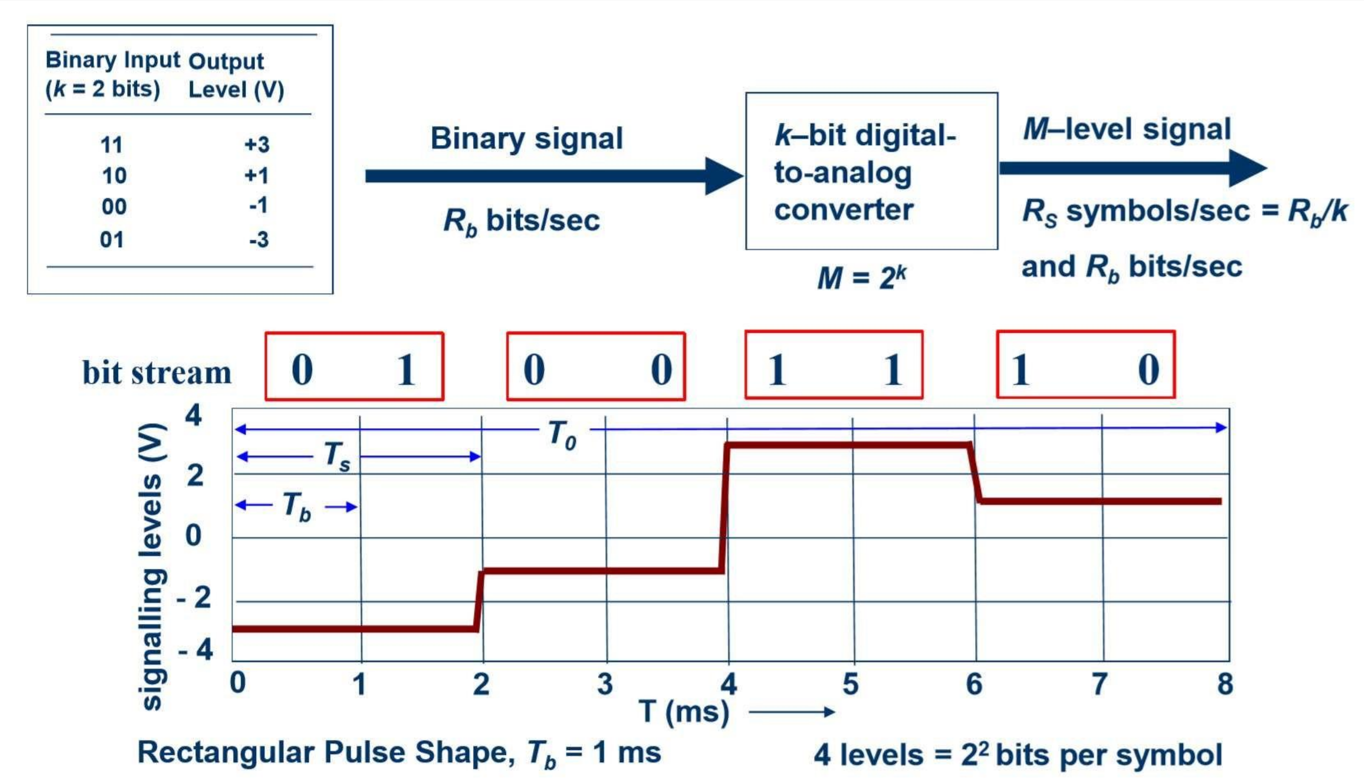
Communication Theory II

Lecture 4: Bandpass Transmission and Modulation

M-ary Signals

- Successive bits are combined to form a longer symbol.
- Number of signal levels is $M = 2^k$, where k is number of bits per symbol.
- Symbol or baud rate $R_S = R_b/k$, where R_b is bit rate.
- Symbol period $T_S = 1/ R_S = k/R_b = kT_b$, where $T_b = 1/ R_b$ is bit period.
- Note: $k = \log_2 M$
- Symbol duration $T = T_b \log_2 M$, where T_b *is the bit duration."*

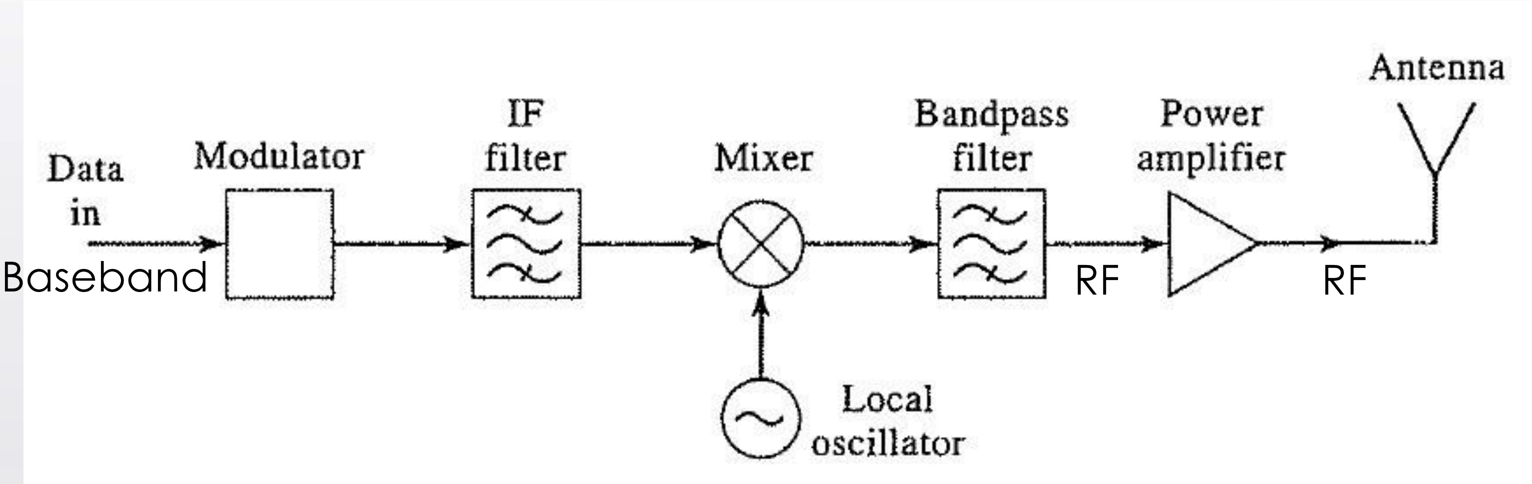
M-ary Signals (cont.)



