EN1020 Circuits, Signals and Systems
Topic 05 - Two-Port Networks

Upeka Premaratne

Department of Electronic and Telecommunication Engineering

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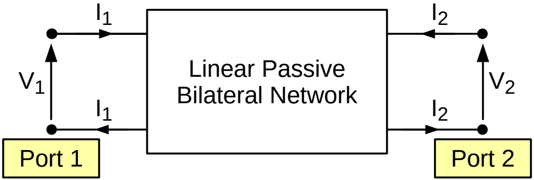
Outline

Introduction

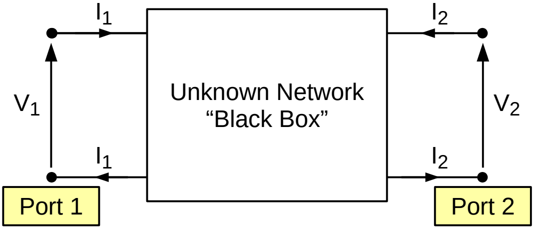
Introduction

- Two-port networks is a method of describing the behavior of linear passive bilateral circuit with two port connections in terms of parameters obtained from measurements
  - Makes it unnecessary to thoroughly analyze the circuit



Black Box Approach

- A further advantage of using two-port networks is that it can be used to approximate an unknown circuit based entirely on measurements
- These parameters can then be used to calculate a multitude of important system parameters related to transfer operations



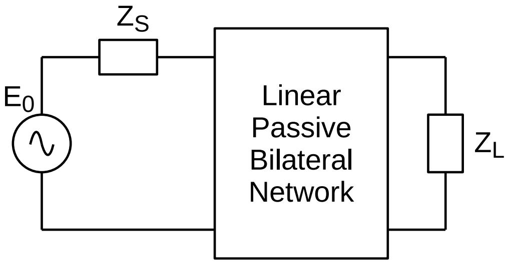
System Transfer Operations

System Transfer Operations

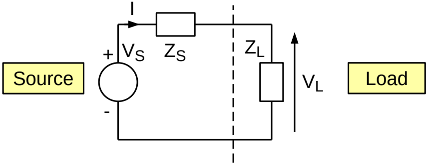
- A transfer operation from the input to the output is often the main requirement of an electrical or electronic circuit
- For example
  - A passive filter (voltage transfer)
  - A pre-amplifier (voltage transfer)
  - A power amplifier (power transfer through current transfer)
- Therefore, realization of or analysis of a circuit to improve a transfer operation is a common design requirement
  - Two-port networks provide a convenient description for this purpose for most passive and small signal active networks

Source and Load Connection Model

- The objective is to commonly maximize delivery of
  1. The input to the system
  2. The system output to the load
- Applicable to active networks as well



Voltage Transfer

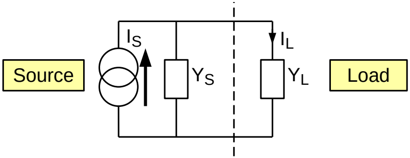


- Voltage transfer to the load

$$V_L = \frac{Z_L V_S}{Z_L + Z_S}$$

- To maximize voltage transfer ($V_L \to V_S$)
  - Typically requires $|Z_L| \gg |Z_S|$

Current Transfer

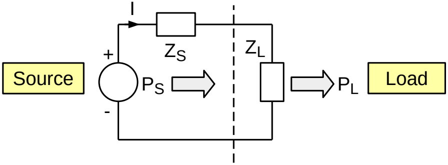


- Current transfer to the load

$$I_L = \frac{Y_L I_S}{Y_L + Y_S}$$

- To maximize current transfer $(I_L \to I_S)$
  - Requires $|Y_L| \gg |Y_S|$
  - In most cases unrealistic

Power Transfer



- Power from the source is delivered to the load
- Since $P = VI$ different from current or voltage transfer
- Two cases have to be considered
  1. Resistive load (no reactance)
  2. Complex load (resistive and reactive)

