Chapter 3

Sampling

3.1 INTRODUCTION

Most discrete-time signals come from *sampling* a continuous-time signal, such as speech and audio signals, radar and sonar data, and seismic and biological signals. The process of converting these signals into digital form is called *analog-to-digital* (A/D) conversion. The reverse process of reconstructing an analog signal from its samples is known as *digital-to-analog* (D/A) conversion. This chapter examines the issues related to A/D and D/A conversion. Fundamental to this discussion is the *sampling theorem*, which gives precise conditions under which an analog signal may be uniquely represented in terms of its samples.

3.2 ANALOG-TO-DIGITAL CONVERSION

An A/D converter transforms an analog signal into a digital sequence. The input to the A/D converter, $x_a(t)$, is a real-valued function of a continuous variable, t. Thus, for each value of t, the function $x_a(t)$ may be any real number. The output of the A/D is a *bit stream* that corresponds to a discrete-time sequence, $x(n)$, with an amplitude that is quantized, for each value of n, to one of a finite number of possible values. The components of an A/D converter are shown in Fig. 3-1. The first is the sampler, which is sometimes referred to as a *continuous-to-discrete* (C/D) converter, or *ideal A/D converter*. The sampler converts the continuous-time signal $x_a(t)$ into a discrete-time sequence $x(n)$ by extracting the values of $x_a(t)$ at integer multiples of the sampling period, T_s,

$$x(n) = x_a(nT_s)$$

Because the samples $x_a(nT_s)$ have a continuous range of possible amplitudes, the second component of the A/D converter is the quantizer, which maps the continuous amplitude into a discrete set of amplitudes. For a uniform quantizer, the quantization process is defined by the number of bits and the quantization interval Δ. The last component is the encoder, which takes the digital signal $\hat{x}(n)$ and produces a sequence of binary codewords.

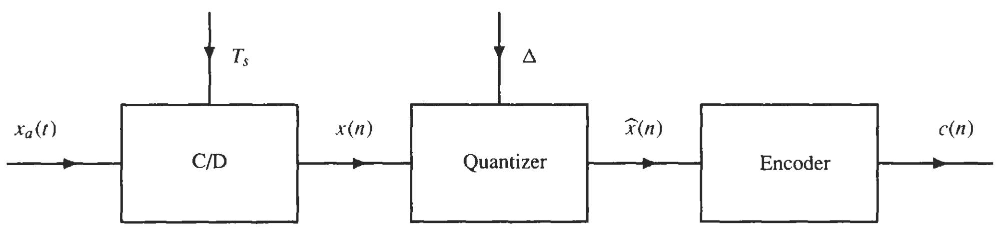


Fig. 3-1. The components of an analog-to-digital converter.

3.2.1 Periodic Sampling

Typically, discrete-time signals are formed by *periodically sampling* a continuous-time signal

$$x(n) = x_a(nT_s) \tag{3.1}$$

The sample spacing T_s is the sampling period, and $f_s = 1/T_s$ is the sampling frequency in samples per second. A convenient way to view this sampling process is illustrated in Fig. 3-2(a). First, the continuous-time signal is multiplied by a periodic sequence of impulses,

$$s_a(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

to form the *sampled signal*

$$x_s(t) = x_a(t)s_a(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

Then, the sampled signal is converted into a discrete-time signal by mapping the impulses that are spaced in time by T_s into a sequence $x(n)$ where the sample values are indexed by the integer variable n:

$$x(n) = x_a(nT_s)$$

This process is illustrated in Fig. 3-2(*b*).

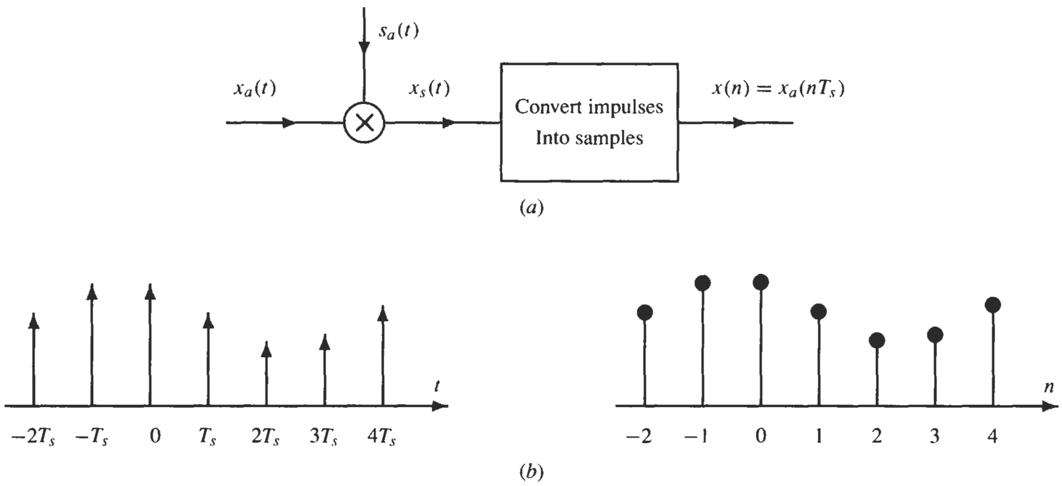


Fig. 3-2. Continuous-to-discrete conversion. (*a*) A model that consists of multiplying $x_a(t)$ by a sequence of impulses, followed by a system that converts impulses into samples. (*b*) An example that illustrates the conversion process.

The effect of the C/D converter may be analyzed in the frequency domain as follows. Because the Fourier transform of $\delta(t - nT_s)$ is $e^{-jn\Omega T_s}$, the Fourier transform of the sampled signal $x_s(t)$ is

$$X_s(j\Omega) = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-jn\Omega T_s} \tag{3.2}$$

Another expression for $X_s(j\Omega)$ follows by noting that the Fourier transform of $s_a(t)$ is

$$S_a(j\Omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_s)$$

where $\Omega_s = 2\pi/T_s$ is the sampling frequency in radians per second. Therefore,

$$X_s(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * S_a(j\Omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(j\Omega - jk\Omega_s)$$

Finally, the discrete-time Fourier transform of $x(n)$ is

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x_a(nT_s)e^{-jn\omega} \tag{3.3}$$

Comparing Eq. (*3.3*) with Eq. (*3.2*), it follows that

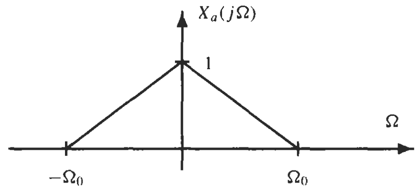
$$X(e^{j\omega}) = X_s(j\Omega)|_{\Omega=\omega/T_s} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right) \tag{3.4}$$

Thus, $X(e^{j\omega})$ is a frequency-scaled version of $X_s(j\Omega)$, with the scaling defined by

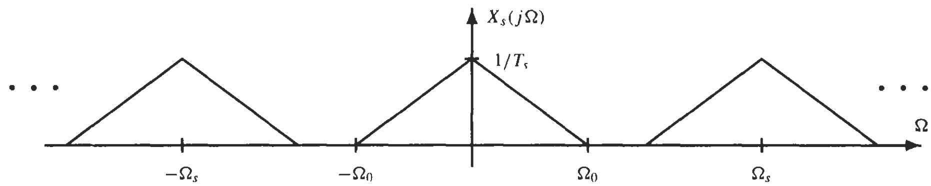
$$\omega = \Omega T_s$$

This scaling, which makes $X(e^{j\omega})$ periodic with a period of 2π, is a consequence of the time-scaling that occurs when $x_s(t)$ is converted to $x(n)$.

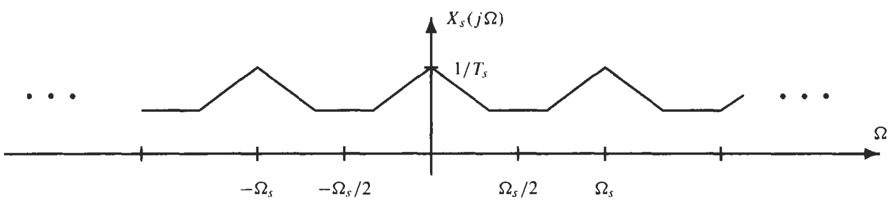
EXAMPLE 3.2.1 Suppose that $x_a(t)$ is strictly bandlimited so that $X_a(j\Omega) = 0$ for $|\Omega| > \Omega_0$ as shown in the figure below.



If $x_a(t)$ is sampled with a sampling frequency $\Omega_s \geq 2\Omega_0$, the Fourier transform of $x_s(t)$ is formed by periodically replicating $X_a(j\Omega)$ as illustrated in the figure below.



However, if $\Omega_s < 2\Omega_0$, the shifted spectra $X_a(j\Omega - jk\Omega_s)$ overlap, and when these spectra are summed to form $X_s(j\Omega)$, the result is as shown in the figure below.



This overlapping of spectral components is called *aliasing*. When aliasing occurs, the frequency content of $x_a(t)$ is corrupted, and $X_a(j\Omega)$ cannot be recovered from $X_s(j\Omega)$.

As illustrated in Example 3.2.1, if $x_a(t)$ is strictly bandlimited so that the highest frequency in $x_a(t)$ is Ω_0, and if the sampling frequency is greater than $2\Omega_0$,

$$\Omega_s \geq 2\Omega_0$$

no aliasing occurs, and $x_a(t)$ may be uniquely recovered from its samples $x_a(nT_s)$ with a low-pass filter. The following is a statement of the famous Nyquist sampling theorem:

Sampling Theorem: If $x_a(t)$ is strictly bandlimited,

$$X_a(j\Omega) = 0 \qquad |\Omega| > \Omega_0$$

then $x_a(t)$ may be uniquely recovered from its samples $x_a(nT_s)$ if

$$\Omega_s = \frac{2\pi}{T_s} \geq 2\Omega_0$$

The frequency Ω_0 is called the *Nyquist frequency*, and the minimum sampling frequency, $\Omega_s = 2\Omega_0$, is called the *Nyquist rate*.

Because the signals that are found in physical systems will never be strictly bandlimited, an analog *anti-aliasing* filter is typically used to filter the signal prior to sampling in order to minimize the amount of energy above the Nyquist frequency and to reduce the amount of aliasing that occurs in the A/D converter.

3.2.2 *Quantization and Encoding*

A quantizer is a nonlinear and noninvertible system that transforms an input sequence $x(n)$ that has a continuous range of amplitudes into a sequence for which each value of $x(n)$ assumes one of a finite number of possible values. This operation is denoted by

$$\hat{x}(n) = Q[x(n)]$$

The quantizer has $L+1$ *decision levels* $x_1, x_2, \ldots, x_{L+1}$ that divide the amplitude range for $x(n)$ into L *intervals*

$$I_k = [x_k, x_{k+1}] \qquad k = 1, 2, \ldots, L$$

For an input $x(n)$ that falls within interval I_k, the quantizer assigns a value within this interval, $\hat{x}_k$, to $x(n)$. This process is illustrated in Fig. 3-3.

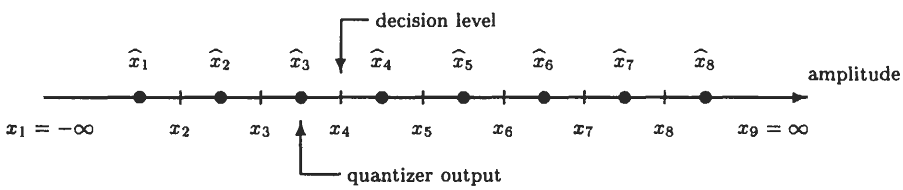


Fig. 3-3. A quantizer with nine decision levels that divide the input amplitudes into eight quantization intervals and eight possible quantizer outputs, $\hat{x}_k$.

Quantizers may have quantization levels that are either uniformly or nonuniformly spaced. When the quantization intervals are uniformly spaced,

$$\Delta = x_{k+1} - x_k$$

Δ is called the *quantization step size* or the *resolution* of the quantizer, and the quantizer is said to be a *uniform* or *linear* quantizer.[1] The number of levels in a quantizer is generally of the form

$$L = 2^{B+1}$$

in order to make the most efficient use of a $(B+1)$-bit binary code word. A 3-bit uniform quantizer in which the quantizer output is *rounded* to the nearest quantization level is illustrated in Fig. 3-4. With $L = 2^{B+1}$ quantization levels and a step size Δ, the *range* of the quantizer is

$$R = 2^{B+1} \cdot \Delta$$

Therefore, if the quantizer input is bounded,

$$|x(n)| \le X_{\max}$$

the range of possible input values may be covered with a step size

$$\Delta = \frac{X_{\max}}{2^B}$$

With rounding, the quantization error

$$e(n) = Q[x(n)] - x(n)$$

[1]In some applications, such as speech coding, the quantizer levels are *adaptive* (i.e., they change with time).

will be bounded by

$$-\frac{\Delta}{2} < e(n) < \frac{\Delta}{2}$$

However, if $|x(n)|$ exceeds $X_{\max}$, then $x(n)$ will be *clipped*, and the quantization error could be very large.

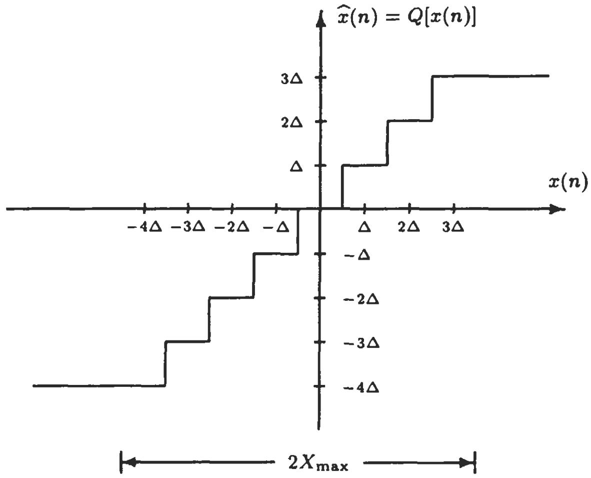


Fig. 3-4. A 3-bit uniform quantizer.

A useful model for the quantization process is given in Fig. 3-5. Here, the quantization error is assumed to be an additive noise source. Because the quantization error is typically not known, the quantization error is described statistically. It is generally assumed that $e(n)$ is a sequence of random variables where

1. The statistics of $e(n)$ do not change with time (the quantization noise is a stationary random process).
2. The quantization noise $e(n)$ is a sequence of *uncorrelated* random variables.
3. The quantization noise $e(n)$ is *uncorrelated* with the quantizer input $x(n)$.
4. The probability density function of $e(n)$ is uniformly distributed over the range of values of the quantization error.

Although it is easy to find cases in which these assumptions do not hold (e.g., if $x(n)$ is a constant), they are generally valid for rapidly varying signals with fine quantization (Δ small).

