Tutorial (05)

Solutions

(01)

(g)

$$\frac{(1+i)}{(1-i)} - (1+2i)(2+2i) + \frac{(3+i)}{(1+i)}$$

$$= \frac{(1+i)^2}{(1-i)(1+i)} - (1+2i)(2+2i) + \frac{(3+i)(1-i)}{(1+i)(1-i)}$$

$$= i - 2 - 2i - 4i + 4 + \frac{3-1-3i-i}{2}$$

$$= i + 2 - 6i + \frac{2-4i}{2}$$

$$= 2 - 5i + 1 - 2i$$

$$= 3 - 7i$$

(02)

(02)

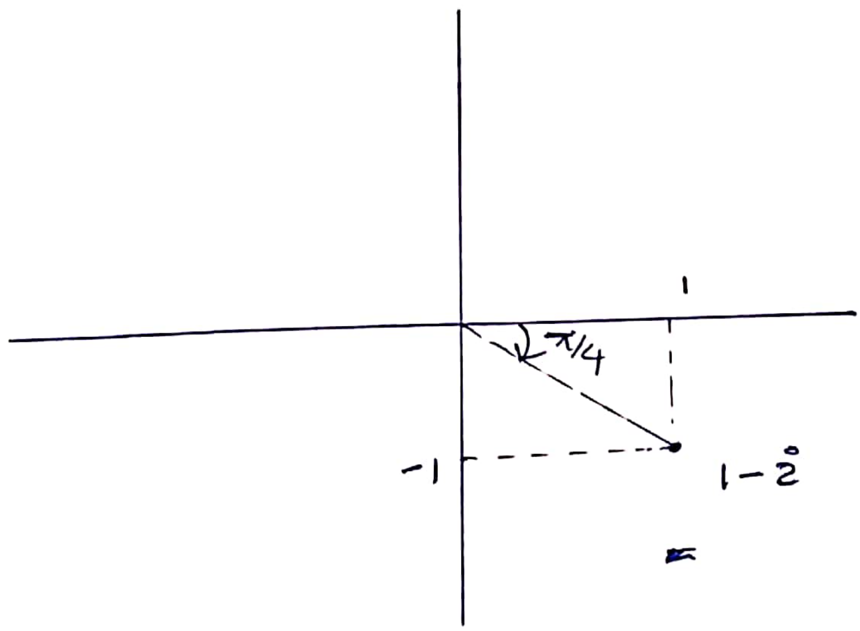
(a)

$$\text{let } z = \left(\frac{3-i}{2+i}\right)^2$$

$$= \left(\frac{(3-i)(2-i)}{(2+i)(2-i)}\right)^2$$

$$= \left(\frac{6+i^2-5i}{5}\right)^2$$

$$= (1-i)^2$$



$$1-i = \sqrt{2}\left\{\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right\}$$

$$= \sqrt{2}\, e^{i\left(-\frac{\pi}{4}\right)}$$

$$\therefore z = (\sqrt{2})^2 e^{i(-\frac{\pi}{2})}$$ (03)

$$= 2\left\{\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right\}$$ //

~~#12~~

(b) let $z = \dfrac{(1-i)}{(1+i)(1+\sqrt{3}\,i)}$

$$= \frac{(1-i)(1-i)(1-\sqrt{3}\,i)}{(1+i)(1-i)(1+\sqrt{3}\,i)(1-\sqrt{3}\,i)}$$

$$= \frac{[1+i^2-2i](1-\sqrt{3}\,i)}{(1-i^2)(1-3i^2)}$$

$$= \frac{-2i(1-\sqrt{3}\,i)}{8}$$

$$= \frac{1}{4}(-\sqrt{3}-i)$$

$$= \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$= \frac{1}{2}\left\{-\cos\left(\pi+\frac{\pi}{6}\right) + i \sin\left(\pi+\frac{\pi}{6}\right)\right\}$$

$$= \frac{1}{2}\left\{-\cos\left(\frac{5\pi}{6}\right) - \cancel{i}\left(\frac{5\pi}{6}\right)\right\}.$$

(03)

(a) $(1+i)^6 = \left[\sqrt{2}\left\{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right\}\right]^{6}$ (04)

$= 8\left\{\cos\left(\frac{6\pi}{4}\right) + i\sin\left(\frac{6\pi}{4}\right)\right\}$

$= 8\left\{\cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right)\right\}$