Bandpass Transmission

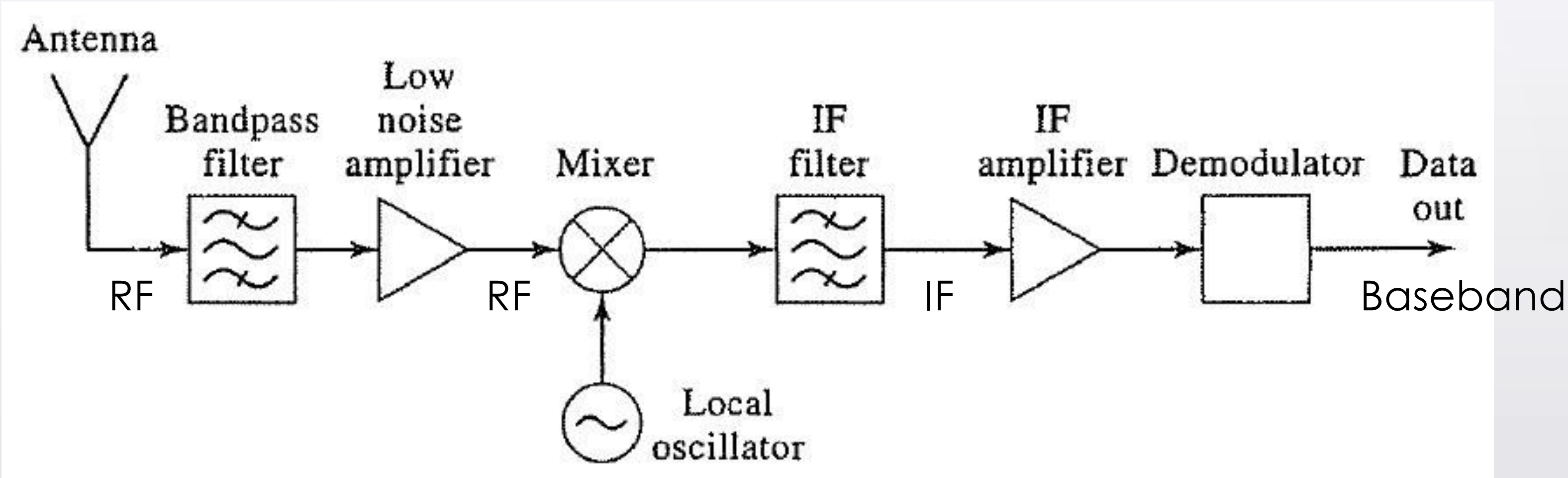
- Bandpass modulation can provide other important benefits:
  - If more than one signal utilizes a single channel, modulation may be used to separate the different signals, for example **frequency-division multiplexing (FDM)**.
  - Modulation can be used to minimize the effects of interferences, such as **spread spectrum** modulation, which requires a system bandwidth much larger than the minimum bandwidth of the message.
  - Modulation can also be used to place a signal in a frequency band where design requirements, such as filtering and amplification, can be easily met. This is the case when **radio frequency (RF)** signals are converted to an **intermediate frequency (IF)** in a receiver.

Bandpass Transmission- Transmitter



RF = Radio Frequency
IF = Intermediate Frequency

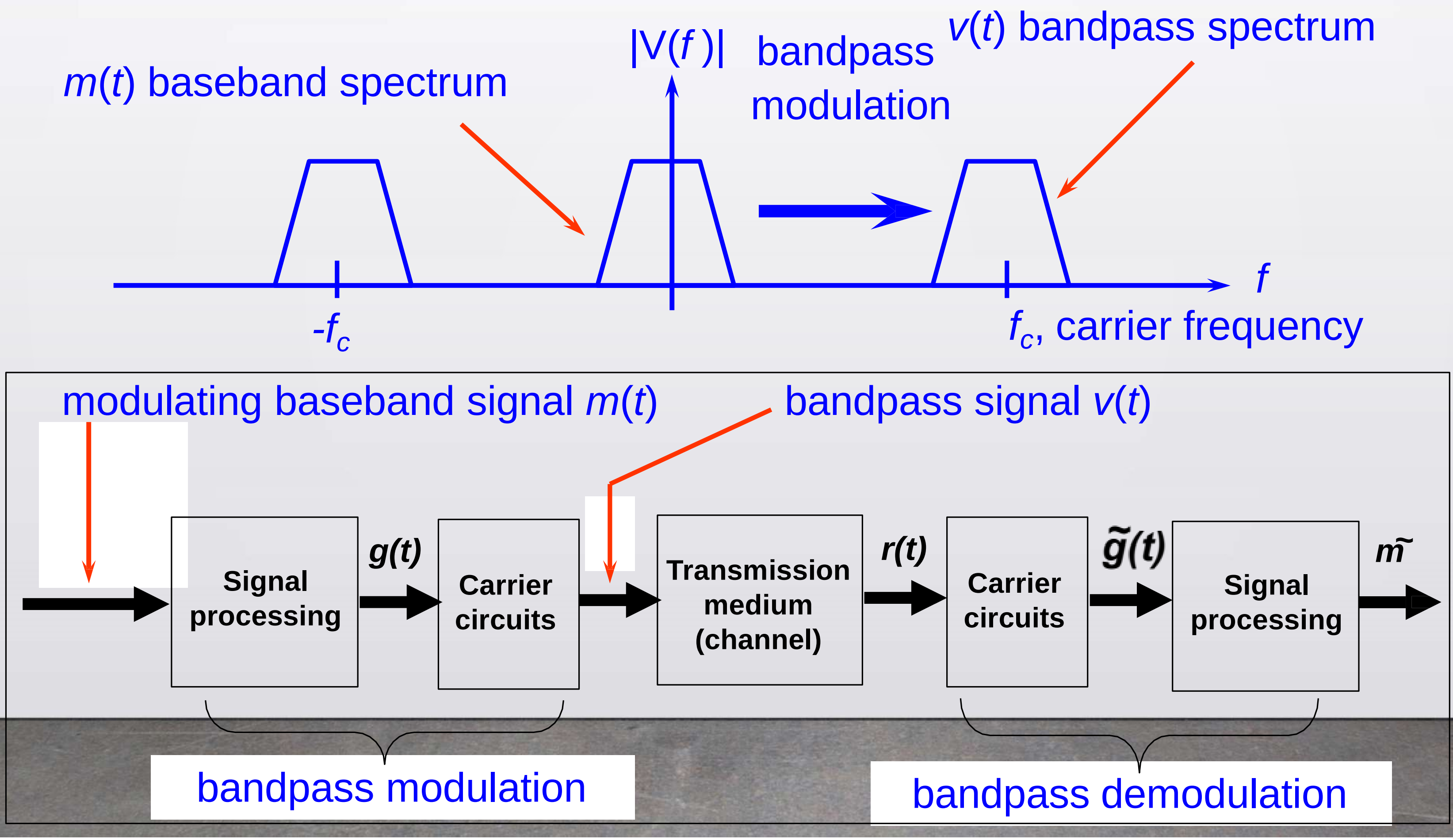
Bandpass Transmission- Receiver



RF = Radio Frequency

IF = Intermediate Frequency

Bandpass Modulation



Bandpass Modulation

- Carrier: $$v(t) = A\cos(2\pi f_c t + \phi)$$

- The 3 essential parameters:
  - Amplitude value $A(t)$ —Amplitude Modulation
  - Frequency value $f(t)$ —Frequency Modulation
  - Phase value $\phi(t)$ —Phase Modulation

- **Digital modulation:** amplitude, frequency, and/or phase are used to represent a digital state.

