1.14 Integration

1.14.1 Area

We begin by attempting to solve the **area problem**:

Find the area of the region S that lies under the curve $y = f(x)$ from a to b.

This means that S, illustrated in Figure 33, is bounded by the graph of a continuous function f [where $f(x) \geq 0$], the vertical lines $x = a$ and $x = b$, and the $x-$axis.

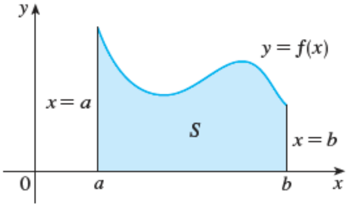


Figure 33 : $S = \{(x, y) \mid a \leq x \leq b, 0 \leq y \leq f(x)\}$

In trying to solve the area problem we have to ask ourselves: What is the meaning of the word area?

This question is easy to answer for regions with straight sides.

For a rectangle, the area is defined as the product of the length and the width.

The area of a triangle is half the base times the height.

The area of a polygon is found by dividing it into triangles (as in Figure 34) and adding the areas of the triangles.

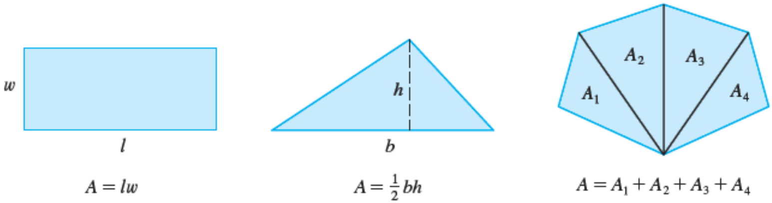


Figure 34

However, it isn't so easy to find the area of a region with curved sides.

We all have an intuitive idea of what the area of a region is.

But part of the area problem is to make this intuitive idea precise by giving an exact definition of area.

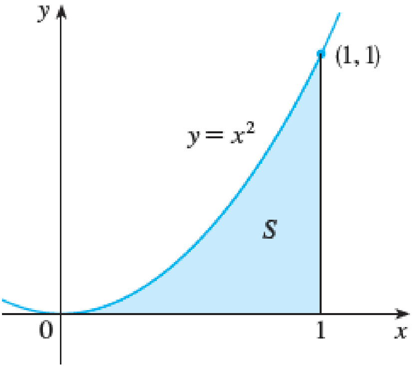
Recall that in defining a tangent we first approximated the slope of the tangent line by slopes of secant lines and then we took the limit of these approximations.

We pursue a similar idea for areas.

We first approximate the region S by rectangles and then we take the limit of the areas of these rectangles as we increase the number of rectangles.

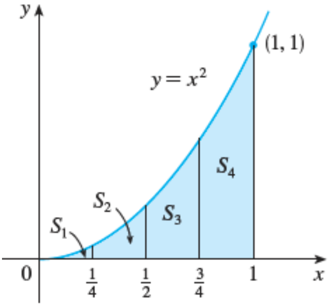
Illustrative Example

Use rectangles to estimate the area under the parabola $y = f(x)$ from 0 to 1.

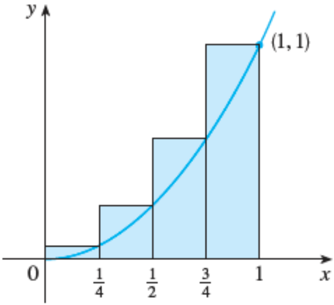


We first notice that the area of S must be somewhere between 0 and 1 because S is contained in a square with side length 1, but we can certainly do better than that.

Suppose we divide S into four strips S_1, S_2, S_3, and S_4 by drawing the vertical lines $x = \frac{1}{4}$, $x = \frac{1}{2}$, and $x = \frac{3}{4}$ as in Figure (a).



(a)



(b)

We can approximate each strip by a rectangle that has the same base as the strip and whose height is the same as the right edge of the strip [see Figure (b)].

In other words, the heights of these rectangles are the values of the function $f(x) = x^2$ at the right end -points of the sub intervals $\left[0, \frac{1}{4}\right]$, $\left[\frac{1}{4}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{3}{4}\right]$ and $\left[\frac{3}{4}, 1\right]$.

Each rectangle has width $\frac{1}{4}$ and the heights are $\left(\frac{1}{4}\right)^2$, $\left(\frac{1}{2}\right)^2$, $\left(\frac{3}{4}\right)^2$ and 1^2.

