Infinite Impulse Response (IIR) DIGITAL FILTERS

LECTURE 4

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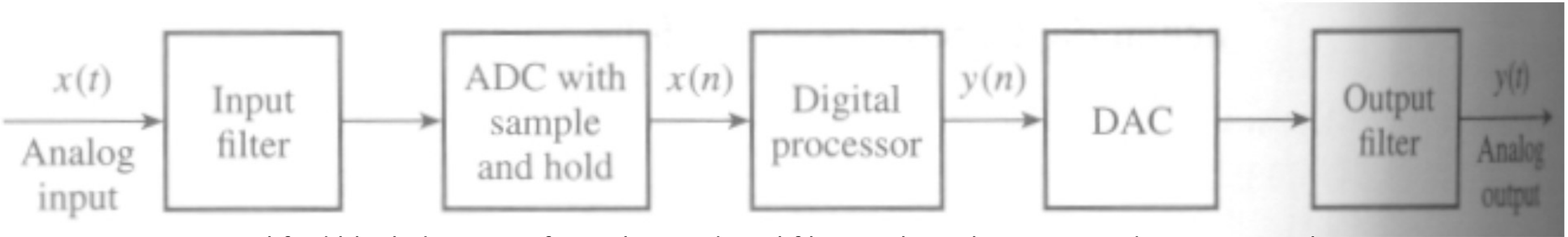
Digital Filters - Introduction

- A digital filter is a mathematical algorithm implemented in hardware and/or software that operates on a digital input signal to produce a digital output signal for the purpose of achieving a filtering objective.
- Digital filters operate on digitized analog signals or just numbers, representing some variable, stored in a computer memory.



A simplified block diagram of a real-time digital filter with analog input and output signals

Applications - Digital Filters

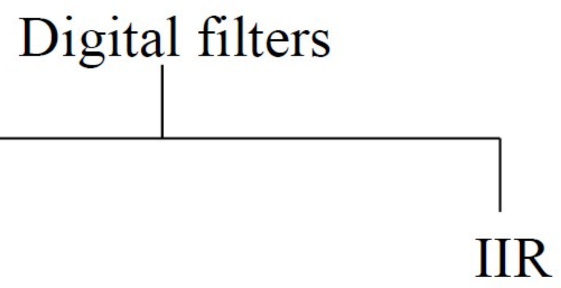
Digital filters are used in applications like;

- data compression
- biomedical signal processing
- speech and image processing
- data transmission
- digital audio
- telephone echo cancellation etc.

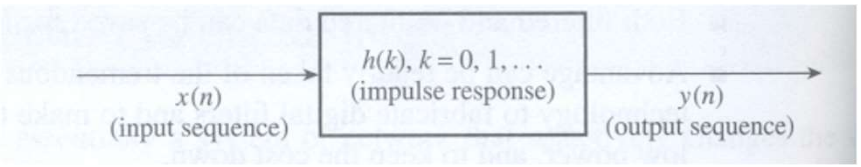
Advantages of digital filters over analog filters

- Truly linear phase response.
- Performance does not vary with environmental changes, for example thermal variations.
- The frequency response can be automatically adjusted if the filter is implemented using a programmable processor.
- Digital filters can be used at very low frequencies.

Types of filters



A digital filter in general can be represented by its impulse response, h(k) (k=0,1,..) as in the following figure:



A conceptual representation of a digital filter

IIR Filter

The input and output signals to the filter are related by the convolution sum:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) \qquad \text{IIR}$$

