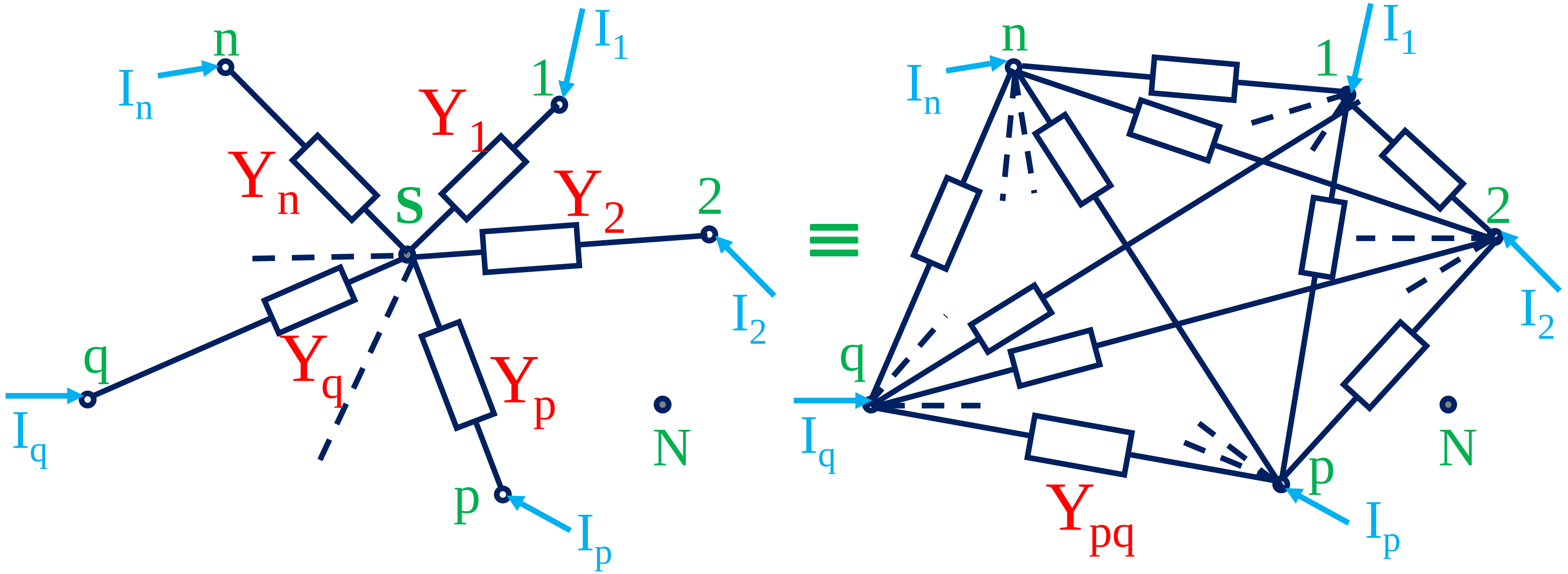
EE 1102 – Fundamentals of Electrical Engineering

Network Theorems



$$Y_{pq} = \frac{Y_{pS} . Y_{qS}}{\sum_{r=1}^{n} Y_{rS}}$$

Prof Rohan Lucas

Learning outcomes

After successful completion of the module, you should be able to

1. Analyze a circuit using Network simplification theorems (Superposition, Thevenin's, Norton's, Compensation, Millmann's, Nodal-mesh transformation)
2. Convert voltage sources to current sources, and vice versa.
3. Perform basic nodal and mesh analysis on basic circuits
4. Obtain the maximum power transfer to a load
5. Determine the peak, average and rms values of waveforms

Outline Syllabus

Overview of Electrical Engineering (2 hrs)

Role of Electrical Engineer, Introduction to Generation, Transmission, Distribution and Utilisation.

SI Units (2 hrs)

Basic and supplementary units, Derived units, Symbols

Basic DC circuit analysis (4 hrs)

Circuit elements, Circuit laws, Circuit solutions with DC.

Network Theorems (4 hrs)

Ohm's Law, Kirchhoff's Laws and network theorems. Introduction to nodal and mesh analysis

Alternating Current theory (8 hrs)

Sinusoidal waveform, phasor and complex representation, Impedance and Admittance, Power and Energy, Power factor. Solution of simple R, L, and C circuit problems by phasor and complex variables.

Electrostatic and Electromagnetic theory (4hrs)

Basic Laws, Calculation of field and force

Electrical Installations (4 hrs)

Fuses, miniature circuit breakers, earth leakage circuit breakers, residual current circuit breakers, earthing, electric shock. Wiring regulations, basic domestic installations.

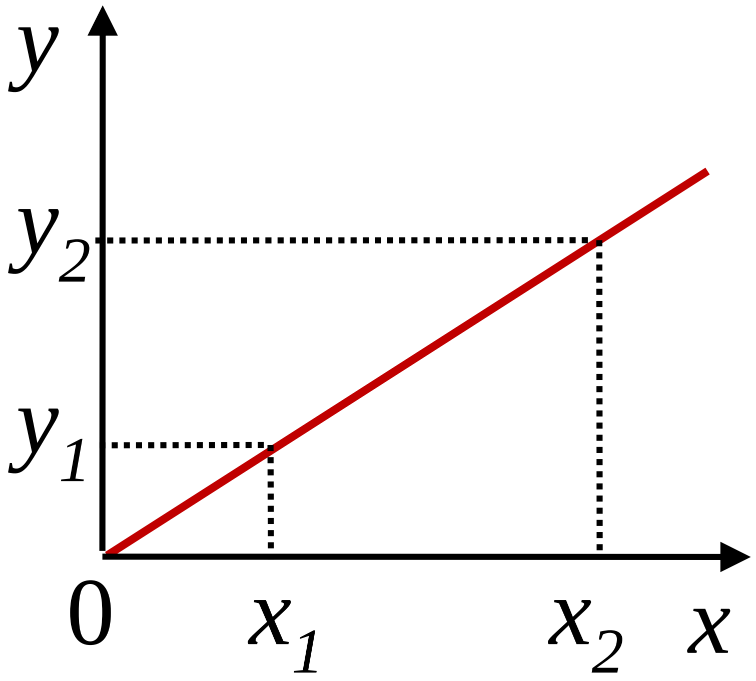
3.0 Network Theorems

3.1 Basic Concepts

Any circuit can be analysed using Ohm's Law and Kirchoff's laws.

To handle complexity, theorems developed.

Applicable to *circuits* with *linear elements*.



In linear circuit,

if variable $\boldsymbol{x_1} \rightarrow \boldsymbol{y_1}$ and $\boldsymbol{x_2} \rightarrow \boldsymbol{y_2}$

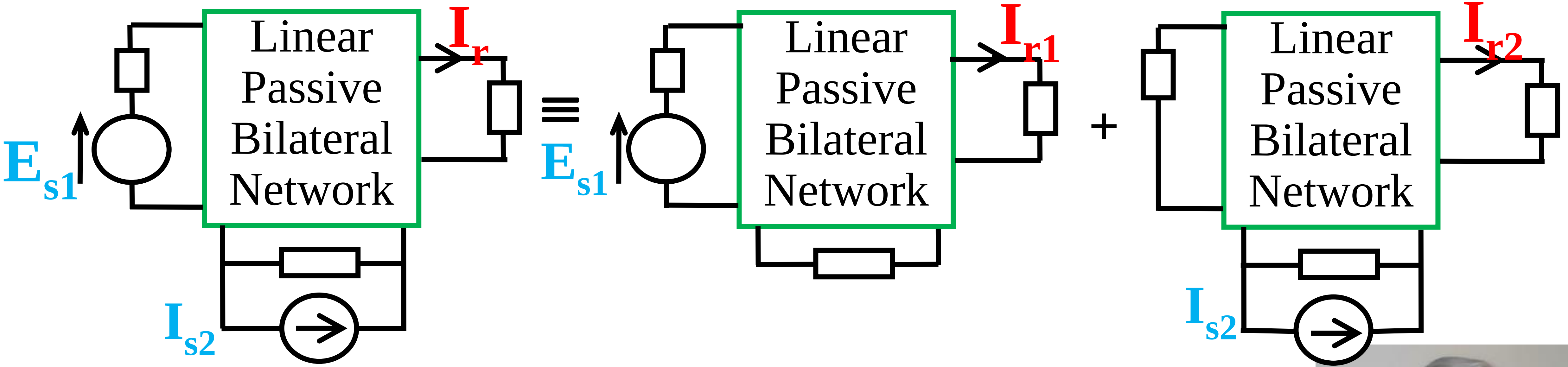
then, $\boldsymbol{k_1\, x_1 + k_2\, x_2 \rightarrow k_1\, y_1 + k_2\, y_2}$

where $\boldsymbol{k_1}$ and $\boldsymbol{k_2}$ are constants.

especially, input $\boldsymbol{x_1 + x_2} \rightarrow$ output $\boldsymbol{y_1 + y_2}$

3.1.1 Superposition theorem

Voltage across (or current through) an element in a linear circuit is algebraic sum of voltages across (or currents through) that element due to each independent source acting alone [i.e. with all other sources replaced by their internal impedance.]



