Communication Theory II

Lecture 3: Baseband Pulse Transmission :

Nyquist's Criterion for Distortion less Baseband Binary

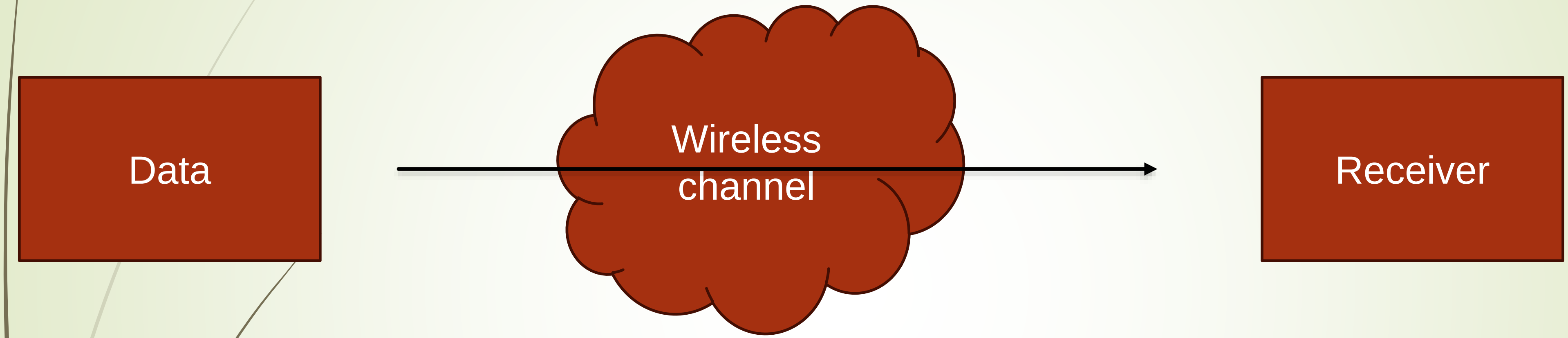
Transmission Ideal Nyquist Channel

Raised Cosine Spectrum

Digital data

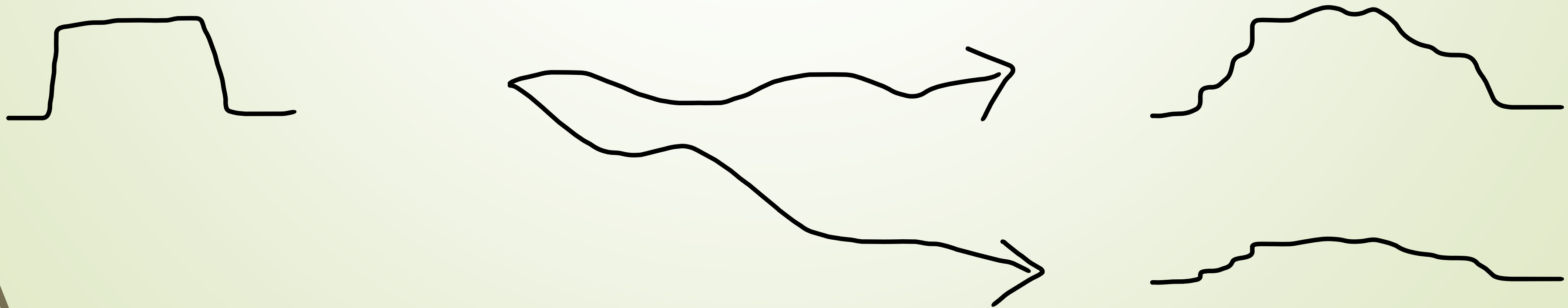
- Digital data have a broad spectrum with a significant low-frequency content.
- To transmit digital data, a low-pass channel is required to accommodate the essential frequency content of the data stream.
- However, the channel often deviates from an ideal low-pass filter, leading to dispersion in its frequency response.
- This dispersion causes intersymbol interference (ISI), where each received pulse is influenced by adjacent pulses.
- ISI is a common form of interference that results in bit errors in the reconstructed data stream at the receiver output.

Error possibility

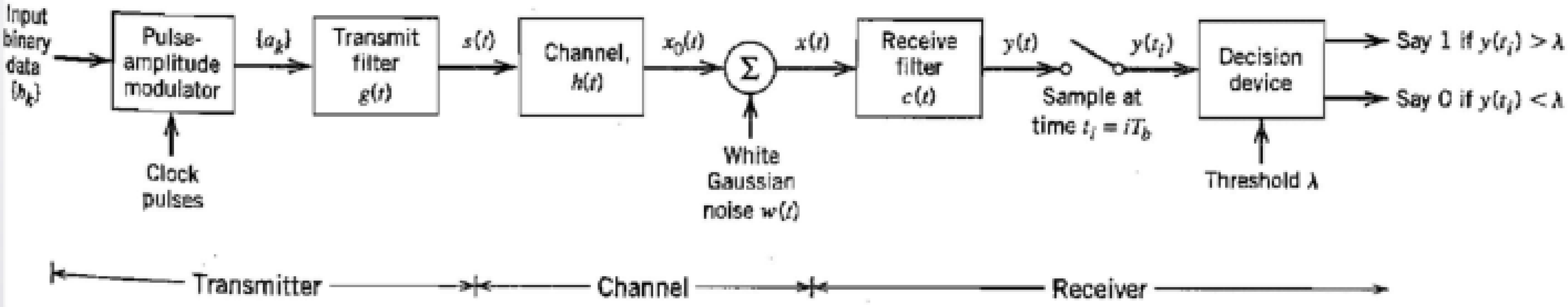


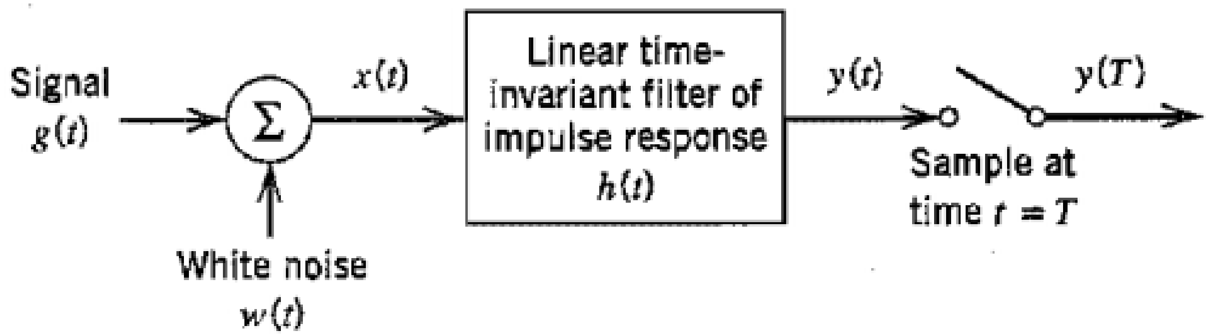
- Channel Noise
- Inter-symbol interference (ISI)

Goal: Detection of a pulse



Detection



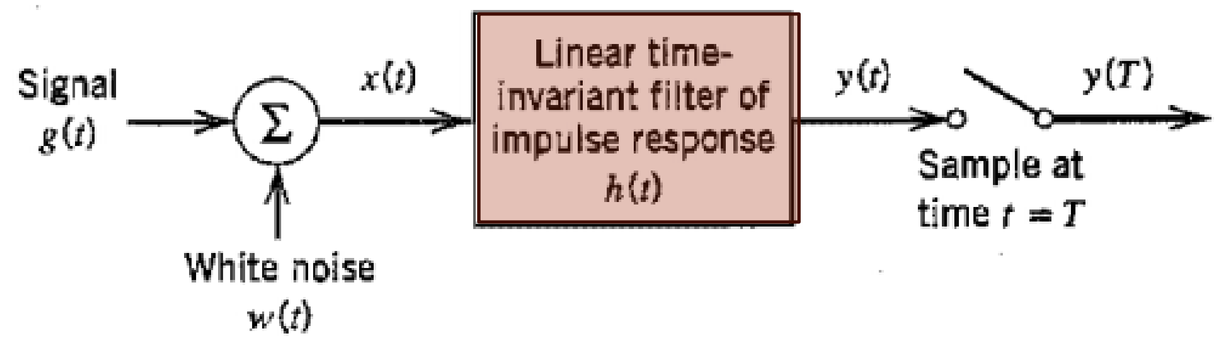


Linear receiver.

$$x(t) = g(t) + w(t), \qquad 0 \le t \le T$$

pulse signal g(t) corrupted by additive channel noise
T-an arbitrary observation interval

Detection



Linear receiver.

$$x(t) = g(t) + w(t), \qquad 0 \le t \le T$$

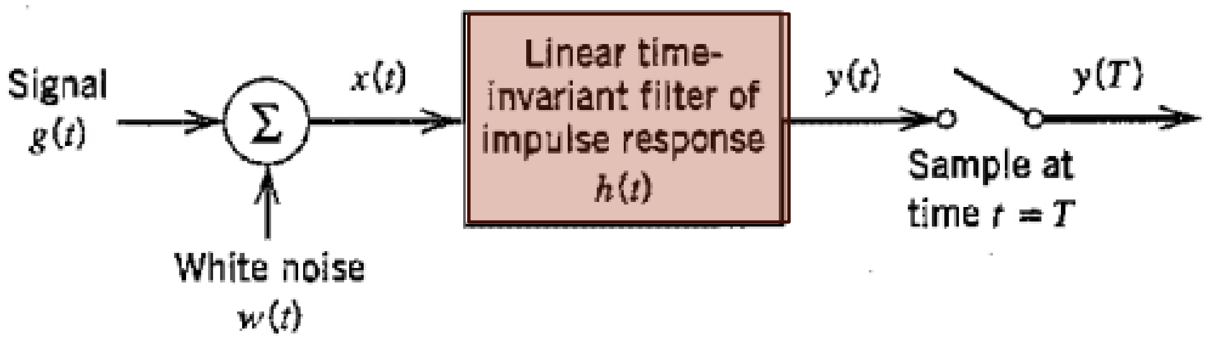
pulse signal g(t) corrupted by additive channel noise
T-an arbitrary observation interval

- It is assumed that the receiver has knowledge of the waveform of the pulse signal *g(t)*
- Receiver :- Detection of pulse signal g(t) with the presence of Noise (source of uncertainty)
- Design:- Optimize the design of the filter so as to minimize the effects of noise at the filter output in some statistical sense, and thereby enhance the detection of the pulse signal

$$y(t) = g_o(t) + n(t)$$

Instantaneous power in the output signal $g_0(t)$, measured at time T ?