Digital Bandpass Modulation

- There are three basic digital modulation schemes:
  - Amplitude-Shift Keying (ASK)

$$v(t) = A(t)\cos(2\pi f_c t + \phi)$$

  - Frequency-Shift Keying (FSK)

$$v(t) = A\cos(2\pi f(t)t + \phi)$$

  - Phase-Shift Keying (PSK)

$$v(t) = A\cos(2\pi f_c t + \phi(t))$$

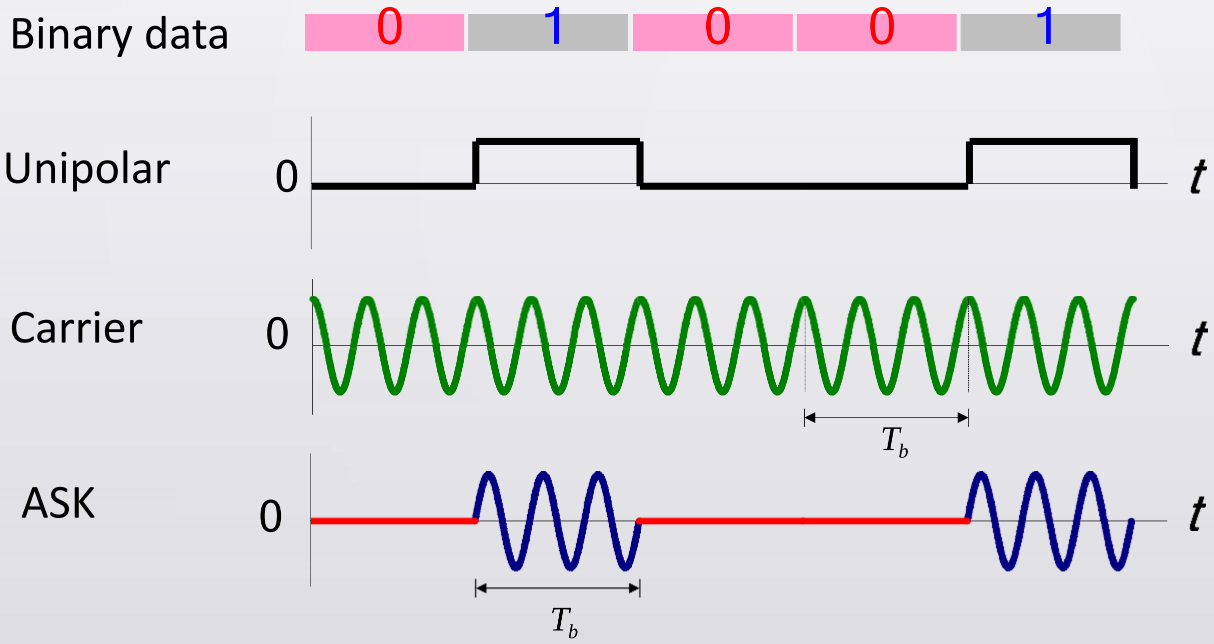
Amplitude Shift Keying (ASK)

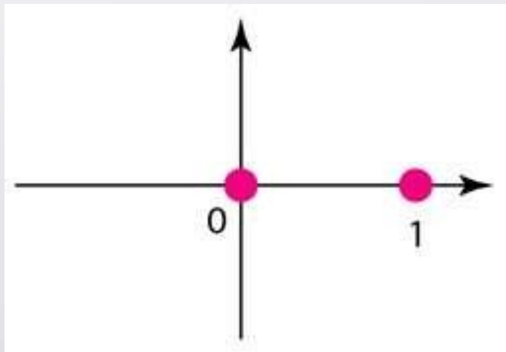
- Binary symbol "1" is represented by transmitting a sinusoidal carrier wave of fixed amplitude and fixed frequency for the bit duration T_b seconds.
- Binary symbol "0" is represented by switching off the carrier for T_b seconds.
- Also known as on-off keying (OOK).
- Modulated signal representation:

$$s_1(t) = A_c \cos(2\pi f_c t) \qquad \text{for symbol 1}$$
$$s_0(t) = 0 \qquad \text{for symbol 0}$$

Binary ASK

- For binary ASK, a unipolar line code is used to represent the data at baseband.





Constellation diagram

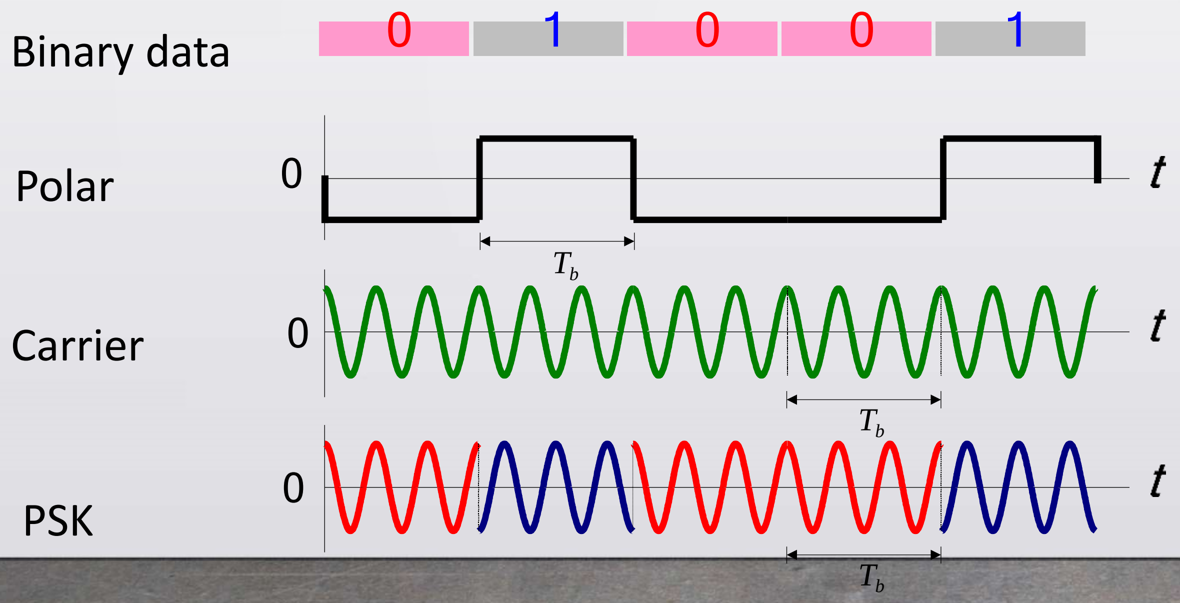
Phase Shift Keying (PSK)

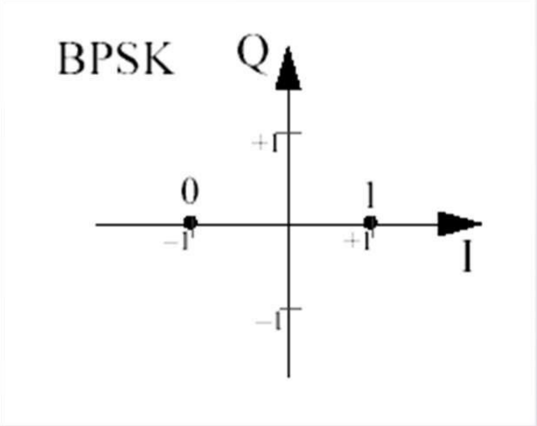
- In a PSK system, a sinusoidal carrier wave of fixed amplitude and fixed frequency is used to represent both binary symbols "0" and "1", except that whenever symbol "0" is transmitted the carrier phase is shifted by 180 degrees.
- Also known as phase-reversal keying (PRK).
- Modulated signal representation:

$$
\begin{aligned}
s_1(t) &= A_c \cos(2\pi f_c t) && \text{for symbol 1} \\
s_0(t) &= -A_c \cos(2\pi f_c t) && \text{for symbol 0} \\
&= A_c \cos(2\pi f_c t + \pi)
\end{aligned}
$$

Binary PSK

- The binary data are used to create a polar line code.
- Applying the polar line code signal, together with the carrier, to a product modulator.





