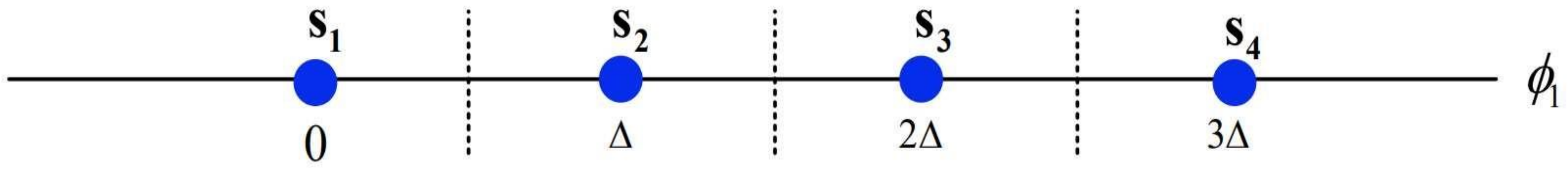
- The basis function has unit energy.

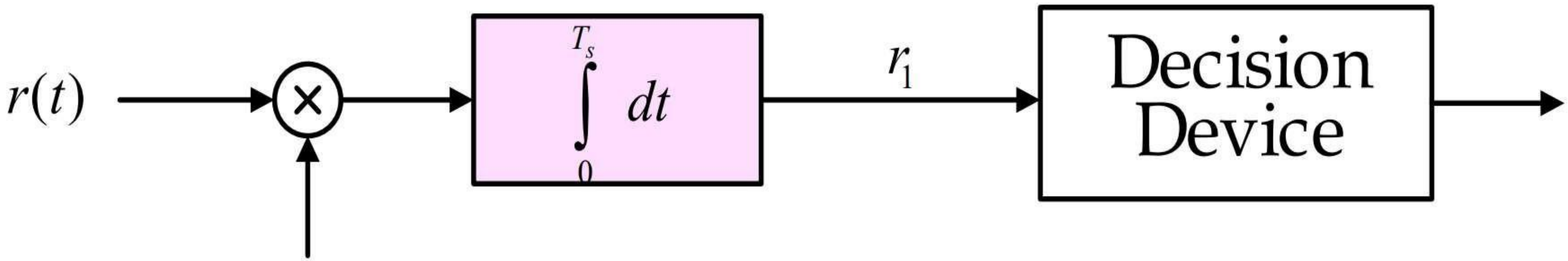
- If $A_i = (\mathrm{i} - 1)\Delta$, then

$$s_i(t) = (i-1)\Delta\phi_1(t), \quad \text{where} \quad i = 1, 2, \ldots, M$$

- Where Δ is the distance between two adjacent signals, see next slide.

M-ary ASK with M = 4





M-ary ASK

- The average probability of symbol error for M-ary PSK is

$$p(e) = \sum_{i=1}^{M} p(e \mid \mathbf{s}_i)\, p(\mathbf{s}_i)$$

  - $p(\mathbf{s}_i)$ **The probability that symbol $\mathbf{s}_i$ was sent.**
  - $p(e \mid \mathbf{s}_i)$ **The probability of decision error when symbol $\mathbf{s}_i$ was sent.**