Peak pulse signal to noise ratio



Linear receiver.

$$y(t) = g_o(t) + n(t)$$

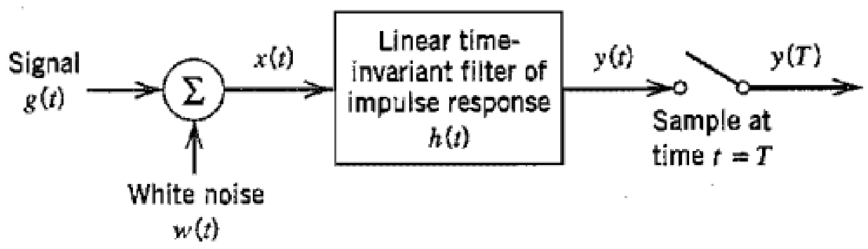
$$\eta = \frac{|g_o(T)|^2}{E[n^2(t)]}$$

instantaneous power in the output signal at t=T

average noise power

Objective:- Maximize peak signal to noise ratio

Peak pulse signal to noise ratio



Linear receiver.

$$y(t) = g_o(t) + n(t)$$

$$\eta = \frac{|g_o(T)|^2}{E[n^2(t)]}$$

g(t) ➔ G (f)
h(t) ➔ H (f)

$$g_o(t) = \int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi f t)\,df$$

filter output is sampled at time t = T, signal power (assume no noise)

$$|g_o(T)|^2 = \left|\int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi f T)\,df\right|^2$$

filter output due to the noise
Power spectral density of n(t) is given by

$$S_N(f) = \frac{N_0}{2}|H(f)|^2$$

The average power of the output noise

$$E[n^2(t)] = \int_{-\infty}^{\infty} S_N(f)\,df = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df$$

Peak pulse signal to noise ratio

$$\eta = \frac{|g_o(T)|^2}{E[n^2(t)]}$$

$$\eta = \frac{\left|\int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)\,df\right|^2}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df}$$

Based on Schwarz's inequality

$$\left|\int_{-\infty}^{\infty} \phi_1(x)\phi_2(x)\,dx\right|^2 \le \int_{-\infty}^{\infty} |\phi_1(x)|^2\,dx \int_{-\infty}^{\infty} |\phi_2(x)|^2\,dx$$

$$\phi_1(x) = k\phi_2^*(x)$$

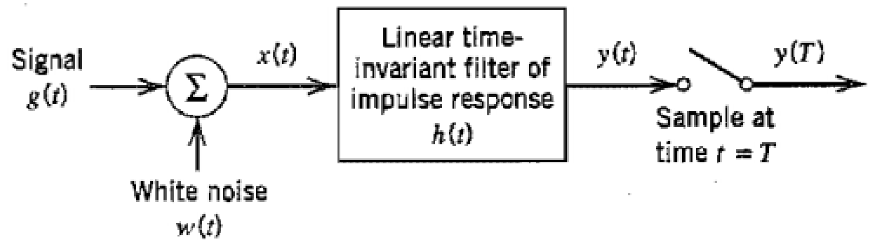
$$\left|\int_{-\infty}^{\infty} H(f)G(f)\exp(j2\pi fT)\,df\right|^2 \le \int_{-\infty}^{\infty} |H(f)|^2\,df \int_{-\infty}^{\infty} |G(f)|^2\,df$$

$$\eta \le \frac{\int_{-\infty}^{\infty} |H(f)|^2\,df \int_{-\infty}^{\infty} |G(f)|^2\,df}{\frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2\,df}$$

$$\eta \le \frac{2}{N_0}\int_{-\infty}^{\infty} |G(f)|^2\,df$$

Matched filter

$$\eta \le \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 \, df$$



Linear receiver.

Does not depend on the frequency response H(f)

only on the signal energy and the noise power spectral density

$$\eta_{max} = \frac{2}{N_0} \int_{-\infty}^{\infty} |G(f)|^2 \, df$$

$$H_{opt}(f) = kG^*(f) \exp(-j2\pi fT)$$

G*(f) complex conjugate of the Fourier transform of the input signal g(t)
k-scaling factor

$$h_{opt}(t) = k \int_{-\infty}^{\infty} G^*(f) \exp[-j2\pi f(T - t)] \, df$$

$$\begin{aligned} h_{opt}(t) &= k \int_{-\infty}^{\infty} G(-f) \exp[-j2\pi f(T - t)] \, df \\ &= k \int_{-\infty}^{\infty} G(f) \exp[j2\pi f(T - t)] \, df \\ &= kg(T - t) \end{aligned}$$

for a real signal g(t) we have G*(f) = G(-f).

Matched filter

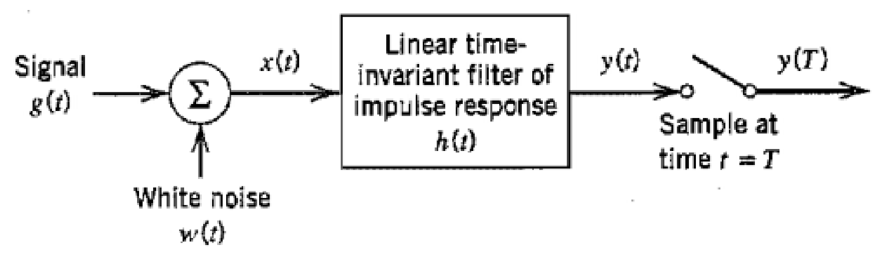
$$h_{opt}(t) = kg(T - t)$$

Time-reversed and delayed version of the input signal

Matched to the input signal

Matched filter

- The derivation of the matched filter assumes that the input noise, denoted as w(t), is stationary and white.
- The assumption further specifies that the input noise has zero mean and a power spectral density of No/2.
- It is important to note that no assumptions were made regarding the statistics of the channel noise, represented as w(t).



Linear receiver.

Matched filter

$$h_{opt}(t) = kg(T - t)$$

$$\eta_{max} = \frac{(kE)^2}{(k^2 N_0 E/2)} = \frac{2E}{N_0}$$

$$E = \int_{-\infty}^{\infty} g^2(t)dt = \int_{-\infty}^{\infty} |G(f)|^2 df$$

time-reversed and delayed version of the input signal

⬇

Matched to the input signal

⬇

Matched filter

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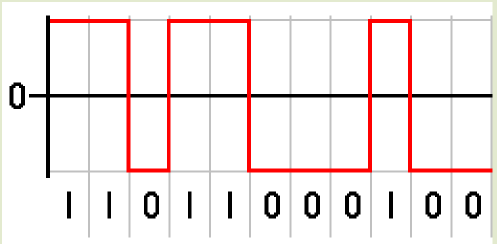
- Dependence on the waveform of the Input waveform g(t) has been removed by the matched filter. ➔ all signals that has the same energy are equally effective.

- E/No ➔ Signal energy to noise spectral density ratio

Error due to Noise

- To proceed with the analysis, consider a binary PCM system based on polar nonreturn-to-zero (NRZ) signaling.

- Symbols 1 and 0 are represented by positive and negative rectangular pulses of equal amplitude and equal duration



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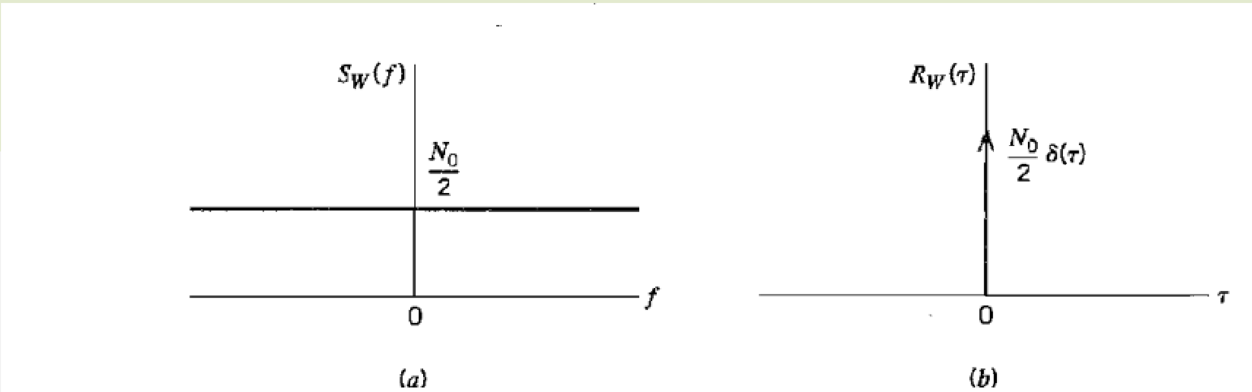
- The channel noise is modeled as additive **white Gaussian** noise w(t) of zero mean and power spectral density $N_O/2$;

uniform power spectral density across the frequency band for the information system



Characteristics of white noise. (a) Power spectral density. (b) Autocorrelation function.

normal distribution in the time domain with an average time domain value of zero

Error due to Noise

- The Gaussian assumption is needed for later calculations. In the signaling interval O < t < Tb the received signal written as:

$$x(t) = \begin{cases} +A + w(t), & \text{symbol 1 was sent} \\ -A + w(t), & \text{symbol 0 was sent} \end{cases}$$

- Tb- bit duration, A- transmit pulse amplitude
- Assumptions:- receiver has acquired knowledge of the starting and ending times of each transmitted pulse; in other words, the receiver has prior knowledge of the pulse shape, but not its polarity.
- Given the noisy signal x(t), the receiver is required to make a decision in each signaling interval as to whether the transmitted symbol is a 1 or a 0.

