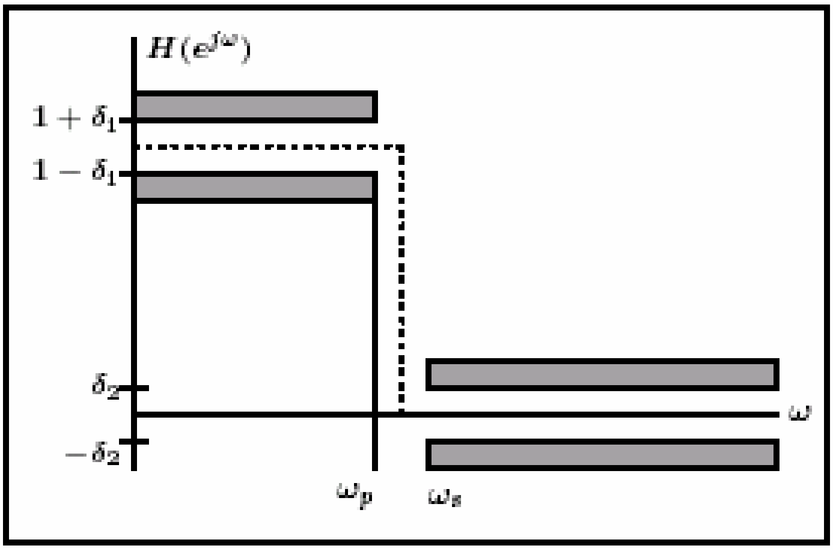
Optimum Approximation of FIR Filters

- **FIR filters with the narrowest transition bandwidth are equiripple:**



Let's consider a zero-phase, type I, filter:

$$H(e^{j\omega}) = \sum_{n=-L}^{L} h[n]e^{-j\omega n} = h[0] + \sum_{n=1}^{L} 2h[n]\cos(n\omega), \text{ where } L = \frac{M}{2}$$

Which can be rewrite as:

$$H(e^{j\omega}) = h[0] + \sum_{n=1}^{L} 2h[n]\cos(\omega n) = \sum_{k=0}^{L} a_k(\cos\omega)^k$$

- The terms *cos(wn)* have been expressed as a sum of *cosw*

The ripples have extrema where the derivative goes to zero:

$$\begin{aligned} \frac{d}{d\omega} H(e^{j\omega}) &= \sum_{k=0}^{L} -a_k k (\cos\omega)^{k-1} \sin\omega \\ &= -\sin\omega \sum_{k=0}^{L-1} (k+1) a_{k+1} (\cos\omega)^k \end{aligned}$$

- Because of the *sinω* term, the derivative is zero at $\omega = 0$ and $\omega = \pi$.
- The remaining trigonometric polynomial has order $(L - 1)$, so it has at most $(L - 1)$ distinct roots.

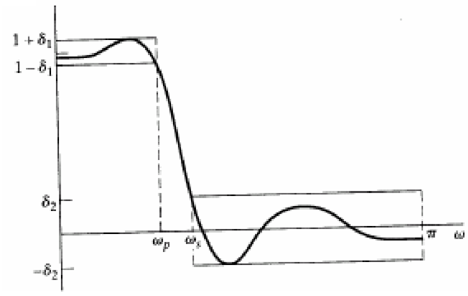
- It can be shown that if we have either L or $L+1$ zero-slope points this will yield minimum ripples and the ripples will be equal:

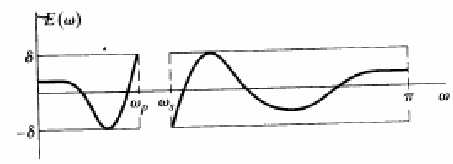
Alternation Theorem *Let* $E(\omega) \triangleq W(\omega)\left[H_d(e^{j\omega}) - H(e^{j\omega})\right]$, *where* $W(\omega)$ *is a continuous weighting function and* $H_d(e^{j\omega})$ *is the desired filter frequency response. Now, define the set of* K *alternation values* $\{\omega_i\}$ *such that* $\omega_1 < \omega_2 < \cdots < \omega_K$ *and such that* $E(\omega_i) = -E(\omega_{i+1}) = \pm\max|E(\omega)|$ *for* $i = 1, 2, \ldots, K$. *Then,* $H(e^{j\omega})$ *is the unique* L*th order polynomial that minimizes* $\max|E(\omega)|$ *if and only if* $E(\omega)$ *exhibits at least* $L + 2$ *alternations.*