- When each symbol is sent with the same probability,

$$p(\mathbf{s}_i) = \frac{1}{M}$$

M-ary ASK with M = 8

- **Assume that $\mathbf{s}_i$ was sent, the receiver produces,**

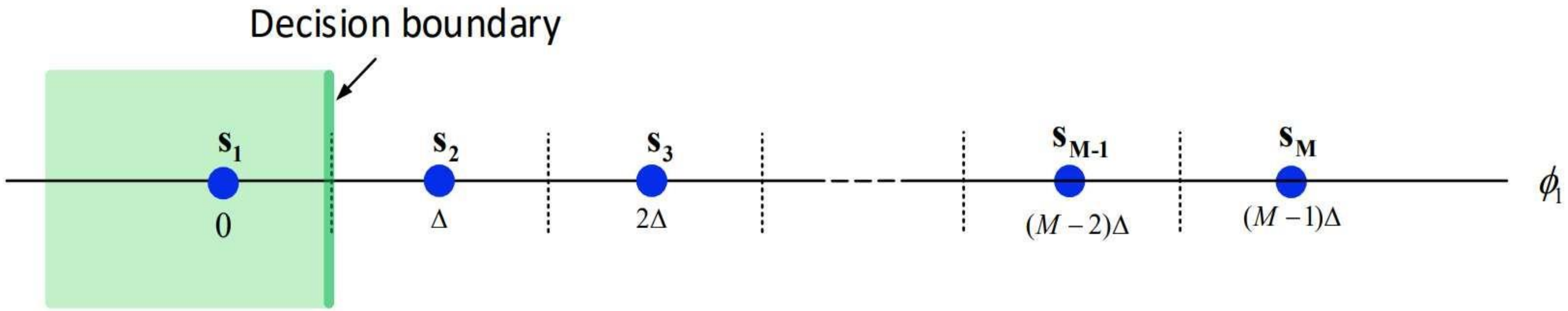
$$r_1 = (i-1)\Delta + N$$

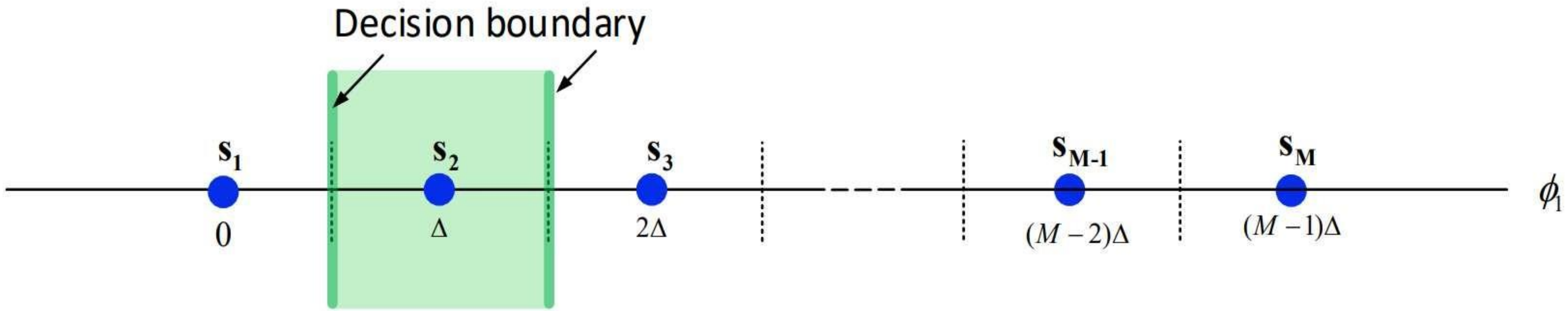
- N **is an independent Gaussian random variable with zeros mean and variance** $\sigma^2 = \frac{N_0}{2}$.

$$f_{r_1}(r_1 \mid \mathbf{s_i}) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left[-\frac{\left(r_1 - (i-1)\Delta\right)^2}{2\sigma^2} \right]$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{\left(r_1 - (i-1)\Delta\right)^2}{N_0} \right]$$

M-ary ASK with M = 8





$\mathbf{s}_1$ and $\mathbf{s}_M$ have different decision regions from $\mathbf{s}_2, \mathbf{s}_3, \ldots, \mathbf{s}_{M-1}$

M-ary ASK

For $\mathbf{s_1}$ and $\mathbf{s_M}$,

$$p(e\,|\,\mathbf{s_1}) = \int_{\Delta/2}^{\infty} f_{r_1}(r_1\,|\,\mathbf{s}_1)dr = \int_{\Delta/2}^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{r_1^2}{N_0}\right]dr = Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

$$p(e\,|\,\mathbf{s_M}) = \int_{-\infty}^{(M-1.5)\Delta} f_{r_1}(r_1\,|\,\mathbf{s}_M)dr$$

$$= \int_{-\infty}^{(M-1.5)\Delta} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{(r_1-(M-1)\Delta)^2}{N_0}\right]dr$$

$$= Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) = p(e\,|\,\mathbf{s_1})$$

