1.7 Limit of Functions

1.7.1 The Concept of Limit

Let's investigate the behavior of the function f defined by

$$f(x) = \frac{(x-1)}{(x^2-1)}$$

for values of x near 1.

The following table gives values of $f(x)$ for values of x close to 1 but not equal to 1.

$x < 1$	$f(x)$	$x > 1$	$f(x)$
0.5	0.666667	1.5	0.400000
0.9	0.526316	1.1	0.476190
0.99	0.502513	1.01	0.497512
0.999	0.500250	1.001	0.499750
0.9999	0.500025	1.0001	0.499975
$\downarrow$ 1	$\downarrow$ 0.5	$\downarrow$ 1	$\downarrow$ 0.5

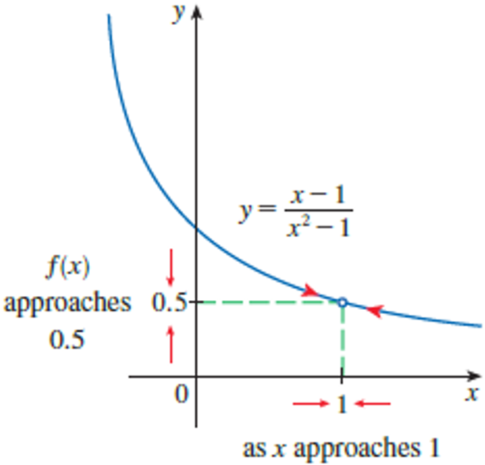


Figure 23

From the table and the graph of $f(x)$, we see that the closer x is to 1 (on either side of 1), the closer $f(x)$ is to 0.5.

In fact, it appears that we can make the values of $f(x)$ as close as we like to 0.5 by taking x sufficiently close to 1.

We express this by saying

$$\text{“the limit of the function } f(x) = \frac{(x-1)}{(x^2-1)}$$

$$\text{as } x \text{ approaches } 1 \text{ is equal to } 0.5.\text{”}$$

The notation for this is

$$\lim_{x \to 1} \frac{(x-1)}{(x^2-1)} = 0.5$$

1.7.2 Definition : Limit of a Function

Definition

Suppose $f(x)$ is defined when x is near the number a. Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L" if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

1.7.3 One-Sided Limits

Consider the Heaviside function H is defined by

$$H(t) = \begin{cases} 0 & ;\text{if t<0} \\ 1 & ;\text{if t>0.} \end{cases}$$

Its graph is shown in Figure 24.

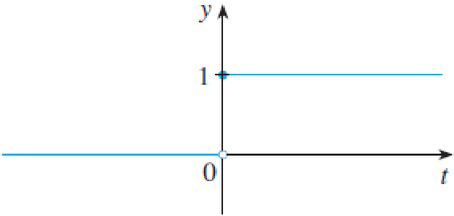


Figure 24 : The Heaviside function

As t approaches 0 from the left, $H(t)$ approaches 0.

As t approaches 0 from the right, $H(t)$ approaches 1.

There is no single number that $H(t)$ approaches as t approaches 0.

Therefore,

$$\lim_{t \to 0} H(t) \text{ does not exist.}$$

We noticed that $H(t)$ approaches 0 as t approaches 0 from the left and $H(t)$ approaches 1 as t approaches 0 from the right.

We indicate this situation symbolically by writing

$$\lim_{t \to 0^-} H(t) = 0$$

and

$$\lim_{t \to 0^+} H(t) = 1.$$

The notation $t \to 0^-$ indicates that we consider only values of t that are less than 0.

Likewise, $t \to 0^+$ indicates that we consider only values of t that are greater than 0.

1.7.4 Definition : One-Sided Limits

Definition

We write

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of $f(x)$ as x approaches a [or the limit of $f(x)$ as x approaches a from the left] is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a with x less than a.

1.7.4 Definition : One-Sided Limits

Definition

Similarly, if we require that x be greater than a, we get "the right-hand limit of $f(x)$ as x approaches a is equal to L" and we write

$$\lim_{x \to a^+} f(x) = L$$

Thus the notation $x \to a^+$ means that we consider only x greater than a.

These definitions are illustrated in Figure 25.