Error due to Noise

- The structure of the receiver used to perform this decision-making process is shown in Figure. It consists of a matched filter followed by a sampler, and then finally a decision device.

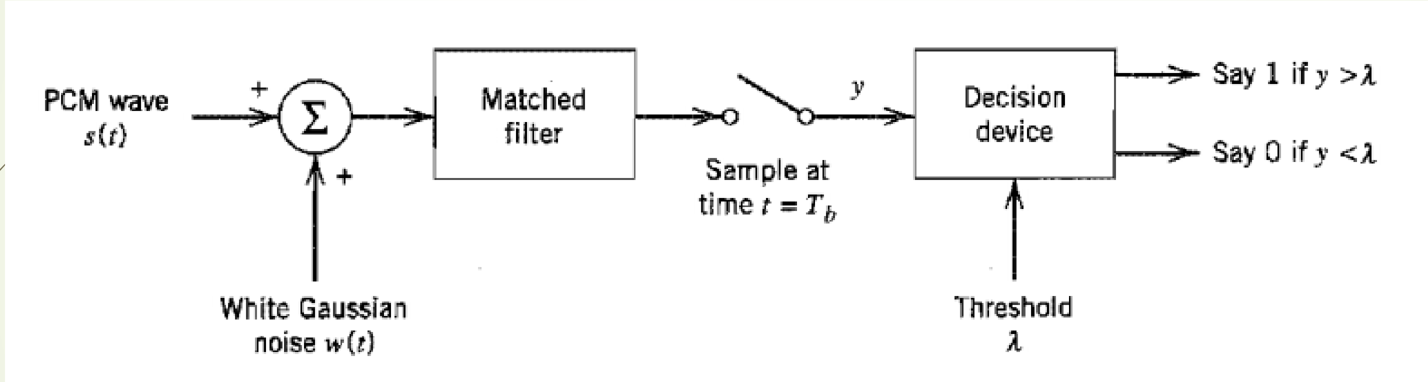


Figure: Receiver for baseband transmission of binary-encoded PCM wave using polar NRZ signaling.

- ➢ The filter is matched to a rectangular pulse of amplitude A and duration T_b, exploiting the bit-timing information available to the receiver.
- ➢ The resulting matched filter output is sampled at the end of each signaling interval

Error due to Noise

- The sample value y is compared to a preset *threshold* A in the decision device.

- If the threshold is exceeded, the receiver makes a decision in favor of symbol 1; if not, a decision is made in favor of symbol 0.

- There are two possible kinds of error to be considered:

1. Symbol 1 is chosen when a 0 was actually transmitted; we refer to this error as an *error of the first kind.*

2. Symbol 0 is chosen when a 1 was actually transmitted; we refer to this error as an *error o f the second kind.*

Error due to Noise

Suppose that symbol 0 was sent

$$x(t) = -A + w(t), \qquad 0 \le t \le T_b$$

The matched filter output, sampled at time t = T

$$y = \int_0^{T_b} x(t)\,dt \qquad\qquad = -A + \frac{1}{T_b}\int_0^{T_b} w(t)\,dt$$

which represents the sample value of a random variable *Y.*

- The random variable Y is Gaussian distributed with a mean of $-A$.
- The variance of the random variable Y is

$$\begin{aligned}
\sigma_Y^2 &= E[(Y + A)^2] \\
&= \frac{1}{T_b^2} E\left[\int_0^{T_b}\int_0^{T_b} w(t)w(u)\,dt\,du\right] \\
&= \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} E[w(t)w(u)]\,dt\,du \\
&= \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b} R_W(t, u)\,dt\,du
\end{aligned}$$

where $R_W(t, u)$ is the autocorrelation function of the white noise $w(t)$ with a power spectral density $N_0/2$, we have

$$R_W(t, u) = \frac{N_0}{2}\,\delta(t - u)$$

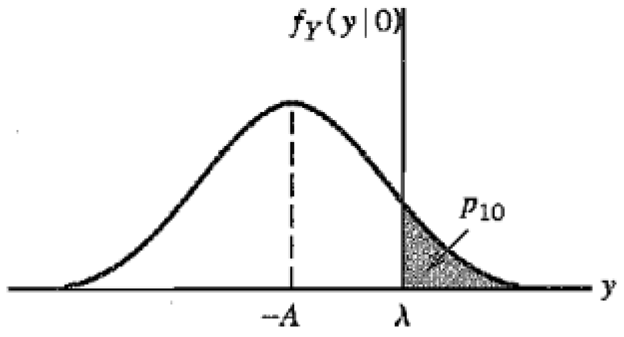
Error due to Noise

$$\sigma_Y^2 = \frac{1}{T_b^2}\int_0^{T_b}\int_0^{T_b}\frac{N_0}{2}\delta(t-u)\,dt\,du$$
$$= \frac{N_0}{2T_b}$$

The conditional probability density function of the random variable y

$$f_Y(y|0) = \frac{1}{\sqrt{\pi N_0/T_b}}\exp\left(-\frac{(y+A)^2}{N_0/T_b}\right)$$

conditional probability of error, ***given*** *that symbol 0* ***was sent***



$$p_{10} = P(y > \lambda|\text{symbol 0 was sent})$$
$$= \int_\lambda^\infty f_Y(y|0)\,dy$$
$$= \frac{1}{\sqrt{\pi N_0/T_b}}\int_\lambda^\infty \exp\left(-\frac{(y+A)^2}{N_0/T_b}\right)dy$$

In the absence of noise, the matched filter output y sampled at time t = Tb is equal to - *A*. When noise is present, y occasionally assumes a value greater than λ, in which case an error is made,=.

Error due to Noise

*conditional probability of error, **given** that symbol 0 **was sent***

$$p_{10} = P(y > \lambda \mid \text{symbol 0 was sent})$$
$$= \int_{\lambda}^{\infty} f_Y(y \mid 0)\, dy$$
$$= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y+A)^2}{N_0/T_b}\right) dy$$

$$\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} \exp(-z^2)\, dz$$

complementary error function

which is closely related to the Gaussian distribution. For large positive values of u, we have the following *upper bound* on the complementary error function:

$$\operatorname{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi} u}$$

New variable

$$z = \frac{y + A}{\sqrt{N_0/T_b}}$$

$$p_{10} = \frac{1}{\sqrt{\pi}} \int_{(A+\lambda)/\sqrt{N_0/T_b}}^{\infty} \exp(-z^2)\, dz$$
$$= \frac{1}{2} \operatorname{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right)$$

Error due to Noise

*conditional probability of error, **given** that symbol 1 **was sent***

$$f_Y(y|1) = \frac{1}{\sqrt{\pi N_0/T_b}} \exp\left(-\frac{(y-A)^2}{N_0/T_b}\right)$$

$$\begin{aligned} p_{01} &= P(y < \lambda | \text{symbol 1 was sent}) \\ &= \int_{-\infty}^{\lambda} f_Y(y|1)\, dy \\ &= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{-\infty}^{\lambda} \exp\left(-\frac{(y-A)^2}{N_0/T_b}\right) dy \end{aligned}$$

New variable

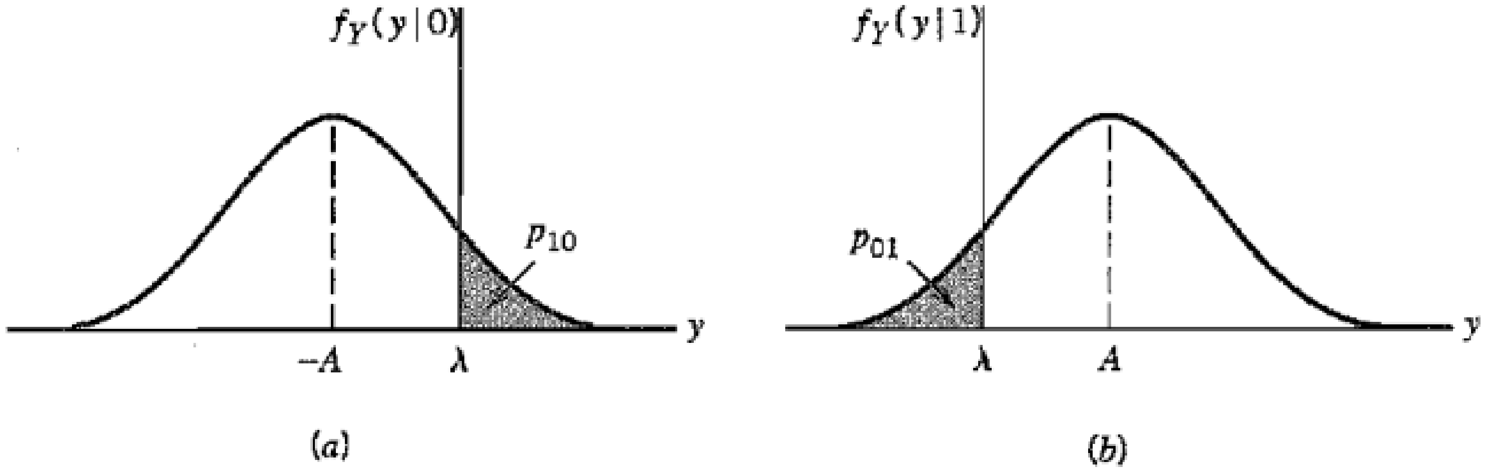
$$z = \frac{A-y}{\sqrt{N_0/T_b}}$$

$$\begin{aligned} p_{01} &= \frac{1}{\sqrt{\pi}} \int_{(A-\lambda)/\sqrt{N_0/T_b}}^{\infty} \exp(-z^2)\, dz \\ &= \frac{1}{2} \operatorname{erfc}\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right) \end{aligned}$$

