3: TRANSMISSION LINES

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CHAPTER OVERVIEW

3: Transmission Lines

[m0085_Reflection_Coefficient_for_Various_Terminations]

We now consider values of Γ that arise for commonly-encountered terminations.

Matched Load ($Z_L = Z_0$). In this case, the termination may be a device with impedance Z_0, or the termination may be another transmission line having the same characteristic impedance. When $Z_L = Z_0$, $\Gamma = 0$ and there is no reflection.

Open Circuit. An "open circuit" is the absence of a termination. This condition implies $Z_L \to \infty$, and subsequently $\Gamma \to +1$. Since the *current* reflection coefficient is $-\Gamma$, the reflected current wave is 180° out of phase with the incident current wave, making the total current at the open circuit equal to zero, as expected.

Short Circuit. "Short circuit" means $Z_L = 0$, and subsequently $\Gamma = -1$. In this case, the phase of Γ is 180°, and therefore, the potential of the reflected wave cancels the potential of the incident wave at the open circuit, making the total potential equal to zero, as it must be. Since the *current* reflection coefficient is $-\Gamma = +1$ in this case, the reflected current wave is in phase with the incident current wave, and the magnitude of the total current at the short circuit non-zero as expected.

Purely Reactive Load. A purely reactive load, including that presented by a capacitor or inductor, has $Z_L = jX$ where X is reactance. In particular, an inductor is represented by $X > 0$ and a capacitor is represented by $X < 0$. We find

$$\Gamma = \frac{-Z_0 + jX}{+Z_0 + jX}$$

The numerator and denominator have the same magnitude, so $|\Gamma| = 1$. Let ϕ be the phase of the denominator ($+Z_0 + jX$). Then, the phase of the numerator is $\pi - \phi$. Subsequently, the phase of Γ is $(\pi - \phi) - \phi = \pi - 2\phi$. Thus, we see that the phase of Γ is no longer limited to be 0° or 180°, but can be any value in between. The phase of reflected wave is subsequently shifted by this amount.

Other Terminations. Any other termination, including series and parallel combinations of any number of devices, can be expressed as a value of Z_L which is, in general, complex-valued. The associated value of $|\Gamma|$ is limited to the range 0 to 1. To see this, note:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1}$$

Note that the smallest possible value of $|\Gamma|$ occurs when the numerator is zero; i.e., when $Z_L = Z_0$. Therefore, the smallest value of $|\Gamma|$ is zero. The largest possible value of $|\Gamma|$ occurs when $Z_L/Z_0 \rightarrow \infty$ (i.e., an open circuit) or when $Z_L/Z_0 = 0$ (a short circuit); the result in either case is $|\Gamma| = 1$. Thus,

$$0 \leq |\Gamma| \leq 1$$

3.1: Introduction to Transmission Lines

> A transmission line is a structure intended to transport electromagnetic signals or power.

A rudimentary transmission line is simply a pair of wires with one wire serving as a datum (i.e., a reference; e.g., “ground”) and the other wire bearing an electrical potential that is defined relative to that datum. Transmission lines having random geometry, such as the test leads shown in Figure $3.1.1$, are useful only at very low frequencies and when loss, reactance, and immunity to electromagnetic interference (EMI) are not a concern.

Figure $3.1.1$: These leads used to connect test equipment to circuits in a laboratory are a very rudimentary form of transmission line, suitable only for very low frequencies. (Public Domain; Dmitry G)

However, many circuits and systems operate at frequencies where the length or cross-sectional dimensions of the transmission line may be a significant fraction of a wavelength. In this case, the transmission line is no longer “transparent” to the circuits at either end. Furthermore, loss, reactance, and EMI *are* significant problems in many applications. These concerns motivate the use of particular types of transmission lines, and make it necessary to understand how to properly connect the transmission line to the rest of the system.

In electromagnetics, the term “transmission line” refers to a structure which is intended to support a *guided wave*. A guided wave is an electromagnetic wave that is contained within or bound to the line, and which does not radiate away from the line. This condition is normally met if the length and cross-sectional dimensions of the transmission line are small relative to a wavelength – say $\lambda/100$ (i.e., 1% of the wavelength). For example, two randomly-arranged wires might serve well enough to carry a signal at $f = 10$ MHz over a length $l = 3$ cm, since l is only 0.1% of the wavelength $\lambda = c/f = 30$ m. However, if l is increased to 3 m, *or* if f is increased to 1 GHz, then l is now 10% of the wavelength. In this case, one should consider using a transmission line that forms a proper guided wave.

Preventing unintended radiation is not the only concern. Once we have established a guided wave on a transmission line, it is important that power applied to the transmission line be delivered to the circuit or device at the other end and not reflected back into the source. For the random wire $f = 10$ MHz, $l = 3$ cm example above, there is little need for concern, since we expect a phase shift of roughly $0.001 \cdot 360^{\circ} = 0.36^{\circ}$ over the length of the transmission line, which is about 0.72° for a round trip. So, to a good approximation, the entire transmission line is at the same electrical potential and thus transparent to the source and destination. However, if l is increased to 3 m, *or* if f is increased to 1 GHz, then the associated round-trip phase shift becomes 72°. In this case, a reflected signal traveling in the opposite direction will add to create a total electrical potential, which varies in both magnitude and phase with position along the line. Thus, the impedance looking toward the destination via the transmission line will be different than the impedance looking toward the destination directly (Section 3.15 gives the details). The modified impedance will depend on the cross-sectional geometry, materials, and length of the line.

Cross-sectional geometry and materials also determine the loss and EMI immunity of the transmission line.

Summarizing:

> Transmission lines are designed to support guided waves with controlled impedance, low loss, and a degree of immunity from EMI.

3.2: Types of Transmission Lines

Two common types of transmission line are **coaxial line** (Figure $3.2.1$) and **microstrip line** (Figure $3.2.2$). Both are examples of *transverse electromagnetic* (TEM) transmission lines. A TEM line employs a single electromagnetic wave "mode" having electric and magnetic field vectors in directions perpendicular to the axis of the line, as shown in Figures $3.2.3$ and $3.2.4$. TEM transmission lines appear primarily in radio frequency applications.

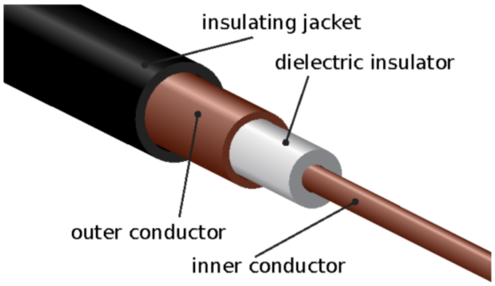


Figure $3.2.1$: Structure of a coaxial transmission line. (CC BY 3.0 (modified)).

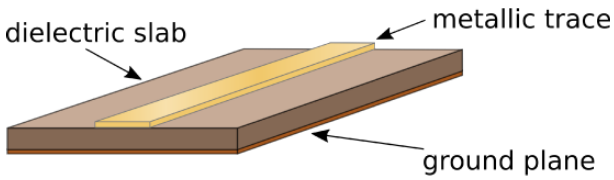


Figure $3.2.2$: Structure of a microstrip transmission line. (CC BY SA 3.0 (modified))

> TEM transmission lines such as coaxial lines and microstrip lines are designed to support a single electromagnetic wave that propagates along the length of the transmission line with electric and magnetic field vectors perpendicular to the direction of propagation.