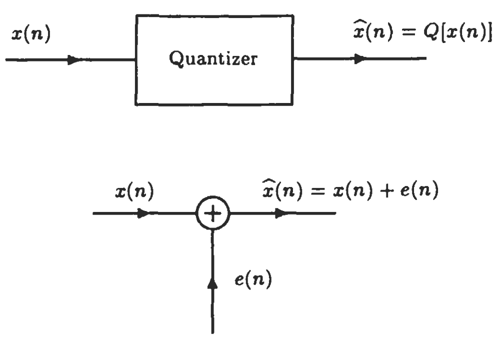


Fig. 3-5. A quantization noise model.

With rounding, the quantization noise is uniformly distributed over the interval $[-\Delta/2, \Delta/2]$, and the quantization noise power (the variance) is

$$\sigma_e^2 = \frac{\Delta^2}{12}$$

With a step size

$$\Delta = \frac{X_{\max}}{2^B}$$

and a signal power σ_x^2, the signal-to-quantization noise ratio, in decibels (dB), is

$$\text{SQNR} = 10\log\frac{\sigma_x^2}{\sigma_e^2} = 6.02B + 10.81 - 20\log\frac{X_{\max}}{\sigma_x} \tag{3.5}$$

Thus, the signal-to-quantization noise ratio increases approximately 6 dB for each bit.

The output of the quantizer is sent to an *encoder*, which assigns a unique binary number *(codeword)* to each quantization level. Any assignment of codewords to levels may be used, and many coding schemes exist. Most digital signal processing systems use the two's-complement representation. In this system, with a $(B+1)$ bit codeword,

$$c = [b_0, b_1, \ldots, b_B]$$

the leftmost or most significant bit, b_0, is the sign bit, and the remaining bits are used to represent either binary integers or fractions. Assuming binary fractions, the codeword $b_0b_1b_2\cdots b_B$ has the value

$$x = (-1)b_0 + b_1 2^{-1} + b_2 2^{-2} + \cdots + b_B 2^{-B}$$

An example is given below for a 3-bit codeword.

Binary Symbol	Numeric Value
0 1 1	$\frac{3}{4}$
0 1 0	$\frac{1}{2}$
0 0 1	$\frac{1}{4}$
0 0 0	0
1 1 1	$-\frac{1}{4}$
1 1 0	$-\frac{1}{2}$
1 0 1	$-\frac{3}{4}$
1 0 0	-1

3.3 DIGITAL-TO-ANALOG CONVERSION

As stated in the sampling theorem, if $x_a(t)$ is strictly bandlimited so that $X_a(j\Omega) = 0$ for $|\Omega| > \Omega_0$, and if $T_s < \pi/\Omega_0$, then $x_a(t)$ may be uniquely reconstructed from its samples $x(n) = x_a(nT_s)$. The reconstruction process involves two steps, as illustrated in Fig. 3-6. First, the samples $x(n)$ are converted into a sequence of impulses,

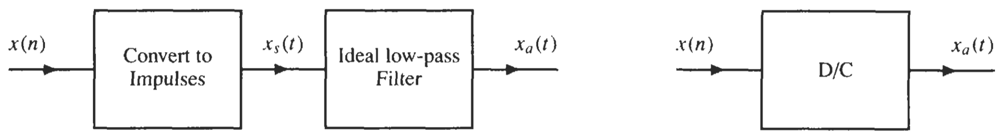
$$x_s(t) = \sum_{n=-\infty}^{\infty} x(n)\delta(t - nT_s)$$

and then $x_a(t)$ is filtered with a *reconstruction filter*, which is an ideal low-pass filter that has a frequency response given by

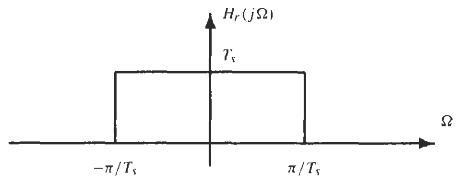
$$H_r(j\Omega) = \begin{cases} T_s & |\Omega| \le \dfrac{\pi}{T_s} \\ 0 & |\Omega| > \dfrac{\pi}{T_s} \end{cases}$$

This system is called an *ideal discrete-to-continuous* (D/C) converter. Because the impulse response of the reconstruction filter is

$$h_r(t) = \frac{\sin(\pi t/T_s)}{\pi t/T_s}$$



(a)



(b)

Fig. 3-6. (a) A discrete-to-continuous converter with an ideal low-pass reconstruction filter. (b) The frequency response of the ideal reconstruction filter.

the output of the filter is

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n)h_r(t - nT_s) = \sum_{n=-\infty}^{\infty} x(n)\frac{\sin \pi(t - nT_s)/T_s}{\pi(t - nT_s)/T_s} \tag{3.6}$$

This *interpolation formula* shows how $x_a(t)$ is reconstructed from its samples $x(n) = x_a(nT_s)$. In the frequency domain, the interpolation formula becomes

$$\begin{aligned} X_a(j\Omega) &= \sum_{n=-\infty}^{\infty} x(n)H_r(j\Omega)e^{-jn\Omega T_s} \\ &= H_r(j\Omega) \sum_{n=-\infty}^{\infty} x(n)e^{-jn\Omega T_s} = H_r(j\Omega)X(e^{j\Omega T_s}) \end{aligned} \tag{3.7}$$

which is equivalent to

$$X_a(j\Omega) = \begin{cases} T_s X(e^{j\Omega T_s}) & |\Omega| < \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases} \tag{3.8}$$

Thus, $X(e^{j\omega})$ is frequency scaled ($\omega = \Omega T_s$), and then the low-pass filter removes all frequencies in the periodic spectrum $X(e^{j\Omega T_s})$ above the cutoff frequency $\Omega_c = \pi/T_s$.

Because it is not possible to implement an ideal low-pass filter, many D/A converters use a *zero-order hold* for the reconstruction filter. The impulse response of a zero-order hold is

$$h_0(t) = \begin{cases} 1 & 0 \le t \le T_s \\ 0 & \text{otherwise} \end{cases}$$

and the frequency response is

$$H_0(j\Omega) = e^{-j\Omega T_s/2}\frac{\sin(\Omega T_s/2)}{\Omega/2}$$

After a sequence of samples $x_a(nT_s)$ has been converted to impulses, the zero-order hold produces the staircase approximation to $x_a(t)$ shown in Fig. 3-7. With a zero-order hold, it is common to postprocess the output with a *reconstruction compensation filter* that approximates the frequency response

$$H_c(j\Omega) = \begin{cases} \dfrac{\Omega T_s/2}{\sin(\Omega T_s/2)} e^{j\Omega T_s/2} & |\Omega| \le \dfrac{\pi}{T_s} \\ 0 & |\Omega| > \dfrac{\pi}{T_s} \end{cases}$$

Fig. 3-7. The use of a zero-order hold to interpolate between the samples in $x_s(t)$.

so that the cascade of $H_0(e^{j\omega})$ with $H_c(e^{j\omega})$ approximates a low-pass filter with a gain of T_s over the passband. Figure 3-8 shows the magnitude of the frequency response of the zero-order hold and the magnitude of the frequency response of the ideal reconstruction compensation filter. Note that the cascade of $H_c(j\Omega)$ with the zero-order hold is an ideal low-pass filter.

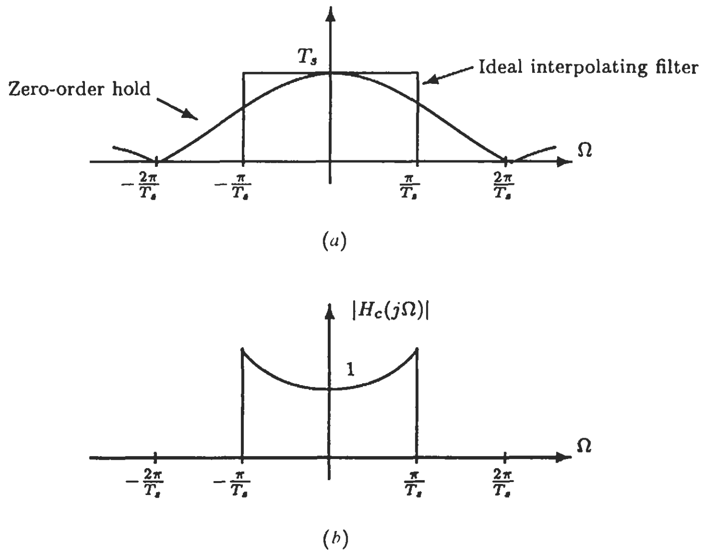


Fig. 3-8. (*a*) The magnitude of the frequency response of a zero-order hold compared to the ideal reconstruction filter. (*b*) The ideal reconstruction compensation filter.

3.4 DISCRETE-TIME PROCESSING OF ANALOG SIGNALS

One of the important applications of A/D and D/A converters is the processing of analog signals with a discrete-time system. In the ideal case, the overall system, shown in Fig. 3-9, consists of the cascade of a C/D converter, a discrete-time system, and a D/C converter. Thus, we are assuming that the sampled signal is not quantized and that an ideal low-pass filter is used for the reconstruction filter in the D/C converter. Because the input $x_a(t)$ and the output $y_a(t)$ are analog signals, the overall system corresponds to a continuous-time system. To analyze this system, note that the C/D converter produces the discrete-time signal $x(n)$, which has a DTFT given by

$$X(e^{j\omega}) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right)$$

If the discrete-time system is linear and shift-invariant with a frequency response $H(e^{j\omega})$,

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = H(e^{j\omega})\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right)$$

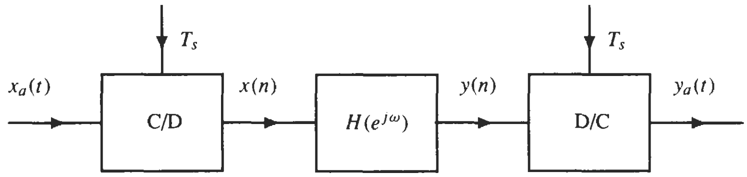


Fig. 3-9. Processing an analog signal using a discrete-time system.

Finally, the D/C converter produces the continuous-time signal $y_a(t)$ from the samples $y(n)$ as follows:

$$y_a(t) = \sum_{n=-\infty}^{\infty} y(n) \frac{\sin \pi(t - nT_s)/T_s}{\pi(t - nT_s)/T_s}$$

Either using Eq. (*3.7*) or by taking the DTFT directly, in the frequency domain this relationship becomes

$$Y_a(j\Omega) = H_r(j\Omega)Y(e^{j\Omega T_s}) = H_r(j\Omega)H(e^{j\Omega T_s})X(e^{j\Omega T_s})$$

or

$$Y_a(j\Omega) = H_r(j\Omega)H(e^{j\Omega T_s})\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(j\Omega - j\frac{2\pi k}{T_s}\right)$$

If $x_a(t)$ is bandlimited with $X_a(j\Omega) = 0$ for $|\Omega| > \pi/T_s$, the low-pass filter $H_r(j\Omega)$ eliminates all terms in the sum except the first one, and

$$Y_a(j\Omega) = \begin{cases} H(e^{j\Omega T_s})X_a(j\Omega) & |\Omega| \le \dfrac{\pi}{T_s} \\ 0 & |\Omega| > \dfrac{\pi}{T_s} \end{cases}$$

Therefore, the overall system behaves as a linear time-invariant continuous-time system with an effective frequency response

$$H_a(j\Omega) = \begin{cases} H(e^{j\Omega T_s}) & |\Omega| \le \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases} \tag{3.9}$$

Just as a continuous-time system may be implemented in terms of a discrete-time system, it is also possible to implement a discrete-time system in terms of a continuous-time system as illustrated Fig. 3-10. The signal $x_a(t)$ is related to the sequence values $x(n)$ as follows:

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin \pi(t - nT_s)/T_s}{\pi(t - nT_s)/T_s}$$

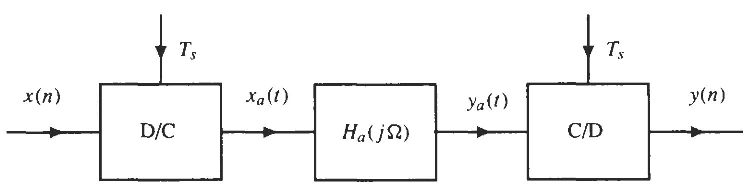


Fig. 3-10. Processing a discrete-time signal using a continuous-time system.

Because $x_a(t)$ is bandlimited, $y_a(t)$ is also bandlimited and may be represented in terms of its samples as follows:

$$y_a(t) = \sum_{n=-\infty}^{\infty} y(n) \frac{\sin \pi(t - nT_s)/T_s}{\pi(t - nT_s)/T_s}$$

The relationship between the Fourier transform of $x_a(t)$ and the DTFT of $x(n)$ is

$$X_a(j\Omega) = \begin{cases} T_s X(e^{j\Omega T_s}) & |\Omega| < \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

and the relationship between the Fourier transforms of $x_a(t)$ and $y_a(t)$ is

$$Y_a(j\Omega) = \begin{cases} H_a(j\Omega)X_a(j\Omega) & |\Omega| < \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

Therefore,
$$Y(e^{j\omega}) = \frac{1}{T_s} Y_a\left(\frac{j\omega}{T_s}\right) \qquad |\omega| < \pi$$

and the frequency response of the equivalent discrete-time system is

$$H(e^{j\omega}) = H_a\left(\frac{j\omega}{T_s}\right) \qquad |\omega| < \pi \tag{3.10}$$

3.5 SAMPLE RATE CONVERSION

In many practical applications of digital signal processing, one is faced with the problem of changing the sampling rate of a signal. The process of converting a signal from one rate to another is called *sample rate conversion.* There are two ways that sample rate conversion may be done. First, the sampled signal may be converted back into an analog signal and then resampled. Alternatively, the signal may be *resampled* in the digital domain. This approach has the advantage of not introducing additional distortion in passing the signal through an additional D/A and A/D converter. In this section, we describe how sample rate conversion may be performed digitally.

