Note

* If A is an $m \times n$ matrix, $m = n$ then A is called a square matrix.

* An $m \times n$ matrix with all components equals to zero is called the $m \times n$ zero matrix.

Def^n^ (Equality of Matrices).

Two matrices A & B are equal if

* they have same size and
* corresponding components are equal.

i.e. $A = (a_{ij})_{m \times n}$, $B = (b_{ij})_{m \times n}$.

Then $A = B$ if

$a_{ij} = b_{ij}$, for all i, j.

Matrix Operation

Addition of Matrices

Let $A = (a_{ij})$ & $B = (b_{ij})$ be two $m \times n$ matrices. Then the sum of A & B is the $m \times n$ matrix $A+B$ given by

$$A+B = (a_{ij} + b_{ij})_{m \times n}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & & & \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}_{m \times n}.$$

(*) ~~addition~~ add the corresponding components of A & B.

Multiplication of a Matrix by a Scalar

If $A = (a_{ij})_{m \times n}$ is a matrix & if λ is a scalar, then the $m \times n$ matrix λA is given by;

$$\lambda A = (\lambda a_{ij})_{m \times n}$$

$$= \begin{pmatrix} \lambda a_{11} & \lambda a_{12} & \cdots & \lambda a_{1n} \\ \lambda a_{21} & \lambda a_{22} & \cdots & \lambda a_{2n} \\ \vdots & & & \vdots \\ \lambda a_{m1} & \lambda a_{m2} & \cdots & \lambda a_{mn} \end{pmatrix}_{m \times n}$$

~~Product of Two Matrices~~

~~If $A = (a_{ij})_{m\times n}$ is a matrix & if λ is a scalar, then the $m\times n$ matrix~~

Product of Two Matrices

Let $A = (a_{ij})_{m\times n}$ matrix, and $B = (b_{ij})_{n\times p}$ matrix. Then the product of A & B is an $m\times p$ matrix $C = (c_{ij})_{m\times p}$.

where $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \dots + a_{in}b_{nj}$.

$$= \sum_{k=1}^{n} a_{ik}b_{kj}$$

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* If the number of colums of A is equal to the number of rows of B, then A & B are said to be <u>compatible under multiplication</u>.

<u>Ex</u>. If $A = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ & $B = \begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix}$

Then calculate AB & BA.

$$AB = \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}_{2\times 2} \begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix}_{2\times 2}$$

$$AB = \begin{pmatrix} 1.3 + 3.5 & 1(-2) + 3(6) \\ (-2)3 + 4.5 & (-2)(-2) + 4.6 \end{pmatrix}_{2\times 2} = \begin{pmatrix} 18 & 16 \\ 14 & 28 \end{pmatrix}_{2\times 2}.$$

$$BA = \begin{pmatrix} 3 & -2 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 1 \\ -7 & 39 \end{pmatrix}_{2\times 2}$$

Is $AB = BA$? No.

Note

(x) In generaly Matrix product do not commute.

i.e. $AB \neq BA$ in general.

Let A, B & C are matrices compatible under following operations and let α & β are scalars. Then,

(i) $A + B = B + A$ (Commutative under addition).

(ii) $(A+B)+C = A+(B+C)$ (associative under addition).

(iii) $\alpha(A+B) = \alpha A + \alpha B$ (distributive law for scalar multiplication).

(iv) $(\alpha+\beta)A = \alpha A + \beta A$

(v) $A(BC) = (AB)C$

(associative under multiplication)

(vi) $A(B+C) = AB + AC$

(left distributive law).

(vii) $(A+B)C = AC + BC$

(right distributive law).

(viii) $\alpha(\beta A) = \beta(\alpha A) = (\alpha\beta)A$

(ix) $\alpha(AB) = (\alpha A)B = A(\alpha B)$.

(x) $(\alpha A)(\beta B) = \alpha\beta AB$.

Transpose Matrix

Let $A = (a_{ij})_{m \times n}$ matrix. Then the transpose of A, written A^T, is the $n \times m$ matrix obtained by interchanging the rows and columns of A.

i.e. If $A = (a_{ij})_{m \times n}$, then

$$A^T = (a_{ji})_{n \times m}.$$

Propetries

(i) $(A^T)^T = A$

(ii) $(AB)^T = B^T A^T$

(iii) $(A+B)^T = A^T + B^T$

(iv) $(\alpha A)^T = \alpha A^T$, α - scalar.