Not all transmission lines exhibit TEM field structure. In non-TEM transmission lines, the electric and magnetic field vectors that are not necessarily perpendicular to the axis of the line, and the structure of the fields is complex relative to the field structure of TEM lines. An example of a transmission line that exhibits non-TEM field structure is the waveguide (see example in Figure $3.2.5$). Waveguides are most prevalent at radio frequencies, and tend to appear in applications where it is important to achieve very low loss or where power levels are very high. Another example is common "multimode" optical fiber (Figure $3.2.6$). Optical fiber exhibits complex field structure because the wavelength of light is very small compared to the cross-section of the fiber, making the excitation and propagation of non-TEM waves difficult to avoid. (This issue is overcome in a different type of optical fiber, known as "single mode" fiber, which is much more difficult and expensive to manufacture.)

> Higher-order transmission lines, including radio-frequency waveguides and multimode optical fiber, are designed to guide waves that have relatively complex structure.

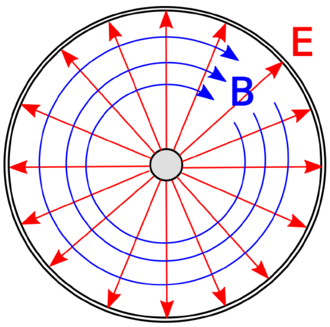


Figure $3.2.3$: Structure of the electric and magnetic fields within coaxial line. In this case, the wave is propagating away from the viewer.

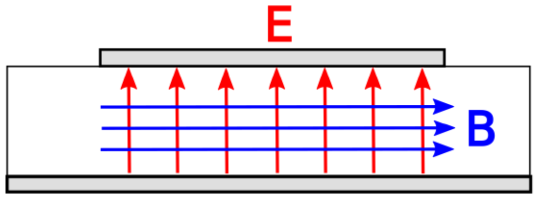


Figure 3.2.4: Structure of the electric and magnetic fields within microstrip line. (The fields *outside* the line are possibly significant, complicated, and not shown.) In this case, the wave is propagating away from the viewer. (CC BY SA 3.0 Unported).

Figure 3.2.5: A network of radio frequency waveguides in an air traffic control radar. (CC BY SA 2.0 Germany)

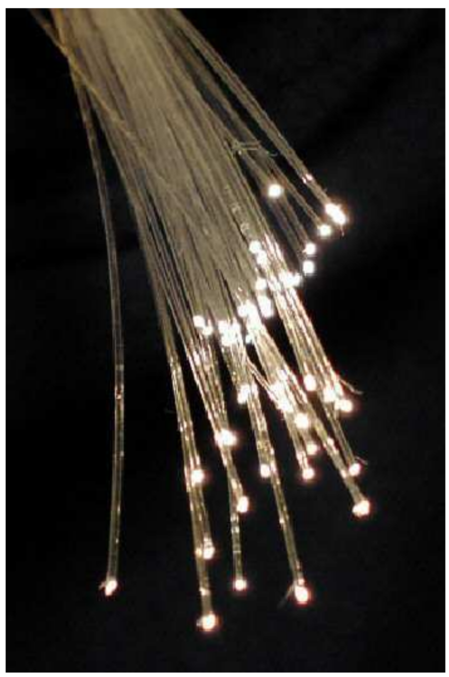


Figure 3.2.6: Strands of optical fiber.

3.3: Transmission Lines as Two-Port Devices

Figure $3.3.1$ shows common ways to represent transmission lines in circuit diagrams. In each case, the source is represented using a Thévenin equivalent circuit consisting of a voltage source V_S in series with an impedance Z_S.[1] In transmission line analysis, the source may also be referred to as the *generator*. The termination on the receiving end of the transmission line is represented, without loss of generality, as an impedance Z_L. This termination is often referred to as the *load*, although in practice it can be any circuit that exhibits an input impedance of Z_L.

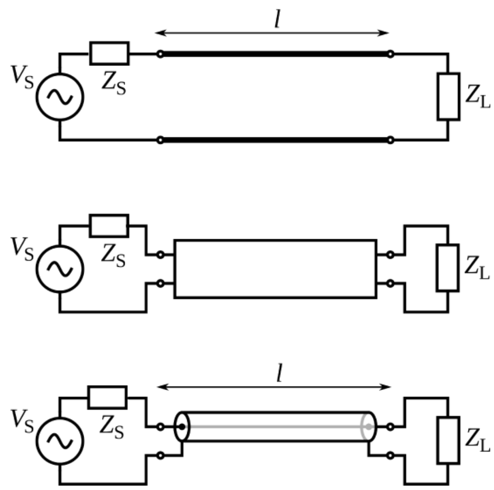


Figure $3.3.1$: Symbols representing transmission lines: Top: As a generic two-conductor direct connection. Middle: As a generic two-port "black box." Bottom: As a coaxial cable. © CC BY SA 3.0 Unported (modified)

The two-port representation of a transmission line is completely described by its length l along with some combination of the following parameters:

- Phase propagation constant β, having units of rad/m. This parameter also represents the wavelength in the line through the relationship $\lambda = 2\pi/\beta$. (See Sections 1.3 and 3.8 for details.)
- Attenuation constant α, having units of 1/m. This parameter quantifies the effect of loss in the line. (See Section 3.8 for details.)
- Characteristic impedance Z_0, having units of Ω. This is the ratio of potential ("voltage") to current when the line is perfectly impedance-matched at both ends. (See Section 3.7 for details.)

These parameters depend on the materials and geometry of the line.

Note that a transmission line is typically not transparent to the source and load. In particular, the load impedance may be Z_L, but the impedance presented to the source may or may not be equal to Z_L. (See Section 3.15 for more on this concept.) Similarly, the source impedance may be Z_S, but the impedance presented to the load may or may not be equal to Z_S. The effect of the transmission line on the source and load impedances will depend on the parameters identified above.

1. For a refresher on this concept, see "Additional Reading" at the end of this section.↩

3.4: Lumped-Element Model

It is possible to ascertain the relevant behaviors of a transmission line using elementary circuit theory applied to a differential-length lumped-element model of the transmission line. The concept is illustrated in Figure $3.4.1$, which shows a generic transmission line aligned with its length along the z axis. The transmission line is divided into segments having small but finite length Δz. Each segment is modeled as an identical two-port having the equivalent circuit representation shown in Figure $3.4.2$. The equivalent circuit consists of 4 components as follows:

- The resistance $R'\Delta z$ represents the series-combined ohmic resistance of the two conductors. This should account for *both* conductors since the current in the actual transmission line must flow through both conductors. The prime notation reminds us that R' is resistance *per unit length*; i.e., Ω/m, and it is only after multiplying by length that we get a resistance in Ω.
- The conductance $G'\Delta z$ represents the leakage of current directly from one conductor to the other. When $G'\Delta z > 0$, the resistance between the conductors is less than infinite, and therefore, current may flow between the conductors. This amounts to a loss of power separate from the loss associated with R' above. G' has units of S/m. Further note that G' is *not* equal to $1/R'$ as defined above. G' and R' are describing entirely different physical mechanisms (and in principle *either* could be defined as either a resistance or a conductance).
- The capacitance $C'\Delta z$ represents the capacitance of the transmission line structure. Capacitance is the tendency to store energy in electric fields and depends on the cross-sectional geometry and the media separating the conductors. C' has units of F/m.
- The inductance $L'\Delta z$ represents the inductance of the transmission line structure. Inductance is the tendency to store energy in magnetic fields, and (like capacitance) depends on the cross-sectional geometry and the media separating the conductors. L' has units of H/m.