3.5.1 *Sample Rate Reduction by an Integer Factor*

Suppose that we would like to reduce the sampling rate by an integer factor, M. With a new sampling period $T_s' = MT_s$, the resampled signal is

$$x_d(n) = x_a(nT_s') = x_a(nMT_s) = x(nM)$$

Therefore, reducing the sampling rate by an integer factor M may be accomplished by taking every Mth sample of $x(n)$. The system for performing this operation, called a *down-sampler*, is shown in Fig. 3-11(a). Down-sampling generally results in aliasing. Specifically, recall that the DTFT of $x(n) = x_a(nT_s)$ is

$$X(e^{j\omega}) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a\left(j\frac{\omega}{T_s} - j\frac{2\pi k}{T_s}\right)$$

Similarly, the DTFT of $x_d(n) = x(nM) = x_a(nMT_s)$ is

$$X_d(e^{j\omega}) = \frac{1}{MT_s} \sum_{r=-\infty}^{\infty} X_a\left(j\frac{\omega}{MT_s} - j\frac{2\pi r}{MT_s}\right)$$

Note that the summation index r in the expression for $X_d(e^{j\omega})$ may be expressed as

$$r = i + kM$$

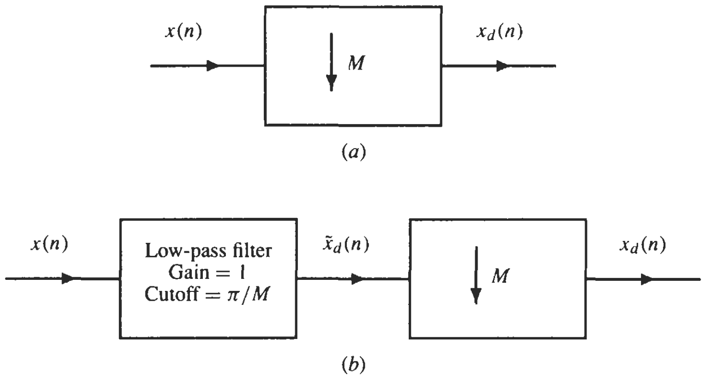


Fig. 3-11. (*a*) Down-sampling by an integer factor M. (*b*) Decimation by a factor of M, where $H(e^{j\omega})$ is a low-pass filter with a cutoff frequency $\omega_c = \pi/M$.

where $-\infty < k < \infty$ and $0 \le i \le M-1$. Therefore, $X_d(e^{j\omega})$ may be expressed as

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{i=0}^{M-1}\left[\frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(j\frac{\omega}{MT_s} - j\frac{2\pi k}{T_s} - j\frac{2\pi i}{MT_s}\right)\right]$$

The term inside the square brackets is

$$X\left(e^{j(\omega-2\pi i)/M}\right) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a\left(j\frac{(\omega-2\pi i)}{MT_s} - j\frac{2\pi k}{T_s}\right)$$

Thus, the relationship between $X(e^{j\omega})$ and $X_d(e^{j\omega})$ is

$$X_d(e^{j\omega}) = \frac{1}{M}\sum_{k=0}^{M-1} X\left(e^{j(\omega-2\pi k)/M}\right) \tag{3.11}$$

Therefore, in order to prevent aliasing, $x(n)$ should be filtered prior to down-sampling with a low-pass filter that has a cutoff frequency $\omega_c = \pi/M$. The cascade of a low-pass filter with a down-sampler illustrated in Fig. 3-11(*b*) is called a *decimator*.

3.5.2 Sample Rate Increase by an Integer Factor

Suppose that we would like to increase the sampling rate by an integer factor L. If $x_a(t)$ is sampled with a sampling frequency $f_s = 1/T_s$, then

$$x(n) = x_a(nT_s)$$

To increase the sampling rate by an integer factor L, it is necessary to extract the samples

$$x_i(n) = x_a\left(\frac{nT_s}{L}\right)$$

from $x(n)$. The samples of $x_i(n)$ for values of n that are integer multiples of L are easily extracted from $x(n)$ as follows:

$$x_i(nL) = x(n)$$

Shown in Fig. 3-12(a) is an *up-sampler* that produces the sequence

$$\tilde{x}_i(n) = \begin{cases} x(n/L) & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{otherwise} \end{cases}$$

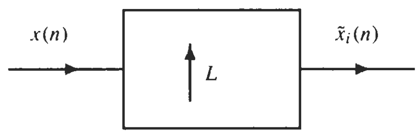
In other words, the up-sampler expands the time scale by a factor of L by inserting $L-1$ zeros between each sample of $x(n)$. In the frequency domain, the up-sampler is described by

$$\tilde{X}_i(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \tilde{x}_i(n) e^{-jn\omega} = \sum_{n=-\infty}^{\infty} x(n) e^{-jnL\omega}$$

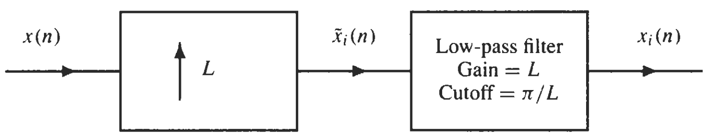
or

$$\tilde{X}_i(e^{j\omega}) = X(e^{jL\omega}) \tag{3.12}$$

Therefore, $X(e^{j\omega})$ is simply scaled in frequency. After up-sampling, it is necessary to remove the frequency scaled *images* of $X_a(j\Omega)$, except those that are at integer multiples of 2π. This is accomplished by filtering $\tilde{x}_i(n)$



(a)



(b)

Fig. 3-12. (a) Up-sampling by an integer factor L. (b) Interpolation by a factor of L.

with a low-pass filter that has a cutoff frequency of π/L and a gain of L. In the time domain, the low-pass filter interpolates between the samples at integer multiples of L as shown in Fig. 3-13. The cascade of an up-sampler with a low-pass filter shown in Fig. 3-12(b) is called an *interpolator*. The interpolation process in the frequency domain is illustrated in Fig. 3-14.

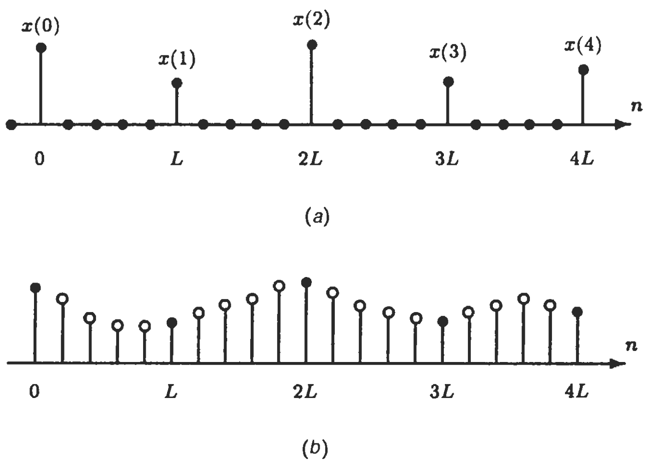


Fig. 3-13. (a) The output of the up-sampler. (b) The interpolation between the samples $\tilde{x}_i(n)$ that is performed by the low-pass filter.

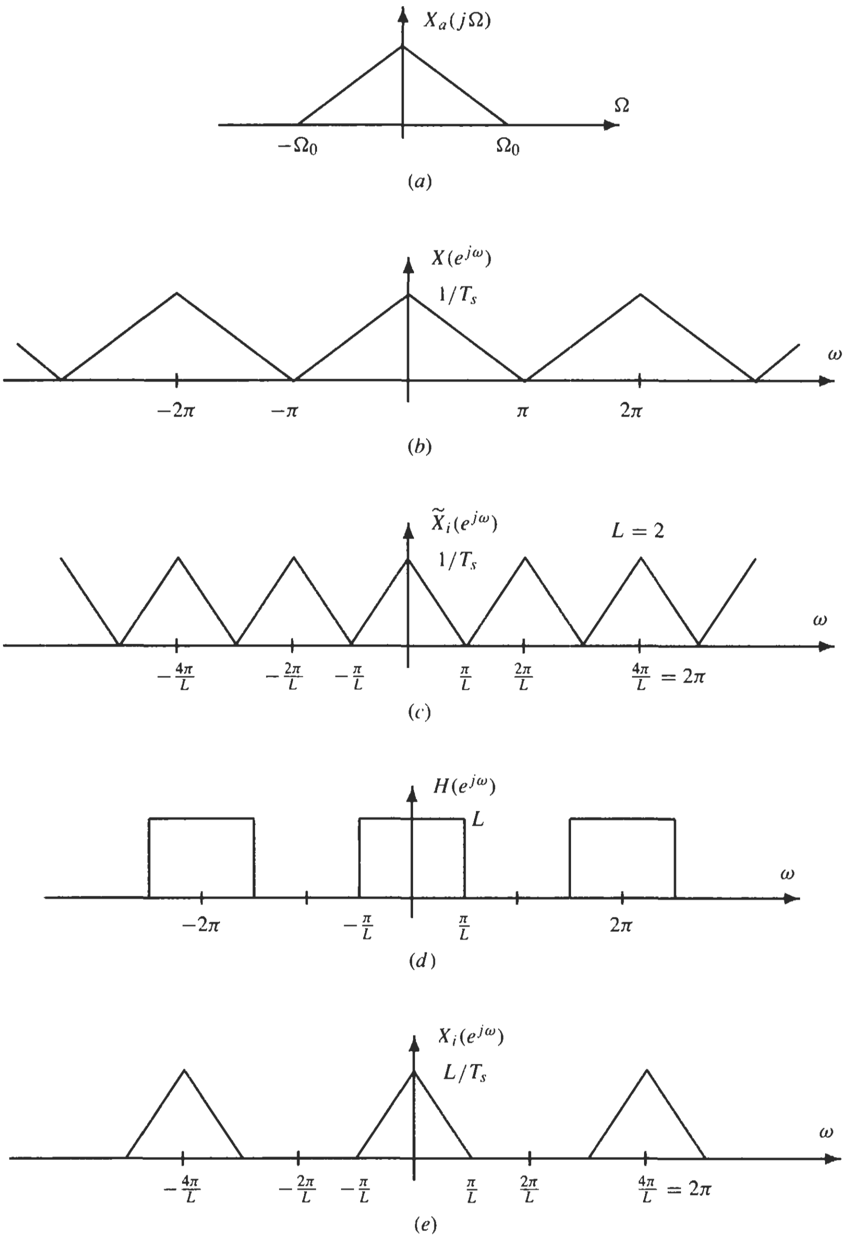


Fig. 3-14. Frequency domain illustration of the process of interpolation. (*a*) The continuous-time signal. (*b*) The DTFT of the sampled signal $x(n) = x_a(nT_s)$. (*c*) The DTFT of the up-sampler output. (*d*) The ideal low-pass filter to perform the interpolation. (*e*) The DTFT of the interpolated signal.

3.5.3 Sample Rate Conversion by a Rational Factor

The cascade of a decimator that reduces the sampling rate by a factor of M with an interpolator that increases the sampling rate by vital factor of L results in a system that changes the sampling rate by a rational factor of L/M. This cascade is illustrated in Fig. 3-15(*a*). Because the cascade of two low-pass filters with cutoff frequencies π/M and π/L is equivalent to a single low-pass filter with a cutoff frequency

$$\omega_c = \min\left\{\frac{\pi}{M}, \frac{\pi}{L}\right\}$$

the sample rate converter may be simplified as illustrated in Fig. 3-15(*b*).

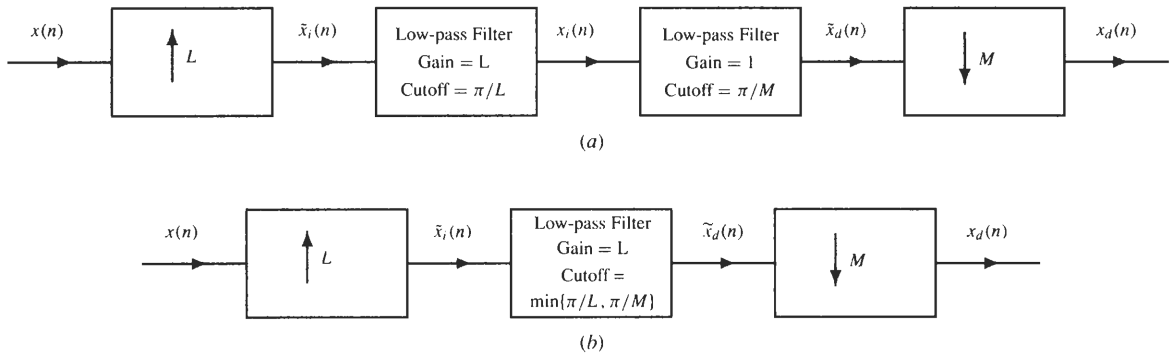


Fig. 3-15. (*a*) Cascade of an interpolator and a decimator for changing the sampling rate by a rational factor L/M. (*b*) A simplified structure that results when the two low-pass filters are combined.

EXAMPLE 3.5.1 Suppose that a signal $x_a(t)$ has been sampled with a sampling frequency of 8 kHz and that we would like to derive the discrete-time signal that would have been obtained if $x_a(t)$ had been sampled with a sampling frequency of 10 kHz. Thus, we would like to change the sampling rate by a factor of

$$\frac{L}{M} = \frac{10}{8} = \frac{5}{4}$$

This may be accomplished by up-sampling $x(n)$ by a factor of 5, filtering the up-sampled signal with a low-pass filter that has a cutoff frequency $\omega_c = \pi/5$ and a gain of 5, and then down-sampling the filtered signal by a factor of 4.

Solved Problems

A/D and D/A Conversion

3.1 Consider the discrete-time sequence

$$x(n) = \cos\left(\frac{n\pi}{8}\right)$$

Find two different continuous-time signals that would produce this sequence when sampled at a frequency of $f_s = 10$ Hz.

A continuous-time sinusoid

$$x_a(t) = \cos(\Omega_0 t) = \cos(2\pi f_0 t)$$

that is sampled with a sampling frequency of f_s results in the discrete-time sequence

$$x(n) = x_a(nT_s) = \cos\left(2\pi \frac{f_0}{f_s} n\right)$$

However, note that for any integer k,

$$\cos\left(2\pi \frac{f_0}{f_s} n\right) = \cos\left(2\pi \frac{f_0 + kf_s}{f_s} n\right)$$

Therefore, any sinusoid with a frequency

$$f = f_0 + kf_s$$

will produce the same sequence when sampled with a sampling frequency f_s. With $x(n) = \cos(n\pi/8)$, we want

$$2\pi \frac{f_0}{f_s} = \frac{\pi}{8}$$

or

$$f_0 = \tfrac{1}{16} f_s = 625 \text{ Hz}$$

Therefore, two signals that produce the given sequence are

$$x_1(t) = \cos(1250\pi t)$$

and

$$x_2(t) = \cos(21250\pi t)$$

3.2 If the Nyquist rate for $x_a(t)$ is Ω_s, what is the Nyquist rate for each of the following signals that are derived from $x_a(t)$?

(*a*) $\dfrac{dx_a(t)}{dt}$

(*b*) $x_a(2t)$

(*c*) $x_a^2(t)$

(*d*) $x_a(t)\cos(\Omega_0 t)$

(*a*) The Nyquist rate is equal to twice the highest frequency in $x_a(t)$. If

$$y_a(t) = \frac{dx_a(t)}{dt}$$

then

$$Y_a(j\Omega) = j\Omega X_a(j\Omega)$$

Thus, if $X_a(j\Omega) = 0$ for $|\Omega| > \Omega_0$, the same will be true for $Y_a(j\Omega)$. Therefore, the Nyquist frequency is not changed by differentiation.

(*b*) The signal $y_a(t) = x_a(2t)$ is formed from $x_a(t)$ by *compressing* the time axis by a factor of 2. This results in an *expansion* of the frequency axis by a factor of 2. Specifically, note that

$$\begin{aligned} Y_a(j\Omega) &= \int_{-\infty}^{\infty} y_a(t) e^{-j\Omega t}\,dt = \int_{-\infty}^{\infty} x_a(2t) e^{-j\Omega t}\,dt \\ &= \int_{-\infty}^{\infty} \tfrac{1}{2} x_a(\tau) e^{-j\Omega\tau/2}\,d\tau = \tfrac{1}{2} X_a\left(\frac{j\Omega}{2}\right) \end{aligned}$$

Consequently, if the Nyquist frequency for $x_a(t)$ is Ω_s, the Nyquist frequency for $y_a(t)$ will be $2\Omega_s$.