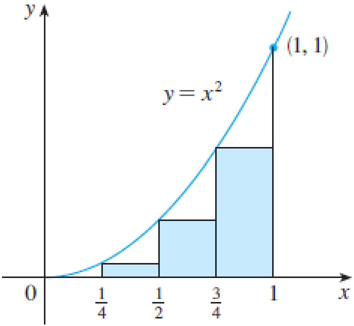
If we let R_4 be the sum of the areas of these approximating rectangles, we get

$$R_4 = \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{1}{2}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\cdot 1^2 = \frac{15}{32} = 0.46875$$

From Figure (b) we see that the area A of S is less than R_4, so

$$A < 0.46875$$

Instead of using the rectangles in Figure (b) we could use the smaller rectangles in Figure (c) whose heights are the values of f at the left endpoints of the sub intervals. (The leftmost rectangle has collapsed because its height is 0.)



(c)

The sum of the areas of these approximating rectangles is

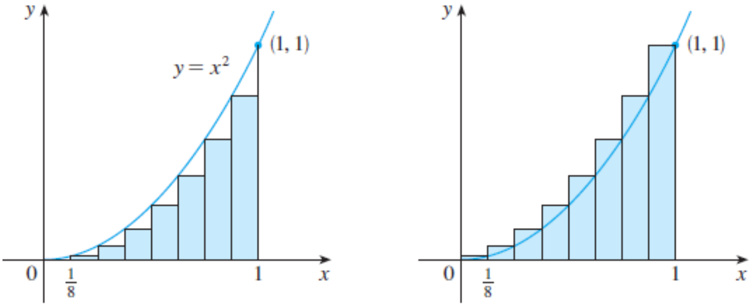
$$L_4 = \frac{1}{4} \cdot 0^2 + \frac{1}{4} \cdot \left(\frac{1}{4}\right)^2 + \frac{1}{4} \cdot \left(\frac{1}{2}\right)^2 + \frac{1}{4} \cdot \left(\frac{3}{4}\right)^2 = \frac{7}{32} = 0.21875$$

We see that the area of S is larger than L_4, so we have lower and upper estimates for A:

$$0.21875 < A < 0.46875$$

We can repeat this procedure with a larger number of strips.

Figure below shows what happens when we divide the region S into eight strips of equal width.



(a) Using left endpoints (b) Using right endpoints

Approximating S with eight rectangles

By computing the sum of the areas of the smaller rectangles (L_8) and the sum of the areas of the larger rectangles (R_8), we obtain better lower and upper estimates for A:

$$0.2734375 < A < 0.3984375$$

So one possible answer to the question is to say that the true area of S lies somewhere between 0.2734375 and 0.3984375.

We could obtain better estimates by increasing the number of strips.

The table shows the results of similar calculations (with a computer) using n rectangles whose heights are found with left endpoints (L_n) or right endpoints (R_8).

n	L_n	R_n
10	0.2850000	0.3850000
20	0.3087500	0.3587500
30	0.3168519	0.3501852
50	0.3234000	0.3434000
100	0.3283500	0.3383500
1000	0.3328335	0.3338335

By using 50 strips that the area lies between 0.3234 and 0.3434.

With 1000 strips we narrow it down even more: A lies between 0.3328335 and 0.3338335.

A good estimate is obtained by averaging these numbers:

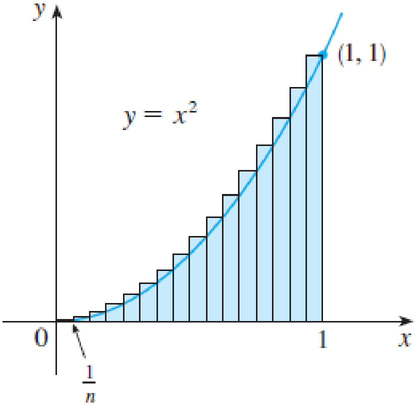
$$A \approx 0.3333335.$$

From the values in the table, it looks as if R_n is approaching $\frac{1}{3}$ as n increases.

Let's show

$$\lim_{n\to\infty} R_n = \frac{1}{3}.$$

R_n is the sum of the areas of the n rectangles in Figure.



Each rectangle has width $\frac{1}{n}$.

The heights are the values of the function $f(x) = x^2$ at the points $\frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \ldots, \frac{n}{n}$.

Then the heights are $\left(\frac{1}{n}\right)^2, \left(\frac{2}{n}\right)^2, \left(\frac{3}{n}\right)^2, \ldots, \left(\frac{n}{n}\right)^2$.