M-ary ASK

For $\mathbf{s_2}, \mathbf{s_3}, \ldots, \mathbf{s_{M-1}}$,

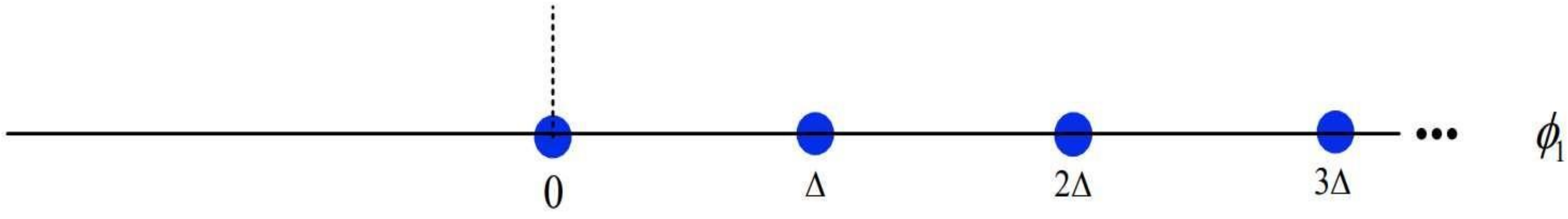
$$p(e \mid \mathbf{s_i}) = 1 - \int_{(i-1.5)\Delta}^{(i-0.5)\Delta} f_{r_1}(r_1 \mid \mathbf{s}_i) dr = 2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

$$p(e) = \sum_{i=1}^{M} p(e \mid \mathbf{s_i}) p(\mathbf{s_i})$$

$$= \frac{1}{M}\left[2 \times Q\left(\frac{\Delta}{\sqrt{2N_0}}\right) + (M-2) \times 2Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)\right]$$

$$= \frac{2(M-1)}{M} Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

M-ary ASK



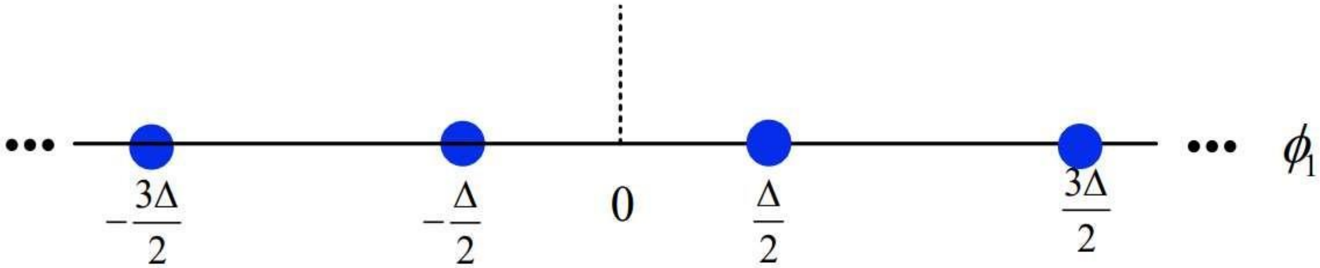
$$E_s = \frac{0^2 + \Delta^2 + (2\Delta)^2 + (3\Delta)^2 + \ldots + ((M-1)\Delta)^2}{M} = \frac{(M-1)(2M-1)\Delta^2}{6}$$

M-ary ASK

- **If we may change the signals to become**

$$s_i(t) = (2i - 1 - M)\frac{\Delta}{2}\phi_1(t), \quad \text{where} \quad i = 1, 2, \ldots, M$$

- **The energy per symbol E_s can be lower without any change to P_e.**



$$E_s = \frac{1}{M}\left(\frac{\Delta}{2}\right)^2 \sum_{i=1}^{M}(2i - 1 - M)^2 = \frac{\Delta^2}{4M}\left(\frac{M}{3}(M^2 - 1)\right) = \frac{(M^2 - 1)\Delta^2}{12}$$

M-ary ASK

$$E_b = \frac{E_s}{\log_2(M)} = \frac{(M^2-1)\Delta^2}{12\log_2(M)} \qquad \Delta = \sqrt{\frac{E_b 12\log_2(M)}{M^2-1}}$$

