Comparing the Performance of Queues

M/M/1/K Queue

Lecture 6

Communication Theory III

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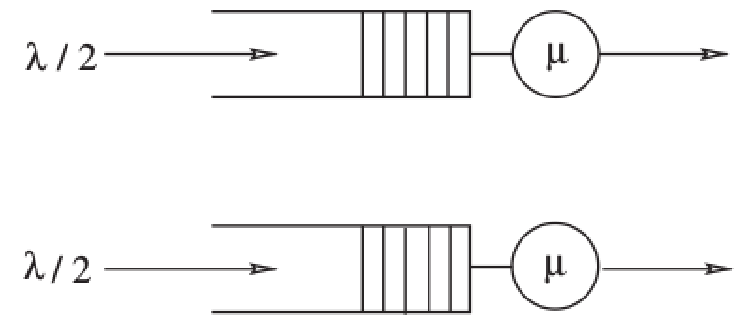
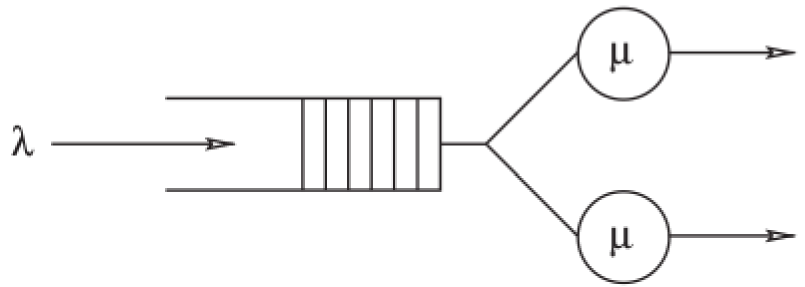
2 of M/M/1 Queues and M/M/2 Queue



(a) Separate queue system



(b) Common queue

Performance of two M/M/1 Queues

- Arrival rate $\lambda/2$ and service rate μ.
- $\rho = (\lambda/2)/\mu = \lambda/(2\mu)$
- Mean Number for 1 M/M/1 Queue = $\rho/(1-\rho)$
- Mean Number for 2 M/M/1 Queues = $2\rho/(1-\rho)$
- The average response time can now be found using Little's law for 2 M/M/1 Queues:

$$E[R_1] = \frac{1}{\lambda} E[N_1] = \frac{1}{\lambda} \frac{2\rho}{(1-\rho)} = \frac{2}{2\mu - \lambda}.$$

Performance of M/M/2 Queue

- The mean number of customers in an M/M/c queue with arrival rate λ and service rate μ per server is given by,

$$E[N_2] = \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^c \lambda\mu}{(c-1)!(c\mu-\lambda)^2} p_0 \quad \text{with} \quad \frac{\lambda}{c\mu} = \rho \quad \text{or} \quad \lambda/\mu = c\rho$$

- With c = 2.

$$\begin{aligned} L_2 = E[N_2] = \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^2 \lambda\mu}{(2\mu-\lambda)^2} p_0 &= \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^2(\lambda/\mu)}{(1/\mu^2)(2\mu-\lambda)^2} p_0 \\ &= \frac{\lambda}{\mu} + \frac{(\lambda/\mu)^3}{(2-\lambda/\mu)^2} p_0 \\ &= 2\rho + \frac{(2\rho)^3}{(2-2\rho)^2} p_0. \end{aligned}$$

Performance of M/M/2 Queue

- The probability of the system being empty, p_0 , is computed as:

$$p_0 = \left[1 + \sum_{n=1}^{c-1} \frac{(c\rho)^n}{n!} + \frac{(c\rho)^c}{c!}\left(\frac{1}{1-\rho}\right)\right]^{-1}$$

$$= \left[1 + 2\rho + \frac{(2\rho)^2}{2!}\frac{1}{1-\rho}\right]^{-1} = \frac{1-\rho}{1+\rho}.$$

- Thus,

$$L_2 = 2\rho + \frac{8\rho^3(1-\rho)}{4(1-\rho)^2(1+\rho)} = \frac{2\rho(1-\rho)(1+\rho)+2\rho^3}{(1-\rho)(1+\rho)} = \frac{2\rho}{1-\rho^2}.$$

- Using Little's formula

$$E[R_2] = \frac{1}{\lambda}E[N_2] = \frac{2\rho/\lambda}{1-\rho^2} = \frac{1/\mu}{1-\rho^2} = \frac{4\mu}{4\mu^2-\lambda^2}.$$

