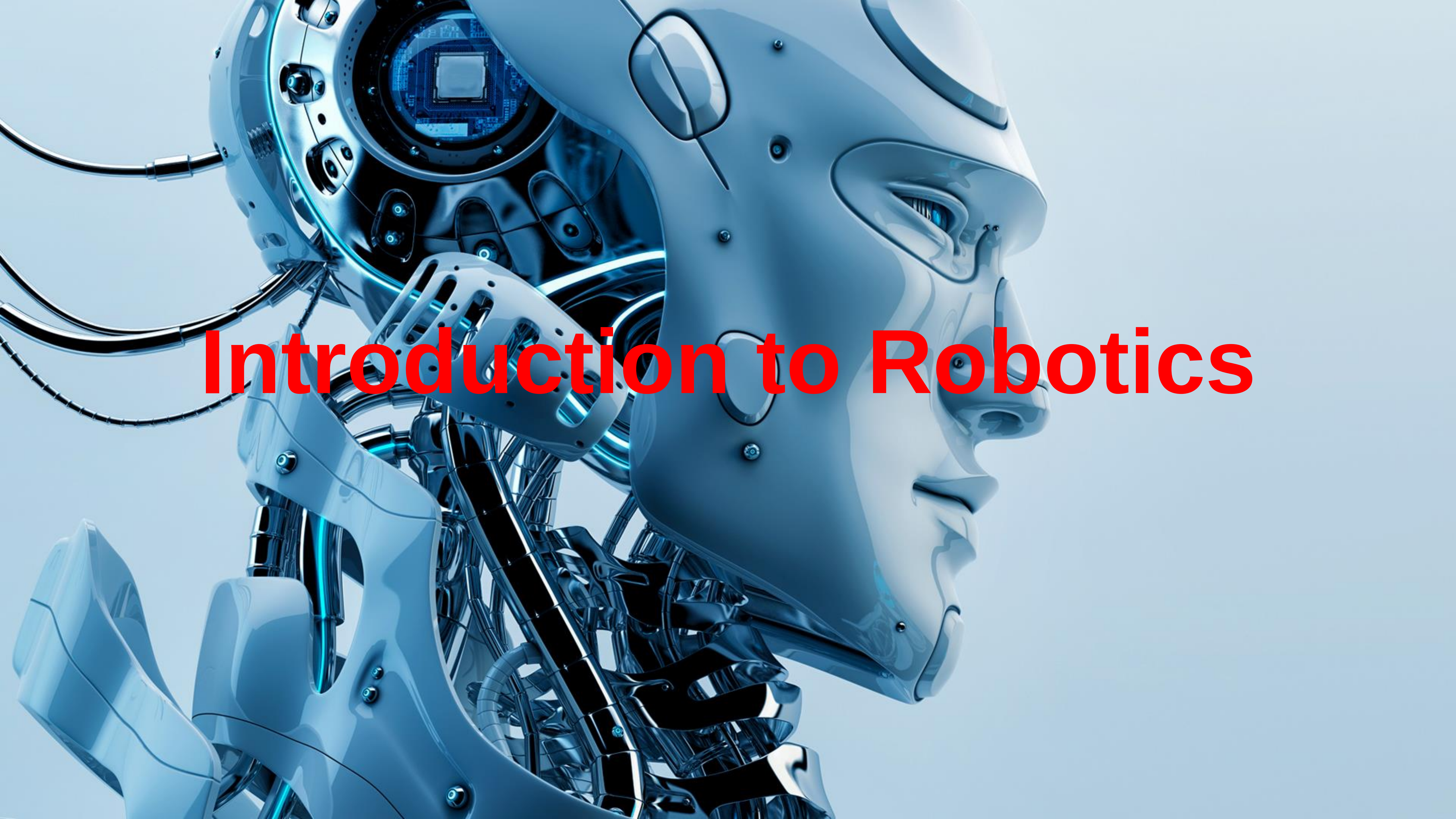
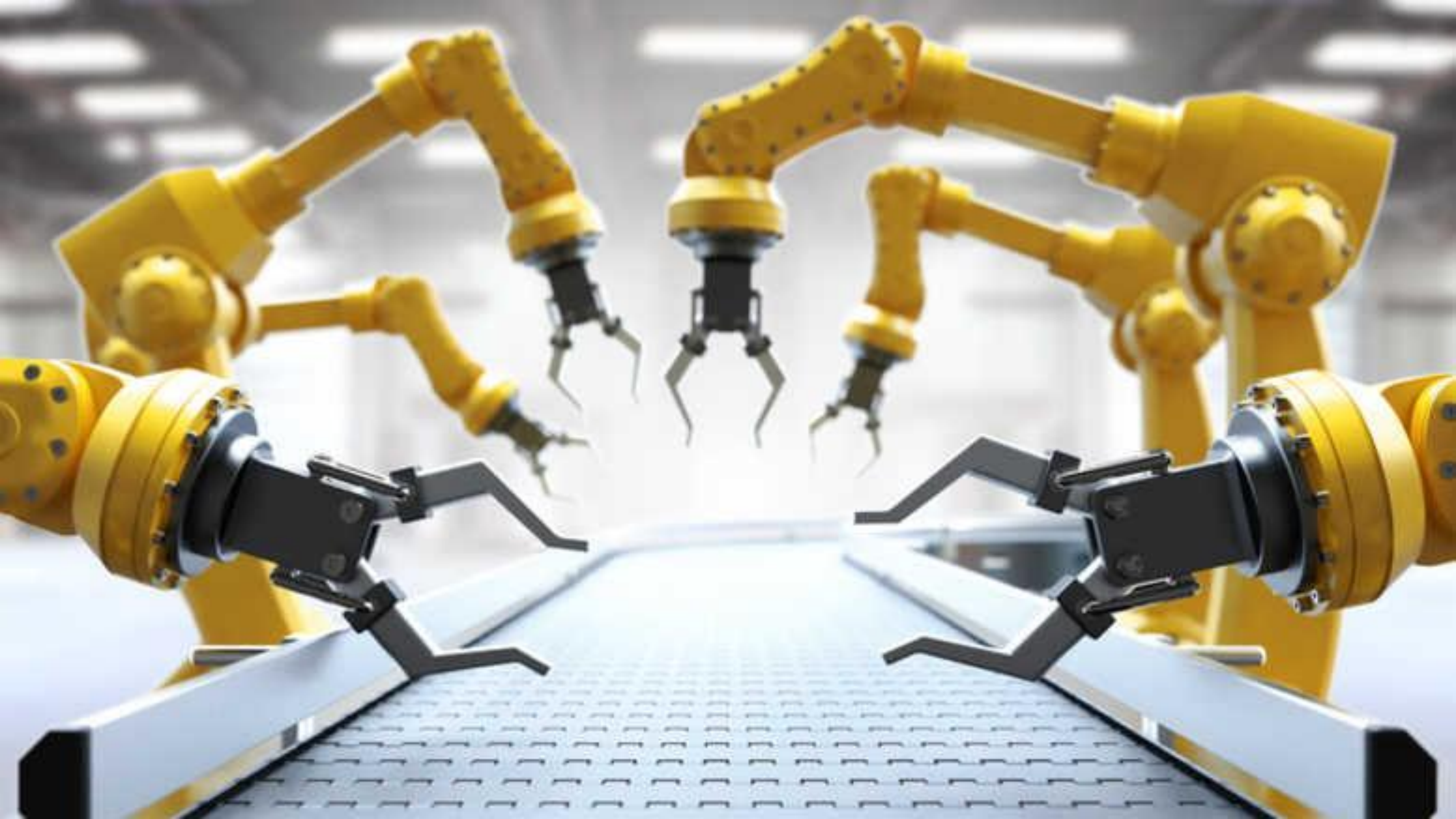
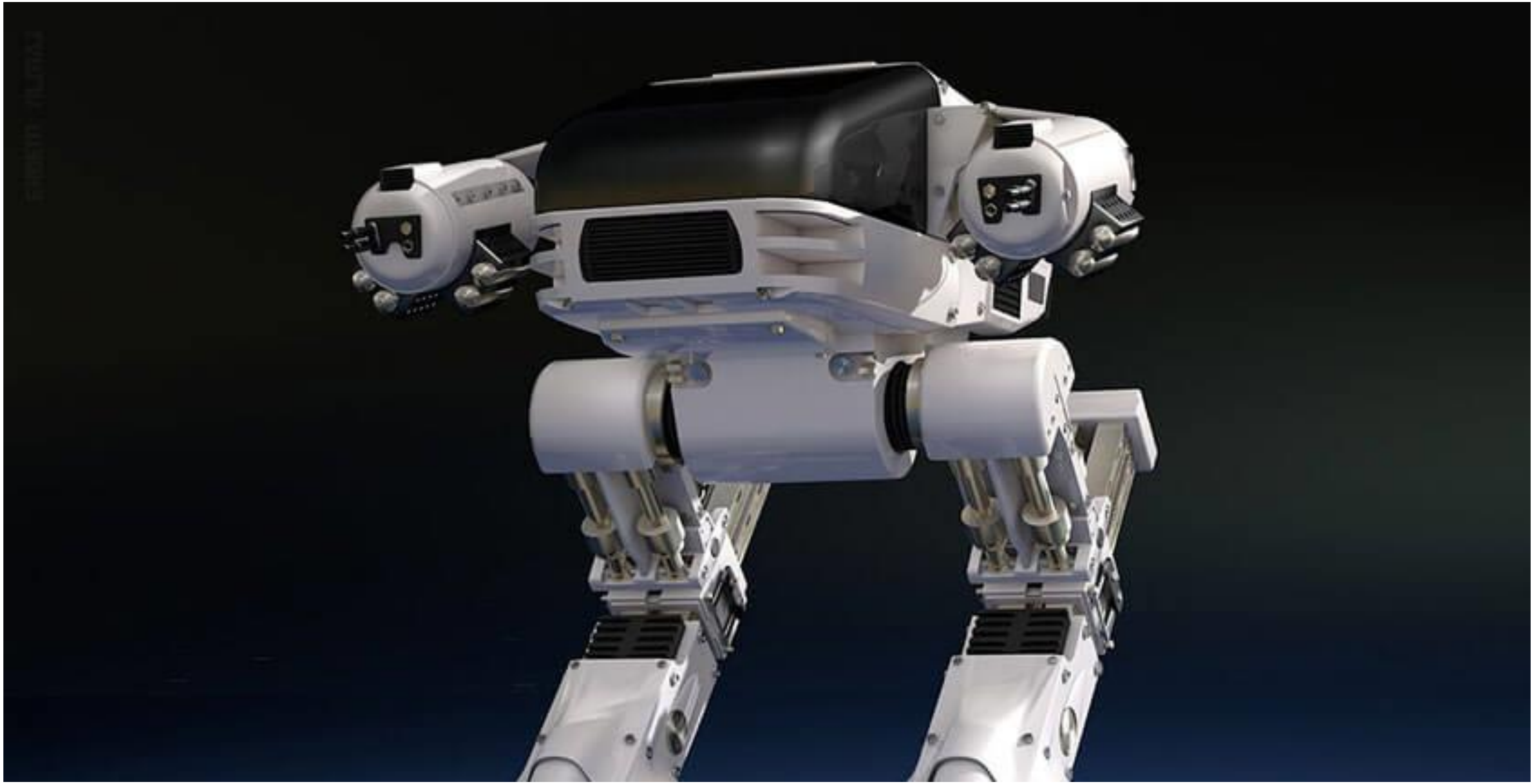
# Introduction to Robotics

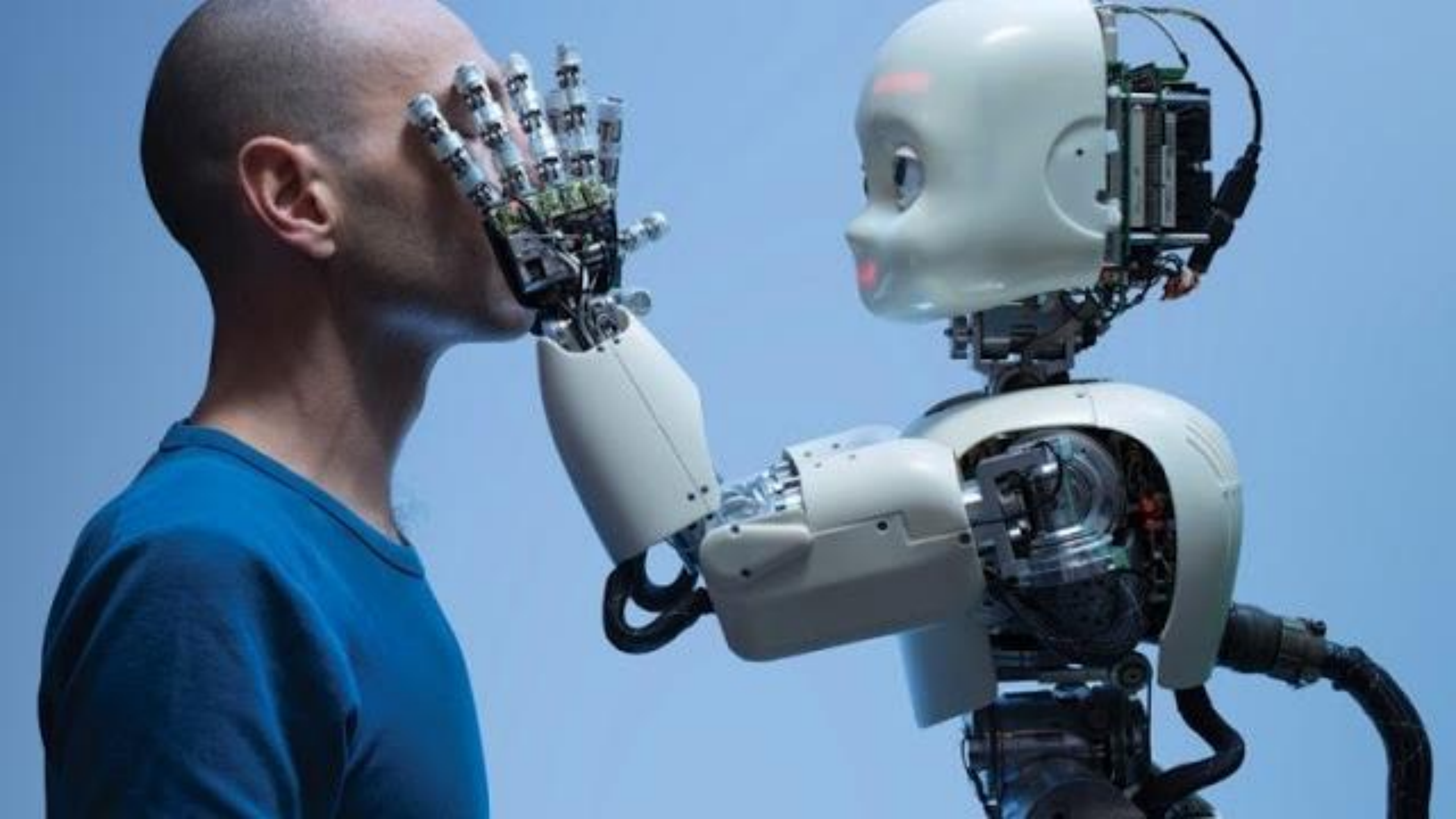


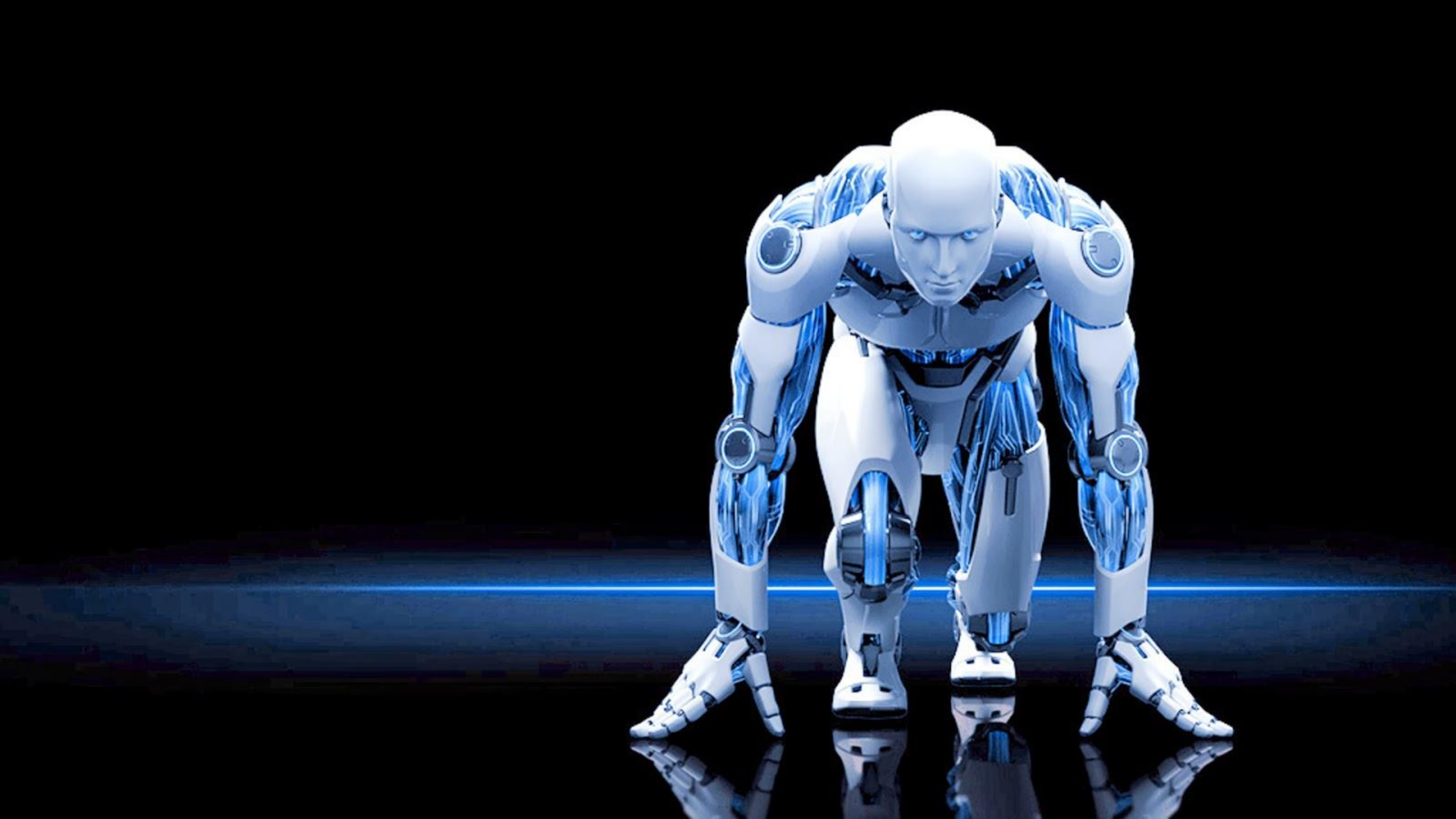
# **SYLLABUS(CAS 30/ EXAMINATION 70)**

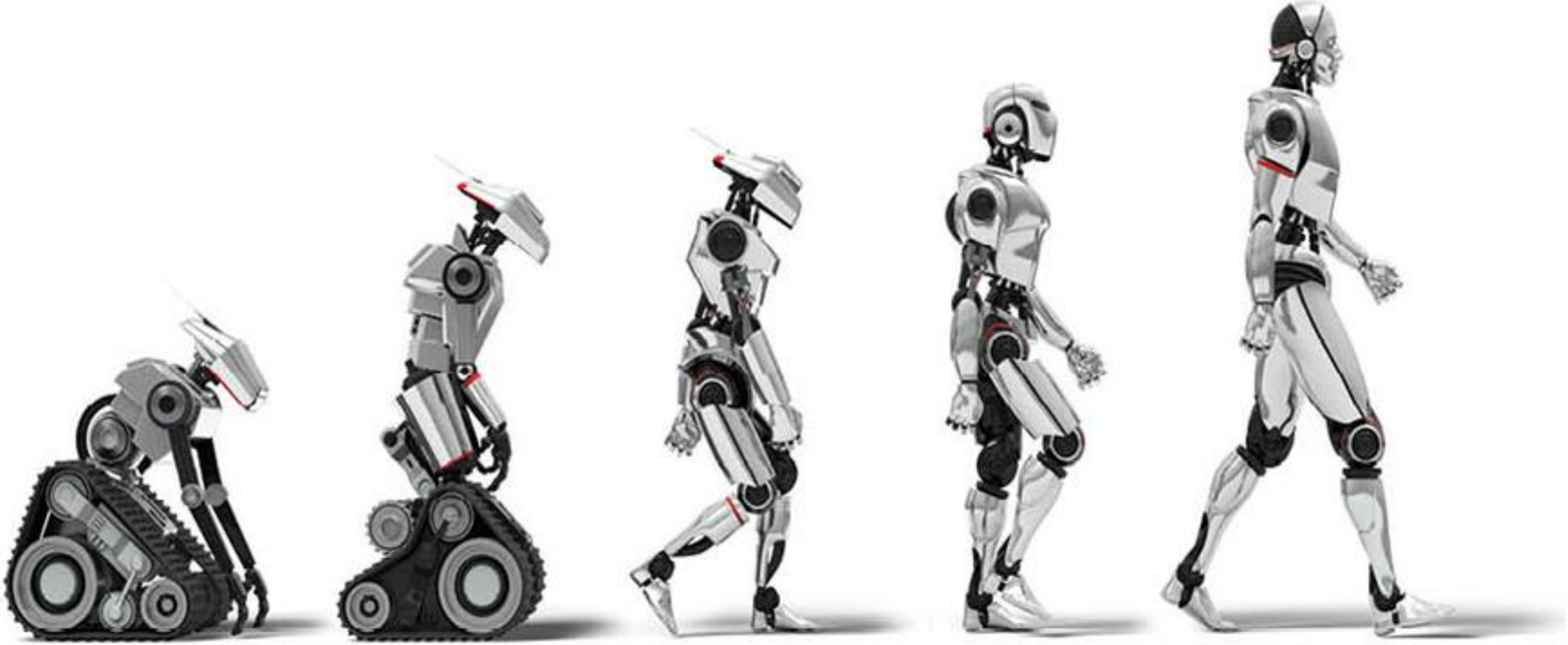
- Introduction
- Mathematics of Robot Manipulators
  - Forward Kinematics & Inverse Kinematics
  - Homogeneous transformation , DH Parameters
- Differential Motion
- Trajectory Planning
- Sensors for Robots
- Drive Systems for Robots
- Application orient robotic system design