*conditional probability of error, **given** that symbol 0 **was sent***

$$\begin{aligned} p_{10} &= P(y > \lambda | \text{symbol 0 was sent}) \\ &= \int_{\lambda}^{\infty} f_Y(y|0)\, dy \\ &= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y+A)^2}{N_0/T_b}\right) dy \end{aligned}$$

Error due to Noise



Noise analysis of PCM system. (*a*) Probability density function of random variable *Y* at matched filter output when 0 is transmitted. (*b*) Probability density function of *Y* when 1 is transmitted.

$$
\begin{aligned}
p_{10} &= P(y > \lambda \,|\, \text{symbol 0 was sent}) \\
&= \int_{\lambda}^{\infty} f_Y(y|0)\, dy \\
&= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{\lambda}^{\infty} \exp\left(-\frac{(y+A)^2}{N_0/T_b}\right) dy
\end{aligned}
$$

$$
\begin{aligned}
p_{01} &= P(y < \lambda \,|\, \text{symbol 1 was sent}) \\
&= \int_{-\infty}^{\lambda} f_Y(y|1)\, dy \\
&= \frac{1}{\sqrt{\pi N_0/T_b}} \int_{-\infty}^{\lambda} \exp\left(-\frac{(y-A)^2}{N_0/T_b}\right) dy
\end{aligned}
$$

Error due to Noise

Average probability of symbol error

$p_{10} = P(y > \lambda \mid \text{symbol 0 was sent})$

$p_{01} = P(y < \lambda \mid \text{symbol 1 was sent})$

Mutually exclusive events, in probability theory, are events that cannot occur simultaneously.

$$\begin{aligned} P_e &= p_0 p_{10} + p_1 p_{01} \\ &= \frac{p_0}{2}\,\mathrm{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right) + \frac{p_1}{2}\,\mathrm{erfc}\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right) \end{aligned}$$

formulating an optimum *threshold* that minimizes P_e

P_e is in fact a function of the **threshold**

Using Leibniz's rule

$$\frac{d}{du}\,\mathrm{erfc}(u) = -\frac{1}{\sqrt{\pi}}\exp(-u^2)$$

$$\frac{\mathrm{d}Pe}{\mathrm{d}\lambda} = 0$$

$$\lambda_{\mathrm{opt}} = \frac{N_0}{4AT_b}\log\left(\frac{p_0}{p_1}\right)$$

Error due to Noise

For the special case when symbols 1 and 0 are equiprobable, we have

$$p_1 = p_0 = \frac{1}{2}$$

$$\lambda_{\text{opt}} = \frac{N_0}{4AT_b} \log\left(\frac{p_0}{p_1}\right) \Rightarrow \lambda_{\text{opt}} = 0$$

For this special case we also have

$$p_{01} = p_{10} \Rightarrow \text{Binary symmetric}$$

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\frac{A}{\sqrt{N_0/T_b}}\right)$$

transmitted signal energy per *bit* is defined by $E_b = A^2 T_b$

$$P_e = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

transmitted signal energy per *bit*

noise spectral density

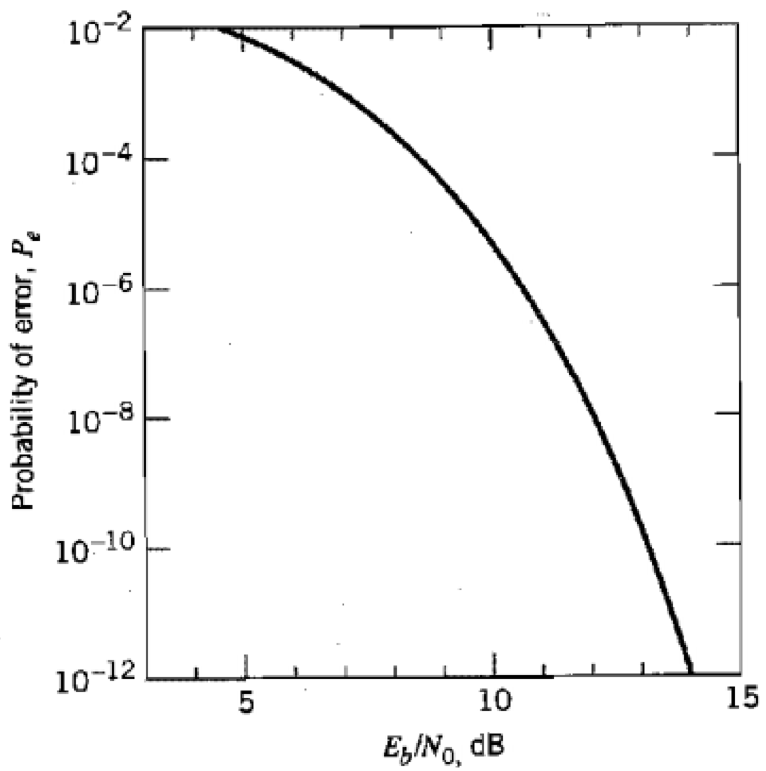
Error due to Noise

$$\operatorname{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi}u}$$

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Upper bound the average probability of symbol error for the PCM receiver

$$P_e < \frac{\exp(-E_b/N_0)}{2\sqrt{\pi E_b/N_0}}$$



Probability of error in a PCM receiver.

- As the ratio Eb/No (energy per bit to noise power spectral density) increases, Pe decreases rapidly.
- Increasing the ratio Eb/No results in a significant reduction in error probability.
- Even a small increase in transmitted signal energy can lead to nearly error-free reception of binary pulses.

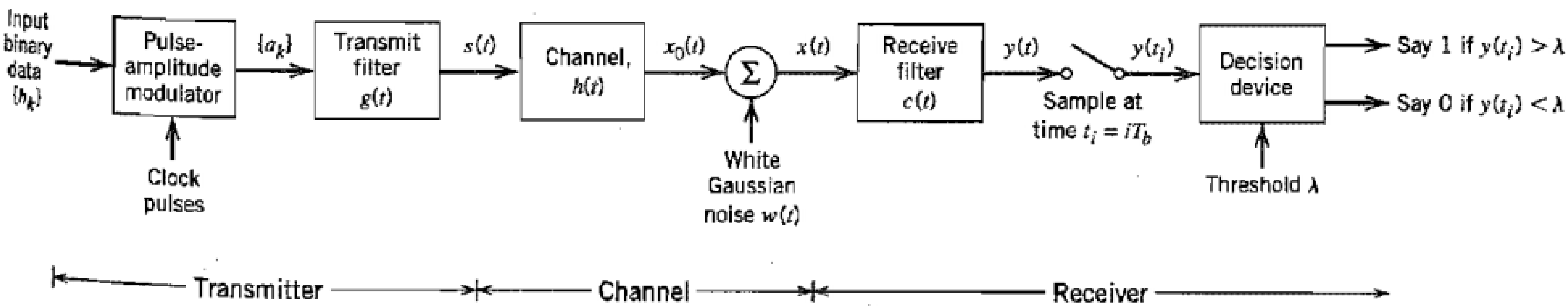
Intersymbol interference (ISI)

Baseband binary PAM system

$$a_k = \begin{cases} +1 & \text{if symbol } b_k \text{ is } 1 \\ -1 & \text{if symbol } b_k \text{ is } 0 \end{cases}$$

The sequence of short pulses so produced is applied to a *transmit filter* of impulse response g(t), producing the transmitted signal

$$s(t) = \sum a_k g(t - kT_b)$$



Baseband binary data transmission system.

Intersymbol interference (ISI)

The receive filter output is written as

$$y(t) = \mu \sum_{k} a_k p(t - kT_b) + n(t)$$

μ is a scaling factor

$$\mu p(t) = g(t) \star h(t) \star c(t)$$

We assume that the pulse p(t) is *normalized* $p(0) = 1$

$$\mu P(f) = G(f)H(f)C(f)$$

The receive filter output *y(t)* is sampled at fime ti = iTb (with i taking on integer values)

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k p[(i - k)T_b] + n(t_i)$$

$$= \mu a_i + \underbrace{\boxed{\mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k p[(i - k)T_b] + n(t_i)}}_{\text{ISI}}$$

Nyquist's Criterion for Distortion less Baseband Binary Transmission

$$y(t_i) = \mu \sum_{k=-\infty}^{\infty} a_k p[(i-k)T_b] + n(t_i)$$

$$= \mu a_i + \boxed{\mu \sum_{\substack{k=-\infty \\ k \neq i}}^{\infty} a_k p[(i-k)T_b] + n(t_i)}$$

$$p(iT_b - kT_b) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

Satisfies the above conditions the receiver output simplifies to (ignoring the noise term)

$$y(t_i) = \mu a_i \qquad \text{for all } i$$

Nyquist's Criterion for Distortion less Baseband Binary Transmission

$$p(iT_b - kT_b) = \begin{cases} 1, & i = k \\ 0, & i \neq k \end{cases}$$

sampling in the time domain produces periodicity in the frequency domain

Consider then the sequence of samples $\{p(nT_b)\}$, where, $n = 0, \pm 1, \pm 2, \cdots$

$$P_\delta(f) = R_b \sum_{n=-\infty}^{\infty} P(f - nR_b)$$

Rb = 1/Tb is the bit *rate* in bits per second (b/s);

$P_\delta(f)$ Fourier transform of an infinite periodic sequence of delta functions of period Tb, whose individual areas are weighted by the respective sample values of p(t).