(*c*) When two signals are multiplied, their Fourier transforms are convolved. Therefore, if

$$y_a(t) = x_a^2(t)$$

then

$$Y_a(j\Omega) = \frac{1}{2\pi} X_a(j\Omega) * X_a(j\Omega)$$

Thus, the highest frequency in $y_a(t)$ will be twice that of $x_a(t)$, and the Nyquist frequency will be $2\Omega_s$.

(*d*) Modulating a signal by $\cos(\Omega_0 t)$ shifts the spectrum of $x_a(t)$ up and down by Ω_0. Therefore, the Nyquist frequency for $y_a(t) = \cos(\Omega_0 t) x_a(t)$ will be $\Omega_s + 2\Omega_0$.

3.3 Let $h_a(t)$ be the impulse response of a causal continuous-time filter with a system function

$$H_a(s) = \frac{s+a}{(s+a)^2+b^2}$$

Thus, $H_a(s)$ has a zero at $s = -a$ and a pair of poles at $s = -a \pm jb$. By sampling $h_a(t)$ we form a discrete-time filter with a unit sample response

$$h(n) = h_a(nT_s)$$

Find the frequency response $H(e^{j\omega})$ of the discrete-time filter.

To find the frequency response $H(e^{j\omega})$, it is necessary to find the impulse response of the analog filter, $h_a(t)$, sample the impulse response,

$$h(n) = h_a(nT_s)$$

and then find the discrete-time Fourier transform,

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)e^{-jn\omega}$$

To find the impulse response, we first perform a partial fraction expansion of $H_a(s)$ as follows:

$$H_a(s) = \frac{A}{s+(a+jb)} + \frac{B}{s+(a-jb)} \tag{3.13}$$

The constant A is

$$A = [(s+a+jb)H_a(s)]_{s=-a-jb} = \left.\frac{s+a}{s+(a-jb)}\right|_{s=-a-jb} = \tfrac{1}{2}$$

Similarly, for B we have

$$B = [(s+a-jb)H_a(s)]_{s=-a+jb} = \left.\frac{s+a}{s+(a+jb)}\right|_{s=-a+jb} = \tfrac{1}{2}$$

Therefore,

$$H_a(s) = \frac{\frac{1}{2}}{s+(a+jb)} + \frac{\frac{1}{2}}{s+(a-jb)}$$

Another way to find the constants A and B would be to write Eq. (3.13) over a common denominator,

$$H_a(s) = \frac{s+a}{(s+a)^2+b^2} = \frac{A(s+a-jb)+B(s+a+jb)}{(s+a)^2+b^2}$$

and equate the polynomial coefficients in the numerators of $H_a(s)$:

$$A+B = 1$$
$$A(a-jb)+B(a+jb) = a$$

Solving these two equations for A and B gives the same result as before. From the partial fraction expansion of $H_a(s)$, the impulse response may be found using the Laplace transform pair

$$e^{-\alpha t}u(t) \Leftrightarrow \frac{1}{s+\alpha}$$

Specifically, we have

$$h_a(t) = \tfrac{1}{2}e^{(-a-jb)t}u(t) + \tfrac{1}{2}e^{(-a+jb)t}u(t) = e^{-at}\cos(bt)u(t)$$

Sampling $h_a(t)$, we have

$$h(n) = h_a(nT_s) = e^{-anT_s}\cos(bnT_s)u(n)$$

Finally, for the frequency response we have

$$\begin{aligned}H(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} h(n)e^{-jn\omega} = \sum_{n=0}^{\infty} e^{-anT_s}\cos(bnT_s)e^{-jn\omega}\\ &= \sum_{n=0}^{\infty} \tfrac{1}{2}e^{(-a-jb)nT_s}e^{-jn\omega} + \sum_{n=0}^{\infty} \tfrac{1}{2}e^{(-a+jb)nT_s}e^{-jn\omega}\\ &= \sum_{n=0}^{\infty} \tfrac{1}{2}(e^{-aT_s})^n e^{-jn(\omega+bT_s)} + \sum_{n=0}^{\infty} \tfrac{1}{2}(e^{-aT_s})^n e^{-jn(\omega-bT_s)}\end{aligned}$$

Note that in order for these sums to converge, and for the frequency response to exist, it is necessary that

$$|e^{-aT_s}| < 1$$

or, because $T_s > 0$, we must have $a > 0$. In other words, the poles of $H_a(s)$ must lie in the left-half s-plane or, equivalently, $h_a(t)$ must be a stable filter. With $a > 0$ we have

$$H(e^{j\omega}) = \frac{\frac{1}{2}}{1 - e^{(-a-jb)T_s}e^{-j\omega}} + \frac{\frac{1}{2}}{1 - e^{(-a+jb)T_s}e^{-j\omega}}$$

which, after combining over a common denominator and simplifying, gives

$$H(e^{j\omega}) = \frac{1 - e^{-aT_s}\cos(bT_s)e^{-j\omega}}{1 - 2e^{-aT_s}\cos(bT_s)e^{-j\omega} + e^{-2aT_s}e^{-j2\omega}}$$

3.4 A continuous-time filter has a system function

$$H_a(s) = \frac{1}{s+1}$$

If $h_a(t)$ is sampled to form a discrete-time system with a unit sample response

$$h(n) = h_a(nT_s)$$

find the value for T_s so that $H(e^{j\omega})$ at $\omega = \pi/2$ is down 6 dB from its maximum value at $\omega = 0$, that is,

$$10\log\frac{|H(e^{j\pi/2})|^2}{|H(e^{j0})|^2} = -6$$

The impulse response of the continuous-time system is

$$h_a(t) = e^{-t}u(t)$$

When sampled with a sampling period T_s, the resulting unit sample response is

$$h(n) = h_a(nT_s) = e^{-nT_s}u(n)$$

and the frequency response is

$$H(e^{j\omega}) = \sum_{n=0}^{\infty} e^{-nT_s}e^{-jn\omega} = \sum_{n=0}^{\infty} e^{-(T_s+j\omega)n} = \frac{1}{1 - e^{-T_s}e^{-j\omega}}$$

With

$$|H(e^{j0})|^2 = \frac{1}{(1-e^{-T_s})^2}$$

and

$$|H(e^{j\pi/2})|^2 = \frac{1}{1+e^{-2T_s}}$$

it follows that we want

$$10 \log \frac{|H(e^{j\pi/2})|^2}{|H(e^{j0})|^2} = 10 \log \frac{(1-e^{-T_s})^2}{1+e^{-2T_s}} = -6$$

or

$$\frac{(1-e^{-T_s})^2}{1+e^{-2T_s}} = 10^{-0.6} = 0.2512$$

Thus, we have

$$1 - 2e^{-T_s} + e^{-2T_s} = 0.2512\,[1 + e^{-2T_s}]$$

or

$$0.7488e^{-2T_s} - 2e^{-T_s} + 0.7488 = 0$$

which is a quadratic equation in e^{-T_s}. Solving for the roots of this quadratic equation, we find

$$e^{-T_s} = \frac{1}{2(0.7488)}\left[2 \pm \sqrt{4 - 4(0.7488)^2}\right] = \frac{1}{0.7488}[1 \pm 0.6628] = 2.2206,\ 0.4503$$

Taking the natural logarithm, and selecting the positive value for T_s, we have

$$T_s = 0.7978$$

3.5 A continuous-time signal $x_a(t)$ is bandlimited with $X_a(j\Omega) = 0$ for $|\Omega| > \Omega_0$. If $x_a(t)$ is sampled with a sampling frequency $\Omega_s \geq 2\Omega_0$, how is the energy in $x(n)$,

$$E_d = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

related to the energy in $x_a(t)$,

$$E_a = \int_{-\infty}^{\infty} |x_a(t)|^2\, dt$$

and the sampling period T_s?

Using Parseval's theorem, the energy in the analog signal $x_a(t)$ may be expressed in the frequency domain as follows:

$$E_a = \int_{-\infty}^{\infty} |x_a(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X_a(j\Omega)|^2\, d\Omega$$

Because $x_a(t)$ is bandlimited with $X_a(j\Omega) = 0$ for $|\Omega| > \Omega_0$,

$$E_a = \frac{1}{2\pi}\int_{-\Omega_0}^{\Omega_0} |X_a(j\Omega)|^2\, d\Omega$$

Sampling $x_a(t)$ at or above the Nyquist rate results in a sequence $x(n)$ with a discrete-time Fourier transform

$$X(e^{j\omega}) = \begin{cases} \dfrac{1}{T_s} X_a\!\left(\dfrac{j\omega}{T_s}\right) & |\omega| \leq \Omega_0 T_s \\ 0 & \Omega_0 T_s < |\omega| \leq \pi \end{cases}$$

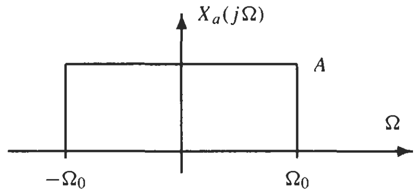
Therefore, the energy in $x(n)$, using Parseval's theorem, is

$$E_d = \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2\, d\omega$$
$$= \frac{1}{2\pi}\int_{-\Omega_0 T_s}^{\Omega_0 T_s} \frac{1}{T_s^2}\left|X_a\left(\frac{j\omega}{T_s}\right)\right|^2 d\omega$$
$$= \frac{1}{2\pi T_s}\int_{-\Omega_0}^{\Omega_0} |X_a(ju)|^2\, du = \frac{1}{T_s}E_a$$

and we have

$$E_d = \frac{1}{T_s}E_a$$

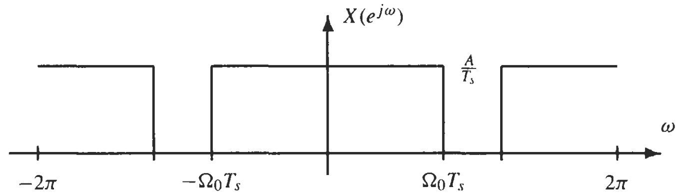
As a check on this result, suppose that $x_a(t)$ is a bandlimited signal with a spectrum shown in the figure below.



The energy in $x_a(t)$ is

$$E_a = \frac{1}{2\pi}A^2 \cdot 2\Omega_0 = \frac{A^2\Omega_0}{\pi}$$

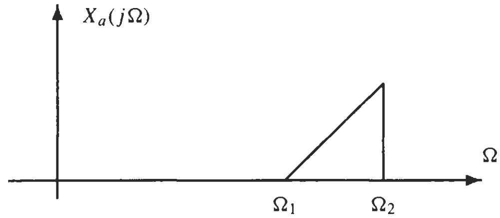
When sampled with a sampling frequency $\Omega_s \geq 2\Omega_0$, the DTFT of the sampled signal is as shown in the following figure:



Therefore, the energy in $x(n)$ is

$$E_d = \frac{1}{2\pi}\left(\frac{A}{T_s}\right)^2 \cdot 2\,\Omega_0 T_s = \frac{A^2\Omega_0}{\pi T_s} = \frac{1}{T_s}E_a$$

3.6 A complex bandpass analog signal $x_a(t)$ has a Fourier transform that is nonzero over the frequency range $[\Omega_1, \Omega_2]$ as shown in the figure below.



The signal is sampled to produce the sequence $x(n) = x_a(nT_s)$.

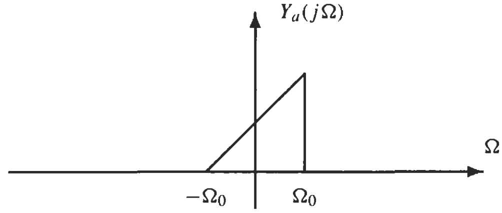
(*a*) What is the smallest sampling frequency that can be used so that $x_a(t)$ may be recovered from its samples $x(n)$?

(*b*) For this minimum sampling frequency, find the interpolation formula for $x_a(t)$ in terms of $x(n)$.

(*a*) Because the highest frequency in $x_a(t)$ is Ω_2, the Nyquist rate is $2\Omega_2$. However, note that if $x_a(t)$ is modulated with a complex exponential of frequency $(\Omega_2 + \Omega_1)/2$,

$$y_a(t) = x_a(t)e^{-j(\Omega_2+\Omega_1)t/2}$$

then $y_a(t)$ is a (complex) low-pass signal with a spectrum shown in the following figure:



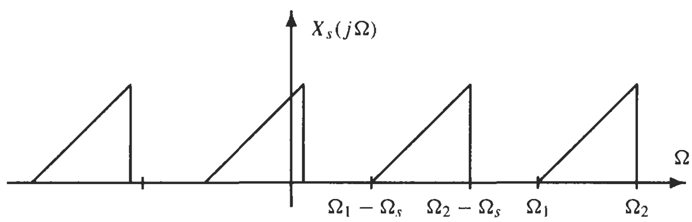
where $\Omega_0 = (\Omega_2 - \Omega_1)/2$. Thus, the Nyquist rate for $y_a(t)$ is $2\Omega_0 = \Omega_2 - \Omega_1$, which suggests that $x_a(t)$ may be uniquely reconstructed from its samples $x_a(nT_s)$ provided that

$$T_s \leq \frac{\pi}{\Omega_2 - \Omega_1}$$

If $x_a(t)$ is sampled with a sampling frequency Ω_s, the spectrum of the sampled signal is

$$X_s(j\Omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(j\Omega - jk\Omega_s)$$

as illustrated below.



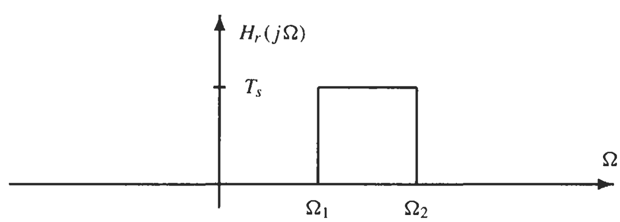
In order for there to be no interference between the shifted spectra, it is necessary that

$$\Omega_2 - \Omega_s \leq \Omega_1$$

or

$$\Omega_s \geq \Omega_2 - \Omega_1$$

If this condition is satisfied, $x_a(t)$ may be uniquely reconstructed from $x_s(t)$ using a bandpass filter with a frequency response as shown below.



(*b*) With a sampling frequency $\Omega_s = \Omega_2 - \Omega_1$, the reconstruction filter is a complex bandpass filter with an impulse response

$$h_a(t) = T_s\frac{\sin(\Omega_s t/2)}{\pi t}e^{-j(\Omega_2+\Omega_1)t/2}$$

Therefore, the output of the reconstruction filter, which produces the complex bandpass signal $x_a(t)$, is

$$x_a(t) = \sum_{n=-\infty}^{\infty} x(n)h_r(t - nT_s) = T_s \sum_{n=-\infty}^{\infty} x(n)\frac{\sin \Omega_s(t - nT_s)/2}{\pi(t - nT_s)} e^{-j(\Omega_2+\Omega_1)(t-nT_s)/2}$$

3.7 Given a real-valued bandpass signal $x_a(t)$ with $X_a(f) = 0$ for $|f| < f_1$ and $|f| > f_2$, the Nyquist sampling theorem says that the minimum sampling frequency is $f_s = 2f_2$. However, in some cases, the signal may be sampled at a lower rate.