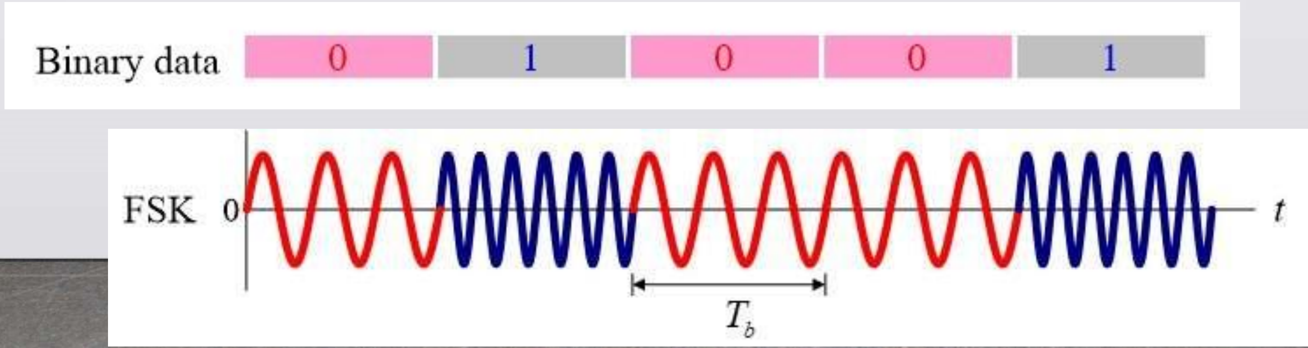
Constellation diagram

Frequency Shift Keying (FSK)

- In an FSK system, two sinusoidal waves of the same amplitude but different frequencies are used to represent binary symbols "0" and "1".

- Modulated signal representation:

$$s_1(t) = A_c \cos(2\pi f_1 t) \quad \text{for symbol 1}$$
$$s_0(t) = A_c \cos(2\pi f_0 t) \quad \text{for symbol 0}$$



Binary FSK

- For binary FSK,

$$s_1(t) = A_c \cos(2\pi f_1 t) \quad \text{for symbol 1}$$
$$s_0(t) = A_c \cos(2\pi f_0 t \quad \text{for symbol 0}$$

- If we define the carrier frequency f_c as the midpoint between f_0 and f_1, that is

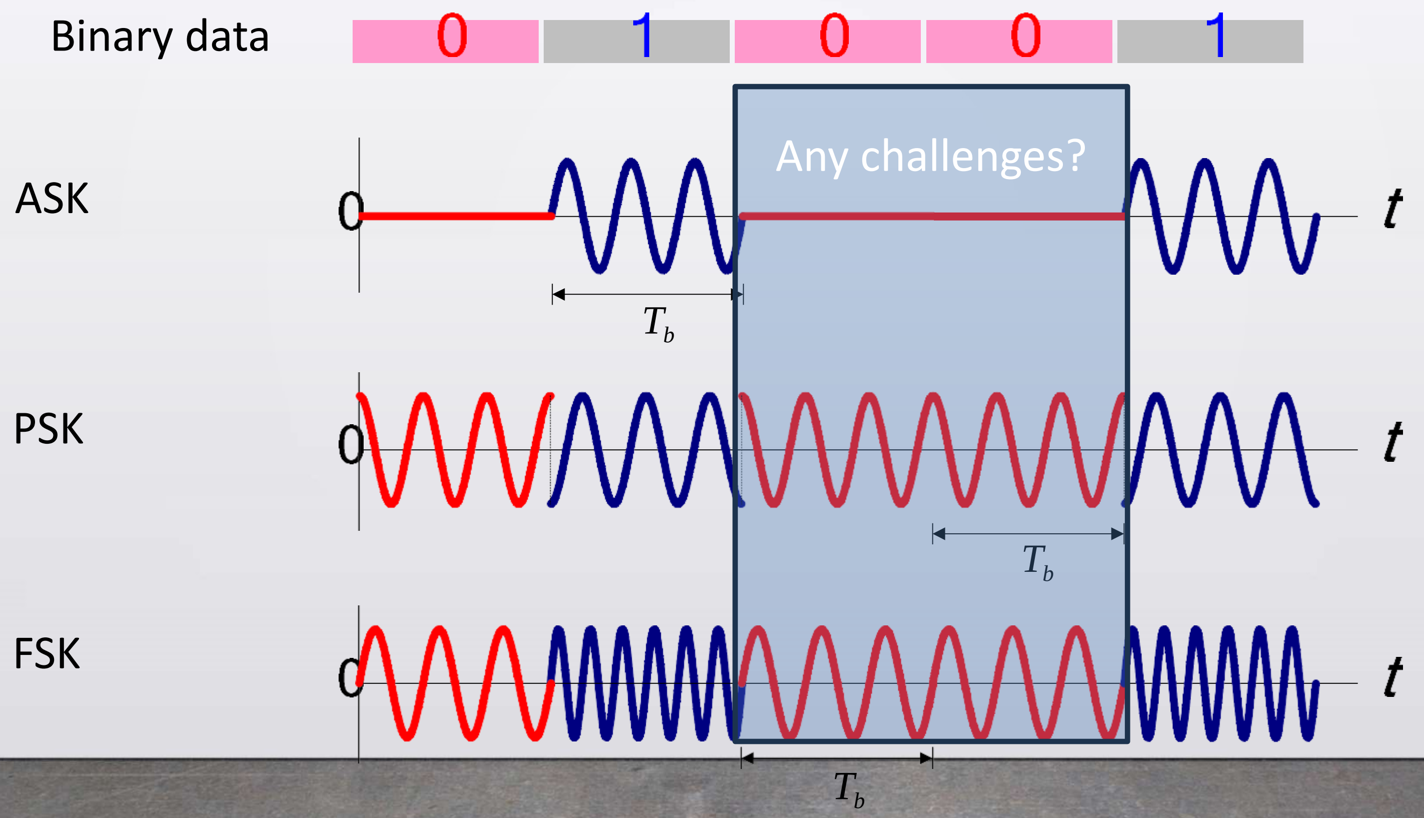
$$f_c = (f_1 + f_0)/2$$

$$\Delta f = (f_1 - f_0)/2$$

- Also define as

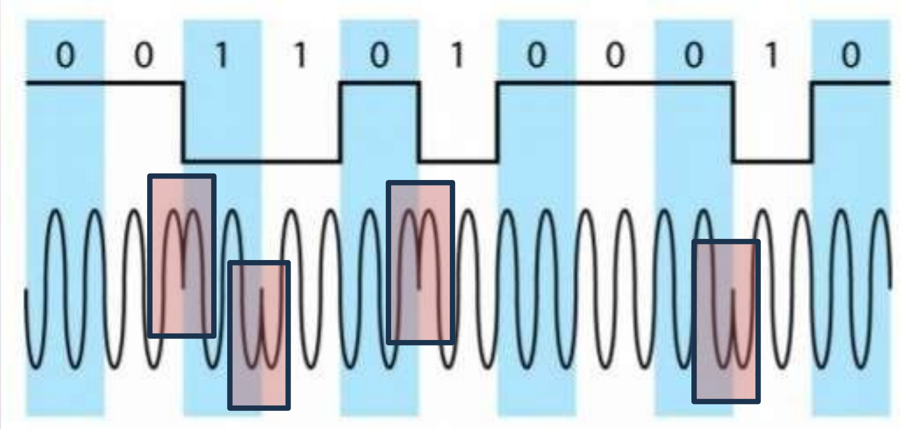
$$s_1(t) = A_c \cos(2\pi(f_c + \Delta f)t) \quad \text{for symbol 1}$$
$$s_0(t) = A_c \cos(2\pi(f_c - \Delta f)t \quad \text{for symbol 0}$$

Digital Bandpass Modulation



Differential Phase Shift Keying (DPSK)

- In a DPSK system, a sinusoidal carrier wave of fixed amplitude and fixed frequency is used to represent both binary symbols "0" and "1", except the phase of the carrier is changed for a binary "1", and there is no phase change for a binary "0".