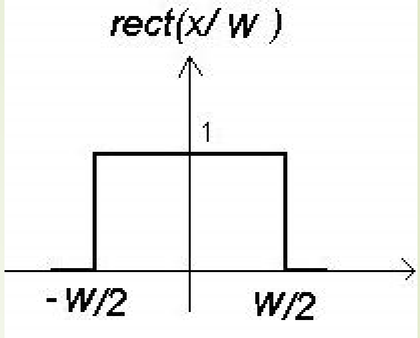
$$P_\delta(f) = \int_{-\infty}^{\infty} \sum_{m=-\infty}^{\infty} [p(mT_b)\, \delta(t - mT_b)] \exp(-j2\pi ft)\, dt$$

Let the integer $m = i - k$. Then, $i = k$ corresponds to $m = 0$, and likewise $i \neq k$ corresponds to $m \neq 0$.

m=0, we get

$$P_\delta(f) = \int_{-\infty}^{\infty} p(0)\, \delta(t) \exp(-j2\pi ft)\, dt$$
$$= p(0)$$

Intersymbol interference (ISI)



Condition for zero intersymbol interference

$$\sum_{n=-\infty}^{\infty} P(f - nR_b) = T_b$$

We may now state the *Nyquist criterion*[4] *for distortionless baseband transmission* in the absence of noise: *The frequency function $P(f)$ eliminates intersymbol interference for samples taken at intervals T_b provided that it satisfies Equation*

P(f) refers to overall system.

$$\mu P(f) = G(f)H(f)C(f)$$

Ideal Nyquist channel

The simplest way of satisfying above Equation is to specify the frequency function *P(f)* be in the form of a *rectangular function*

$$P(f) = \begin{cases} \dfrac{1}{2W}, & -W < f < W \\ 0, & |f| > W \end{cases}$$
$$= \frac{1}{2W}\,\mathrm{rect}\left(\frac{f}{2W}\right)$$

where $\mathrm{rect}(f)$ stands for a *rectangular function* of unit amplitude and unit support centered on $f = 0$, and the overall system bandwidth W is defined by

$$W = \frac{R_b}{2} = \frac{1}{2T_b}$$

Ideal Nyquist channel

$$P(f) = \begin{cases} \dfrac{1}{2W}, & -W < f < W \\ 0, & |f| > W \end{cases}$$
$$= \frac{1}{2W}\operatorname{rect}\left(\frac{f}{2W}\right)$$

$$p(t) = \frac{\sin(2\pi Wt)}{2\pi Wt}$$
$$= \operatorname{sinc}(2Wt)$$

$R_B = 2W$ is called the *Nyquist rate,* and W is called the *Nyquist bandwidth.*

$$W = \frac{1}{2T_b} = \frac{R_b}{2}$$

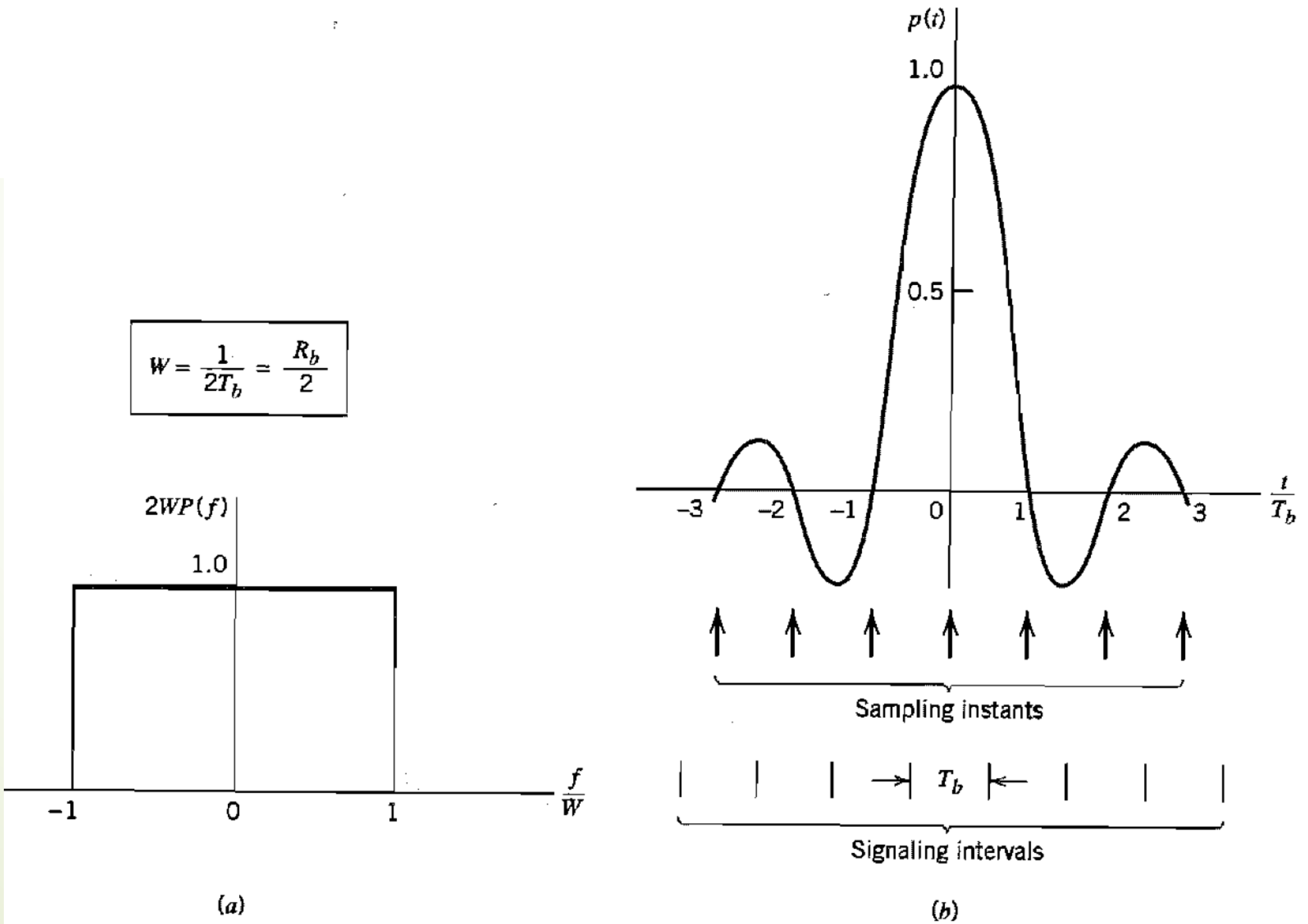
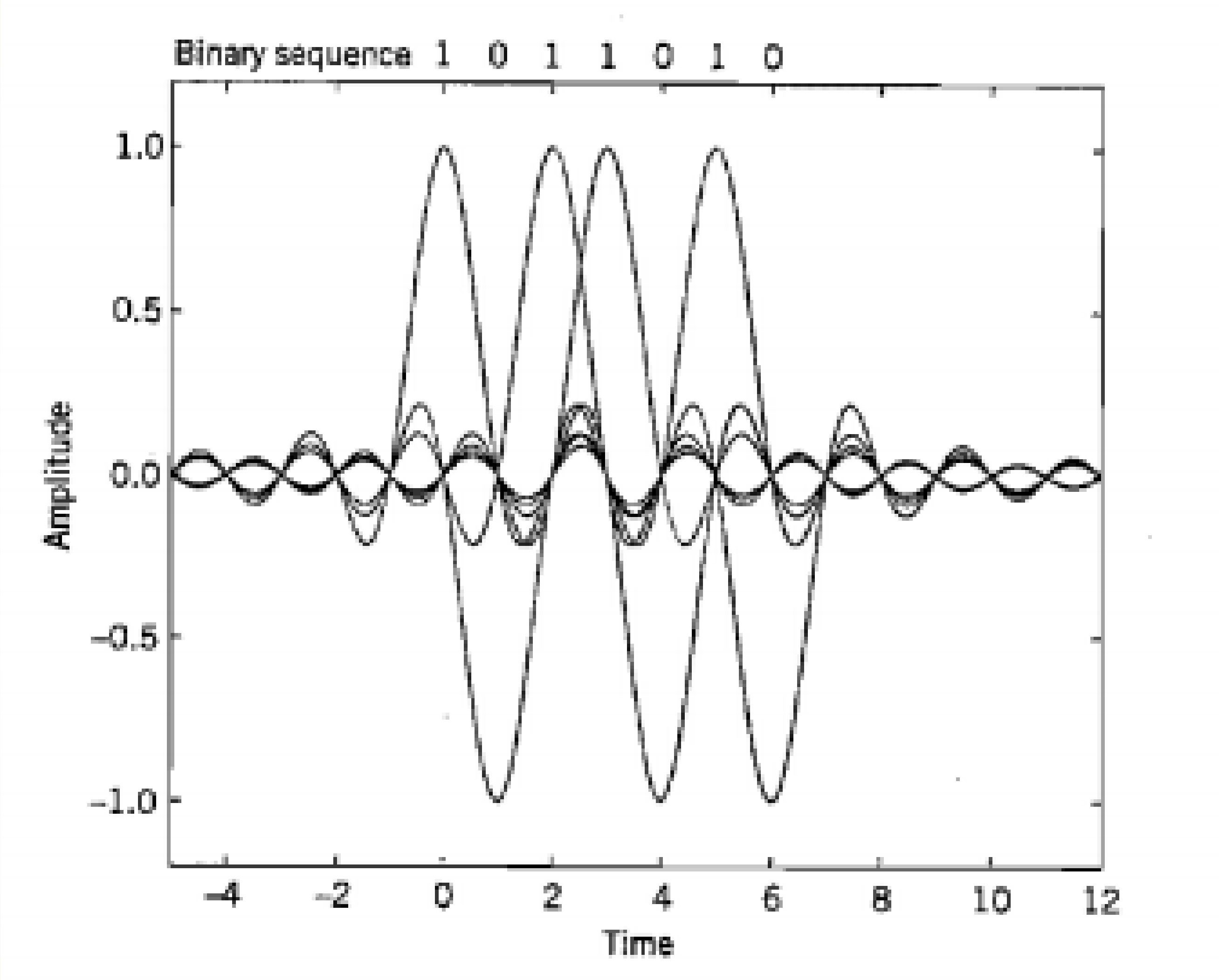


FIGURE 4.8 (*a*) Ideal magnitude response. (*b*) Ideal basic pulse shape.

