



Introduction

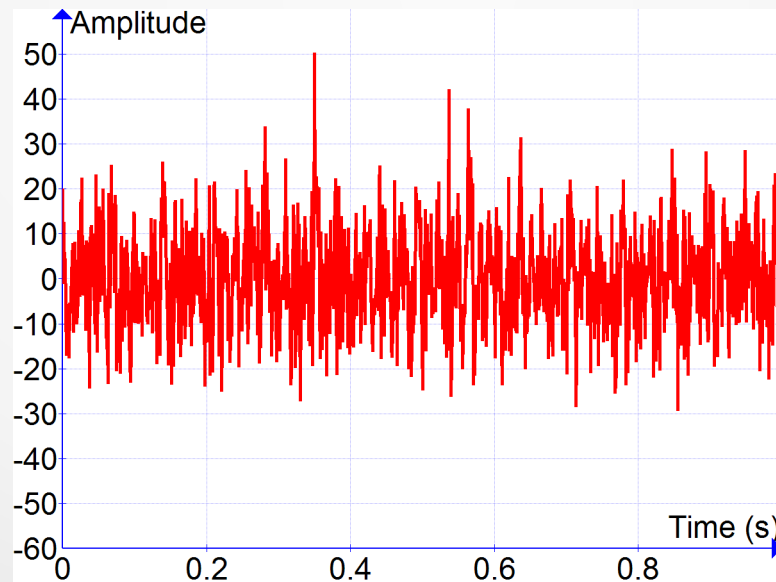
Random Signals & Processes

Lecture 1

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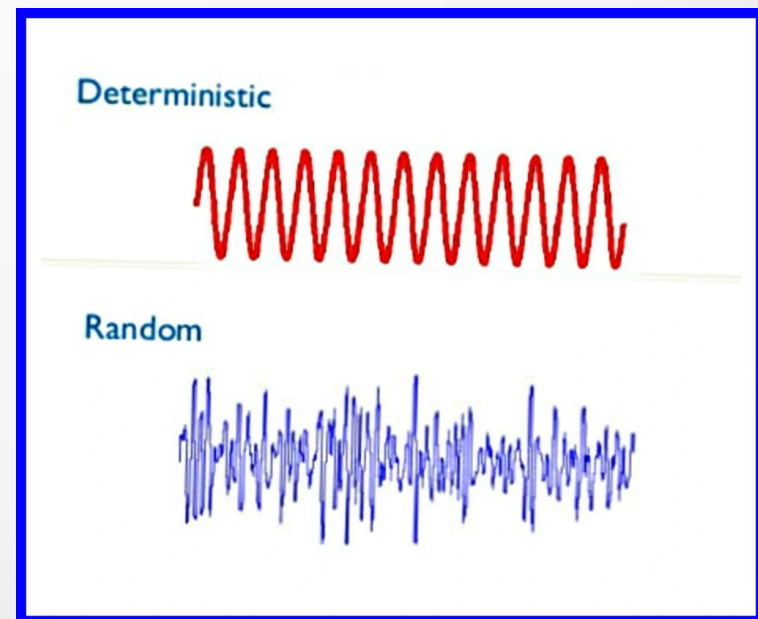
What is a Signal?

- A signal is a physical quantity propagating through space and time that “contains” information about an event. It propagates from one physical location to another through some medium.



Random Signal Vs Deterministic Signal

- Signals which can be **defined exactly by a mathematical formula are known as deterministic signals.**
- A signal is said to be non-deterministic **if there is uncertainty with respect to its value at some instant of time.**
- Non-deterministic signals are random in nature hence they are called random signals.



What is a Random Process?



Random process includes random variable(s) that evolves in time by some random mechanism (the time variable can be replaced by a space variable, or some other variable, in application). Or a collection of random variables.

The variable can have a discrete set of values at a given time, or a continuum of values may be available.