Robustness to phase changes: DPSK is more robust against phase shifts than traditional PSK. In traditional PSK, the receiver needs to maintain synchronization with the transmitter's carrier phase. Any phase deviation can result in errors. In DPSK, the phase difference between consecutive symbols is used for modulation. This means that absolute phase information is not required, making DPSK more resilient to phase shifts caused by channel impairments or synchronization errors.

Signal Energy Per Bit, E_b

For **BPSK** signals,

$$E_b = \int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} \left[A_c \cos(2\pi f_c t)\right]^2 dt$$

$$= \int_0^{T_b} s_0^2(t)dt = \int_0^{T_b} \left[A_c \cos(2\pi f_c t + \pi)\right]^2 dt$$

$$= \frac{A_c^2 T_b}{2}$$

For **FSK** signals,

$$E_b = \int_0^{T_b} s_1^2(t)dt = \int_0^{T_b} \left[A_c \cos(2\pi f_1 t)\right]^2 dt$$

$$= \int_0^{T_b} s_0^2(t)dt = \int_0^{T_b} \left[A_c \cos(2\pi f_0 t)\right]^2 dt$$

$$= \frac{A_c^2 T_b}{2}$$

$$E_b = \frac{A_c^2 T_b}{2}$$

We may then write $A_c = \sqrt{\dfrac{2E_b}{T_b}}$ and

$$s_1(t) = A_c \cos(2\pi f_c t) \quad \text{as} \quad s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

Coherent Binary PSK

- In a coherent binary PSK system,

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

$$s_0(t) = \sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t + \pi) = -\sqrt{\frac{2E_b}{T_b}} \cos(2\pi f_c t)$$

- One basis function of unit energy can be used to represent both signals.

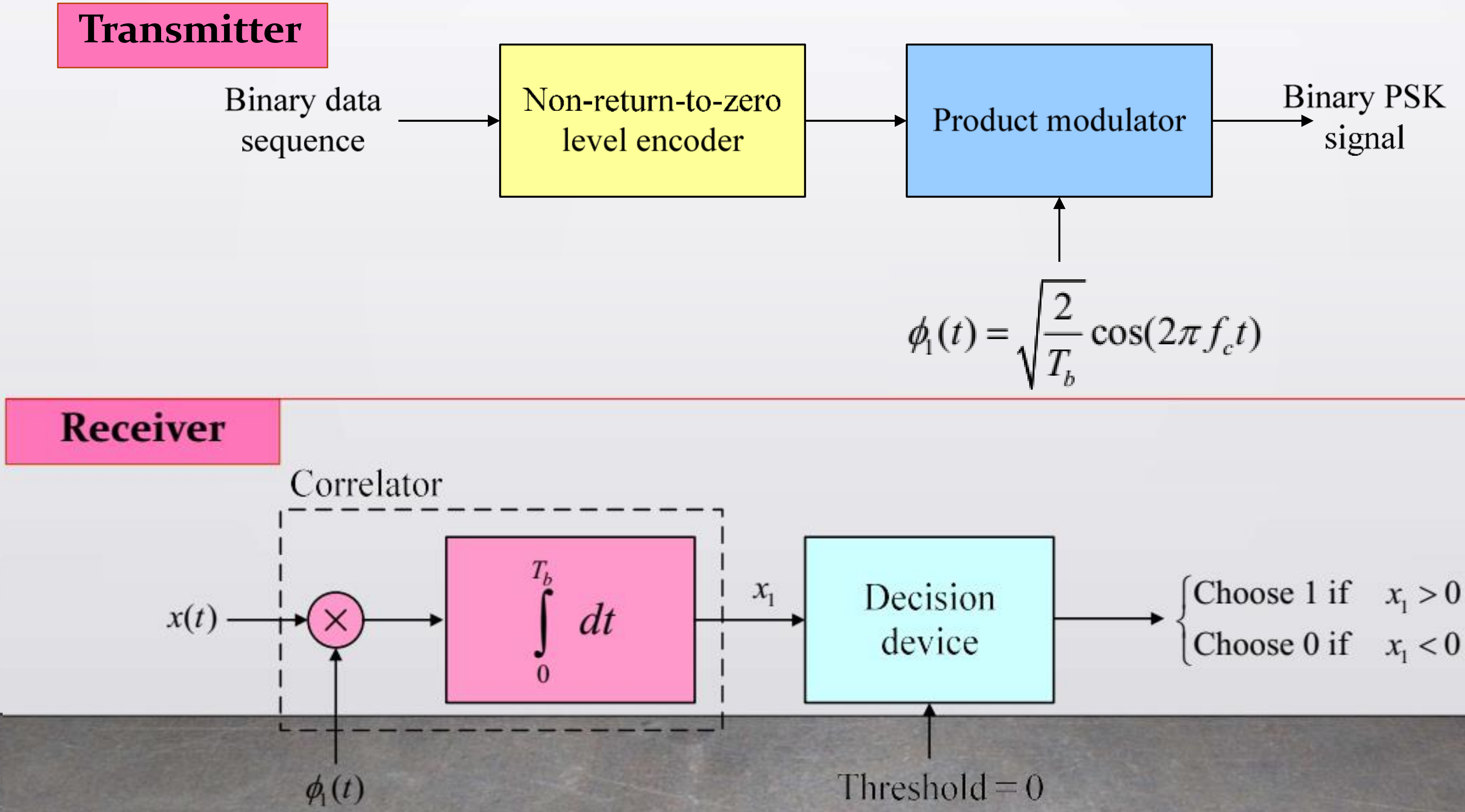
$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos(2\pi f_c t)$$

