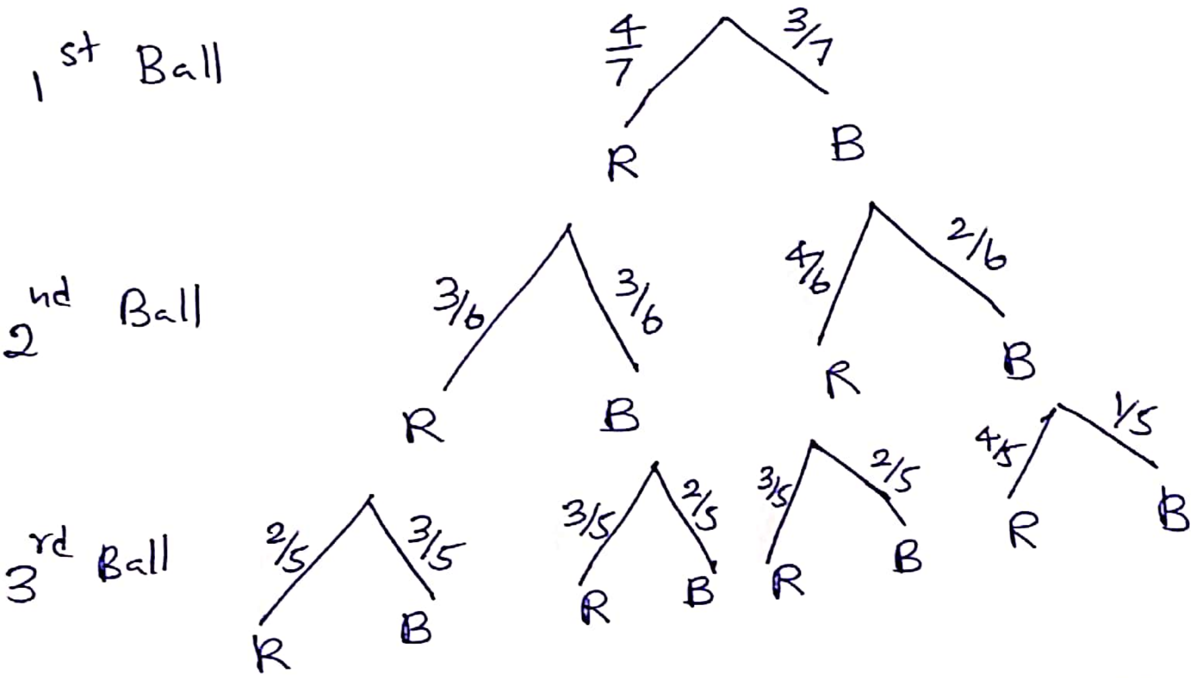
Mathematics – MA 1103
Tutorial 06 – Solutions

(01)