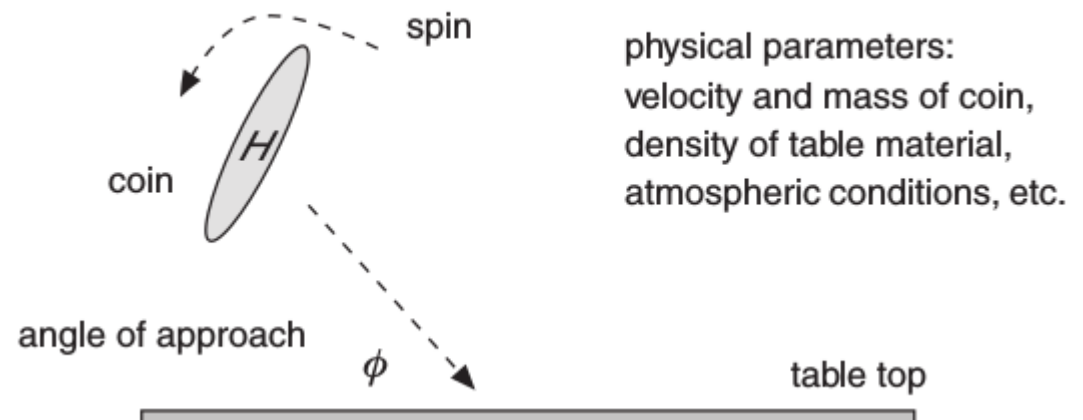
Random Process is also known as also called a stochastic process

What is randomness

- A lack of complete information about a physical process.

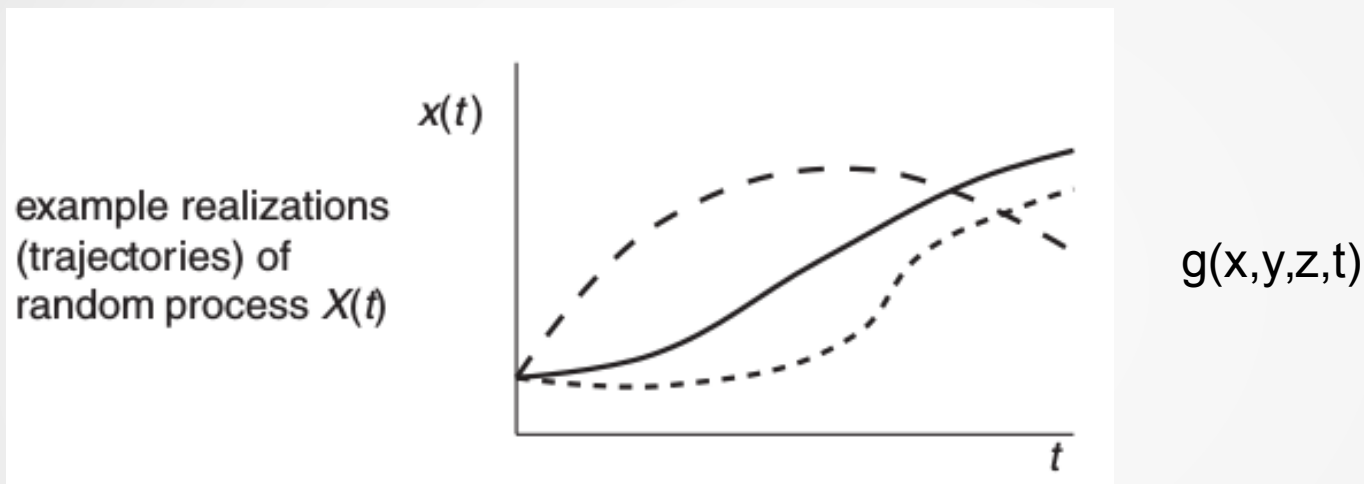
Example: Tossing a coin [H,T]

- Required information to predict exact outcome:
 1. Velocity of the coin just before impact
 2. The angle of approach ϕ with respect to the table top
 3. The mass of the coin
 4. Atmospheric conditions (temperature, humidity, and so on), and other relevant physical parameters



Modelling the Process

- Ex. Tornado in the x-direction as a function of time



Note : A **random process** is denoted by $X(t)$ (**uppercase letter**) and a realization of the process by $x(t)$ (**lowercase letter**).

The **outcome at a specific time t_0** is also random: it is a random variable $X(t_0)$, and a particular outcome at that time is $x(t_0)$.

A **random process** is a collection of random variables that are indexed by **time**, for one realization $x(t)$ and for one outcome $x(t_0)$ at $t = t_0$.

Example Related to Communication

- Task: Recover the sender's message (voice, data, video)
 1. The signal
 2. The characteristic of a radio channel
 3. The interfering signals
 4. Thermal noise introduced at the antenna and the front-end RF amplifier.