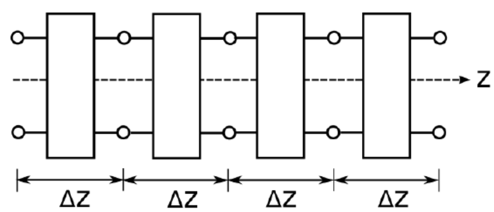


Figure $3.4.1$: Interpretation of a transmission line as a cascade of discrete series-connected two-ports.

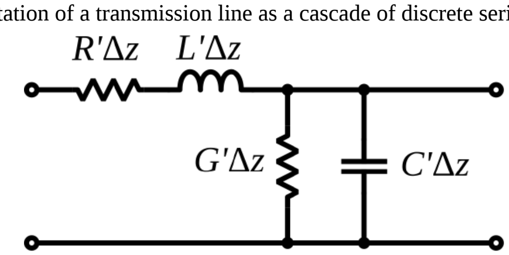


Figure $3.4.1$: Lumped-element equivalent circuit model for each of the two-ports in Figure $3.4.2$. (CC BY SA 3.0 Unported (modified))

In order to use the model, one must have values for R', G', C', and L'. Methods for computing these parameters are addressed elsewhere in this book.

3.5: Telegrapher's Equations

In this section, we derive the equations that govern the potential $v(z,t)$ and current $i(z,t)$ along a transmission line that is oriented along the z axis. For this, we will employ the lumped-element model developed in Section 3.4.

To begin, we define voltages and currents as shown in Figure $3.5.1$.

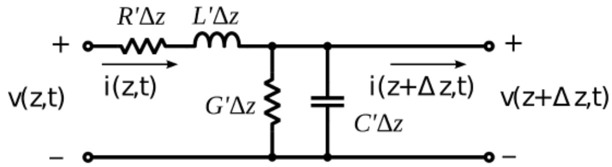


Figure $3.5.1$: Lumped-element equivalent circuit transmission line model, annotated with sign conventions for potentials and currents. (© CC BY SA 3.0 Unported (modified))

We assign the variables $v(z,t)$ and $i(z,t)$ to represent the potential and current on the left side of the segment, with reference polarity and direction as shown in the figure. Similarly we assign the variables $v(z+\Delta z,t)$ and $i(z+\Delta z,t)$ to represent the potential and current on the right side of the segment, again with reference polarity and direction as shown in the figure. Applying Kirchoff's voltage law from the left port, through $R'\Delta z$ and $L'\Delta z$, and returning via the right port, we obtain:

$$\begin{gathered} v(z,t) - (R'\Delta z)\, i(z,t) - (L'\Delta z)\,\frac{\partial}{\partial t} i(z,t) \\ -v(z+\Delta z,t) = 0 \end{gathered}$$

Moving terms referring to current to the right side of the equation and then dividing through by Δz, we obtain

$$\begin{gathered} -\frac{v(z+\Delta z,t) - v(z,t)}{\Delta z} = \\ R'\; i(z,t) + L'\;\frac{\partial}{\partial t} i(z,t) \end{gathered}$$

Then taking the limit as $\Delta z \to 0$:

$$\boxed{-\frac{\partial}{\partial z} v(z,t) = R'\; i(z,t) + L'\;\frac{\partial}{\partial t} i(z,t)} \tag{3.5.1}$$

Applying Kirchoff's current law at the right port, we obtain:

$$\begin{gathered} i(z,t) - (G'\Delta z)\, v(z+\Delta z,t) - (C'\Delta z)\,\frac{\partial}{\partial t} v(z+\Delta z,t) \\ -i(z+\Delta z,t) = 0 \end{gathered}$$

Moving terms referring to potential to the right side of the equation and then dividing through by Δz, we obtain

$$\begin{gathered} -\frac{i(z+\Delta z,t) - i(z,t)}{\Delta z} = \\ G'\; v(z+\Delta z,t) + C'\;\frac{\partial}{\partial t} v(z+\Delta z,t) \end{gathered}$$

Taking the limit as $\Delta z \to 0$:

$$\boxed{-\frac{\partial}{\partial z} i(z,t) = G'\; v(z,t) + C'\;\frac{\partial}{\partial t} v(z,t)} \tag{3.5.2}$$

> Equations $3.5.1$ and $3.5.2$ are the *telegrapher's equations*. These coupled (simultaneous) differential equations can be solved for $v(z,t)$ and $i(z,t)$ given R', G', L', C' and suitable boundary conditions.

The time-domain telegrapher's equations are usually more than we need or want. If we are only interested in the response to a sinusoidal stimulus, then considerable simplification is possible using phasor representation.[1] First we define phasors $\widetilde{V}(z)$ and $\tilde{I}(z)$ through the usual relationship:

$$v(z,t) = \text{Re}\left\{\widetilde{V}(z)\, e^{j\omega t}\right\}$$

$$i(z,t) = \text{Re}\left\{\tilde{I}\,(z)\, e^{j\omega t}\right\}$$

Now we see:

$$\begin{aligned}\frac{\partial}{\partial z}v(z,t) &= \frac{\partial}{\partial z}\text{Re}\left\{\widetilde{V}(z)\, e^{j\omega t}\right\} \\ &= \text{Re}\left\{\left[\frac{\partial}{\partial z}\widetilde{V}(z)\right]\, e^{j\omega t}\right\}\end{aligned}$$

In other words, $\partial v(z,t)/\partial z$ expressed in phasor representation is simply $\partial \widetilde{V}(z)/\partial z$; and

$$\begin{aligned}\frac{\partial}{\partial t}i(z,t) &= \frac{\partial}{\partial t}\text{Re}\left\{\tilde{I}\,(z)\, e^{j\omega t}\right\} \\ &= \text{Re}\left\{\frac{\partial}{\partial t}\left[\tilde{I}\,(z)e^{j\omega t}\right]\right\} \\ &= \text{Re}\left\{\left[j\omega\tilde{I}\,(z)\right]\, e^{j\omega t}\right\}\end{aligned}$$

In other words, $\partial i(z,t)/\partial t$ expressed in phasor representation is $j\omega\tilde{I}\,(z)$. Therefore, Equation 3.5.1 expressed in phasor representation is:

$$\boxed{-\frac{\partial}{\partial z}\widetilde{V}(z) = \left[R' + j\omega L'\right]\; \tilde{I}\,(z)} \tag{3.5.3}$$

Following the same procedure, Equation 3.5.2 expressed in phasor representation is found to be:

$$\boxed{-\frac{\partial}{\partial z}\tilde{I}\,(z) = \left[G' + j\omega C'\right]\; \widetilde{V}(z)} \tag{3.5.4}$$

Equations 3.5.3 and 3.5.4 are the telegrapher’s equations in phasor representation.