$= -8i$

(d) $(1+\sqrt{3}i)^{2011}$

$1+\sqrt{3}i = 2e^{i\pi/3}$

$(1+\sqrt{3}i)^{2011} = \cancel{3^{2011}} = 2^{2011} e^{i(2011\times\pi/3)}$

$= 2^{2011} e^{i\left(2011\frac{\pi}{3}\right)}$

$= 2^{2011} e^{i\left(670\pi i + \frac{\pi i}{3}\right)}$

$= 2^{2011} e^{i\left(\frac{\pi i}{3}\right)}$

$= 2^{2011}(1+\sqrt{3}i)$

(04)

(b) let $z = \dfrac{3 + 2i \sin\theta}{1 - 2i \sin\theta}$

$$\therefore z = \frac{(3 + 2i \sin\theta)(1 + 2i \sin\theta)}{(1 - 2i \sin\theta)(1 + 2i \sin\theta)}$$

$$= \frac{(3 - 4\sin^2\theta) + 8i \sin\theta}{(1 + 4\sin^2\theta)}$$

$$\therefore z = \left(\frac{3 - 4\sin^2\theta}{1 + 4\sin^2\theta}\right) + \left(\frac{8\sin\theta}{1 + 4\sin^2\theta}\right)i$$

if z is real

$$\frac{8\sin\theta}{1 + 4\sin^2\theta} = 0$$

$$\therefore \sin\theta = 0 = \sin 0$$

$\therefore$ the general solution, $\quad n \in \mathbb{Z}$

$$\theta = n\pi + (-1)^n (0) \quad ;$$

$$\therefore \theta = n\pi$$

If z is imaginary (06)

$$\frac{3-4\sin^2\theta}{1+4\sin^2\theta} = 0$$

$$\therefore \sin^2\theta = \frac{3}{4}$$

$$\therefore \sin\theta = \pm\frac{\sqrt{3}}{2}$$

$$\therefore \theta = n\pi \pm (-1)^n \frac{\pi}{3} \; ; \quad n \in \mathbb{Z}$$

(05)

(a) (i)

$$z^3 = 27$$

$$z^3 - 27 = 0$$

$$z^3 - 3^3 = 0$$

$$(z-3)(z^2 + 3z + 9) = 0$$

$\therefore\ z = 3$ or $z^2 + 3z + 9 = 0$

$$\therefore\ z = \frac{-3 \pm \sqrt{9 - 4\times1\times9}}{2\times1}$$

$$= \frac{-3 \pm \sqrt{27}\ i}{2}$$

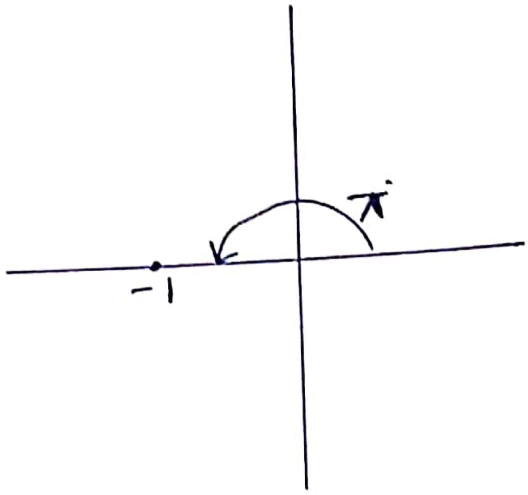
$$= \frac{-3 \pm 3\sqrt{3}\ i}{2}$$

$\therefore\ z = 3$ or

$z = \dfrac{-3 + 3\sqrt{3}\ i}{2}$ or

$$z = \frac{-3 - 3\sqrt{3}\ i}{2}$$

05) (a) (v) $(z-1)^4 = -1$



let $z - 1 = w$

$\therefore \quad w^4 = -1$

$\therefore$ let $w = re^{i\theta}$

$\therefore \quad w^4 = r^4 e^{i4\theta}$

$\therefore \quad |w^4| = r^4 = |-1| = 1$

$\arg(w^4) = 4\theta = \pi + 2k\pi \; ; \; k \in \mathbb{Z}$

$\therefore \quad r = 1$

$\theta = \frac{\pi}{4} + k\frac{\pi}{2} \; ; \; k \in \mathbb{Z}$

when $k = 0$; $\theta = \frac{\pi}{4}$

$k = 1$; $\theta = \frac{3\pi}{4}$

$k = 2$; $\theta = \frac{5\pi}{4}$

$k = 3$; $\theta = \frac{7\pi}{4}$

$k = 4$; $\theta = \frac{9\pi}{4} = \frac{9\pi}{4} - 2\pi = \frac{\pi}{4}$

$\therefore \; w = re^{i\theta}$ where $r = 1$ and $\theta = \frac{\pi}{4}$, or $\frac{3\pi}{4}$ or $\frac{5\pi}{4}$ or $\frac{7\pi}{4}$