Resistive Load

$$
\begin{aligned}
I &= \frac{V_S}{R_S + R_L} \\
P_L &= V_L I = \left(\frac{R_L V_S}{R_S + R_L}\right)\left(\frac{V_S}{R_S + R_L}\right) \\
&= \frac{R_L V_S^2}{(R_S + R_L)^2} \\
\frac{d}{dR_L} P_L &= \frac{V_S^2}{(R_S + R_L)^4}\left[(R_S + R_L)^2 - 2R_L(R_S + R_L)\right] = 0 \\
\Rightarrow R_S &= R_L \text{ (For maximum power transfer)}
\end{aligned}
$$

Complex Load

- Take $Z_S = R_S + jX_S$ and $Z_L = R_L + jX_L$

$$
\begin{aligned}
I &= \frac{V_S}{(R_S + R_L) + j(X_S + X_L)} \\
P_L &= |I|^2 R_L = \frac{V_S^2 R_L}{\sqrt{(R_S + R_L)^2 + (X_S + X_L)^2}}
\end{aligned}
$$

- To find the maximum dissipated P_L has to be differentiated w.r.t. both R_L and X_L

$$
\frac{\partial}{\partial R_L} P_L = \frac{V_S^2 \left[\left[(R_S + R_L)^2 + (X_S + X_L)^2 \right] - 2R_L(R_S + R_L) \right]}{\left[(R_S + R_L)^2 + (X_S + X_L)^2 \right]^2}
$$

Complex Load (Contd..)

$$\frac{\partial}{\partial X_L} P_L = -\frac{V_S^2\left[-2R_L(X_S+X_L)\right]}{\left[(R_S+R_L)^2+(X_S+X_L)^2\right]^2}$$

- $\frac{\partial}{\partial X_L} P_L = 0$ results in $X_S = -X_L$
- $X_S = -X_L$ with $\frac{\partial}{\partial R_L} P_L = 0$ results in $R_S = R_L$
- Therefore, $Z_S = Z_L^*$ for maximum power transfer

Power Transfer Efficiency

- Maximum power transfoer gives the maximum power that can be *extracted* from the load
- The power efficiency is given by

$$\eta = \frac{P_L}{P_S} = \frac{V_L I}{V_S I} = \frac{R_L}{(R_L + R_S)}$$

- At maximum power transfer the system is only 50% efficient
  - Half the power is lost in the source as heat
- For maximum power transfer efficiency $R_S \ll R_L$
  - Same conditions for maximizing voltage transfer
  - However, there are often practical limitations

Power Transfer Efficiency (Contd..)

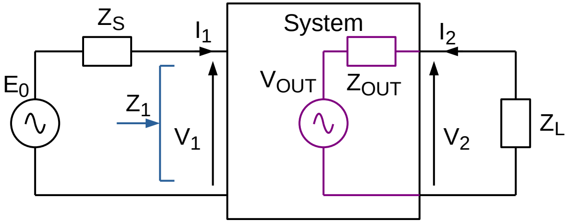
- The importance of maximum power transfer and high power transfer efficiency depend on the application
  - Driving a motor or a power supply will require maximum power transfer efficiency to minimize power wastage
  - A sensitive sensor that provides a small power output may require maximum power to be extracted for the best result
  - Audio power amplifiers seek to maximize power output (that is why speaker impedances have to be matched)

Two-Port Networks

Introduction

- The circuit is described in terms of a set of four parameters
- Each set is obtained differently
  1. Impedance parameters (z or open circuit parameters)
  2. Admittace parameters (y or short circuit parameters)
  3. Hybrid parameters
  4. ABCD parameters (cascade parameters)
- All of these sets of parameters
  - Conveniently describe the behavior of the circuit
  - Are interrelated with one another
  - Have specific applications

Terminated Two-Port Network Model



- Port 1 is taken as the input and Port 2 is taken as the output
- The source impedance and input impedance $z_{IN} = z_1$ forms a potential divider for the source
- The output impedance Z_{OUT} forms a Thevenin impedance with the output "source" V_{OUT}