- We may write the following relations

$$s_1(t) = \sqrt{E_b}\,\phi_1(t), \qquad 0 \le t < T_b$$

$$s_0(t) = -\sqrt{E_b}\,\phi_1(t), \qquad 0 \le t < T_b$$

Coherent Binary PSK



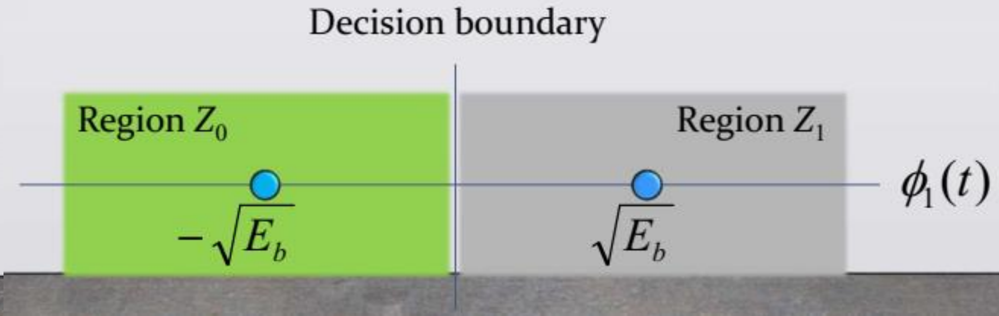
Coherent Binary PSK

- A coherent binary PSK system is therefore characterized by having a signal space that is one-dimensional, with a signal constellation consisting of two message points.
- The coordinates of the message points are

$$s_{11} = \int_0^{T_b} s_0(t)\phi_1(t)dt = \sqrt{E_b}$$

and

$$s_{01} = \int_0^{T_b} s_0(t)\phi_1(t)dt = -\sqrt{E_b}$$



Coherent Binary PSK

- To realize a rule for making a decision in favor of symbol "1" or symbol "0", we partition the signal space into two regions:
  - The set of points closets to message point "1" at $\sqrt{E_b}$
  - The set of points closets to message point "0" at $-\sqrt{E_b}$
- This is accomplished by constructing the midpoint of the line joining these two message points, and then marking off the appropriate decision regions, Z_0 and Z_1.
- The decision rule is simply to decide that signal $s_1(t)$ was transmitted if the received signal point falls in region Z_1, and decide that signal $s_0(t)$ was transmitted if the received signal point falls in region Z_0.

Coherent Binary PSK

- To calculate the probability of making an error of the first kind,

$$Z_1 : \quad 0 < x_1 < \infty \qquad\qquad x_1 = \int_0^{T_b} x(t)\phi_1(t)dt$$

- When symbol “0” is transmitted, the likelihood function is defined by

$$f_{X_1}(x_1 | 0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 - s_{01})^2 \right]$$

$$= \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2 \right]$$

- The conditional probability of the receiver deciding in favor of symbol “1”, given that symbol “0” was transmitted, is therefore

$$P_{e0} = \int_0^{\infty} f_{X_1}(x_1 | 0)dx_1 = \int_0^{\infty} \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_1 + \sqrt{E_b})^2 \right] dx_1$$

Coherent Binary PSK

- And changing the variable of integration from x_1 to z,

$$z = \frac{1}{\sqrt{N_0}}(x_1 + \sqrt{E_b})$$

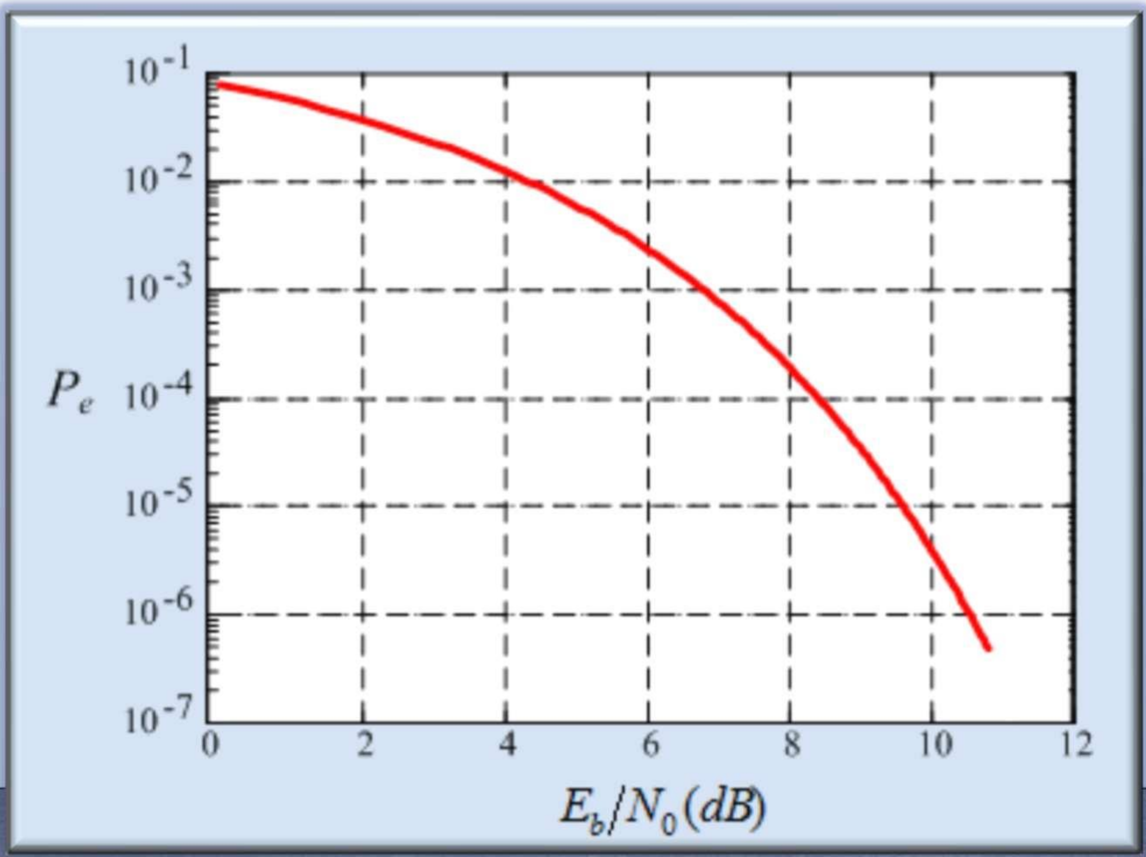
- we may rewrite the previous Eq. of P_{e0} in the compact form

$$P_{e0} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} \exp\left[-z^2\right] dz = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

- For an error of the second kind, P_{e1} has the same value as P_{e0}.
- Averaging the conditional error probabilities P_{e1} and P_{e0}, we obtain the bit error rate for coherent binary PSK

$$\boxed{P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

Coherent Binary PSK



$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Example

- The average power received in a BPSK transmission is 10 mW, and the bit period is 100 μs. If the noise power spectral density is 0.1 μJ, and coherent detection is used, determine the bit error rate.

$$E_b = P_R T_b = (10\times10^{-3})\times(100\times10^{-6}) = 10^{-6}\ \text{J}$$

$$\frac{E_b}{N_0} = \frac{10^{-6}}{10^{-7}} = 10$$

$$\text{BER} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{10}\right) = \frac{1}{2}(7.74\times10^{-6}) = 3.87\times10^{-6}$$

Coherent Binary FSK

- In a coherent binary FSK system,

$$s_1(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_1 t) \qquad s_0(t) = \sqrt{\frac{2E_b}{T_b}}\cos(2\pi f_0 t)$$

- Two orthonormal basis functions of unit energy are used to represent both signals.

$$\phi_1(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_1 t) \qquad \phi_0(t) = \sqrt{\frac{2}{T_b}}\cos(2\pi f_0 t)$$