Axioms of probability

Sample space

- Mathematical abstraction of the collection of all possible experimental outcomes

Event

- An event is a set of sample points. Usually events are denoted by capital letters.

Probability measure

- An assignment of real numbers to the events defined on sample space.

Sigma Field

- An algebra for events called a sigma field is defined,

A collection \mathcal{F} of subsets of Ω is called a σ -field, if it satisfies the following properties:

1. $\Phi \in \mathcal{F}$
2. if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$
3. if $A_1, A_2, A_3, \dots \in \mathcal{F}$, then

$$\bigcup_{m=1}^{\infty} A_m \in \mathcal{F}.$$

Examples

1. The smallest σ -field associated with Ω is $F = \{\emptyset, \Omega\}$.
2. If A is a subset of Ω , then $F = \{\emptyset, A, A^c, \Omega\}$ is a σ -field.
3. Consider a sample space defined by $\Omega = \{a, b, c, d\}$. A set $C = \{\{a\}, \{b\}\}$ is a subset of Ω , but it is not a field.

The complement of a simple event $\{a\}$ is $\{a\}^c = \{b, c, d\}$.

Similarly, $\{b\}^c = \{a, c, d\}$.

$\{a\} \cup \{b\} = \{a, b\}$ and $(\{a\} \cup \{b\})^c = \{c, d\}$.

Then collecting all these subsets of Ω , we find that

- $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{b, c, d\}, \{a, c, d\}, \Omega\}$ is the smallest σ -field containing all the elements of C .

Probability measure and probability space

- A probability measure P defined on (Ω, \mathcal{F}) is a function that maps any element of \mathcal{F} into $[0, 1]$ such that,

(a) $P[\emptyset] = 0, P[\Omega] = 1$;

(b) if $A_1, A_2, \dots \in \mathcal{F}$ and $A_m \cap A_n = \emptyset$ ($m \neq n$), then

$$P \left[\bigcup_{m=1}^{\infty} A_m \right] = \sum_{m=1}^{\infty} P[A_m].$$

*The triple (Ω, \mathcal{F}, P) is called a **probability space**.*

Example

- Experiment of tossing two coins with possibly biased coins.
- The sample space of tossing the first coin (coin “1”) is denoted $\Omega_1 = \{h, t\}$. Its σ -field is $F_1 = \{\emptyset, \{h\}, \{t\}, 1\}$.
- $P_1[\emptyset] = 0$, $P_1[\{h\}] = p_1$, $P_1[\{t\}] = 1 - p_1$, and $P_1[1] = 1$
where p_1 is a fixed real number in the interval $[0, 1]$
- The sample space of tossing the first coin (coin “1”) is denoted $\Omega_2 = \{h, t\}$. Its σ -field is $F_2 = \{\emptyset, \{h\}, \{t\}, 1\}$
- $P_2[\emptyset] = 0$, $P_2[\{h\}] = p_2$, $P_2[\{t\}] = 1 - p_2$, and $P_2[1] = 1$
where p_2 is a fixed real number in the interval $[0, 1]$

Example

- The sample space of the experiment of tossing the two coins is the Cartesian product of the two sample spaces defined above.

$$\Omega = \Omega_1 \times \Omega_2 = \{(\omega_1, \omega_2) : \omega_1 \in \Omega_1, \omega_2 \in \Omega_2\}.$$

Bernoulli trials and Bernoulli's theorem

- **Repeated independent trials** are called Bernoulli trials if there are **only two possible outcomes** for each trial and **their probabilities remain the same throughout the trials**.
- The sample space of each individual trial is :
 $\Omega = \{s, f\}$
- probability of the simple event $\{s\}$ denoted by p

$$P[\{s\}] = p, \quad 0 \leq p \leq 1,$$

probability of the event $\{f\}$ is given by

$$P[\{f\}] = 1 - p \triangleq q,$$

- The sample space for an experiment consisting of two independent Bernoulli trials,

$$\Omega^2 = \Omega \times \Omega = \{(ss), (sf), (fs), (ff)\}.$$

Bernoulli distribution for two trials is given by $\{p^2, pq, qp, q^2\}$

Sample space for n Bernoulli trials is the n^{th} -fold Cartesian product of Ω :

$$\Omega^n = \Omega \times \Omega \times \cdots \times \Omega = \{(ss \dots s), (ss \dots f), \dots, (ff \dots s), (ff \dots f)\}.$$

the probability of the outcome $ssf \dots fsf$ is given by:

$$P[\{ssf \dots fsf\}] = p p q \dots q p q = p^k q^{n-k}$$

where k is the number of successes and $n - k$ is that of failures in a given outcome of n Bernoulli trials