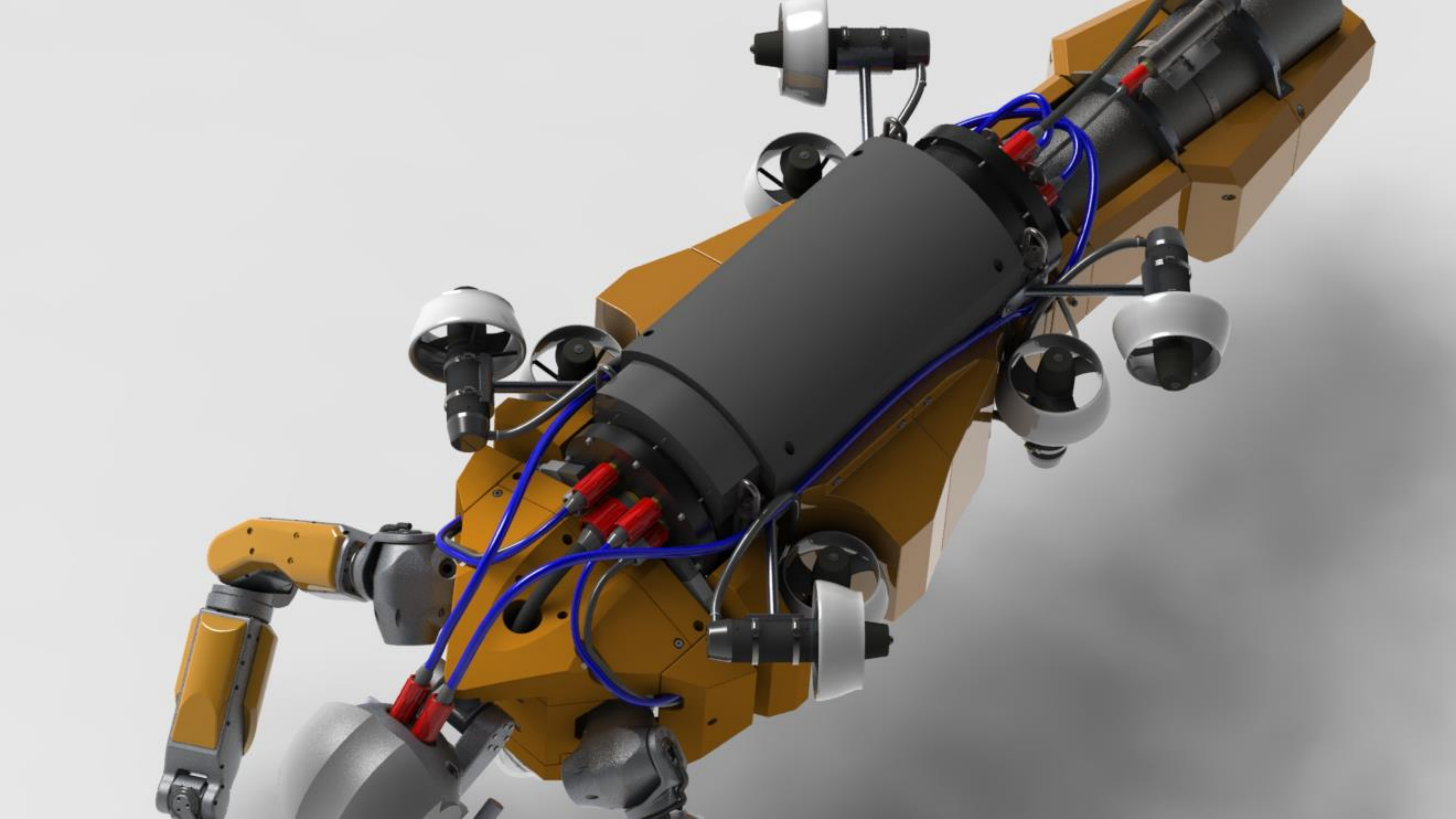










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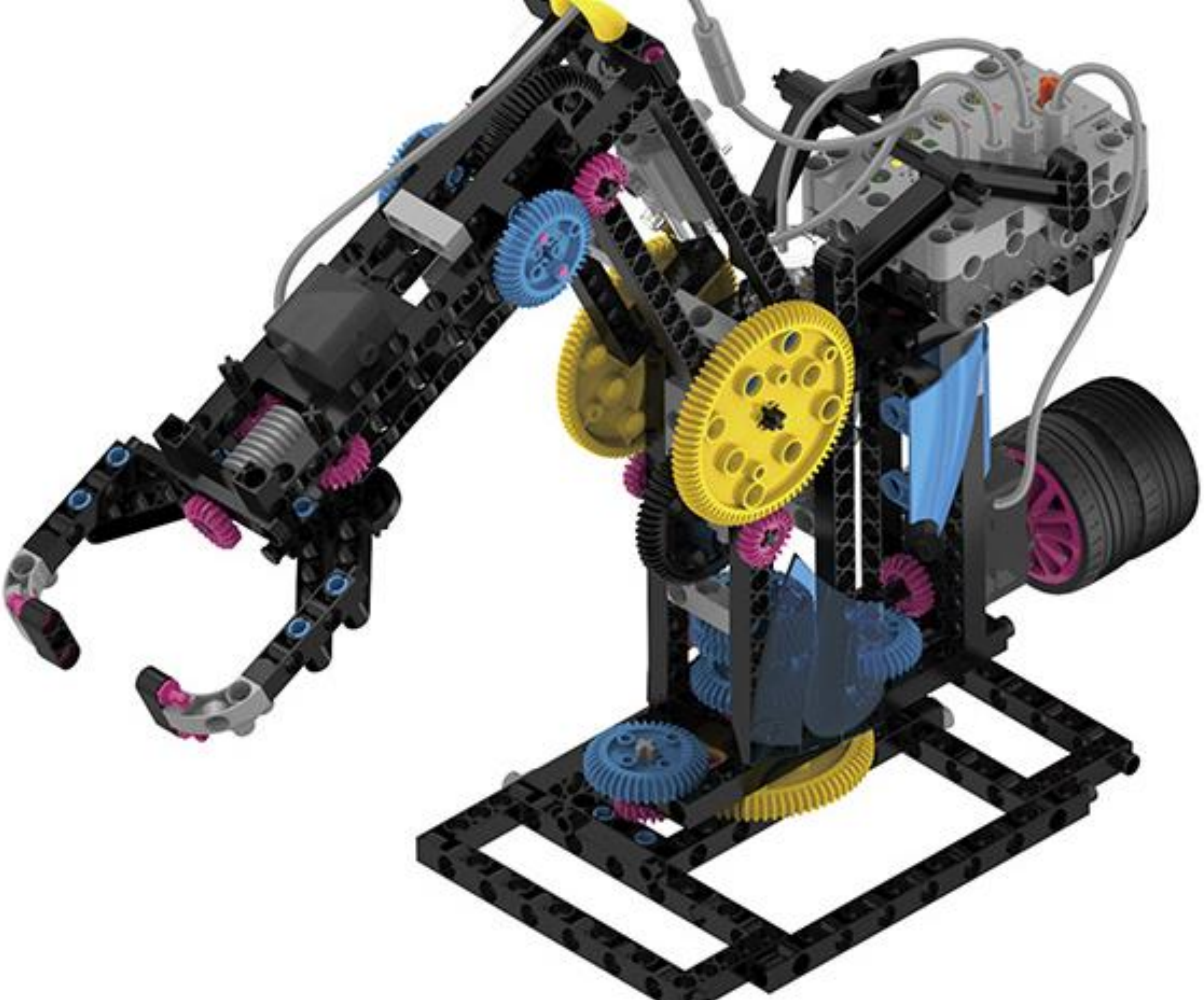
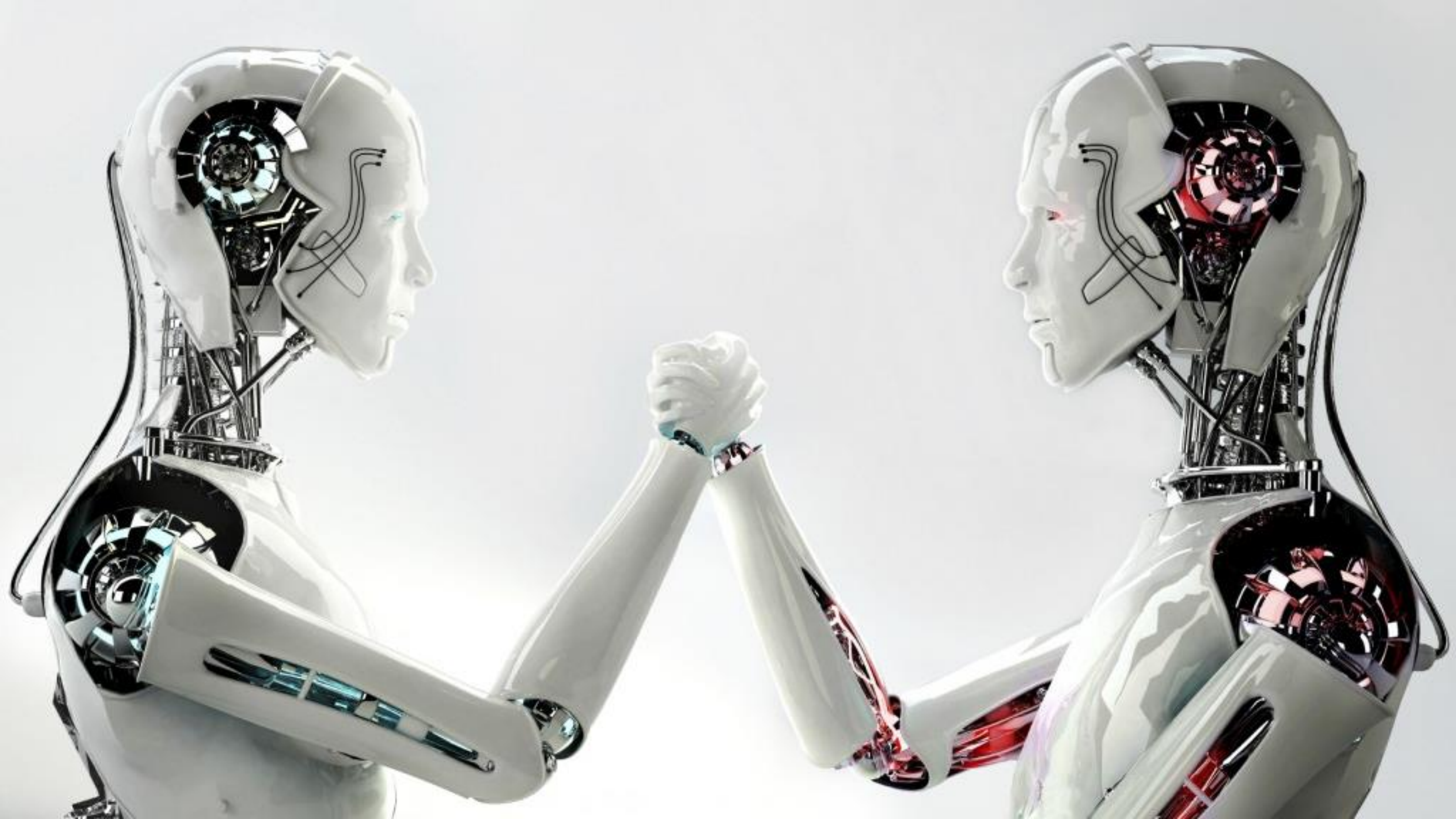




# Introduction

- Understanding the complexity of robots and their applications requires knowledge of
  - Electrical Engineering, Mechanical Engineering, Systems and Industrial Engineering, Computer Science, Economics, and Mathematics.
- In addition , robotics encompasses several areas **not covered in the syllabus** such as
  - locomotion, including Wheeled and Legged Robots, Flying and Swimming Robots, Grasping, Artificial Intelligence, Computer Architectures, Programming Languages, and Computer-aided Design.

Industrial engineering is an inter-disciplinary profession that is concerned with the optimization of complex processes, systems, or organizations by developing, improving and implementing integrated systems of people, money, knowledge, information, equipment, energy and materials.

# History of Robotics

- The term robot was first introduced into our vocabulary by the Czech playwright **Karel Capek** in his 1920 **Play** Rossum's Universal Robots,
- The word **Robota** being the Czech word for **Work**.
- Since then the term has been applied to a great variety of mechanical devices, such as teleoperators, underwater vehicles, autonomous land rovers, etc
- Virtually anything that operates with **some degree of autonomy, usually under computer control**, has at some point been called a **robot**.

# History of Robotics…

- You are studying ;the term robot will mean a computer controlled industrial **manipulator** of the type shown in Figure
- This type of robot is essentially a **mechanical arm operating under computer control.**


**ABB IRB6600 Robot**

# History of Robotics…

In robotics a manipulator is a device used to manipulate materials without direct contact.

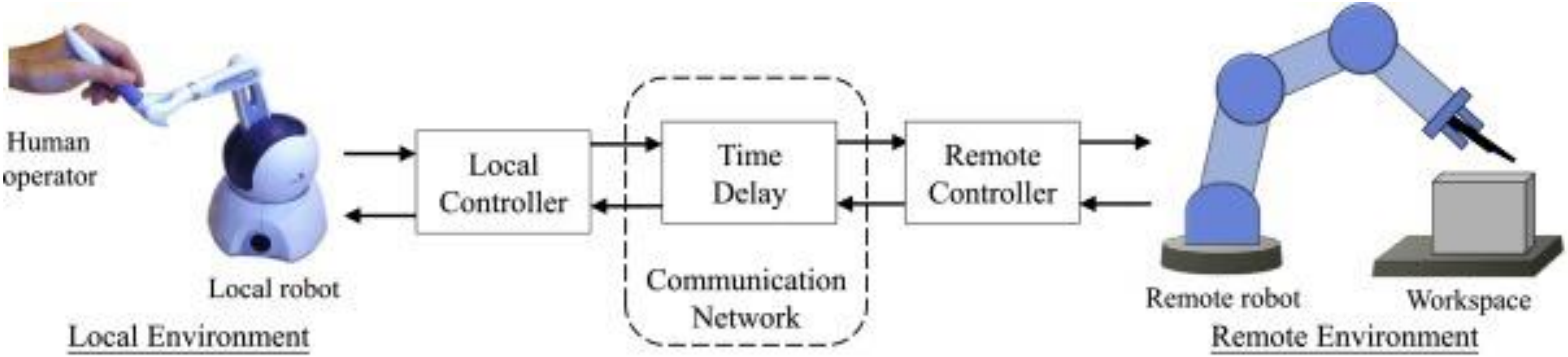
- An official definition of such a robot comes from the **Robot Institute of America** (RIA):
- *A robot is a Reprogrammable Multifunctional manipulator designed to move material, parts, tools, or specialized devices through variable programmed motions for the performance of a variety of tasks*.
- The key element in the above definition is the ***programmability*** of robots.

# History of Robotics…

- Among the advantages often cited in favor of the introduction of robots are
  - Decreased labor costs,
  - Increased Precision and Productivity,
  - Increased flexibility compared with specialized machines,
  - and more humane working conditions as dull, Repetitive, or Hazardous jobs are performed by robots.
- The robot, was born out of the marriage of two earlier technologies: that of teleoperators and numerically controlled milling machines.

# History of Robotics…

- Teleoperators, or master-slave devices, were developed during the second world war to handle radioactive materials.



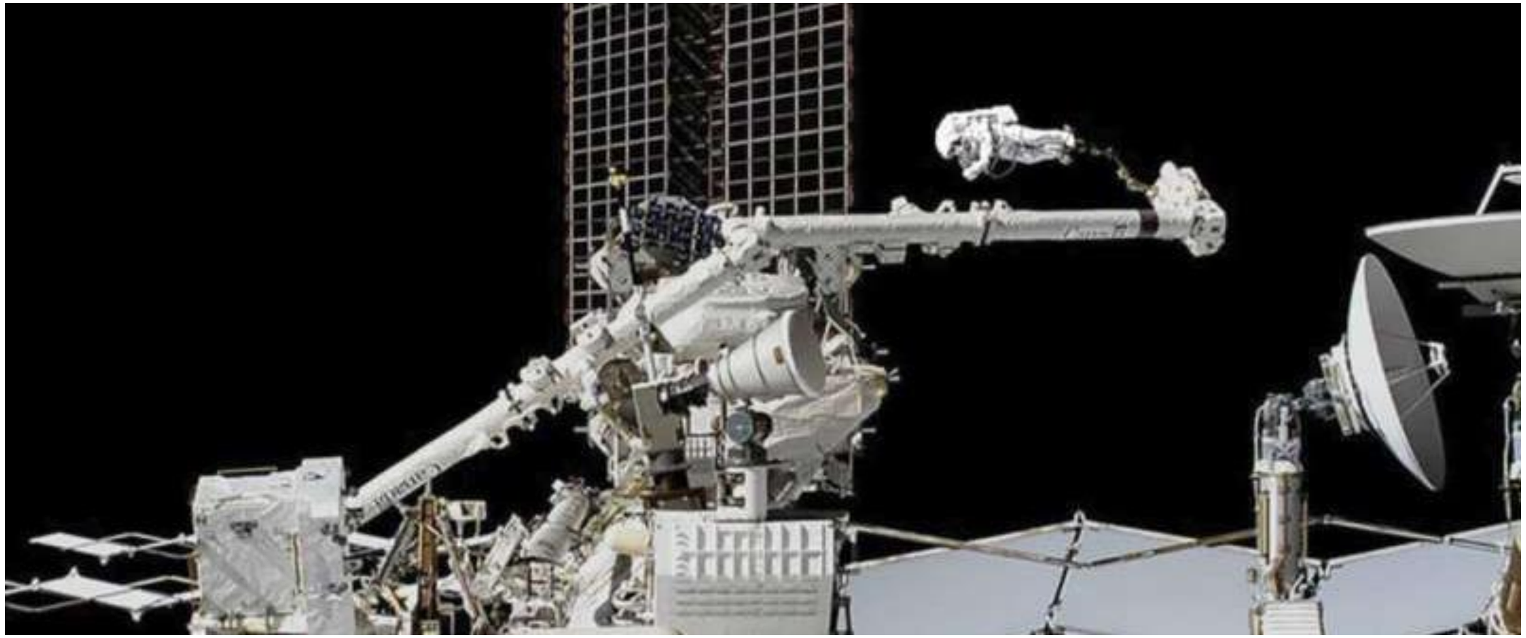
# History of Robotics…

- Computer Numerical Control (CNC) was developed because of the **high precision** required in the **machining** of certain items, such as components of high performance aircraft.
- The first robots essentially combined the **Mechanical Linkages of the Teleoperator** with the **Autonomy and programmability** of **CNC machines**.

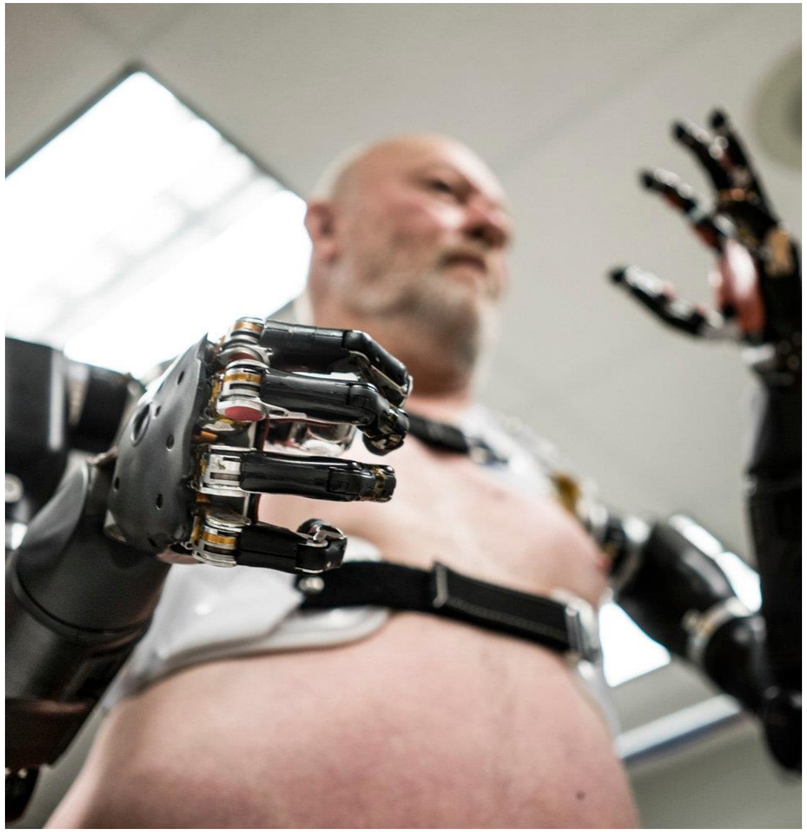
# History of Robotics…

- Important applications of robots are by no means limited to ***those industrial jobs where the robot is directly replacing a human worker.***
- There are many other applications of robotics in areas where the use of humans is **impractical or undesirable**.
- These are **undersea** and **planetary exploration**, **satellite retrieval** and repair, the **defusing of explosive devices**, and work in radioactive environments.
- **Prostheses,** such as **artificial limbs**, are themselves robotic devices requiring methods of analysis and design similar to those of industrial manipulators.
- Exoskeletons