(a) Suppose that $f_1 = 8$ kHz and $f_2 = 10$ kHz. Make a sketch of the discrete-time Fourier transform of $x(n) = x_a(nT_s)$ if $f_s = 1/T_s = 4$ kHz.

(b) Define the bandwidth of the bandpass signal to be

$$B = f_2 - f_1$$

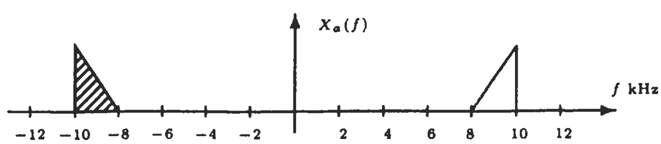
and the center frequency to be

$$f_c = \frac{f_2 + f_1}{2}$$

Show that if $f_c > B/2$ and f_2 is an integer multiple of the bandwidth B, no aliasing will occur if $x_a(t)$ is sampled at a sampling frequency $f_s = 2B$.

(c) Repeat part (b) for the case in which f_2 is not an integer multiple of the bandwidth B.

(a) Let $x_a(t)$ have a spectrum as shown in the figure below.



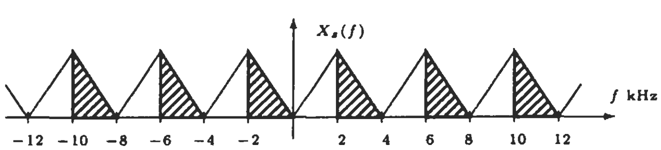
The spectrum of the sampled signal

$$x_s(t) = \sum_{n=-\infty}^{\infty} x_a(nT_s)\delta(t - nT_s)$$

is

$$X_s(f) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X_a(f - kf_s)$$

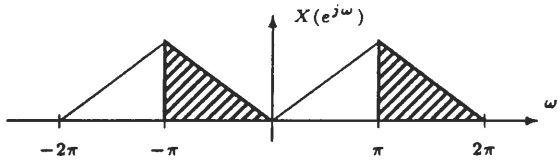
which is formed by shifting $X_a(f)$ by integer multiples of the sampling frequency and summing. With $f_s =$ 4 kHz, we have the spectrum sketched below.



Note that $X_a(f)$ is not aliased. Therefore, with the appropriate processing of $x_s(t)$, the signal $x_a(t)$ may be recovered from its samples. Finally, the DTFT of the discrete-time sequence $x(n) = x_a(nT_s)$ is

$$X(e^{j\omega}) = X_s\left(\frac{j\omega}{T_s}\right)$$

which is sketched below.



(*b*) If f_2 is an integer multiple of B, we may express f_1 and f_2 as follows:

$$f_1 = (l-1)B \qquad f_2 = lB$$

With a sampling frequency of $f_s = 2B$, the sampled signal has a spectrum

$$X_s(f) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X_a(f - 2kB)$$

Because $X_a(f)$ is nonzero only for $(l-1)B < |f| < lB$, there is only one term in the sum that contributes to $X_s(f)$ in the frequency range $0<f<B$ and only one term that contributes to the frequency range $-B < f < 0$ (draw a picture as in part (*a*) to see this clearly). Therefore, there is no aliasing, and $x_a(t)$ may be sampled without aliasing if a sampling frequency $f_s = 2B$.

(*c*) If f_2 is not an integer multiple of B, we may always increase B until this is the case. Specifically, let

$$k = \left\lfloor \frac{f_2}{B} \right\rfloor$$

where $\lfloor \cdot \rfloor$ is defined to be the "integer part." Now, if we simply increase B to B' where

$$k = \frac{f_2}{B'}$$

we have the case described in part (*b*) where f_2 is an integer multiple of the bandwidth. Thus, $x_a(t)$ may be sampled without aliasing a sampling frequency of

$$f_s = 2B' = \frac{2f_2}{\lfloor f_2/B \rfloor}$$

3.8 Determine the minimum sampling frequency for each of the following bandpass signals:

(*a*) $x_a(t)$ is real with $X_a(f)$ nonzero only for 9 kHz $< |f| <$ 12 kHz.

(*b*) $x_a(t)$ is real with $X_a(f)$ nonzero only for 18 kHz $< |f| <$ 22 kHz.

(*c*) $x_a(t)$ is complex with $X_a(f)$ nonzero only for 30 kHz $< f <$ 35 kHz.

(*a*) For this signal, the bandwidth is $B = f_2 - f_1 = 3$ kHz, and $f_2 = 12 = 4B$ is an integer multiple of B. Therefore, the minimum sampling frequency is $f_s = 2B = 6$ kHz.

(*b*) For this signal, $B = 4$ kHz and $f_2 = 22$, which is not an integer multiple of B. With $\lfloor f_2/B \rfloor = 5$, if we let $B' = f_2/5 = 4.4$, f_2 is an integer multiple of B', and $x_a(t)$ may be sampled with a sampling frequency of $f_s = 2B' = 8.8$ kHz.

(*c*) For a complex bandpass signal with a spectrum that is nonzero for $f_1 < f < f_2$, the minimum sampling frequency is $f_s = f_2 - f_1$. Thus, for this signal, $f_s = 5$ kHz.

3.9 How many bits are needed in an A/D converter if we want a signal-to-quantization noise ratio of at least 90 dB? Assume that $x_a(t)$ is gaussian with a variance σ_x^2, and that the range of the quantizer extends from $-3\sigma_x$ to $3\sigma_x$; that is, $X_{\max} = 3\sigma_x$ (with this value for $X_{\max}$, only about one out of every 1000 samples will exceed the quantizer range).

For a $(B+1)$-bit quantizer, the signal-to-quantization noise ratio is

$$\text{SQNR} = 6.02B + 10.81 - 20\log\frac{X_{\max}}{\sigma_x}$$

With $X_{\max} = 3\sigma_x$ this becomes

$$\text{SQNR} = 6.02B + 10.81 - 20\log 3 = 6.02B + 10.81 - 9.54 = 6.02B + 1.27$$

If we want a signal-to-quantization noise ratio of 90 dB, we require

$$B = \frac{90 - 1.27}{6.02} = 14.74$$

or $B + 1 = 16$ bits.

3.10 An image is to be sampled with a signal-to-quantization noise ratio of at least 80 dB. Unlike many other signals, the image samples are nonnegative. Assume that the sampling device is calibrated so that the sampled image intensities fall within the range from 0 to 1. How many bits are needed to achieve the desired signal-to-quantization noise ratio?

For a bipolar signal with amplitudes that fall within the range $[-X_{\max}, X_{\max}]$, the signal-to-quantization noise ratio is

$$\text{SQNR} = 6.02B + 10.81 - 20\log\frac{X_{\max}}{\sigma_x}$$

For a nonnegative signal that is confined to the interval [0, 1], the signal-to-quantization noise ratio is equivalent to the bipolar case if we set $X_{\max} = 0.5$. If we assume that the intensities of the image are uniformly distributed over the interval [0, 1],

$$\sigma_x^2 = \tfrac{1}{12}$$

Therefore,
$$\text{SQNR} = 6.02B + 10.81 - 20\log\frac{\sqrt{12}}{2} = 6.02B + 6.03$$

and for a signal-to-quantization noise ratio of 80 dB, we require

$$B = \frac{80 - 6.03}{6.02} = 12.29$$

or $B + 1 = 14$ bits.

3.11 Suppose that we have a set of unquantized samples, $x(n)$, that are nonnegative for all n. A method for quantizing $x(n)$ that is often used in speech processing is as follows. First, we form the sequence

$$y(n) = \log[x(n)]$$

Then $y(n)$ is quantized with a $(B+1)$-bit uniform quantizer,

$$\hat{y}(n) = Q[y(n)] = y(n) + e(n)$$

The quantized signal samples are then obtained by exponentiating $\hat{y}(n)$,

$$\hat{x}(n) = \exp\{\hat{y}(n)\}$$

Show that if $e(n)$ is small, the signal-to-quantization noise ratio is independent of the signal power.

With
$$\hat{y}(n) = Q[y(n)] = y(n) + e(n) = \log[x(n)] + e(n)$$

we have, for $\hat{x}(n)$,

$$\hat{x}(n) = \exp\{\log[x(n)] + e(n)\} = x(n) \cdot \exp\{e(n)\}$$

If $e(n) \ll 1$, we may use the expansion

$$\exp\{e(n)\} \approx 1 + e(n)$$

to write

$$\hat{x}(n) = x(n)[1 + e(n)] = x(n) + f(n)$$

where $f(n) = x(n)e(n)$ is a (signal-dependent) quantization noise. If we assume that the quantization noise $e(n)$ is statistically independent of $x(n)$,

$$E\{f^2(n)\} = E\{x^2(n)\} \cdot E\{e^2(n)\}$$

and the signal-to-quantization noise ratio is

$$\text{SQNR} = 10 \log \frac{E\{x^2(n)\}}{E\{f^2(n)\}} = -10 \log E\{e^2(n)\}$$

which is independent of the signal power.

Discrete-Time Processing of Analog Signals

3.12 A continuous-time signal $x_a(t)$ is to be filtered to remove frequency components in the range $5\text{ kHz} \le f \le 10\text{ kHz}$. The maximum frequency present in $x_a(t)$ is 20 kHz. The filtering is to be done by sampling $x_a(t)$, filtering the sampled signal, and reconstructing an analog signal using an ideal D/C converter. Find the minimum sampling frequency that may be used to avoid aliasing, and for this minimum sampling rate, find the frequency response of the ideal digital filter $H(e^{j\omega})$ that will remove the desired frequencies from $x_a(t)$.

Because the highest frequency in $x_a(t)$ is 20 kHz, the minimum sampling frequency to avoid aliasing is $f_s = 40$ kHz. The relationship between the continuous frequency variable Ω and the discrete frequency variable ω is given by

$$\omega = \Omega T_s$$

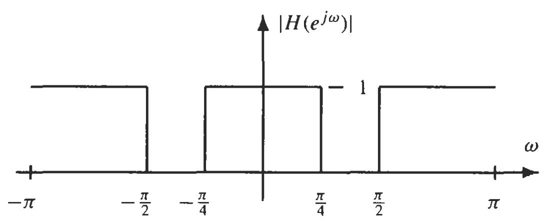
or

$$\omega = 2\pi \frac{f}{f_s}$$

Therefore, the frequency range $5\text{ kHz} \le f \le 10\text{ kHz}$ corresponds to a digital frequency range

$$\frac{\pi}{4} \le \omega \le \frac{\pi}{2}$$

and the desired digital filter is a bandstop filter that has a frequency response as illustrated in the figure below.



3.13 A major problem in the recording of electrocardiograms (ECGs) is the appearance of unwanted 60-Hz interference in the output. The causes of this power line interference include magnetic induction, displacement currents in the leads on the body of the patient, and equipment interconnections. Assume that the bandwidth of the signal of interest is 1 kHz, that is,

$$X_a(f) = 0 \qquad |f| > 1000\text{ Hz}$$

The analog signal is converted into a discrete-time signal with an ideal A/D converter operating using a sampling frequency f_s. The resulting signal $x(n) = x_a(nT_S)$ is then processed with a discrete-time system that is described by the difference equation

$$y(n) = x(n) + ax(n-1) + bx(n-2)$$

The filtered signal, $y(n)$, is then converted back into an analog signal using an ideal D/A converter. Design a system for removing the 60-Hz interference by specifying values for f_s, a, and b so that a 60-Hz signal of the form

$$w_a(t) = A\sin(120\pi t)$$

will not appear in the output of the D/A converter.

The signal that is to have the 60-Hz noise removed is bandlimited to 1000 Hz. Therefore, in order to avoid aliasing when the signal is sampled, we require a sampling frequency

$$f_s \geq 2000$$

Using the minimum rate of 2000 Hz, note that a 60-Hz signal $w_a(t) = \sin(120\pi t)$ becomes

$$w(n) = w_a(nT_s) = \sin\left(\frac{120\pi n}{2000}\right) = \sin(n\omega_0)$$

where $\omega_0 = 0.06\pi$. Recall that complex exponentials are eigenfunctions of linear shift-invariant systems. Therefore, if the input to an LSI system is $x(n) = e^{jn\omega_0}$, the output is

$$y(n) = H(e^{j\omega_0})e^{jn\omega_0}$$

Because

$$w(n) = \frac{e^{jn\omega_0} - e^{-jn\omega_0}}{2j}$$

$w(n)$ will be removed from $x(n)$ if we design a filter so that $H(e^{j\omega})$ is equal to zero at $\omega = \pm\omega_0$. Because $H(e^{j\omega})$ is a second-order filter with a frequency response

$$H(e^{j\omega}) = 1 + ae^{-j\omega} + be^{-j2\omega}$$

it may be factored as follows:

$$H(e^{j\omega}) = (1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})$$

Therefore, $H(e^{j\omega})$ will be zero for $\omega = \pm\omega_0$ if $\alpha = e^{j\omega_0}$ and $\beta = e^{-j\omega_0}$. In this case, we have

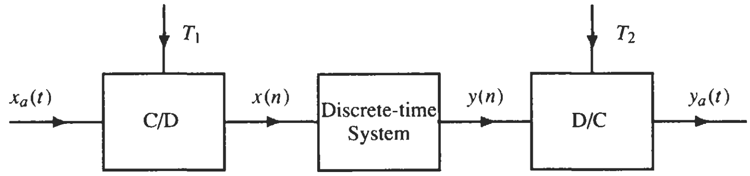
$$H(e^{j\omega}) = 1 - 2(\cos\omega_0)e^{-j\omega} + e^{-j2\omega}$$

Thus, our requirements are that

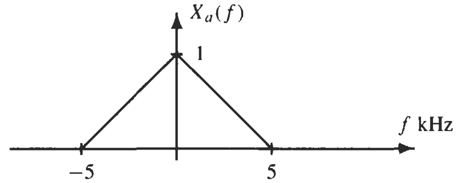
$$a = -2\cos\omega_0 = -2\cos(0.06\pi) \qquad b = 1$$

and $f_s = 2000$.

3.14 The following system is used to process an analog signal with a discrete-time system.



Suppose that $x_a(t)$ is bandlimited with $X_a(f) = 0$ for $|f| > 5$ kHz as shown in the figure below,



and that the discrete-time system is an ideal low-pass filter with a cutoff frequency of $\pi/2$.

(*a*) Find the Fourier transform of $y_a(t)$ if the sampling frequencies are $f_1 = f_2 = 10$ kHz.

(*b*) Repeat for $f_1 = 20$ kHz and $f_2 = 10$ kHz.

(*c*) Repeat for $f_1 = 10$ kHz and $f_2 = 20$ kHz.