- **Therefore, the symbol error rate of M-ary ASK**

$$p(e) = \frac{2(M-1)}{M} Q\left(\frac{\Delta}{\sqrt{2N_0}}\right)$$

SER = [(M-1)/M*math.erfc(math.sq

$$= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{E_b}{N_0}\frac{6\log_2(M)}{M^2-1}}\right)$$

$$= \frac{(M-1)}{M} erfc\left(\sqrt{\frac{E_b}{N_0}\frac{3\log_2(M)}{M^2-1}}\right)$$

M-ary QAM

- In M-ary QAM, during each signaling interval of duration T, one of the following M possible signals is sent

$$s_i(t) = A_{I,i}\sqrt{\frac{2}{T}}\cos\left(2\pi f_c t\right) + A_{Q,i}\sqrt{\frac{2}{T}}\sin\left(2\pi f_c t\right),$$

where $i = 1,2,\ldots,M$

- The carrier frequency $f_c = n_c/T$ for some fixed integer n_c

M-ary QAM

- Each $s_i(t)$ may be expanded in terms of two basis functions $\phi_1(t)$ and $\phi_2(t)$, defined as

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos\left(2\pi f_c t\right), \qquad 0 \le t \le T$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin\left(2\pi f_c t\right), \qquad 0 \le t \le T$$

- Both basis functions have unit energy.