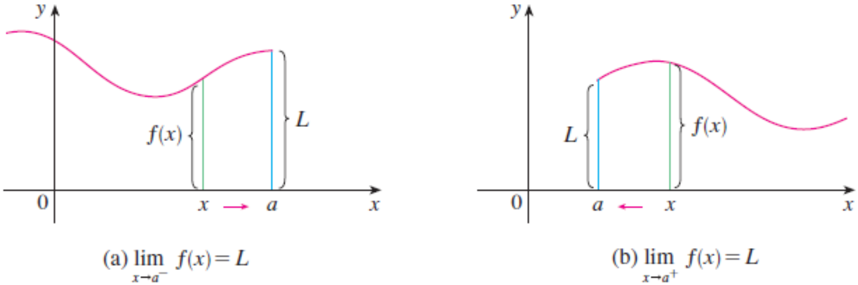


Figure 25

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L.$$

Example (10)

Show that the limit $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

1.7.5 Infinite Limits

Let find the limit $\lim_{x\to 0} \dfrac{1}{x^2}$ if it exists.

As x becomes close to 0, x^2 also becomes close to 0, and $\dfrac{1}{x^2}$ becomes very large.

x	$\dfrac{1}{x^2}$
± 1	1
± 0.5	4
± 0.2	25
± 0.1	100
± 0.05	400
± 0.01	10,000
± 0.001	1,000,000

In fact, it appears from the graph of the function $f(x) = \frac{1}{x^2}$ shown in Figure 26 that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0.

Thus the values of $f(x)$ do not approach a number, so $\lim_{x \to 0} \left(\frac{1}{x^2} \right)$ does not exist.

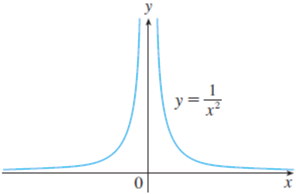


Figure 26

To indicate this kind of behavior, we use the notation

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

- This does not mean that we are regarding ∞ as a number.

Nor does it mean that the limit exists.

It simply expresses the particular way in which the limit does not exist: $\dfrac{1}{x^2}$ can be made as large as we like by taking x close enough to 0.

- In general, we write symbolically

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of $f(x)$ tend to become larger and larger (or "increase without bound") as x becomes closer and closer to a.

1.7.6 Definition : Infinite Limits

Definition

Let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to a.

1.7.6 Definition : Infinite Limits

Definition

Similarly, let f be a function defined on both sides of a, except possibly at a itself. Then

$$\lim_{x \to a} f(x) = -\infty$$

means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to a, but not equal to a.

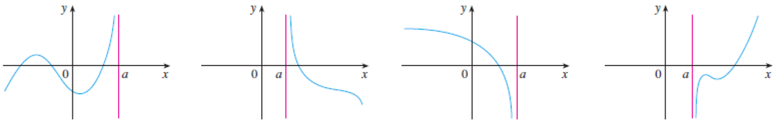
Similar definitions can be given for the one-sided infinite limits

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty$$

$$\lim_{x \to a^+} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

remembering that $x \to a^-$ means that we consider only values of x that are less than a, and similarly $x \to a^+$ means that we consider only $x > a$.

Illustrations of these four cases are given in Figure 27.



(a) $\lim_{x \to a^-} f(x) = \infty$ (b) $\lim_{x \to a^+} f(x) = \infty$ (c) $\lim_{x \to a^-} f(x) = -\infty$ (d) $\lim_{x \to a^+} f(x) = -\infty$

Figure 27

1.7.7 Definition : Vertical Asymptote

Definition

The vertical line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

- ☆ $\lim_{x \to a} f(x) = \infty$
- ☆ $\lim_{x \to a} f(x) = -\infty$
- ☆ $\lim_{x \to a^+} f(x) = \infty$
- ☆ $\lim_{x \to a^+} f(x) = -\infty$
- ☆ $\lim_{x \to a^-} f(x) = \infty$
- ☆ $\lim_{x \to a^-} f(x) = -\infty$

1.8 Continuity

1.8.1 Definition : Continuity of a Function at a Number

Definition

A function f is continuous at a number a if

$$\lim_{x \to a} f(x) = f(a).$$

Notice that Definition implicitly requires three things if $f(x)$ is continuous at a:

1. $f(a)$ is defined. (that is, a is in the domain of f)
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$.