since $z = w+1$, we have

$$z_1 = 1 + e^{i\frac{\pi}{4}} = 1 + \left\{\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right\}$$
$$= 1 + \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$
$$\therefore z_1 = \left(1 + \frac{1}{\sqrt{2}}\right) + i\left(\frac{1}{\sqrt{2}}\right)$$

or

$$z_2 = 1 + e^{i\frac{3\pi}{4}} = 1 + \left\{\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right\}$$
$$= 1 + \left\{\cos\left(\pi - \frac{\pi}{4}\right) + i\sin\left(\pi - \frac{\pi}{4}\right)\right\}$$
$$= 1 + \left\{-\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right\}$$
$$= 1 - \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$$
$$= \left(1 - \frac{1}{\sqrt{2}}\right) + i\left(\frac{1}{\sqrt{2}}\right)$$

or

$$z_3 = 1 + e^{i\frac{5\pi}{4}} = \left(1 - \frac{1}{\sqrt{2}}\right) - i\left(\frac{1}{\sqrt{2}}\right)$$

or

$$z_4 = 1 + e^{i\frac{7\pi}{4}} = \left(1 + \frac{1}{\sqrt{2}}\right) - i\left(\frac{1}{\sqrt{2}}\right).$$

(06) (a)

(iv) $|z| + |z-4| = 6$

let $z = x + iy$

$\therefore \sqrt{x^2+y^2} + \sqrt{(x-4)^2 + y^2} = 6$

$$(x-4)^2 + y^2 = \left(6 - \sqrt{x^2+y^2}\right)^2$$

$$x^2 - 8x + 16 + y^2 = 36 - 12\sqrt{x^2+y^2} + x^2+y^2$$

$$12\sqrt{x^2+y^2} = 8x + 20$$

$$3\sqrt{x^2+y^2} = 2x+5$$

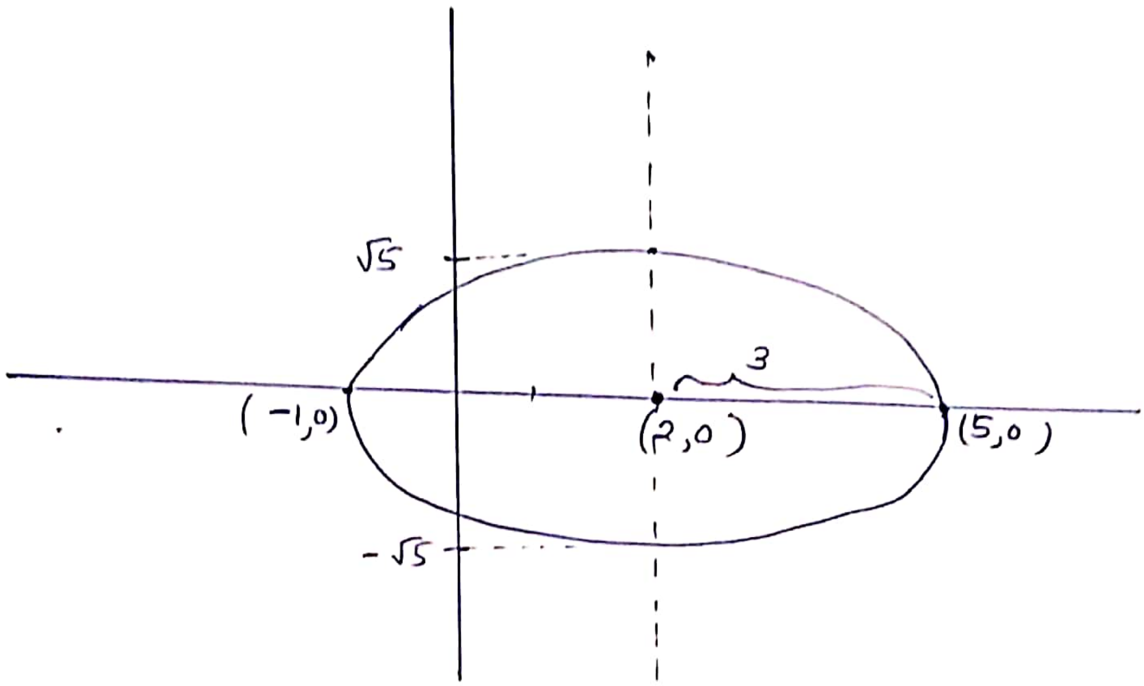
$$9(x^2+y^2) = (2x+5)^2 = 4x^2+20x+25$$

$$5x^2 - 20x + 9y^2 = 25$$

$$5(x-2)^2 + 9y^2 = 45$$

$$\frac{(x-2)^2}{9} + \frac{y^2}{5} = 1$$

$$\frac{(x-2)^2}{3^2} + \frac{(y-0)^2}{(\sqrt{5})^2} = 1$$



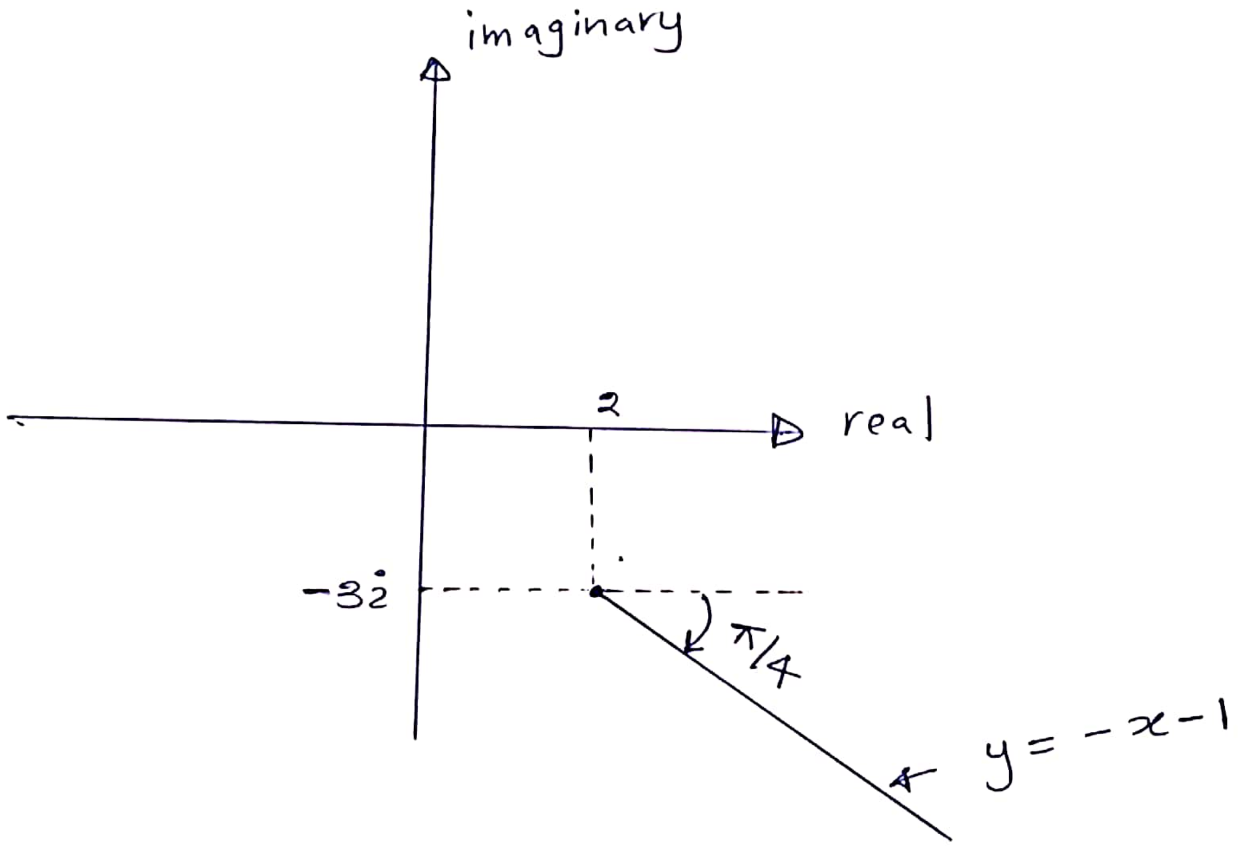
$\therefore$ the locus is an ellipse.

(06)

(b) (ii)

$$\arg(z - 2 + 3i) = -\frac{\pi}{4}$$

$$\arg(z - (2 - 3i)) = -\frac{\pi}{4}$$



let $z = x + iy$

$$\therefore \arg((x-2) + (y+3)i) = -\frac{\pi}{4}$$

$$\therefore \quad \frac{y+3}{x-2} = -\tan\left(\frac{\pi}{4}\right) = -1$$

$$y + 3 = -x + 2$$

$$\therefore \quad y = -x - 1$$