
Chapter - 01

Differentiation and Integration

Independent Study Material No. 01

Combinations of Functions

★ Two functions f and g can be combined to form new functions $f + g$, $f - g$, fg , and f/g in a manner similar to the way we add, subtract, multiply, and divide real numbers.

★ The sum and difference functions are defined by

$$(f + g)(x) = f(x) + g(x) \text{ and } (f - g)(x) = f(x) - g(x).$$

★ If the domain of f is A and the domain of g is B , then the domain of $f + g$ is the intersection $A \cap B$ because both $f(x)$ and $g(x)$ have to be defined

★ For example, the domain of $f(x) = \sqrt{x}$ is $A = [0, \infty)$ and the domain of $g(x) = \sqrt{2-x}$ is $B = (-\infty, 2]$, so the domain of $(f + g)(x) = \sqrt{x} + \sqrt{2-x}$ is $A \cap B = [0, 2]$.

★ Similarly, the product and quotient functions are defined by

$$(fg)(x) = f(x)g(x)$$

and

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}.$$

★ The domain of fg is $A \cap B$ and the domain of f/g is $\{x \in A \cap B \mid g(x) \neq 0\}$.

★ For instance, if $f(x) = x^2$ and $g(x) = x - 1$, then the domain of the rational function $\frac{x^2}{(x-1)}$ is $\{x \mid x \neq 1\}$, or $(-\infty, 1) \cup (1, \infty)$.

★ There is another way of combining two functions to obtain a new function. For example, suppose that $y = f(u) = \sqrt{u}$ and $u = g(x) = x^2 + 1$. Since y is a function of u and u is, in turn, a function of x , it follows that y is ultimately a function of x .

We compute this by substitution:

$$y = f(u) = f(g(x)) = f(x^2 + 1) = \sqrt{x^2 + 1}$$

The procedure is called **composition** because the new function is composed of the two given functions f and g .

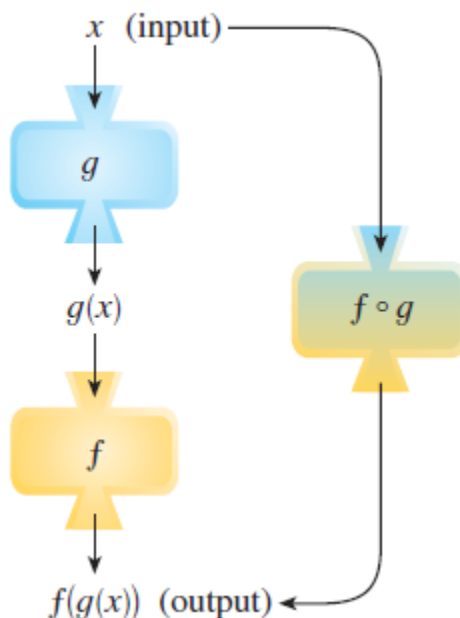
- ★ In general, given any two functions f and g , we start with a number x in the domain of g and calculate $g(x)$. If this number $g(x)$ is in the domain of f , then we can calculate the value of $f(g(x))$.
- ★ Notice that the output of one function is used as the input to the next function.
- ★ The result is a new function $h(x) = f(g(x))$ obtained by substituting g into f . It is called the **composition** (or **composite**) of f and g and is denoted by $f \circ g$ (“ f circle g ”).

Definition:

Given two functions f and g , the **composite function** $f \circ g$ (also called the **composition of f and g**) is defined by

$$(f \circ g)(x) = f(g(x)).$$

The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f . In other words, $(f \circ g)(x)$ is defined whenever both $g(x)$ and $f(x)$ are defined.



The $f \circ g$ machine is composed of the g machine (first) and then the f machine.

Example 01 If $f(x) = x^2$ and $g(x) = x - 3$, find the composite functions $f \circ g$ and $g \circ f$.

Answer:

$$(f \circ g) = f(g(x)) = f(x - 3) = (x - 3)^2$$

$$(g \circ f) = g(f(x)) = g(x^2) = x^2 - 3$$

■ In general, $f \circ g \neq g \circ f$.

Example 02 If $f(x) = \sqrt{x}$ and $g(x) = \sqrt{2-x}$, find each of the following functions and their domains.

(i) $f \circ g$

(ii) $g \circ f$

(iii) $f \circ f$

(iv) $g \circ g$

■ It is possible to take the composition of three or more functions. For instance, the composite function $f \circ g \circ h$ is found by first applying h , then g , and then f as follows:

$$(f \circ g \circ h)(x) = f(g(h(x)))$$

Example 03 Find $f \circ g \circ h$ if $f(x) = \frac{x}{(x+1)}$, $g(x) = x^{10}$ and $h(x) = x + 3$.

In calculus it is often useful to be able to **decompose** a complicated function into simpler ones, as in the following example.

Example 04 Given $F(x) = \cos^2(x + 9)$, find the functions f , g and h such that $F = f \circ g \circ h$.

Answer:

Since $F(x) = \cos^2(x + 9)$, the formula for F says: First add 9, then take the cosine of the result, and finally square. So we let

$$h(x) = x + 9 ; \quad g(x) = \cos x ; \quad f(x) = x^2$$

Then,

$$(f \circ g \circ h)(x) = f(g(h(x))) = f(g(x + 9)) = f(\cos(x + 9)) = (\cos(x + 9))^2 = \cos^2(x + 9) = F(x)$$