Example: Lowpass filter

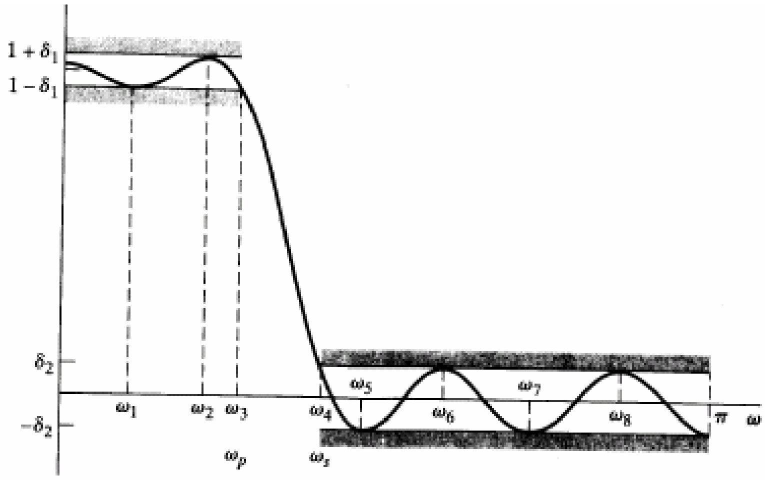
$$H_d(e^{j\omega}) = \begin{cases} 1, & 0 \le \omega \le \omega_p, \\ 0, & \omega_s \le \omega \le \pi. \end{cases}$$

$$W(\omega) = \begin{cases} \frac{1}{K} & 0 \le \omega \le \omega_p, \\ 1, & \omega_s \le \omega \le \pi, \end{cases}$$





An optimal lowpass, type I, filter with M=7:



In the above figure, we see:
- 9 alternation points.

- 8 of these are zero-slope extrema
- 2 are at the edge of the transition region.

Note:
The slope of the filter curve is zero at both $\omega = 0$ and $\omega = \pi$, but we only have an alternation at π.

Optimal filter properties:

- Equiripple in both passband and stopband, except possibly at 0 and π.

- If $L + 2$ or $L + 3$ alternations are found in a filter frequency response, this filter is the **unique** filter that gives the minimum approximation error.

- If there are $L + 3$ alternations, the filter is referred to as extra-ripple.
- Alternations always happen at ω_s and ω_p.
- Alternations may happen at 0 and/or π.

A Note on Type II, III, and IV Systems

For Type II:

$$H(e^{j\omega}) = \cos(\omega/2) \sum_{k=0}^{(M-1)/2} a_k (\cos\omega)^k$$

This is because $M = 2L+1$ is odd, and we have a zero at $\omega=\pi$.

For Type III:

$$H(e^{j\omega}) = \sin(\omega) \sum_{k=0}^{M/2-1} a_k (\cos\omega)^k$$

Note that in this type we have two zeros at $\omega=0$ and $\omega=\pi$.

And finally for type IV:

$$H(e^{j\omega}) = \sin(\omega/2) \sum_{k=0}^{(M-1)/2} a_k (\cos\omega)^k$$

All four different types can be compactly expressed as:

$$H(e^{j\omega}) = Q(\omega)A(\omega),$$

where

$$Q(\omega) = \begin{cases} 1 & \text{for type I}, \\ \cos(\omega/2) & \text{for type II}, \\ \sin(\omega) & \text{for type III}, \\ \sin(\omega/2) & \text{for type IV}, \end{cases}$$

and

$$A(\omega) = \sum_{k=0}^{L} a_k (\cos\omega)^k.$$

We can now rewrite the weighting function as:

$$E(\omega) = W(\omega)\left[H_d(e^{j\omega}) - Q(\omega)A(\omega)\right] = W(\omega)Q(\omega)\left[\frac{H_d(e^{j\omega})}{Q(\omega)} - A(\omega)\right].$$

Therefore, in order to design type II, III, and IV systems, all we have to do is modify our weighting function and desired response!