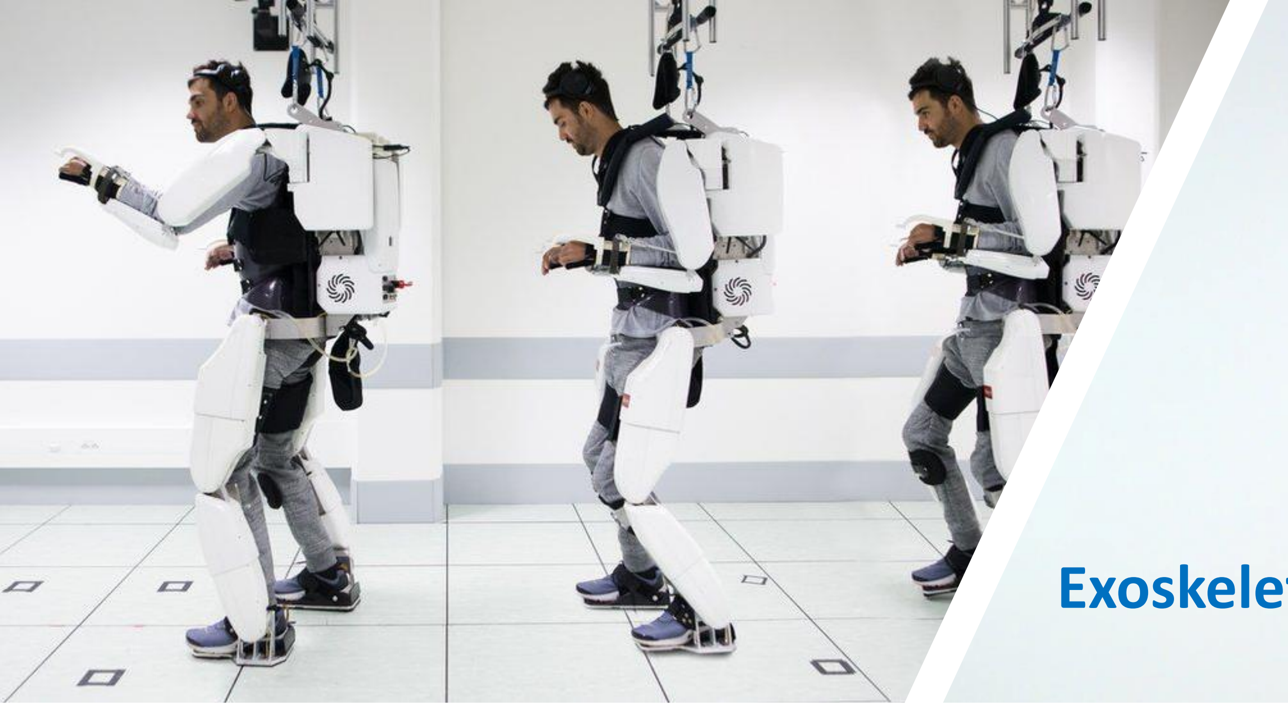
# Category Robots based on Applications



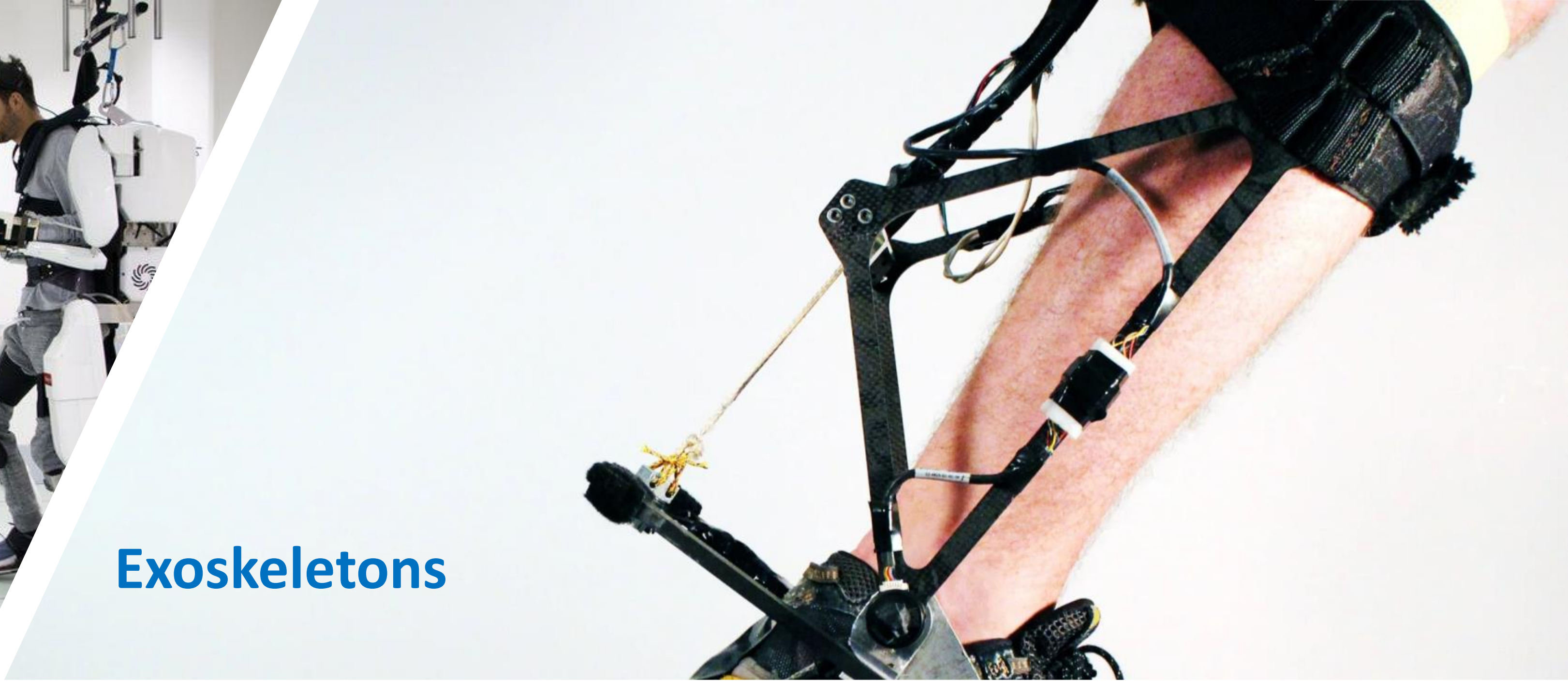
# Satellite Retrieval Robot

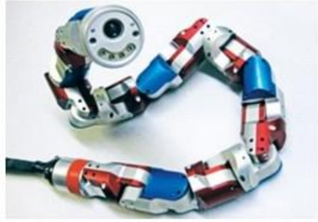


# prostheses



# Exoskeletons




# Societal Impact of Robots

Home service

Health care

Transportation & Logistics

Structure examination

Extraterrestrial exploration

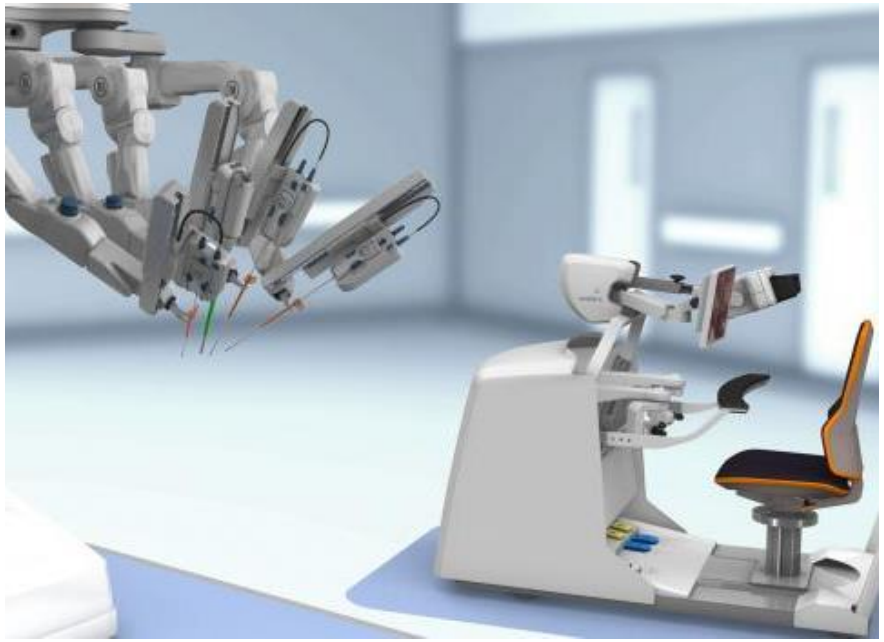
Search & rescue

Defense & public safety

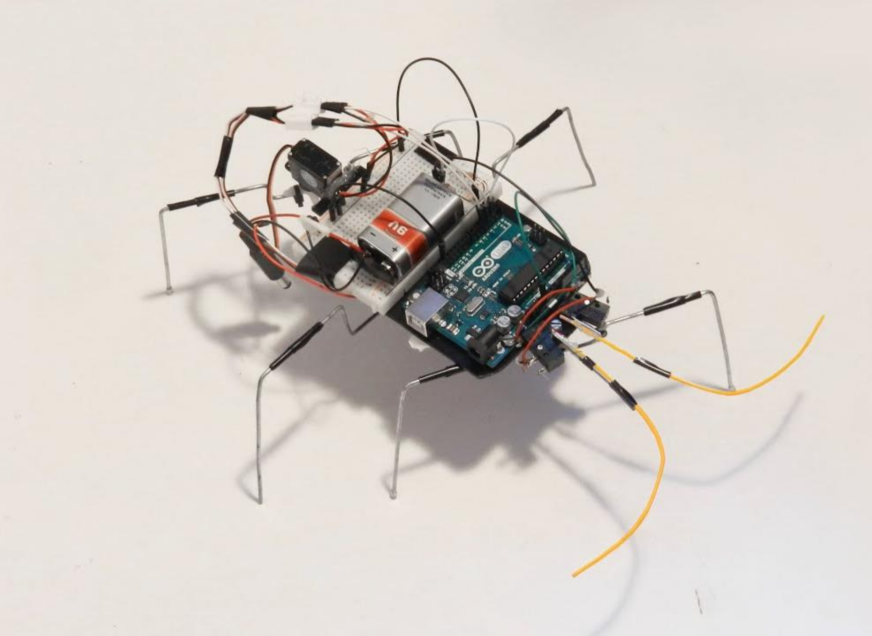
Environmental monitoring

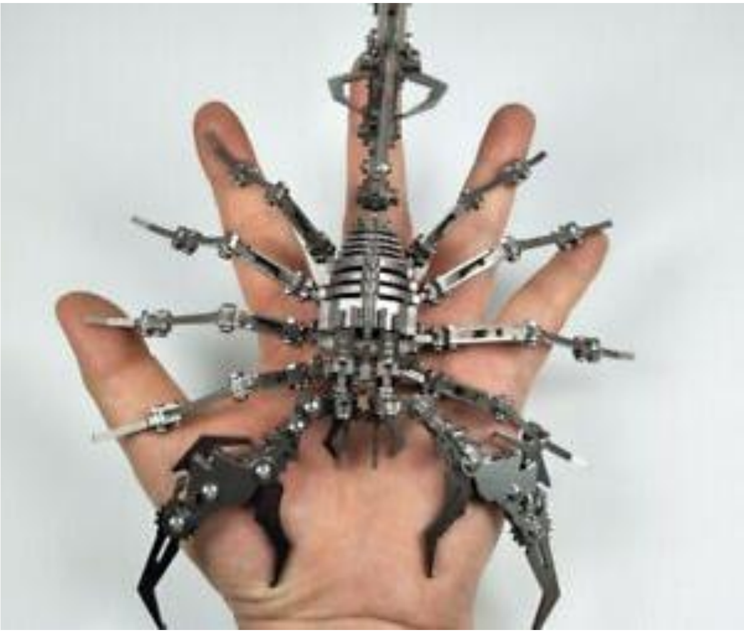
Forestry

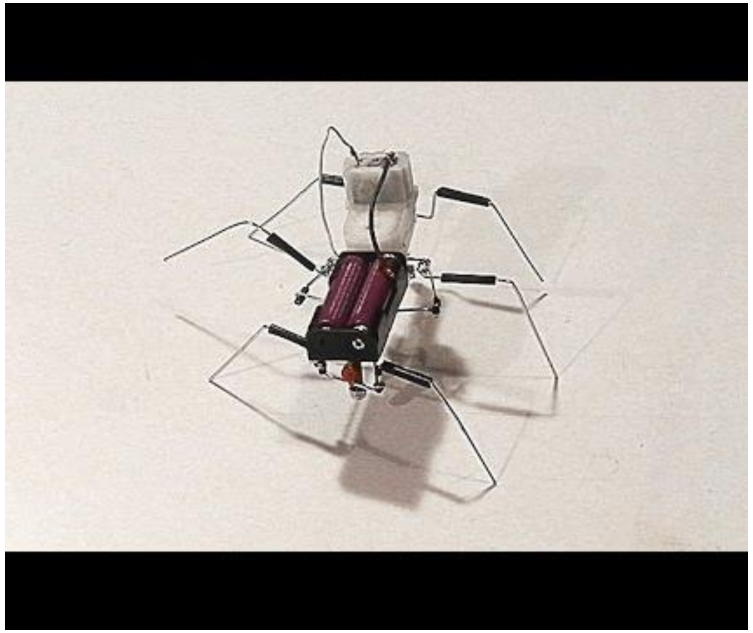
Agriculture

# Robotic Surgery





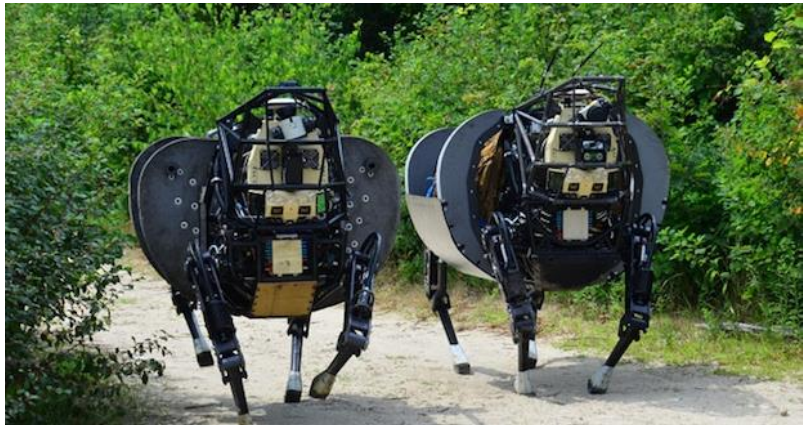


# Insect robots


# Agriculture Robots

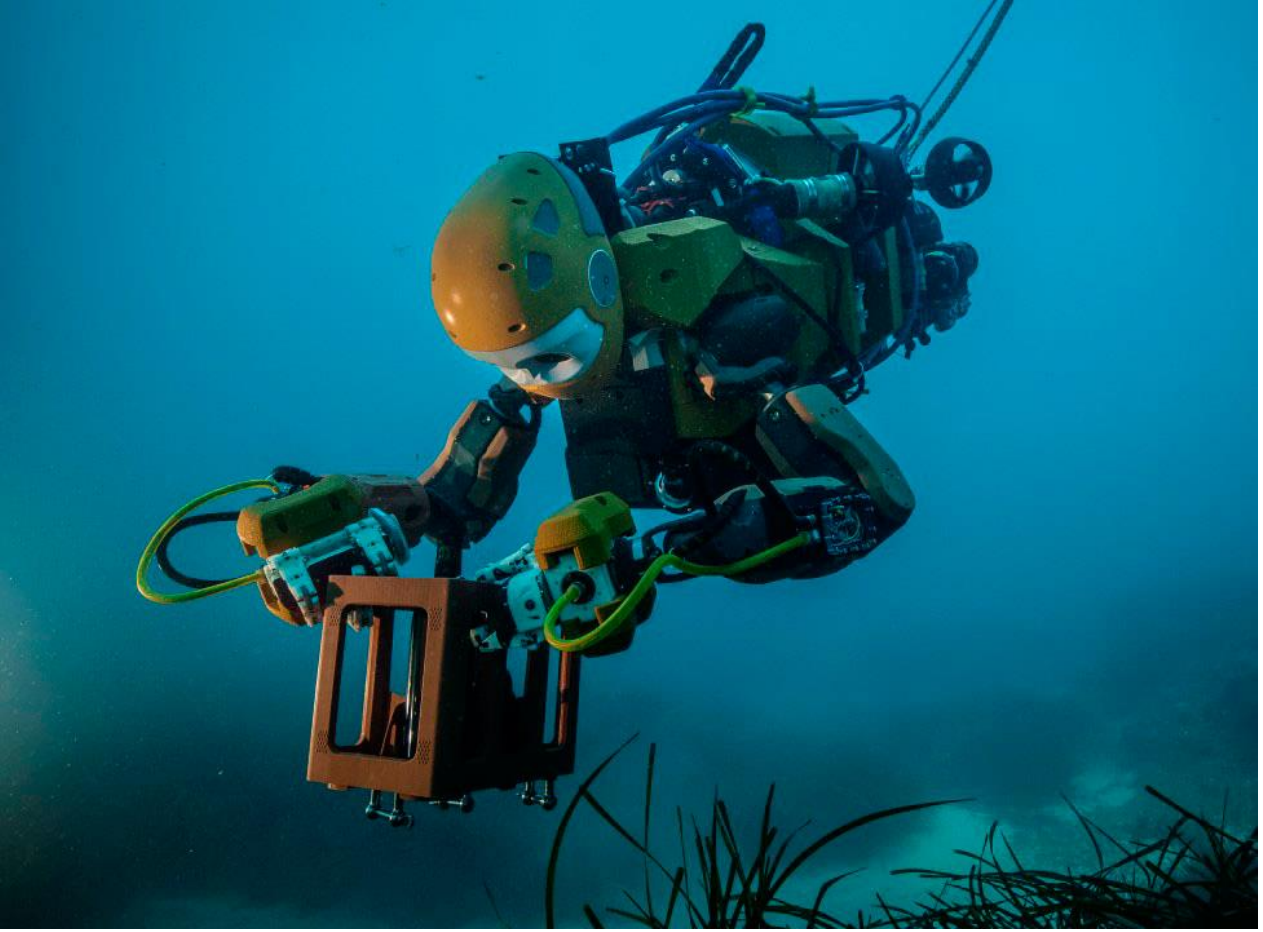
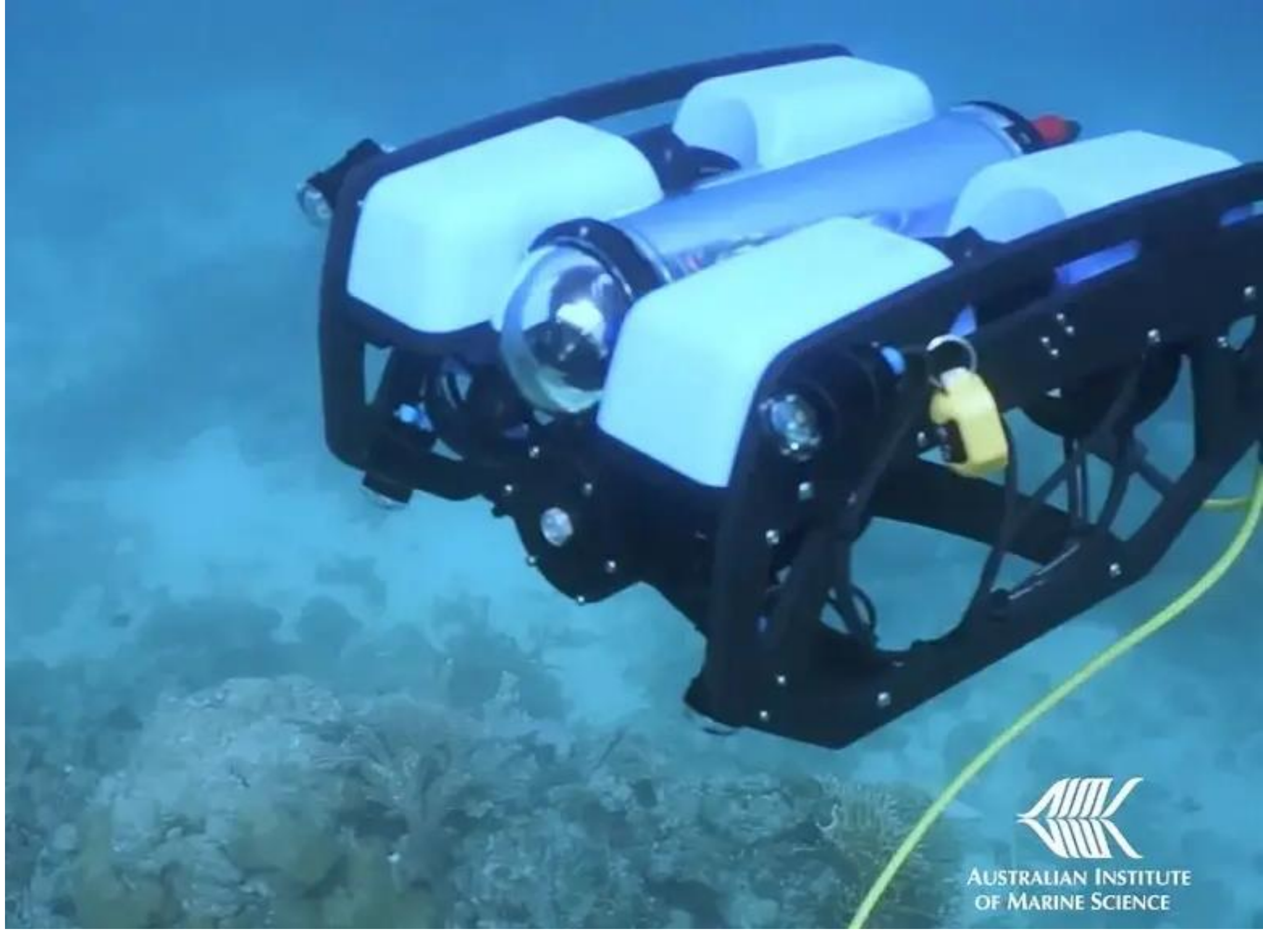



