Functions of Random Variables & Random Process

Random Signals & Processes

Lecture 5

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Introduction

- In many engineering applications, the input X to a given system is a random variable, and thus the corresponding output Y is also a random variable.
- The input–output relation is characterized by a known deterministic function $Y = g(X)$.

- Then, given the pdf $f_X(x)$ (or PMF if X is a discrete RV), it is required to find the PDF $f_Y(y)$ (or PMF) of the output RV.

Function of one random variable

Let X be a random variable with PDF $f_X(x)$ and let g(·) be a function that maps from R to R.
Then Y = g(X) is also a random variable.

$$\begin{aligned} F_Y(y) &= P[Y \leq y] = P[g(X) \leq y] \\ &= P[X \in \mathcal{D}_y], \end{aligned}$$

where $\mathcal{D}_y$ is the domain in the real line $\mathbb{R} = \{-\infty < x < \infty\}$ that is mapped to the range $\{-\infty < g(x) \leq y\}$; i.e.,

Function of two random variables

Consider the case where Z is a function of two RVs X and Y , which have joint PDF $f_{X\,Y}(x, y)$:

$$Z = g(X, Y).$$

$$F_Z(z) = P[g(X, Y) \leq z] = P[(X, Y) \in \mathcal{D}_z]$$

Distributions derived from the normal distribution

The normal distribution plays a central role in the mathematical theory of st
1. the normal distribution often describes a variety of physical quantities ob
Ex:In a communication system, for example, a received waveform is
often a superposition of a desired signal waveform and (unwanted) noise p

2. Mathematical tractability of the normal distribution.
Ex: sums of independent normal RVs are themselves normally distributed.

Rayleigh Distribution

- The Rayleigh and Rice distributions are primarily used by communication engineers.
- Assume that X and Y are independent normal variables with zero mean and common variance σ^2 . We define a new R

$$R = \sqrt{X^2 + Y^2}, \quad R \geq 0.$$

Then the PDF of the RV R is

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad r \geq 0.$$

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Rayleigh Distribution

The joint PDF of (X, Y) is given by

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{r^2}{2\sigma^2}\right).$$

Applications:

- Models magnitude variations of a signal (called fading) in a wireless communication chan
- Speckle noise in ultrasound images.
- Model for wind speed variations.

Rice distribution

- Assume that the independent normal RVs X and Y have nonzero means μ_X and μ_Y with common variance $\sigma_y{}^{2.}$ The PDF of R,

$$f_R(r) = \frac{r\, e^{-(r^2+\mu^2)/2\sigma^2}}{2\pi\sigma^2} I_0\left(\frac{r\mu}{\sigma^2}\right), \quad r \geq 0,$$