The principal advantage of these equations over the time-domain versions is that we no longer need to contend with derivatives with respect to time – only derivatives with respect to distance remain. This considerably simplifies the equations.

1. For a refresher on phasor analysis, see Section 1.5.↩

3.6: Wave Equation for a TEM Transmission Line

Consider a TEM transmission line aligned along the z axis. The phasor form of the Telegrapher's Equations (Section 3.5) relate the potential phasor $\widetilde{V}(z)$ and the current phasor $\tilde{I}(z)$ to each other and to the lumped-element model equivalent circuit parameters R', G', C', and L'. These equations are

$$-\frac{\partial}{\partial z}\widetilde{V}(z) = [R' + j\omega L']\ \tilde{I}(z) \tag{3.6.1}$$

$$-\frac{\partial}{\partial z}\tilde{I}(z) = [G' + j\omega C']\ \widetilde{V}(z) \tag{3.6.2}$$

An obstacle to using these equations is that we require both equations to solve for either the potential or the current. In this section, we reduce these equations to a single equation – a *wave equation* – that is more convenient to use and provides some additional physical insight.

We begin by differentiating both sides of Equation 3.6.1 with respect to z, yielding:

$$-\frac{\partial^2}{\partial z^2}\widetilde{V}(z) = [R' + j\omega L']\ \frac{\partial}{\partial z}\tilde{I}(z)$$

Then using Equation 3.6.2 to eliminate $\tilde{I}(z)$, we obtain

$$-\frac{\partial^2}{\partial z^2}\widetilde{V}(z) = -[R' + j\omega L'][G' + j\omega C']\ \widetilde{V}(z)$$

This equation is normally written as follows:

$$\boxed{\frac{\partial^2}{\partial z^2}\widetilde{V}(z) - \gamma^2\ \widetilde{V}(z) = 0} \tag{3.6.3}$$

where we have made the substitution:

$$\gamma^2 = (R' + j\omega L')(G' + j\omega C')$$

The principal square root of γ^2 is known as the *propagation constant*:

$$\gamma \triangleq \sqrt{(R' + j\omega L')(G' + j\omega C')} \tag{3.6.4}$$

> The *propagation constant* γ (units of m^{-1}) captures the effect of materials, geometry, and frequency in determining the variation in potential and current with distance on a TEM transmission line.

Following essentially the same procedure but beginning with Equation 3.6.2, we obtain

$$\boxed{\frac{\partial^2}{\partial z^2}\tilde{I}(z) - \gamma^2\ \tilde{I}(z) = 0} \tag{3.6.5}$$

> Equations 3.6.3 and 3.6.5 are the *wave equations* for $\widetilde{V}(z)$ and $\tilde{I}(z)$, respectively.

Note that both $\widetilde{V}(z)$ and $\tilde{I}(z)$ satisfy the *same* linear homogeneous differential equation. This does *not* mean that $\widetilde{V}(z)$ and $\tilde{I}(z)$ are equal. Rather, it means that $\widetilde{V}(z)$ and $\tilde{I}(z)$ can differ by no more than a multiplicative constant. Since $\widetilde{V}(z)$ is potential and $\tilde{I}(z)$ is current, that constant must be an impedance. This impedance is known as the *characteristic impedance* and is determined in Section 3.7.

The general solutions to Equations 3.6.3 and 3.6.5 are

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \tag{3.6.6}$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \tag{3.6.7}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are complex-valued constants. It is shown in Section 3.8 that Equations 3.6.6 and 3.6.7 represent sinusoidal waves propagating in the $+z$ and $-z$ directions along the length of the line. The constants may represent sources, loads, or simply discontinuities in the materials and/or geometry of the line. The values of the constants are determined by boundary conditions; i.e., constraints on $\widetilde{V}(z)$ and $\tilde{I}(z)$ at some position(s) along the line.

The reader is encouraged to verify that the Equations 3.6.6 and 3.6.7 are in fact solutions to Equations 3.6.3 and 3.6.5, respectively, for any values of the constants V_0^+, V_0^-, I_0^+, and I_0^-.

3.7: Characteristic Impedance

Characteristic impedance is the ratio of voltage to current for a wave that is propagating in single direction on a transmission line. This is an important parameter in the analysis and design of circuits and systems using transmission lines. In this section, we formally define this parameter and derive an expression for this parameter in terms of the equivalent circuit model introduced in Section 3.4.

Consider a transmission line aligned along the z axis. Employing some results from Section 3.6, recall that the phasor form of the wave equation in this case is

$$\frac{\partial^2}{\partial z^2}\widetilde{V}(z) - \gamma^2\,\widetilde{V}(z) = 0 \tag{3.7.1}$$

where

$$\gamma \triangleq \sqrt{(R' + j\omega L')(G' + j\omega C')} \tag{3.7.2}$$

Equation 3.7.1 relates the potential phasor $\widetilde{V}(z)$ to the equivalent circuit parameters R', G', C', and L'. An equation of the same form relates the current phasor $\tilde{I}(z)$ to the equivalent circuit parameters:

$$\frac{\partial^2}{\partial z^2}\tilde{I}(z) - \gamma^2\,\tilde{I}(z) = 0 \tag{3.7.3}$$

Since both $\widetilde{V}(z)$ and $\tilde{I}(z)$ satisfy the *same* linear homogeneous differential equation, they may differ by no more than a multiplicative constant. Since $\widetilde{V}(z)$ is potential and $\tilde{I}(z)$ is current, that constant can be expressed in units of impedance. Specifically, this is the *characteristic impedance*, so-named because it depends only on the materials and cross-sectional geometry of the transmission line – i.e., things which determine γ – and not length, excitation, termination, or position along the line.

To derive the characteristic impedance, first recall that the general solutions to Equations 3.7.1 and 3.7.3 are

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \tag{3.7.4}$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \tag{3.7.5}$$

where V_0^+, V_0^-, I_0^+, and I_0^- are complex-valued constants whose values are determined by boundary conditions; i.e., constraints on $\widetilde{V}(z)$ and $\tilde{I}(z)$ at some position(s) along the line. Also, we will make use of the telegrapher's equations (Section 3.5):

$$-\frac{\partial}{\partial z}\widetilde{V}(z) = [R' + j\omega L']\ \tilde{I}(z) \tag{3.7.6}$$

$$-\frac{\partial}{\partial z}\tilde{I}(z) = [G' + j\omega C']\ \widetilde{V}(z) \tag{3.7.7}$$

We begin by differentiating Equation 3.7.4 with respect to z, which yields

$$\frac{\partial}{\partial z}\widetilde{V}(z) = -\gamma\left[V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right]$$

Now we use this to eliminate $\partial\widetilde{V}(z)/\partial z$ in Equation 3.7.6, yielding

$$\gamma\left[V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right] = [R' + j\omega L']\ \tilde{I}(z)$$