# Flying Robots

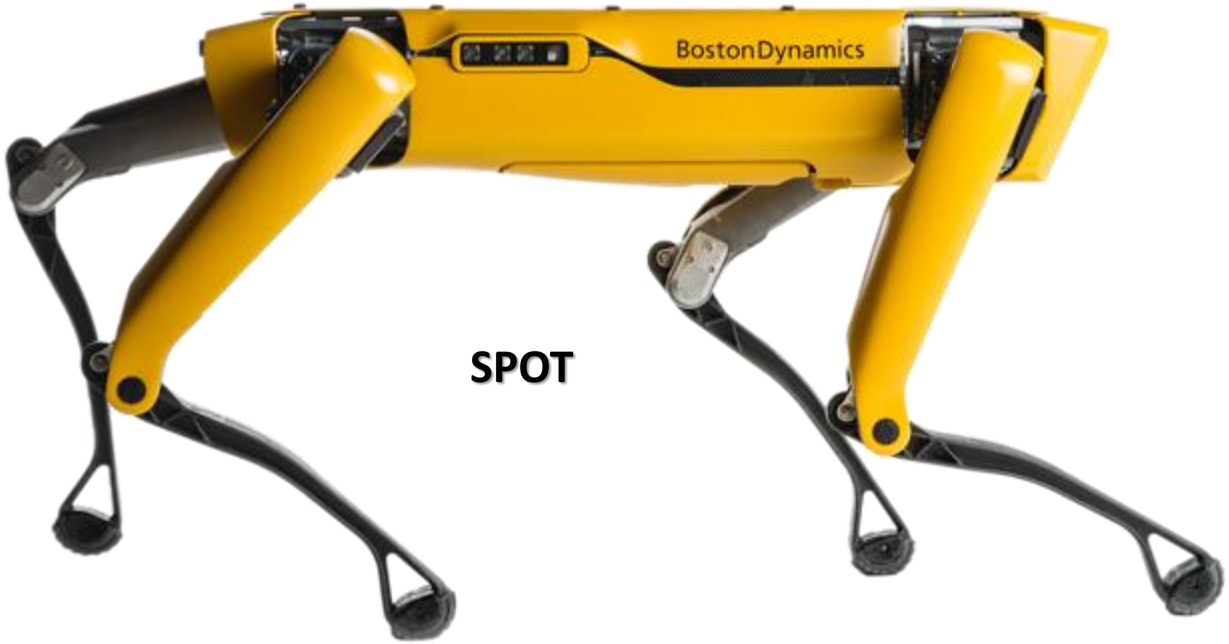




# Military Robots

# Humanoid Robots


# Underwater Robots

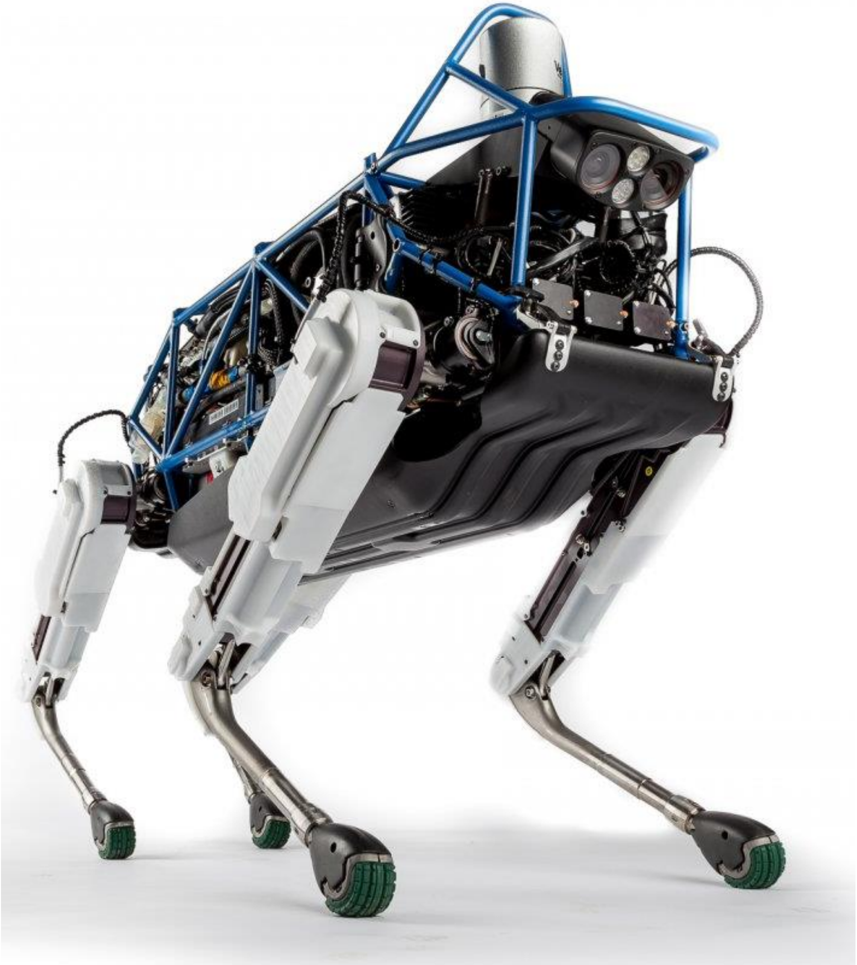



SPOT

*HANDLE*

*PICK*

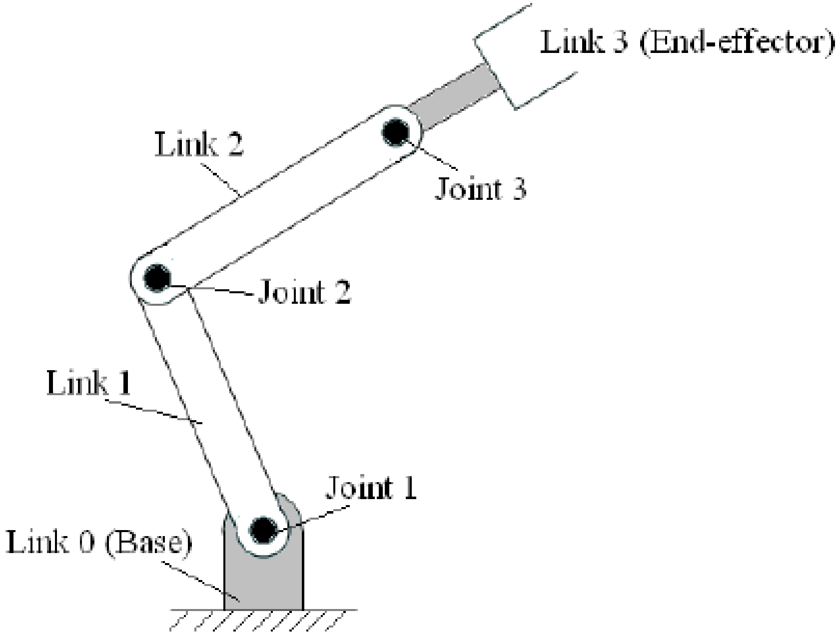
*SPOT CLASSIC (2015)*



***ATLAS***

# Components and Structure of Robots

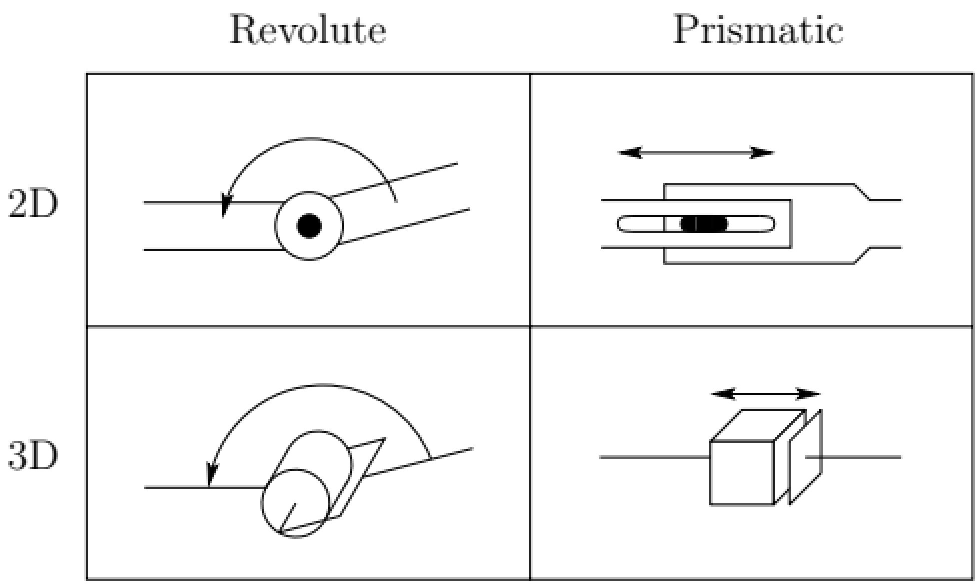
- Robot **Manipulators** are composed of **links** connected by **joints** into a **kinematic chain**.
- Joints are typically **rotary** (revolute) or **linear** (prismatic).
- A revolute joint is like a hinge and allows relative rotation between two links.
- A prismatic joint allows a linear relative motion between two links.

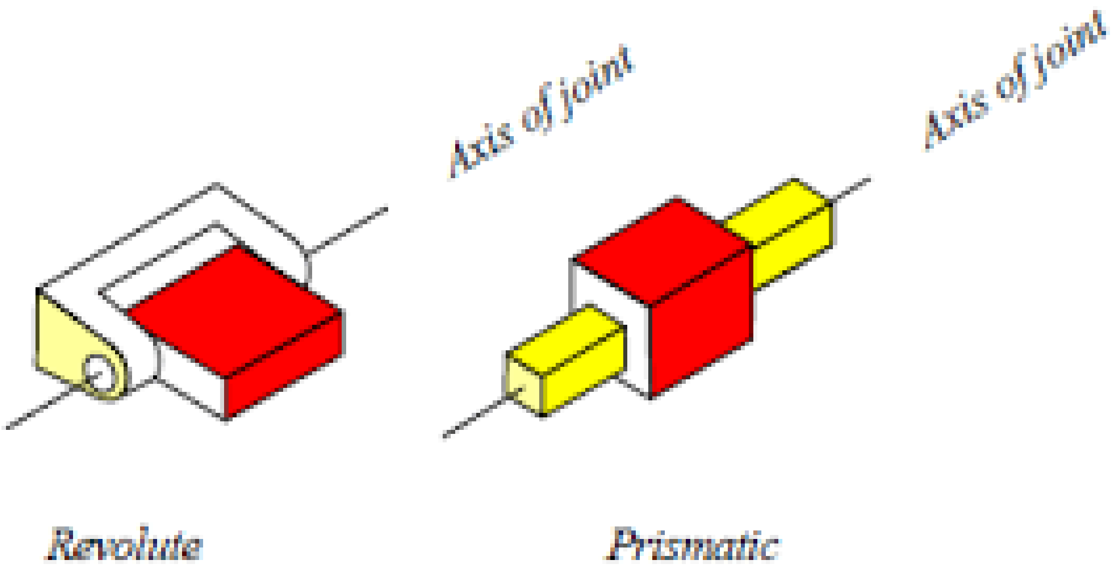



Kinematics is a branch of classical mechanics that describes the **motion of points, bodies , and systems of bodies** without considering the forces that caused the motion.

# Components and Structure of Robots…

- We use the convention **(R)** for representing revolute joints and **(P)** for prismatic joints and draw them as shown in Figure

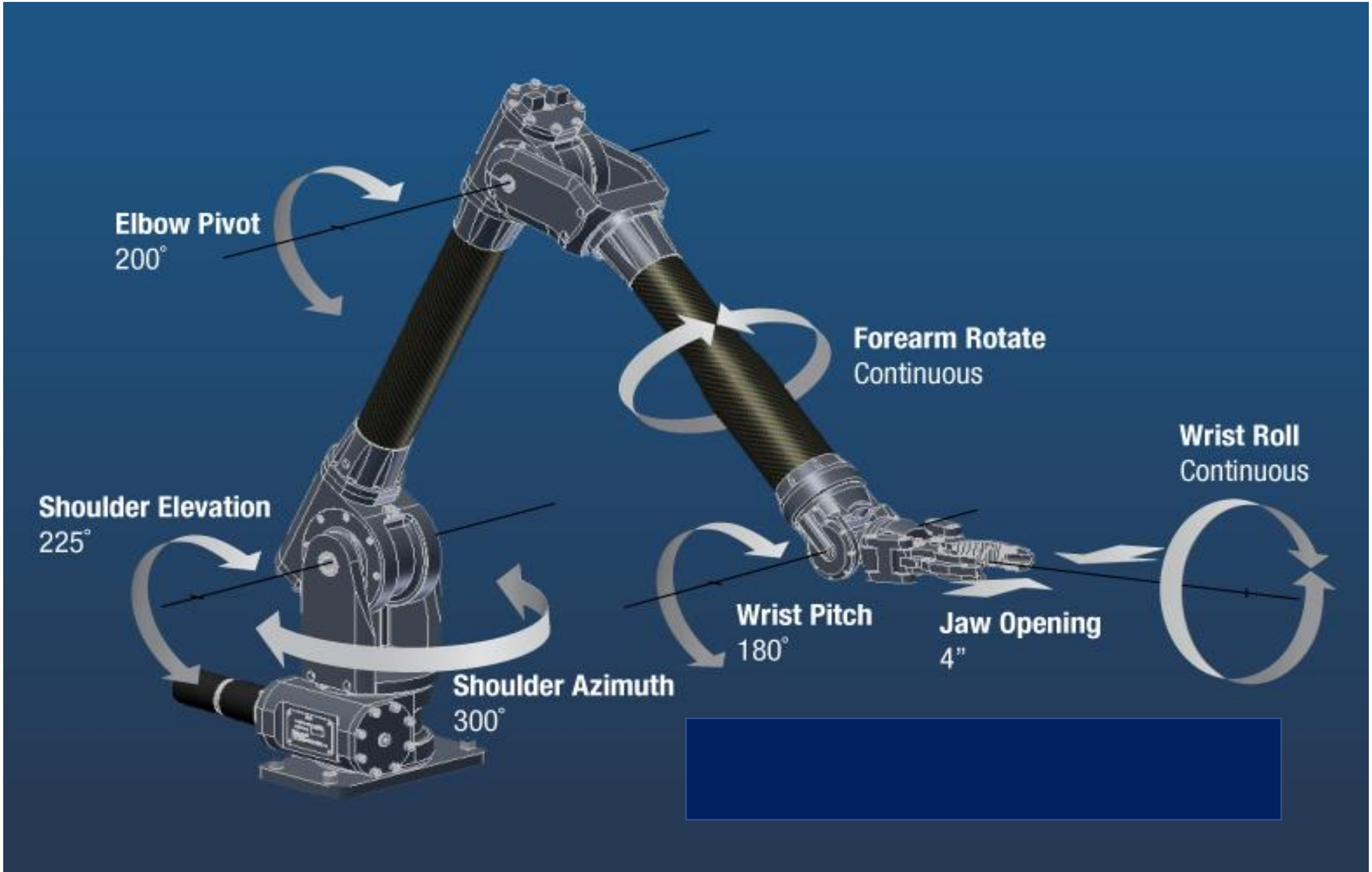


# Degrees of Freedom and Workspace

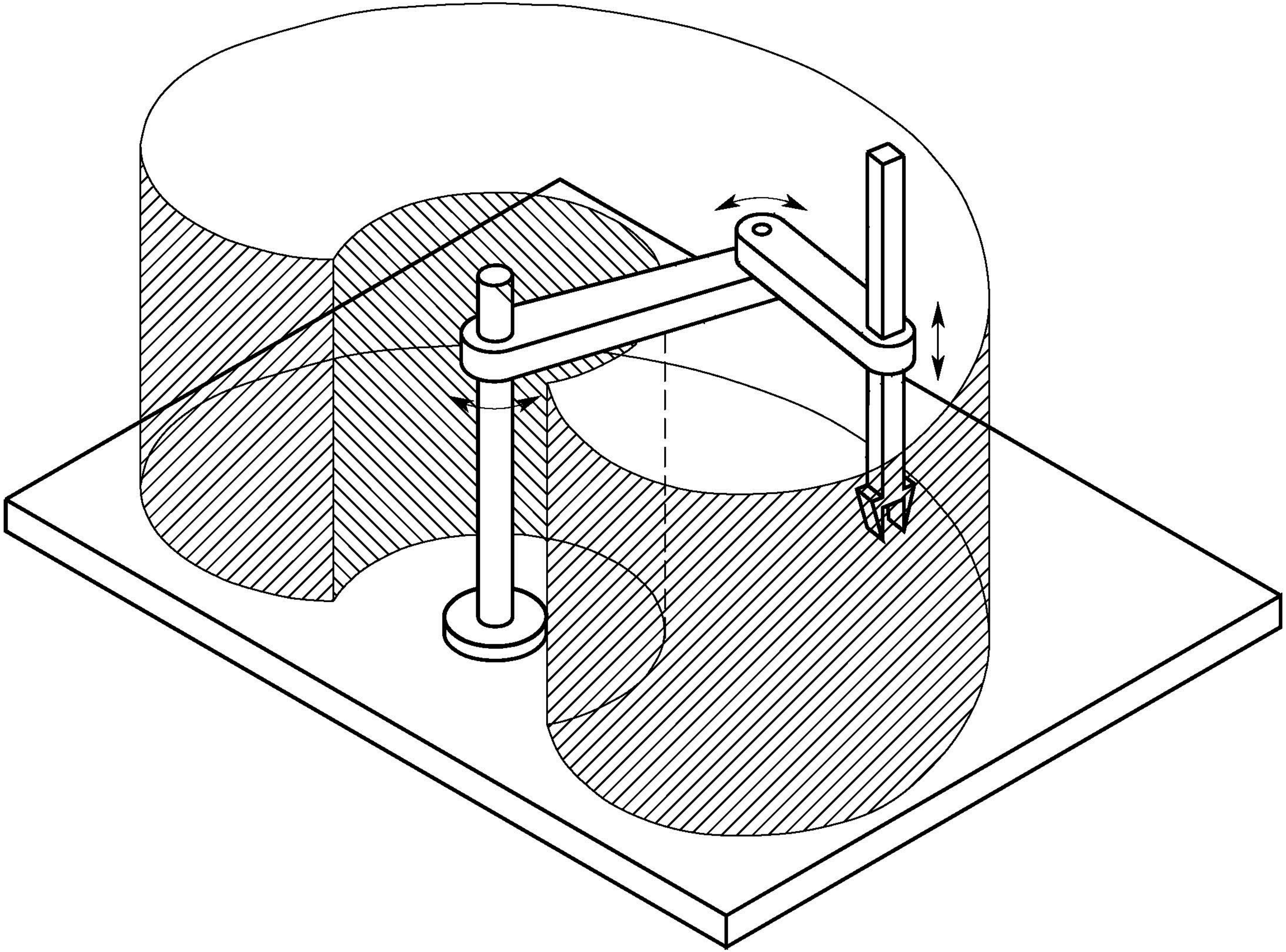
- The number of **Joints** determines the degrees-of-freedom (DOF) of the manipulator.
- Typically, a manipulator should possess at least **six** independent **DOF**: three for **positioning** and three for **orientation**.
- With fewer than six DOF the **arm cannot reach every point** in its work environment with arbitrary orientation.
- Certain applications such as reaching around or behind obstacles require more than six DOF.
- A manipulator having more than six links is referred to as a kinematically redundant manipulator.

***The degree of freedom (DOF) of a mechanical system is the number of independent parameters that define its configuration.***


Elbow Pivot
200°
Forearm Rotate
Continuous
Wrist Roll
Continuous
Shoulder Elevation
225°
Wrist Pitch
180°
Jaw Opening
4"
Shoulder Azimuth
300°

# Degrees of Freedom and Workspace…

- The **workspace** of a manipulator is the total volume swept out by the end-effector as the manipulator executes all possible motions.
- The workspace is constrained by the geometry of the manipulator as well as mechanical constraints on the joints.
- The workspace is often broken down into a **reachable** workspace and a **dextrous** workspace.
- The reachable workspace is the entire set of points reachable by the manipulator, whereas the **dextrous** workspace consists of those points that the manipulator can reach with an **arbitrary** orientation of the **end effector**.



# Classification of Robots

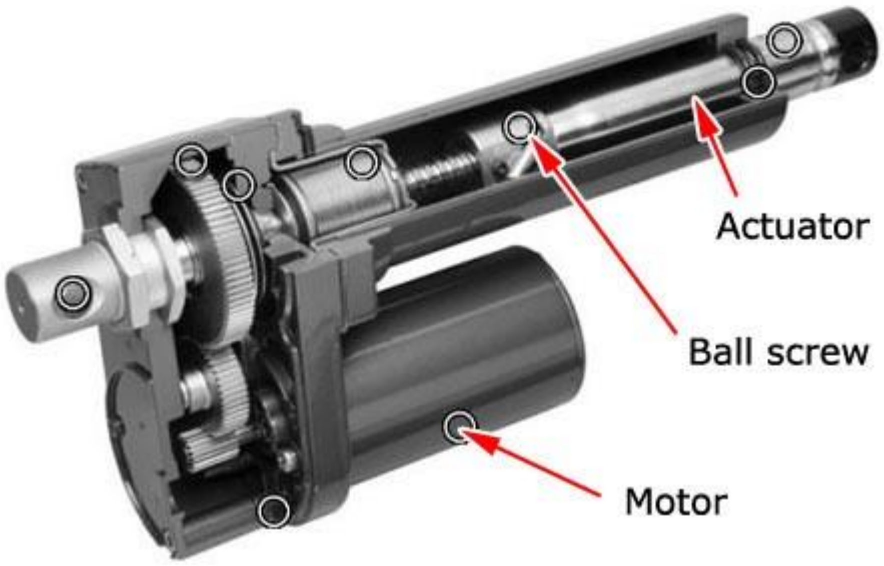
- Robot manipulators can be classified by several criteria, such as their
- **power source**,
- or way in which the **joints are actuated**,
- their **geometry**,
- or kinematic **structure**,
- their intended **application** area,
- or their method of **control**.

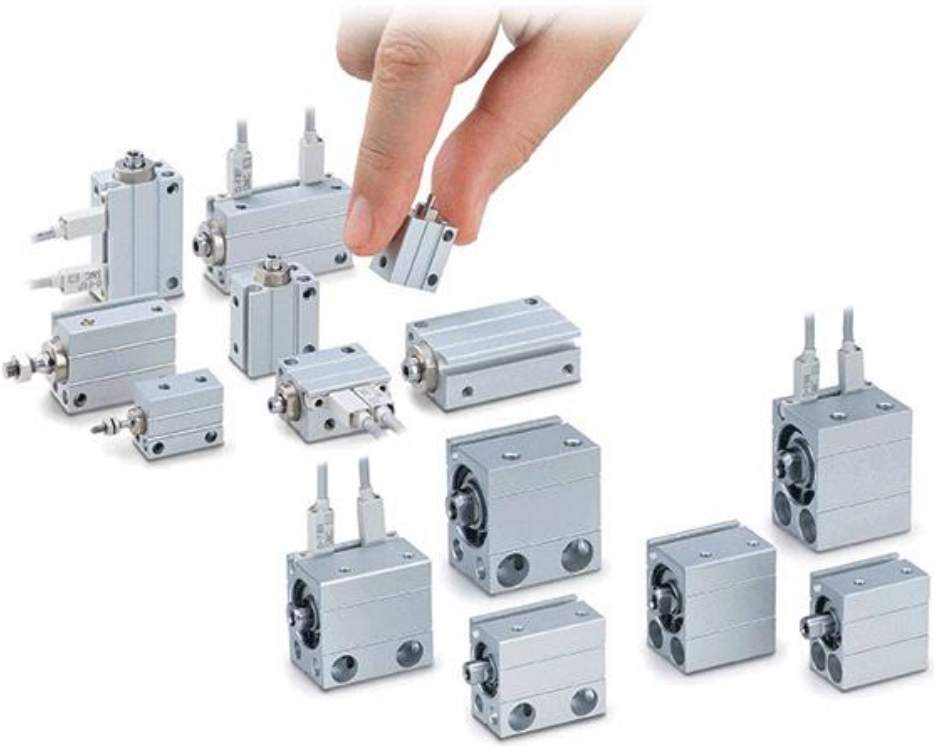
- Power Source
  - Typically, robots are either **electrically**, **hydraulically**, or **pneumatically** powered.
  - Hydraulic actuators are unbeatable in their speed of response and torque producing capability.
  - Hydraulic robots are used primarily for **lifting heavy loads**.
  - The drawbacks of hydraulic robots are that they tend to leak hydraulic fluid,
  - require much more peripheral equipment, such as pumps, which also requires more maintenance, and they are noisy.