Impedance Parameters

- Impedance parameters are obtained as ratios when one port is open circuited (i.e., $I_1 = 0$ or $I_2 = 0$)
- Feed impedance parameters

$$\begin{aligned} z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} \\ z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} \end{aligned}$$

  - z_{11} is the feed impedance of Port 1 when Port 2 is OC
  - z_{22} is the feed impedance of Port 2 when Port 1 is OC
  - If the network is symmetric $z_{11} = z_{22}$

Impedance Parameters (Contd..)

- Trans-impedance parameters

$$
\begin{aligned}
z_{12} &= \left.\frac{V_1}{I_2}\right|_{I_1=0} \\
z_{21} &= \left.\frac{V_2}{I_1}\right|_{I_2=0}
\end{aligned}
$$

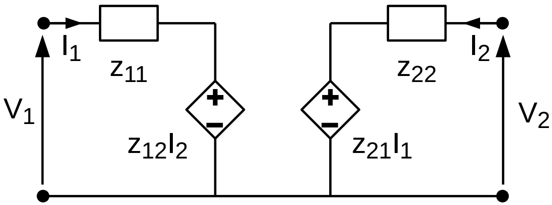
  - z_{12} is the TI of the OC voltage of Port 1 and current of Port 2
  - z_{21} is the TI of the OC voltage of Port 2 and current of Port 1
  - If the network is reciprocal $z_{12} = z_{21}$

Impedance Parameters (Contd..)

- The parameter matrix is given by

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ I_2 \end{pmatrix}$$

- Two-port model



Impedance Parameters (Contd..)

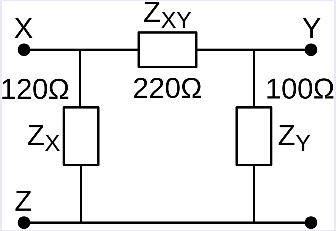
- For a terminated network

$$
\begin{aligned}
z_{IN} &= z_{11} - \left(\frac{z_{12} z_{21}}{z_{22} + z_L} \right) \\
z_{OUT} &= z_{22} - \left(\frac{z_{12} z_{21}}{z_{11} + z_S} \right) \\
A_I &= \frac{I_2}{I_1} = \frac{-z_{21}}{z_{22} + z_L} \\
A_V &= \frac{V_2}{V_1} = \frac{z_{21} z_L}{z_{11} z_L + |Z|} \\
A_S &= \frac{V_2}{V_S} = \frac{z_{21} z_L}{(z_{11} + z_S)(z_{22} + z_L) - z_{12} z_{21}}
\end{aligned}
$$

Impedance Parameters (Contd..)

Example:

1. Find the impedance parameter matrix of the following network
2. What will be the overall gain when the network is terminated with a load of $100\ \Omega$ and connected to a source with an impedance of $10\ \Omega$?



Impedance Parameters (Contd..)

- Impedance parameter matrix

$$Z = \frac{\begin{pmatrix} z_X(z_Y + z_{XY}) & z_X z_Y \\ z_X z_Y & z_Y(z_X + z_{XY}) \end{pmatrix}}{z_X + z_{XY} + z_Y}$$

- After substitution

$$Z = \begin{pmatrix} 87.27 & 27.27 \\ 27.27 & 77.27 \end{pmatrix}$$

- $A_S = 0.165$

Admittance Parameters

- Admittance parameters are obtained as ratios when one port is short circuited (i.e., $V_1 = 0$ or $V_2 = 0$)
- Feed admittance parameters

$$
\begin{aligned}
y_{11} &= \left.\frac{I_1}{V_1}\right|_{V_2=0} \\
y_{22} &= \left.\frac{I_2}{V_2}\right|_{V_1=0}
\end{aligned}
$$

  - y_{11} is the feed admittance of Port 1 when Port 2 is SC
  - y_{22} is the feed admittance of Port 2 when Port 1 is SC
  - If the network is symmetric $y_{11} = y_{22}$

Admittance Parameters (Contd..)