M-ary QAM

- Each $s_i(t)$ is represented as

$$s_i(t) = A_{I,i}\phi_1(t) + A_{Q,i}\phi_2(t),$$

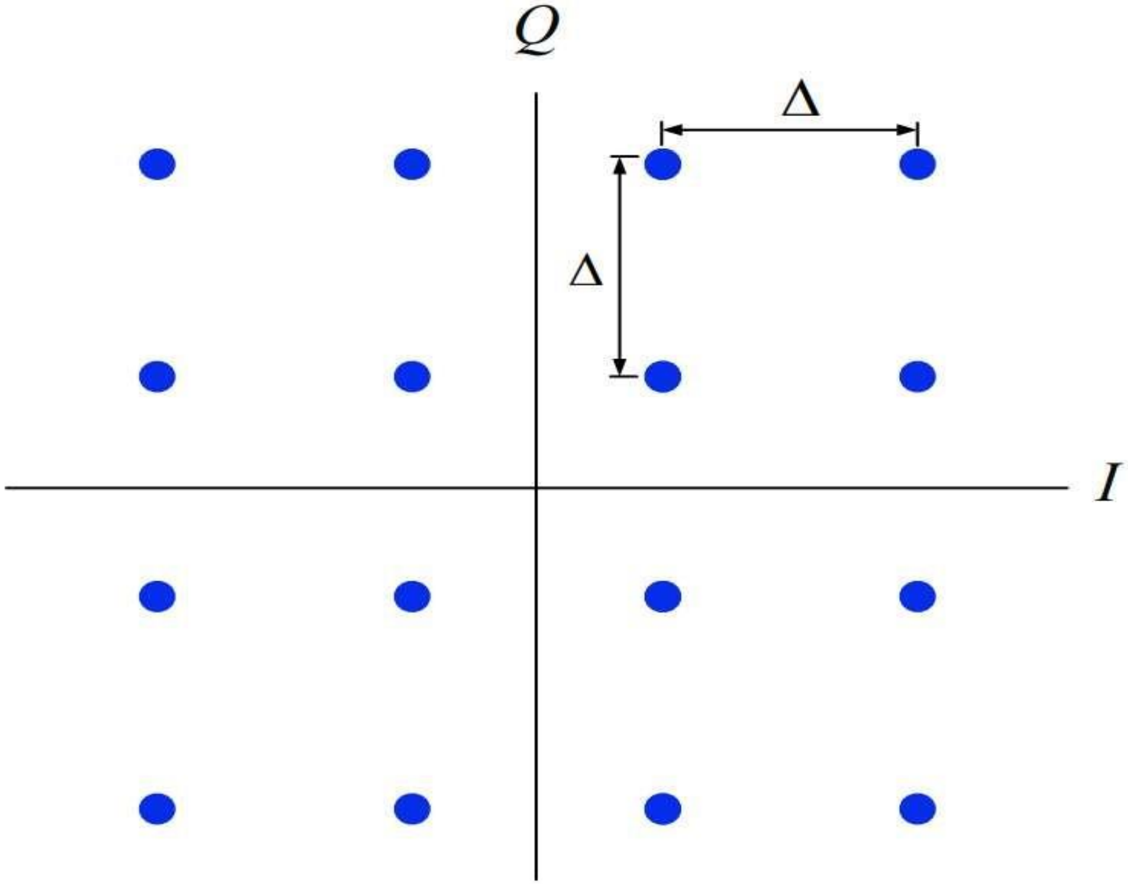
- Energy of symbol *i* is

$$E_i = A_{I,i}^2 + A_{\mathrm{Q},i}^2$$