(*a*) When the sampling frequencies of the C/D and D/C converters are the same, and $x_a(t)$ is bandlimited with $X_a(j\Omega) = 0$ for $|\Omega| > \pi/T_1$, this system is equivalent to an analog filter with a frequency response

$$H_a(j\Omega) = \begin{cases} H(e^{j\Omega T_1}) & |\Omega| < \dfrac{\pi}{T_1} \\ 0 & \text{else} \end{cases}$$

Therefore, if $H(e^{j\omega})$ is a low-pass filter with a cutoff frequency $\pi/2$, the cutoff frequency of $H_a(j\Omega)$, denoted by Ω_0, is given by

$$\Omega_0 T_1 = \frac{\pi}{2}$$

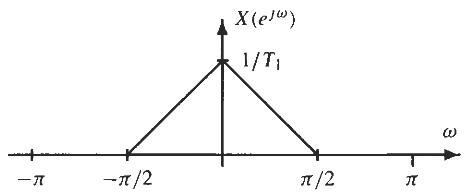
or

$$2\pi f_0 \cdot T_1 = \frac{\pi}{2}$$

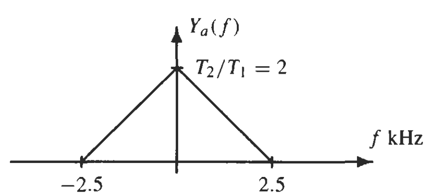
Thus,

$$f_0 = \tfrac{1}{4} f_1 = 2500 \text{ Hz}$$

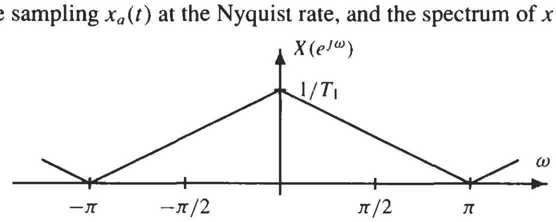
(*b*) When the sampling frequencies of the C/D and D/C are different, it is best to plot the spectrum of the signals as they progress through the system. With $X_a(f)$ as shown above, the discrete-time Fourier transform of $x(n)$ is



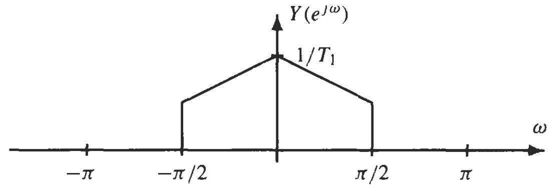
Because the cutoff frequency of the discrete-time low-pass filter is $\pi/2$, $y(n) = x(n)$, and the output of the D/C converter is as plotted below.



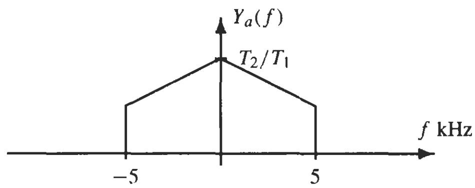
(*c*) With $f_1 = 10$ kHz, we are sampling $x_a(t)$ at the Nyquist rate, and the spectrum of $x(n)$ is



and the output of the low-pass filter is as shown below.



Therefore, the spectrum of $y_a(t)$ is as follows:



3.15 Consider the system in Fig. 3-9 for implementing a continuous-time system in terms of a discrete-time system. Assume that the input to the C/D converter is bandlimited to $\Omega_0 = \Omega_s/2$ and that the unit sample response of the discrete-time system is

$$h(n) = \delta(n) - 0.9\delta(n-1)$$

Find the overall frequency response of this system.

Assuming bandlimited inputs with $X_a(j\Omega) = 0$ for $|\Omega| > \Omega_s/2$, the output $Y_a(j\Omega)$ is related to the input $X_a(j\Omega)$ as follows:

$$Y_a(j\Omega) = H_a(j\Omega)X(j\Omega)$$

where
$$H_a(j\Omega) = \begin{cases} H(e^{j\Omega T_s}) & |\Omega| < \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

Because the frequency response of the discrete-time system is

$$H(e^{j\omega}) = 1 - 0.9e^{-j\omega}$$

then
$$H_a(j\Omega) = \begin{cases} 1 - 0.9e^{-j\Omega T_s} & |\Omega| < \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

3.16 Consider the system shown in Fig. 3-9 for implementing a continuous-time system in terms of a discrete-time system. Assuming that the input signals $x_a(t)$ are bandlimited so that $X_a(f) = 0$ for $|f| > 10$ kHz, find the discrete-time system that produces the output

$$Y_a(f) = \begin{cases} |f|X_a(f) & 2000 \le |f| \le 8000 \\ 0 & \text{otherwise} \end{cases}$$

For bandlimited inputs, the system in Fig. 3-9 is a linear shift-invariant system with an effective frequency response equal to

$$H_a(j\Omega) = \begin{cases} H(e^{j\Omega T_s}) & |\Omega| < \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

The system that we would like to realize has a frequency response

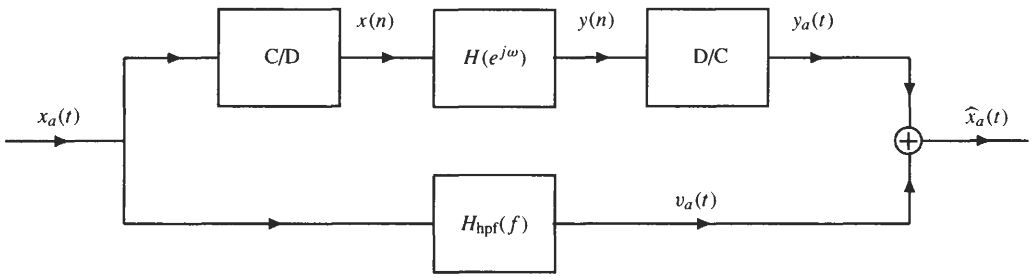
$$H_a(j\Omega) = \begin{cases} |\Omega| & 4000\pi \le |\Omega| \le 16000\pi \\ 0 & \text{otherwise} \end{cases}$$

If we assume a sampling frequency $f_s = 20$ kHz, the frequency response of the discrete-time system should be

$$H(e^{j\omega}) = \begin{cases} \left|\dfrac{\omega}{T_s}\right| & 0.2\pi \le |\omega| \le 0.8\pi \\ 0 & \text{otherwise} \end{cases}$$

where $T_s = 1/20000$.

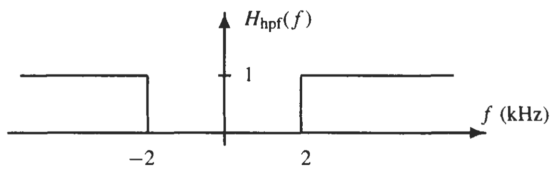
3.17 Diagrammed in the figure below is a hybrid digital-analog network.



The discrete-time system $H(e^{j\omega})$ is a low-pass filter

$$H(e^{j\omega}) = \begin{cases} A & |\omega| \le \omega_0 \\ 0 & \text{else} \end{cases}$$

and the analog system $H_{\text{hpf}}(f)$ is a high-pass filter with a frequency response as shown below.



The input $x_a(t)$ is bandlimited to 4 kHz, and the sampling frequencies of the ideal C/D and D/C converters are 10 kHz. Find the value for A and ω_0 that will result in perfect reconstruction of $x_a(t)$,

$$\hat{x}_a(t) = x_a(t)$$

Because $x_a(t)$ is bandlimited to 4 kHz, the upper branch of this hybrid system acts as an ideal analog low-pass filter with a frequency response

$$H_{\text{lpf}}(f) = \begin{cases} A & |f| \le \dfrac{\omega_0}{2\pi T_s} \\ 0 & |f| > \dfrac{\omega_0}{2\pi T_s} \end{cases}$$

Because the analog network is a high-pass filter with a cutoff frequency of 4 kHz, and

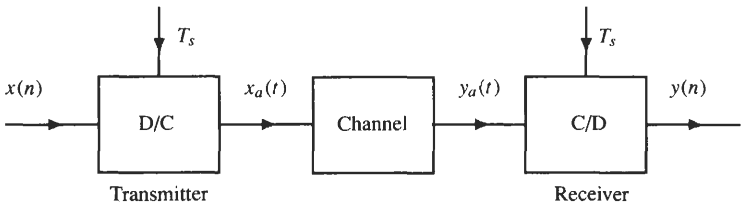
$$\hat{X}_a(f) = Y_a(f) + V_a(f)$$

$\hat{x}_a(t)$ will be equal to $x_a(t)$ provided that $A = 1$ and

$$\frac{\omega_0}{2\pi T_s} = 2000$$

or $\omega_0 = 0.4\pi$.

3.18 A digital sequence $x(n)$ is to be transmitted across a linear time-invariant bandlimited channel as illustrated in the figure below.



The transmitter is a D/C converter, and the receiver simply samples the received waveform $y_a(t)$:

$$y(n) = y_a(nT_s)$$

Assume that the channel may be modeled as an ideal low-pass filter with a cutoff frequency of 4 kHz:

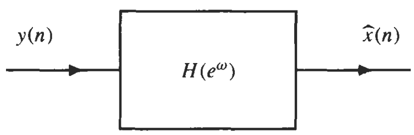
$$G_a(j\Omega) = \begin{cases} 1 & |\Omega| \le 2\pi(4000) \\ 0 & |\Omega| > 2\pi(4000) \end{cases}$$

(*a*) Assuming an ideal C/D and D/C, and perfect synchronization between the transmitter and receiver, what values of T_s (if any) will guarantee that $y(n) = x(n)$?

(*b*) Suppose that the D/C is nonideal. Specifically, suppose that $x(n)$ is first converted to an impulse train and then a zero-order hold is used to perform the "interpolation" between the sample values. In other words, the impulse response of the interpolating filter is a pulse of duration T_s:

$$h_a(t) = \begin{cases} 1 & 0 \le t \le T_s \\ 0 & \text{otherwise} \end{cases}$$

Because the received sequence $y(n)$ will no longer be equal to $x(n)$, in order to improve the performance of the receiver, the received samples are processed with a digital filter as shown below.



Find the frequency response of the filter that should be used to filter $y(n)$.

(*a*) The output of the D/C converter is a bandlimited signal $x_a(t)$ with a Fourier transform that is equal to zero for $|f| > f_s/2$. Because $x_a(t)$ is passed through a bandlimited channel that rejects all frequencies greater than 4 kHz, in order for there to be no distortion at the receiver, it is necessary that

$$\frac{f_s}{2} < 4000$$

or

$$f_s < 8000$$

Thus, the C/D and D/C converters must operate at a rate less than 8 kHz.

(*b*) In order to get the maximum amount of data through the channel per unit of time, we will let T_s be the minimum sampling period,

$$T_s = \tfrac{1}{8000}$$

When the reconstruction filter in the D/C converter is a zero-order hold, the frequency response of the discrete-time system that relates the input sequence $x(n)$ to the reconstructed sequence $y(n)$ is

$$H(e^{j\omega}) = H_a\left(\frac{j\omega}{T_s}\right) \qquad |\omega| < \pi$$

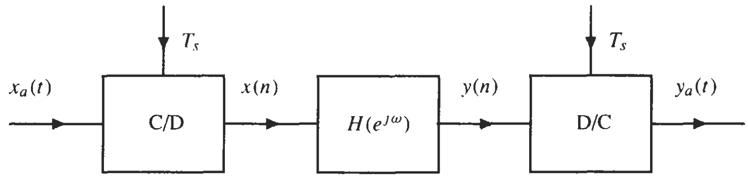
where
$$H_a(j\Omega) = \begin{cases} e^{-j\Omega T_s/2}\dfrac{\sin(\Omega T_s/2)}{\Omega/2} & |\Omega| < \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

Therefore,
$$H(e^{j\omega}) = T_s e^{-j\omega/2}\frac{\sin(\omega/2)}{\omega/2} \qquad |\omega| < \pi$$

and the discrete-time filter for processing $y(n)$ to remove the distortion introduced by the zero-order hold should approximate the response

$$G(e^{j\omega}) = \frac{\omega/2}{\sin(\omega/2)}e^{j\omega/2} \qquad |\omega| < \pi$$

3.19 Consider the following system for processing a continuous-time signal with a discrete-time system:



The frequency response of the discrete-time filter is

$$H(e^{j\omega}) = \frac{2\left(\frac{1}{3} - e^{-j\omega}\right)}{1 - \frac{1}{3}e^{-j\omega}}$$

If $f_s = 2$ kHz and $x_a(t) = \sin(1000\pi t)$, find the output $y_a(t)$.

Sampling $x_a(t) = \sin(1000\pi t)$ with a sampling frequency $f_s = 2000$ produces the discrete-time sequence

$$x(n) = x_a(nT_s) = \sin(1000\pi nT_s) = \sin\left(\frac{n\pi}{2}\right)$$

This sequence is then filtered with the discrete-time filter

$$H(e^{j\omega}) = \frac{2\left(\frac{1}{3} - e^{-j\omega}\right)}{1 - \frac{1}{3}e^{-j\omega}}$$

Because $x(n)$ is a sinusoid, the response is

$$y(n) = A\sin\left(\frac{n\pi}{2} + \phi\right)$$

where A and ϕ are the magnitude and phase, respectively, of the frequency response at $\omega = \pi/2$. With

$$|H(e^{j\omega})|^2 = 4\frac{\left(\frac{1}{3} - e^{-j\omega}\right)}{\left(1 - \frac{1}{3}e^{-j\omega}\right)} \cdot \frac{\left(\frac{1}{3} - e^{j\omega}\right)}{\left(1 - \frac{1}{3}e^{j\omega}\right)} = 4\frac{\frac{10}{9} - \frac{2}{3}\cos\omega}{\frac{10}{9} - \frac{2}{3}\cos\omega} = 4$$

it follows that $|H(e^{j\omega})| = 2$. We may evaluate the phase as follows:

$$\begin{aligned} H(e^{j\omega}) &= 2\frac{\frac{1}{3} - e^{-j\omega}}{1 - \frac{1}{3}e^{-j\omega}} \cdot \frac{1 - \frac{1}{3}e^{j\omega}}{1 - \frac{1}{3}e^{j\omega}} \\ &= 2\frac{\frac{2}{3} - \frac{1}{9}e^{j\omega} - e^{-j\omega}}{\left|1 - \frac{1}{3}e^{-j\omega}\right|^2} = 2\frac{\frac{2}{3} - \frac{10}{9}\cos\omega + j\frac{8}{9}\sin\omega}{\left|1 - \frac{1}{3}e^{-j\omega}\right|^2} \end{aligned}$$

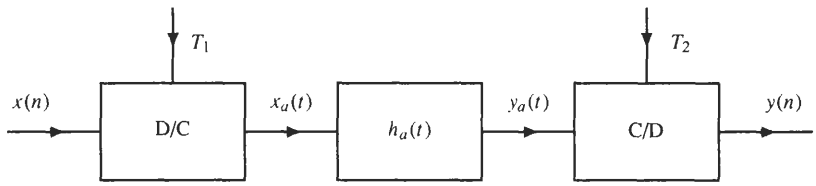
Therefore,
$$\phi_h(\omega) = \tan^{-1} \frac{\frac{8}{9}\sin\omega}{\frac{2}{3} - \frac{10}{9}\cos\omega}$$

which, when evaluated at $\omega = \pi/2$, gives

$$\phi_h(\omega)|_{\omega=\pi/2} = \tan^{-1}\frac{8/9}{2/3} = \tan^{-1}\frac{4}{3} = 0.2952\pi$$

Thus,
$$y(n) = 2\sin\left(\tfrac{\pi}{2}[n + 0.5903]\right)$$

3.20 Consider the following system consisting of an ideal D/C converter, a linear time-invariant filter, and an ideal C/D converter.