$$I_r = I_{r1} + I_{r2}$$

Example

From sub-circuit 1,

$$I_{s1} = \frac{12}{1+2//(5+17)} = 4.23529\text{A},$$

$$\therefore I_{r1} = \frac{2}{2+(5+17)} \times 4.23529 = 0.35294\text{A}$$

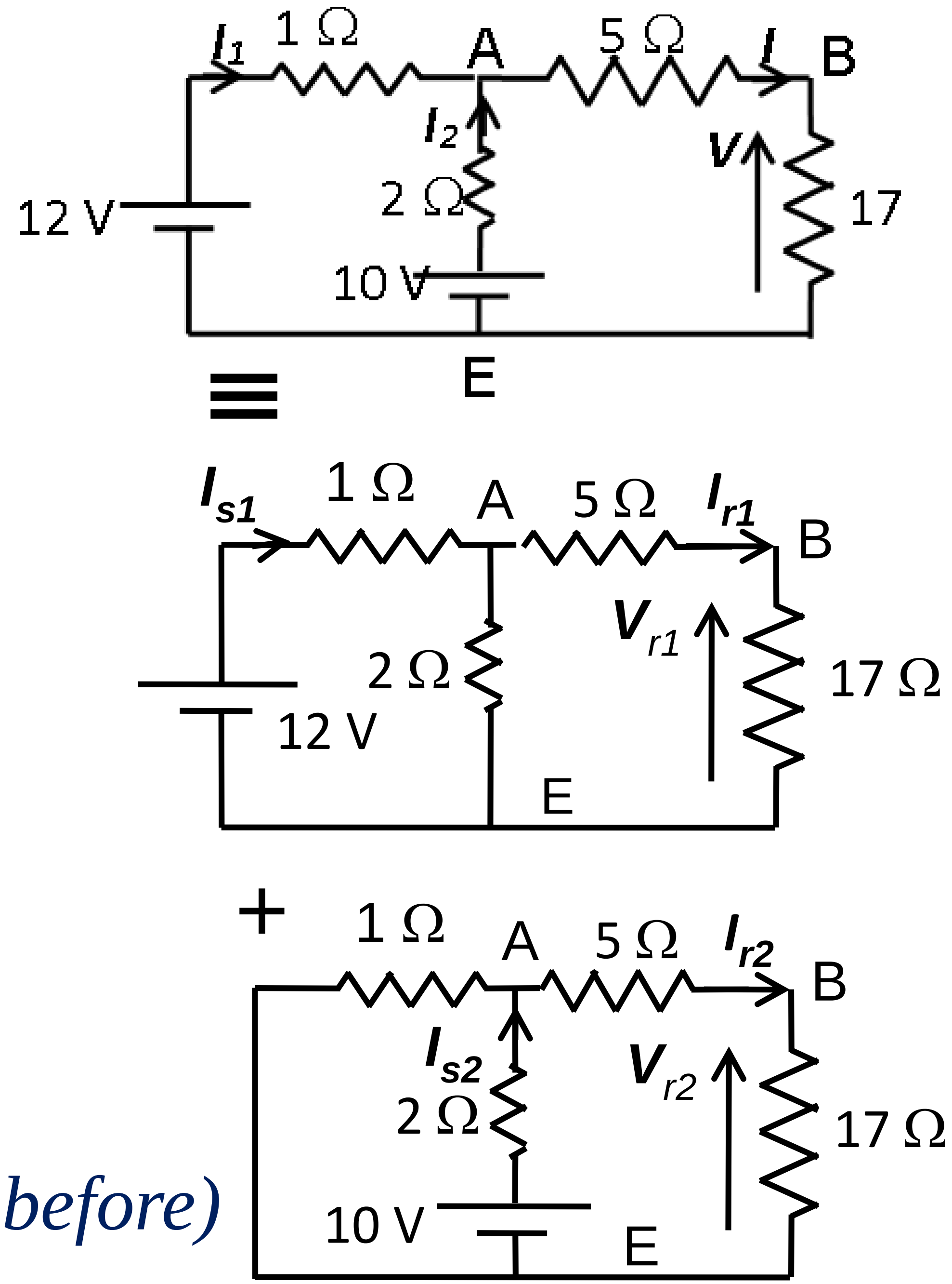
similarly,

$$I_{s2} = \frac{10}{2+1//(5+17)} = 3.38235\text{A},$$

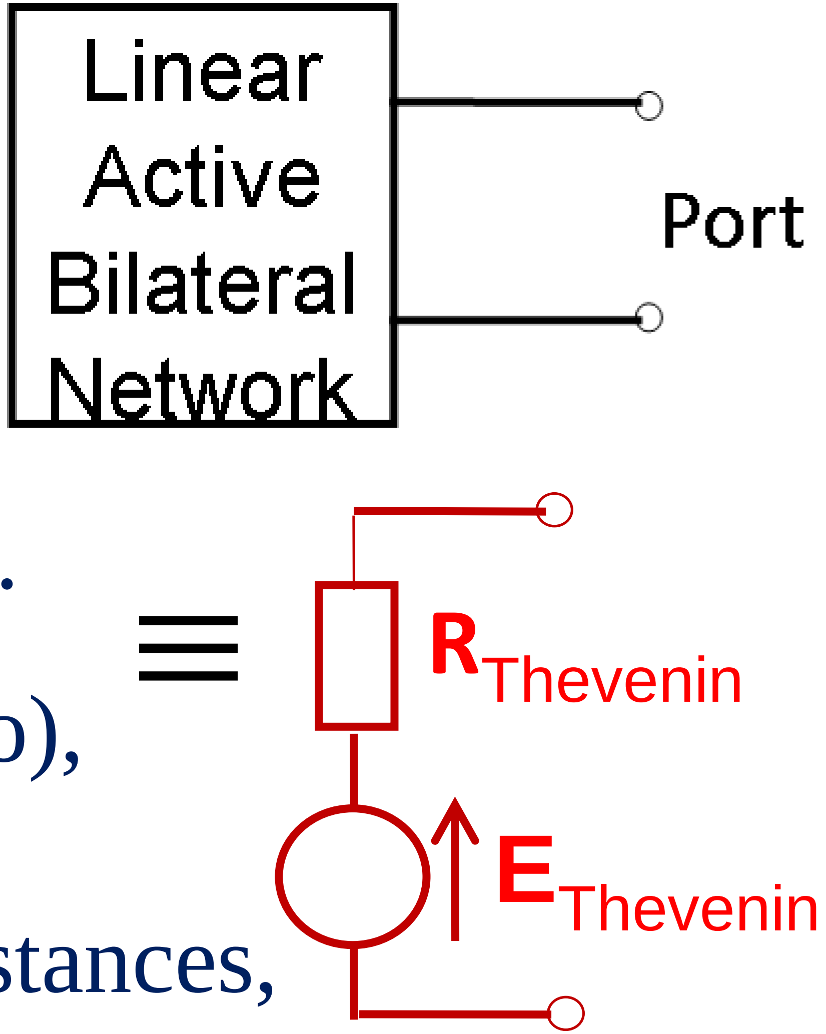
$$\therefore I_{r2} = \frac{1}{1+(5+17)} \times 3.38235 = 0.14706\text{A}$$

$\therefore$ from superposition theorem, $\boldsymbol{I = I_{r1} + I_{r2}}$

$= 0.35294 + 0.14706 = \mathbf{0.5\ A}$ *(same result as before)*



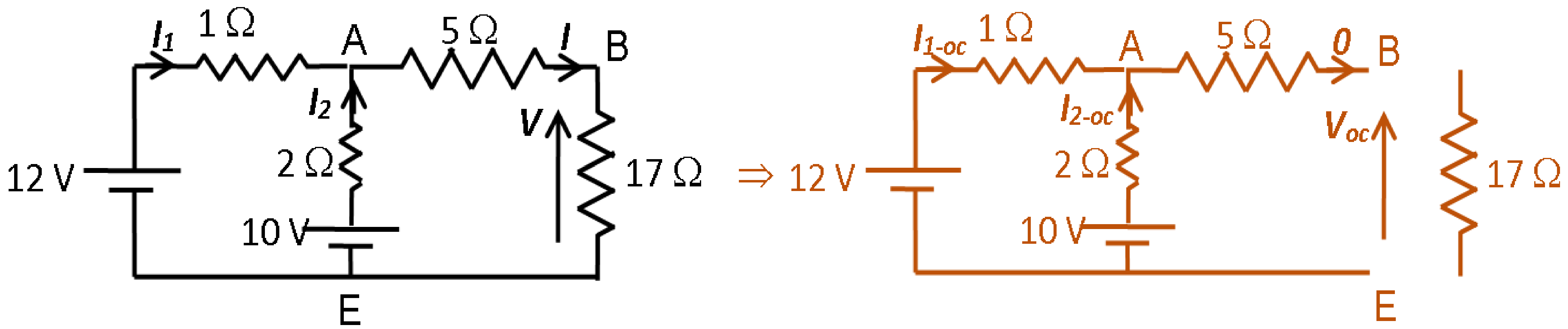
3.1.2 Thevenin's Theorem



Linear, active, bilateral, single-port network has an equivalent circuit having a voltage source $\mathbf{E}_{\text{thevenin}}$ and series resistance $\mathbf{R}_{\text{thevenin}}$ (or impedance $\mathbf{Z}_{\text{thevenin}}$).

If port is kept on open circuit (current zero),
then $\mathbf{E}_{\text{thevenin}} = V_{\text{open-circuit}}$
If all sources are replaced by internal resistances,
then $\mathbf{R}_{\text{thevenin}} = R_{\text{input}}$
= Resistance seen into port from outside

Example



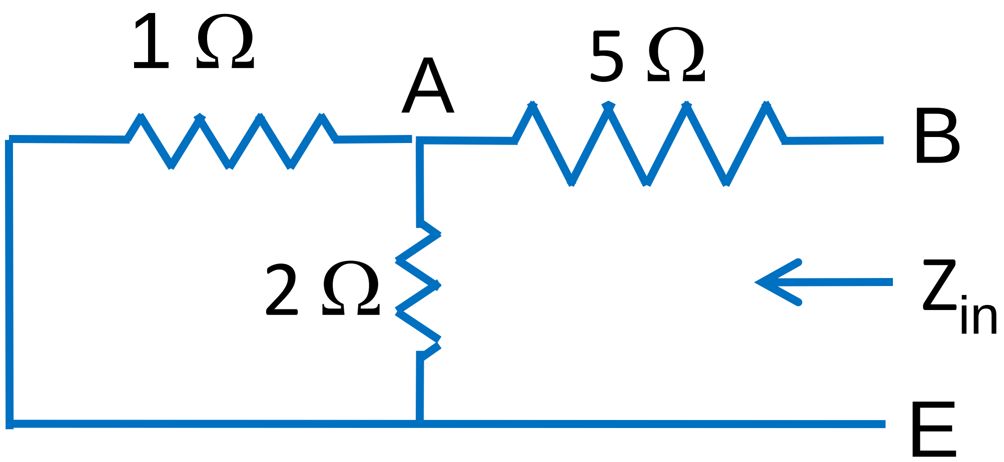
To determine unknown ***V*** and/or ***I*** at port BE

open circuit port BE (temporarily disconnect 17 Ω resistor).

Then, $\boldsymbol{I_{1\text{-}oc}} = -\,\boldsymbol{I_{2\text{-}oc}} = (12 - 10)/(1 + 2) = 0.66667$ A ,

and $\boldsymbol{V_{AE\text{-}oc}} = 12 - 1 \times 0.66667 = 11.33333$ V

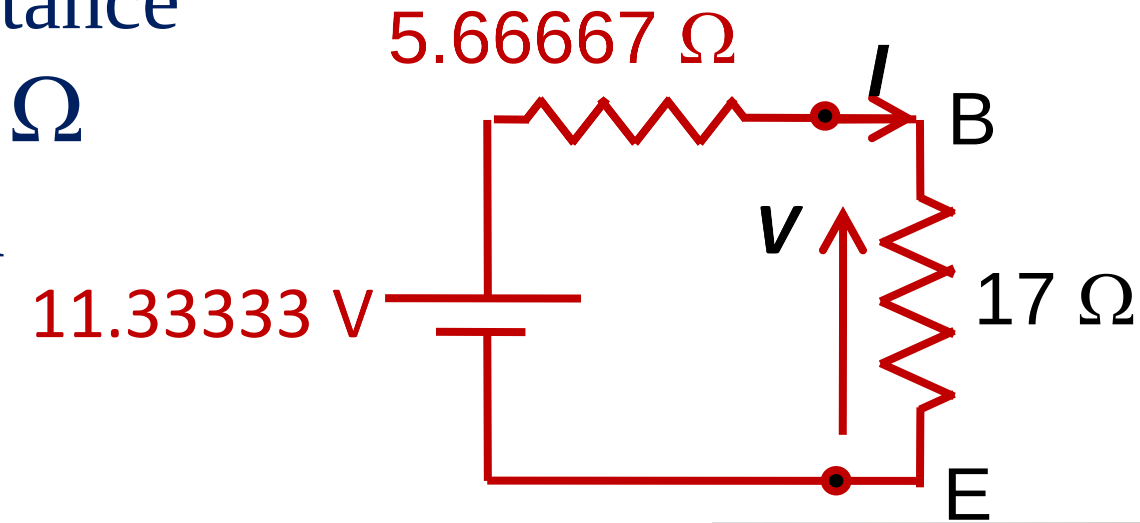
$\therefore\ \boldsymbol{V}_{\text{thevenin}} = \boldsymbol{V_{oc}} = 11.33333 - 5 \times 0 = 11.33333$ V



With sources replaced by internal resistance

$$Z_{thevenin} = \boldsymbol{Z_{in}} = 5 + 1//2 = 5.66667\ \Omega$$

∴ original circuit may be replaced by Thevenin's equivalent circuit



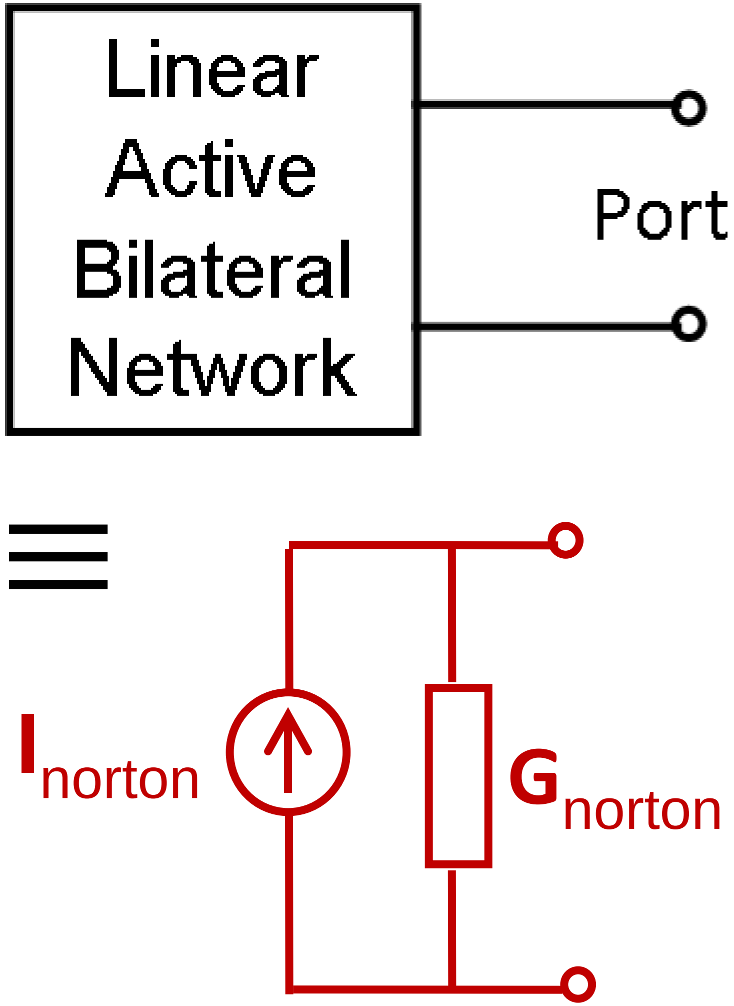
$$\therefore \boldsymbol{I} = 11.33333/(5.66667+17)$$