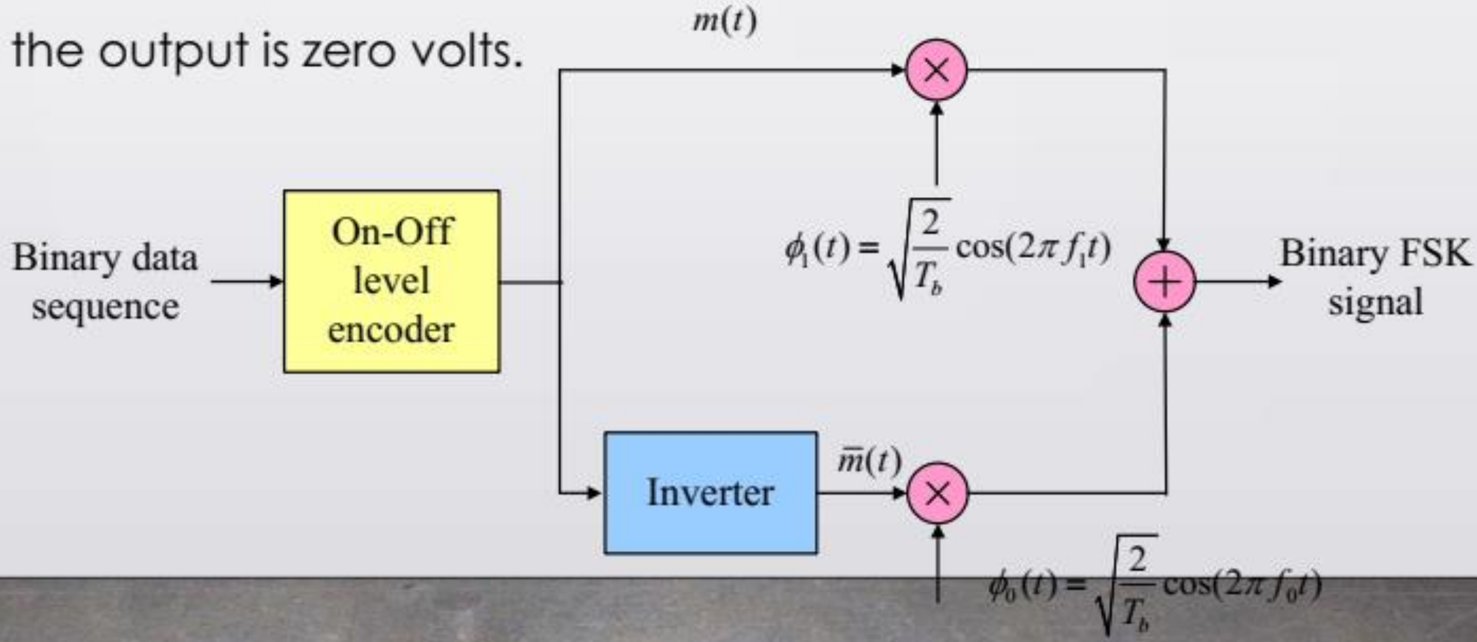
- We may write the following relations

$$s_1(t) = \sqrt{E_b}\,\phi_1(t), \quad 0 \le t < T_b$$

$$s_0(t) = \sqrt{E_b}\,\phi_0(t), \quad 0 \le t < T_b$$

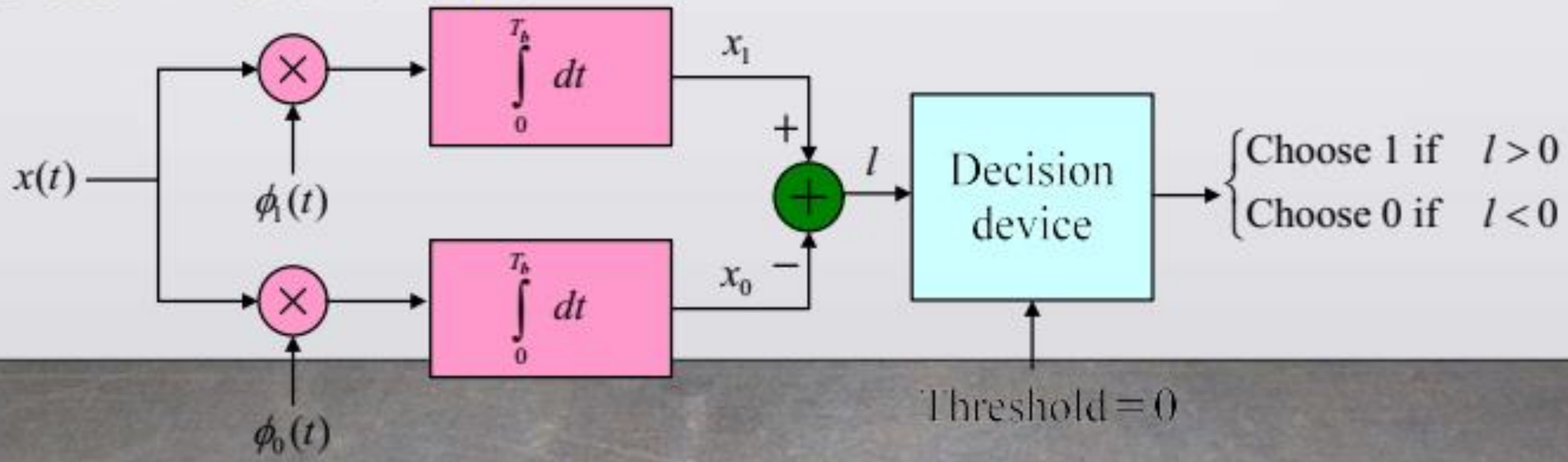
Binary FSK transmitter

- For on-off encoder, if the binary data sequence is symbol 1, the output is a constant amplitude of $\sqrt{E_b}$
- If it is symbol 0, the output is zero volts.



Coherent binary FSK receiver

- The receiver consists of two correlators with a common input, which are supplied with locally generated coherent reference signals $\phi_1(t)$ and $\phi_0(t)$.
- The correlator outputs are then subtracted one from the other, and the resulting difference l, is compared with a threshold of zero.
- If $l > 0$, the receiver decides in favor of "1", and if $l < 0$, it decides in favor of "0".



Coherent binary FSK receiver

- A coherent binary FSK system is therefore characterized by having a signal space that is two-dimensional, with a signal constellation consisting of two message points: one on each axis.

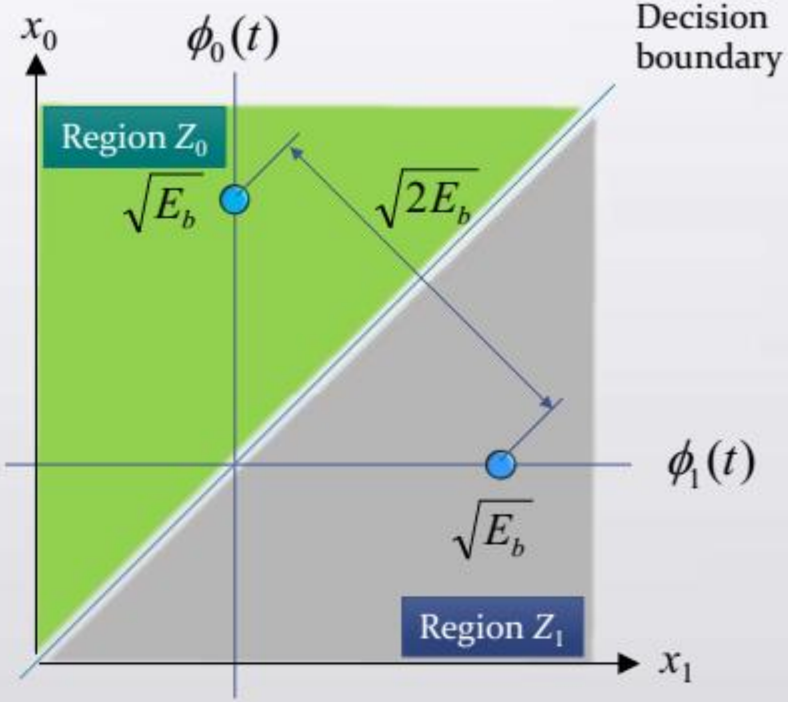
- The signal outputs of correlators are

$$x_1 = \int_0^{T_b} x(t)\phi_1(t)dt$$

$$x_0 = \int_0^{T_b} x(t)\phi_0(t)dt$$

Coherent binary FSK receiver

- The observation space is partitioned into two decision regions, labeled Z_0 and Z_1.
- The decision boundary separating region Z_0 and Z_1 is the perpendicular bisector of the line joining the two message points.