The continuous-time system $h_a(t)$ is an ideal low-pass filter with a frequency response

$$H_a(f) = \begin{cases} 1 & |f| \le 10\,\text{kHz} \\ 0 & \text{otherwise} \end{cases}$$

(*a*) If $T_1 = T_2 = 10^{-4}$, find an expression relating the output $y(n)$ to the input $x(n)$.

(*b*) If $T_1 = \left(\frac{1}{4}\right) \times 10^{-4}$ and $T_2 = 10^{-4}$, find $y(n)$ when

$$x(n) = \left[\frac{\sin(n\pi/2)}{n\pi/2}\right]^2$$

(*a*) When $T_1 = T_2$, this system behaves as a linear shift-invariant discrete-time system with a frequency response

$$H(e^{j\omega}) = H_a\left(\frac{j\omega}{T_1}\right) \qquad |\omega| < \pi$$

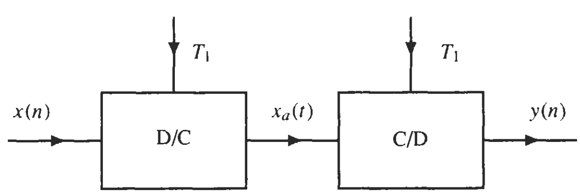
Because $H_a(j\Omega) = 1$ for $|\Omega| < 2\pi \cdot 10^4$,

$$H(e^{j\omega}) = 1 \qquad |\omega| < \pi$$

and
$$h(n) = \delta(n)$$

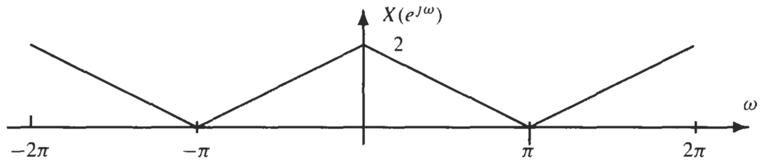
Therefore, $y(n) = x(n)$.

Another way to analyze this system is to note that the output of the D/C converter, $x_a(t)$, is bandlimited to $f = 5$ kHz. Because $H_a(f)$ is an ideal low-pass filter with a cutoff frequency 10 kHz, $y_a(t) = x_a(t)$. Therefore, this system is equivalent to the one shown below.

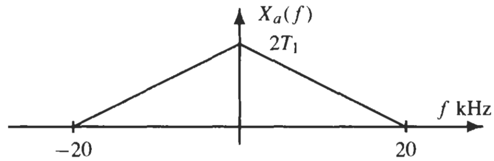


Because an ideal D/C converter followed by an ideal D/C converter is the identity system, $y(n) = x(n)$.

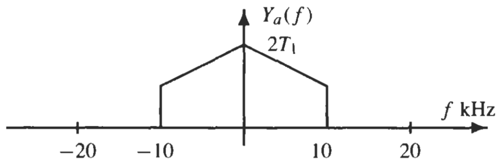
(*b*) When $T_1 \ne T_2$, this system is, in general, no longer a linear shift-invariant system. However, we may analyze this system in the frequency domain as follows. First, note that the DTFT of $x(n)$ is as illustrated in the following figure:



Thus, the output of the D/C converter is a bandlimited signal that has a Fourier transform as shown in the following figure:



The analog low-pass filter removes all frequencies in $x_a(t)$ above 10 kHz to produce a signal $y_a(t)$ that has a Fourier transform as shown below.



Because the highest frequency in $y_a(t)$ is 10 kHz, the Nyquist rate is 20 kHz. However, the sampling frequency of the C/D converter is 10 kHz, so $y_a(t)$ will be aliased. The DTFT of $y(n)$ is related to $Y_a(j\Omega)$ as follows:

$$Y(e^{j\omega}) = \frac{1}{T_2} \sum_{k=-\infty}^{\infty} Y_a\left(j\frac{\omega}{T_2} - j\frac{2\pi k}{T_2}\right)$$

Summing the shifted and scaled transforms yields

$$Y(e^{j\omega}) = \tfrac{3}{4} \qquad |\omega| < \pi$$

Therefore,

$$y(n) = \tfrac{3}{4}\delta(n)$$

Sample Rate Conversion

3.21 Suppose that a discrete-time sequence $x(n)$ is bandlimited so that

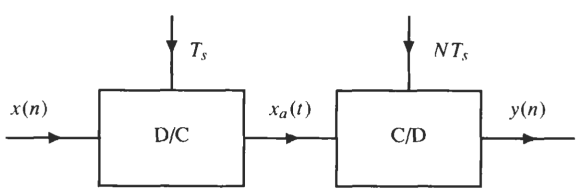
$$X(e^{j\omega}) = 0 \qquad 0.3\pi < |\omega| < \pi$$

This sequence is then sampled to form the sequence

$$y(n) = x(nN)$$

where N is an integer. Find the largest value for N for which $x(n)$ may be uniquely recovered from $y(n)$.

The easiest way to view this problem is as illustrated below.



Converting $x(n)$ into a continuous-time signal with an ideal D/C converter with a sampling frequency f_s produces a continuous-time signal $x_a(t)$ that is bandlimited to $f_0 = 0.3 \cdot f_s/2$. Therefore, $x_a(t)$ may be sampled, without aliasing, if we use a sampling frequency $f_s' \geq 2f_0 = 0.3f_s$, or

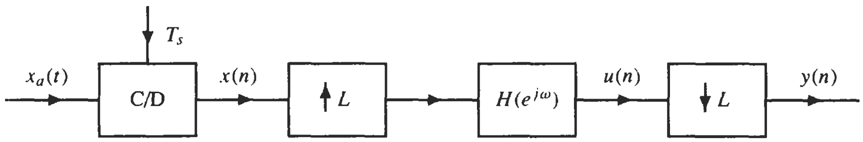
$$T_s' < \frac{T_s}{0.3} = 3.33\bar{3}T_s$$

Therefore, if $T_s' = 3T_s$,

$$y(n) = x_a(3nT_s) = x(3n)$$

and $x(n)$ may be uniquely recovered from $y(n)$. Thus, $N = 3$.

3.22 Consider the following system:



Assume that $X_a(f) = 0$ for $|f| > 1/T_s$ and that

$$H(e^{j\omega}) = \begin{cases} e^{-j\omega} & |\omega| \leq \dfrac{\pi}{L} \\ 0 & \dfrac{\pi}{L} < |\omega| \leq \pi \end{cases}$$

How is the output of the discrete-time system, $y(n)$, related to the input signal $x_a(t)$?

In this system, the bandlimited signal $x_a(t)$ is sampled, without aliasing, to produce the sampled signal $x(n) = x_a(nT_s)$. Up-sampling $x(n)$ by a factor of L, and filtering with an ideal low-pass filter with a cutoff frequency $\omega_c = \pi/L$, produces the signal

$$w(n) = x_a\left(\frac{nT_s}{L}\right)$$

that is, a signal that is sampled with a sampling frequency Lf_s. However, because the low-pass filter has linear phase with a group delay of one sample, the interpolated up-sampled signal is delayed by 1. Therefore, the output of the low-pass filter is

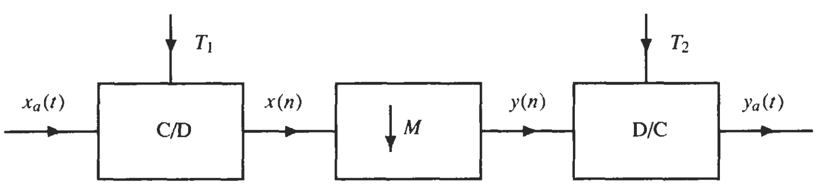
$$u(n) = w(n-1) = x_a\left([n-1]\frac{T_s}{L}\right)$$

Down-sampling by L then produces the output

$$y(n) = u(Ln) = w(Ln-1) = x_a\left(nT_s - \frac{T_s}{L}\right)$$

Thus, $y(n)$ corresponds to samples of $x_a(t - t_0)$ where $t_0 = T_s/L$.

3.23 Consider the system shown in the figure below.

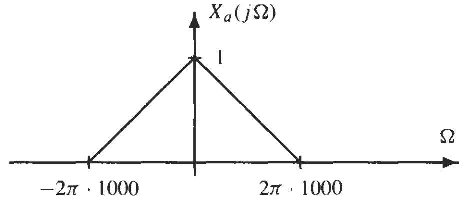


Assume that the input is bandlimited, $X_a(j\Omega) = 0$ for $|\Omega| > 2\pi \cdot 1000$.

(*a*) What constraints must be placed on M, T_1, and T_2 in order for $y_a(t)$ to be equal to $x_a(t)$?

(*b*) If $f_1 = f_2 = 20$ kHz and $M = 4$, find an expression for $y_a(t)$ in terms of $x_a(t)$.

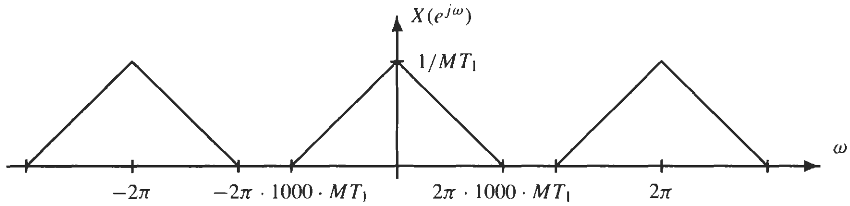
(*a*) Suppose that $x_a(t)$ has a Fourier transform as shown in the figure below.



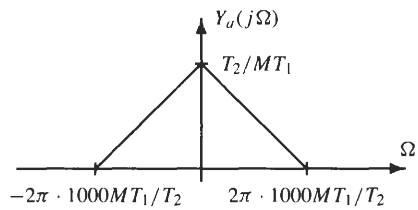
Because $y(n) = x(Mn) = x_a(nMT_1)$, in order to prevent $x(n)$ from being aliased, it is necessary that

$$MT_1 < \tfrac{1}{2000}$$

If this constraint is satisfied, the output of the down-sampler has a DTFT as shown below.



Going through the D/C converter produces the signal $y_a(t)$, which has the Fourier transform shown below.



Therefore, in order for $y_a(t)$ to be equal to $x_a(t)$, we require that

1. $MT_1 \le 1/2000$ in order to avoid aliasing.
2. $T_2 = MT_1$ to prevent frequency scaling.

(*b*) With $T_1 = T_2 = 1/20000$ and $M = 4$, note that

$$MT_1 = \tfrac{1}{5000} < \tfrac{1}{2000}$$

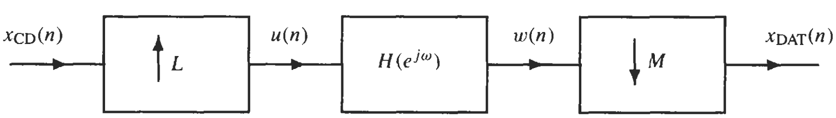
Therefore, there is no aliasing. Thus, as we see from the figure above,

$$Y_a(j\Omega) = \tfrac{1}{4}X_a\left(\frac{j\Omega}{4}\right)$$

or

$$y_a(t) = x_a(4t)$$

3.24 Digital audio tape (DAT) drives have a sampling frequency of 48 kHz, whereas a compact disk (CD) player operates at a rate of 44.1 kHz. In order to record directly from a CD onto a DAT, it is necessary to convert the sampling rate from 44.1 to 48 kHz. Therefore, consider the following system for performing this sample rate conversion:



Find the smallest possible values for L and M and find the appropriate filter $H(e^{j\omega})$ to perform this conversion.

Given that $48000 = 2^7 \cdot 3 \cdot 5^3$ and $44100 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$, to change the sampling rate we require

$$\frac{L}{M} = \frac{2^7 \cdot 3 \cdot 5^3}{2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2} = \frac{2^5 \cdot 5}{3 \cdot 7^2} = \frac{160}{147}$$

Therefore, if we up-sample by $L = 160$ and then down-sample by $M = 147$, we achieve the desired sample rate conversion. The low-pass filter that we require is one that has a cutoff frequency

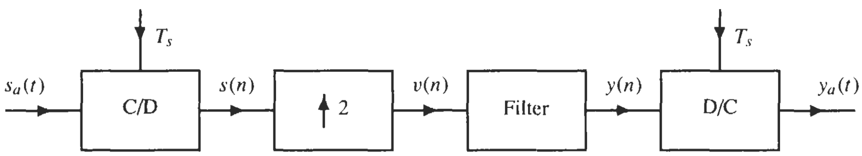
$$\omega_c = \min\left(\frac{\pi}{L}, \frac{\pi}{M}\right) = \frac{\pi}{160}$$

and the gain of the filter should be equal to $L = 160$.

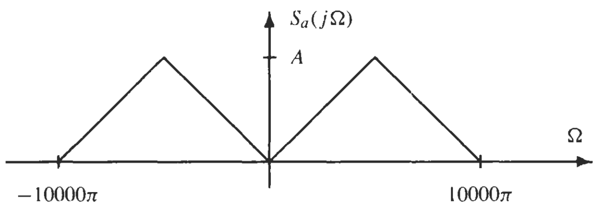
3.25 Suppose that we would like to slow a segment of speech to one-half its normal speed. The speech signal $s_a(t)$ is assumed to have no energy outside of 5 kHz, and is sampled at a rate of 10 kHz, yielding the sequence

$$s(n) = s_a(nT_s)$$

The following system is proposed to create the slowed-down speech signal.



Assume that $S_a(j\Omega)$ is as shown in the following figure:



(*a*) Find the spectrum of $v(n)$.

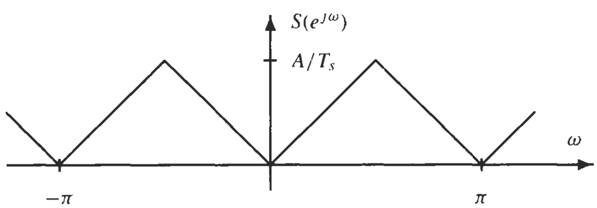
(*b*) Suppose that the discrete-time filter is described by the difference equation

$$y(n) = v(n) + \tfrac{1}{2}[v(n-1) + v(n+1)]$$

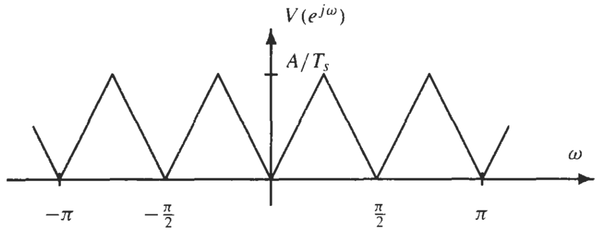
Find the frequency response of the filter and describe its effect on $v(n)$.