HYDRAULIC ACTUATORS

# Classification of Robots…

- Robots driven by DC- or AC-servo motors
- increasingly popular since they are cheaper, cleaner and quieter.
- Pneumatic robots are inexpensive and simple
- but cannot be controlled precisely

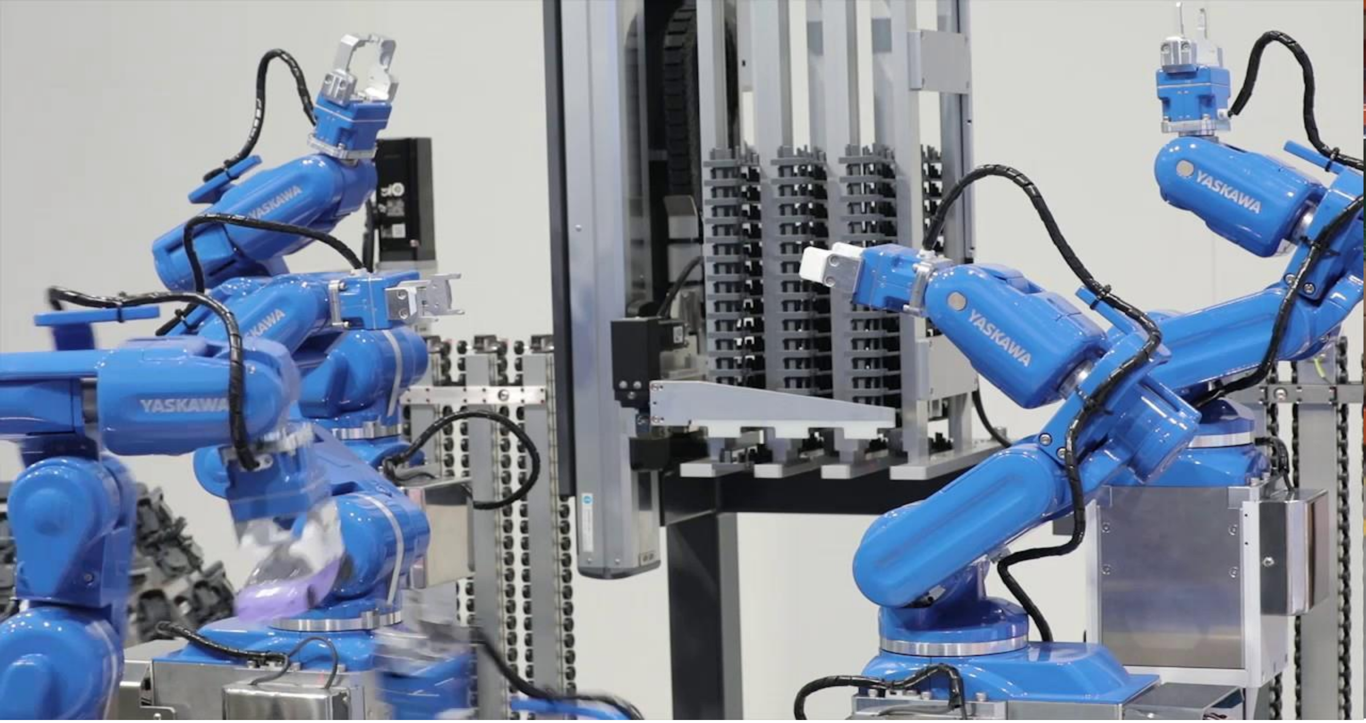


# Application Area

- The largest projected area of future application of robots is in **assembly**.
- Therefore, robots are often classified by application into **assembly** and **non-assembly** robots.
- Assembly robots tend to be small, electrically driven
- The main **non assembly** application areas :
  - welding,
  - spray painting,
  - material handling, and
  - machine loading and unloading.

# Assembly Robots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MITSUBISHI


ABB


# Method of Control

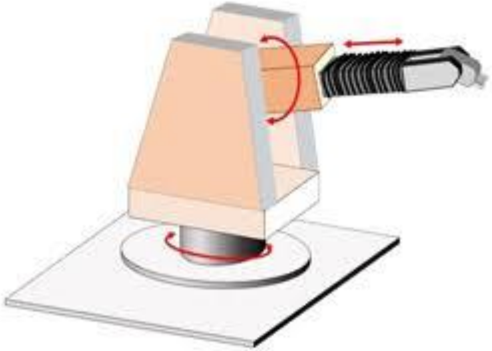
- Robots are classified by control method into servo and non-servo robots.
- Servo controlled robots are further classified according to the method that the controller uses to guide the end-effector
- The simplest type of robot in this class is the point-to-point robot.
- Such robots are usually taught a series of points
- The points are then stored and played back.
- Point-to -point robots are severely limited in their range of applications.

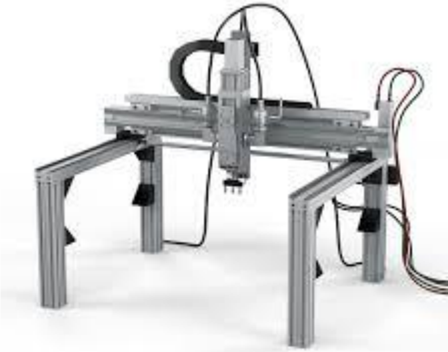
# Method of Control..

- In continuous path robots, the entire path of the end-effector can be controlled.
- For example, the robot end-effector can be taught to follow a straight line between two points or even to follow a contour such as a welding seam.
- In addition, the velocity and/or acceleration of the end-effector can often be controlled.

# Geometry

- Most industrial manipulators have **six** or fewer degrees-of-freedom.
- These manipulators are usually classified kinematically on the basis of the first three joints of the arm, with the wrist being described separately.
- The majority of these manipulators fall into one of five geometric types: **articulate** (**RRR**), **spherical** (**RRP**), **SCARA** (**RRP**), **cylindrical** (**RPP**), or **cartesian** (**PPP**).
- **Each of these five configurations are serial link robots**



# Common Kinematic Arrangements

# Articulated Configuration (RRR)

- The articulated manipulator is also called a revolute, or anthropomorphic manipulator.

ABB IRB1400

- A common revolute joint design is the parallelogram linkage
- .


Motoman SK16

# elbow manipulator

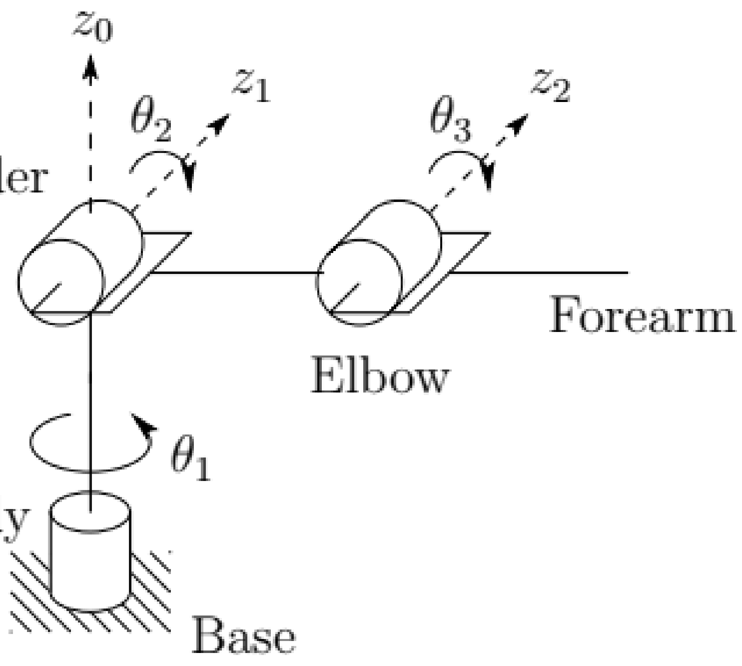



Figure 1.5: Structure of the elbow manipulator.

# Workspace

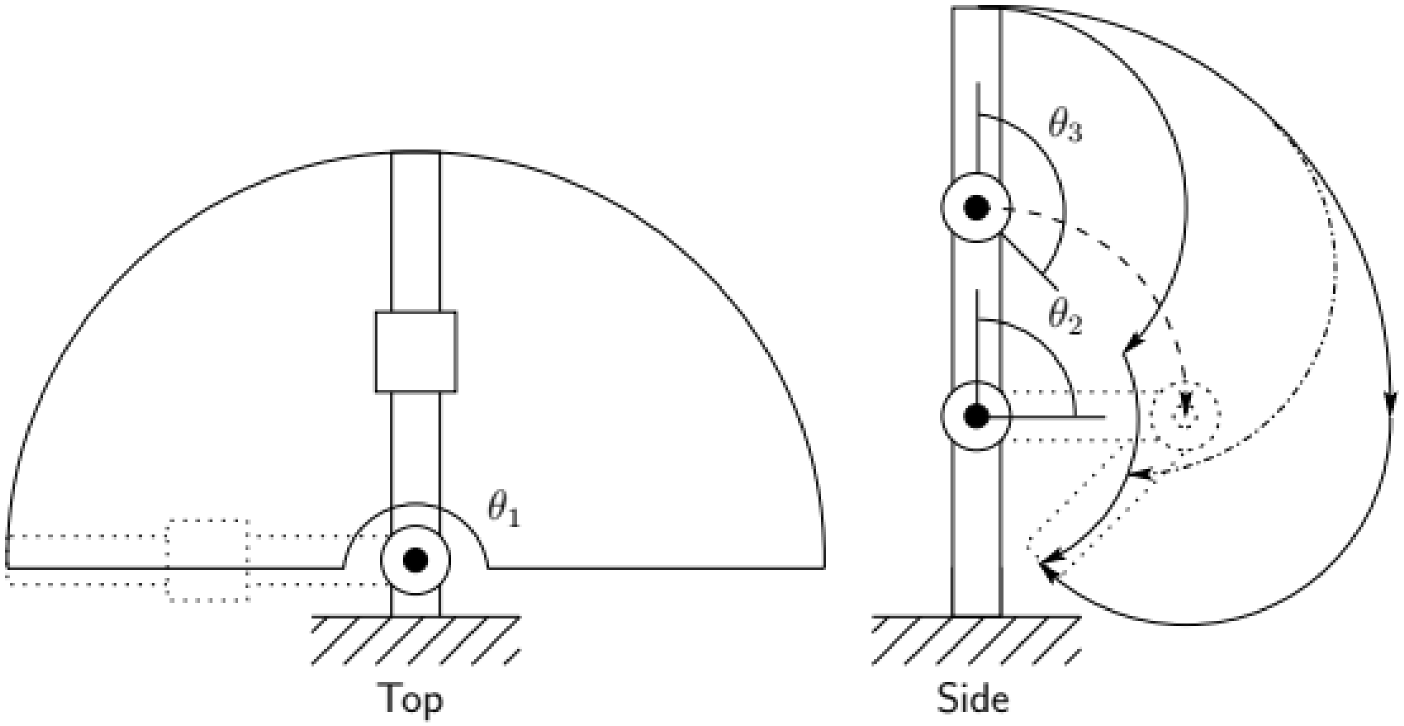



Figure 1.6: Workspace of the elbow manipulator.

# Spherical Configuration (RRP)

- The term spherical configuration :
- at the spherical coordinates defining the position of the end-effector

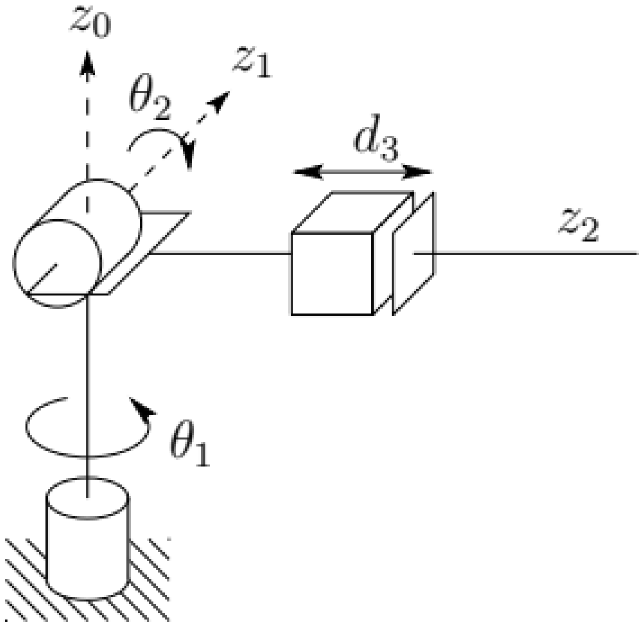



Figure 1.7: The spherical manipulator configuration.

# SCARA Configuration (RRP)

- SCARA (for Selective Compliant Articulated Robot for Assembly)
- is tailored for assembly operations.

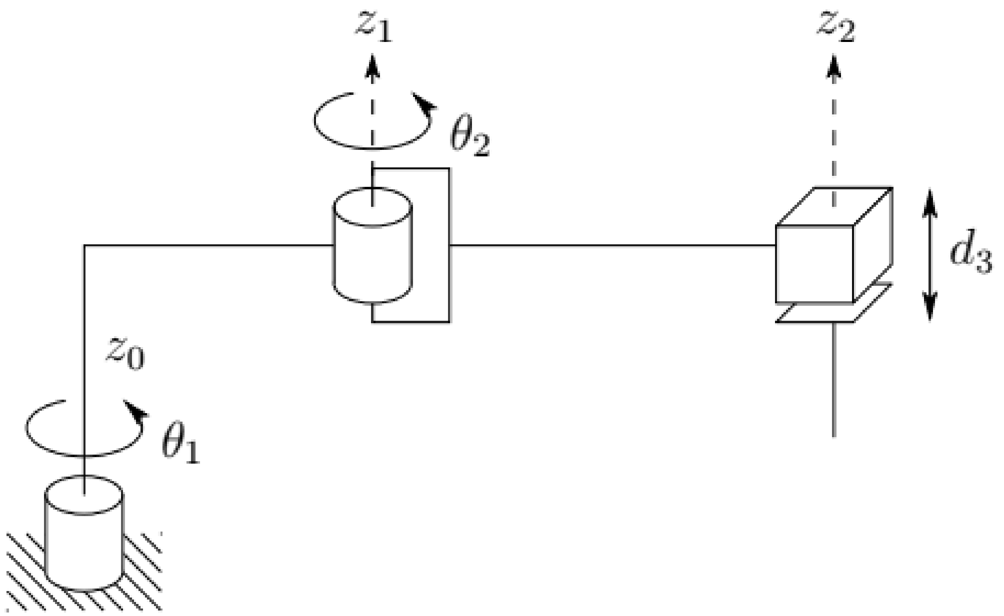



Figure 1.10: The SCARA (Selective Compliant Articulated Robot for Assembly).


SCARA

# Cylindrical Configuration (RPP)

- The first joint is revolute and produces a rotation about the base,
- while the second and third joints are prismatic.
- As the name suggests, the joint variables are the cylindrical coordinates of the end-effector with respect to the base.

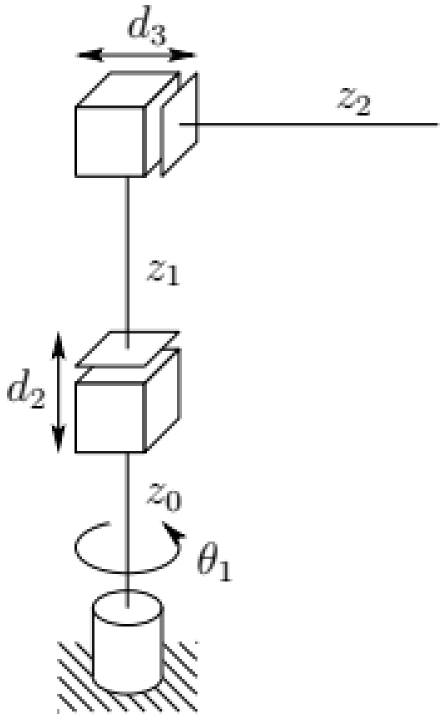



Figure 1.13: The cylindrical manipulator configuration.

Figure 1.14: The Seiko RT3300 Robot.

# Spatial descriptions and transformations

# INTRODUCTION

- Robotic manipulation, by definition,
  - implies that parts and tools will be moved around in space by some sort of mechanism.
- This leads to a need for **representing positions and orientations of parts**, **of tools, and of the mechanism itself**
- To define and manipulate mathematical quantities that represent position and orientation,
- We define coordinate systems…..

# Description of a position

$$^{A}P = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}.$$

- we can locate **any point** in the universe with a **3 x 1** position vector.
- **$^{A}P$ ,** means that the components of **$^{A}P$** have numerical values that indicate distances along the axes of {A}
- *Each of these distances along an axis can be thought of as the result of projecting the vector onto the corresponding axis.*

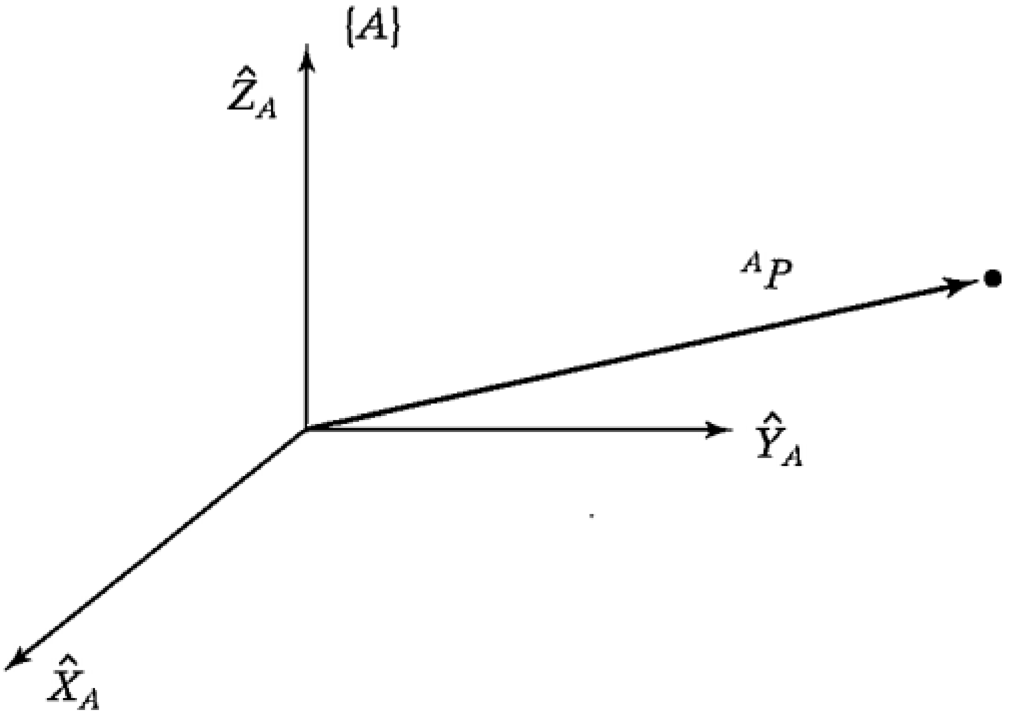



FIGURE 2.1: Vector relative to frame (example).

# Description of an orientation

- it necessary not only to represent a point in space and also orientation of a body in space.
- complete location of the hand is still not specified until its orientation is also given.
- *In order to describe the orientation of a body,*
- *we will attach a coordinate system to the body*
- *and then give a description of this coordinate system relative to the reference system*

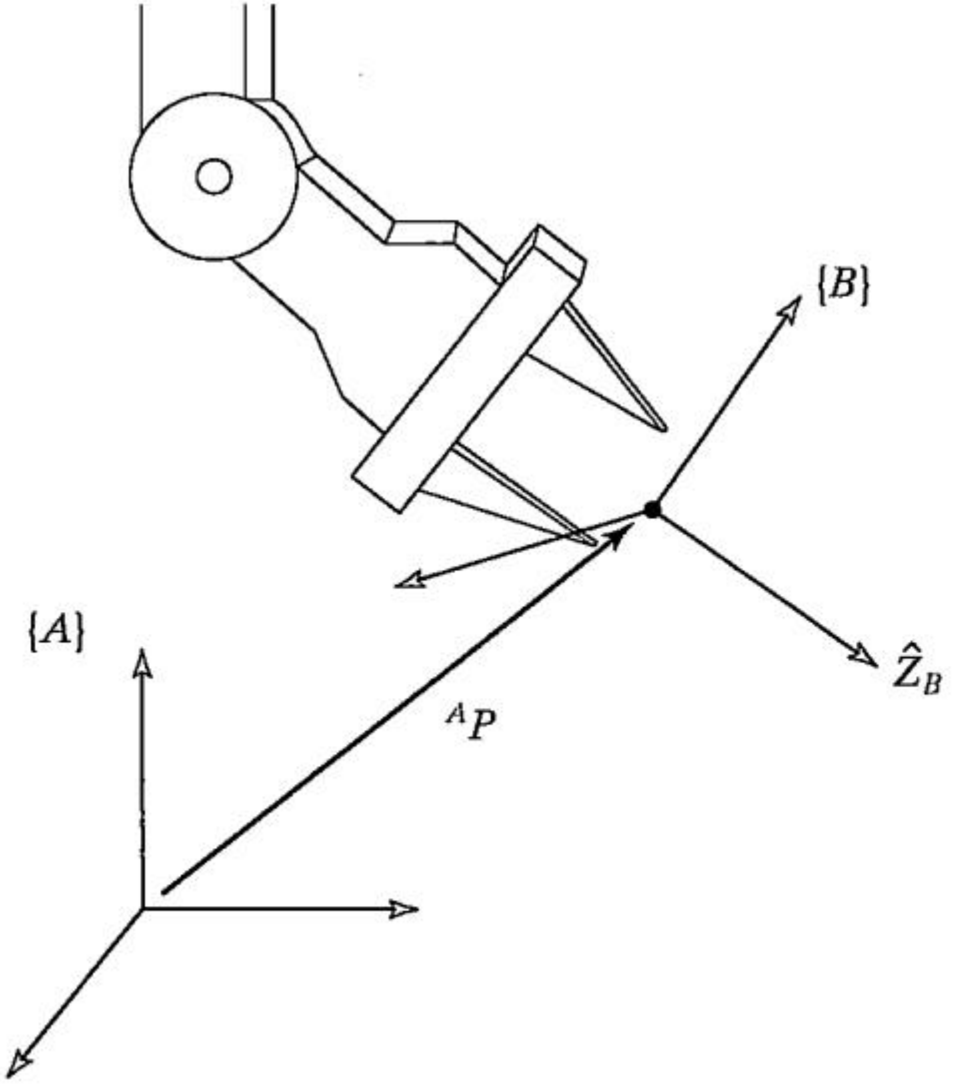



FIGURE 2.2: Locating an object in position and orientation.