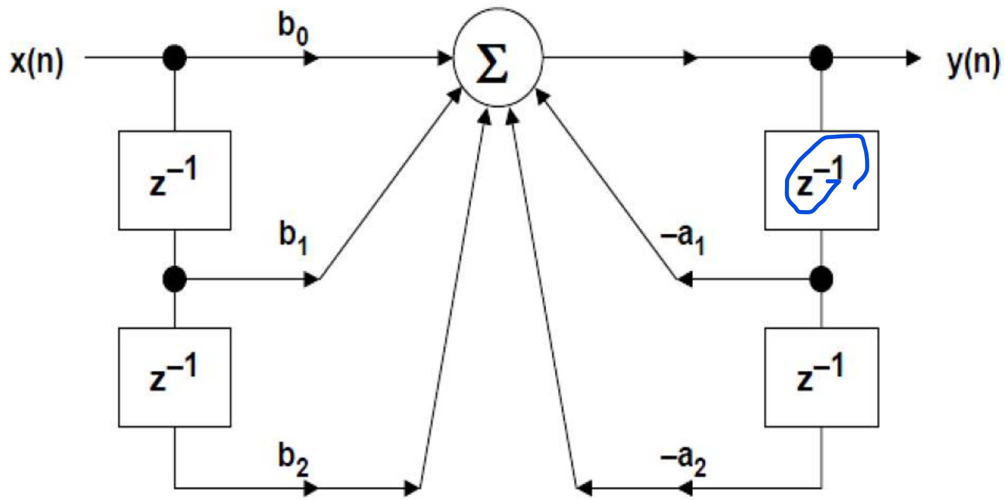
Because of the infinite length of the impulse response of the IIR filter, the IIR filtering equation can be expressed in a recursive form as:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=0}^{N} b_k x(n-k) - \sum_{k=1}^{M} a_k y(n-k)$$

where a_k and b_k are the coefficients of the filter

IIR Filter

The current output sample, y(n), is a function of past outputs as well as present and past input samples, *that is the IIR is a feedback system of some sort*.



Simple IIR filter block diagram

The transfer functions of IIR filters

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{1 + \sum_{k=1}^{M} a_k z^{-k}}$$

IIR

Comparison of Digital and Analog Filters

Digital filter	*Analog filter*
1. It operates on digital samples (or sampled version) of the signal.	1. It operates on analog signals (or actual signals).
2. It is governed (or defined) by linear difference equations.	2. It is governed (or defined) by linear differential equations.
3. It consists of adders, multipliers, and delay elements implemented in digital logic (either in hardware or software or both).	3. It consists of electrical components like resistors, capacitors, and inductors.
4. In digital filters, the filter coefficients are designed to satisfy the desired frequency response.	4. In analog filters, the approximation problem is solved to satisfy the desired frequency response.

IIR filter (calculating a_k and b_k)

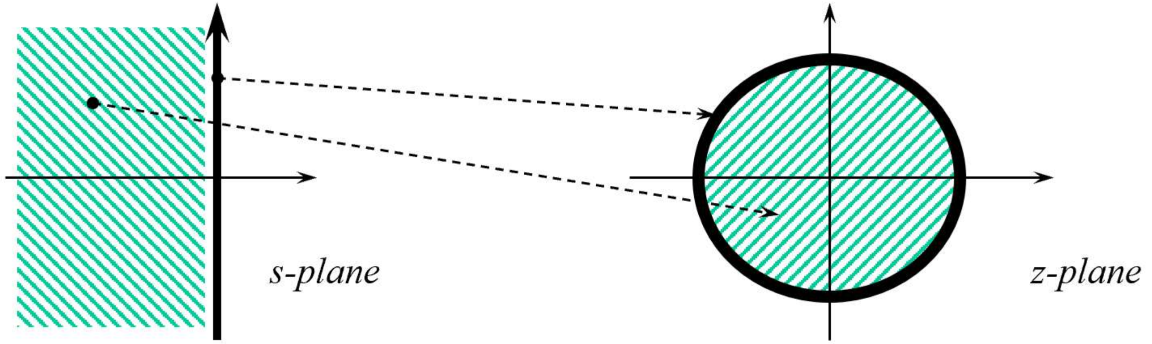
Calculations of IIR filter coefficients are traditionally based on the transformation of known analog filter characteristics into equivalent digital filters.

- The two basic methods used are **impulse invariant** and **bilinear transformation** methods.
- The impulse invariant method is good for simulating analog systems, but the bilinear method is best for frequency selective IIR filters.