Solving the above equation for $\tilde{I}(z)$ yields:

$$\tilde{I}(z) = \frac{\gamma}{R' + j\omega L'}\left[V_0^+ e^{-\gamma z} - V_0^- e^{+\gamma z}\right]$$

Comparing this to Equation 3.7.5, we note

$$I_0^+ = \frac{\gamma}{R' + j\omega L'}\ V_0^+$$

$$I_0^- = \frac{-\gamma}{R' + j\omega L'}\ V_0^-$$

We now make the substitution

$$Z_0 = \frac{R' + j\omega L'}{\gamma} \tag{3.7.8}$$

and observe

$$\boxed{\frac{V_0^+}{I_0^+} = \frac{-V_0^-}{I_0^-} \triangleq Z_0}$$

As anticipated, we have found that coefficients in the equations for potentials and currents are related by an impedance, namely, Z_0.Characteristic impedance can be written entirely in terms of the equivalent circuit parameters by substituting Equation 3.7.2 into Equation 3.7.8, yielding:

$$\boxed{Z_0 = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}}$$

The characteristic impedance Z_0 (Ω) is the ratio of potential to current in a wave traveling in a single direction along the transmission line.

Take care to note that Z_0 is *not* the ratio of $\widetilde{V}(z)$ to $\tilde{I}(z)$ in general; rather, Z_0 relates only the potential and current waves traveling in the *same* direction.

Finally, note that transmission lines are normally designed to have a characteristic impedance that is completely real-valued – that is, with no imaginary component. This is because the imaginary component of an impedance represents energy *storage* (think of capacitors and inductors), whereas the purpose of a transmission line is energy *transfer*.

3.8: Wave Propagation on a TEM Transmission Line

In Section 3.6, it is shown that expressions for the phasor representations of the potential and current along a transmission line are

$$\widetilde{V}(z) = V_0^+ e^{-\gamma z} + V_0^- e^{+\gamma z} \tag{3.8.1}$$

$$\tilde{I}(z) = I_0^+ e^{-\gamma z} + I_0^- e^{+\gamma z} \tag{3.8.2}$$

where γ is the propagation constant and it assumed that the transmission line is aligned along the z axis. In this section, we demonstrate that these expressions represent sinusoidal waves, and point out some important features. Before attempting this section, the reader should be familiar with the contents of Sections 3.4, 3.6 and 3.7. A refresher on fundamental wave concepts (Section 1.3) may also be helpful.

We first define real-valued quantities α and β to be the real and imaginary components of γ; i.e.,

$$\alpha \triangleq \text{Re}\{\gamma\}$$

$$\beta \triangleq \text{Im}\{\gamma\}$$

and subsequently

$$\gamma = \alpha + j\beta$$

Then we observe

$$e^{\pm\gamma z} = e^{\pm(\alpha+j\beta)z} = e^{\pm\alpha z}\, e^{\pm j\beta z}$$

It may be easier to interpret this expression by reverting to the time domain:

$$\text{Re}\left\{e^{\pm\gamma z} e^{j\omega t}\right\} = e^{\pm\alpha z}\cos(\omega t \pm \beta z)$$

Thus, $e^{-\gamma z}$ represents a damped sinusoidal wave traveling in the $+z$ direction, and $e^{+\gamma z}$ represents a damped sinusoidal wave traveling in the $-z$ direction.

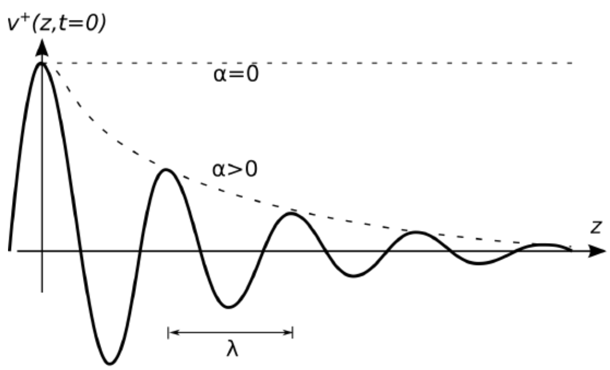


Figure 3.8.1: The potential $v^+(z,t)$ of the wave traveling in the $+z$ direction at $t=0$ for $\psi=0$.

Let's define $\widetilde{V}^+(z)$ and $\tilde{I}^+(z)$ to be the potential and current associated with a wave propagating in the $+z$ direction. Then:

$$\widetilde{V}^+(z) \triangleq V_0^+ e^{-\gamma z}$$

or equivalently in the time domain:

$$\begin{aligned} v^+(z,t) &= \text{Re}\left\{\widetilde{V}^+(z) e^{j\omega t}\right\} \\ &= \text{Re}\left\{V_0^+ e^{-\gamma z} e^{j\omega t}\right\} \\ &= \left|V_0^+\right| e^{-\alpha z}\cos(\omega t - \beta z + \psi) \end{aligned}$$

where ψ is the phase of V_0^+. Figure $3.8.1$ shows $v^+(z,t)$. From fundamental wave theory we recognize

> $\beta \triangleq \text{Im}\{\gamma\}$ (rad/m) is the *phase propagation constant*, which is the rate at which phase changes as a function of distance.

Subsequently the wavelength in the line is

$$\boxed{\lambda = \frac{2\pi}{\beta}}$$

Also we recognize:

> $\alpha \triangleq \text{Re}\{\gamma\}$ (1/m) is the *attenuation constant*, which is the rate at which magnitude diminishes as a function of distance.

Sometimes the units of α are indicated as “Np/m” (“nepers” per meter), where the term “neper” is used to indicate the units of the otherwise unitless real-valued exponent of the constant e.

Note that $\alpha = 0$ for a wave that does not diminish in magnitude with increasing distance, in which case the transmission line is said to be *lossless*. If $\alpha > 0$ then the line is said to be *lossy* (or possibly “low loss” if the loss can be neglected), and in this case the rate at which the magnitude decreases with distance increases with α.

Next let us consider the speed of the wave. To answer this question, we need to be a bit more specific about what we mean by “speed.” At the moment, we mean phase velocity; that is, the speed at which a point of constant phase seems to move through space. In other words, what distance Δz does a point of constant phase traverse in time Δt? To answer this question, we first note that the phase of $v^{+}(z,t)$ can be written generally as

$$\omega t - \beta z + \phi$$

where ϕ is some constant. Similarly, the phase at some time Δt later and some point Δz further along can be written as

$$\omega(t+\Delta t) - \beta(z+\Delta z) + \phi$$

The phase velocity v_p is $\Delta z/\Delta t$ when these two phases are equal; i.e., when

$$\omega t - \beta z + \phi = \omega(t+\Delta t) - \beta(z+\Delta z) + \phi$$

Solving for $v_p = \Delta z/\Delta t$, we obtain:

$$\boxed{v_p = \frac{\omega}{\beta}}$$

Having previously noted that $\beta = 2\pi/\lambda$, the above expression also yields the expected result

$$\boxed{v_p = \lambda f}$$

> The phase velocity $v_p = \omega/\beta = \lambda f$ is the speed at which a point of constant phase travels along the line.