$$R = \sqrt{X^2 + Y^2}$$

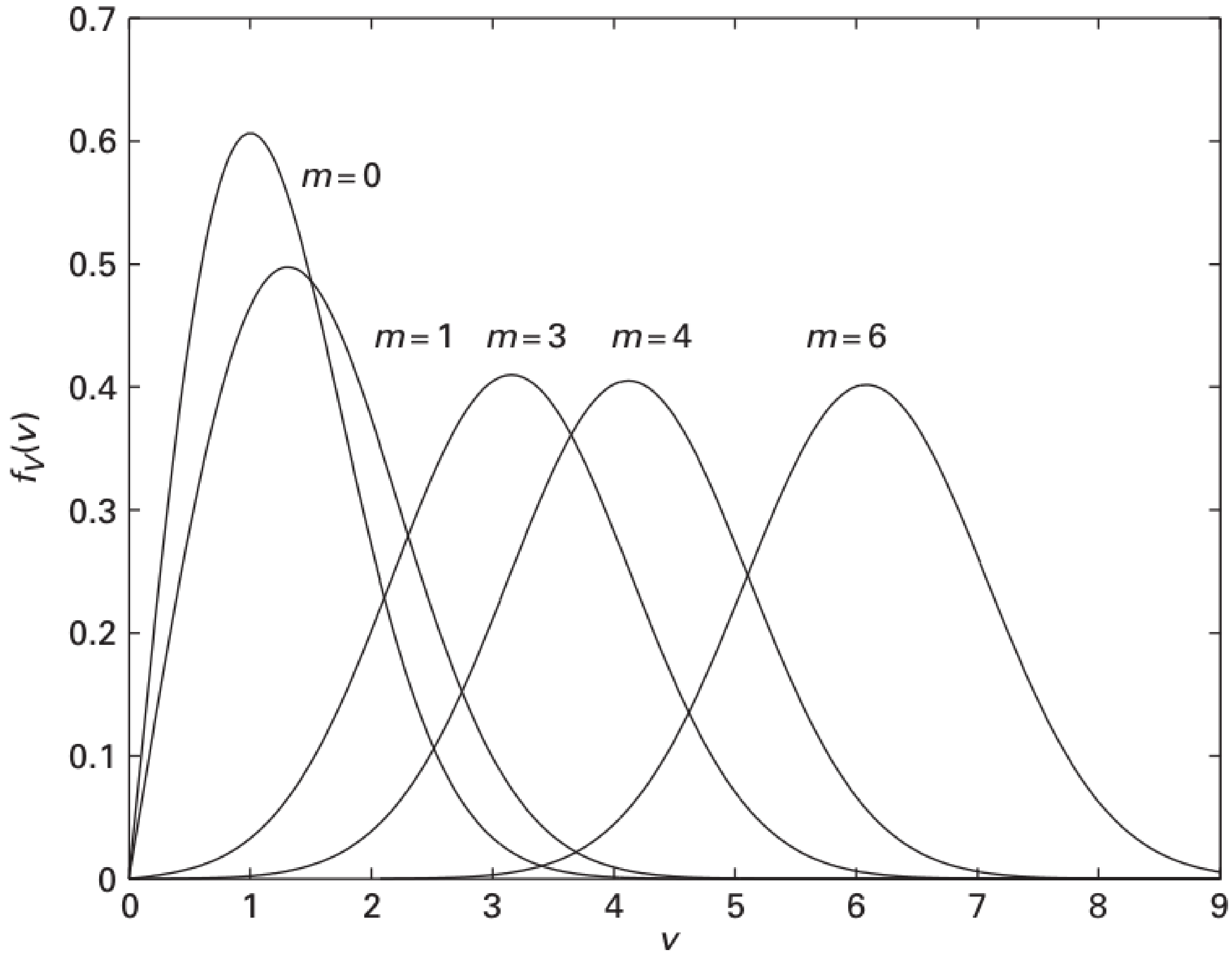
$$\mu = \sqrt{\mu_X^2 + \mu_Y^2},$$

$$I_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{x\cos\phi}\, d\phi, \quad -\infty < x < \infty,$$

- After normalizing the amplitude by σ , the distribution of this normalized amplitude RV,
- V = R/σ

$$f_V(v) = \frac{v\, e^{-(v^2+m^2)/2}}{2\pi} I_0(vm), \quad v \geq 0,$$

where $m = \mu/\sigma$.



The normalized Rice distribution for $m = 0, 1, 3, 4$, and 6.

Random process

Introduction

- For situations in which the time dependency of a set of probability functions is important.
- A random process is the **mathematical model of an empirical process** whose development is governed by probability law.
- Random processes provides useful models for the studies of such diverse fields as statistical physics, communication and control, time series analysis, population growth, and management sciences

Random process

Examples

1. a noise process that accompanies a signal process and should be suppressed or filtered out so that the signal can be recovered reliably and accurately.

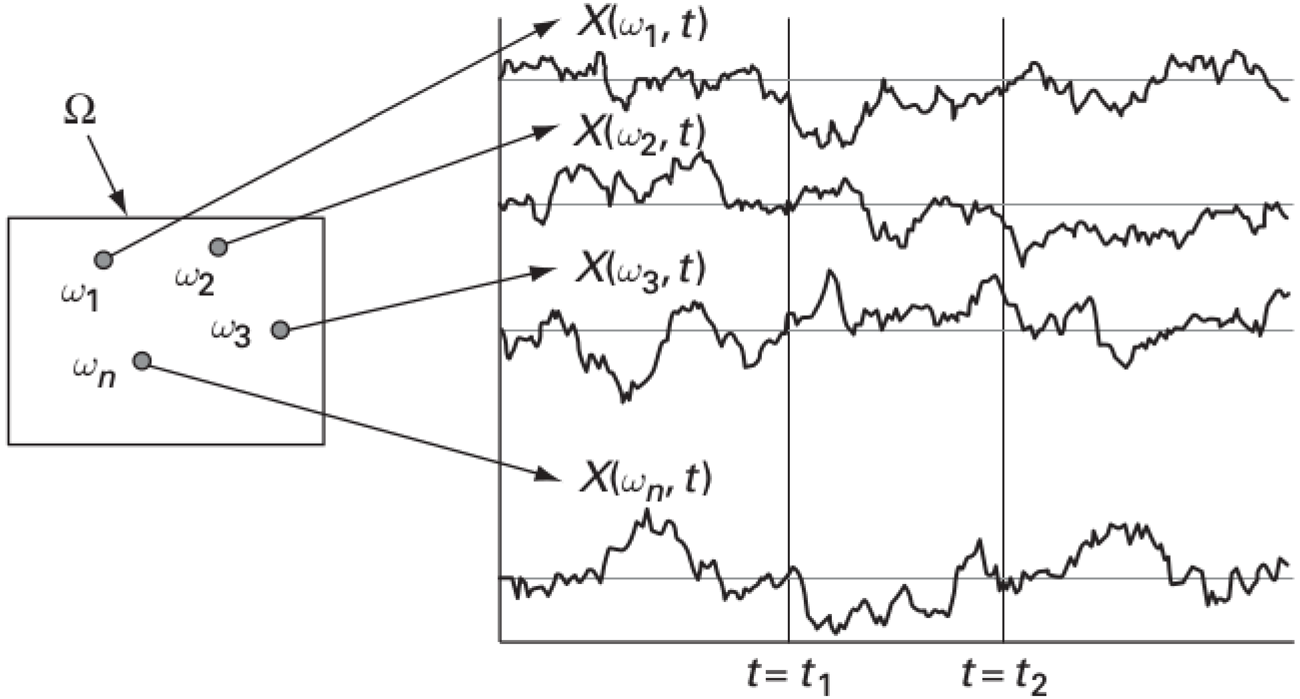
2. The amount of outstanding packets yet to be processed at a network router or switch, which may lead to undesirable packet loss due to buffer overflows if not properly attended to in time.