Thus

$$
\begin{aligned}
R_n =& \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \frac{1}{n}\left(\frac{3}{n}\right)^2 + \dots + \frac{1}{n}\left(\frac{n}{n}\right)^2 \\
=& \frac{1}{n} \cdot \frac{1}{n^2}\left(1^2 + 2^2 + 3^2 + \dots + n^2\right) \\
=& \frac{1}{n^3}\left(1^2 + 2^2 + 3^2 + \dots + n^2\right) \\
=& \frac{1}{n^3} \cdot \frac{n(n+1)(n+2)}{6} \\
=& \frac{(n+1)(n+2)}{6n^2}
\end{aligned}
$$

$$
\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty} \frac{(n+1)(n+2)}{6n^2} \\
&= \lim_{n\to\infty} \frac{1}{6}\left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right) \\
&= \lim_{n\to\infty} \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right) \\
&= \frac{1}{6}\cdot 1\cdot 2 = \frac{1}{3}
\end{aligned}
$$

Similarly, the lower approximating sums also approach $\frac{1}{3}$, that is,

$$\lim_{n\to\infty} L_n = \frac{1}{3}$$

As n increases, both L_n and R_n become better and better approximations to the area of S.

Therefore we define the area A to be the limit of the sums of the areas of the approximating rectangles, that is,

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} L_n = \frac{1}{3}$$

Let's apply the idea of Illustrative example to the more general region S.

We start by subdividing S into n strips $S_1, S_2, \ldots, S_n$ of equal width as in Figure 35.

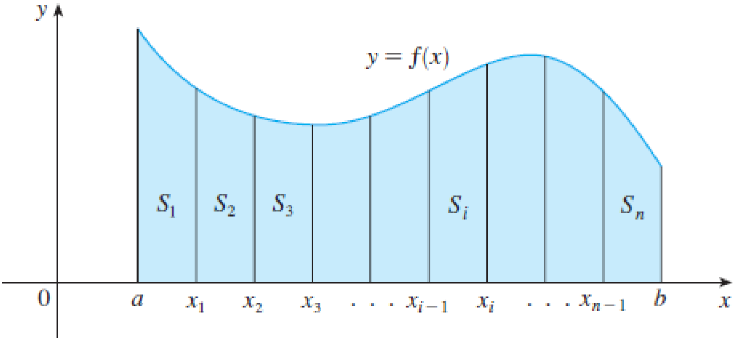


Figure 35

The width of the interval $[a, b]$ is $b - a$, so the width of each of the n strips is

$$\Delta x = \frac{b-a}{n}$$

These strips divide the interval $[a, b]$ into n subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots, [x_{n-1}, x_n]$$

where $x_0 = a$ and $x_n = b$.

The right endpoints of the subintervals are

$$x_1 = a + \Delta x$$

$$x_2 = a + 2\Delta x$$

$$x_3 = a + 3\Delta x$$

$$\vdots$$

Let's approximate the i^{th} strip S_i by a rectangle with width Δx and height $f(x_i)$, which is the value of f at the right endpoint (see Figure 36).

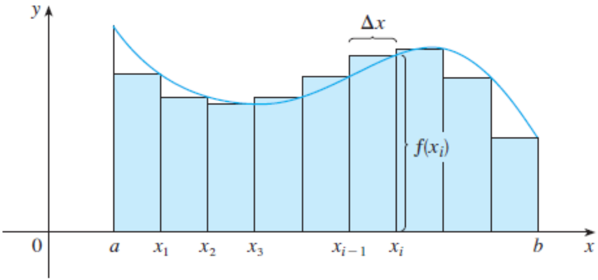


Figure 36

Then the area of the i^{th} rectangle is $f(x_i)\Delta x$.

What we think of intuitively as the area of S is approximated by the sum of the areas of these rectangles, which is

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + ... + f(x_n)\Delta x$$

Notice that this approximation appears to become better and better as the number of strips increases, that is, as $n \to \infty$.

Therefore we define the area A of the region S in the following way.

Definition

The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$
\begin{aligned}
A &= \lim_{n\to\infty} R_n \\
&= \lim_{n\to\infty} [f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x + \ldots + f(x_n)\Delta x]
\end{aligned}
$$

It can also be shown that we get the same value if we use left endpoints:

$$
\begin{aligned}
A &= \lim_{n\to\infty} L_n \\
&= \lim_{n\to\infty} [f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x + \ldots + f(x_{n-1})\Delta x]
\end{aligned}
$$

In fact, instead of using left endpoints or right endpoints, we could take the height of the i^{th} rectangle to be the value of f at any number x_i^* in the i^{th} subinterval $[x_{i-1}, x_i]$.

We call the numbers x_1^*, x_2^*, . . . , x_n^* the sample points.

Figure 37 shows approximating rectangles when the sample points are not chosen to be endpoints.