Returning now to consider the current associated with the wave traveling in the $+z$ direction:

$$\tilde{I}^{+}(z) = I_0^{+} e^{-\gamma z}$$

We can rewrite this expression in terms of the characteristic impedance Z_0, as follows:

$$\tilde{I}^{+}(z) = \frac{V_0^{+}}{Z_0} e^{-\gamma z}$$

Similarly, we find that the current $\tilde{I}^{-}(z)$ associated with $\widetilde{V}^{-}(z)$ for the wave traveling in the $-z$ direction is

$$\tilde{I}^{-}(z) = \frac{-V_0^{-}}{Z_0} e^{+\gamma z}$$

The negative sign appearing in the above expression emerges as a result of the sign conventions used for potential and current in the derivation of the telegrapher’s equations (Section 3.5). The physical significance of this change of sign is that wherever the potential of the wave traveling in the $-z$ direction is positive, then the current at the same point is flowing in the $-z$ direction.

It is frequently necessary to consider the possibility that waves travel in both directions simultaneously. A very important case where this arises is when there is reflection from a discontinuity of some kind; e.g., from a termination which is not perfectly impedance-matched. In this case, the total potential $\widetilde{V}(z)$ and total current $\tilde{I}(z)$ can be expressed as the general solution to the wave equation; i.e., as the sum of the "incident" ($+z$-traveling) wave and the reflected ($-z$-traveling) waves:

$$\widetilde{V}(z) = \widetilde{V}^{+}(z) + \widetilde{V}^{-}(z)$$

$$\tilde{I}(z) = \tilde{I}^{+}(z) + \tilde{I}^{-}(z)$$

The existence of waves propagating simultaneously in both directions gives rise to a phenomenon known as a *standing wave*. Standing waves and the calculation of the coefficients V_0^- and I_0^- due to reflection are addressed in Sections 3.13 and 3.12 respectively.

3.9: Lossless and Low-Loss Transmission Lines

Quite often the loss in a transmission line is small enough that it may be neglected. In this case, several aspects of transmission line theory may be simplified. In this section, we present these simplifications.

First, recall that "loss" refers to the reduction of magnitude as a wave propagates through space. In the lumped-element equivalent circuit model (Section 3.4), the parameters R' and G' of the represent physical mechanisms associated with loss. Specifically, R' represents the resistance of conductors, whereas G' represents the undesirable current induced between conductors through the spacing material. Also recall that the propagation constant γ is, in general, given by

$$\gamma \triangleq \sqrt{(R'+j\omega L')(G'+j\omega C')}$$

With this in mind, we now define "low loss" as meeting the conditions:

$$R' \ll \omega L' \tag{3.9.1}$$

$$G' \ll \omega C' \tag{3.9.2}$$

When these conditions are met, the propagation constant simplifies as follows:

$$\begin{aligned}\gamma &\approx \sqrt{(j\omega L')(j\omega C')} \\ &= \sqrt{-\omega^2 L'C'} \\ &= j\omega\sqrt{L'C'}\end{aligned}$$

and subsequently

$$\alpha \triangleq \operatorname{Re}\{\gamma\} \approx 0 \qquad \text{(low-loss approx.)} \tag{3.9.3}$$

$$\beta \triangleq \operatorname{Im}\{\gamma\} \approx \omega\sqrt{L'C'} \qquad \text{(low-loss approx.)} \tag{3.9.4}$$

$$v_p = \omega/\beta \approx \frac{1}{\sqrt{L'C'}} \qquad \text{(low-loss approx.)} \tag{3.9.5}$$

Similarly:

$$Z_0 = \sqrt{\frac{R'+j\omega L'}{G'+j\omega C'}} \approx \sqrt{\frac{L'}{C'}} \quad \text{(low-loss approx.)}$$

Of course if the line is strictly lossless (i.e., $R' = G' = 0$) then these are not approximations, but rather the exact expressions.

In practice, these approximations are quite commonly used, since practical transmission lines typically meet the conditions expressed in Inequalities 3.9.1 and 3.9.2 and the resulting expressions are much simpler. We further observe that Z_0 and v_p are approximately independent of frequency when these conditions hold.

However, also note that "low loss" does not mean "no loss," and it is common to apply these expressions even when R' and/or G' is large enough to yield significant loss. For example, a coaxial cable used to connect an antenna on a tower to a radio near the ground typically has loss that is important to consider in the analysis and design process, but nevertheless satisfies Equations 3.9.1 and 3.9.2. In this case, the low-loss expression for β is used, but α might not be approximated as zero.

3.10: Coaxial Line

Coaxial transmission lines consists of metallic inner and outer conductors separated by a spacer material as shown in Figure 3.10.1. The spacer material is typically a low-loss dielectric material having permeability approximately equal to that of free space ($\mu \approx \mu_0$) and permittivity ϵ_s that may range from very near ϵ_0 (e.g., air-filled line) to 2–3 times ϵ_0. The outer conductor is alternatively referred to as the "shield," since it typically provides a high degree of isolation from nearby objects and electromagnetic fields. Coaxial line is single-ended[1] in the sense that the conductor geometry is asymmetric and the shield is normally attached to ground at both ends. These characteristics make coaxial line attractive for connecting single-ended circuits in widely-separated locations and for connecting antennas to receivers and transmitters.

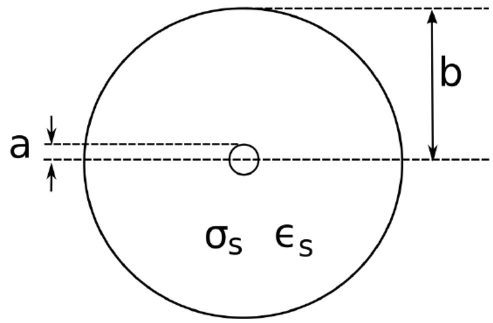


Figure 3.10.1: Cross-section of a coaxial transmission line, indicating design parameters.

Coaxial lines exhibit TEM field structure as shown in Figure 3.10.2

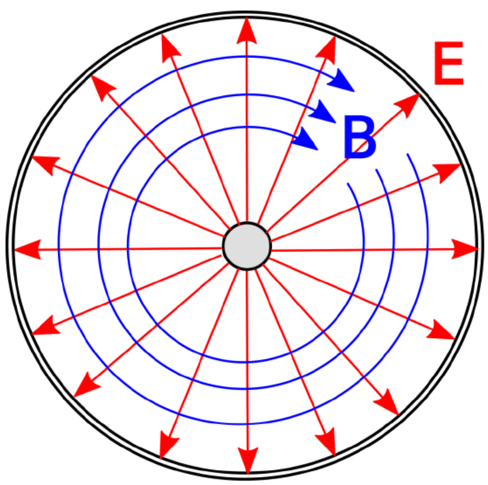


Figure 3.10.2: Structure of the electric and magnetic fields within coaxial line. In this case, the wave is propagating away from the viewer.