Ideal Nyquist channel

$$p(t) = \frac{\sin(2\pi W t)}{2\pi W t}$$
$$= \mathrm{sinc}(2Wt)$$



Practical difficulties encountered with the ideal Nyquist channel

- It requires that the magnitude characteristic of *P(f)* be flat from -W to W and zero elsewhere
- This is physically unrealizable
  - Sharp cutoff. However, this cannot be physically realized because they would require an infinite amount of time and resources.

Raised Cosine Spectrum

- Overcome the practical difficulties encountered with the ideal Nyquist channel by extending the bandwidth from the minimum value W = $R_b/2$ to an adjustable value between W and 2W. Where Rb is the Nyquist rate.

- Restrict the frequency band of interest to [- W, W], as shown by

$$P(f) + P(f - 2W) + P(f + 2W) = \frac{1}{2W}, \qquad -W \leq f \leq W$$

Raised Cosine Spectrum

- A particular form of *P(f)* that embodies many desirable features is provided by a *raised cosine spectrum.*
- This frequency response consists of a *flat* portion and *a roll off* portion that has a sinusoidal form, as follows:

$$P(f) = \begin{cases} \dfrac{1}{2W}, & 0 \le |f| < f_1 \\ \dfrac{1}{4W}\left\{1 - \sin\left[\dfrac{\pi(|f| - W)}{2W - 2f_1}\right]\right\}, & f_1 \le |f| < 2W - f_1 \\ 0, & |f| \ge 2W - f_1 \end{cases}$$

Raised Cosine Spectrum

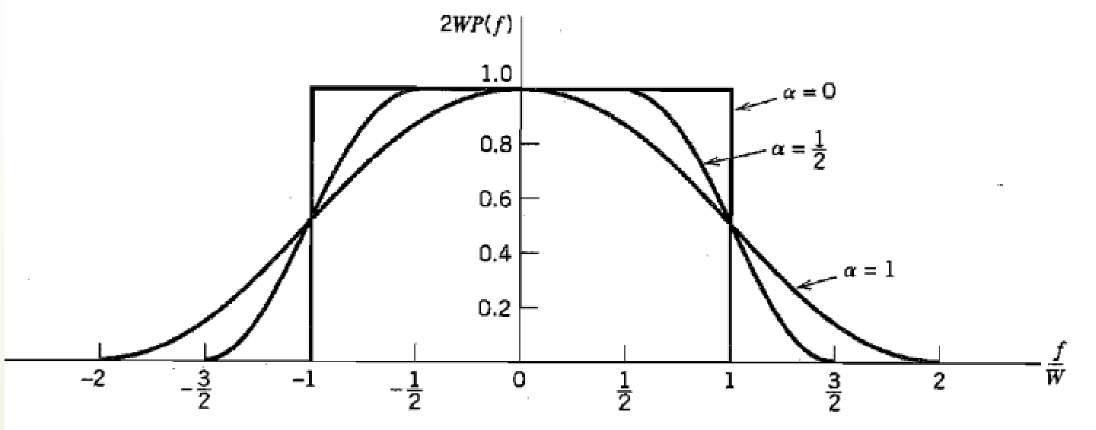
- The frequency parameter f1, and bandwidth W are related by

$$\alpha = 1 - \frac{f_1}{W}$$

- The parameter *a* is called the *rolloff factor;* it indicates the *excess bandwidth* over the ideal solution, W. Specifically, the transmission bandwidth BT is defined by

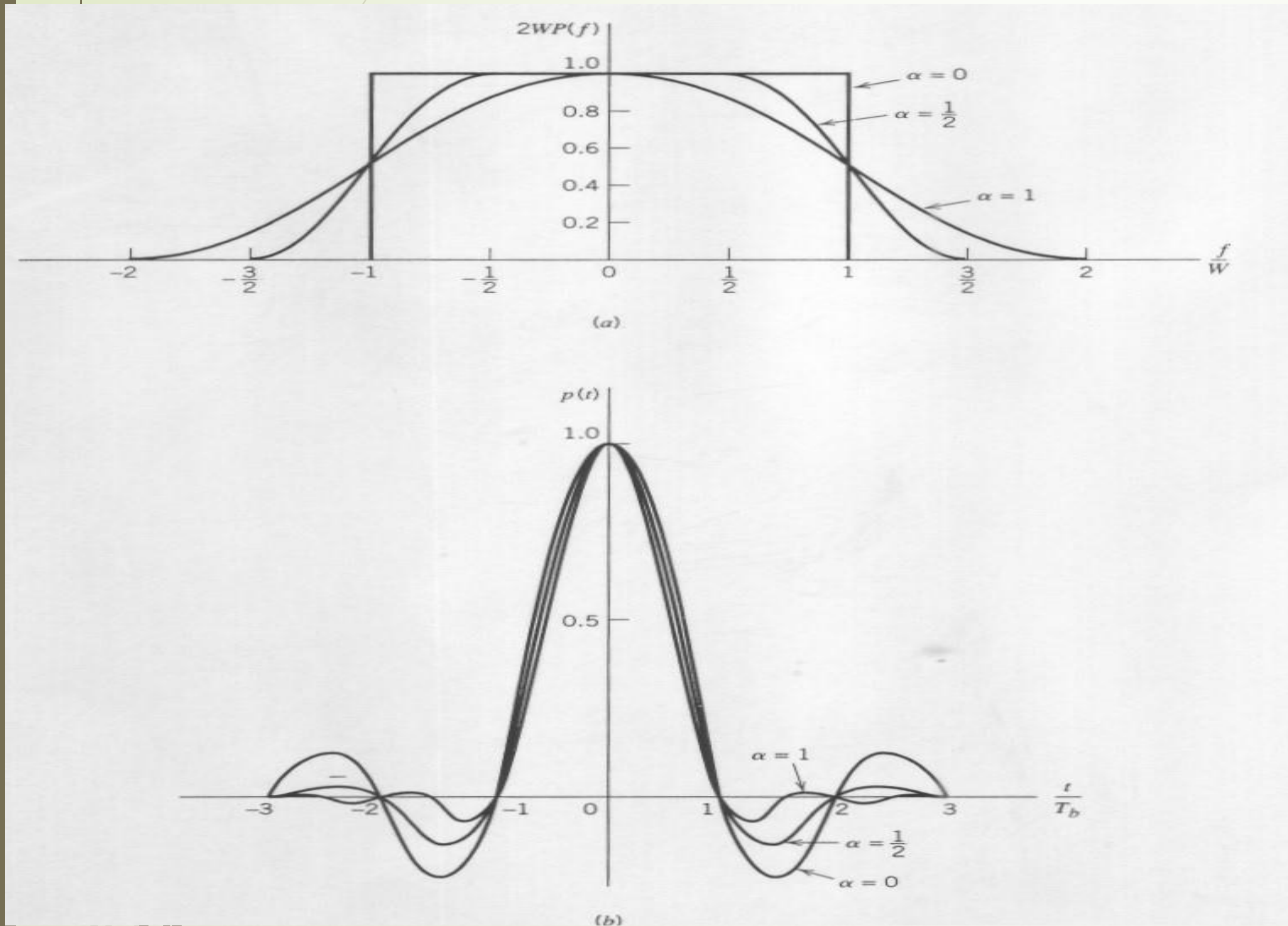
$$\begin{aligned} B_T &= 2W - f_1 \\ &= W(1 + \alpha) \end{aligned}$$

- The frequency response P(f), normalized by multiplying it by 2W, is plotted in Figure for three values of *a,* namely, 0, 0.5, and 1.



(a)

Raised Cosine Spectrum



$$P(f) = \begin{cases} \dfrac{1}{2W}, & 0 \le |f| < f_1 \\ \dfrac{1}{4W}\left\{1 - \sin\left[\dfrac{\pi(|f| - W)}{2W - 2f_1}\right]\right\}, & f_1 \le |f| < 2W - \\ 0, & |f| \ge 2W - \end{cases}$$

$$p(t) = (\operatorname{sinc}(2Wt))\left(\frac{\cos(2\pi\alpha Wt)}{1 - 16\alpha^2 W^2 t^2}\right)$$

Responses for different roll-off factors.
(a) Frequency response. (b) Time response.