# Contd ….

- We denote the unit vectors giving the principal directions of coordinate system {B } as $\hat{X}_B$, $\hat{Y}_B$, and $\hat{Z}_B$.
- When written in terms of coordinate system {**A**}, they are called

$$^{A}\hat{X}_B, \ ^{A}\hat{Y}_B, \text{ and } ^{A}\hat{Z}_B$$

- if we stack these three unit vectors together as the columns of a 3 x 3 matrix $^{A}\hat{X}_B$, $^{A}\hat{Y}_B$, $^{A}\hat{Z}_B$.
- We will call this matrix a **rotation** matrix,
- because this particular rotation matrix **describes {B } relative to {A},** we name it with the notation

$$^{A}_{B}R$$

- the rows of the matrix are the unit vectors of {A} **expressed in {B};**

$$ {}^A_BR = \left[ \, {}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B \, \right] = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}. $$

$$ {}^A_BR = \left[ \, {}^A\hat{X}_B \quad {}^A\hat{Y}_B \quad {}^A\hat{Z}_B \, \right] = \begin{bmatrix} \boxed{\hat{X}_B \cdot \hat{X}_A \quad \hat{Y}_B \cdot \hat{X}_A \quad \hat{Z}_B \cdot \hat{X}_A} \\ \hat{X}_B \cdot \hat{Y}_A \quad \hat{Y}_B \cdot \hat{Y}_A \quad \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A \quad \hat{Y}_B \cdot \hat{Z}_A \quad \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}. $$

$$ {}^{A}_{B}R = \left[ {}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}\hat{Z}_B \right] = \begin{bmatrix} {}^{B}\hat{X}_A^T \\ {}^{B}\hat{Y}_A^T \\ {}^{B}\hat{Z}_A^T \end{bmatrix}. $$

*Hence,* ${}^{B}_{A}R$*, the description of frame {A} relative to {B}, is given by the transpose of ; that is,*

$$ {}^{B}_{A}R = {}^{A}_{B}R^T. $$

- *the inverse of a rotation matrix is equal to its transpose*

$$ {}^{A}_{B}R^T \, {}^{A}_{B}R = \begin{bmatrix} {}^{A}\hat{X}_B^T \\ {}^{A}\hat{Y}_B^T \\ {}^{A}\hat{Z}_B^T \end{bmatrix} \left[ {}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}\hat{Z}_B \right] = I_3, \Rightarrow {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^T. $$

# Description of a frame

- *the point whose position we will describe is chosen as the origin of the body-attached frame.*

$$\{B\} = \{{}^A_B R, {}^A P_{BORG}\}.$$

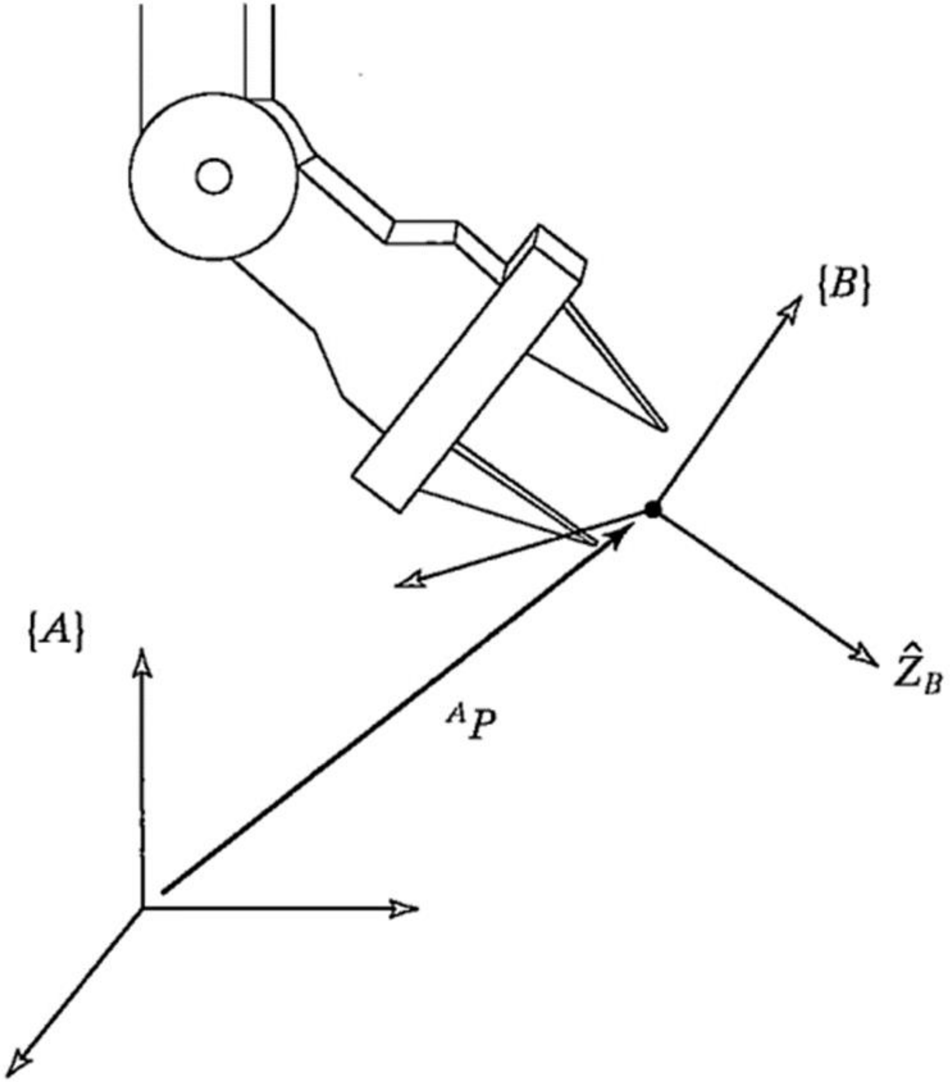



FIGURE 2.2: Locating an object in position and orientation.

- a frame can be used as a description of one coordinate system relative to another
- frame =representing both position and orientation

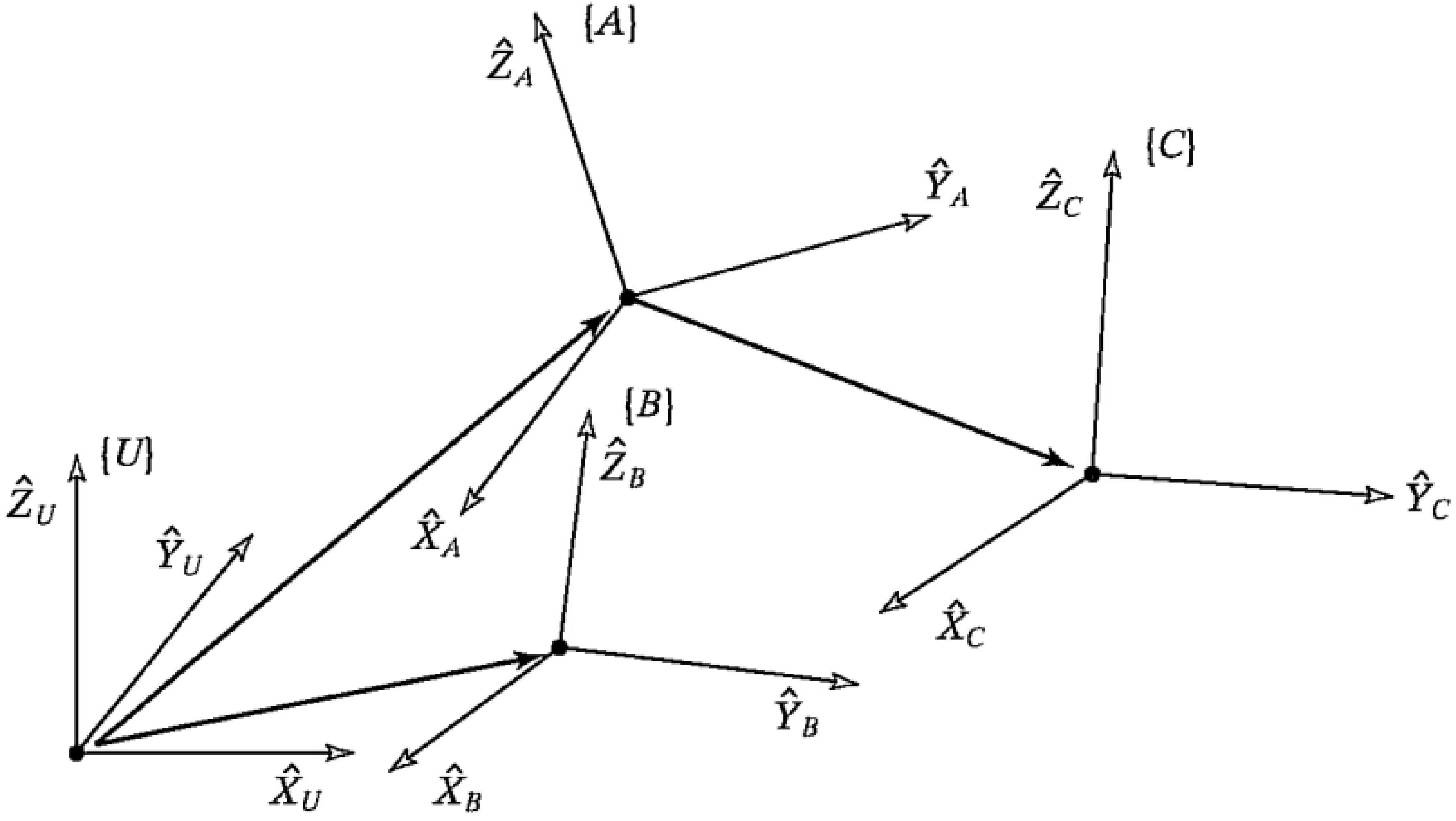
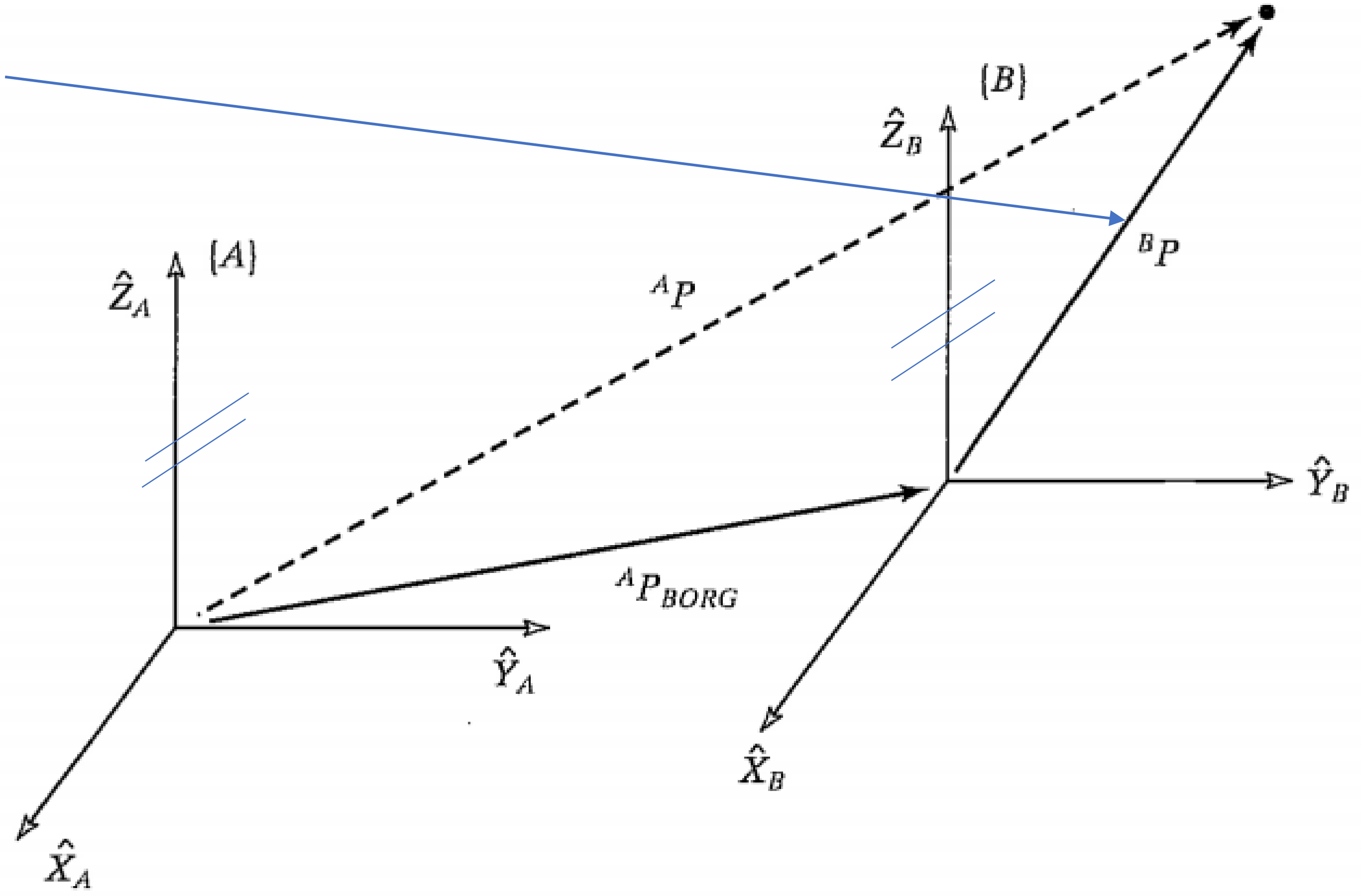



FIGURE 2.3: Example of several frames.

*Mappings: changing descriptions from frame to frame*

# Mappings involving translated frames

- A position defined by the vector ${}^{B}P$ We wish to express this point in space in terms of frame **{A}**,
- when {A} has the same orientation as {B}: {B} differs from {A} only by a translation,
- ${}^{A}P_{BORG}$
- A vector that locates the origin of {B} relative to {A}




$${}^{A}P = {}^{B}P + {}^{A}P_{BORG}.$$

FIGURE 2.4: Translational mapping.

# Mappings involving rotated frames

- the columns of a rotation matrix all have unit magnitude, and, these unit vectors are orthogonal.
- As we saw earlier, a result of this is

$$ {}^{A}_{B}R = {}^{B}_{A}R^{-1} = {}^{B}_{A}R^{T}. $$

- the columns of ${}^{A}_{B}\mathbf{R}$ are the unit vectors of {B} written in {A}, the rows of are the unit vectors of {A} written in {B}.

$$ {}^{A}_{B}R = \left[ \, {}^{A}\hat{X}_{B} \;\; {}^{A}\hat{Y}_{B} \;\; {}^{A}\hat{Z}_{B} \, \right] = \begin{bmatrix} {}^{B}\hat{X}^{T}_{A} \\ {}^{B}\hat{Y}^{T}_{A} \\ {}^{B}\hat{Z}^{T}_{A} \end{bmatrix}. $$

$$ {}^{A}_{B}R = \left[ \, {}^{A}\hat{X}_{B} \;\; {}^{A}\hat{Y}_{B} \;\; {}^{A}\hat{Z}_{B} \, \right] = $$

$$ \begin{bmatrix} \hat{X}_{B} \cdot \hat{X}_{A} & \hat{Y}_{B} \cdot \hat{X}_{A} & \hat{Z}_{B} \cdot \hat{X}_{A} \\ \hat{X}_{B} \cdot \hat{Y}_{A} & \hat{Y}_{B} \cdot \hat{Y}_{A} & \hat{Z}_{B} \cdot \hat{Y}_{A} \\ \hat{X}_{B} \cdot \hat{Z}_{A} & \hat{Y}_{B} \cdot \hat{Z}_{A} & \hat{Z}_{B} \cdot \hat{Z}_{A} \end{bmatrix}. $$

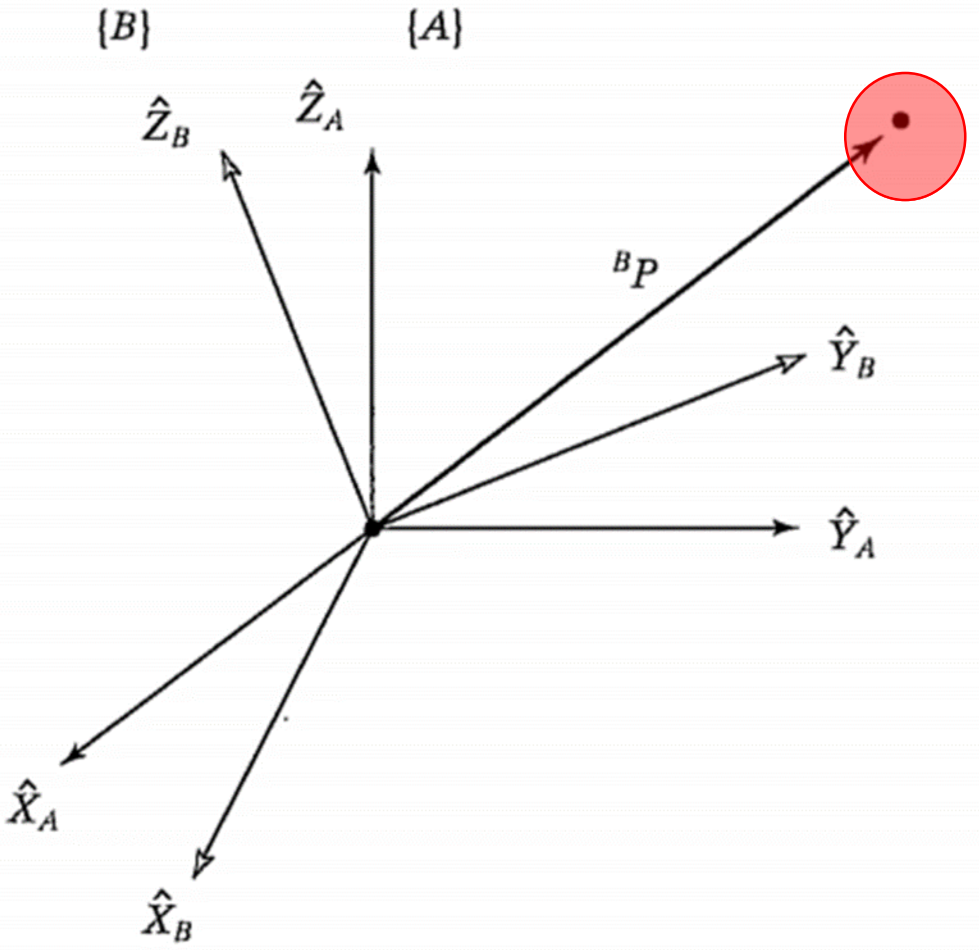



FIGURE 2.5: Rotating the description of a vector.

$$^{A}px = {}^{B}\hat{X}_A \cdot {}^{B}P,$$

$$^{A}p_y = {}^{B}\hat{Y}_A \cdot {}^{B}P,$$

$$^{A}p_z = {}^{B}\hat{Z}_A \cdot {}^{B}P.$$

$$^{A}P = {}^{A}_{B}R\, {}^{B}P.$$

# Example

- Figure shows a frame {B} that is rotated relative to frame {A} about Z by 30 degrees. Here, Z is pointing out of the page.

$$ {}^{A}_{B}R = \left[\, {}^{A}\hat{X}_B \quad {}^{A}\hat{Y}_B \quad {}^{A}\hat{Z}_B \,\right] = \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}. $$

$$ {}^{A}_{B}R = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}. $$

$$ {}^{B}P = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix}, $$

$$ {}^{A}P = {}^{A}_{B}R \, {}^{B}P = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}. $$

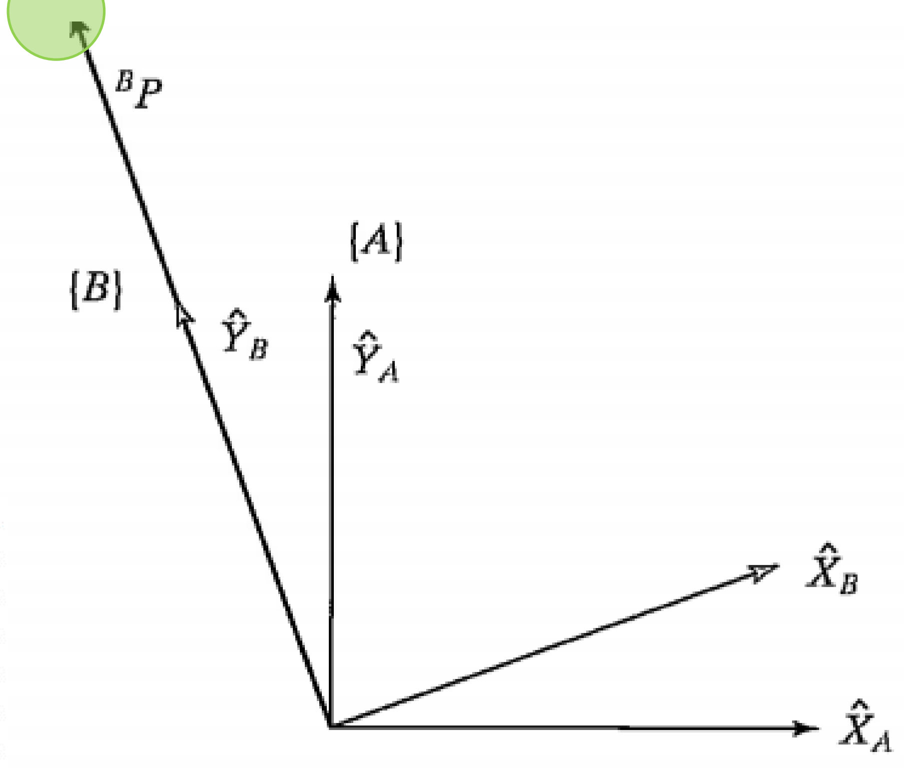



FIGURE 2.6: $\{B\}$ rotated 30 degrees about $\hat{Z}$.

