INVERSE KINEMATICS

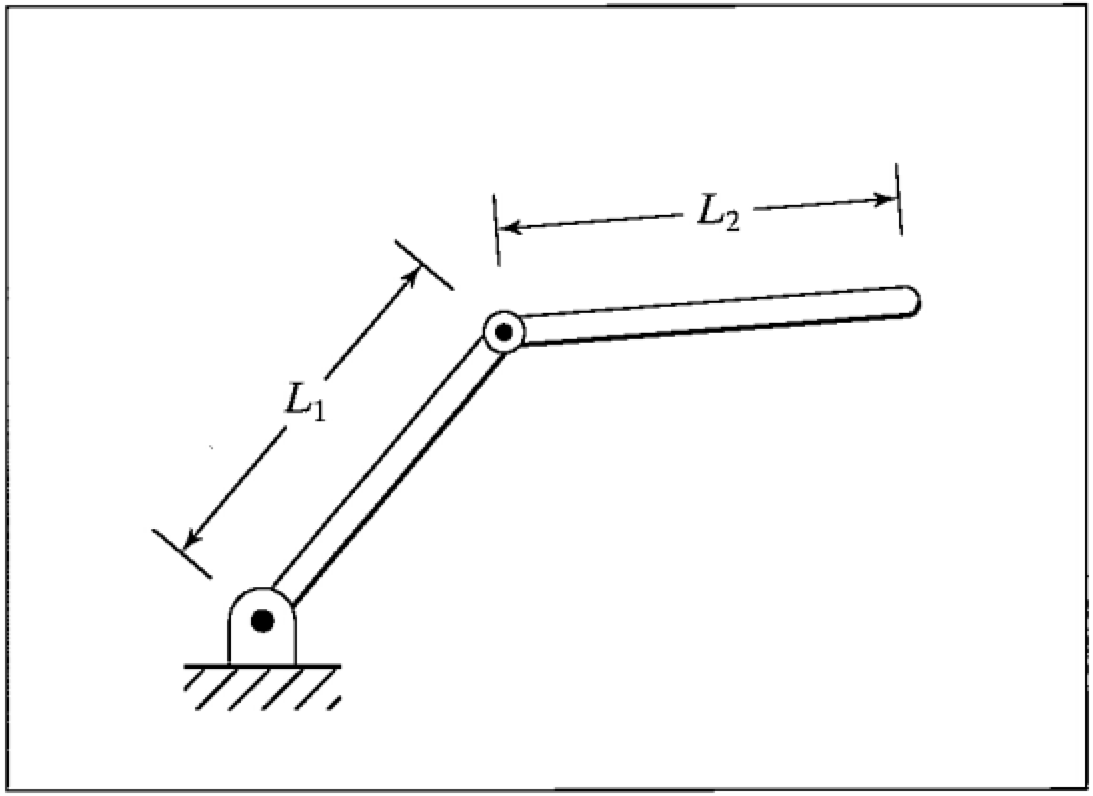
Introduction

- Given the desired position and orientation of the tool relative to the station, how do we compute the set of joint angles which will achieve this desired result
- Solving the problem of finding the required **joint angles** to place the tool frame, {T}, relative to the station frame, {S}
- First, frame transformations are performed to find the wrist frame, {W}, relative to the base frame, {B},
- and then the inverse kinematics are used to solve for the *joint angles*.

Existence of solutions

- For a solution to exist, the specified goal point must lie within the workspace.
- **Dextrous** workspace is that volume of space that the robot end-effector can reach with all orientations. That is, at each point in the dexterous workspace, the end-effector can be arbitrarily oriented.
- The **reachable workspace** is that volume of space that the robot can reach in at least one orientation.
- Clearly, the dextrous workspace is a subset of the reachable workspace.

- If *l1= l2,* then the reachable workspace consists of a disc of radius *2l1*
- *The dexterous workspace consists of only a single point, the origin*
- *If l1≠l2, then there is no dexterous workspace*
- *and the reachable workspace becomes a ring of outer radius l1+l2 and inner radius l1-l2*