- ☆ The definition says that f is continuous at a if $f(x)$ approaches $f(a)$ as x approaches a.
- ☆ If f is defined near a, we say that f is discontinuous at a (or f has a discontinuity at a) if f is not continuous at a.

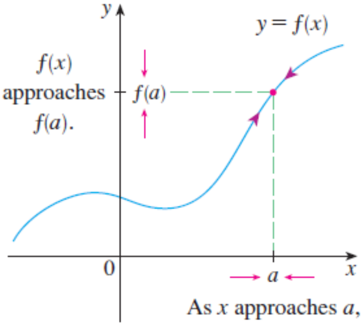


Figure 28

Example (11)

Where are each of the following functions discontinuous?

(i) $f(x) = \dfrac{x^2 - x - 2}{x - 2}.$

(ii) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

(iii) $f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2} & \text{if } x \neq 2 \\ 1 & \text{if } x = 2. \end{cases}$

1.8.2 Definition : Continuity from Right and Left

Definition

A function f is continuous from the right at a number a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at a if

$$\lim_{x \to a^-} f(x) = f(a).$$

1.8.3 Definition : Continuity on an Interval

Definition

A function f is continuous on an interval if it is continuous at every number in the interval.

(If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or continuous from the left.)

Example (12)

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous on the interval $[-1, 1]$.

■ If functions $f(x)$ and $g(x)$ are continuous at a and c is a constant, then the following functions are also continuous at a.

1. $f \pm g$
2. cf
3. fg
4. $\dfrac{f}{g}$; if $g(a) \neq 0$

■ If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.

Example (13)

Use the definition of continuity and the properties of limits to show that the function is continuous at the given number a.

(i) $f(x) = \dfrac{x^2 + 5x}{2x + 1}\ ;\ a = 2$

(ii) $f(x) = 2\sqrt{3x^2 + 1}; a = 1$

■ The following types of functions are continuous at every number in their domains:

- ☆ Polynomials
- ☆ Rational functions
- ☆ Root functions
- ☆ Trigonometric functions
- ☆ Inverse trigonometric functions
- ☆ Exponential functions
- ☆ Logarithmic functions

1.9 Derivatives of Functions

1.9.1 Definition : Derivative of a Function at a Number

Definition

The derivative of a function f at a number a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

- If we write $x = a + h$, then we have $h = x - a$ and h approaches 0 if and only if x approaches a. Therefore an equivalent way of stating the definition of the derivative,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Example (14)

Find the derivative of the function $f(x) = x^2 - 8x + 9$ at the number a.

- The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$, the derivative of f at a.

- If we use the point-slope form of the equation of a line, we can write an equation of the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$:

$$y - f(a) = f'(a)(x - a)$$

Example (15)

Find an equation of the tangent line to the parabola $y = x^2 - 8x + 9$ at the point $(3, -6)$.

1.9.2 Rates of Change

Suppose y is a quantity that depends on another quantity x. Thus y is a function of x and we write $y = f(x)$. If x changes from x_1 to x_2, then the change in x (also called the **increment** of x) is

$$\Delta x = x_2 - x_1$$

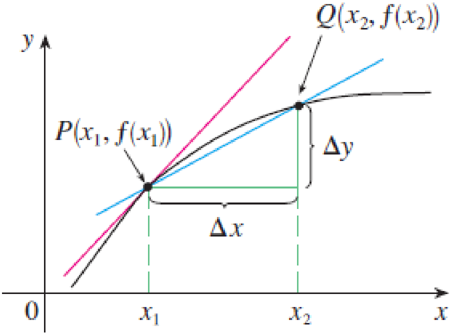
and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1).$$

The difference quotient

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

is called the average rate of change of y with respect to x over the interval $[x_1, x_2]$ and can be interpreted as the slope of the secant line PQ in Figure 29.



average rate of change $= m_{PQ}$

instantaneous rate of change $=$
slope of tangent at P

Figure 29

If we consider the average rate of change over smaller and smaller intervals by letting x_2 approach x_1 and therefore letting Δx approach 0. The limit of these average rates of change is called the (instantaneous) rate of change of y with respect to x at $x = x_1$, which is interpreted as the slope of the tangent to the curve $y = f(x)$ at $P(x_1, f(x_1))$:

$$\text{Instantaneous rate of change} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

- The derivative $f'(a)$ is the instantaneous rate of change of $y = f(x)$ with respect to x when $x = a$.