- Trans-admittance parameters

$$
\begin{aligned}
y_{12} &= \left.\frac{I_1}{V_2}\right|_{V_1=0} \\
y_{21} &= \left.\frac{I_2}{V_1}\right|_{V_2=0}
\end{aligned}
$$

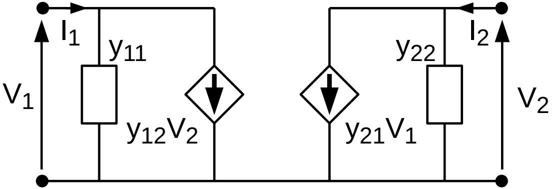
  - y_{12} is the TA of the SC current of Port 1 and voltage of Port 2
  - y_{21} is the TI of the SC current of Port 2 and voltage of Port 1
  - If the network is reciprocal $y_{12} = y_{21}$

Admittance Parameters (Contd..)

- The parameter matrix is given by

$$\begin{pmatrix} I_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}$$

- Relates to impedance parameters according to $Y = Z^{-1}$
- Two-port model



Admittance Parameters (Contd..)

- For a terminated network

$$
\begin{aligned}
y_{IN} &= y_{11} - \left(\frac{y_{12}y_{21}}{y_{22} + y_L}\right) \\
y_{OUT} &= y_{22} - \left(\frac{y_{12}y_{21}}{y_{11} + y_S}\right) \\
A_I &= \frac{I_2}{I_1} = \frac{y_{21}y_L}{y_{11}y_L + |Y|} \\
A_V &= \frac{V_2}{V_1} = \frac{-y_{21}}{y_{22} + y_L} \\
A_S &= \frac{V_2}{V_S} = \frac{y_{21}y_L}{(y_{11} + y_S)(y_{22} + y_L) - y_{12}y_{21}}
\end{aligned}
$$

Hybrid Parameters

- Hybrid parameters contain a mixture of ratios
  - Also inverse hybrid parameters aka g parameters ($G = H^{-1}$)
- Both are used for the small signal analysis of amplifiers
  - The small signal corresponds to a linearization
- Based on the following parameter matrix

$$\begin{pmatrix} V_1 \\ I_2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} I_1 \\ V_2 \end{pmatrix}$$

- Input impedance when Port 2 is SC

$$h_{11} = \left. \frac{V_1}{I_1} \right|_{V_2=0}$$

Hybrid Parameters (Contd..)

- Reciprocal of voltage gain when Port 1 is OC

$$h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$

- Current gain when Port 2 is SC

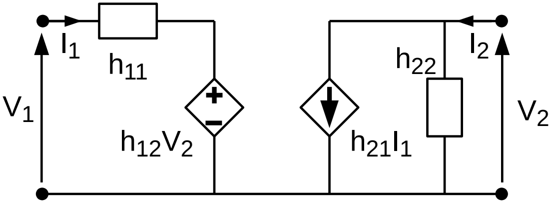
$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}$$

- Output admittance when Port 1 is OC

$$h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0}$$

Hybrid Parameters (Contd..)

- If the network is reciprocal $h_{12} = -h_{21}$
- When the network is symmetrical $h_{11}h_{22} - h_{12}h_{21} = 1$
- Two-port model



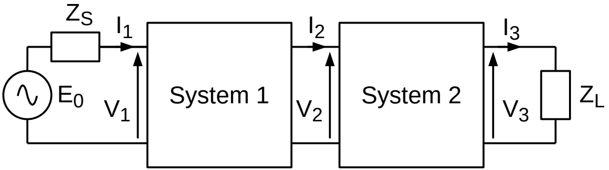
Hybrid Parameters (Contd..)