# Mappings involving general frames

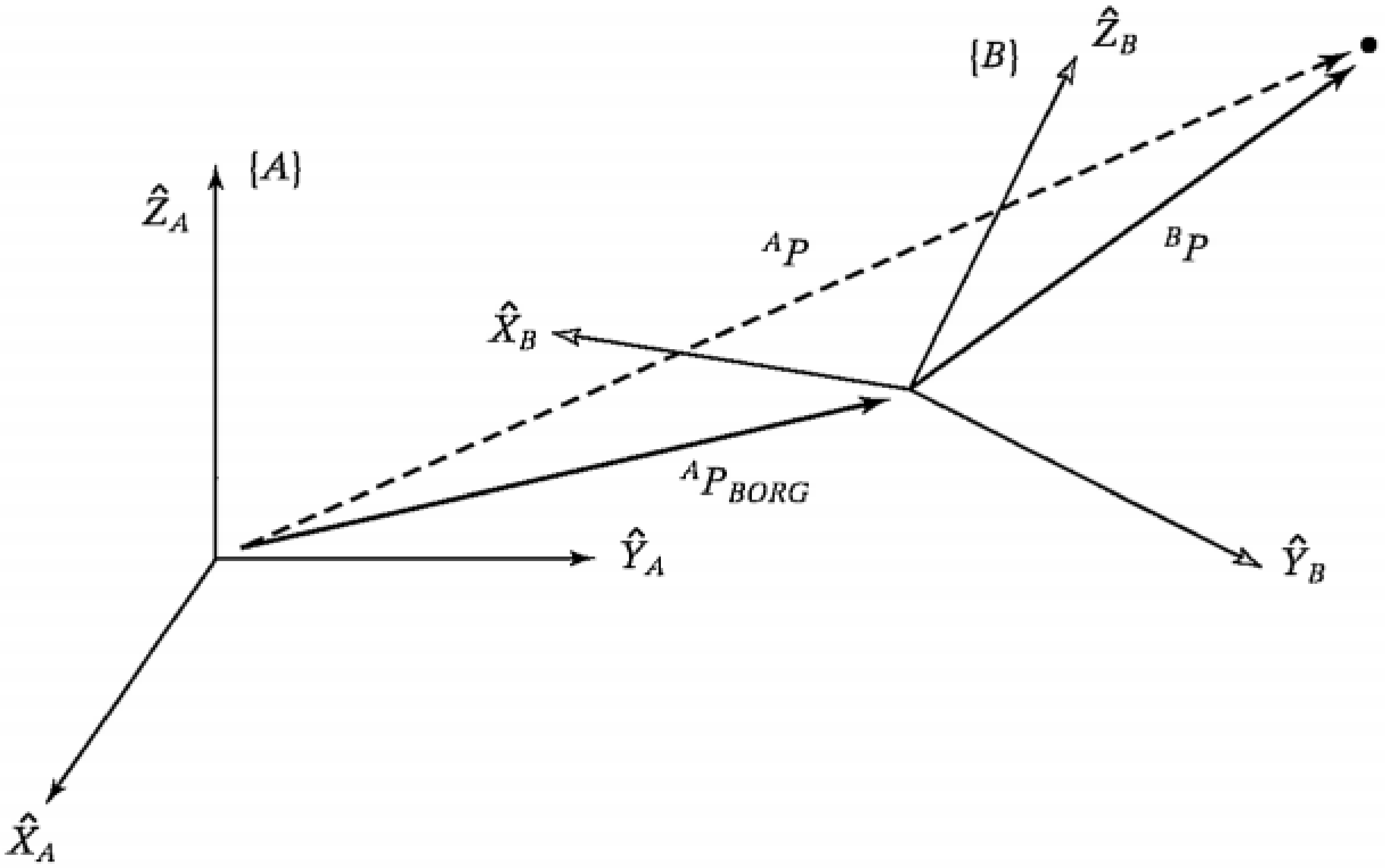



FIGURE 2.7: General transform of a vector.

# Mappings involving general frames…

- We can first change ${}^{B}P$ to its description relative to an intermediate frame that has the same orientation as {A},
- but whose origin is coincident with the origin of {B}.
- This is done by pre multiplying by ${}^{A}_{B}R$

$$ {}^{A}P = {}^{A}_{B}R \, {}^{B}P + {}^{A}P_{BORG}. $$

- Equation describes a general transformation mapping of a vector from its description in one frame to a description in a second frame.

$$ {}^{A}P = {}^{A}_{B}T \, {}^{B}P. $$

# Mappings involving general frames…

- *We define a 4 x 4 matrix operator and use 4 x 1 position vectors,*

$$\begin{bmatrix} {}^{A}P \\ 1 \end{bmatrix} = \left[\begin{array}{ccc|c} & {}^{A}_{B}R & & {}^{A}P_{BORG} \\ \hline 0 & 0 & 0 & 1 \end{array}\right] \begin{bmatrix} {}^{B}P \\ 1 \end{bmatrix}.$$

1. a "1" is added as the last element of the $4 \times 1$ vectors;
2. a row "[0 0 0 1]" is added as the last row of the $4 \times 4$ matrix.

$${}^{A}P = {}^{A}_{B}R \, {}^{B}P + {}^{A}P_{BORG}$$

$$1 = 1.$$

**This 4 x 4 matrix is called a homogeneous transform.**

Figure 2.8 shows a frame $\{B\}$, which is rotated relative to frame $\{A\}$ about $\hat{Z}$ by 30 degrees, translated 10 units in $\hat{X}_A$, and translated 5 units in $\hat{Y}_A$. Find ${}^AP$, where ${}^BP = [3.0\ 7.0\ 0.0]^T$ .

The definition of frame $\{B\}$ is

$$ {}^A_BT = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

$$ {}^BP = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix}, $$

$$ \boxed{\begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}}. $$

$$ {}^AP = {}^A_BT\ {}^BP = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}. $$

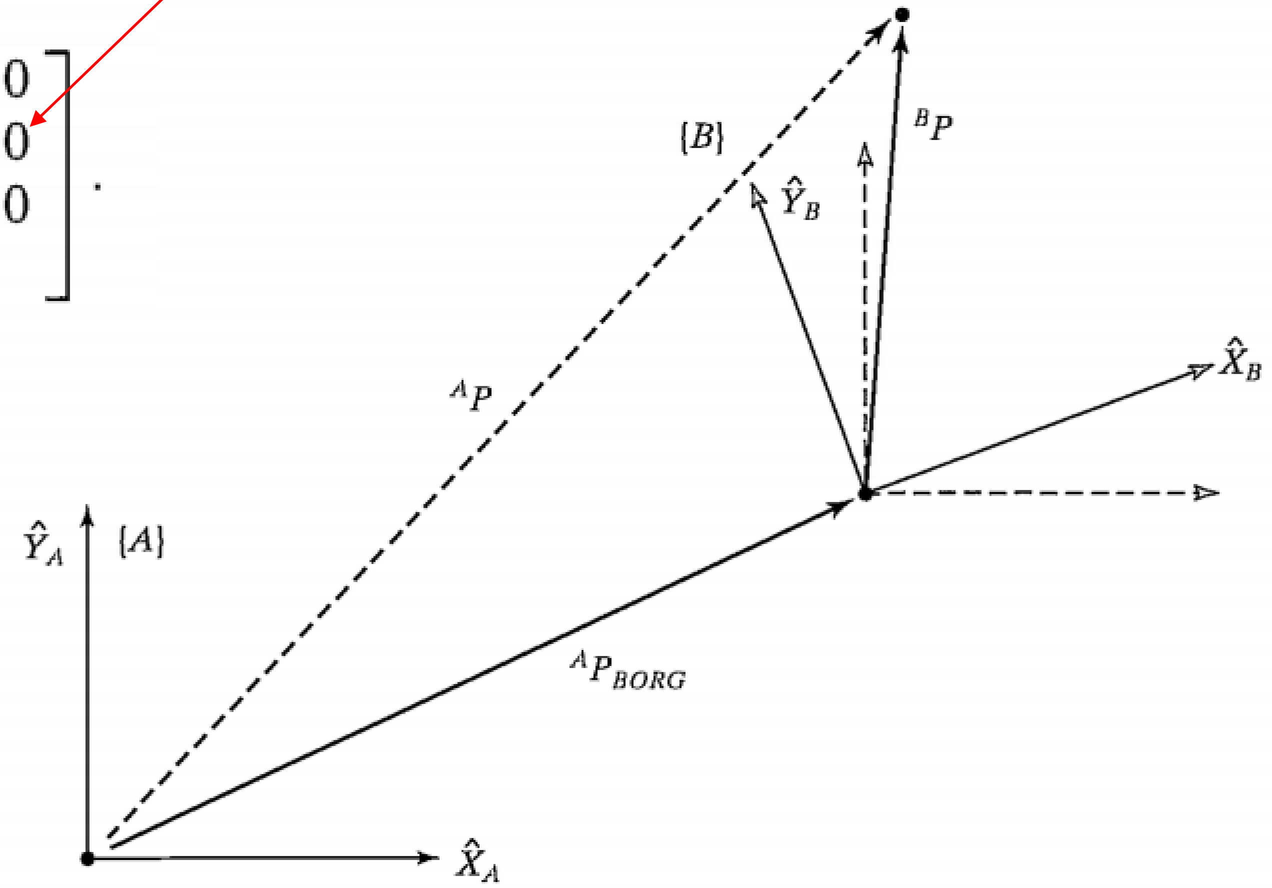



FIGURE 2.8: Frame $\{B\}$ rotated and translated.

# Operators: translations, rotations, and transformations

Figure 2.10 shows a vector ${}^{A}P_1$. We wish to compute the vector obtained by rotating this vector about $\hat{Z}$ by 30 degrees. Call the new vector ${}^{A}P_2$.

$$R_z(30.0) = \begin{bmatrix} 0.866 & -0.500 & 0.000 \\ 0.500 & 0.866 & 0.000 \\ 0.000 & 0.000 & 1.000 \end{bmatrix}.$$

$${}^{A}P_1 = \begin{bmatrix} 0.0 \\ 2.0 \\ 0.0 \end{bmatrix},$$

$${}^{A}P_2 = R_z(30.0)\,{}^{A}P_1 = \begin{bmatrix} -1.000 \\ 1.732 \\ 0.000 \end{bmatrix}.$$

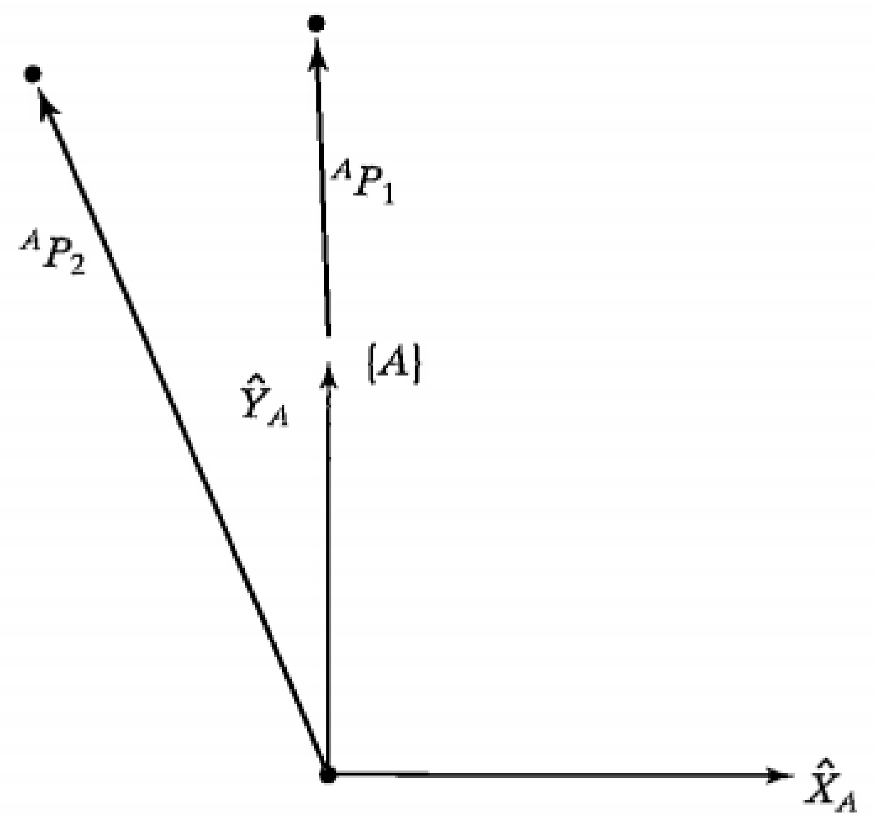



FIGURE 2.10: The vector ${}^{A}P_1$ rotated 30 degrees about $\hat{Z}$.

Figure 2.11 shows a vector ${}^{A}P_1$. We wish to rotate it about $\hat{Z}$ by 30 degrees and translate it 10 units in $\hat{X}_A$ and 5 units in $\hat{Y}_A$. Find ${}^{A}P_2$, where ${}^{A}P_1 = [3.0\ 7.0\ 0.0]^T$.

The operator $T$, which performs the translation and rotation, is

$$T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 10.0 \\ 0.500 & 0.866 & 0.000 & 5.0 \\ 0.000 & 0.000 & 1.000 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{2.34}$$

$${}^{A}P_1 = \begin{bmatrix} 3.0 \\ 7.0 \\ 0.0 \end{bmatrix},$$

$${}^{A}P_2 = T\, {}^{A}P_1 = \begin{bmatrix} 9.098 \\ 12.562 \\ 0.000 \end{bmatrix}.$$

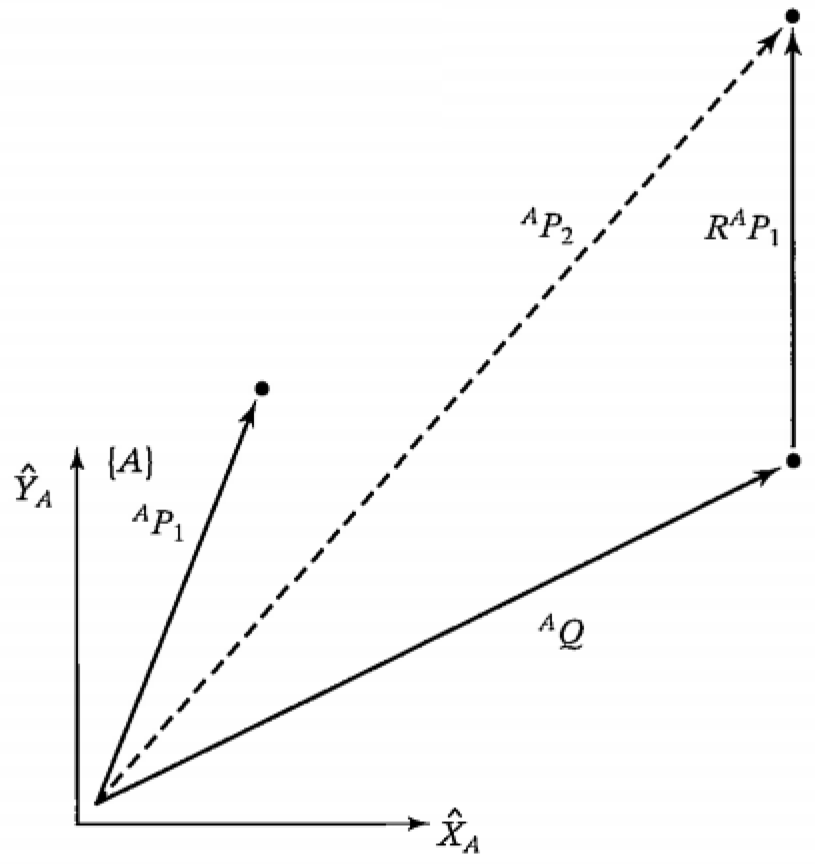


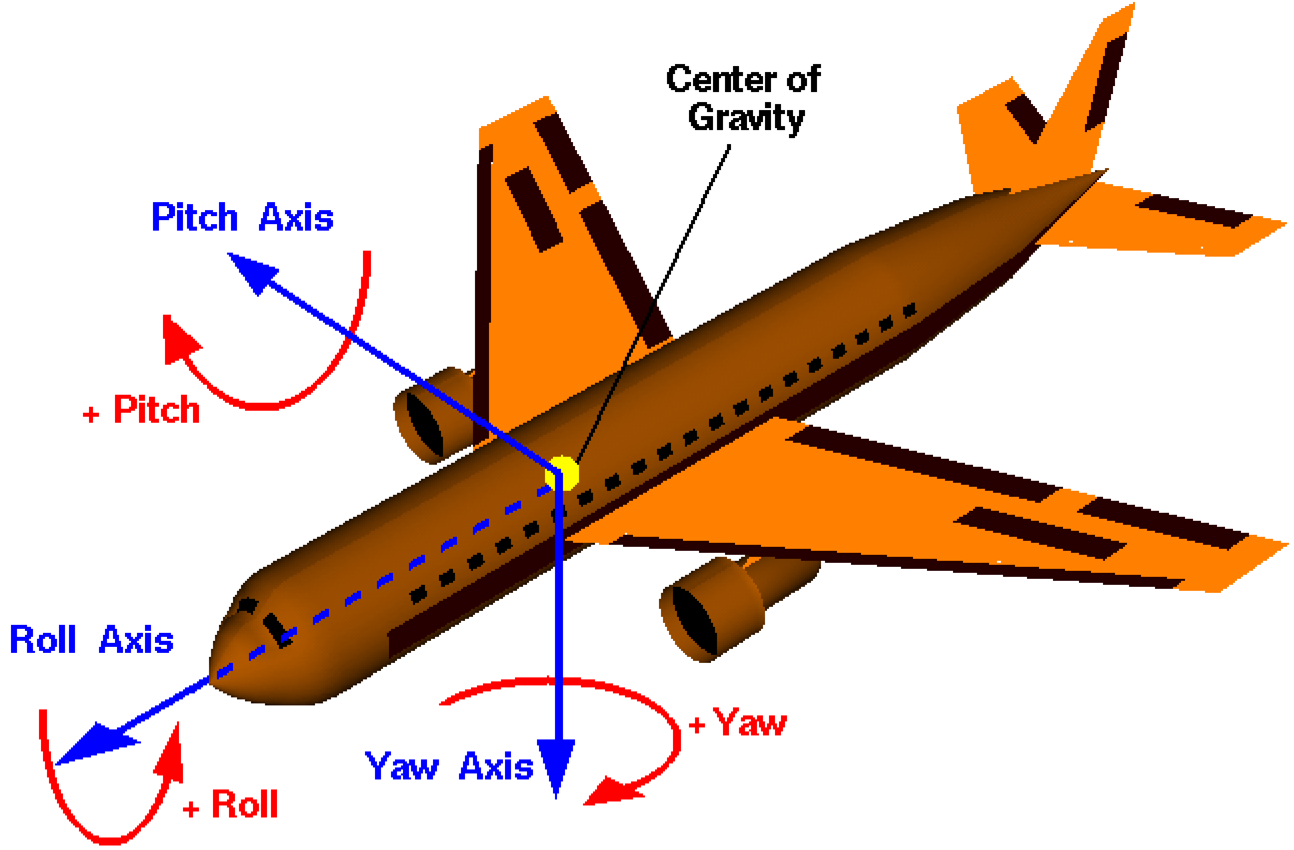

FIGURE 2.11: The vector ${}^{A}P_1$ rotated and translated to form ${}^{A}P_2$.

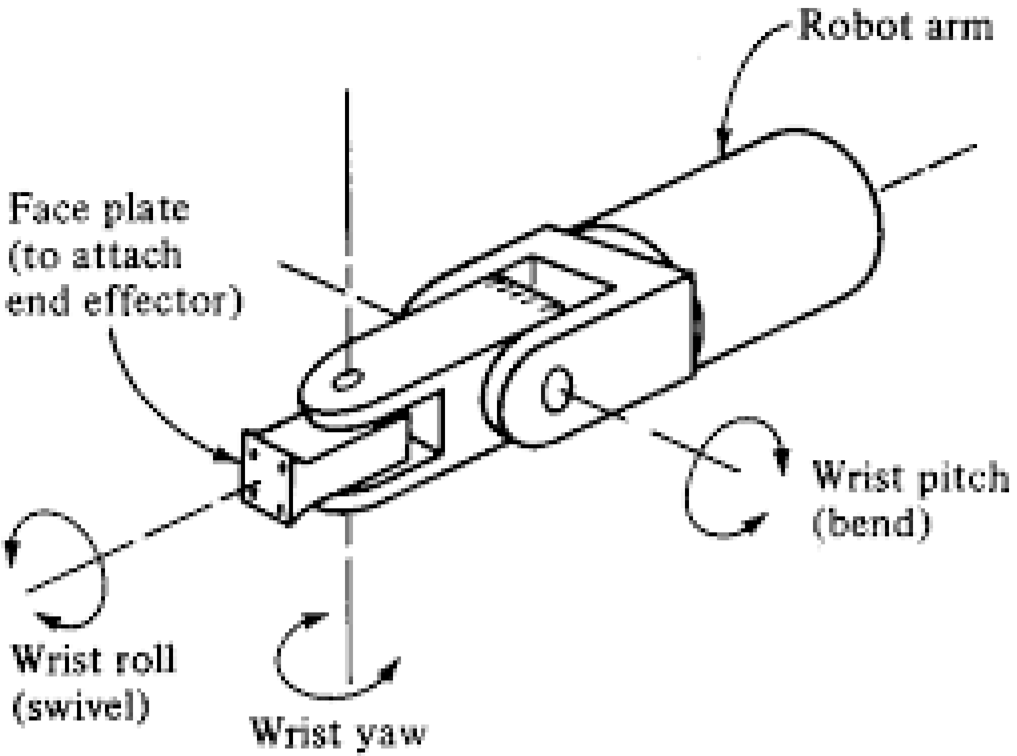
# X - Y - Z fixed angles

One method of describing the orientation of a frame $\{B\}$ is as follows:

> Start with the frame coincident with a known reference frame $\{A\}$. Rotate $\{B\}$ first about $\hat{X}_A$ by an angle $\gamma$, then about $\hat{Y}_A$ by an angle $\beta$, and, finally, about $\hat{Z}_A$ by an angle $\alpha$.