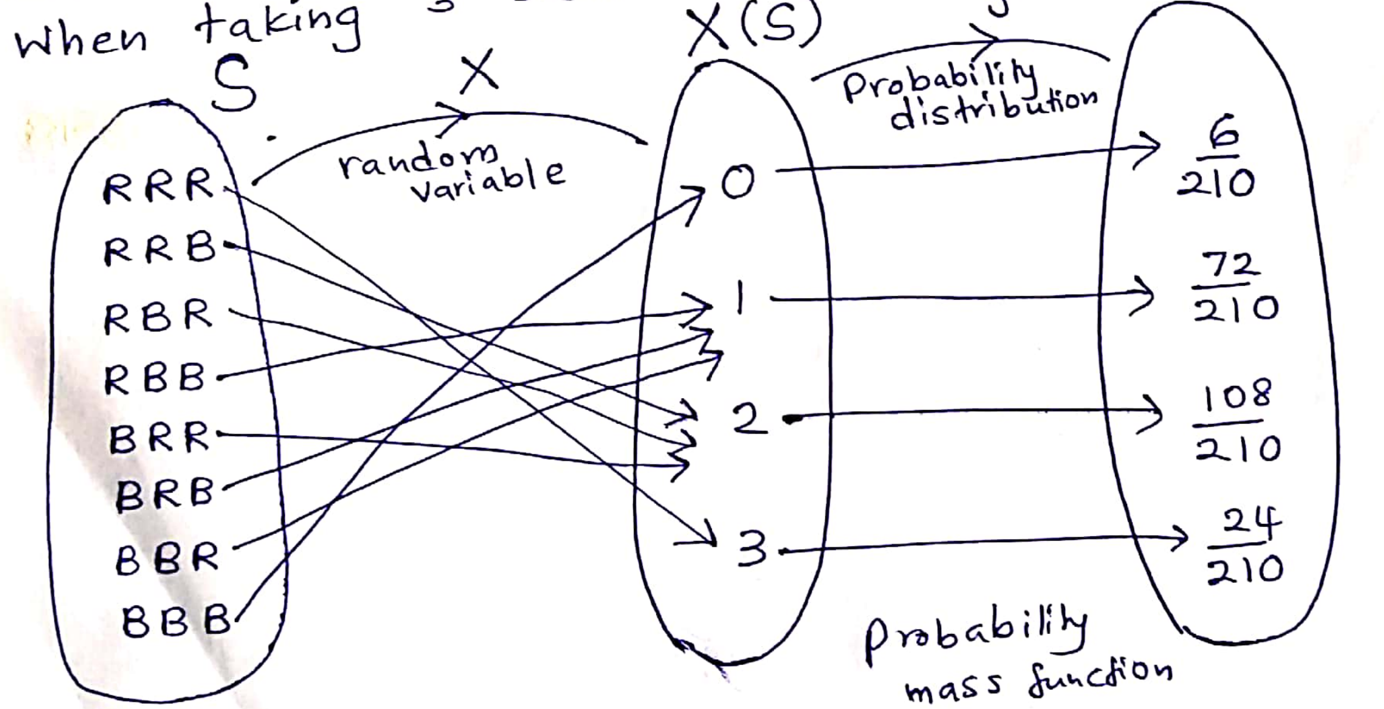
Sample Space :

R – 4
B – 3



$S = \{RRR, RRB, RBR, RBB, BRR, BRB, BBR, BBB\}$

Let X be the number of red balls drawn when taking 3 balls out without replacement.



$X(S) = \{0, 1, 2, 3\} \leftarrow$ values X can assume (02)

let $x \in X(S)$

$X=0$; $\{BBB\} = E_1$

$X=1$; $\{RBB, BRB, BBR\} = E_2$

$X=2$; $\{RRB, RBR, BRR\} = E_3$

$X=3$; $\{RRR\} = E_4$

$$P(X=0) = P(E_1) = \frac{3}{7} \times \frac{2}{6} \times \frac{1}{5} = \frac{6}{210} = f(0)$$

$$P(X=1) = P(E_2)$$
$$= \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5} + \frac{3}{7} \times \frac{4}{6} \times \frac{2}{5} + \frac{3}{7} \times \frac{2}{6} \times \frac{4}{5}$$
$$= \frac{24 + 24 + 24}{210}$$
$$= \frac{72}{210} = f(1)$$

$$P(X=2) = P(E_3)$$
$$= \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} + \frac{4}{7} \times \frac{3}{6} \times \frac{3}{5} + \frac{3}{7} \times \frac{4}{6} \times \frac{3}{5}$$
$$= \frac{108}{210} = f(2)$$

$$P(X=3) = P(E_4)$$
$$= \frac{4}{7} \times \frac{3}{6} \times \frac{2}{5}$$
$$= \frac{24}{210} = f(3)$$

$X(S)$ is discrete. $\therefore$ X is a discrete (03)
random variable. $\therefore$ it has a probability
distribution or probability mass function $f(x)$.