$= 0.4999998 =$ **0.5 A** *(same as before)*

and $\boldsymbol{V} = 17 \times 0.5 =$ **8.5 V** *(same as before)*

3.1.3 Norton's Theorem

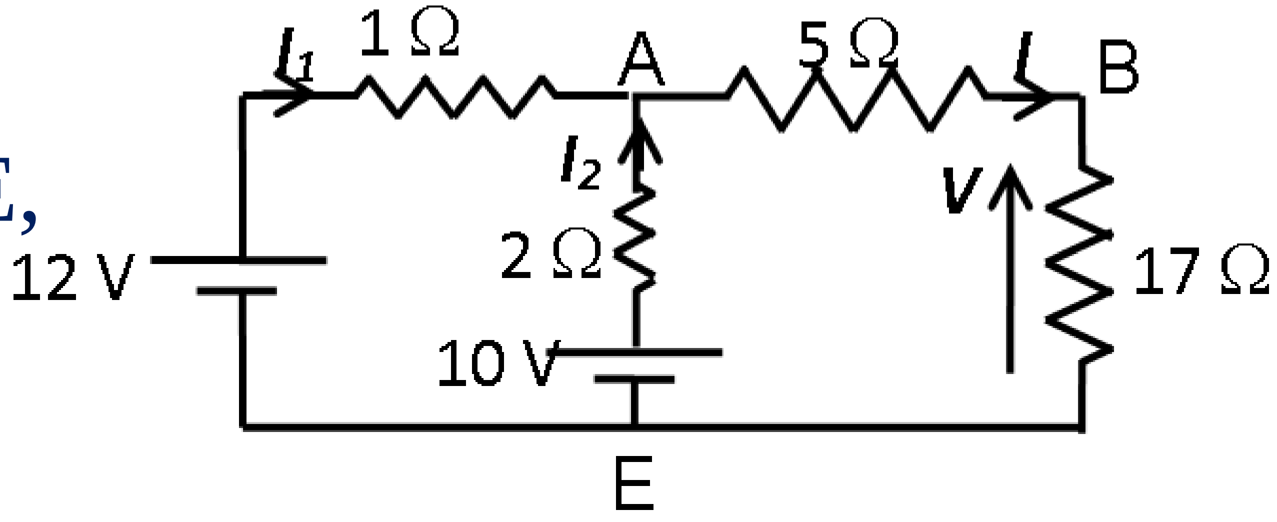
A linear active bilateral single-port network has an equivalent circuit having a current source $\mathbf{I}_{norton}$ and shunt conductance $\mathbf{G}_{norton}$ (or admittance $\mathbf{Y}_{norton}$). Norton's theorem is the dual theorem of Thevenin's theorem.



If port is kept on short circuit (voltage zero), then $\mathbf{I}_{norton} = I_{short\text{-}circuit}$ and

if all sources replaced by internal resistances then $\mathbf{Y}_{norton} = Y_{input}$, Note: $\mathbf{Y}_{input} = 1/Z_{input}$

Example



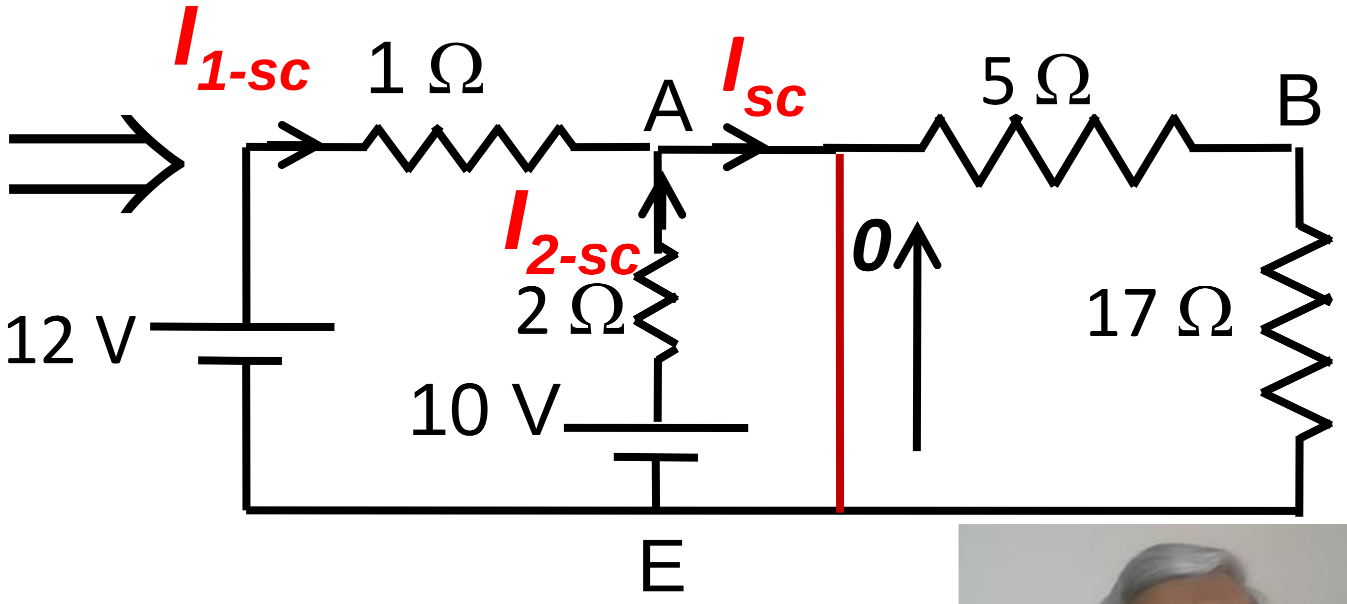
For equivalent circuit across AE, short circuit AE as shown.

Then $\boldsymbol{I_{sc} = I_{1\text{-}sc} + I_{2\text{-}sc}}$

and $I_{1\text{-sc}} = 12/1 = 12$ A,

$I_{2\text{-sc}} = 10/2 = 5$ A,

$\Rightarrow$

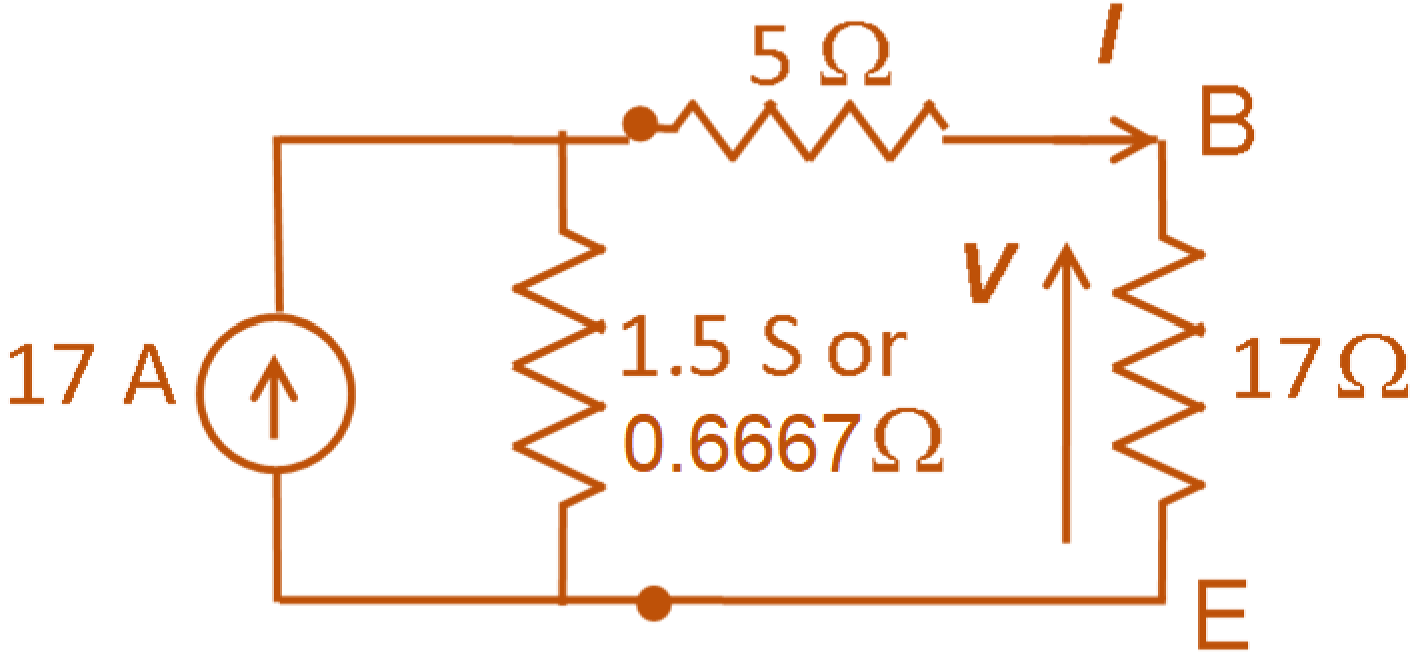


so that $I_{sc} = 12 + 5 = 17$ A

i.e. $I_{norton} = 17$ A.

- $G_{norton} = 1/2 + 1/1 = 1.5$ S

Norton's equivalent circuit is



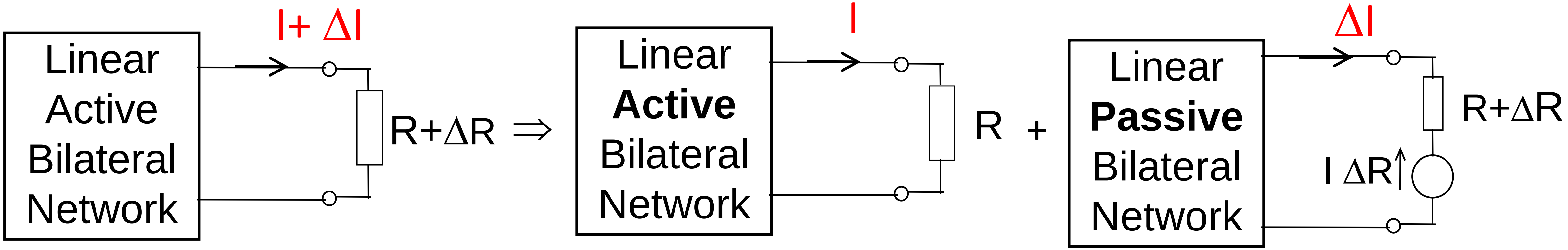
The current I is given by

$$17\times\frac{0.6667}{0.6667+5+17}=0.50002=0.5\,\text{A}$$

(same result as before)

3.1.4 Compensation Theorem

Theorem to find changes, when component is changed by small amount without recalculating full network.



In branch carrying a current ***I*****,** if impedance changed by ΔR, then change in all other currents obtained by inserting voltage source -IΔZ and replacing all sources with internal resistance.

Example

From earlier problem,

I = 0.5 A

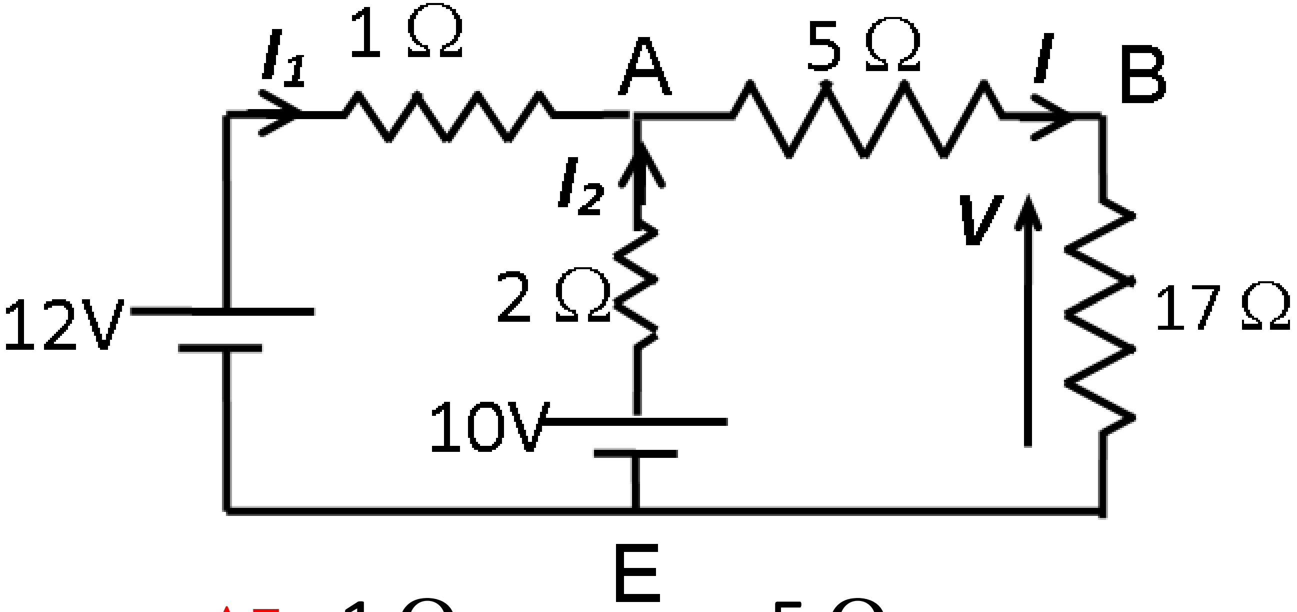
If 17 Ω resistor changed by a small amount 1 Ω to 18 Ω.