(*c*) What is $Y_a(j\Omega)$ in terms of $X_a(j\Omega)$? Does $y_a(t)$ correspond to slowed-down speech?

(*a*) Since $s_a(t)$ is sampled at the Nyquist rate, the DTFT of the sampled speech signal, $s(n)$, is as follows:



Up-sampling by a factor of 2 scales the frequency axis of $S(e^{j\omega})$ by a factor of two as shown below.



(*b*) The unit sample response of the discrete-time filter is

$$h(n) = \tfrac{1}{2}\delta(n+1) + \delta(n) + \tfrac{1}{2}\delta(n-1)$$

which has a frequency response

$$H(e^{j\omega}) = 1 + \cos\omega$$

To see the effect of this filter on $v(n)$, note that due to the up-sampling, $v(n) = 0$ for n odd. Therefore, with

$$y(n) = v(n) + \tfrac{1}{2}v(n-1) + \tfrac{1}{2}v(n+1)$$

it follows that

$$y(n) = \begin{cases} v(n) & n \text{ odd} \\ \tfrac{1}{2}v(n-1) + \tfrac{1}{2}v(n+1) & n \text{ even} \end{cases}$$

Thus, the even-index values of $v(n)$ are unchanged, and the odd-index values are the average of the two neighboring values. As a result, $h(n)$ performs a linear interpolation between the values of $v(n)$.

(*c*) The output of the DC converter, $y_a(t)$, has a Fourier transform

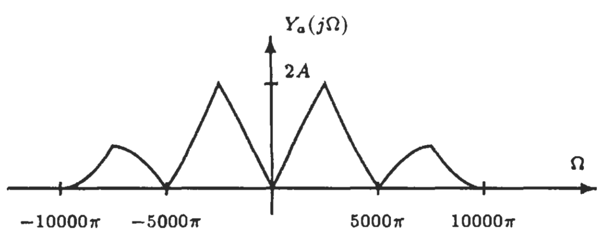
$$Y_a(j\Omega) = \begin{cases} T_s Y(e^{j\Omega T_s}) & |\Omega| < \pi/T_s \\ 0 & \text{otherwise} \end{cases}$$

Since $$Y(e^{j\omega}) = H(e^{j\omega})V(e^{j\omega}) = (1 + \cos\omega)V(e^{j\omega})$$

and $$V(e^{j\omega}) = S(e^{j2\omega})$$

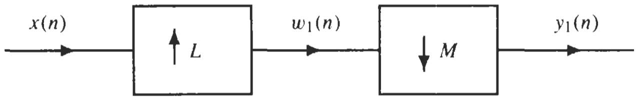
then $$Y_a(j\Omega) = \begin{cases} T_s(1 + \cos\Omega T_s)S(e^{j2\Omega T_s}) & |\Omega| < 10000\pi \\ 0 & \text{otherwise} \end{cases}$$

which is the product of $(1 + \cos\Omega T_s)$ and $T_s S(e^{j2\Omega T_s})$ as illustrated below.

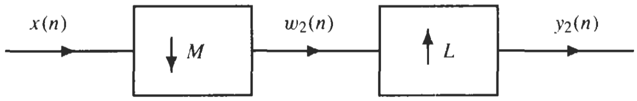


Thus, $y_a(t)$ does not correspond to slowed-down speech due to the images of $s_a(t)$ that occur in the frequency range $5000\pi < |\Omega| < 10000\pi$ and the nonideal linear interpolator. Note that a better approximation would be to use a DC converter with a sampling rate of $2T_s$ to eliminate the images.

3.26 Shown in the figure below are two different ways of cascading an up-sampler with a down-sampler.



(a)



(b)

(*a*) If $M = L$, show that the two systems are not identical.

(*b*) Under what conditions will the two systems be identical?

(*a*) In the first system, which consists of an up-sampler followed by a down-sampler, note that $w_1(n)$ is a sequence that is formed by inserting $L - 1$ zeros between each value of $x(n)$. The down-sampler then extracts every Lth value of $w_1(n)$, thus producing the output

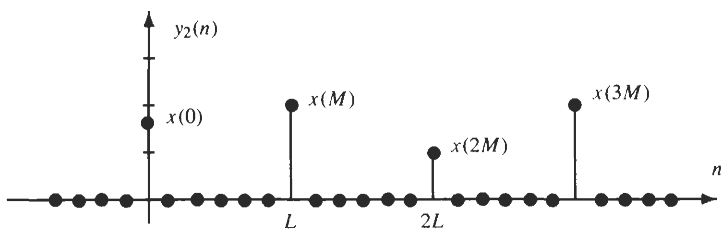
$$y_1(n) = x(n)$$

In the second system, however, the down-sampler extracts every Lth sample of $x(n)$ and discards the rest. The up-sampler then inserts $L - 1$ zeros between each value of $w_2(n)$. Thus,

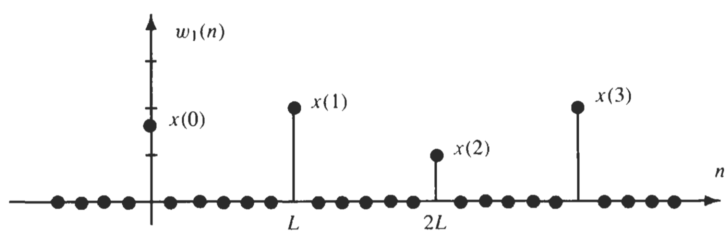
$$y_2(n) = \begin{cases} x\left(\dfrac{nM}{L}\right) & n = 0, \pm L, \pm 2L, \dots \\ 0 & \text{else} \end{cases}$$

Therefore, the two systems are not the same.

(*b*) In order to analyze these systems when $L \neq M$, note that $y_2(n)$ in the second system has the form shown in the following figure:



On the other hand, the sequence $w_1(n)$ in the first system is as shown below.



Note that $y_1(n)$ is formed by extracting every Mth value of $w_1(n)$,

$$y_1(n) = w_1(nM)$$

Clearly, $$y_1(kL) = w_1(kML) = x(kM)$$

so $$y_1(kL) = y_2(kL)$$

However, in order for $y_1(n)$ to be equal to $y_2(n)$, we require that

$$y_1(n) = w_1(nM) = 0 \qquad n \neq kL$$

This will be true if and only if M and L are relatively prime.

Supplementary Problems

A/D and D/A Conversion

3.27 Find two different continuous-time signals that will produce the sequence

$$x(n) = \cos(0.15n\pi)$$

when sampled with a sampling frequency of 8 kHz.

3.28 If the Nyquist rate for $x_a(t)$ is Ω_s, find the Nyquist rate for (a) $x^2(2t)$, (b) $x(t/3)$, (c) $x(t) * x(t)$.

3.29 A continuous-time signal $x_a(t)$ is known to be uniquely recoverable from its samples $x_a(nT_s)$ when $T_s = 1$ ms. What is the highest frequency in $X_a(f)$?

3.30 Suppose that $x_a(t)$ is bandlimited to 8 kHz (that is, $X_a(f) = 0$ for $|f| > 8000$). (a) What is the Nyquist rate for $x_a(t)$? (b) What is the Nyquist rate for $x_a(t)\cos(2\pi \cdot 1000t)$?

3.31 Let $x_a(t) = \cos(650\pi t) + 2\sin(720\pi t)$. (a) What is the Nyquist rate for $x_a(t)$? (b) If $x_a(t)$ is sampled at twice the Nyquist rate, what are the frequencies of the sinusoids in the sampled sequence?

3.32 If a continuous-time filter with an impulse response $h_a(t)$ is sampled with a sampling frequency of f_s, what happens to the cutoff frequency ω_c of the discrete-time filter as f_s is increased?

3.33 A complex bandpass signal $x_a(t)$ with $X_a(f)$ nonzero for 10 kHz $< f <$ 12 kHz is sampled at a sampling rate of 2 kHz. The resulting sequence is

$$x(n) = \delta(n)$$

What is $x_a(t)$?

3.34 If the highest frequency in $x_a(t)$ is $f = 8$ kHz, find the minimum sampling frequency for the bandpass signal $y_a(t) = x_a(t)\cos(\Omega_0 t)$ if (a) $\Omega_0 = 2\pi \cdot 20 \cdot 10^3$ and (b) $\Omega_0 = 2\pi \cdot 24 \cdot 10^3$.

3.35 The continuous-time signal $x_a(t) = 7.25\cos(2000\pi t)$ is sampled at a sampling frequency of 8 kHz and quantized with a resolution $\Delta = 0.02$. How many bits are required in the A/D converter to avoid clipping $x_a(t)$?

3.36 Suppose that we want to sample the signal $x_a(t)$ with a 12-bit quantizer, where $x_a(t)$ is assumed to be gaussian with a variance σ_x^2. What is the signal-to-quantization noise ratio if we want the range of the quantizer to extend from $-3\sigma_x$ to $3\sigma_x$?

3.37 Suppose that an analog waveform is sampled with a sampling frequency of 10 kHz and that $x_a(t)$ contains a strong 60-Hz interference signal. If the only information in $x_a(t)$ of interest is in the frequency band above

60 Hz, the interference may be eliminated with a discrete-time high-pass filter that has a frequency response of the form

$$H(e^{j\omega}) = \begin{cases} 0 & |\omega| < \omega_c \\ 1 & \omega_c \le \omega \le \pi \end{cases}$$

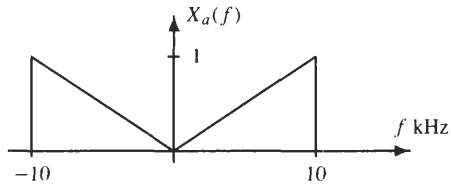
What is the smallest cutoff frequency ω_c that may be used and still eliminate the 60-Hz interference?

3.38 *True or False*: If $x(n)$ has a discrete-time Fourier transform that is equal to zero for $\pi/4 < |\omega| < \pi$,

$$x(n) = \sum_{k=-\infty}^{\infty} x(4k) \frac{\sin[\pi(n-4k)/4]}{\pi(n-4k)/4}$$

Discrete-Time Processing of Analog Signals

3.39 The system shown in Fig. 3-9 may be used to process an analog signal with a discrete-time system. Assume that $x_a(t)$ is bandlimited with $X_a(f) = 0$ for $|f| > 10$ kHz as shown in the figure below.



If the discrete-time system is an ideal low-pass filter with a cutoff frequency of $\pi/4$, find the Fourier transform of $y_a(t)$ when (*a*) $f_1 = 20$ kHz and $f_2 = 10$ kHz and (*b*) $f_1 = 10$ kHz and $f_2 = 20$ kHz.

3.40 For bandlimited input signals, the system shown in Fig. 3-10 is a linear time-invariant continuous-time system. If

$$y(n) = \tfrac{1}{2}y(n-1) + x(n)$$

find the frequency response of the equivalent continuous-time system.

3.41 For bandlimited input signals, the system shown in Fig. 3-10 is a linear time-invariant continuous-time system. If the overall system is to be a differentiator,

$$y_a(t) = \frac{d}{dt}x_a(t)$$

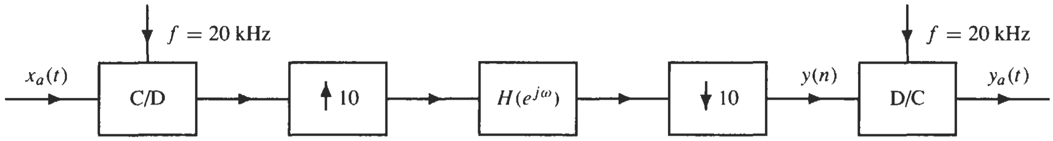
how should the frequency response of the discrete-time system be defined?

Sample Rate Conversion

3.42 The up-sampler and down-sampler are components that are found in interpolators and decimators, respectively. Are these systems linear? Are they shift-invariant?

3.43 A sequence $x(n)$ corresponds to samples of a bandlimited signal using a sampling frequency of 10 kHz. However, the sequence should have been sampled using a sampling frequency $f_s = 12$ kHz. Design a system for digitally changing the sampling rate.

3.44 A signal $x_a(t)$ that is bandlimited to 10 kHz is processed by the following system:



If

$$H(e^{j\omega}) = \begin{cases} e^{-4j\omega} & |\omega| < \dfrac{\pi}{10} \\ 0 & \text{otherwise} \end{cases}$$

express the output $y_a(t)$ in terms of the input $x_a(t)$.

Answers to Supplementary Problems

3.27 $x_1(t) = \cos(1200\pi t)$ and $x_2(t) = \cos(17200\pi t)$.

3.28 (*a*) $4\Omega_s$. (*b*) $\Omega_s/3$. (*c*) Ω_s.

3.29 500 Hz.

3.30 (*a*) 16 kHz. (*b*) 18 kHz.

3.31 (*a*) 720 kHz. (*b*) $\omega_1 = 65\pi/142$ and $\omega_2 = \pi/2$.

3.32 ω_c decreases.

3.33 $x_a(t) = \dfrac{1}{2000}\dfrac{\sin(2000\pi t)}{\pi t}e^{j2\pi(11000)t}$.

3.34 (*a*) 56 kHz. (*b*) 32 kHz.

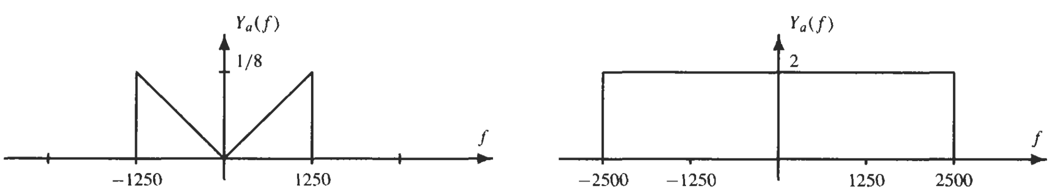
3.35 10 bits.

3.36 73.51 dB.

3.37 $\omega_c = 0.012\pi$.

3.38 True.

3.39



(*a*) (*b*)

3.40 $$H_a(j\Omega) = \begin{cases} \dfrac{1}{1 - \frac{1}{2}e^{-j\Omega T_s}} & |\Omega| < \dfrac{\pi}{T_s} \\ 0 & \text{otherwise} \end{cases}$$

3.41 $H(e^{j\omega}) = j\omega/T_s$ for $|\omega| < \pi$.

3.42 Both are linear and shift-varying.

3.43 Up-sample by $L = 6$, filter with a low-pass filter that has a cutoff frequency of $\omega_c = \pi/6$ and a gain of 6, and down-sample by $M = 5$.

3.44 $y_a(t) = x_a(t - 4T_s/10) = x_a(t - 2 \cdot 10^{-5})$.