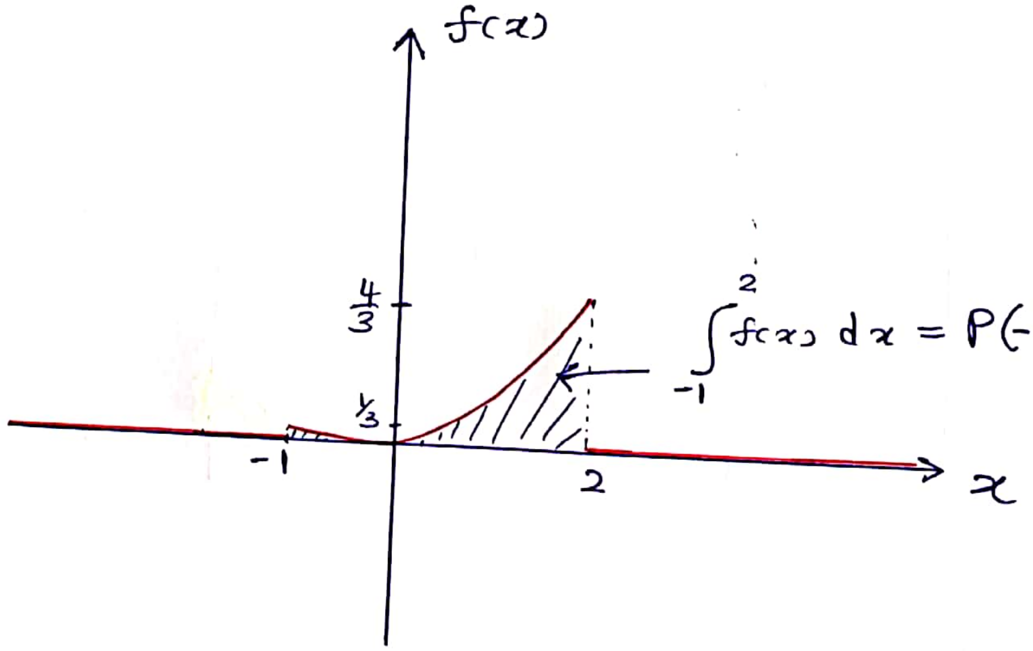
$\therefore$ The probability distribution of X is as follows:

x	0	1	2	3
$f(x)$	$\frac{6}{210}$	$\frac{72}{210}$	$\frac{108}{210}$	$\frac{24}{210}$

* When $X(S)$ is continuous, then X is a Continuous Random Variable and hence $f(x)$ is called as probability density function.

(02) (04)

$$f(x) = \begin{cases} \frac{x^2}{3} & \text{if } -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$



(a) for all $x \in \mathbb{R}$, $f(x) \geq 0$

$$\int_{-\infty}^{\infty} f(x)\,dx = \underbrace{\int_{-\infty}^{-1} f(x)\,dx}_{0} + \int_{-1}^{2} f(x)\,dx + \underbrace{\int_{2}^{\infty} f(x)\,dx}_{0}$$

$$= \int_{-1}^{2} f(x)\,dx$$

$$= \int_{-1}^{2} \frac{x^2}{3}\,dx = \frac{1}{3}\left[\frac{x^3}{3}\right]_{-1}^{2} = 1$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\therefore$ $f(x)$ is a pdf.

(b) $P(0 < X \leq 1) = P(0 < X < 1)$

$$= \int_0^1 f(x)\, dx = \int_0^1 \frac{x^2}{3}\, dx$$

$$= \frac{1}{9}$$

(c) $F(x) = P(X \leq x) \quad ; \quad x \in (-\infty, \infty)$

$$= \int_{-\infty}^{x} f(t)\, dt$$

Case I :- $x < -1$

since $f(x) = 0$, $\quad F(x) = 0$

case II :- $-1 < x < -2$

$$F(x) = \int_{-\infty}^{x} f(t)\, dt$$

$$= \underbrace{\int_{-\infty}^{-1} f(t)\, dt}_{= 0} + \int_{-1}^{x} f(t)\, dt$$

$$= \int_{-1}^{x} f(t)\, dt = \int_{-1}^{x} \frac{t^2}{3}\, dt = \left[\frac{t^3}{9}\right]_{-1}^{x}$$

$$\therefore F(x) = \frac{x^3 + 1}{9}$$

Case III :- $x > 2$

$$F(x) = \int_{-\infty}^{x} f(t)\,dt$$

$$= \underbrace{\int_{-\infty}^{-1} f(t)\,dt}_{0} + \int_{-1}^{2} f(t)\,dt + \underbrace{\int_{2}^{x} f(t)\,dt}_{0}$$

$$= \int_{-1}^{2} f(t)\,dt$$

$$= 1$$

$$\therefore \quad F(x) = \begin{cases} 0 & \text{if } x \leq -1 \\ \dfrac{x^3+1}{9} & \text{if } -1 < x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

$$P(a < X < b) = F(b) - F(a)$$

$$\therefore \quad P(0 < X \leq 1) = F(1) - F(0)$$

$$= \frac{1^3+1}{9} - \left\{\frac{0^3+1}{9}\right\}$$

$$= \frac{1}{9}$$

(d)