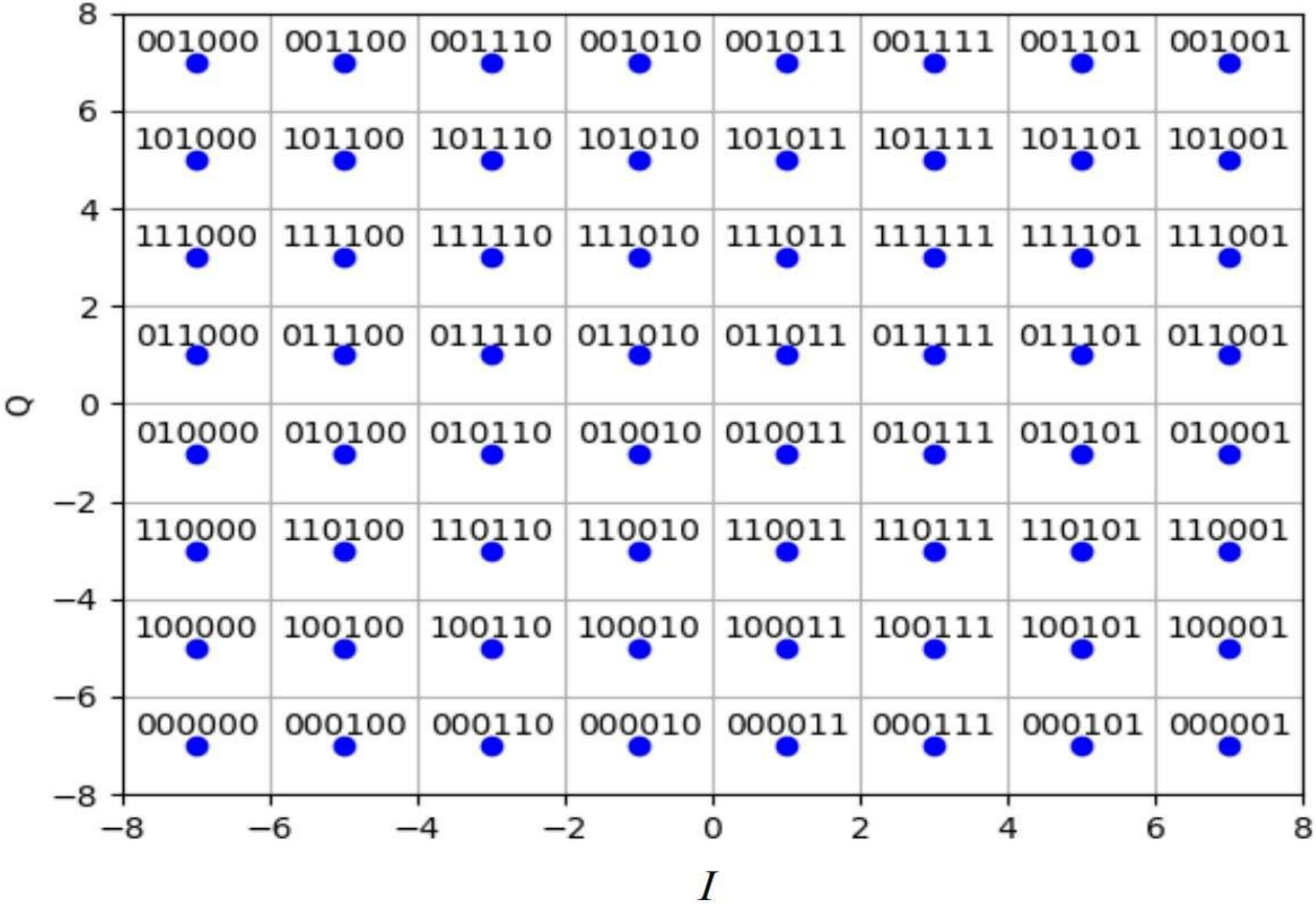
- Energy per symbol of QAM is

$$E_S = \frac{1}{M}\sum_{i=1}^{M} E_i = \frac{1}{M}\sum_{i=1}^{M}\left(A_{I,i}^2 + A_{\mathrm{Q},i}^2\right)$$