Performance of single superserver working twice as fast

- Use the M/M/1 queue with arrival rate λ and service rate 2μ.

$$L_3 = E[N_3] = \frac{\rho}{1-\rho} \quad \text{and} \quad E[R_3] = \frac{1/2\mu}{1-\lambda/2\mu} = \frac{1}{2\mu-\lambda}$$

Comparison of the three systems

$$E[N_1] = \frac{2\rho}{1-\rho} \quad \geq \quad E[N_2] = \frac{2\rho}{1-\rho} \cdot \frac{1}{1+\rho} \quad \geq \quad E[N_3] = \frac{\rho}{1-\rho},$$

$$E[R_1] = \frac{2}{2\mu-\lambda} \quad \geq \quad E[R_2] = \frac{1/\mu}{1-\rho^2} \quad \geq \quad E[R_3] = \frac{1}{2\mu-\lambda},$$

- Let $\alpha = 2\rho/(1 - \rho)$ and $\beta = 2/(2\mu - \lambda)$, then $\alpha = \lambda\beta$

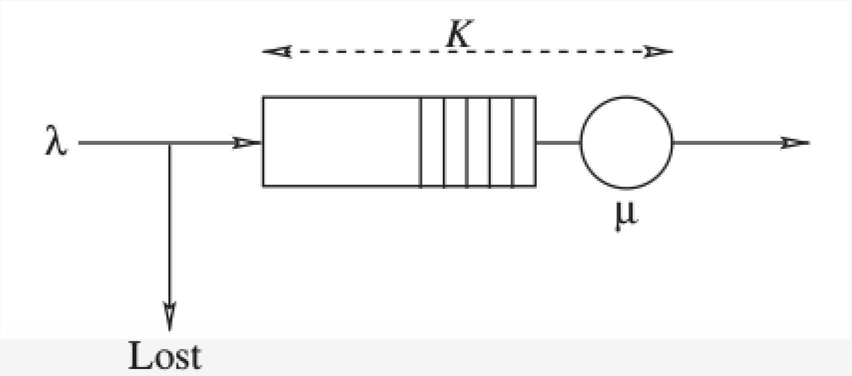
$$E[N_1] = \alpha \quad \geq \quad E[N_2] = \alpha \cdot \frac{1}{1+\rho} \quad \geq \quad E[N_3] = \alpha/2,$$

$$E[R_1] = \beta \quad \geq \quad E[R_2] = \beta \cdot \frac{1}{1+\rho} \quad \geq \quad E[R_3] = \beta/2.$$

What can you say about the three Queues?

Finite-Capacity Systems—The M/M/1/K Queue

- Customers arrive according to a Poisson process at rate λ and receive service that is exponentially distributed with a mean service time of 1/μ from a single server. At most K customers are allowed into the system.



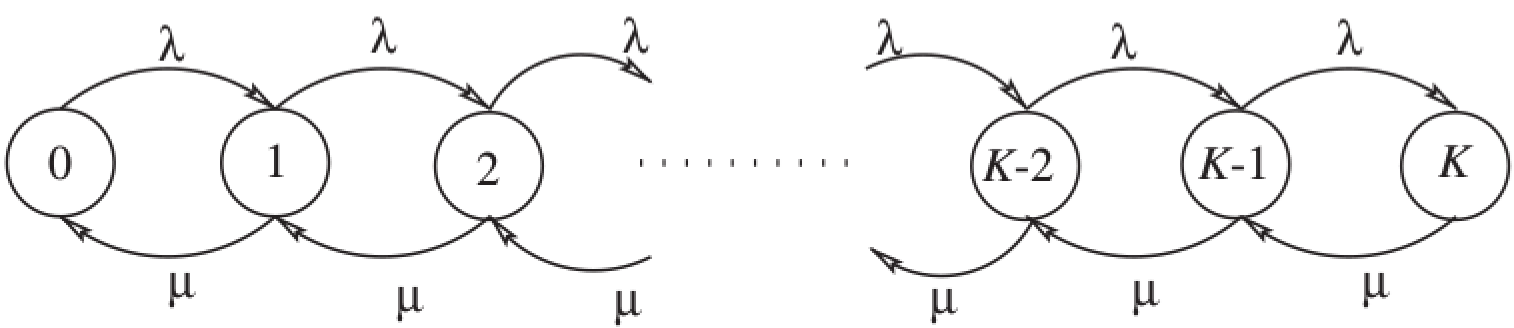
- A customer who arrives after system is filled, will be dropped out.
- **<u>In communication systems, rejected customers are called “lost” customers and the system is called a “loss” system.</u>**