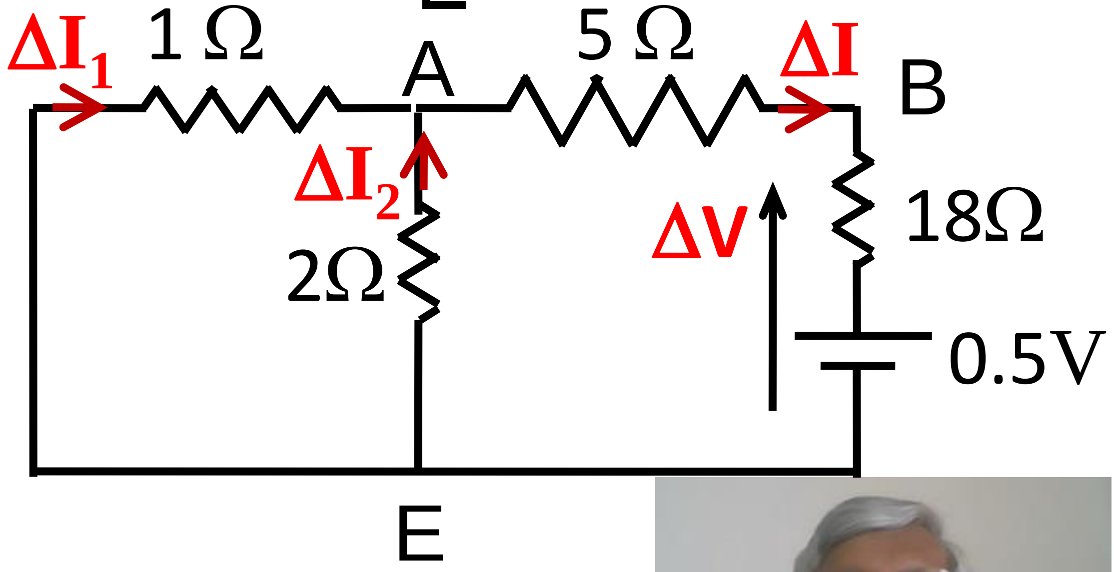
$\Delta R = +1\ \Omega$,

$-I\,\Delta R = -0.5 \times 1 = -0.5\ \text{V}$

$$\Delta I = -\frac{\mathbf{0.5}}{\mathbf{18 + 5 + 2 // 1}} = \frac{\mathbf{-0.5}}{\mathbf{23.6667}} = \mathbf{-0.02113}$$

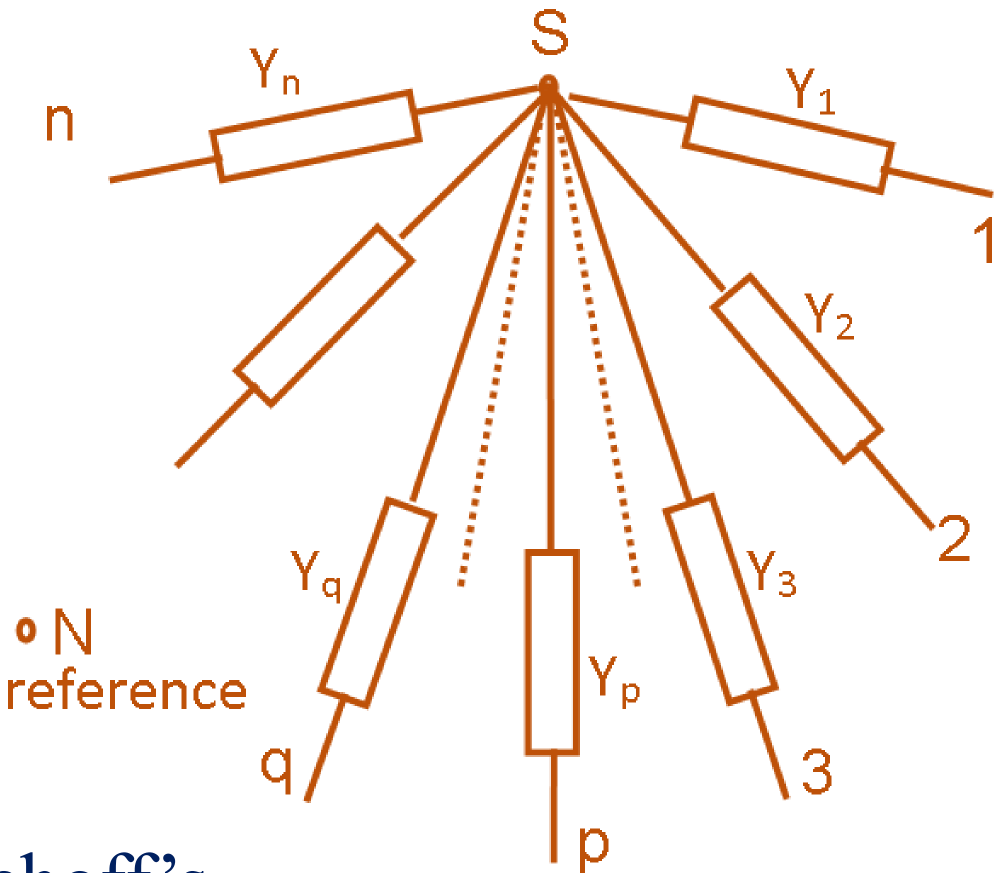
$\therefore\ I = 0.5 - 0.02113 = \mathbf{0.4789\ A}$





3.1.5 Millmann's Theorem

If Y_1, Y_2, Y_3 ….Y_p…. Y_q, …Y_n are connected at a common point S, and free-end voltages with respect to reference N are V_{1N}, V_{2N}, V_{3N} ….V_{pN}…. V_{qN}, …V_{nN}, then voltage of S with respect to N is



$$\boldsymbol{V_{SN}} = \frac{\sum_{p=1}^{n} \boldsymbol{Y_p V_{pN}}}{\sum_{p=1}^{n} \boldsymbol{Y_p}}$$

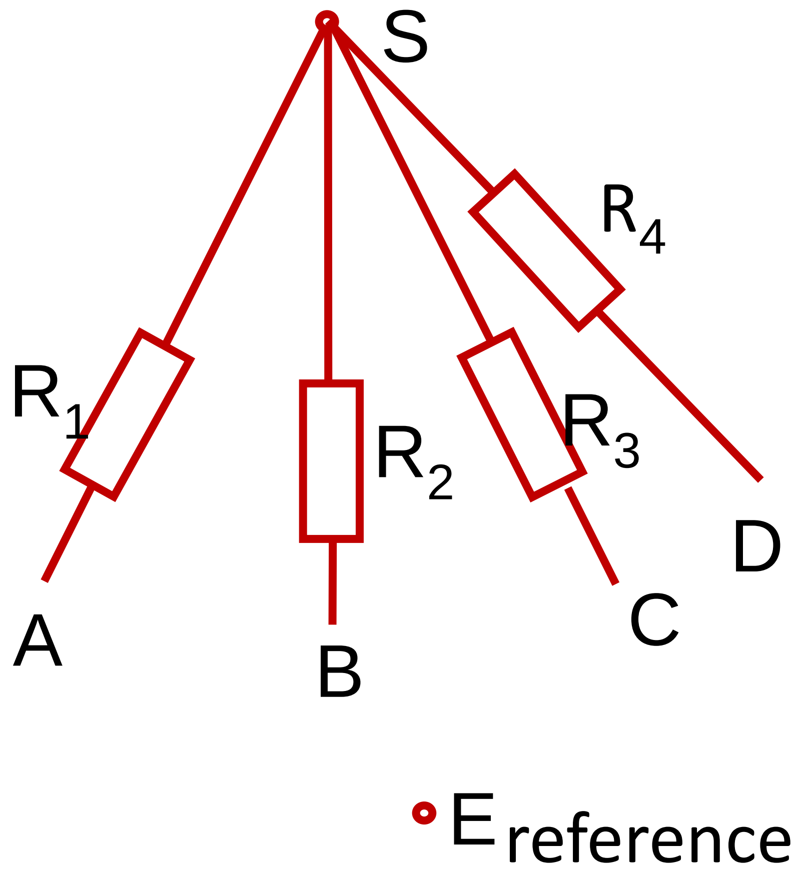
Proof is based on Kirchoff's Current Law at node S

$$\sum_{p=1}^{n} I_p = 0 \text{, and } I_p = Y_p \left(V_{pN} - V_{SN}\right)$$

Example

Four resistances, 2 Ω, 1 Ω, 4 Ω, and 2.5 Ω are connected in star at a common point S.

If potentials of other ends of with respect to earth E are V_{AE} = 100 V, V_{BE} = 80 V, V_{CE} = 60 V and V_{DE} =120 V, find the potential V_{SE} of the star point S with respect to earth E.

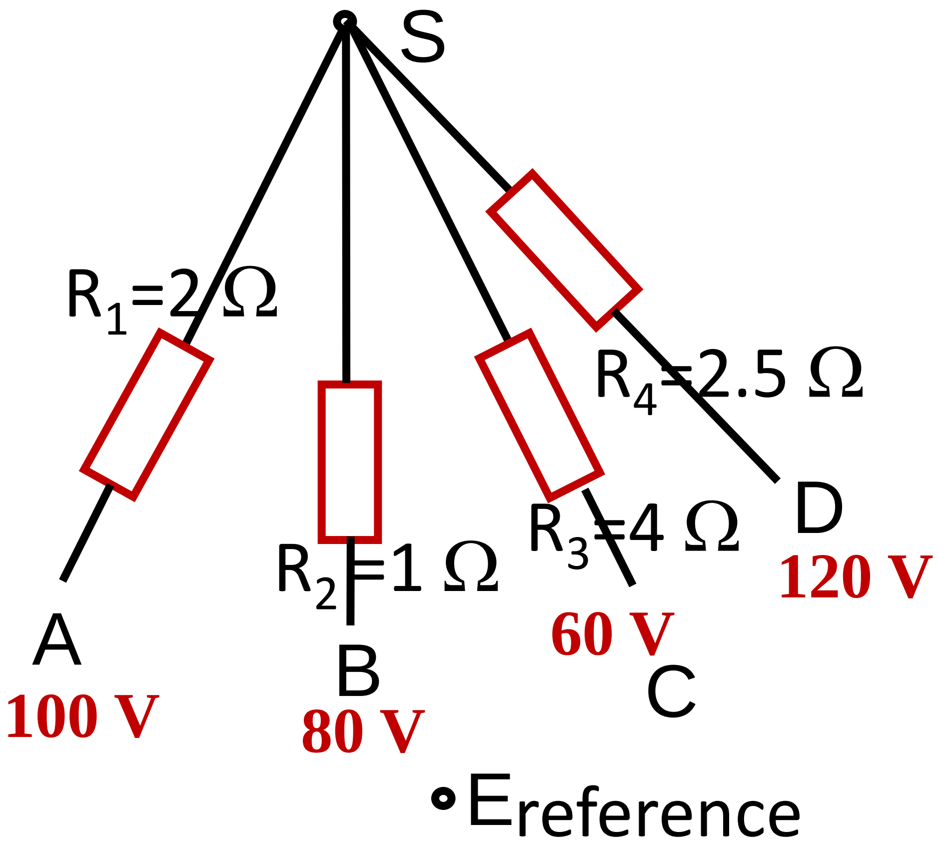


Solution

Using Millmann's theorem,

$$V_{SE} = \frac{\frac{1}{2} \times 100 + \frac{1}{1} \times 80 + \frac{1}{4} \times 60 + \frac{1}{2.5} \times 120}{\frac{1}{2} + \frac{1}{1} + \frac{1}{4} + \frac{1}{2.5}}$$

$$V_{SE} = \frac{50+80+15+48}{0.5+1+0.25+0.4} = \frac{193}{2.15} = 89.77\text{V}$$