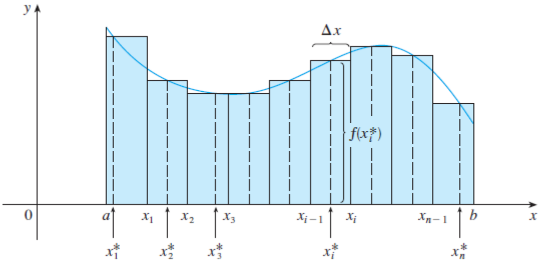


Figure 37

So a more general expression for the area of S is

$$A = \lim_{n \to \infty} [f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + ... + f(x_n^*)\Delta x]$$

We use sigma notation to write sums with many terms more compactly.

For instance,

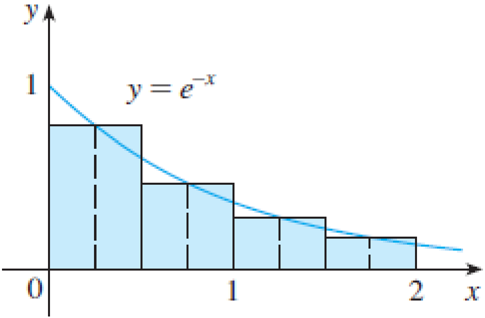
$$\sum_{i=1}^{n} f(x_i^*)\Delta x = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + ... + f(x_n^*)\Delta x$$

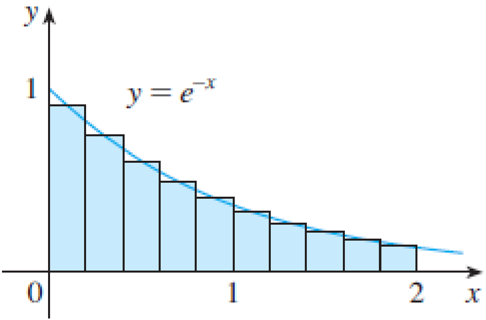
Example (19)

Let A be the area of the region that lies under the graph of $f(x) = e^{-x}$ between $x = 0$ and $x = 2$.

(i) Using right endpoints, find an expression for A as a limit. Do not evaluate the limit.

(ii) Estimate the area by taking the sample points to be midpoints and using four subintervals and then ten subintervals.


y
1
$y = e^{-x}$
0
1
2
x


y
1
$y = e^{-x}$
0
1
2
x

1.14.2 The Definite Integral

Definition

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = \dfrac{(b-a)}{n}$. We let $x_0(=a), x_1, x_2, \ldots, x_n(=b)$ be the endpoints of these subintervals and we let $x_1^*, x_2^*, \ldots, x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i^{th} subinterval $[x_{i-1}, x_i]$. Then the definite integral of f from a to b is

$$\int_a^b f(x)\, dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

provided that this limit exists and gives the same value for all possible choices of sample points. If it does exist, we say that f is integrable on $[a, b]$.

Note:

- ☆ The symbol $\int$ was introduced by Leibniz and is called an integral sign. It is an elongated S and was chosen because an integral is a limit of sums.
- ☆ In the notation $\int_a^b f(x)\, dx$, $f(x)$ is called the integrand and a and b are called the limits of integration; a is the lower limit and b is the upper limit.
- ☆ For now, the symbol dx has no meaning by itself; $\int_a^b f(x)\, dx$ is all one symbol.
- ☆ The dx simply indicates that the independent variable is x.
- ☆ The procedure of calculating an integral is called integration.

Note:

- ☆ The definite integral $\int_a^b f(x)\,dx$ is a number; it does not depend on x.
- ☆ In fact, we could use any letter in place of x without changing the value of the integral:

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(y)\,dy$$

- ☆ The sum

$$\sum_{i=1}^{n} f(x_i^*)\Delta x$$

is called a Riemann sum.

Note:

- ✯ If f happens to be positive, then the Riemann sum can be interpreted as a sum of areas of approximating rectangles (see Figure 1).
- ✯ According to the definition, the definite integral $\int_a^b f(x)\,dx$ can be interpreted as the area under the curve $y = f(x)$ from a to b. (See Figure 2.)

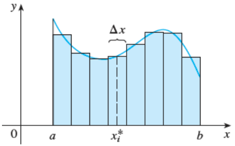


FIGURE 1
If $f(x) \geqslant 0$, the Riemann sum $\sum f(x_i^*)\,\Delta x$ is the sum of areas of rectangles.

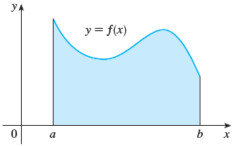


FIGURE 2
If $f(x) \geqslant 0$, the integral $\int_a^b f(x)\,dx$ is the area under the curve $y = f(x)$ from a to b.

