Chapter - 01

Differentiation and Integration

Independent Study Material No. 04

Table of Indefinite Integrals

1. $\int cf(x)\ dx = c\int f(x)\ dx$
2. $\int [f(x)+g(x)]\ dx = \int f(x)\ dx + \int g(x)\ dx$
3. $\int k\ dx = kx + C$
4. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C\ ; (n \neq -1)$
5. $\int \frac{1}{x}\ dx = \ln|x| + C$
6. $\int e^x\ dx = e^x + C$
7. $\int b^x\ dx = \frac{b^x}{\ln b} + C$
8. $\int \sin x\ dx = -\cos x + C$
9. $\int \cos x\ dx = \sin x + C$
10. $\int \sec^2 x\ dx = \tan x + C$
11. $\int \csc^x\ dx = -\cot x + C$
12. $\int \sec x \tan x\ dx = \sec x + C$
13. $\int \csc x \cot x\ dx = -\csc x + C$
14. $\int \frac{1}{x^2+a^2}\ dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

15. $\displaystyle\int \frac{1}{\sqrt{a^- x^2}}\, dx = \sin^{-1}\left(\frac{x}{a}\right) + C \;; a > 0$

16. $\displaystyle\int \sinh x \; dx = \cosh x + C$

17. $\displaystyle\int \cosh x \; dx = \sinh x + C$

$\boxed{\text{Practice Problems:}}$

$\displaystyle\int_{-2}^{3} (x^2 - 3)\, dx$

$\displaystyle\int_{-2}^{0} \left(\tfrac{1}{2}t^4 + \tfrac{1}{4}t^3 - t\right) dt$

$\displaystyle\int \left(5 + \tfrac{2}{3}x^2 + \tfrac{3}{4}x^3\right) dx$ $\qquad$ $\displaystyle\int_{0}^{3} (1 + 6w^2 - 10w^4)\, dw$

$\displaystyle\int \left(u^6 - 2u^5 - u^3 + \tfrac{2}{7}\right) du$ $\qquad$ $\displaystyle\int_{0}^{2} (2x - 3)(4x^2 + 1)\, dx$

$\displaystyle\int (u + 4)(2u + 1)\, du$ $\qquad$ $\displaystyle\int \sqrt{t}\,(t^2 + 3t + 2)\, dt$ $\qquad$ $\displaystyle\int_{0}^{\pi} (5e^x + 3\sin x)\, dx$

$\displaystyle\int \frac{1 + \sqrt{x} + x}{x}\, dx$ $\qquad$ $\displaystyle\int_{1}^{4} \left(\frac{4 + 6u}{\sqrt{u}}\right) du$

$\displaystyle\int \left(x^2 + 1 + \frac{1}{x^2 + 1}\right) dx$ $\qquad$ $\displaystyle\int_{0}^{1} x\left(\sqrt[3]{x} + \sqrt[4]{x}\right) dx$

$\displaystyle\int (\sin x + \sinh x)\, dx$ $\qquad$ $\displaystyle\int \left(\frac{1 + r}{r}\right)^2 dr$ $\qquad$ $\displaystyle\int_{1}^{2} \left(\frac{x}{2} - \frac{2}{x}\right) dx$

$\displaystyle\int (2 + \tan^2\theta)\, d\theta$ $\qquad$ $\displaystyle\int \sec t\,(\sec t + \tan t)\, dt$ $\qquad$ $\displaystyle\int_{0}^{1} (x^{10} + 10^x)\, dx$

$\displaystyle\int 2^t(1 + 5^t)\, dt$ $\qquad$ $\displaystyle\int \frac{\sin 2x}{\sin x}\, dx$ $\qquad$ $\displaystyle\int_{0}^{\pi/4} \frac{1 + \cos^2\theta}{\cos^2\theta}\, d\theta$

Techniques of Integration

1. Integration by Parts

Let $u = f(x)$ and $v = g(x)$. Then the differentials are $du = f'(x)\ dx$ and $dv = g'(x)\ dx$, so, by the Substitution Rule, the formula for integration by parts is

$$\int u = uv - \int v\ du.$$

Illustrative Example 01 Find $\displaystyle\int x \sin x\ dx$.