Multiple solutions

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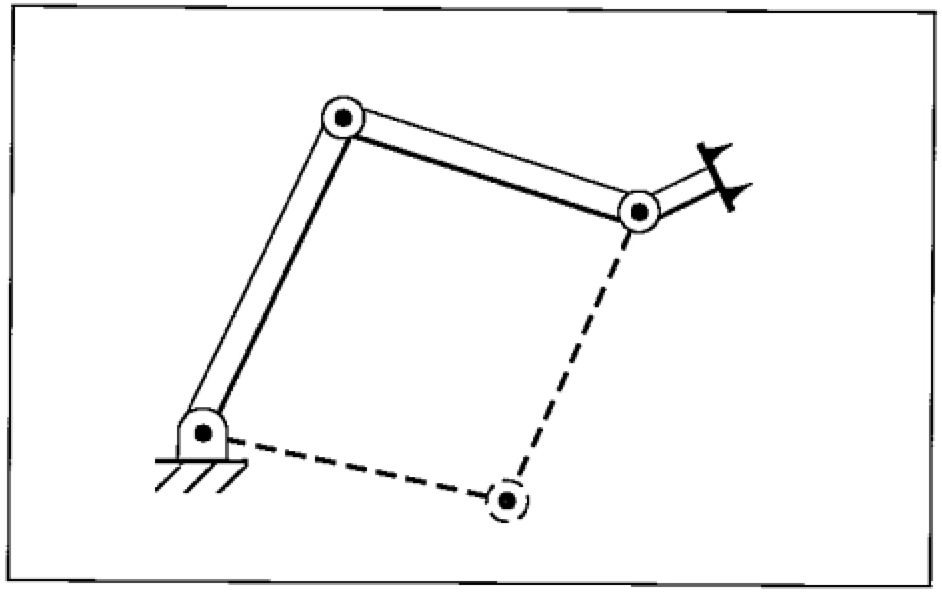
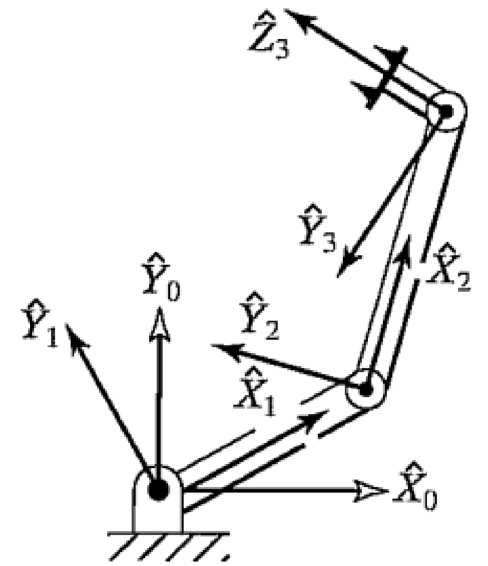


: Three-link manipulator. Dashed lines indicate a second solution.

ALGEBRAIC VS. GEOMETRIC

$$ {}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1 + l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$



i	$\alpha_i - 1$	$\alpha_i - 1$	d_i	θ_i
1	0	0	0	θ_1
2	0	L_1	0	θ_2
3	0	L_2	0	θ_3

- we will assume a transformation with the structure

$$ {}^{B}_{W}T = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

=

$$ {}^{B}_{W}T = {}^{0}_{3}T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1 + l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. $$

$$ c_\phi = c_{123}, \tag{4.8} $$

$$ s_\phi = s_{123}, \tag{4.9} $$

$$ x = l_1c_1 + l_2c_{12}, \tag{4.10} $$

$$ y = l_1s_1 + l_2s_{12}. \tag{4.11} $$

- If we square both (4.10) and (4.11) and add them, we obtain

$$x^2 + y^2 = l_1^2 + l_2^2 + 2l_1 l_2 c_2, \tag{4.12}$$

where we have made use of

$$\begin{aligned} c_{12} &= c_1 c_2 - s_1 s_2, \\ s_{12} &= c_1 s_2 + s_1 c_2. \end{aligned} \tag{4.13}$$

- Solving (4.12) for c_2, we obtain

$$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}. \tag{4.14}$$

$$s_2 = \pm\sqrt{1 - c_2^2}. \tag{4.15}$$

Finally, we compute θ_2, using the two-argument arctangent routine[1]:

$$\theta_2 = \mathrm{Atan2}(s_2, c_2). \tag{4.16}$$

Having found θ_2, we can solve (4.10) and (4.11) for θ_1. We write (4.10) and (4.11) in the form

$$x = k_1 c_1 - k_2 s_1, \tag{4.17}$$

$$y = k_1 s_1 + k_2 c_1, \tag{4.18}$$

where

$$\begin{aligned} k_1 &= l_1 + l_2 c_2, \\ k_2 &= l_2 s_2. \end{aligned} \tag{4.19}$$

If

$$r = +\sqrt{k_1^2 + k_2^2} \tag{4.20}$$

and

$$\gamma = \mathrm{Atan2}(k_2, k_1),$$

then

$$k_1 = r\cos\gamma,$$
$$k_2 = r\sin\gamma. \tag{4.21}$$

Equations (4.17) and (4.18) can now be written as

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1, \tag{4.22}$$
$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1, \tag{4.23}$$

so

$$\cos(\gamma + \theta_1) = \frac{x}{r}, \tag{4.24}$$
$$\sin(\gamma + \theta_1) = \frac{y}{r}. \tag{4.25}$$

$$\gamma + \theta_1 = \mathrm{Atan2}\left(\frac{y}{r}, \frac{x}{r}\right) = \mathrm{Atan2}(y, x), \tag{4.26}$$

and so

$$\theta_1 = \mathrm{Atan2}(y, x) - \mathrm{Atan2}(k_2, k_1). \tag{4.27}$$

Finally, from (4.8) and (4.9), we can solve for the sum of $\theta 1$through $\theta 3$:

$$\theta_1 + \theta_2 + \theta_3 = \mathrm{Atan2}(s_\phi, c_\phi) = \phi. \tag{4.28}$$

Geometric solution