Design of IIR Filter By Bilinear Transformation Method

- This transformation is a one-to-one mapping from the s-domain to the z-domain.
- That is, the bilinear transformation is a conformal mapping that transforms the imaginary axis of s-plane into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components.

$s = \sigma + j\Omega$



Design of IIR Filter By Bilinear Transformation Method

- In this mapping, all points in the left half of s-plane are mapped inside the unit circle in the z-plane, and all points in the right half of s-plane are mapped outside the unit circle in the z-plane.
- So the transformation of a stable analog filter results in stable digital filter.
- The bilinear transformation can be obtained by using the trapezoidal formula for the numerical integration.

Bilinear Transformation Method

Let the system function of the analog filter be $H_a(s) = \dfrac{b}{s+a}$

The differential equation describing the above analog filter can be obtained as:

$$H_a(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

or

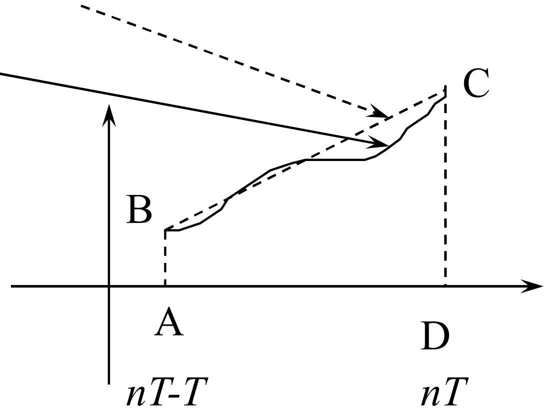
$$sY(s) + aY(s) = bX(s)$$

Taking inverse Laplace transform on both sides, we get

$$\frac{dy(t)}{dt} + a\,y(t) = bx(t)$$

Bilinear Transformation. It is based on the relationship

$$y(nT) - y(nT - T) = \int_{nT-T}^{nT} \dot{y}(t)dt \cong \underbrace{\left[\dot{y}(nT) + \dot{y}(nT - T)\right]\frac{T}{2}}_{area\ ABCD}$$



Take the z-Transform of both sides:

$$Y(z) - z^{-1}Y(z) = \frac{T}{2}\left[Y_1(z) + z^{-1}Y_1(z)\right]$$

Bilinear Transformation Method

Integrating the above equation between the limits $(nT - T)$ and nT, we have

$$\int_{nT-T}^{nT} \frac{dy(t)}{dt}\,dt + a\int_{nT-T}^{nT} y(t)\,dt = b\int_{nT-T}^{T} x(t)\,dt$$

The trapezoidal rule for numeric integration is expressed as:

$$\int_{nT-T}^{nT} a(t)\,dt = \frac{T}{2}[a(nT) + a(nT-T)]$$

Therefore, we get

$$y(nT) - y(nT-T) + a\frac{T}{2}y(nT) + a\frac{T}{2}y(nT-T) = b\frac{T}{2}x(nT) + b\frac{T}{2}x(nT-T)$$

Taking z-transform, we get

$$Y(z)[1 - z^{-1}] + a\frac{T}{2}[1 + z^{-1}]\,Y(z) = b\frac{T}{2}[1 + z^{-1}]\,X(z)$$

Bilinear Transformation Method

Therefore, the system function of the digital filter is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{b}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} + a}$$

Comparing this with the analog filter system function $H_a(s)$ we get

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) = \frac{2}{T}\left(\frac{z-1}{z+1}\right)$$

Rearranging, we can get

$$z = \frac{1+\frac{T}{2}s}{1-\frac{T}{2}s}$$

This is the relation between analog and digital poles in bilinear transformation. So to convert an analog filter function into an equivalent digital filter function, just put

$$s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \text{ in } H_a(s)$$

Bilinear Transformation Method

The general characteristic of the mapping $z = e^{sT}$ may be obtained by putting $s = \sigma + j\Omega$ and expressing the complex variable z in the polar form as $z = re^{j\omega}$ in the above equation for s.