μ_X - mean of X

σ_X^2 - Variance of X

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-1}^{2} x \cdot \frac{x^2}{3}\, dx$$

$$= \frac{5}{4}$$

$$\sigma_X^2 = \int_{-\infty}^{\infty} (x-\mu_X)^2 f(x)\, dx$$

Now let's find $\mu_{g(X)}$ and $\sigma_{g(X)}^2$.

$$g(X) = 4\cancel{X}+3$$

$\therefore$ $g(x) = 4x+3$ ← p.d.f. of $g(X)$

$$\therefore \mu_{g(X)} = E(g(X)) = \int_{-\infty}^{\infty} f(x) \cdot g(x)\, dx$$

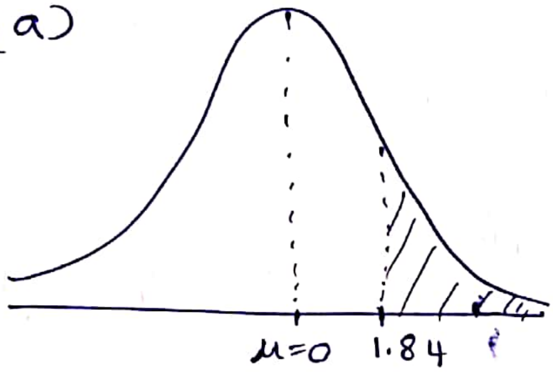
$$= \int_{-1}^{2} f(x) \cdot g(x)\, dx$$

$$= \int_{-1}^{2} \frac{x^3}{3} \cdot (4x+3)\, dx = 8$$

$\therefore \mu_{g(X)} = 8.$

(06)

(a)



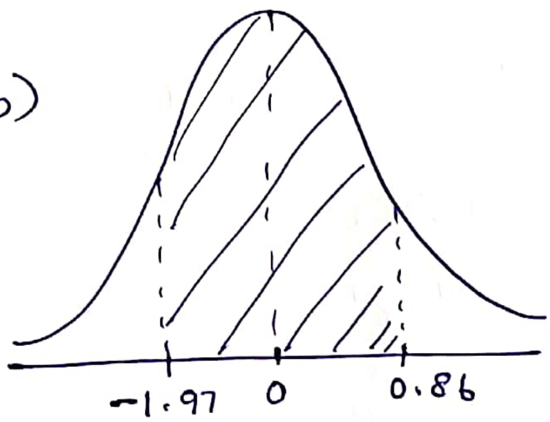
$P(Z > 1.84)$

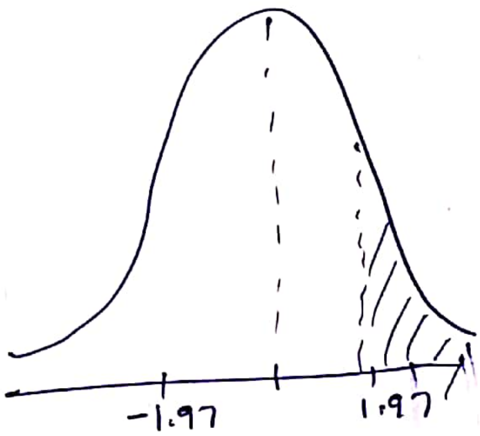
$= 1 - P(Z < 1.84)$

$= 1 - 0.9671 = 0.0329$

$\therefore P(Z > 1.84) = 3.29\%$

(b)





$P(-1.97 < Z < 0.86)$

$= P(Z < 0.86) - P(Z < -1.97)$

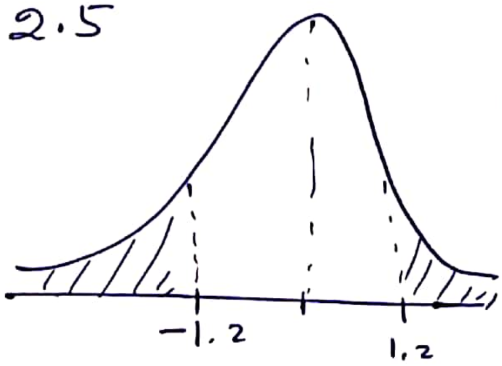
$= P(Z < 0.86) - \{1 - P(Z < 1.97)\}$

$= 0.8051 - \{1 - 0.9756\}$

$= 0.7807$

(09) X - normally distributed (09)