Expressions for the equivalent circuit parameters C' and L' for coaxial lines can be obtained from basic electromagnetic theory. It is shown in Section 5.24 that the capacitance per unit length is

$$C' = \frac{2\pi\epsilon_s}{\ln(b/a)} \tag{3.10.1}$$

where a and b are the radii of the inner and outer conductors, respectively. Using analysis shown in Section 7.14, the inductance per unit length is

$$L' = \frac{\mu_0}{2\pi}\ln\left(\frac{b}{a}\right) \tag{3.10.2}$$

The loss conductance G' depends on the conductance σ_s of the spacer material, and is given by

$$G' = \frac{2\pi\sigma_s}{\ln(b/a)}$$

This expression is derived in Section 6.5.

The resistance per unit length, R', is relatively difficult to quantify. One obstacle is that the inner and outer conductors typically consist of different materials or compositions of materials. The inner conductor is not necessarily a single homogeneous material; instead, the inner conductor may consist of a variety of materials selected by trading-off between conductivity, strength, weight, and cost. Similarly, the outer conductor is not necessarily homogeneous; for a variety of reasons, the outer conductor may instead be a metal mesh, a braid, or a composite of materials. Another complicating factor is that the resistance of the conductor varies significantly with frequency, whereas C', L', and G' exhibit relatively little variation from their electro- and magnetostatic values. These factors make it difficult to devise a single expression for R' that is both as simple as those shown above for the other parameters and generally applicable. Fortunately, it turns out that the low-loss conditions $R' \ll \omega L'$ and $G' \ll \omega C'$ are often applicable,[2] so that R' and G' are important only if it is necessary to compute loss.

Since the low-loss conditions are often met, a convenient expression for the characteristic impedance is obtained from Equations 3.10.1 and 3.10.2 for L' and C' respectively:

$$\begin{aligned} Z_0 &\approx \sqrt{\frac{L'}{C'}} \text{(low-loss)} \\ &= \frac{1}{2\pi}\sqrt{\frac{\mu_0}{\epsilon_s}} \ln\frac{b}{a} \end{aligned}$$

The spacer permittivity can be expressed as $\epsilon_s = \epsilon_r\epsilon_0$ where ϵ_r is the relative permittivity of the spacer material. Since $\sqrt{\mu_0/\epsilon_0}$ is a constant, the above expression is commonly written

$$Z_0 \approx \frac{60\ \Omega}{\sqrt{\epsilon_r}} \ln\frac{b}{a} \quad \text{(low-loss)}$$

Thus, it is possible to express Z_0 directly in terms of parameters describing the geometry (a and b) and material (ϵ_r) used in the line, without the need to first compute the values of components in the lumped-element equivalent circuit model.

Similarly, the low-loss approximation makes it possible to express the phase velocity ν_p directly in terms the spacer permittivity:

$$\begin{aligned} \nu_p &\approx \frac{1}{\sqrt{L'C'}} \text{ (low-loss)} \\ &= \frac{c}{\sqrt{\epsilon_r}} \end{aligned}$$

since $c \triangleq 1/\sqrt{\mu_0\epsilon_0}$. In other words, the phase velocity in a low-loss coaxial line is approximately equal to the speed of electromagnetic propagation in free space, divided by the square root of the relative permittivity of the spacer material. Therefore, the phase velocity in an air-filled coaxial line is approximately equal to speed of propagation in free space, but is reduced in a coaxial line using a dielectric spacer.

Example 3.10.1: RG-59 Coaxial Cable

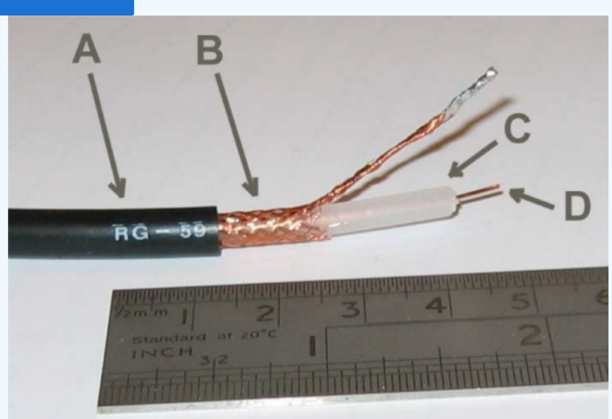


Figure 3.10.3: RG-59 coaxial line. A: Insulating jacket. B: Braided outer conductor. C: Dielectric spacer. D: Inner conductor (© CC BY SA 3.0; Arj)

RG-59 is a very common type of coaxial line. Figure 3.10.3 shows a section of RG-59 cut away so as to reveal its structure. The radii are $a \cong 0.292$ mm and $b \approx 1.855$ mm (mean), yielding $L' \approx 370$ nH/m. The spacer material is polyethylene having

$\epsilon_r \cong 2.25$, yielding $C' \approx 67.7$ pF/m. The conductivity of polyethylene is $\sigma_s \cong 5.9 \times 10^{-5}$ S/m, yielding $G' \approx 200\ \mu$S/m. Typical resistance per unit length R' is on the order of $0.1\ \Omega$/m near DC, increasing approximately in proportion to the square root of frequency.

From the above values, we find that RG-59 satisfies the low-loss criteria $R' \ll \omega L'$ for $f \gg 43$ kHz and $G' \ll \omega C'$ for $f \gg 470$ kHz. Under these conditions, we find $Z_0 \approx \sqrt{L'/C'} \cong 74\ \Omega$. Thus, the ratio of the potential to the current in a wave traveling in a single direction on RG-59 is about $74\ \Omega$.

The phase velocity of RG-59 is found to be $v_p \approx 1/\sqrt{L'C'} \cong 2 \times 10^8$ m/s, which is about 67% of c. In other words, a signal that takes 1 ns to traverse a distance l in free space requires about 1.5 ns to traverse a length-l section of RG-59. Since $v_p = \lambda f$, a wavelength in RG-59 is 67% of a wavelength in free space.

Using the expression

$$\gamma = \sqrt{(R' + j\omega L')\,(G' + j\omega C')}$$

with $R' = 0.1\ \Omega$/m, and then taking the real part to obtain α, we find $\alpha \sim 0.01\ \mathrm{m}^{-1}$. So, for example, the magnitude of the potential or current is decreased by about 50% by traveling a distance of about 70 m. In other words, $e^{-\alpha l} = 0.5$ for $l \sim 70$ m at relatively low frequencies, and increases with increasing frequency.

Additional Reading:

- "Coaxial cable" on Wikipedia. *Includes descriptions and design parameters for a variety of commonly-encountered coaxial cables.*
- "Single-ended signaling" on Wikipedia.
- Sec. 8.7 ("Differential Circuits") in S.W. Ellingson, *Radio Systems Engineering*, Cambridge Univ. Press, 2016.

1. The references in "Additional Reading" at the end of this section may be helpful if you are not familiar with this concept.↩
2. See Section 3.9 for a reminder about this concept.↩

3.11: Microstrip Line

A microstrip transmission line consists of a narrow metallic trace separated from a metallic ground plane by a slab of dielectric material, as shown in Figure $3.11.1$. This is a natural way to implement a transmission line on a printed circuit board, and so accounts for an important and expansive range of applications. The reader should be aware that microstrip is distinct from *stripline*, which is a very different type of transmission line; see "Additional Reading" at the end of this section for disambiguation of these terms.