- Each of the three rotations takes place about an axis in the fixed reference frame {A}.
- We will call this convention for specifying an orientation X—Y—Z fixed angles.
- Sometimes this convention is referred to as **roll, pitch, yaw angles**,


Center of
Gravity
Pitch Axis
+ Pitch
Roll Axis
+ Roll
Yaw Axis
+ Yaw


Robot arm
Face plate
(to attach
end effector)
Wrist pitch
(bend)
Wrist roll
(swivel)
Wrist yaw

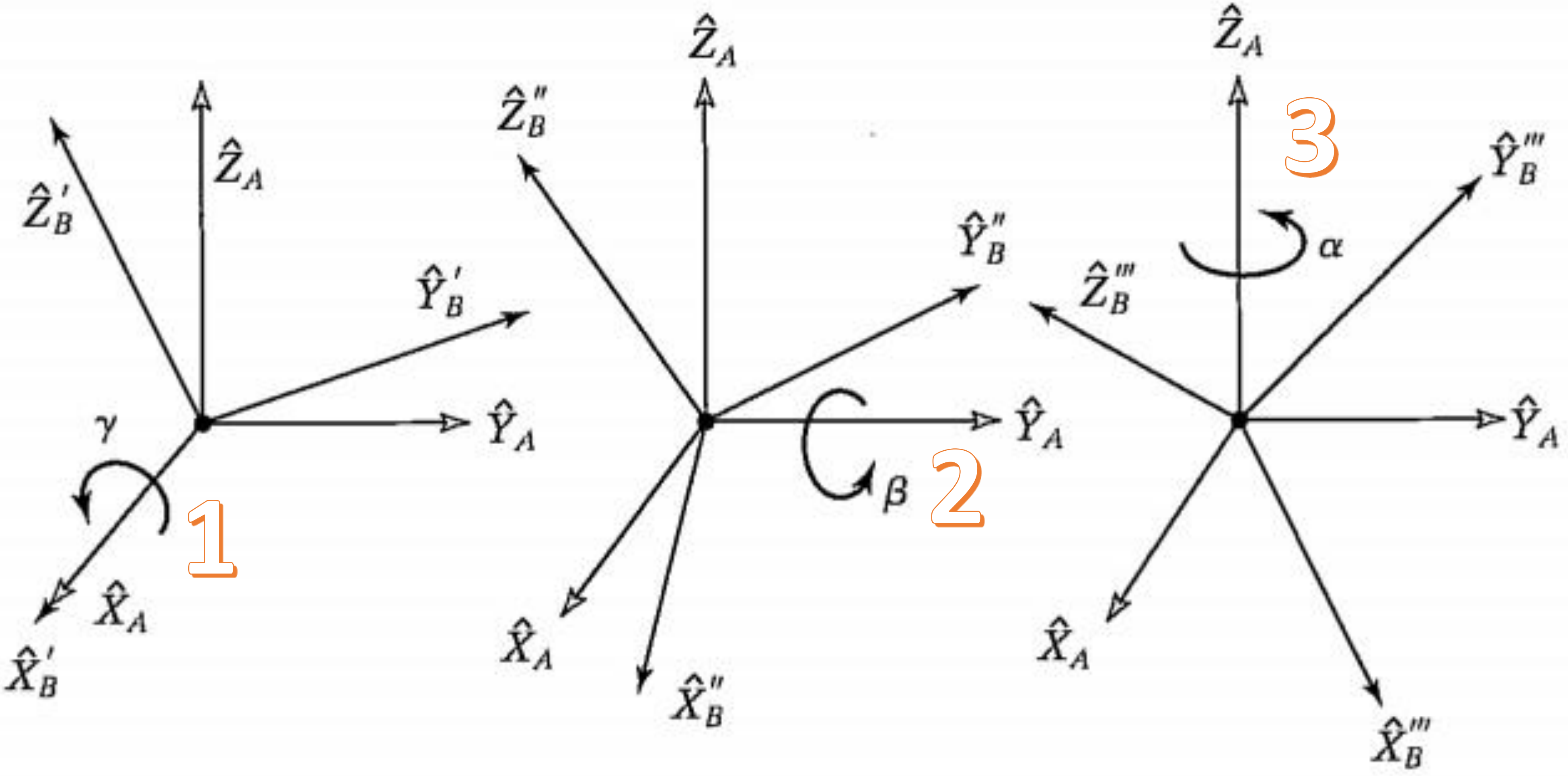



FIGURE 2.17: X–Y–Z fixed angles. Rotations are performed in the order $R_X(\gamma)$, $R_Y(\beta)$, $R_Z(\alpha)$.

- The derivation of the equivalent rotation matrix ${}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha)$,

$$
\begin{aligned}
{}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) &= R_Z(\alpha) R_Y(\beta) R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix}, \qquad (2.63)
\end{aligned}
$$

$$
{}^{A}_{B}R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. \qquad (2.64)
$$

- The inverse problem, that of extracting equivalent X—Y—Z fixed angles from a rotation matrix

$$\beta = \text{Atan2}(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}),$$
$$\alpha = \text{Atan2}(r_{21}/c\beta, r_{11}/c\beta),$$
$$\gamma = \text{Atan2}(r_{32}/c\beta, r_{33}/c\beta),$$

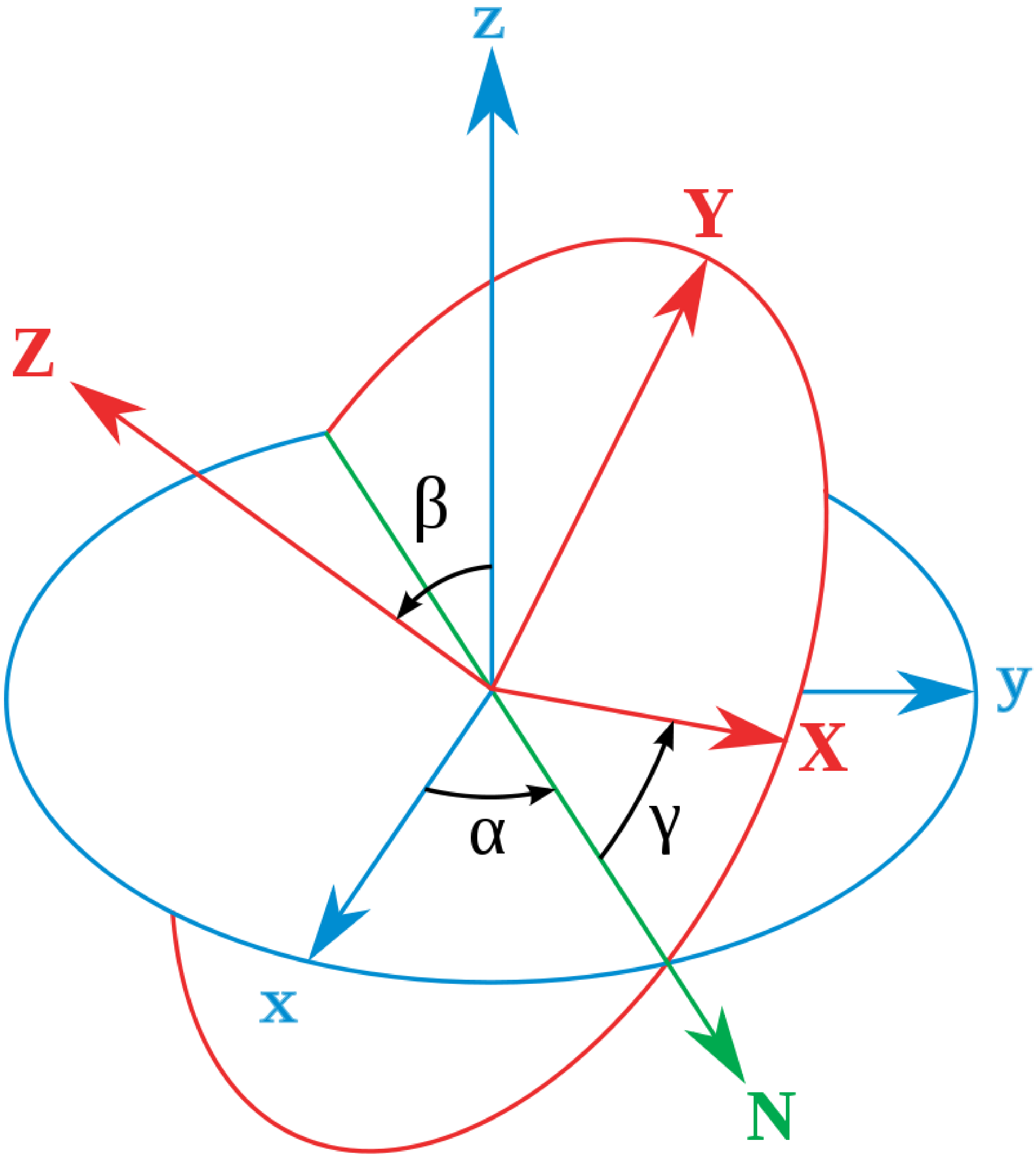
$${}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}.$$

$${}^A_B R_{XYZ}(\gamma, \beta, \alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. \quad (2.64)$$

# Z-Y-X Euler angles

They can also represent the orientation of a mobile frame of reference in physics or the orientation of a general basis in 3-dimensional linear algebra

- Proper Euler angles geometrical definition.
- The xyz (fixed) system is shown in blue, the XYZ (rotated) system is shown in red.
- The line of nodes (N) is shown in green



Another possible description of a frame $\{B\}$ is as follows:

> Start with the frame coincident with a known frame $\{A\}$. Rotate $\{B\}$ first about $\hat{Z}_B$ by an angle $\alpha$, then about $\hat{Y}_B$ by an angle $\beta$, and, finally, about $\hat{X}_B$ by an angle $\gamma$.

- Because the three rotations occur about the axes Z, Y, and Z, we will call this representation Z—Y—Z Euler angles.

*XYX,XZX,YXY,YZY,ZXZ,ZXZ…………….*

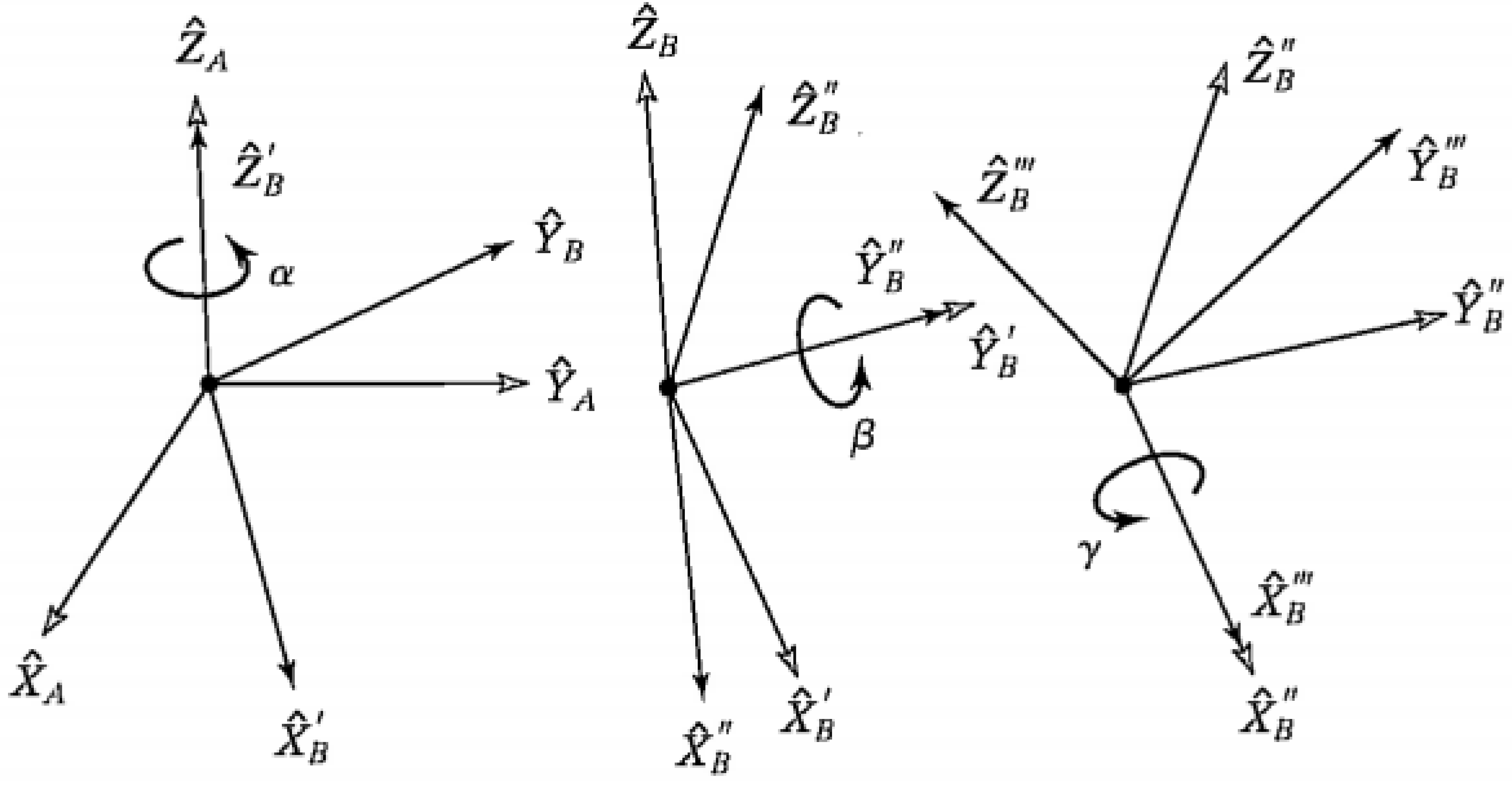



FIGURE 2.18: Z–Y–X Euler angles.

$$
\begin{aligned}
{}^{A}_{B}R_{Z'Y'X'} &= R_Z(\alpha)R_Y(\beta)R_X(\gamma) \\
&= \begin{bmatrix} c\alpha & -s\alpha & 0 \\ s\alpha & c\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\beta & 0 & s\beta \\ 0 & 1 & 0 \\ -s\beta & 0 & c\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma & -s\gamma \\ 0 & s\gamma & c\gamma \end{bmatrix},
\end{aligned} \tag{2.70}
$$

where $c\alpha = \cos\alpha$, $s\alpha = \sin\alpha$, and so on. Multiplying out, we obtain

$$
{}^{A}_{B}R_{Z'Y'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}. \tag{2.71}
$$

$$
{}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.
$$

$$
{}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix},
$$

then, if $\sin\beta \neq 0$, it follows that

$$
\begin{aligned}
\beta &= \mathrm{Atan2}\left(\sqrt{r_{31}^2 + r_{32}^2}, r_{33}\right), \\
\alpha &= \mathrm{Atan2}(r_{23}/s\beta, r_{13}/s\beta), \\
\gamma &= \mathrm{Atan2}(r_{32}/s\beta, -r_{31}/s\beta).
\end{aligned}
$$

The 12 Euler angle sets are given by

$$R_{X'Y'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{X'Z'Y'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Y'X'Z'}(\alpha, \beta, \gamma) = \begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{Y'Z'X'}(\alpha, \beta, \gamma) = \begin{bmatrix} c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Z'X'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix},$$

$$R_{Z'Y'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix},$$

$$R_{X'Y'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

$$R_{X'Z'X'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Y'X'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

$$R_{Y'Z'Y'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma \\ s\beta c\gamma & c\beta & s\beta s\gamma \\ -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{Z'X'Z'}(\alpha,\beta,\gamma)=\begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma & s\alpha s\beta \\ c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha s\beta \\ s\beta s\gamma & s\beta c\gamma & c\beta \end{bmatrix},$$

$$R_{Z'Y'Z'}(\alpha,\beta,\gamma)=\begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix}.$$

The 12 fixed angle sets are given by

$$R_{XYZ}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix},$$

$$R_{XZY}(\gamma,\beta,\alpha)=\begin{bmatrix} c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma \\ s\beta & c\beta c\gamma & -c\beta s\gamma \\ -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{YXZ}(\gamma,\beta,\alpha)=\begin{bmatrix} -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta & s\alpha s\beta c\gamma + c\alpha s\gamma \\ c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta & -c\alpha s\beta c\gamma + s\alpha s\gamma \\ -c\beta s\gamma & s\beta & c\beta c\gamma \end{bmatrix},$$

$$R_{YZX}(\gamma,\beta,\alpha)=\begin{bmatrix} c\beta c\gamma & -s\beta & c\beta s\gamma \\ c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{ZXY}(\gamma,\beta,\alpha)=\begin{bmatrix} s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma & s\alpha c\beta \\ c\beta s\gamma & c\beta c\gamma & -s\beta \\ c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{ZYX}(\gamma,\beta,\alpha)=\begin{bmatrix} c\beta c\gamma & -c\beta s\gamma & s\beta \\ s\alpha s\beta c\gamma + c\alpha s\gamma & -s\alpha s\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta \\ -c\alpha s\beta c\gamma + s\alpha s\gamma & c\alpha s\beta s\gamma + s\alpha c\gamma & c\alpha c\beta \end{bmatrix},$$

$$R_{XYX}(\gamma,\beta,\alpha)=\begin{bmatrix} c\beta & s\beta s\gamma & s\beta c\gamma \\ s\alpha s\beta & -s\alpha c\beta s\gamma + c\alpha c\gamma & -s\alpha c\beta c\gamma - c\alpha s\gamma \\ -c\alpha s\beta & c\alpha c\beta s\gamma + s\alpha c\gamma & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

$$R_{XZX}(\gamma,\beta,\alpha)=\begin{bmatrix} c\beta & -s\beta c\gamma & s\beta s\gamma \\ c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma \\ s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma \end{bmatrix},$$

$$R_{YXY}(\gamma,\beta,\alpha)=\begin{bmatrix} -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta & s\alpha c\beta c\gamma + c\alpha s\gamma \\ s\beta s\gamma & c\beta & -s\beta c\gamma \\ -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta & c\alpha c\beta c\gamma - s\alpha s\gamma \end{bmatrix},$$

# Equivalent angle-axis representation

- With the notation $R_x$ (30.0) we give the description of an orientation by giving an axis, **X**, and an angle, 30.0 degrees. This is an example of an equivalent angle-axis representation.

General direction

- Start with the frame coincident with a known frame {A}; then rotate {B} about the vector $^{A}\boldsymbol{K}$ by an angle $\boldsymbol{\theta}$ according to the right-hand rule.

Vector $\hat{K}$ is sometimes called the equivalent axis of a finite rotation. A general orientation of $\{B\}$ relative to $\{A\}$ may be written as ${}^{A}_{B}R(\hat{K}, \theta)$ or $R_K(\theta)$ and will be called the equivalent angle–axis representation.

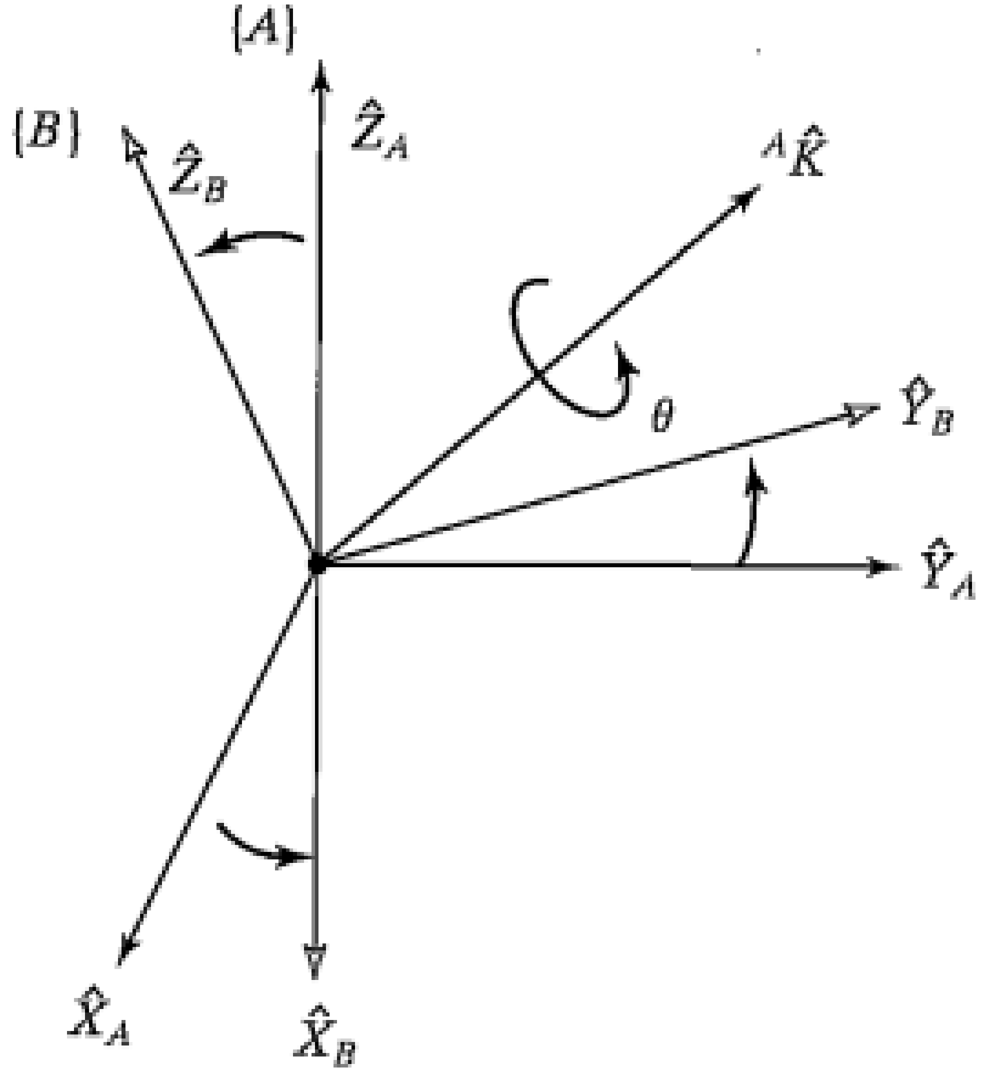



FIGURE 2.19: Equivalent angle–axis representation.

- When the axis of rotation is chosen from among the principal axes of {A}, then the equivalent rotation matrix

$$R_X(\theta) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix},$$

$$R_Y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

$$R_Z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

**2.1** [15] A vector ${}^{A}P$ is rotated about $\hat{Z}_A$ by $\theta$ degrees and is subsequently rotated about $\hat{X}_A$ by $\phi$ degrees. Give the rotation matrix that accomplishes these rotations in the given order.

**2.2** [15] A vector ${}^{A}P$ is rotated about $\hat{Y}_A$ by 30 degrees and is subsequently rotated about $\hat{X}_A$ by 45 degrees. Give the rotation matrix that accomplishes these rotations in the given order.

**2.3** [16] A frame $\{B\}$ is located initially coincident with a frame $\{A\}$. We rotate $\{B\}$ about $\hat{Z}_B$ by $\theta$ degrees, and then we rotate the resulting frame about $\hat{X}_B$ by $\phi$ degrees. Give the rotation matrix that will change the descriptions of vectors from ${}^{B}P$ to ${}^{A}P$.

**2.4** [16] A frame $\{B\}$ is located initially coincident with a frame $\{A\}$. We rotate $\{B\}$ about $\hat{Z}_B$ by 30 degrees, and then we rotate the resulting frame about $\hat{X}_B$ by 45 degrees. Give the rotation matrix that will change the description of vectors from ${}^{B}P$ to ${}^{A}P$.

**2.12** [14] A velocity vector is given by

$$^{B}V = \begin{bmatrix} 10.0 \\ 20.0 \\ 30.0 \end{bmatrix}.$$

Given

$$^{A}_{B}T = \begin{bmatrix} 0.866 & -0.500 & 0.000 & 11.0 \\ 0.500 & 0.866 & 0.000 & -3.0 \\ 0.000 & 0.000 & 1.000 & 9.0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

compute $^{A}V$.