Ex.

$$A = \begin{pmatrix} -2 & 3 & 5 \\ 1 & 2 & 3 \end{pmatrix}_{2\times 3}$$

$$A^T = \begin{pmatrix} 2 & 1 \\ 3 & 2 \\ 5 & 3 \end{pmatrix}_{3\times 2}$$

Identity Matrix

The identity matrix $I_{n\times n}$ is the $n\times n$ matrix with 1's down the main diagonal and 0's everywhere else. i.e.

$$I_{n\times n} = I_n = (a_{ij})_{n\times n} ;$$

where $a_{ij} = \begin{cases} 1 & ; \text{ if } i=j \\ 0 & ; \text{ if } i\neq j \end{cases}$

Note

Let A be any matrix. If A & I are compatible under multiplication, then

$$AI = IA = A.$$

Ex

$$A = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}_{2\times 3}, \quad I_{2\times 2} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2}.$$

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{2\times 2} \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}_{2\times 3}$$

$$IA = \begin{pmatrix} 2 & 3 & 2 \\ 1 & 2 & 3 \end{pmatrix}_{2\times 3} = A$$

The Inverse of a Matrix

Let A & B be $n \times n$ matrices.
Suppose that

$$AB = BA = I.$$

Then B is called the inverse of A and is written as A^{-1}.
We ~~have~~ then have

$$AA^{-1} = A^{-1}A = I.$$

Note.

(*) If A has a inverse, then A is said to be invertible.

(*) A square matrix that is not invertible is called singular.

* The inverse of a non-singular matrix is unique.

Theorem.

(i) If A & B are invertible $n \times n$ matrices. Then AB is invertible and

$$(AB)^{-1} = B^{-1} A^{-1}$$

(ii) Let A is invertible & $\lambda \neq 0$, then

$$(\lambda A)^{-1} = \frac{1}{\lambda} A^{-1}$$

(iii) If A is invertible, then A^T is invertible and

$$(A^T)^{-1} = (A^{-1})^T$$

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(iv) If A is invertible, then A^{-1} is also invertible and

$$(A^{-1})^{-1} = A$$

(v) If A is invertible, then A^n is invertible for $n = 1, 2, 3 - - -$.

$$(A^n)^{-1} = (A^{-1})^n.$$

<u>Ex</u>.

Let $A = \begin{pmatrix} 2 & -3 \\ -4 & 5 \end{pmatrix}$. Then

Compute A^{-1} if it exists.

$$\left(\begin{array}{cc|cc} 2 & 3 & 1 & 0 \\ -4 & 5 & 0 & 1 \end{array}\right) \xrightarrow{r_1 \to \frac{1}{2} r_1} \left(\begin{array}{cc|cc} 1 & -3/2 & 1/2 & 0 \\ -4 & 5 & 0 & 1 \end{array}\right)$$

$$\downarrow r_2 \to r_2 + 4r_1$$

$$\left(\begin{array}{cc|cc} 1 & -3/2 & 1/2 & 0 \\ 0 & 1 & -2 & -1 \end{array}\right) \xleftarrow{r_2 \to -r_2} \left(\begin{array}{cc|cc} 1 & -3/2 & 1/2 & 0 \\ 0 & -1 & 2 & 1 \end{array}\right)$$

$$\xrightarrow{r_1 \to r_1 + \frac{3}{2} r_2} \left(\begin{array}{cc|cc} 1 & 0 & -5/2 & -3/2 \\ 0 & 1 & -2 & -1 \end{array}\right)$$

$$\therefore A^{-1} = \begin{pmatrix} -5/2 & -3/2 \\ -2 & -1 \end{pmatrix}$$

Ex.

Let $A = \begin{pmatrix} 1 & 2 \\ -2 & -4 \end{pmatrix}$. Determine whether A is invertible and, if it is, calculate its inverse.

$$\left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ -2 & -4 & 0 & 1 \end{array}\right) \xrightarrow{r_2 \to r_2 + 2r_1} \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{array}\right)$$

$\Rightarrow A$ is not invertible.

Matrix representation of linear system

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mn} \end{pmatrix}_{m\times n}$$

coefficient matrix.

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}_{n\times 1}, \quad \underline{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}_{m\times 1}$$

Then matrix representation of above system is

$$A\underline{x} = \underline{b}$$

Note

* When A is a square matrix and A is invertible, using matrix operation we can solve linear system.