$\mu = 18$; $\sigma = 2.5$



(a) $P(X < 15)$

$$Z = \frac{X - \mu}{\sigma}$$

$$= \frac{15 - 18}{2.5}$$

$$= -1.2$$

$$\therefore P(X < 15) = P(Z < -1.2)$$

$$= P(Z > 1.2)$$

$$= 1 - P(Z < 1.2)$$

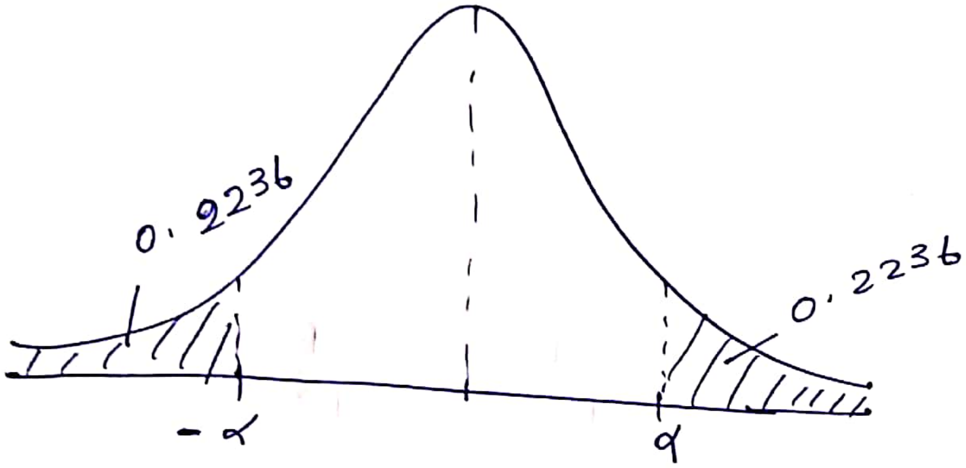
$$= 1 - 0.8849$$

$$= 0.1151$$

$$= 11.51\%$$

(b) $P(X < k) = 0.2236$

$$Z = \frac{X - \mu}{\sigma} = \frac{k - 18}{2.5} = (-\alpha)$$



$$\therefore P(z < \alpha) = 1 - 0.2236 = 0.7764$$

$$\therefore \alpha = 0.76$$

$$\therefore z = -\alpha = -0.76$$

$$\therefore \frac{k - 18}{2.5} = -0.76$$

$$k = 18 - 1.9 = 16.1$$

$$\therefore k = 16.1$$

(07) let X be the resistance of the electrical resistors.

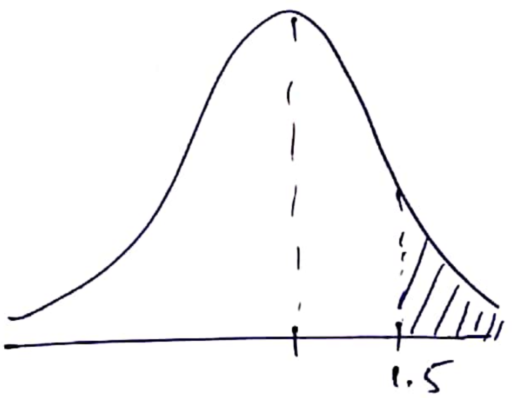
$\therefore \mu_X = 40$; $\sigma_X = 2$

X - normally distributed.

(a) $P(X > 43)$

$$Z = \frac{X - \mu_X}{\sigma_X} = \frac{X - 40}{2} = 1.5$$

$$\therefore P(X > 43) = P(Z > 1.5)$$
$$= 1 - P(Z < 1.5)$$
$$= 1 - 0.9332$$
$$= 0.0668$$
$$= 6.68\%$$



(b) We have to assign a measurement 43 to all resistors such that

$$42.5 < X < 43.5$$

$$\therefore P(X > 43) = P(X > 43.5)$$

$$Z = \frac{X - \mu_X}{\sigma_X} = \frac{43.5 - 40}{2} = 1.75$$

$\therefore P(X > 43.5)$

$= P(Z > 1.75)$

$= 1 - P(Z < 1.75)$

$= 1 - 0.9599$

$= 0.0401$

$\therefore$ 4.01 % of the resistors exceed 43 ohms when measured to the nearest ohm.