Answer:
Let $u = x$ and $dv = \sin x\ dx$. Then, $du = dx$ and $v = -\cos x$

$$\begin{aligned}\int \underbrace{x}_{u}\underbrace{\sin x\ dx}_{dv} &= \underbrace{x}_{u}\underbrace{(-\cos x)}_{v} - \int \underbrace{(-\cos x)}_{v}\underbrace{dx}_{du}\\ &= -x\cos x + \int \cos x\ dx\\ &= -x\cos x + \sin x + C\ ;\ C \text{ is an arbitrary constant.}\end{aligned}$$

Illustrative Example 02 Find $\displaystyle\int \ln x\ dx$.

Answer:
Let $u = \ln x$ and $dv = dx$. Then, $du = \dfrac{1}{x}\ dx$ and $v = x$

$$\begin{aligned}\int \ln x\ dx =& x\ln x - \int x\ \frac{dx}{x}\\ =& x\ln x - \int\ dx\\ =& x\ln x - x + C\ ;\ C \text{ is an arbitrary constant.}\end{aligned}$$

Illustrative Example 03 Find $\displaystyle\int t^2e^t\ dt$.

Answer:

Let $u = t^2$ and $dv = e^t\ dt$. Then, $du = 2t\ dt$ and $v = e^t$

$$\begin{aligned}\int t^2 e^t\ dt =& t^2e^t - 2\int te^t\ dt \\ =& t^2e^t - 2\left[te^t - \int e^t\ dt\right] \\ =& t^2e^t - 2\left[te^t - e^t + C\right] \\ =& t^2e^t - 2te^t + 2e^t + C_0\ ;\ C_0 = -2C \text{ is an arbitrary constant.}\end{aligned}$$

Practice Problems:

$\int x \cos 5x\, dx$ $\quad$ $\int ye^{0.2y}\, dy$

$\int te^{-3t}\, dt$ $\quad$ $\int (x-1)\sin \pi x\, dx$

$\int (x^2+2x)\cos x\, dx$ $\quad$ $\int t^2 \sin \beta t\, dt$

$\int \cos^{-1} x\, dx$ $\quad$ $\int \ln\sqrt{x}\, dx$

$\int t^4 \ln t\, dt$ $\quad$ $\int \tan^{-1} 2y\, dy$

$\int t\csc^2 t\, dt$ $\quad$ $\int x\cosh ax\, dx$

$\int (\ln x)^2\, dx$ $\quad$ $\int \frac{z}{10^z}\, dz$

$\int e^{2\theta}\sin 3\theta\, d\theta$ $\quad$ $\int e^{-\theta}\cos 2\theta\, d\theta$

$\int z^3 e^z\, dz$ $\quad$ $\int x\tan^2 x\, dx$

$\int \frac{xe^{2x}}{(1+2x)^2}\, dx$ $\quad$ $\int (\arcsin x)^2\, dx$

$\int_0^{1/2} x\cos \pi x\, dx$ $\quad$ $\int_0^1 (x^2+1)e^{-x}\, dx$

$\int_0^2 y\sinh y\, dy$ $\quad$ $\int_1^2 w^2 \ln w\, dw$

$\int_1^5 \frac{\ln R}{R^2}\, dR$ $\quad$ $\int_0^{2\pi} t^2 \sin 2t\, dt$

$\int_0^{\pi} x\sin x\cos x\, dx$ $\quad$ $\int_1^{\sqrt{3}} \arctan(1/x)\, dx$

2. Trigonometric Integrals

Strategy for Evaluating $\displaystyle\int \sin^m x \cos^n x \; dx$

(a) If the power of cosine is odd ($n = 2k+1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\begin{aligned}\int \sin^m x \cos^{2k+1} x \; dx &= \int \sin^m x (\cos^2 x)^k \cos x \; dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x \; dx.\end{aligned}$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k+1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x \; dx &= \int (\sin^2 x)^k \cos^n x \sin x \; dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x \; dx.\end{aligned}$$

Then substitute $u = \cos x$.

Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$$

and

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x).$$