1.9.3 Notations

If we use the traditional notation $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x)$$

The symbols D and $\dfrac{dy}{dx}$ are called differentiation operators because they indicate the operation of differentiation, which is the process of calculating a derivative.

1.10 Differentiability

1.10.1 Definition : Differentiability

Definition

A function f is differentiable at a if $f'(a)$ exists. It is differentiable on an open interval (a, b) [or (a, ∞) or $(-\infty, a)$ or $(-\infty, \infty)$] if it is differentiable at every number in the interval.

Theorem

If f is differentiable at a, then f is continuous at a.

1.10.2 Introduction : Indeterminate Forms

Suppose we are trying to analyze the behavior of the function

$$F(x) = \frac{\ln x}{x-1}$$

Although F is not defined when $x = 1$, we need to know how F behaves **near** 1.

In particular, we would like to know the value of the limit

$$\lim_{x \to 1} \frac{\ln x}{x-1}$$

In computing this limit we can't apply the limit of a quotient is the quotient of the limits because the limit of the denominator is 0.

In fact, although the limit $\lim_{x \to 1} \dfrac{\ln x}{x - 1}$ exists, its value is not obvious because both numerator and denominator approach 0 and $\dfrac{0}{0}$ is not defined.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an indeterminate form of type $\frac{0}{0}$.

For rational functions, we can cancel common factors:

$$\lim_{x\to 1} \frac{x^2 - x}{x^2 - 1} = \lim_{x\to 1} \frac{x(x-1)}{(x-1)(x+1)} = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2}$$

But these methods do not work for limits such as $\lim\limits_{x\to 1} \dfrac{\ln x}{x-1}$, so we introduce a systematic method, known as l'Hospital's Rule, for the evaluation of indeterminate forms.

Another situation in which a limit is not obvious occurs when we look for a horizontal asymptote of F and need to evaluate the limit

$$\lim_{x \to \infty} \frac{\ln x}{x - 1}$$

It isn't obvious how to evaluate this limit because both numerator and denominator become large as $x \to \infty$.

There is a struggle between numerator and denominator. If the numerator wins, the limit will be ∞ (the numerator was increasing significantly faster than the denominator); if the denominator wins, the answer will be 0.

Or here may be some compromise, in which case the answer will be some finite positive number.

In general, if we have a limit of the form

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

where both $f(x) \to \infty$ (or $-\infty$) and $g(x) \to \infty$ (or $-\infty$) as $x \to a$,th en the limit may or may not exist and is called an indeterminate form of type $\frac{\infty}{\infty}$.

This type of limit can be evaluated for certain functions, including rational functions, by dividing numerator and denominator by the highest power of x that occurs in the denominator.

For instance,

$$\lim_{x \to \infty} \frac{x^2 - 1}{2x^2 + 1} = \lim_{x \to \infty} \frac{1 - \dfrac{1}{x^2}}{2 + \dfrac{1}{x^2}} = \frac{1 - 0}{2 + 0} = \frac{1}{2}$$

This method does not work for limits such as $\displaystyle\lim_{x \to \infty} \frac{\ln x}{x - 1}$, but l'Hospital's Rule also applies to this type of indeterminate form.

1.10.3 L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I that contains a (except possibly at a). Suppose that

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

or that

$$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty$$

Then,

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

if the limit on the right side exists (or is ∞ or $-\infty$).

Note:

1. L'Hospital's Rule says that the limit of a quotient of functions is equal to the limit of the quotient of their derivatives, provided that the given conditions are satisfied.
   It is especially important to verify the conditions regarding the limits of f and g before using l'Hospital's Rule.
2. L'Hospital's Rule is also valid for one-sided limits and for limits at infinity or negative infinity; that is, "$x \to a$" can be replaced by any of the symbols $x \to a^+$, $x \to a^-$, $x \to \infty$ or $x \to -\infty$.

Example (16)

(i) Find $\lim_{x \to 1} \frac{\ln x}{x - 1}$.

(ii) Calculate $\lim_{x \to \infty} \frac{e^x}{x^2}$.

(iii) Find $\lim_{x \to 0} \frac{\tan x - x}{x^3}$.