* If A is invertible, the system $A\underline{x} = \underline{b}$ has the unique solution $\underline{x} = A^{-1}\underline{b}$.

$$A\underline{x} = \underline{b}$$

$$A^{-1}A\underline{x} = A^{-1}\underline{b}$$

$$I\underline{x} = A^{-1}\underline{b}$$

$$\underline{x} = A^{-1}\underline{b}$$

Ex. Solve following system.

$$2x_1 + 4x_2 + 3x_3 = 6$$

$$x_2 - x_3 = -4$$

$$3x_1 + 5x_2 + 7x_3 = 7$$

Let $A = \begin{pmatrix} 2 & 4 & 3 \\ 0 & 1 & -1 \\ 3 & 5 & 7 \end{pmatrix}_{3\times 3}$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1}$, $\underline{b} = \begin{pmatrix} 6 \\ -4 \\ 7 \end{pmatrix}_{3\times 1}$

We can find $A^{-1} = \begin{pmatrix} 4 & -13/3 & -7/3 \\ -1 & 5/3 & 2/3 \\ -1 & 2/3 & 2/3 \end{pmatrix}$

Then $A\underline{x} = \underline{b}$

$$A^{-1}Ax = A^{-1}\underline{b}$$

$$Ix = A^{-1}\underline{b} \quad (\because A^{-1}A = I)$$

$$\underline{x} = A^{-1}\underline{b} \quad (\because I\underline{x} = \underline{x})$$

$$x = \begin{pmatrix} 4 & -13/3 & -7/3 \\ -1 & 5/3 & 2/3 \\ -1 & 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 6 \\ -4 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}_{3\times 1} = \begin{pmatrix} 25 \\ -8 \\ -4 \end{pmatrix}_{3\times 1} \Rightarrow \left\{ \begin{array}{l} x_1 = 25 \\ x_2 = -8 \\ x_3 = -4 \end{array} \right\}$$

Determinants

The determinant of a square matrix A, denoted by $\det A$ or $|A|$, is a scalar.

Let $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$, then

$$\det A = |A| = a_{11}a_{22} - a_{12}a_{21}$$

Ex

$$\begin{vmatrix} 1 & 3 \\ 5 & 2 \end{vmatrix} = 1(2) - 3(5) = 2 - 15 = -13$$

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Minor of A

Let A be an $n\times n$ matrix and M_{ij} be the $(n-1)\times(n-1)$ matrix obtained from A by deleting the ith row and jth column of A. M_{ij} is called the ijth minor of A.

Ex. If $A = \begin{pmatrix} 2 & -1 & 4 \\ 0 & 1 & 5 \\ 6 & 3 & -4 \end{pmatrix}$

find M_{13} & M_{32}.

$$M_{13} = \begin{pmatrix} 0 & 1 \\ 6 & 3 \end{pmatrix}_{2\times 2}$$

$$M_{32} = \begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}_{2\times 2}$$

Cofactor

Let A be an $(n \times n)$ matrix. ijth cofactor of A, denoted by A_{ij}, is given by,

$$A_{ij} = (-1)^{i+j} |M_{ij}|$$

i.e. the ijth cofactor of A is obtained by taking the determinant of the ijth minor and multiplying it by $(-1)^{i+j}$.

Ex

If $A = \begin{pmatrix} 1 & -3 & 5 & 6 \\ 2 & 4 & 0 & 3 \\ 1 & 5 & 9 & -2 \\ 4 & 0 & 2 & 7 \end{pmatrix}$

find A_{32} & A_{24}

$$A_{32} = (-1)^{3+2} |M_{32}|$$

$$= - \begin{vmatrix} 1 & 5 & 6 \\ 2 & 0 & 3 \\ 4 & 2 & 7 \end{vmatrix} = -8.$$

$$A_{24} = (-1)^{2+4} \begin{vmatrix} 1 & -3 & 3 \\ 1 & 5 & 9 \\ 4 & 0 & 2 \end{vmatrix} = -192$$

Defⁿ (Determinant)

Let A be an $n \times n$ matrix. Then the determinant of A is given by

$$\det(A) = |A|$$

$$= a_{11} A_{11} + a_{12} A_{12} + \dots + a_{1n} A_{1n}$$

$$|A| = \sum_{k=1}^{n} a_{1k} A_{1k}.$$

Note

- Above definition said as expanding by cofactors in the first row of A.

- We can calculate determinant of A expanding by cofactors in any row or column.