Note:

☆ If f takes on both positive and negative values, as in Figure 3, then the Riemann sum is the sum of the areas of the rectangles that lie above the x-axis and the negatives of the areas of the rectangles that lie below the x-axis.

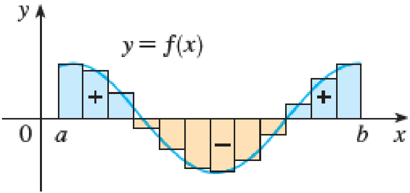


FIGURE 3

$\sum f(x_i^*)\,\Delta x$ is an approximation to the net area.

☆ A definite integral can be interpreted as a net area, that is, a difference of areas:

$$\int_a^b f(x)\,dx = A_1 - A_2$$

where A_1 is the area of the region above the x-axis and below the graph of f, and A_2 is the area of the region below the x-axis and above the graph of f.

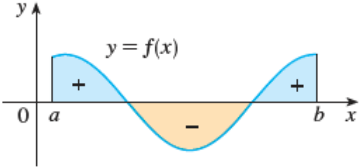


FIGURE 4
$\int_a^b f(x)\,dx$ is the net area.

Theorem

If f is integrable on $[a.b]$, then

$$\int_a^b f(x)\ dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\ \Delta x$$

where $\Delta x = \dfrac{(b-a)}{n}$ *and* $x_i = a + i\Delta x$.

Example (20)

(i) Evaluate the Riemann sum for $f(x) = x^3 - 6x$, taking the sample points to be right endpoints and $a = 0$, $b = 3$, and $n = 6$.

(ii) Evaluate $\displaystyle\int_0^3 (x^3 - 6x)\ dx$

1.14.3 The Midpoint Rule

We often choose the sample point x_i^* to be the right endpoint of the i^{th} subinterval because it is convenient for computing the limit.

But if the purpose is to find an *approximation* to an integral, it is usually better to choose x_i^* to be the midpoint of the interval, which we denote by $\overline{x_i}$.

Any Riemann sum is an approximation to an integral, but if we use midpoints we get the following approximation.

$$\int_a^b f(x)\ dx \approx \sum_{i=1}^{n} f(\overline{x_i})\ \Delta x = \Delta x \left[f(\overline{x_1}) + f(\overline{x_2}) + ... + f(\overline{x_n}) \right]$$

where $\Delta x = \dfrac{(b-a)}{n}$ and

$\overline{x_i} = \dfrac{1}{2}(x_{i-1} + x_i) = \text{Midpoint of } [x_{i-1}, x_i].$

Example (21)

Use the Midpoint Rule with $n = 5$ to approximate $\displaystyle\int_1^2 \frac{1}{x}\,dx$.

1.14.4 The Fundamental Theorem of Calculus

Theorem

Suppose F is continuous on $[a, b]$.

1. *If* $g(x) = \int_a^x f(t)\, dt$, *then* $g'(x) = f(x)$.
2. $\int_a^b f(x)\, dx = F(b) - F(a)$, *where F is any antiderivative of f, that is,* $F' = f$.

Example (22)

Find the derivative of the function $g(x) = \int_0^x \sqrt{1+t^2}\, dt$.

Example (23)

Find $\dfrac{d}{dx}\displaystyle\int_{1}^{x^4} \sec t \; dt.$

Example (24)

Evaluate the integral $\displaystyle\int_1^3 e^x \, dx$.

Example (25)

Find the area under the parabola $y = x^2$ from 0 to 1.

Example (26)

Evaluate $\displaystyle\int_3^6 \frac{dx}{x}$.

1.14.5 Indefinite Integrals

The notation $\int f(x)\,dx$ is traditionally used for an antiderivative of f and is called an indefinite integral.

$$\int f(x)\,dx = F(x) \text{ means } F'(x) = f(x).$$

An indefinite integral is representing an entire family of functions.

1.14.6 The Substitution Rule

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$\int f(g(x))g'(x)\ dx = \int f(u)\ du.$$

Example (27)

Find $\displaystyle\int x^3 \cos\left(x^4 + 2\right)\, dx$.

Example (28)

Find $\displaystyle\int \frac{x}{\sqrt{1-4x^2}}\,dx.$

1.14.7 The Substitution Rule for Definite Integrals

If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

Example (29)

Find $\displaystyle\int_1^2 \frac{dx}{(3-5x)^2}$.

Example (30)

Find $\displaystyle\int_1^e \frac{\ln x}{x}\, dx$.