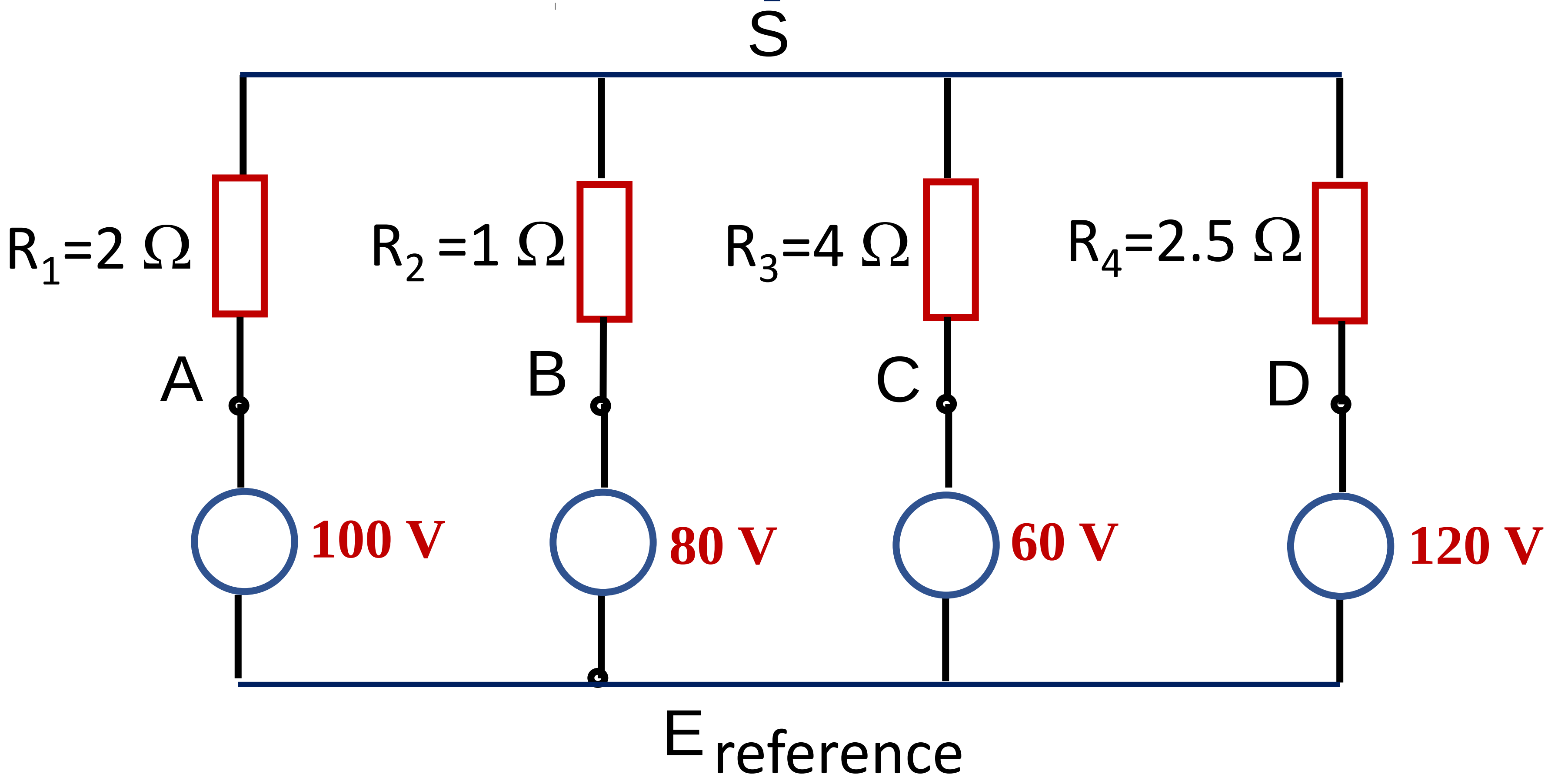


Note: Millmann's theorem can also be applied when a number of generators are connected in parallel to find the equivalent source.

Parallel Generator theorem

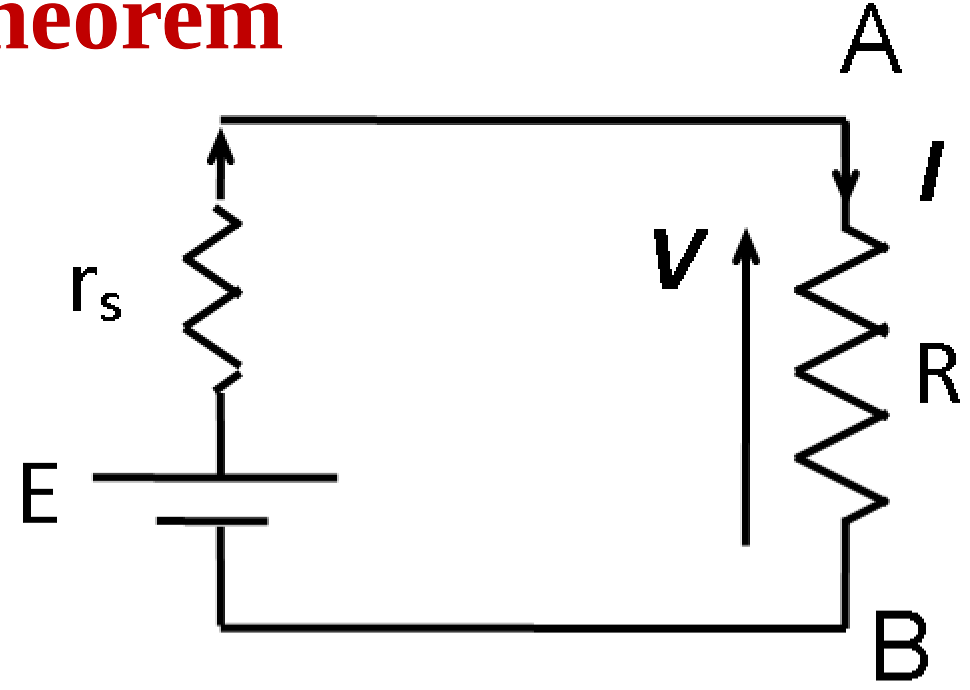
When a number of generators are connected in parallel, Millmann's theorem can be applied to find the equivalent source.

Let us consider the same example to illustrate this.



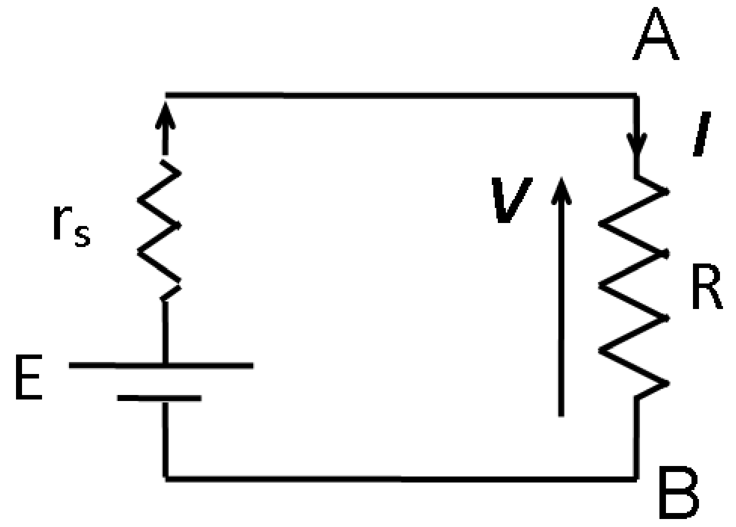
3.1.6 Maximum Power Transfer Theorem

Maximum power that can be supplied from a given source with internal resistance r_s to a purely resistive load R occurs when $R = r_s$.



[A different result occurs in the case of complex loads, where Load impedance becomes equal to conjugate of source impedance – outside scope of this course]

Proof:



Power delivered to load = $P = V.\ I$,

where $I = \dfrac{E}{r_s + R}$, $V = R.I$

$$P = R.I^2 = \frac{E^2}{(R + r_s)^2}.R$$, for maximum power transfer to load

$$\frac{dP}{dR} = 0 = \frac{E^2}{(R + r_s)^4}.\left((R + r_s)^2.1 - R.2(R + r_s)\right)$$

or $R + r_s - 2R = 0$, i.e. $R = r_s$

Under this condition, $V = ½\ E$ and $P = E^2/4r_s$

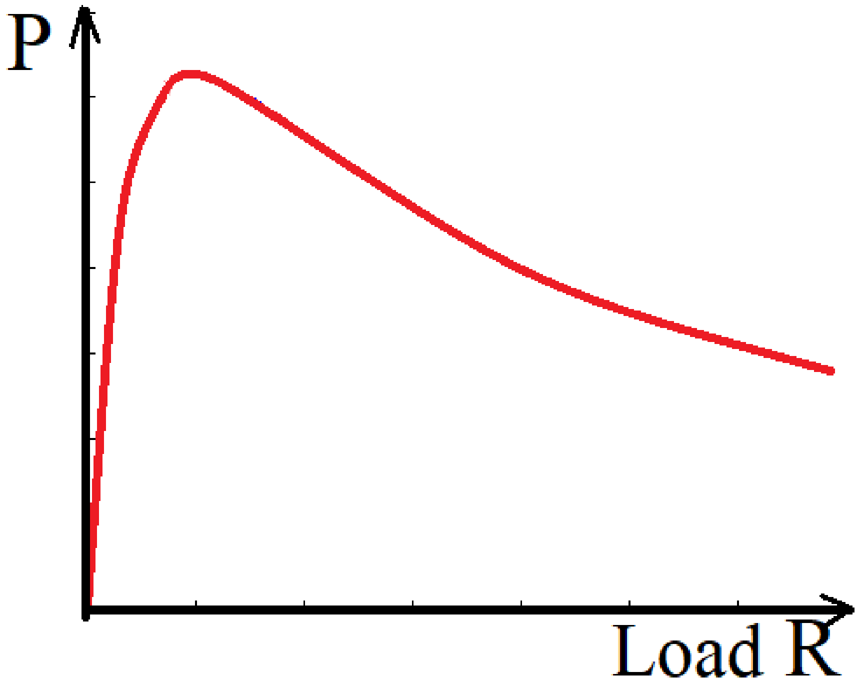
How do we know that this corresponds to maximum power.

We do not need second derivative, but can deduce from physical considerations.

We know that when $R = 0$ we have a short circuit (V=0) and no power is delivered to load (power from source and gets practically wasted)

When $R = \infty$ we have open circuit (I=0) and again no power will be delivered.

Therefore physical maximum magnitude must occur between these two values.



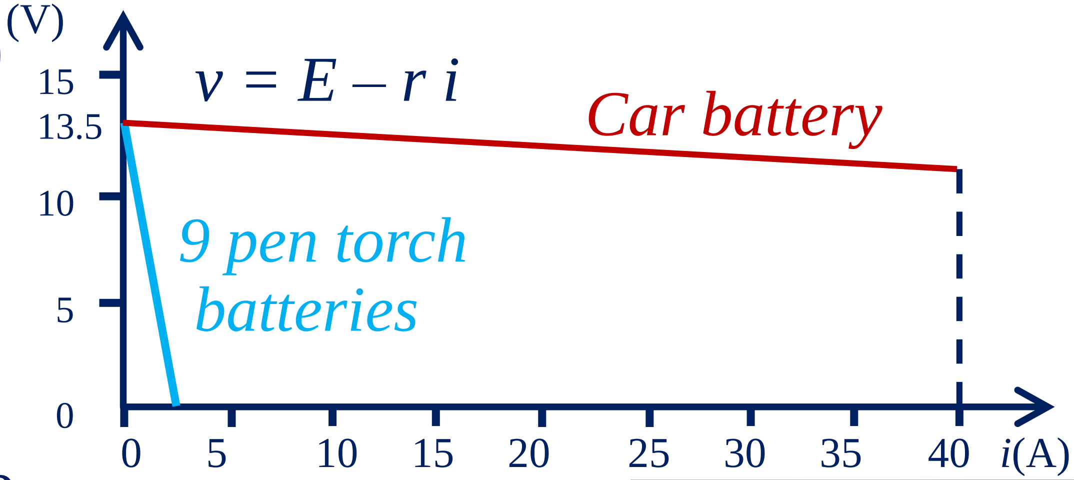
Example

A certain car battery rated at 12V has an internal resistance 0.015 Ω and an open circuit voltage of 13.5V. One morning you are unable to start your car as the battery is dead. You go and buy 9 pen-torch batteries, each with a voltage of 1.5V. Will it start? If not, Why?