Binomial coefficient

- If the order in which the successes occur does not matter, then the number of sample points belonging to this event is equal to the number of combinations of n things taken k at a time.
- This number is referred to as the binomial coefficient and denoted as:

$$\binom{n}{k} \triangleq \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k \times (k-1) \cdots 2 \times 1}.$$

Binomial Distribution

- The set of probabilities $B(k; n, p)$ is called the binomial distribution.

$$B(k; n, p) \triangleq \binom{n}{k} p^k q^{n-k}, \quad k = 0, 1, 2, \dots, n.$$

Joint probability and conditional probability

- Example:

A joint (or compound) experiment that consists of one experiment having possible outcomes A_m ($m = 1, 2, \dots, M$) and another having the possible outcomes B_n ($n = 1, 2, \dots, N$) can be considered as a single experiment having the set of possible outcomes (A_m, B_n) .

- **Probabilities relating to such a combined experiment are known as joint (or compound) probabilities.**
- The joint probability of events A and B is often written as $P[A, B]$ instead of $P[A \cap B]$.

$$0 \leq P[A, B] \leq 1.$$

Conditional Probability

- The conditional probability that event B occurs given that event A occurs is defined as:

$$P[B | A] \triangleq \frac{P[A, B]}{P[A]},$$

Bayes' theorem

- A set of events A_1, A_2, \dots, A_n is called a partition of the sample space if they are a set of mutually exclusive and exhaustive events in ; i.e., $A_1 \cup A_2 \cup \dots \cup A_n = \Omega$

and $A_i \cap A_j = \emptyset$ for $i \neq j$.

- Then we obtain for any event B:

$$\bigcup_{j=1}^n \{B \cap A_j\} = B,$$

and then

$$\sum_{j=1}^n P[B, A_j] = \sum_{j=1}^n P[A_j]P[B | A_j] = P[B].$$

Bayes' theorem

- Let B be an event in a sample space and A_1, A_2, \dots, A_n be a partition of . Then it can be shown that

$$P[A_j | B] = \frac{P[A_j] P[B | A_j]}{P[B]} = \frac{P[A_j] P[B | A_j]}{\sum_{i=1}^n P[B | A_i] P[A_i]}.$$

Example

- Consider some disease and its medical diagnosis test. The following statistics are known about this disease and its medical test.
 - For a person with this disease, the test yields a positive result 99% of the time and a negative result 1%.
 - For a person without this disease, the test yields a negative result 99% of the time and a positive result 1%.
 - Suppose that 1% of the population is infected by this disease and 99% of the population is not.
- Suppose that you have taken this test and, unfortunately, the test result is positive. What is the chance that you are indeed infected by this disease?

Answer

- Let A represent a person's condition with respect to this disease and B represent their test result:
- A = "Not infected by the disease"
- A_c = "Infected by the disease"
- B = "Negative test result"
- B_c = "Positive test result."

$$\begin{aligned}P[A] &= p, \quad P[A^c] = 1 - p; \\P[B|A] &= \alpha, \quad P[B^c|A] = 1 - \alpha; \\P[B^c|A^c] &= \beta, \quad P[B|A^c] = 1 - \beta.\end{aligned}$$

$$\begin{aligned}P[A, B] &= P[A]P[B|A] = p\alpha; \\P[A, B^c] &= p(1 - \alpha); \\P[A^c, B] &= (1 - p)(1 - \beta); \\P[A^c, B^c] &= (1 - p)\beta.\end{aligned}$$

$$P[A^c|B^c] = \frac{P[A^c, B^c]}{P[A, B^c] + P[A^c, B^c]} = \frac{(1 - p)\beta}{p(1 - \alpha) + (1 - p)\beta}.$$

If we substitute $p = \alpha = \beta = 0.99$, then

$$P[A^c|B^c] = \frac{0.01 \times 0.99}{0.99 \times 0.01 + 0.01 \times 0.99} = \frac{0.0099}{0.0198} = 0.5.$$

That is, the probability that you have this disease is 50%.

Thank you!