Random process

Extending the notion of a random variable (RV) as follows:

Assign to each sample point $\omega \in \Omega$ a **real-valued function X (ω, t)**, where **t** is the time parameter or index parameter in some range **T** , which may be, for instance,

- $T = (-\infty, \infty)$ or $T = \{0, 1, 2, . . .\}$



A random process $X(\omega, t)$ as a mapping from a sample point $\omega \in \Omega$ to a real-valued function.

Observe this set of time functions $\{X(\omega,t);\ \omega\in\Omega,\ t\in T\}$ at some instant $t = t_1$.

- Since each point $\omega \in \Omega$ has associated with it both the number $X(\omega,t_1)$ and its probability, the collection of numbers, $\{X(\omega, t_1);\ \omega \in\Omega\}$ forms a random variable.

- By observing the time functions at another time, say at $t = t_2$, another collection of numbers with a possibly different probability measure. This set of time-indexed functions defines a separate RV for each choice of the time.

- A probability system, which is composed of a sample space, a set of real-valued time-indexed functions, and a probability measure, is called a **random process** or a **stochastic process** and is usually denoted by a notation such as $X(t); t \in T$, or simply as $X(t)$, if T is implicitly understood.
- The individual time functions of the random process $X(t)$ are called **sample functions**, and the particular sample function associated with a sample point $\omega \in \Omega$ is denoted as $X(\omega, t); t \in T$.

- The set of all possible sample functions, together with a probability law, is called the **ensemble.**

- By definition, a random process implies the existence of an infinite number of rvs, one for each *t* in some range.

- Thus, the pdf $f_X(t_1)$ $(\cdot)$ of the random variable $X(t_1)$ obtained by observing $X(t)$ at time t_1 .

- Generally, for N time instants $\{t_i : i = 1, 2, . . . , N \}$, we define the N random variables $X_i = X(t_i)$; $i = 1, 2, . . . , N$.

- Then we can speak of the joint PDF of $X_1 , X_2 , . . . , X_N$.

Classification of random processes

Two classes of random processes,

1. Discrete or continous time

2. Point processes or counting processes, where the intervals between points of events are RVs.

Discrete-time versus continuous-time processes

When the time index takes on values from a set of discrete of time instants, $T = \{0, 1, 2, . . .\}$, **the process is said to be a discrete-time random process** or a random sequence, and is often denoted as X_t ; $t \in T$ or X_k ; $k \in T$.

Random Variable versus Random Process

- For a random variable, the outcome of **a random experiment is mapped into a number**

- For a random process, **the outcome of a random experiment is mapped in to a waveform that is a function of time.**

Probabilistic Descriptions

Consider a random process X(t),

For a fixed time t_1, $X(t_1)$ = ***X*** **,** is a r.v., and its cdf F,(x_l; t,) is defined as :

$$F_X(x_1; t_1) = P\{X(t_1) \le x_1\}$$

Joint distribution / second-order distribution

Given t_1 and t_2 $X(t_1) = X_1$ and $X(t_2) = X_2$ represent two r.v.s, their joint distribution is known as the second-order distribution of X(t) and is given by:

$$F_X(x_1, x_2; t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\}$$

Nth Order Probability Distribution Function

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) \leq x_1, \ldots, X(t_n) \leq x_n\}$$

Nth Order Probability Mass Function

$$p_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = P\{X(t_1) = x_1, \ldots, X(t_n) = x_n\}$$

Nth Order Probability Density Function

$$f_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = \frac{\partial^n F_X(x_1, \ldots, x_n; t_1, \ldots, t_n)}{\partial x_1 \cdots \partial x_n}$$