Optimum Filter Design Problem

Herrmann and Hofstetter:

Given specs (M, δ_1, δ_2), determine *the best* ω_s, ω_p, and *h[n]*.

Parks-McClellan (Remez):

Given specs (M, ω_s, ω_p, δ_1/δ_2), determine δ_1 and *h[n]* so that we have L+2 or L+3 externa.

Parks-McClellan (Remez) Algorithm:

From the alternation thorem, the optimum filter will satisfy the set of equations:

$$W(\omega_i)[H_d(e^{j\omega_i}) - A_e(e^{j\omega_i})] = (-1)^{i+1}\delta,$$

where:

$$i = 1, 2, \ldots, (L+2),$$

Here, δ is the optimum value of error is stopband (δ_2).

In the matrix form we will have:

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^L & \dfrac{1}{W(\omega_1)} \\ 1 & x_2 & x_2^2 & \cdots & x_2^L & \dfrac{-1}{W(\omega_2)} \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 1 & x_{L+2} & x_{L+2}^2 & \cdots & x_{L+2}^L & \dfrac{(-1)^{L+2}}{W(\omega_{L+2})} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \delta \end{bmatrix} = \begin{bmatrix} H_d(e^{j\omega_1}) \\ H_d(e^{j\omega_2}) \\ \vdots \\ H_d(e^{j\omega_{L+2}}) \end{bmatrix}$$

where $x_i = \cos\omega_i$

More efficient way is to use polynomial interpolation:

$$\delta = \frac{\sum_{k=1}^{L+2} b_k H_d(e^{j\omega_k})}{\sum_{k=1}^{L+2} \dfrac{b_k(-1)^{k+1}}{W(\omega_k)}},$$

$$b_k = \prod_{\substack{i=1 \\ i\neq k}}^{L+2} \frac{1}{(x_k - x_i)}$$

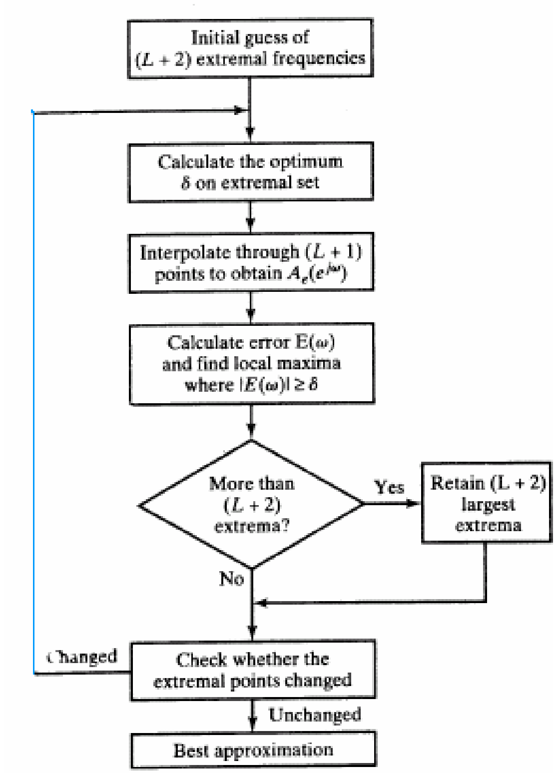
using lagrange interpolation we can obtain:

$$A_e(e^{j\omega}) = P(\cos\omega) = \frac{\sum_{k=1}^{L+1} [d_k/(x - x_k)] C_k}{\sum_{k=1}^{L+1} [d_k/(x - x_k)]},$$

Where:

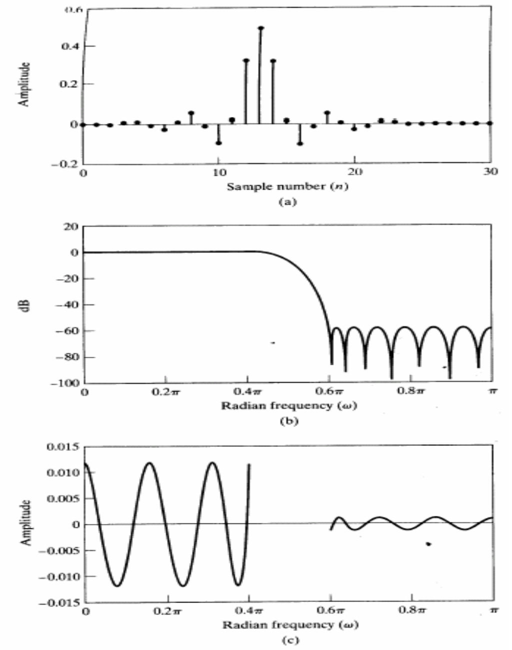
$$C_k = H_d(e^{j\omega_k}) - \frac{(-1)^{k+1}\delta}{W(\omega_k)},$$

$$d_k = \prod_{\substack{i=1 \\ i\neq k}}^{L+1} \frac{1}{(x_k - x_i)} = b_k(x_k - x_{L+2}).$$



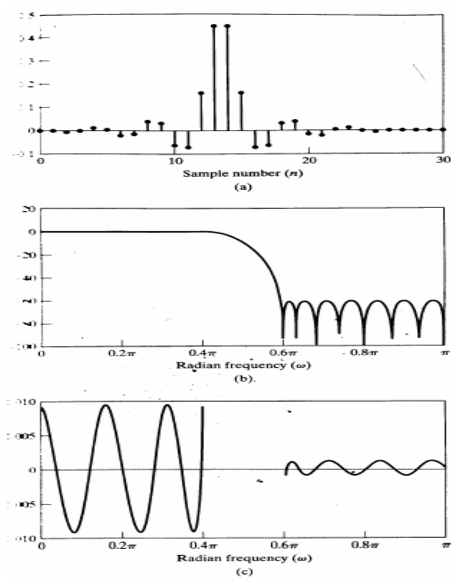
Example1: LPF, TypeI,

with M=26, $\omega_s = 0.6\pi$, $\omega_p = 0.4\pi$, $\delta_1/\delta_2 = 10$

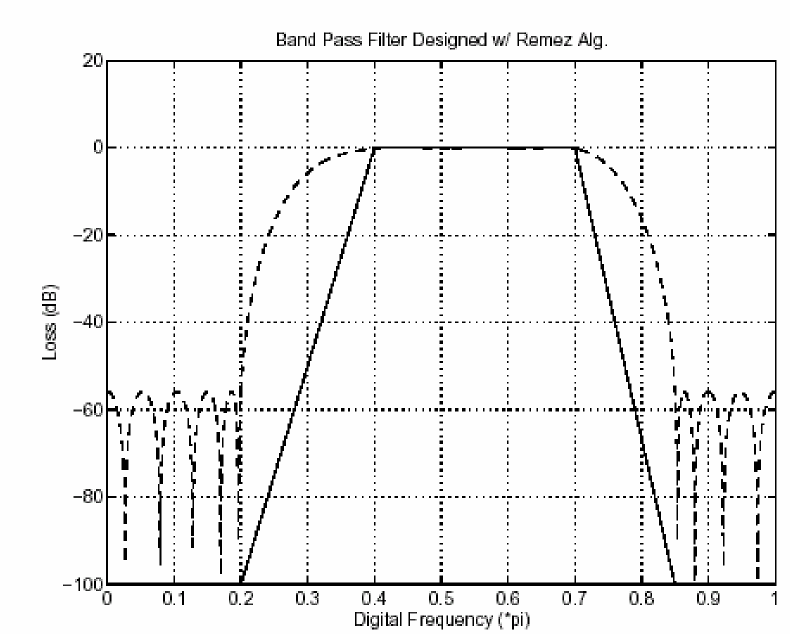


Example2: LPF, TypeII,

with M=27, $\omega_s = 0.6\pi$, $\omega_p = 0.4\pi$, $\delta_1/\delta_2 = 10$



Example 3: bandpass filter



Example 4: another bandpass filter