Raised Cosine Spectrum

The special case with $\alpha = 1$ (i.e., $f_1 = 0$) is known as the *full-cosine rolloff*

$$\alpha = 1 - \frac{f_1}{W}$$

$$P(f) = \begin{cases} \dfrac{1}{2W}, & 0 \le |f| < f_1 \\ \dfrac{1}{4W}\left\{1 - \sin\left[\dfrac{\pi(|f| - W)}{2W - 2f_1}\right]\right\}, & f_1 \le |f| < 2W - f_1 \\ 0, & |f| \ge 2W - f_1 \end{cases}$$

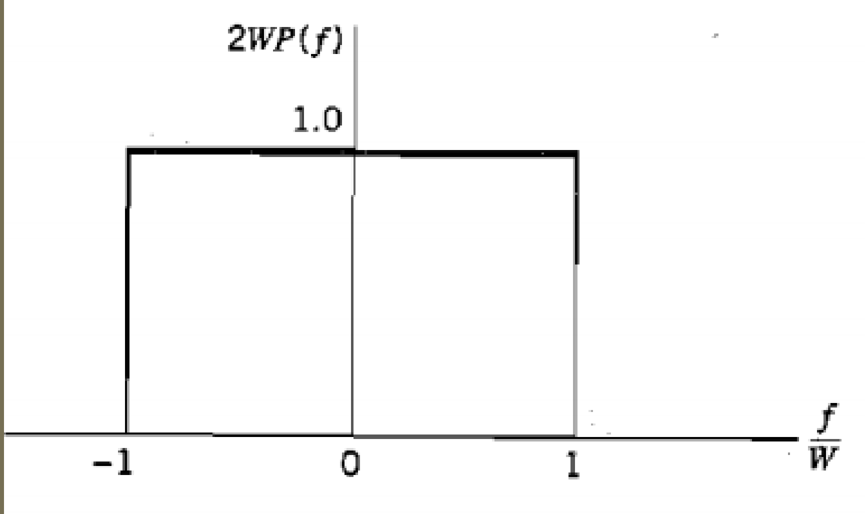
$$p(t) = (\text{sinc}(2Wt))\left(\frac{\cos(2\pi\alpha Wt)}{1 - 16\alpha^2 W^2 t^2}\right)$$

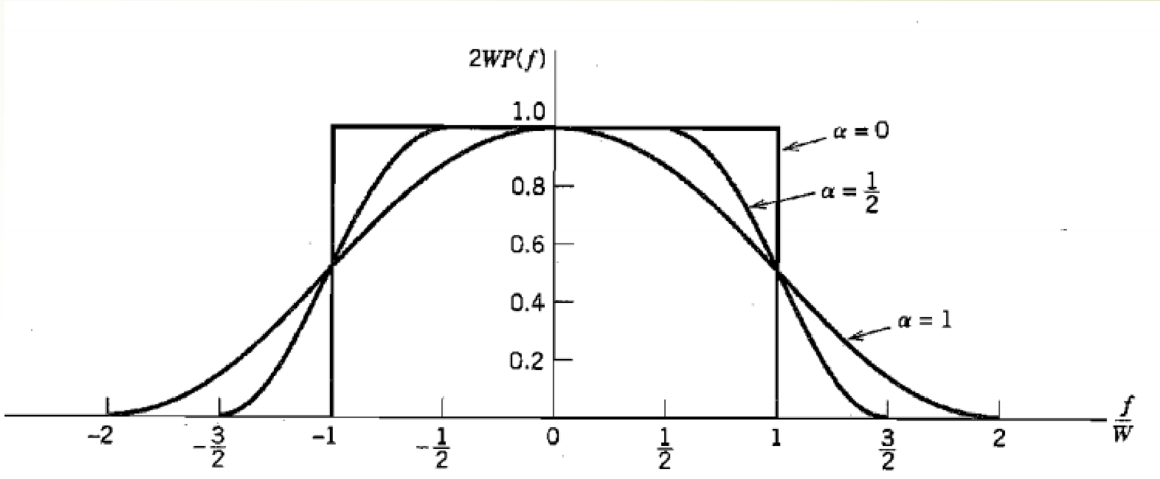
$$P(f) = \begin{cases} \dfrac{1}{4W}\left[1 + \cos\left(\dfrac{\pi f}{2W}\right)\right], & 0 < |f| < 2W \\ 0, & |f| \ge 2W \end{cases}$$

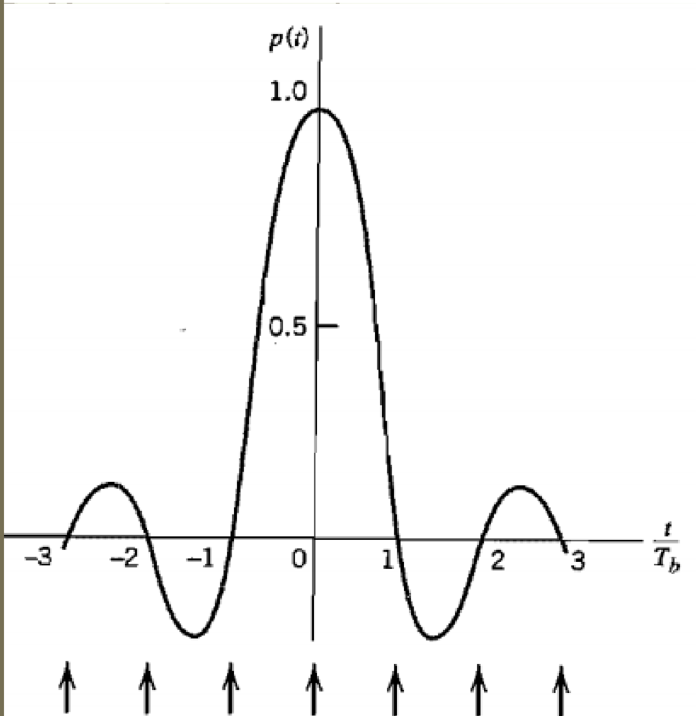
Correspondingly, the time response $p(t)$ simplifies to

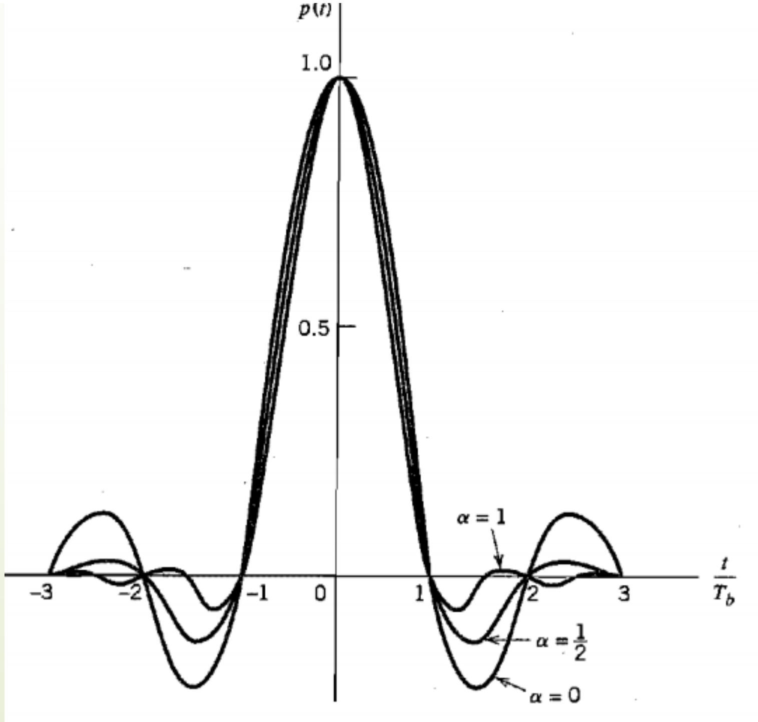
$$p(t) = \frac{\text{sinc}(4Wt)}{1 - 16W^2t^2}$$

Ideal Nyquist Channel and *raised cosine spectrum*









Error Function Table

TABLE A6.6 *The error function*[a]

u	$erf(u)$	u	$erf(u)$
0.00	0.00000	1.10	0.88021
0.05	0.05637	1.15	0.89612
0.10	0.11246	1.20	0.91031
0.15	0.16800	1.25	0.92290
0.20	0.22270	1.30	0.93401
0.25	0.27633	1.35	0.94376
0.30	0.32863	1.40	0.95229
0.35	0.37938	1.45	0.95970
0.40	0.42839	1.50	0.96611
0.45	0.47548	1.55	0.97162
0.50	0.52050	1.60	0.97635
0.55	0.56332	1.65	0.98038
0.60	0.60386	1.70	0.98379
0.65	0.64203	1.75	0.98667
0.70	0.67780	1.80	0.98909
0.75	0.71116	1.85	0.99111
0.80	0.74210	1.90	0.99279
0.85	0.77067	1.95	0.99418
0.90	0.79691	2.00	0.99532
0.95	0.82089	2.50	0.99959
1.00	0.84270	3.00	0.99998
1.05	0.86244	3.30	0.999998

[a]The error function is tabulated extensively in several references; see for example, Abramowitz and Stegun (1965, pp. 297–316).

$$\mathrm{erf}(u) = \frac{2}{\sqrt{\pi}} \int_0^u \exp(-z^2)\, dz$$

$\mathrm{erf}(-u) = -\mathrm{erf}(u)$

This is known as the *symmetry relation.*

The *complementary error function* is defined by

$$\mathrm{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^\infty \exp(-z^2)\, dz$$

which is related to the error function as follows:

$$\mathrm{erfc}(u) = 1 - \mathrm{erf}(u)$$

Table A6.6 gives values of the error function $\mathrm{erf}(u)$ for u in the range 0 to 3.3.

For large positive values of u, we have two simple bounds on $\mathrm{erfc}(u)$, one lower and the other upper, as shown by

$$\frac{\exp(-u^2)}{\sqrt{\pi u}}\left(1 - \frac{1}{2u^2}\right) < \mathrm{erfc}(u) < \frac{\exp(-u^2)}{\sqrt{\pi u}}$$

Example:

T1 carrier system used to multiplex 24 independent voice inputs, based on an 8-bit PCM word. It was shown that the bit duration of the resulting time-division multiplexed signal (including a framing bit) given that T_b=0.647 μs.

- Assuming the use of an *ideal* Nyquist channel, calculate the minimum transmission bandwidth?

Answer

$$B_T = W = \frac{1}{2T_b} = 772 \text{ kHz}$$

- What is the transmission bandwidth for a full-cosine rolloff characteristics with α=1.

Answer

$$B_T = W(1 + \alpha) = 2W = \frac{1}{T_b} = 1.544 \text{ MHz}$$

Example:

$$P_e = p_0 p_{10} + p_1 p_{01}$$
$$= \frac{p_0}{2}\,\mathrm{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right) + \frac{p_1}{2}\,\mathrm{erfc}\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

It is given that threshold $\lambda=0$ and and, $p_0=0.25$ get and expression for Pe.