Coherent binary FSK receiver

- The receiver decides in favor of symbol "1" if the receiver signal point represented by the observation vector $\mathbf{x} = [x_1\, x_0]$ falls inside region Z_1. This occurs when $x_1 > x_0$.
- If on the other hand we have $x_1 < x_0$, the received signal point falls inside region Z_0, the receiver decides in favor of symbol "0".
- On the decision boundary, we have $x_1 = x_0$, in which case the receiver makes a random guess in favor of symbol "0" or "1".

Coherent binary FSK receiver

- Define a new Gaussian random variable L whose sample value l is equal to the difference between x_1 and x_0: $l = x_1 - x_0$
- The mean value of the random variable L depends on which binary symbol was transmitted.

- Given that symbol 1 was transmitted

$$E[L \mid 1] = E[X_1 \mid 1] - E[X_0 \mid 1] = \sqrt{E_b}$$

- Given that symbol 0 was transmitted

$$E[L \mid 0] = E[X_1 \mid 0] - E[X_0 \mid 0] = -\sqrt{E_b}$$

Coherent binary FSK receiver

- Since the random variables X_1 and X_0 are statistically independent, each with a variance equal to $N_0/2$, it follows that

$$\mathrm{var}[L] = \mathrm{var}[X_1] + \mathrm{var}[X_0] = N_0$$

- Suppose that symbol "0" was sent. The conditional probability density function of L is

$$f_L(l\,|\,0) = \frac{1}{\sqrt{2\pi N_0}} \exp\left[-\frac{(l+\sqrt{E_b})^2}{2N_0}\right]$$

Coherent binary FSK receiver

- Since the condition $x_1 > x_0$, or equivalently, $l > 0$, corresponds to the receiver making a decision in favor of symbol "1", we deduce

$$P_{e0} = P(l > 0 \mid \text{symbol 0 was sent}) = \int_0^\infty f_L(l \mid 0)\,dl$$
$$= \frac{1}{\sqrt{2\pi N_0}} \int_0^\infty \exp\left[-\frac{(l + \sqrt{E_b})^2}{2N_0} \right] dl$$

- Put $\frac{l + \sqrt{E_b}}{\sqrt{2N_0}} = z$ then changing the variable of integration from l to z, we obtain

$$P_{e0} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/2N_0}}^{\infty} \exp(-z^2)\,dz = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{2N_0}} \right) = P_{e1} = P_e$$

Example

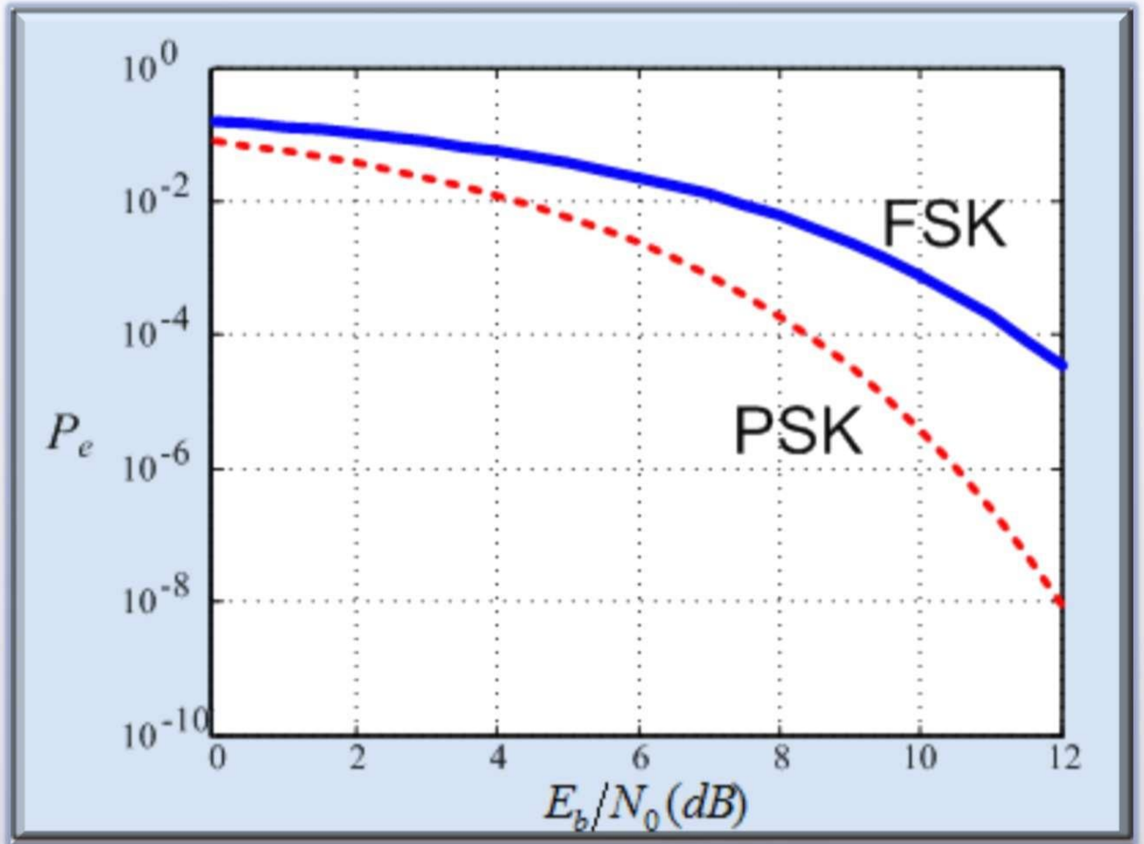
- The average power received in a binary FSK transmission is 10 mW, and the bit period is 100 μs. If the noise power spectral density is 0.1 μJ, and coherent detection is used, determine the bit error rate.

$$E_b = P_R T_b = (10\times10^{-3})\times(100\times10^{-6}) = 10^{-6}\text{ J}$$

$$\frac{E_b}{N_0} = \frac{10^{-6}}{10^{-7}} = 10$$

$$\text{BER} = \frac{1}{2}\text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right) = \frac{1}{2}\text{erfc}\left(\sqrt{5}\right) = \frac{1}{2}(1.565\times10^{-3}) = 7.83\times10^{-4}$$

Binary PSK & FSK



References

- Simon Haykin and Michael Moher, *Communication Systems*, 5th edition, John Wiley & Sons, 2009.
- Simon Haykin, *Digital Communication Systems*, John Wiley & Sons, 2013.

THANK YOU