Solution

To start a large current is required (say 40A)

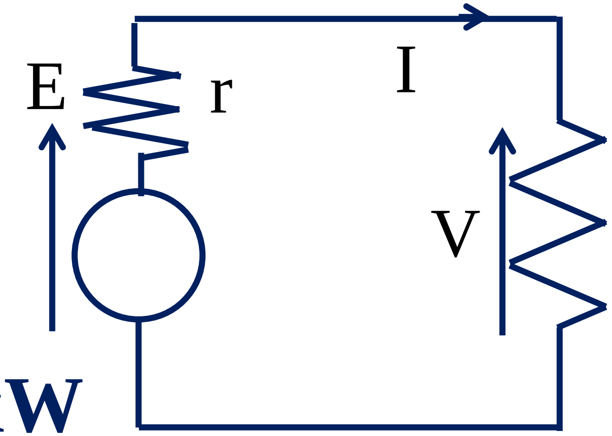
- Pen-torch batteries do not have required power.
  Have a high internal resistance (say 0.5Ω each)
  Voltage would drop to zero without starting
- car battery has low internal resistance (0.05Ω)
  would drop to around 11.5V which is acceptable.



Thus although the voltage is apparently the same, the maximum power deliverable is quite different.

Example

A certain car battery rated at 12V has an internal resistance 0.015 Ω and an open circuit voltage of 13.5V. Determine the maximum power that the battery can supply to a load. Determine also the voltage of load under this condition and the value of load resistance.



Solution

From maximum power transfer theorem,

Load resistance = source resistance = **0.015 Ω**

Maximum power transferred $=13.5^2/(4\times0.015)=$ **2.882kW**

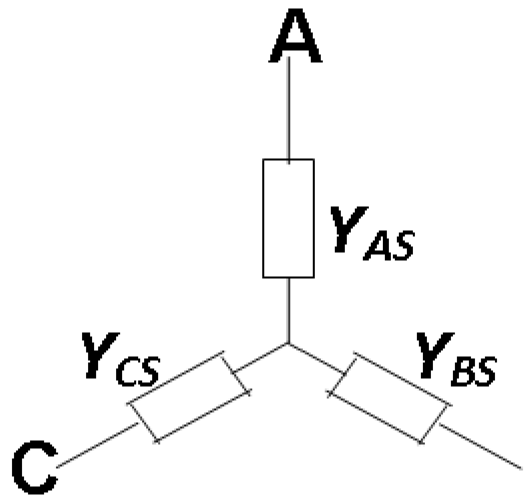
Voltage across load under these conditions =13.5/2 =**6.75V**

In practice, it is not acceptable for voltage to drop to as low as 6.75 V.

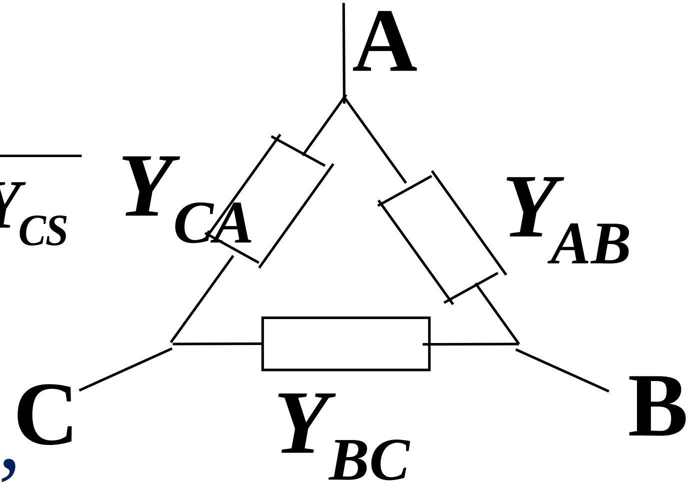
Practical maximum power would be significantly less than 2.882 kW, and would depend on minimum voltage required.

3.1.7 Star-Delta Transformation

Star connected network $\boldsymbol{Y_{AS}}$, $\boldsymbol{Y_{BS}}$, and $\boldsymbol{Y_{CS}}$ connected at S transformed into *delta connected* network of $\boldsymbol{Y_{AB}}$, $\boldsymbol{Y_{BC}}$, and $\boldsymbol{Y_{CA}}$

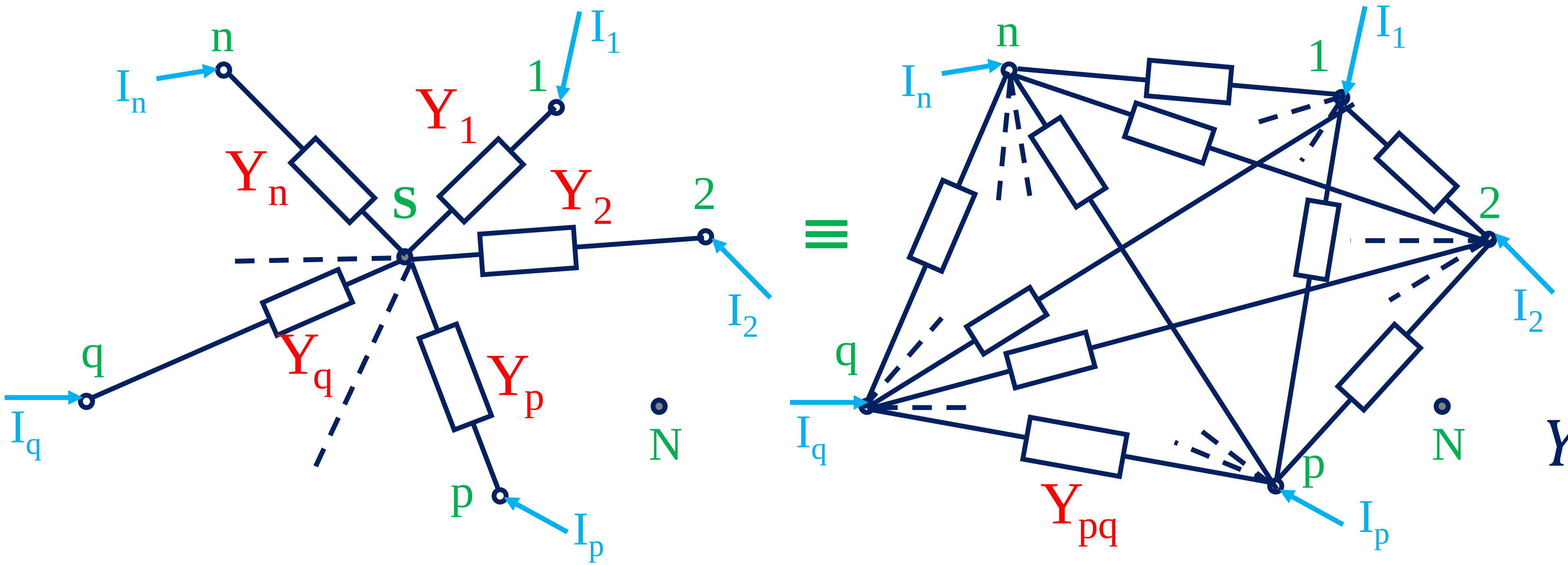


$$Y_{AB} = \frac{Y_{AS}.Y_{BS}}{Y_{AS}+Y_{BS}+Y_{CS}}, Y_{BC} = \frac{Y_{BS}.Y_{CS}}{Y_{AS}+Y_{BS}+Y_{CS}}, Y_{CA} = \frac{Y_{CS}.Y_{AS}}{Y_{AS}+Y_{BS}+Y_{CS}}$$



Note: To find a delta admittance element value, multiply the two values of admittance at nodes on either side in original star-network and divide by sum of three admittances.

Nodal Mesh Transformation theorem (Rosen's Theorem)



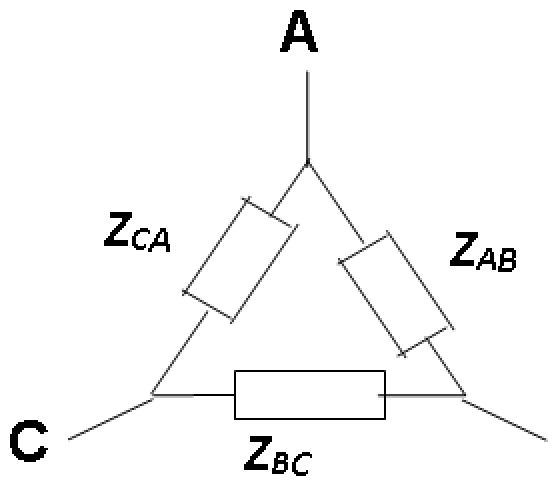
$$Y_{pq} = \frac{Y_{ps} . Y_{qs}}{\sum_{r=1}^{n} Y_{rs}}$$

The more general transformation between star connected network and corresponding mesh connected network with many more elements.

Star-Delta is a special case of this.

Delta-Star Transformation

A *delta connected* network of Z_{AB}, Z_{BC}, and Z_{CA} can be transformed into a *star connected* network of Z_{AS}, Z_{BS}, and Z_{CS} connected at common node S.



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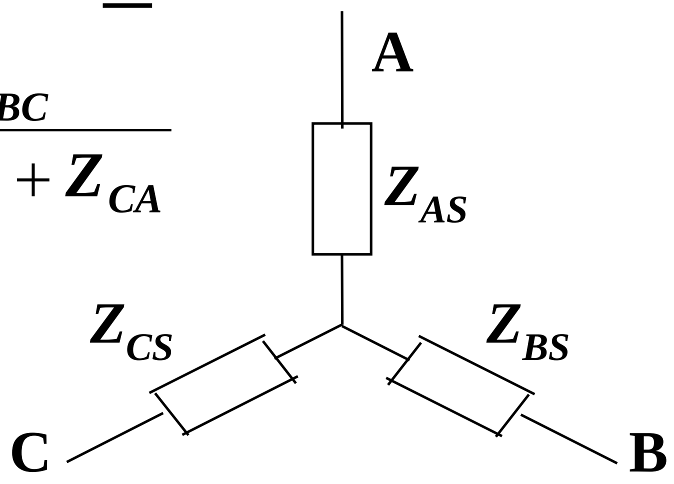


$$Z_{AS} = \frac{Z_{AB}.Z_{CA}}{Z_{AB} + Z_{BC} + Z_{CA}}, \quad Z_{BS} = \frac{Z_{AB}.Z_{BC}}{Z_{AB} + Z_{BC} + Z_{CA}}, \quad Z_{CS} = \frac{Z_{CA}.Z_{BC}}{Z_{AB} + Z_{BC} + Z_{CA}}$$

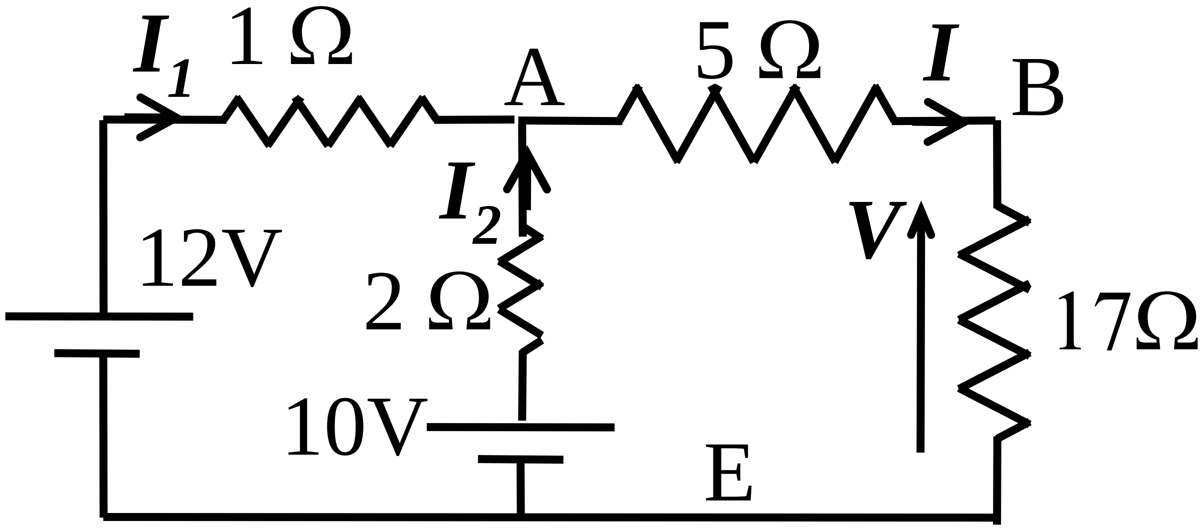
Note: Observe that, the two impedance values on either side of node in the original star-network have to be multiplied and then divided by the sum of the three impedances.