- For a terminated network

$$
\begin{aligned}
z_{IN} &= h_{11} - \left(\frac{h_{12}h_{21}z_L}{1 + h_{22}z_L}\right) \\
z_{OUT} &= \frac{z_S + h_{11}}{h_{22}z_S + |H|} \\
A_I &= \frac{I_2}{I_1} = \frac{h_{21}}{1 + h_{22}z_L} \\
A_V &= \frac{V_2}{V_1} = \frac{-h_{21}z_L}{h_{11} + |H|z_L} \\
A_S &= \frac{V_2}{V_S} = \frac{-h_{21}z_L}{(h_{11} + z_S)(1 + h_{22}z_L) - h_{12}h_{21}z_L}
\end{aligned}
$$

ABCD Parameters

- All sets of parameters up to now were suitable for single stage terminated networks
- ABCD parameters are a special set that can be used for the computational analysis of cascading systems
  - Can model a linear passive component as a 2x2 matrix



ABCD Parameters (Contd..)

- Like hybrid parameters contains a mixture of ratios
- Based on the following parameter matrix with a negative Port 2 current and parameters of a single port per vector

$$\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix}$$

- This can be alternatively expressed as

$$\begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix} = \begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix} \begin{pmatrix} V_1 \\ I_1 \end{pmatrix}$$

where $\mathcal{A} = \mathcal{B}^{-1}$

ABCD Parameters (Contd..)

- Reciprocal of voltage gain when Port 2 is OC

$$a_{11} = \left.\frac{V_1}{V_2}\right|_{I_2=0}$$

- Trans-impedance when Port 2 is SC

$$a_{12} = -\left.\frac{V_1}{I_2}\right|_{V_2=0}$$

- Trans-admittance when Port 2 is OC

$$a_{21} = \left.\frac{I_1}{V_2}\right|_{I_2=0}$$

ABCD Parameters (Contd..)

- Reciprocal of current gain when Port 2 is SC

$$a_{22} = -\frac{I_1}{I_2}\bigg|_{V_2=0}$$

- For a reciprocal network $a_{11}a_{22} - a_{12}a_{21} = 1$
- When symmetric $a_{11} = a_{22}$
- When terminated

$$z_{IN} = \frac{a_{11}z_L + a_{12}}{a_{21}z_L + a_{22}}$$

$$z_{OUT} = \frac{a_{12} + a_{22}z_S}{a_{11} + a_{21}z_S}$$

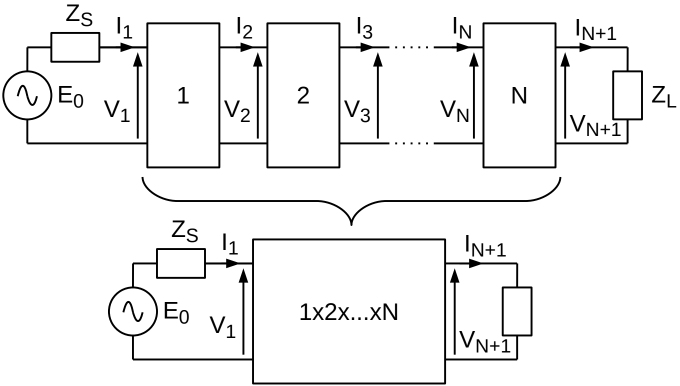
ABCD Parameters (Contd..)

- Gain ratios

$$
\begin{aligned}
A_I &= \frac{I_2}{I_1} = \frac{-1}{a_{21}z_L + a_{22}} \\
A_V &= \frac{V_2}{V_1} = \frac{z_L}{a_{11}z_L + a_{12}} \\
A_S &= \frac{V_2}{V_S} = \frac{z_L}{(a_{11} + a_{21}z_S)z_L + a_{12} + a_{22}z_S}
\end{aligned}
$$

- **Important Note:** when deriving the expressions for a terminated network for ABCD parameters it is necessary to consider the output current as $(-I_2)$ to prevent a violation of the convention of having a current enter the port of a two-port network.