Thus,

$$s = \frac{2}{T}\left(\frac{z-1}{z+1}\right) = \frac{2}{T}\left(\frac{re^{j\omega}-1}{re^{j\omega}+1}\right)$$

or $$s = \frac{2}{T}\,\frac{(re^{j\omega}-1)(re^{-j\omega}+1)}{(re^{j\omega}+1)(re^{-j\omega}+1)} = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

$$\frac{r^2-1}{r^2 + r\left(e^{j\omega} + e^{-j\omega}\right) + 1}$$

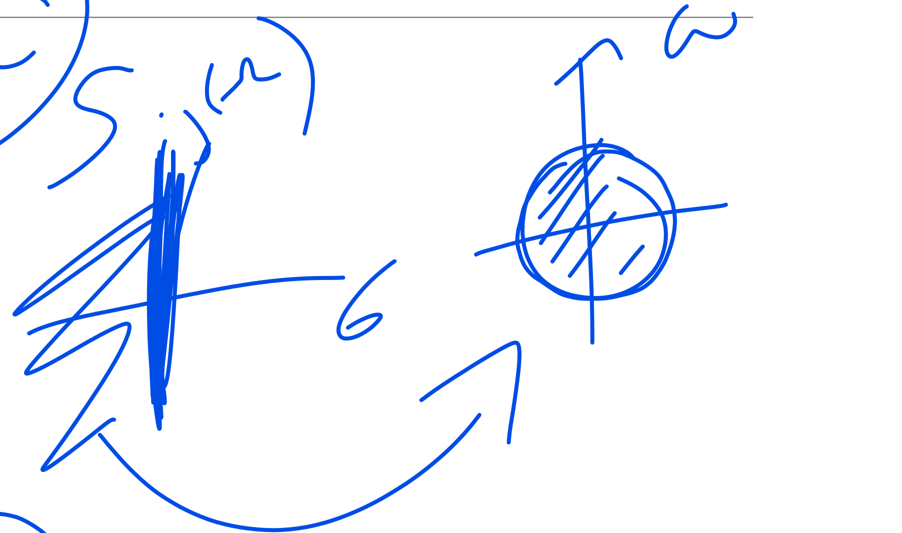
Bilinear Transformation Method

Since $s = \sigma + j\Omega$, we get

$$\sigma = \frac{2}{T}\left[\frac{r^2 - 1}{1 + r^2 + 2r\cos\omega}\right]$$

And

$$\Omega = \frac{2}{T}\left[\frac{2r\sin\omega}{1 + r^2 + 2r\cos\omega}\right]$$



From the above equation for , we observe that if $r < 1$ then $\sigma < 0$ and if $r > 1$, then $\sigma > 0$, and if $r = 1$, then $\sigma = 0$. Hence the left half of the s-plane maps into points inside the unit circle in the z-plane, the right half of the s-plane maps into points outside the unit circle in the z-plane and the imaginary axis of s-plane maps into the unit circle in the z-plane. This transformation results in a stable digital system.

Relation between analog and digital frequencies

On the imaginary axis of s-plane $\sigma = 0$ and correspondingly in the z-plane $r = 1$.

$$\Omega = \frac{2}{T}\left(\frac{2\sin\omega}{1+1+2\cos\omega}\right) = \frac{2}{T}\left(\frac{\sin\omega}{1+\cos\omega}\right)$$

$$= \frac{2}{T}\left(\frac{2\sin\frac{\omega}{2}\cos\frac{\omega}{2}}{1+2\cos^2\omega/2-1}\right) = \frac{2}{T}\tan\frac{\omega}{2}$$

The relation between analog and digital frequencies is:

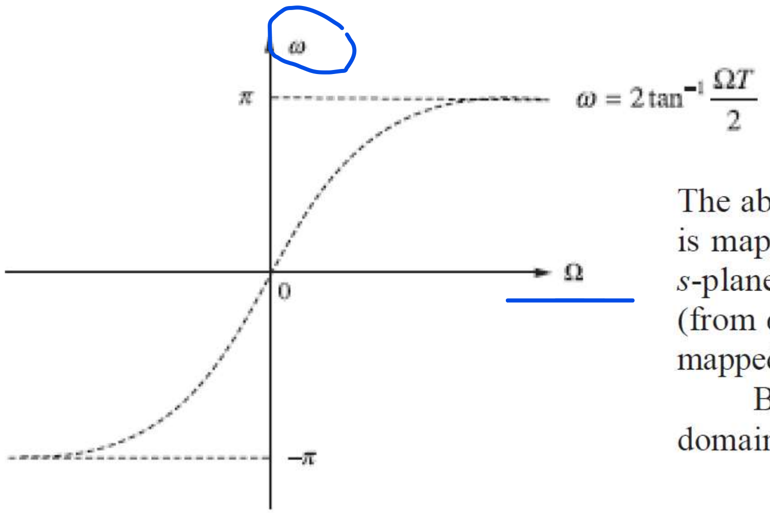
$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$-\pi < \omega < \pi$

or equivalently, we have $= 2\tan\frac{\Omega T}{2}$.

$-\infty < \Omega < \infty$

Relation between analog and digital frequencies

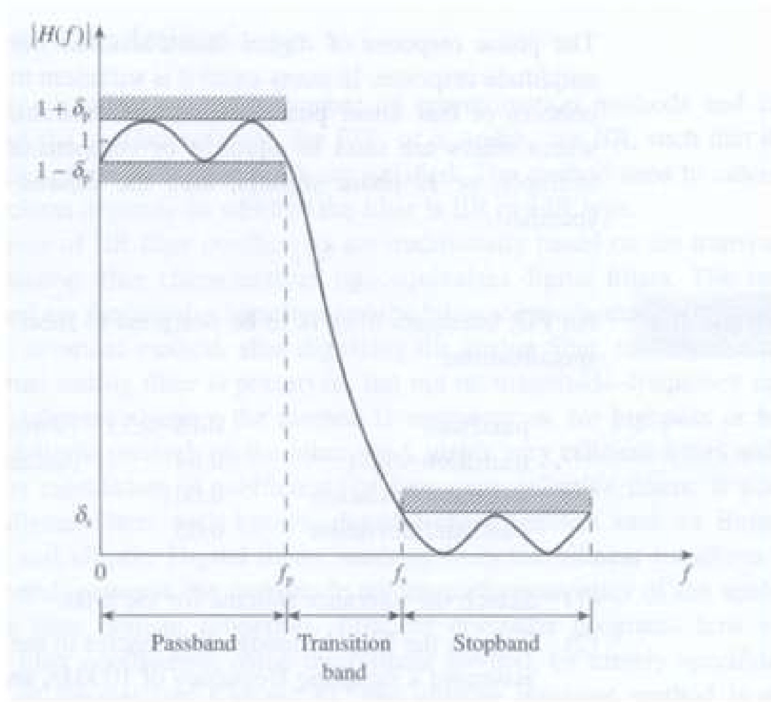


Mapping between Ω and ω in bilinear transformation.

The above relation between analog and digital frequencies shows that the entire range in Ω is mapped only once into the range $-\pi \leq \omega \leq \pi$. The entire negative imaginary axis in the s-plane (from $\Omega = -\infty$ to 0) is mapped into the lower half of the unit circle in z-plane (from $\omega = -\pi$ to 0) and the entire positive imaginary axis in the s-plane (from Ω= α to 0) is mapped into the upper half of unit circle in z-plane (from $\omega = 0$ to $+\pi$).

But as seen in Figure 1, the mapping is non-linear and the lower frequencies in analog domain are expanded in the digital domain, whereas the higher frequencies are

• The characteristics of digital filters are often specified in the frequency domain.



Tolerance scheme for a low pass filter

• The width of the transition band specifies how sharp the filter is.