3.2 Introduction to Mesh and Nodal Analysis

Easy to analyse given network, with lesser number of variables and corresponding number of equations.
Work with either currents only or voltages only.

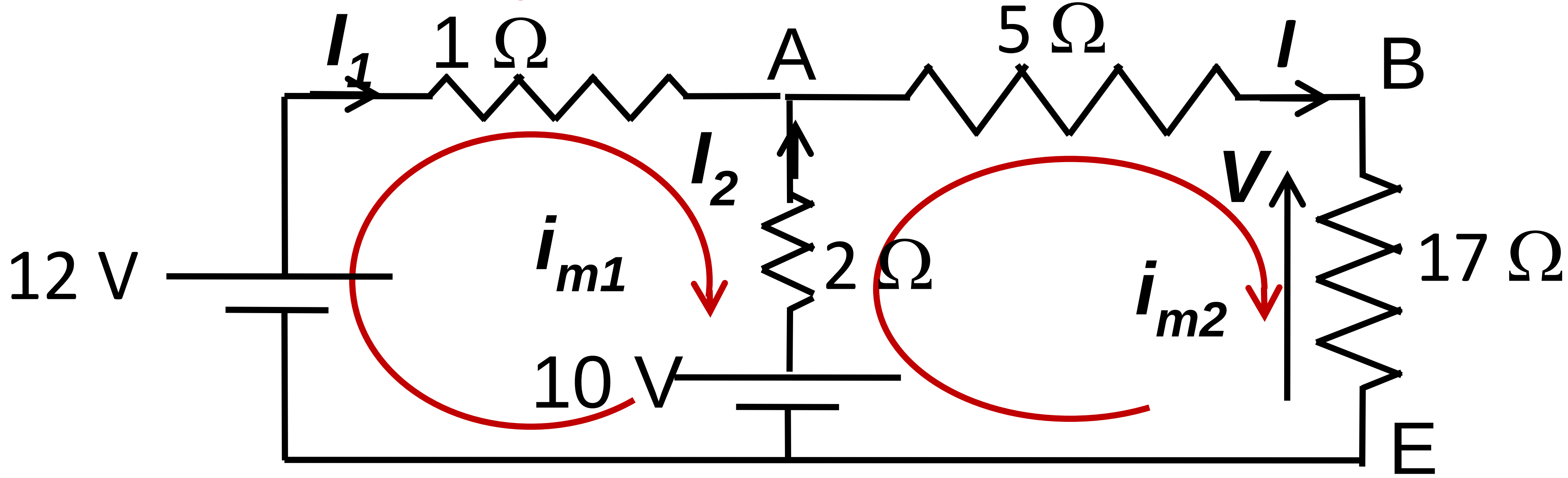
Usual analysis marks only two independent currents I_1 and I_2.
From Kirchoff's current law

$$I = I_1 + I_2$$



Then apply Kirchoff's voltage law equation for the 2 loops.

3.2.1 Mesh Analysis

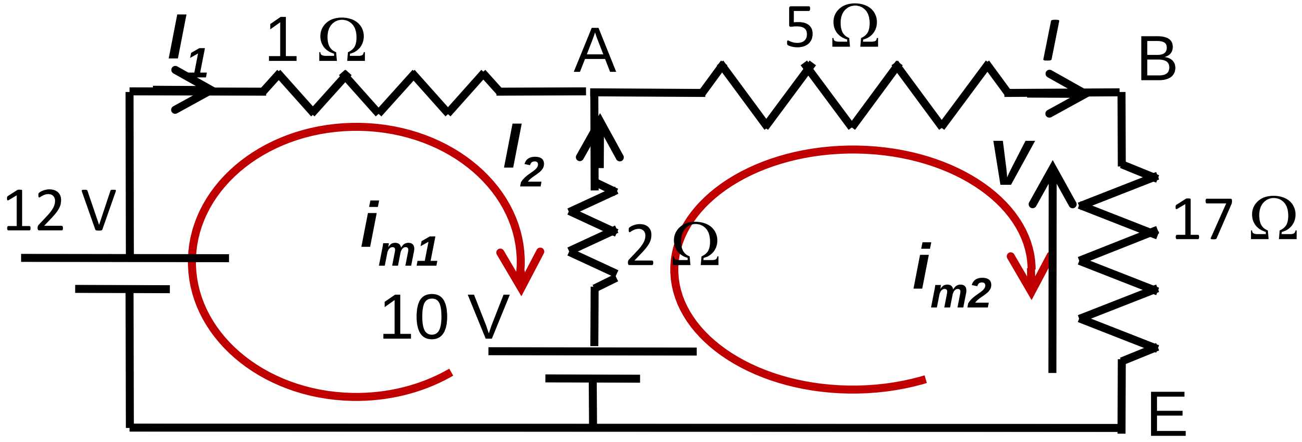


Marks currents differently as mesh currents i_{m1} and i_{m2}.

Branch currents can all be expressed in terms of mesh currents.

$$I_1 = i_{m1},$$
$$I_2 = -i_{m1} + i_{m2},$$
$$I = i_{m2}$$

Kirchoff's voltage law equations written with mesh currents



$$12 = 1\, i_{m1} + 2(i_{m1} - i_{m2}) + 10\,,$$

$$10 = -2(i_{m1} - i_{m2}) + 5\, i_{m2} + 17\, i_{m2}$$

Equations are solved in usual manner to give mesh currents.

$$3\, i_{m1} - 2\, i_{m2} = 2, \quad i_{m1} - 12\, i_{m2} = -5 \rightarrow \; i_{m1} = 1A, \; i_{m2} = 0.5A$$

Using the mesh currents, branch currents may be determined.

$$I_1 = i_{m1} = 1A,$$

$$I_2 = -i_{m1} + i_{m2} = -1 + 0.5 = -0.5A,$$

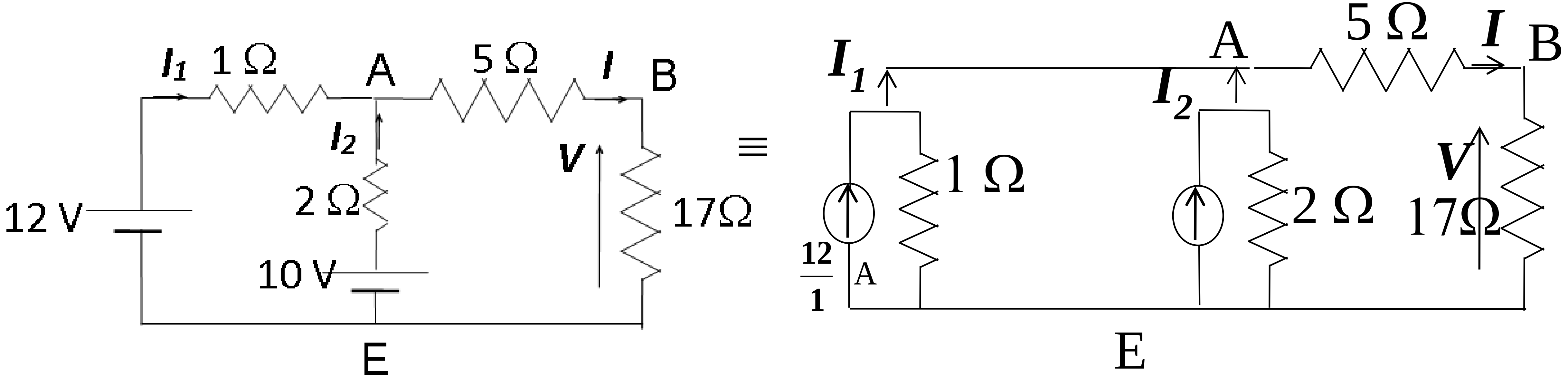
$$I = i_{m2} = 0.5A$$

3.2.2 Nodal Analysis

Basically work with a set of node voltages.

Voltage sources would be replaced by equivalent current sources.

Node E would usually be taken as reference with potential 0.



V_{AE} and V_{BE} are potentials of A and B with respect to E.

Using Ohm's law,

$$I_{1\Omega} = V_{AE}/1$$

$$I_{2\Omega} = V_{AE}/2$$

$$I_{5\Omega} = (V_{AE} - V_{BE})/5$$

$$I_{17\Omega} = V_{BE}/17$$

Applying Kirchoff's current law to node A and to node B,

$$12/1 - V_{AE}/1 + 10/2 - V_{AE}/2 - (V_{AE} - V_{BE})/5 = 0,$$

also $(V_{AE} - V_{BE})/5 = V_{BE}/17$

i.e. $17 - V_{AE} - V_{AE}/2 - V_{AE}/5 + V_{BE}/5 = 0,$

also $V_{AE}/5 - V_{BE}/5 = V_{BE}/17$

These equations are solved to give node voltages at A and B.

The branch currents can then be obtained.

Even node B can be avoided by taking branch (5+17) at A.

$12/1 - \boldsymbol{V_{AE}}/1 + 10/2 - \boldsymbol{V_{AE}}/2 - \boldsymbol{V_{AE}}/22 = 0,$

or $\boldsymbol{17 - V_{AE}(1 + 0.5 + 0.04545) = 0,}$

i.e. $\boldsymbol{V_{AE} = 11.0000\ V}$ (Same result as in first example)

$$\boldsymbol{V_{BE} = 11.0 \times 17/(17+5) = 8.5\ V}$$

$$\boldsymbol{I_{1\Omega} = 11.0/1 = 11.0\ A,}$$

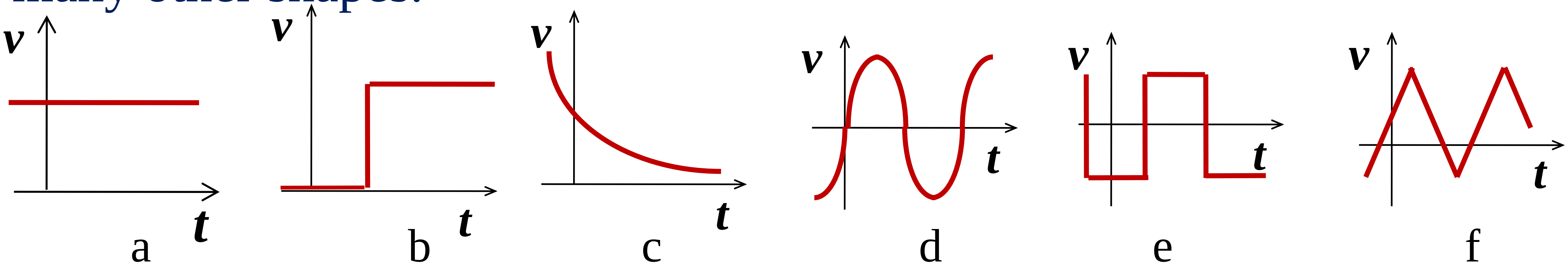
$$\boldsymbol{I_{2\Omega} = 11.0/2 = 5.5\ A}$$

$$\boldsymbol{I_{5\Omega} = (11.0 - 8.5)/5 = 0.5A,}$$

$$\boldsymbol{I_{17\Omega} = 8.5/17 = 0.5A}$$

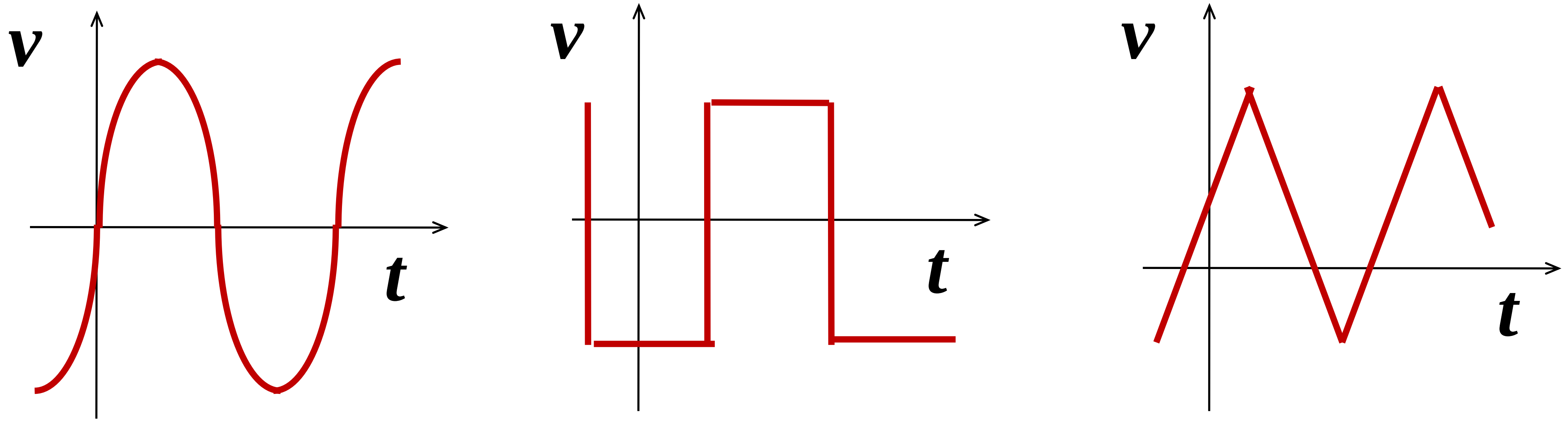
3.3 Introduction to Waveform Analysis

Waveforms of voltage and current can take various forms constant dc value (figure a), a step waveform (figure b), an exponentially decaying shape (figure c), sinusoidal waveform (figure d), a rectangular waveform (figure e), a triangular waveform (figure f) and many other shapes.