16-ary QAM



64-ary QAM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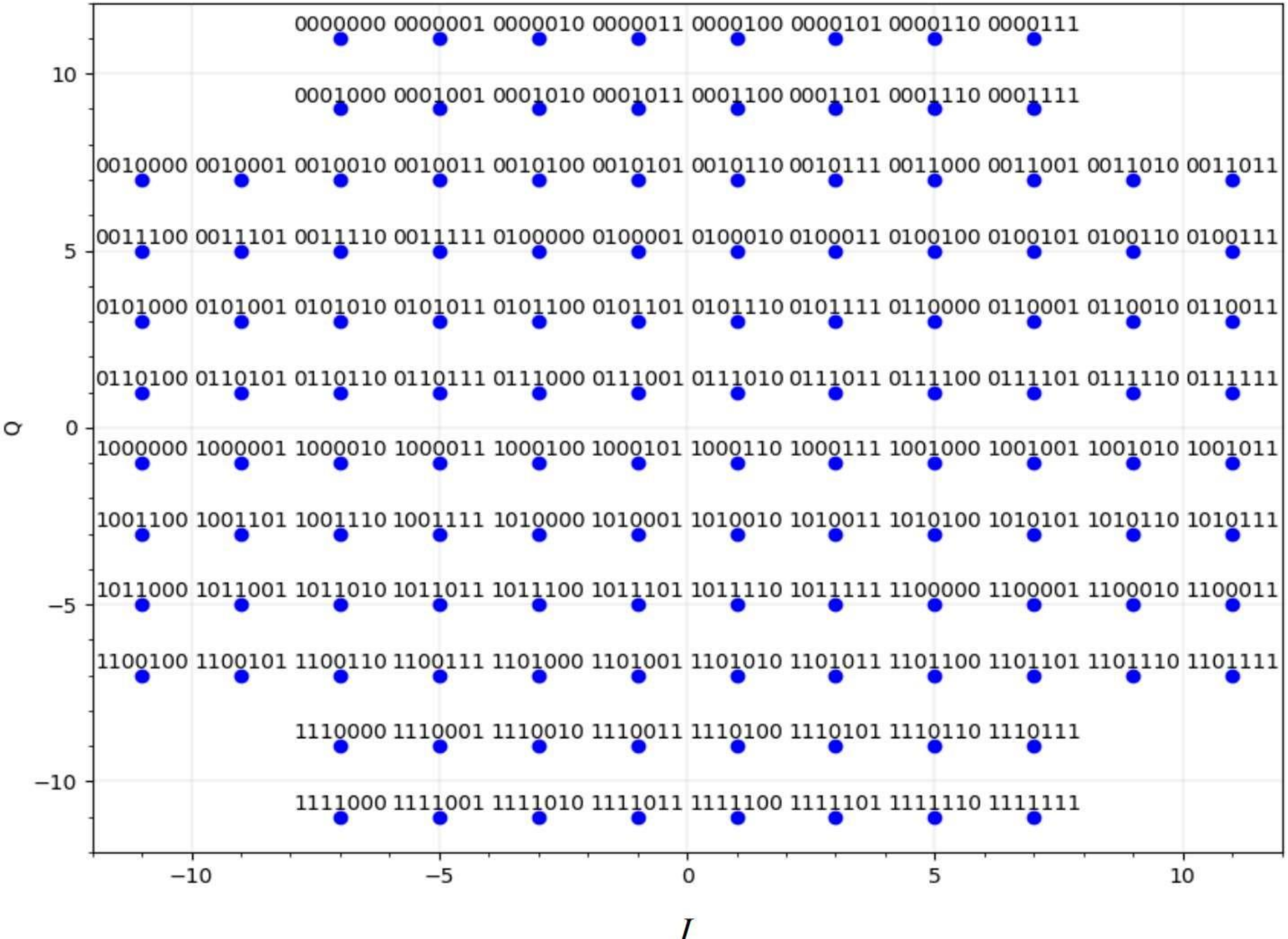
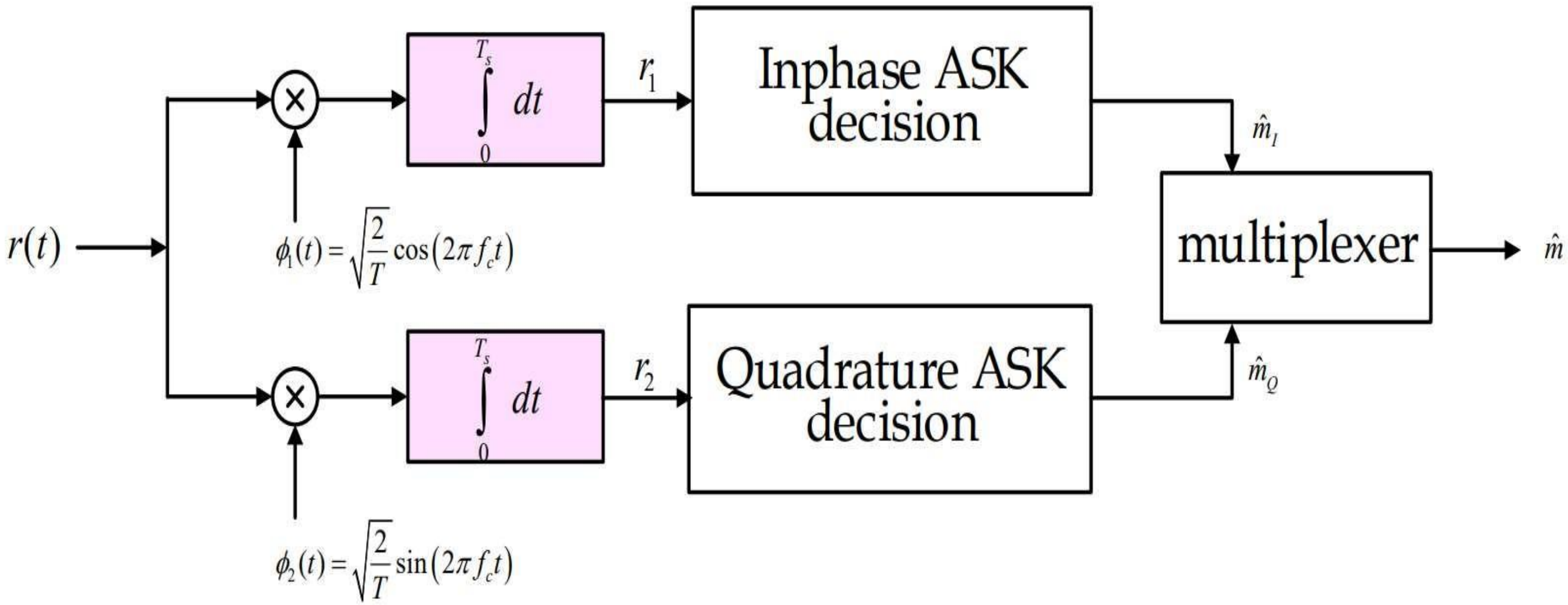
101000 101100 101110 101010 101011 101111 101101 101001
111000 111100 111110 111010 111011 111111 111101 111001
011000 011100 011110 011010 011011 011111 011101 011001
010000 010100 010110 010010 010011 010111 010101 010001
110000 110100 110110 110010 110011 110111 110101 110001
100000 100100 100110 100010 100011 100111 100101 100001
000000 000100 000110 000010 000011 000111 000101 000001
Q
I
8
6
4
2
0
−2
−4
−6
−8