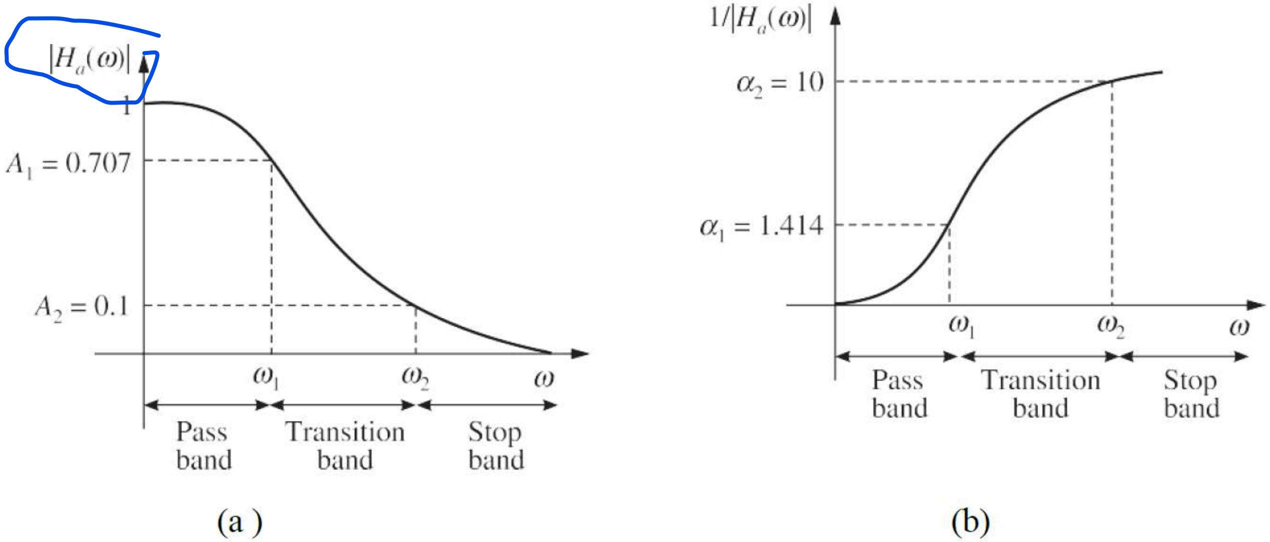
The following are the key parameters of interest:

δ_p	passband deviation
δ_s	stopband deviation
f_p	passband edge frequency
f_s	stopband edge frequency

$$A_s \text{ (stopband attenuation)} = -20\log_{10}\delta_s$$

$$A_p \text{ (passband ripple)} = 20\log_{10}(1+\delta_p)$$

Specifications of the Low-pass Filter



Magnitude response of low–pass filter (a) Gain vs ω and (b) Attenuation vs ω.

Let ω_1 = Passband frequency in rad/s.
ω_2 = Stopband frequency in rad/s.

Let k_1 = Gain in dB at a passband frequency ω_1

k_2 = Gain in dB at a stopband frequency ω_2

The gain can be converted into normal values as follows:

$$20 \log A_1 = k_1 \qquad 20 \log A_2 = k_2$$

$$\log A_1 = k_1/20 \qquad \log A_2 = k_2/20$$

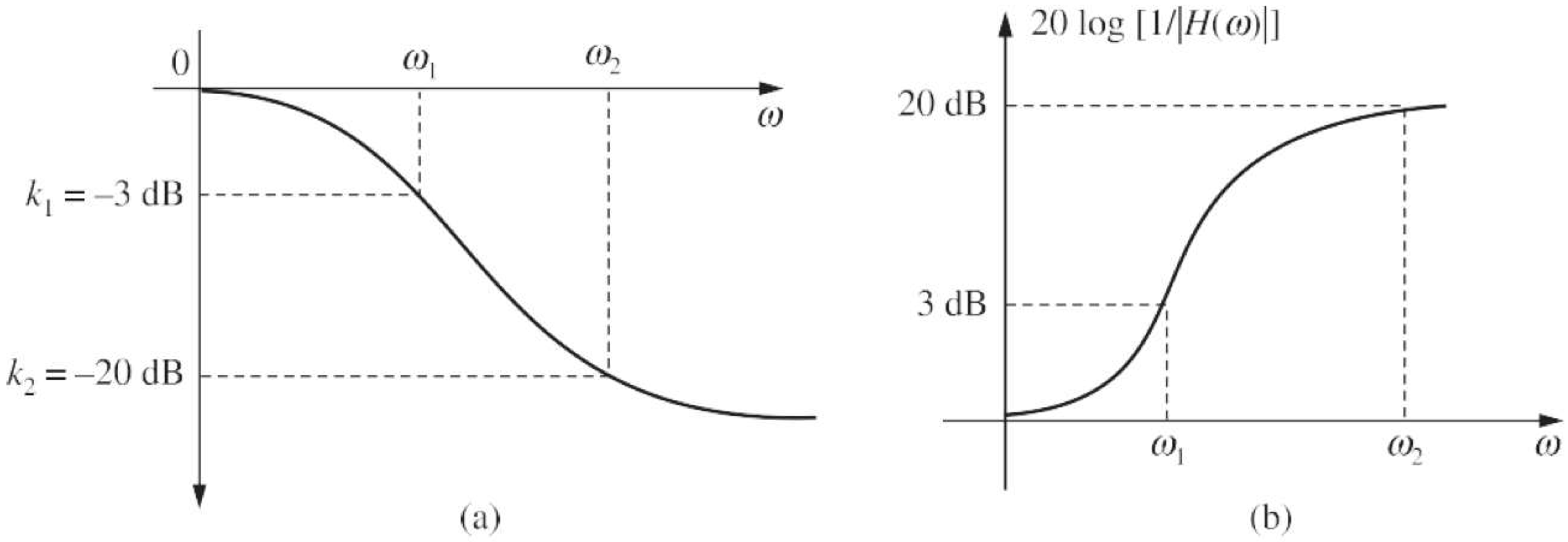
$$A_1 = 10^{k_1/20} \qquad A_2 = 10^{k_2/20}$$

When expressed in dB, the gain and attenuation will have only change in sign because $\log \alpha = \log(1/A) = -\log A$. (Hence when dB is positive it is attenuation and when dB is negative it is gain).

When $A_1 = 0.707$, $k_1 = 20 \log(0.707) = -3.0116 = -3$ dB

When $A_2 = 0.1$, $k_2 = 20 \log(0.1) = -20$ dB

The magnitude response of low-pass filter in terms of dB-attenuation



Magnitude response of low–pass filter (a) dB–Gain vs ω and (b) dB–attenuation vs ω .

Design of Low-Pass Digital Butterworth Filter

- The popular methods of designing IIR digital filter involves the design of equivalent analog filter and then converting the analog filter to digital filter.

- Hence to design a Butterworth IIR digital filter, first an analog Butterworth filter transfer function is determined using the given specifications.

- Then the analog filter transfer function is converted to a digital filter transfer function using either impulse invariant transformation or bilinear transformation.

Analog Butterworth filter

The analog Butterworth filter is designed by approximating the ideal frequency response using an error function. The error function is selected such that the magnitude is maximally flat in the passband and monotonically decreasing in the stopband. (Strictly speaking the magnitude is maximally flat at the origin, i.e., at $\Omega = 0$, and monotonically decreasing with increasing Ω).

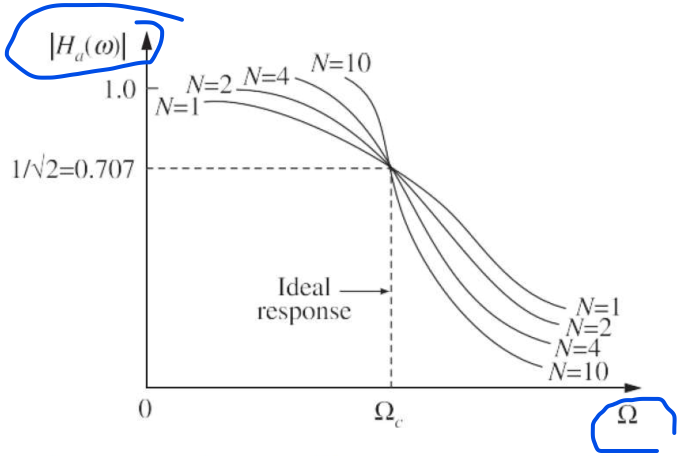
The magnitude response of low-pass filter obtained by this approximation is given by

$$|H_a(\omega)|^2 = \frac{1}{1 + \left(\frac{\Omega}{\Omega_c}\right)^{2N}}$$

where Ω_c is the 3 dB cutoff frequency and N is the order of the filter.