Answer

$$P_e = P_0 P_{10} + P_1 P_{01}$$
$$= \frac{P_0}{2}\,\mathrm{erfc}\left(\frac{A+\lambda}{\sqrt{N_0/T_b}}\right) + P_1\,\mathrm{erfc}\left(\frac{A-\lambda}{\sqrt{N_0/T_b}}\right)$$

$\lambda=0,$
$$P_e = (P_0+P_1)\,\frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sqrt{N_0/T_b}}\right)$$

$$P_0+P_1 = 1$$

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sqrt{N_0/T_b}}\right)$$

Example:

A communication system transmits data at a rate of 1 Mbps over a noisy channel which has bandwith 1 MHz. The channel has a signal-to-noise ratio (SNR) of 3.52 dB. Calculate the probability of symbol error for this system. It is given that and λ=0 and an $P_1=p_0=0.5$.

$$\operatorname{erfc}(u) = 1 - \operatorname{erf}(u)$$

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

Answer

$$SNR = \frac{\text{Signal Power}}{\text{Noise Power}} = \frac{E_b \times \text{data rate}}{0.5 \times N_0 \times Bw}$$

$$= \frac{E_b \times 1\,\text{Mbps}}{0.5 \times N_0 \times 1\,\text{MHz}}$$

$$0.5\,SNR = \frac{E_b}{N_0}$$

$$10 \log(SNR) = 3.52\,\text{dB} \Rightarrow SNR = 10^{(3.52/10)} \approx 1.49$$

$$P_e = \frac{1}{2}\operatorname{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right) = \frac{1}{2}\operatorname{erfc}\left(\sqrt{0.5\,SNR}\right)$$

$$= \frac{1}{2}\operatorname{erfc}(0.745)$$

Examples:

An analog signal is sampled, quantized, and encoded into a binary PCM wave. The number of representation levels used is 128. A synchronizing pulse is added at the end of each code word representing a sample of the analog signal. The resulting PCM wave is transmitted over a channel of bandwidth 12 kHz using a quaternary PAM system with raised-cosine spectrum. The rolloff factor is unity.

(a) Find the rate (b/s) at which information is transmitted through the channel.

(b) Find the rate at which the analog signal is sampled. What is the maximum possible value for the highest frequency component of the analog signal?

Answer

$$B_T = 2W - f_1 = W(1 + \alpha)$$

for unity roll off

$B = \frac{1}{T}$, $T \rightarrow$ pulse length

$T = \frac{1}{12 KHz}$.

quarternary PAM $\Rightarrow$ 2 bit per pulse

$\frac{2 bit}{T} = 24 kbps$

128 quantizing levels $\Rightarrow$ 7 bits

synchronization bit $\Rightarrow$ 1 bit

8 bits

Tx rate $= 24$ Kbps

Sampling rate $= \frac{\text{Tx rate}}{\text{bit per sample}} = \frac{24 \text{ Kbps}}{8 \text{ bit}} = 3 \text{ kHz}$

highest frequency component $\Rightarrow$ 1.5 KHz

(to avoid aliasing)

Example:

A binary PCM system using polar NRZ signaling operates just above the error threshold with an average probability of error equal to 10^{-6}. Suppose that the signaling rate is doubled. Find the new value of the average probability of error. You may use Table A6.6 to evaluate the complementary error function.

Answer

$$P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$$

$$u = \sqrt{E_b/N_0}. \Rightarrow P_e = 10^{-6} = \frac{1}{2}\,\mathrm{erfc}(u) \Rightarrow u = 3.3$$

After the signaling rate is doubled new transmitted signal energy per bit $E_b' = \frac{E_b}{2}$

$$P_e' = \frac{1}{2}\,\mathrm{erfc}\left(\frac{u}{\sqrt{2}}\right) = \frac{1}{2}\,\mathrm{erfc}(2.33) = 10^{-3}.$$

Example:

- A binary PAM wave is to be transmitted over a baseband channel with an absolute maximum bandwidth of 75 kHz. The bit duration is 10 μs. Find a raised-cosine spectrum that satisfies these requirements.

Answer

The raised cosine pulse bandwidth $B = 2W - f_1$, where $W = 1/2T_b$. For this channel, $B = 75$ kHz. For the given bit duration, $W = 50$ kHz. Then,

$$f_1 = 2W - B$$

$$= 25 \text{ kHz}$$

$$\alpha = 1 - f_1/B_T$$

$$= 0.5$$

Example:

A computer puts out binary data at the rate of 56 kb/s. The computer output is transmitted using a baseband binary PAM system that is designed to have a raised-cosine spectrum. Determine the transmission bandwidth required for each of the following rolloff factors: $\alpha = 0.25, 0.5, 0.75, 1.0$.

Answer

$$B_T = 2W - f_1$$
$$= W(1 + \alpha)$$

- In a binary system, each symbol represents one bit. Therefore, the data rate is equal to the symbol rate. In this case, the data rate is given as 56 kb/s.
- Bandwidth = Data Rate / 2= 56 kb/s /2 = 28 kHz
- For α = 0.5: Bandwidth = 56 kb/s * (1 + 0.25) Bandwidth = 28 kHz * 1.25 Bandwidth
- For α = 0.5: Bandwidth = 56 kb/s * (1 + 0.5) Bandwidth = 28 kHz * 1.5 Bandwidth
- For α = 0.75: Bandwidth = 56 kb/s * (1 + 0.75) Bandwidth = 28 kHz * 1.75 Bandwidth
- For α = 1.0: Bandwidth = 56 kb/s * (1 + 1.0) Bandwidth = 28 kHz * 2.0 Bandwidth

Example:

A computer puts out binary data at the rate of 56 kb/s. The computer output is transmitted using a baseband binary PAM system that is designed to have a raised-cosine spectrum. Determine the transmission bandwidth required for each of the following rolloff factors: $\alpha = 0.25, 0.5, 0.75, 1.0$.

Repeat Problem 4.16, given that each set of three successive binary digits in the computer output are coded into one of eight possible amplitude levels, and the resulting signal is transmitted using an eight-level PAM system designed to have a raised-cosine spectrum.

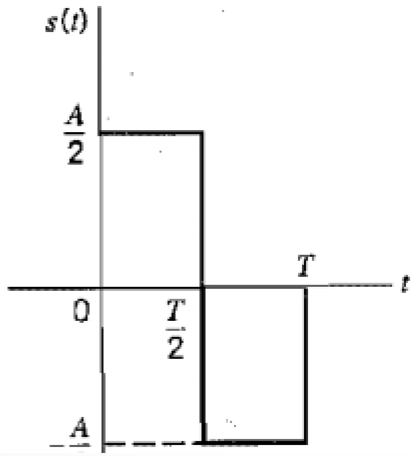
Answer

The use of eight amplitude levels ensures that 3 bits can be transmitted per pulse. The symbol period can be increased by a factor of 3. All four bandwidths in problem 7.12 will be reduced to 1/3 of their binary PAM values.

Example:

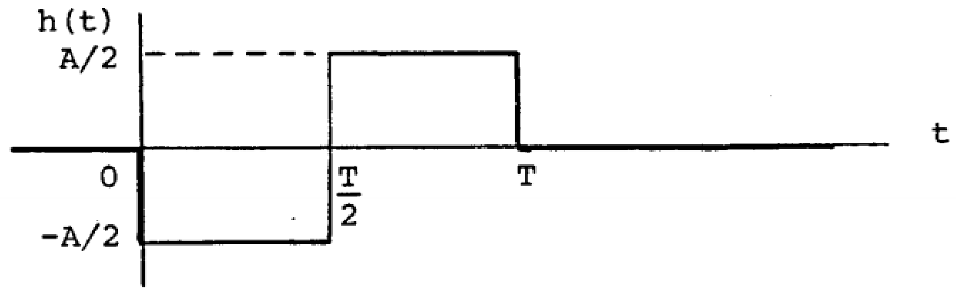
4.1 Consider the signal $s(t)$ shown in Figure P4.1.

(a) Determine the impulse response of a filter matched to this signal and sketch it as a function of time.



Answer

$$h_{\text{opt}}(t) = kg(T - t)$$



Example:

A continuous-time signal is sampled and then transmitted as a PCM signal. The random variable at the input of the decision device in the receiver has a variance of 0.01 volts^2.

(a) Assuming the use of polar NRZ signaling, determine the pulse amplitude that must be transmitted for the average error rate not to exceed 1 bit in 10^8 bits.

(b) If the added presence of interference causes the error rate to increase to 1 bit in 10^6 bits, what is the variance of the interference?

Answer

(a) The average probability of error is $P_e = \frac{1}{2}\,\mathrm{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)$

where $E_b = A^2T_b$. We may rewrite this formula as $P_e = \frac{1}{2}\,\mathrm{erfc}\left(\frac{A}{\sigma}\right)$

where A is the pulse amplitude at $\sigma = \sqrt{N_0T_b}$.

Example:

A PCM system uses a uniform quantizer followed by a 7-bit binary encoder. The bit rate of the system is equal to 50×10^6 b/s.

(a) What is the maximum message bandwidth for which the system operates satisfactorily?

Answer

(a) Let the message bandwidth be W. Then, sampling the message signal at its Nyquist rate, and using an R-bit code to represent each sample of the message signal, we find that the bit duration is

$$T_b = \frac{T_s}{R} = \frac{1}{2WR}$$

The bit rate is

$$\frac{1}{T_b} = 2WR$$

The maximum value of message bandwidth is therefore

$$W_{max} = \frac{50 \times 10^6}{2 \times 7}$$

$$= 3.57 \times 10^6 \text{ Hz}$$

THANK YOU