Tutorial (04)
Solutions

(01)

$$x^2 + 2x - 4 = 0$$

$$\therefore \alpha + \beta = -\frac{b}{a} = \frac{-2}{1} = -2$$

$$\alpha\beta = \frac{c}{a} = \frac{-4}{1} = -4$$

$$\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$$

$$= (-2)^2 - 2(-4)$$

$$= 4 + 8$$

$$= 12$$

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2$$

$$= (12)^2 - 2(-4)^2$$

$$= 144 - 32$$

$$= 112$$

let $A = \alpha^4 + \frac{1}{\beta^2}$

$$B = \beta^4 + \frac{1}{\alpha^2}$$

$$A+B = \left(\alpha^4 + \frac{1}{\beta^2}\right) + \left(\beta^4 + \frac{1}{\alpha^2}\right)$$

$$= \left(\alpha^4 + \beta^4\right) + \left(\frac{1}{\alpha^2} + \frac{1}{\beta^2}\right)$$

$$= \left(\alpha^4 + \beta^4\right) + \frac{\left(\alpha^2 + \beta^2\right)}{(\alpha\beta)^2}$$

$$= 112 + \frac{12}{(-4)^2}$$

$$= 112 + \frac{3}{4}$$

$$= \frac{451}{4}$$

$$AB = \left(\alpha^4 + \frac{1}{\beta^2}\right)\left(\beta^4 + \frac{1}{\alpha^2}\right)$$

$$= \alpha^4\beta^4 + \left(\alpha^2 + \beta^2\right) + \frac{1}{\alpha^2\beta^2}$$

$$= (-4)^4 + 12 + \frac{1}{(-4)^2}$$

$$= 256 + 12 + \frac{1}{16}$$

$$= \frac{4289}{16}$$

$$\therefore \quad x^2 - \frac{451}{4}x + \frac{4289}{16} = 0$$

(02) $x^2 - 4\sqrt{2}\,kx + 2k^4 - 1 = 0$

$\alpha^2 + \beta^2 = 66$

$\alpha + \beta = -\frac{b}{a} = 4\sqrt{2}\,k$

$\alpha\beta = \frac{c}{a} = 2k^4 - 1$

$(\alpha+\beta)^2 = \alpha^2 + \beta^2 + 2\alpha\beta$

$(4\sqrt{2}k)^2 = 66 + 2(2k^4 - 1)$

$32k^2 = 66 + 2(2k^4 - 1)$

$16k^2 = 33 + 2k^4 - 1$

$2k^4 - 16k^2 + 32 = 0$

$k^4 - 8k^2 + 16 = 0$

$(k^2 - 4)^2 = 0$

$k^2 = 4$

$k = 2 \quad (\because k > 0)$

(04)

$$(\alpha+\beta)^3 = \alpha^3 + 3\alpha^2\beta + 3\alpha\beta^2 + \beta^3$$

$$(4\sqrt{2}k)^3 = \alpha^3 + \beta^3 + 3\alpha\beta(\alpha+\beta)$$

$$(8\sqrt{2})^3 = \alpha^3 + \beta^3 + 3(2k^4-1)(4\sqrt{2}k)$$

$$(8\sqrt{2})^3 = \alpha^3 + \beta^3 + 3 \times 31 \times 8\sqrt{2}$$

$$1024\sqrt{2} = \alpha^3 + \beta^3 + 744\sqrt{2}$$

$$\therefore \quad \alpha^3 + \beta^3 = 280\sqrt{2}.$$

(06) $x^3 - 6x^2 + 4x + 12 = 0$

$$\alpha + \beta + \gamma = -\frac{b}{a} = \frac{-(-6)}{1} = 6$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{c}{a} = \frac{4}{1} = 4$$

$$\alpha\beta\gamma = -\frac{d}{a} = -\frac{12}{1} = -12$$

(a) $\therefore \alpha + \beta + \gamma = 6$

(b) $(\alpha + \beta + \gamma)^2 = \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta + 2\beta\gamma + 2\gamma\alpha$

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$$

$$= 6^2 - 2 \times 4$$

$$= 28$$

(c) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$$= \frac{4}{-12}$$

$$= -\frac{1}{3}$$

(08) Let the roots be

$\alpha, 2\alpha, 4\alpha$

$\therefore \alpha + 2\alpha + 4\alpha = -\frac{b}{a}$

$\therefore 7\alpha = -\frac{b}{a} \quad -①$

$2\alpha^2 + 8\alpha^2 + 4\alpha^2 = \frac{c}{a}$

$\therefore 14\alpha^2 = \frac{c}{a} \quad -②$

$8\alpha^3 = -\frac{d}{a} \quad -③$

① × ② ⟹

$7 \times 14\ \alpha^3 = -\frac{b}{a} \times \frac{c}{a}$

$98\alpha^3 = \frac{-bc}{a^2} \quad -④$

④ / ③ ⟹

$\frac{98\alpha^3}{8\alpha^3} = \frac{-bc}{a^2} \times \frac{-a}{d}$

$\frac{49}{4} = \frac{bc}{ad}$

$49ad = 4bc$