Frequency response of the Butterworth filter

The frequency response of Butterworth filter depends on the order N. The magnitude response for different values of N are shown



Magnitude response of Butterworth low–pass filter for various values of *N*.

Design procedure for low-pass digital Butterworth IIR filter

The low-pass digital Butterworth filter is designed as per the following steps:

Let A_1 = Gain at a passband frequency ω_1

A_2 = Gain at a stopband frequency ω_2

Ω_1 = Analog frequency corresponding to ω_1

Ω_2 = Analog frequency corresponding to ω_2

Step 1 Choose the type of transformation, i.e., either bilinear or impulse invariant transformation.

Step 2 Calculate the ratio of analog edge frequencies Ω_2/Ω_1.

For bilinear transformation

$$\Omega_1 = \frac{2}{T}\tan\frac{\omega_1}{2},\ \Omega_2 = \frac{2}{T}\tan\frac{\omega_2}{2} \quad \therefore \frac{\Omega_2}{\Omega_1} = \frac{\tan\omega_2/2}{\tan\omega_1/2}$$

For impulse invariant transformation,

$$\Omega_1 = \frac{\omega_1}{T},\quad \Omega_2 = \frac{\omega_2}{T} \quad \therefore \frac{\Omega_2}{\Omega_1} = \frac{\omega_2}{\omega_1}$$

Step 3 Decide the order N of the filter. The order N should be such that

$$N \ge \frac{1}{2}\frac{\log\left\{\left[\frac{1}{A_2^2}-1\right]\Big/\left[\frac{1}{A_1^2}-1\right]\right\}}{\log\frac{\Omega_2}{\Omega_1}}$$

Choose N such that it is an integer just greater than or equal to the value obtained above.

Design procedure for low-pass digital Butterworth IIR filter

Step 4 Calculate the analog cutoff frequency

$$\Omega_c = \frac{\Omega_1}{\left[\frac{1}{A_1^2} - 1\right]^{1/2N}}$$

Step 5 Determine the transfer function of the analog filter.
Let $H_a(s)$ be the transfer function of the analog filter. When the order N is even, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \prod_{k=1}^{N/2} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

When the order N is odd, for unity dc gain filter, $H_a(s)$ is given by

$$H_a(s) = \frac{\Omega_c}{s + \Omega_c} \prod_{k=1}^{\frac{N-1}{2}} \frac{\Omega_c^2}{s^2 + b_k\Omega_c s + \Omega_c^2}$$

The coefficient b_k is given by

$$b_k = 2\sin\left[\frac{(2k-1)\pi}{2N}\right]$$

For normalized case, $\Omega_c = 1$ rad/s

Design procedure for low-pass digital Butterworth IIR filter

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Design procedure for low-pass digital Butterworth IIR filter

Step 6 Using the chosen transformation, transform the analog filter transfer function $H_a(s)$ to digital filter transfer function $H(z)$.

Step 7 Realize the digital filter transfer function $H(z)$ by a suitable structure.

Properties of Butterworth filters

1. The Butterworth filters are all pole designs (i.e. the zeros of the filters exist at œ).
2. The filter order N completely specifies the filter.
3. The magnitude response approaches the ideal response as the value of N increases.
4. The magnitude is maximally flat at the origin.
5. The magnitude is monotonically decreasing function of ω.
6. At the cutoff frequency **fi**$_c$, the magnitude of normalized Butterworth filter is $1/\sqrt{2}$. Hence the dB magnitude at the cutoff frequency will be 3 dB less than the maximum value.