128-ary QAM



0000000 0000001 0000010 0000011 0000100 0000101 0000110 0000111
0001000 0001001 0001010 0001011 0001100 0001101 0001110 0001111
0010000 0010001 0010010 0010011 0010100 0010101 0010110 0010111 0011000 0011001 0011010 0011011
0011100 0011101 0011110 0011111 0100000 0100001 0100010 0100011 0100100 0100101 0100110 0100111
0101000 0101001 0101010 0101011 0101100 0101101 0101110 0101111 0110000 0110001 0110010 0110011
0110100 0110101 0110110 0110111 0111000 0111001 0111010 0111011 0111100 0111101 0111110 0111111
1000000 1000001 1000010 1000011 1000100 1000101 1000110 1000111 1001000 1001001 1001010 1001011
1001100 1001101 1001110 1001111 1010000 1010001 1010010 1010011 1010100 1010101 1010110 1010111
1011000 1011001 1011010 1011011 1011100 1011101 1011110 1011111 1100000 1100001 1100010 1100011
1100100 1100101 1100110 1100111 1101000 1101001 1101010 1101011 1101100 1101101 1101110 1101111
1110000 1110001 1110010 1110011 1110100 1110101 1110110 1110111
1111000 1111001 1111010 1111011 1111100 1111101 1111110 1111111
10
5
0
−5
−10
Q
−10
−5
0
5
10
I

16-ary QAM



$r(t)$
$\int_0^{T_s} dt$
r_1
Inphase ASK decision
$\hat{m}_I$
multiplexer
$\hat{m}$
$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t)$
$\int_0^{T_s} dt$
r_2
Quadrature ASK decision
$\hat{m}_Q$
$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t)$

M-ary QAM

- Receiver can detect inphase and quadrature signal separately and independently.
- Therefore, symbol error rate of M-ary QAM can be computed easily using the symbol error rate of M-ary ASK as follows.

$$p_{M-QAM}(e) = 1 - P[\text{correct}] = 1 - \left(1 - p_{\sqrt{M}-ASK}(e)\right)^2$$

M-ary QAM

- **Since** $p_{\sqrt{M}-ASK}(e) = \frac{2(\sqrt{M}-1)}{\sqrt{M}} Q\left(\sqrt{\frac{E_b}{N_0}\frac{6\log_2(\sqrt{M})}{M-1}}\right)$

$$= 2\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{E_b}{N_0}\frac{3\log_2(M)}{M-1}}\right)$$

- **we obtain**

$$p_{M-QAM}(e) = 1-\left(1-2\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{E_b}{N_0}\frac{3\log_2(M)}{M-1}}\right)\right)^2$$

$$= 4\left(1-\frac{1}{\sqrt{M}}\right)Q\left(\sqrt{\frac{E_b}{N_0}\frac{3\log_2(M)}{M-1}}\right) - 4\left(1-\frac{1}{\sqrt{M}}\right)^2 Q^2\left(\sqrt{\frac{E_b}{N_0}\frac{3\log_2(M)}{M-1}}\right)$$

M-ary QAM

$$p_{M-QAM}(e) = 2\left(1-\frac{1}{\sqrt{M}}\right)erfc\left(\sqrt{\frac{E_s}{N_0}\frac{3}{2(M-1)}}\right) - \left(1-\frac{1}{\sqrt{M}}\right)^2 erfc^2\left(\sqrt{\frac{E_s}{N_0}\frac{3}{2(M-1)}}\right)$$

$$= 2\left(1-\frac{1}{\sqrt{M}}\right)erfc\left(\sqrt{\frac{E_b}{N_0}\frac{3\log_2(M)}{2(M-1)}}\right) - \left(1-\frac{1}{\sqrt{M}}\right)^2 erfc^2\left(\sqrt{\frac{E_b}{N_0}\frac{3\log_2(M)}{2(M-1)}}\right)$$

THANK YOU