Cascading


Z_S
I_1
I_2
I_3
I_N
I_N+1
E_0
V_1
1
V_2
2
V_3
V_N
N
V_N+1
Z_L
Z_S
I_1
I_N+1
E_0
V_1
1x2x...xN
V_N+1

Cascading (Contd..)

- The main importance of ABCD parameters as a circuit analysis and synthesis tool is the ability to cascade multiple systems and simplify as a matrix multiplication

$$
\begin{aligned}
\begin{pmatrix} V_1 \\ I_1 \end{pmatrix} &= \begin{pmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{pmatrix} \begin{pmatrix} V_2 \\ -I_2 \end{pmatrix} \\
&= \begin{pmatrix} a_{11}^1 & a_{12}^1 \\ a_{21}^1 & a_{22}^1 \end{pmatrix} \begin{pmatrix} a_{11}^2 & a_{12}^2 \\ a_{21}^2 & a_{22}^2 \end{pmatrix} \begin{pmatrix} V_3 \\ -I_3 \end{pmatrix} \\
&= \mathcal{A}_1 \times \mathcal{A}_2 \times \cdots \times \mathcal{A}_N \begin{pmatrix} V_{N+1} \\ -I_{N+1} \end{pmatrix}
\end{aligned}
$$

Cascading (Contd..)

- Series component $\mathcal{A}$ matrix

$$\mathcal{A}_Z = \begin{pmatrix} 1 & Z \\ 0 & 1 \end{pmatrix} \Rightarrow \mathcal{B}_Z = \begin{pmatrix} 1 & -Z \\ 0 & 1 \end{pmatrix}$$

- Shunt component $\mathcal{A}$ matrix

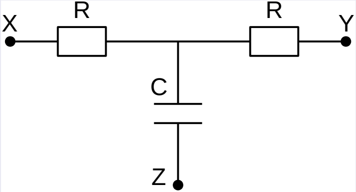
$$\mathcal{A}_Y = \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} \Rightarrow \mathcal{B}_Y = \begin{pmatrix} 1 & 0 \\ -Y & 1 \end{pmatrix}$$

- Using these two matrices an entire circuit can be obtained in its matrix form in the ω or s domain
- The $\mathcal{B}$ form of the matrix allows for the output to be directly obtained without inversion

Cascading (Contd..)

Example:

Obtain the $\mathcal{A}$ and $\mathcal{B}$ matrices of the RC T filter given below



Cascading (Contd..)

- $\mathcal{A}$ matrix

$$\mathcal{A} = \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ j\omega C & 1 \end{pmatrix} \begin{pmatrix} 1 & R \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + j\omega RC & 2R + j\omega R^2 C \\ j\omega C & 1 + j\omega RC \end{pmatrix}$$

- $\mathcal{B}$ matrix

$$\mathcal{B} = \begin{pmatrix} 1 & -R \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -j\omega C & 1 \end{pmatrix} \begin{pmatrix} 1 & -R \\ 0 & 1 \end{pmatrix}$$

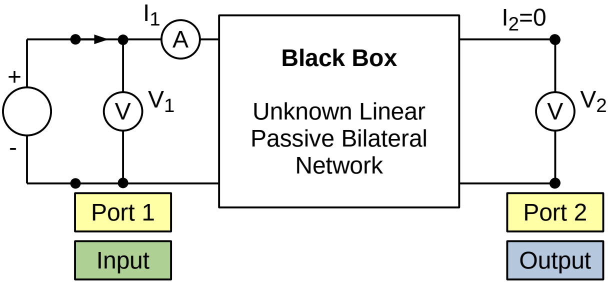
$$= \begin{pmatrix} 1 + j\omega RC & -(2R + j\omega R^2 C) \\ -j\omega C & 1 + j\omega RC \end{pmatrix}$$