Analysis of M/M/1/k Queue

- Poisson arrivals stop as soon as K customers are present. The parameters of the exponential service time distribution are unchanged.

$$\lambda_n = \begin{cases} \lambda, & n < K, \\ 0, & n \geq K, \end{cases}$$

$$\mu_n = \mu \quad \text{for } n = 1, 2, \ldots, K.$$



Analysis of M/M/1/k Queue

$$p_n = p_0 \prod_{i=1}^{n} \frac{\lambda_{i-1}}{\mu_i}, \quad n \geq 1,$$

$$p_1 = \frac{\lambda}{\mu} p_0,$$

$$p_{n+1} = \frac{\lambda + \mu}{\mu} p_n - \frac{\lambda}{\mu} p_{n-1}, \quad 1 \leq n < K - 1,$$

$$p_K = \frac{\lambda}{\mu} p_{K-1}.$$

Analysis of M/M/1/k Queue

- For all possible values of n, and setting ρ = λ/μ,

$$p_n = \left(\frac{\lambda}{\mu}\right)^n p_0 = \rho^n p_0, \quad 0 \le n \le K.$$

$$p_0 = \frac{1}{\sum_{n=0}^{K} \rho^n}.$$

$$p_0 = \begin{cases} (1-\rho)/(1-\rho^{K+1}) & \text{if } \rho \neq 1 \\ 1/(K+1) & \text{if } \rho = 1. \end{cases}$$

- Therefore, for all 0 ≤ n ≤ K ,

$$p_n = \frac{(1-\rho)\rho^n}{1-\rho^{K+1}} \quad \text{if } \rho \neq 1,$$

Analysis of M/M/1/k Queue

- Let $K \to \infty$ and $\rho < 1$.

$$\lim_{K\to\infty} \frac{1-\rho}{1-\rho^{K+1}}\rho^n = (1-\rho)\rho^n,$$

- M/M/1 Queue

Home Work

- Analyse the probabilities of M/M/1/1 Queue

Performance Measures for the M/M/1/K Queue

- Consider the case when ρ = 1, and let L be the expected system size.

$$L = \sum_{n=0}^{K} n p_n \quad \text{and} \quad p_n = \rho^n p_0.$$

$$p_0 = \frac{1-\rho}{1-\rho^{K+1}}$$

$$\begin{aligned} L &= \frac{\rho\left[1-(K+1)\rho^K + K\rho^{K+1}\right]}{(1-\rho)^2}\frac{1-\rho}{1-\rho^{K+1}} \\ &= \frac{\rho\left[1-(K+1)\rho^K + K\rho^{K+1}\right]}{(1-\rho)(1-\rho^{K+1})}. \end{aligned}$$

Performance Measures for the M/M/1/K Queue

- compute L_q

$$L_q = L - (1 - p_0) = L - \left(1 - \frac{1-\rho}{1-\rho^{K+1}}\right) = L - \frac{1-\rho^{K+1} - 1 + \rho}{1-\rho^{K+1}} = L - \frac{\rho(1-\rho^K)}{1-\rho^{K+1}}.$$

Effective arrival rate

- Mean rate of customers actually entering the system denoted by λ' .
- $\lambda' = \lambda(1 - p_K)$

- Using the Little's law,

$$W = \frac{1}{\lambda'} L \quad \text{and} \quad W_q = \frac{1}{\lambda'} L_q .$$

$$W_q = W - \frac{1}{\mu}.$$

Throughput in the M/M/1/K Queue

- In a queueing situation in which customers may be lost, the throughput cannot be defined as being equal to the customer arrival rate. (all arriving customers do not enter the queue).
- The probability that an arriving customer is lost is equal to the probability that there are already K customers in the system, (p_K).
- The probability that the queue is not full, and hence the probability that an arriving customer is accepted into the queueing system is $1 - p_K$.
- Thus the throughput, X ,is given by,

$$X = \lambda(1 - p_K).$$

Throughput in the M/M/1/K Queue

- What can you say about the throughput interms of service rate?

Throughput in the M/M/1/K Queue

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Throughput in the M/M/1/K Queue

- The utilization of the M/M/1/K queue is the probability that the server is busy.

$$U = 1 - p_0 = \frac{1}{\mu} X = \rho(1 - p_K).$$


Thank You!