Waveforms a, b and c are unidirectional whereas d, e and f have positive and negative values. d, e and f are repetitive waveforms (periodic). d and e have mean values which are zero, whereas f has a positive mean value.

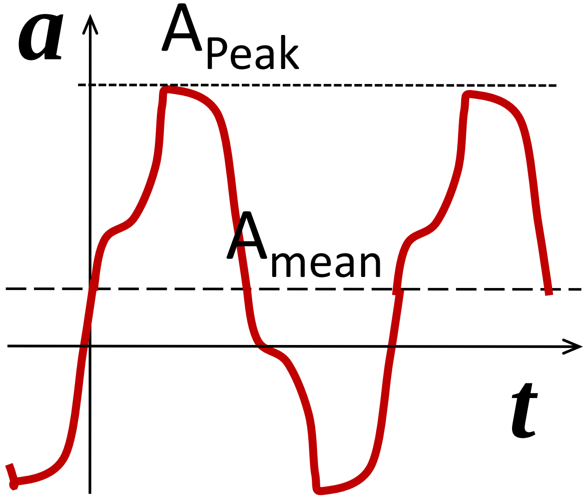
Repetitive waveforms



Repetitive waveforms can be represented by a combination of waveforms

- with mean value zero (alternating component)
- with a positive or negative mean value (direct component).

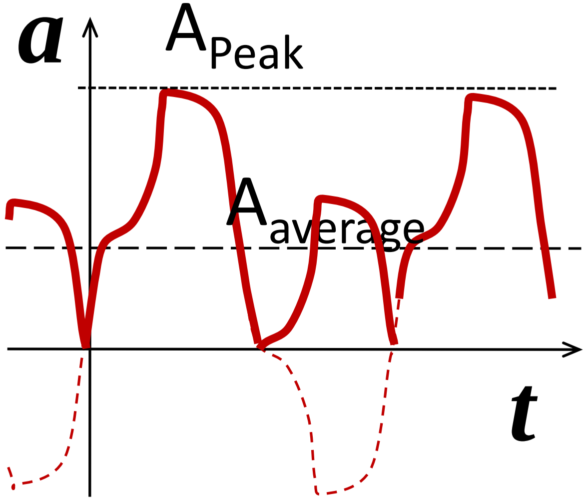
Magnitudes of Waveforms



Peak value of a waveform is not its useful value.

Mean value can be zero for a non-zero waveform.

$$I_{mean} = \frac{1}{T}\int_0^T i(t).dt$$



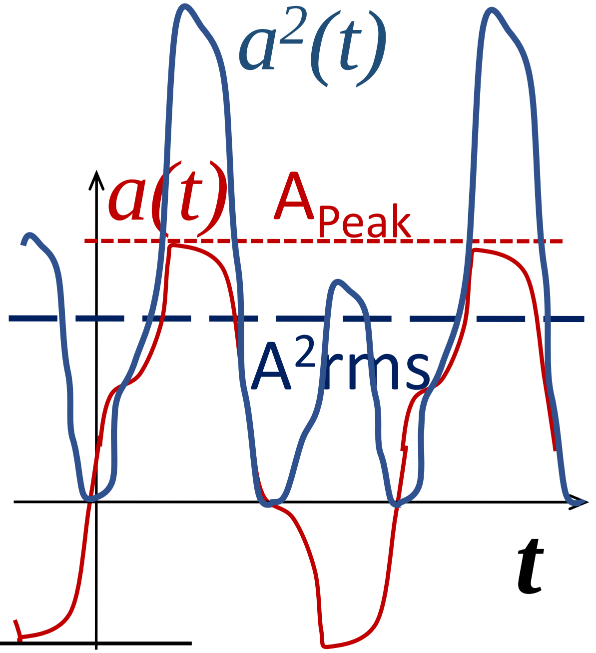
and $$I_{average} = \frac{1}{T}\int_{positive} i(t).dt - \frac{1}{T}\int_{negative} i(t).dt$$

Thus the mean value alone is not useful.

One method commonly used is to invert any negative part of waveform and obtain **average value** of **rectified** waveform.

Effective or rms value

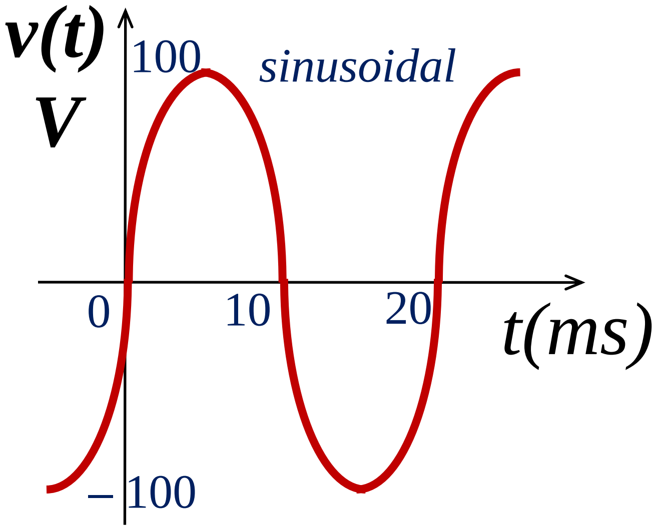
Useful value or **effective value** of an alternating waveform is value with which correct power is obtained. It is defined as ***Average value*** too is not fully indicative of useful value.

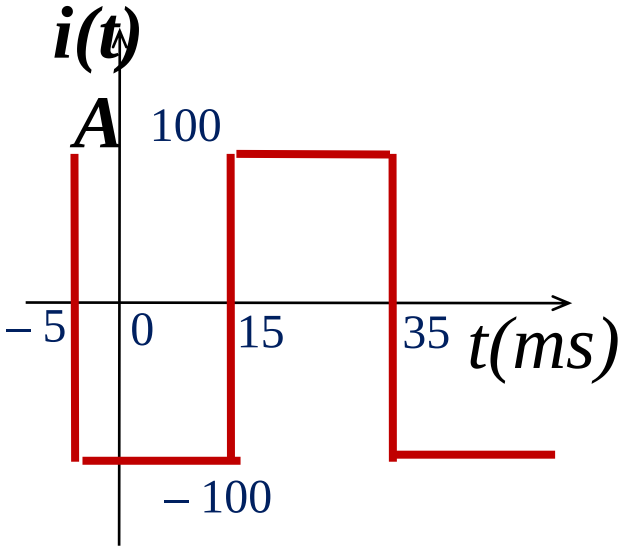


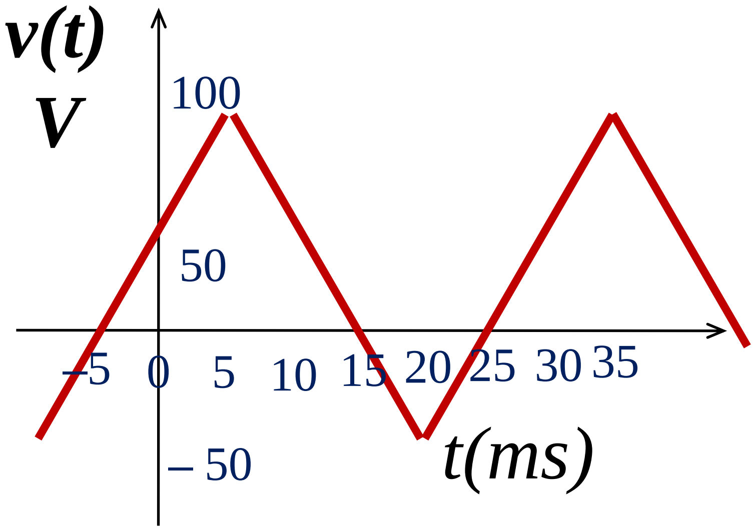
$$R.I_{eff}{}^{2}.T = \int_{0}^{T} R.i^{2}(t).dt \quad \text{or} \quad I_{eff} = \sqrt{\frac{1}{T}\int_{0}^{T} i^{2}(t).dt}$$

Effective value can be obtained by taking the square **root** of the **mean** of the **square**d waveform. Because of the method of obtaining this value, it is called the **root-mean-square** value or **rms** value.

Examples







Determine the peak, mean, average and rms values of the waveforms shown.

Solution

For the sinusoidal waveform shown, period = 20ms = 0.02s

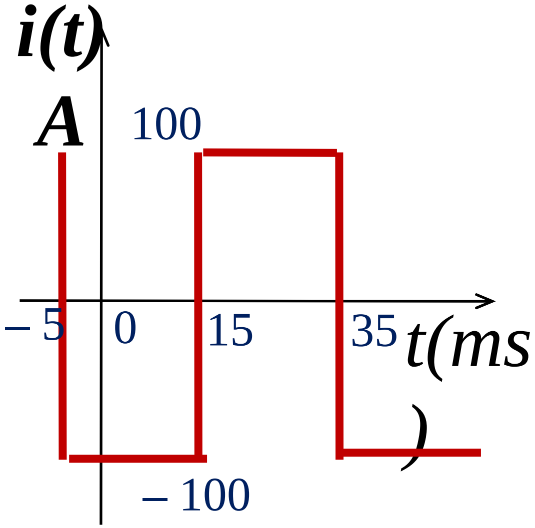
Peak value = 100V

Mean value = 0V

Average value = $(1/0.02) \int 100 \sin 314.16t \, dt = 63.66V$

rms value = $\sqrt{(1/0.02) \int 100^2 \sin^2 314.16t \, dt} = 70.71V$

For the rectangular waveform shown, period = 0.04s



Peak value = **100V**

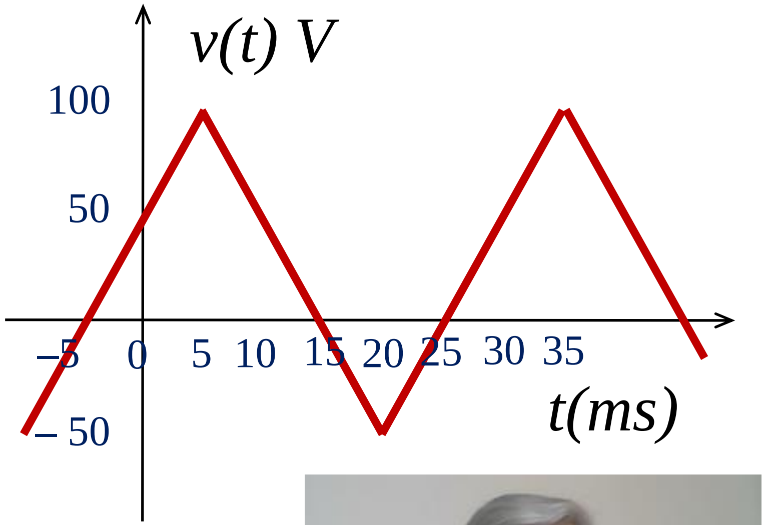
Mean value = **0V**

Average value = (1/0.04) [100×(0.015 – (-0.005)) + 100 ×0.02]

= **100V**

rms value = $\sqrt{(1/0.04)\,[100^2 \times 0.02 + 100^2 \times 0.02]}$ = **100V**

For the triangular waveform shown, period = 0.030s



Positive Peak value =100V, Negative Peak = **– 50V**

Mean value = (1/0.03) [½×100×0.02 – ½×50×0.01] = **25V**

Average value = (1/0.03) [½ 100×0.02 + ½×50×0.01]]

= **41.67V**

Using symmetry, considering half cycle from -10 to 5ms

rms value = $\sqrt{(1/0.015)\int(10000t+50)^2\,dt} = \sqrt{66.67\times(10000t+50)^3/30000}$

= $\sqrt{[2.222\times\{(10)^3 - (-5)^3\}]}$V = **50V**

END OF PRESENTATION