Matrix Estimation

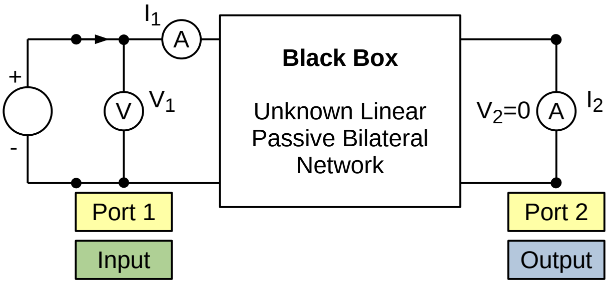
Measurement

- One of the main strengths of the two-port network model is the ability to estimate the matrix from measurements
  - Measured using ammeters and voltmeters
- Requires the following tests for each port
  - The open circuit test (output port open circuited)
  - The short circuit test (output port short circuited)
- The measurement configuration depends upon the input impedance which is initially unknown
  - Critical only at the extremes
  - Therefore, can initially take use to estimate the feed impedance and determine the required configuration

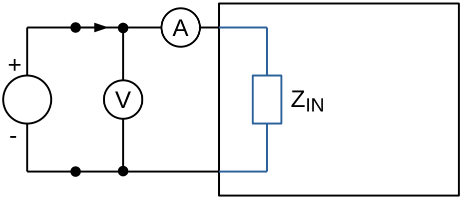
The Open Circuit Test



The Short Circuit Test

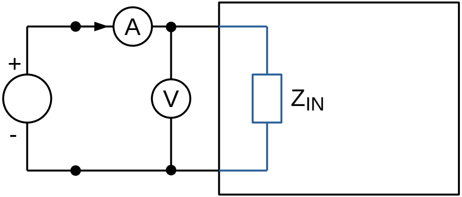


High Impedance Measurement



- This configuration is necessary when the feed impedance is comparable to the large impedance of the voltmeter

Low Impedance Measurement



- This configuration is necessary when the feed impedance is comparable to the small impedance of the ammeter

Example 1

Port 1 of an unknown two-port network consisting of a linear passive bilateral network is subjected to an open circuit and short circuit test. The test results are given as follows,

Test	V_1	I_1	V_2	I_2
OC	12 V	40 mA	8 V	0
SC	12 V	72.3 mA	0	(-47.7) mA

Obtain the ABCD parameters of the network.

Example 1 (Contd..)

$$
\begin{aligned}
a_{11} &= \left.\frac{V_1}{V_2}\right|_{I_2=0} = \frac{12}{8} = 1.5 \\
a_{12} &= -\left.\frac{V_1}{I_2}\right|_{V_2=0} = -\frac{12}{(-0.0477)} = 251.6\ \Omega \\
a_{21} &= \left.\frac{I_1}{V_2}\right|_{I_2=0} = \frac{0.040}{8} = 5\ \mathrm{mS} \\
a_{22} &= -\left.\frac{I_1}{I_2}\right|_{V_2=0} = -\frac{0.0723}{(-0.0477)} = 1.5
\end{aligned}
$$

Example 2

The following open circuit test results were obtained for an unknown linear passive bilateral two-port network.

Port	V_1	I_1	V_2	I_2
1	9 V	45 mA	2.25 V	0
2	6 V	0	9 V	120 mA

Determine the impedance parameters of the network.

Example 2 (Contd..)

$$
\begin{aligned}
z_{11} &= \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{9}{0.045} = 200\ \Omega \\
z_{12} &= \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{6}{0.120} = 50\ \Omega \\
z_{21} &= \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{2.25}{0.045} = 50\ \Omega \\
z_{22} &= \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{9}{0.120} = 75\ \Omega
\end{aligned}
$$

Conclusion

Conclusion

- Two-port networks provide a measurement based description of a linear bilateral passive circuit
  - Does not require a thorough analysis
  - Can be used for analyzing the behavior of the circuit in the frequency or transient (Laplace) domain
  - Specially the transfer of voltage, current or power
- It can also be used to approximate a black box
- Relates to the formal transfer function of a circuit which maps the input to the output