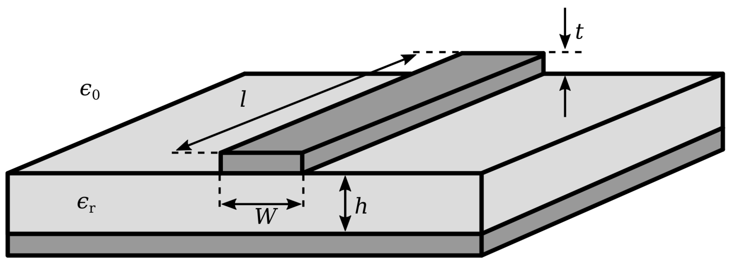


Figure 3.11.1: Microstrip transmission line structure and design parameters. (© CC BY SA 3.0 (modified))

A microstrip line is single-ended[1] in the sense that the conductor geometry is asymmetric and the one conductor – namely, the ground plane – also normally serves as ground for the source and load.

The spacer material is typically a low-loss dielectric material having permeability approximately equal to that of free space ($\mu \approx \mu_0$) and relative permittivity ϵ_r in the range 2 to about 10 or so.

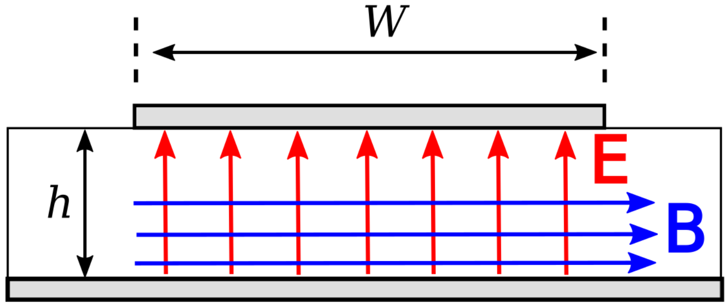


Figure 3.11.2: Structure of the electric and magnetic fields within microstrip line. (The fields *outside* the line are possibly significant, complicated, and not shown.) In this case, the wave is propagating away from the viewer.

A microstrip line nominally exhibits TEM field structure. This structure is shown in Figure $3.11.2$. Note that electric and magnetic fields exist both in the dielectric and in the space above the dielectric, which is typically (but not always) air. This complex field structure makes it difficult to describe microstrip line concisely in terms of the equivalent circuit parameters of the lumped-element model. Instead, expressions for Z_0 directly in terms of h/W and ϵ_r are typically used instead. A variety of these expressions are in common use, representing different approximations and simplifications. A widely-accepted and broadly-applicable expression is:[2]

$$Z_0 \approx \frac{42.4\ \Omega}{\sqrt{\epsilon_r + 1}} \times \ln\left[1 + \frac{4h}{W'}\left(\Phi + \sqrt{\Phi^2 + \frac{1+1/\epsilon_r}{2}\pi^2}\right)\right] \tag{3.11.1}$$

where

$$\Phi \triangleq \frac{14 + 8/\epsilon_r}{11}\left(\frac{4h}{W'}\right)$$

and W' is W adjusted to account for the thickness t of the microstrip line. Typically $t \ll W$ and $t \ll h$, for which $W' \approx W$. Simpler approximations for Z_0 are also commonly employed in the design and analysis of microstrip lines. These expressions are limited in the range of h/W for which they are valid, and can usually be shown to be special cases or approximations of Equation $3.11.1$. Nevertheless, they are sometimes useful for quick "back of the envelope" calculations.

Accurate expressions for wavelength λ, phase propagation constant β, and phase velocity v_p are similarly difficult to obtain for waves in microstrip line. An approximate technique employs a result from the theory of uniform plane waves in unbounded media (Equation 9.2.18 from Section 9.2):

$$\beta = \omega\sqrt{\mu\epsilon}$$

It turns out that the electromagnetic field structure in the space between the conductors is well-approximated as that of a uniform plane wave in unbounded media having the same permeability μ_0 but a different relative permittivity, which we shall assign the symbol $\epsilon_{r,eff}$ (for "*effective* relative permittivity"). Then

$$\begin{aligned}\beta &\approx \omega\sqrt{\mu_0\epsilon_{r,eff}\epsilon_0} \text{ (low-loss microstrip)} \\ &= \beta_0\sqrt{\epsilon_{r,eff}}\end{aligned}$$

In other words, the phase propagation constant in a microstrip line can be approximated as the free-space phase propagation $\beta_0 \triangleq \omega\sqrt{\mu_0\epsilon_0}$ times a correction factor $\sqrt{\epsilon_{r,eff}}$. Then $\epsilon_{r,eff}$ may be crudely approximated as follows:

$$\epsilon_{r,eff} \approx \frac{\epsilon_r + 1}{2}$$

i.e., $\epsilon_{r,eff}$ is roughly the average of the relative permittivity of the dielectric slab and the relative permittivity of free space. The assumption employed here is that $\epsilon_{r,eff}$ is approximately the average of these values because some fraction of the power in the guided wave is in the dielectric, and the rest is above the dielectric. Various approximations are available to improve on this approximation; however, in practice variations in the value of ϵ_r for the dielectric due to manufacturing processes typically make a more precise estimate irrelevant.

Using this concept, we obtain

$$\lambda = \frac{2\pi}{\beta} = \frac{2\pi}{\beta_0\sqrt{\epsilon_{r,eff}}} = \frac{\lambda_0}{\sqrt{\epsilon_{r,eff}}}$$

where λ_0 is the free-space wavelength c/f. Similarly the phase velocity v_p, can be estimated using the relationship

$$v_p = \frac{\omega}{\beta} = \frac{c}{\sqrt{\epsilon_{r,eff}}}$$

i.e., the phase velocity in microstrip is slower than c by a factor of $\sqrt{\epsilon_{r,eff}}$.

Example 3.11.1: 50 Ω Microstrip in FR4 Printed Circuit Boards.

FR4 is a low-loss fiberglass epoxy dielectric that is commonly used to make printed circuit boards (see "Additional Reading" at the end of this section). FR4 circuit board material is commonly sold in a slab having thickness $h \cong 1.575$ mm with $\epsilon_r \cong 4.5$. Let us consider how we might implement a microstrip line having $Z_0 = 50\ \Omega$ using this material. Since h and ϵ_r are fixed, the only parameter remaining to set Z_0 is W. A bit of experimentation with Equation 3.11.1 reveals that $h/W \approx 1/2$ yields $Z_0 \approx 50\ \Omega$ for $\epsilon_r = 4.5$. Thus, W should be about 3.15 mm. The effective relative permittivity is

$$\epsilon_{r,eff} \approx (4.5+1)/2 = 2.75$$

so the phase velocity for the wave guided by this line is about $c/\sqrt{2.75}$; i.e., 60% of c. Similarly, the wavelength of this wave is about 60% of the free space wavelength.

1. The reference in "Additional Reading" at the end of this section may be helpful if you are not familiar with this concept.↩
2. This is from Wheeler 1977, cited in "Additional Reading" at the end of this section.↩

3.12: Voltage Reflection Coefficient

We now consider the scenario shown in Figure 3.12.1. Here a wave arriving from the left along a lossless transmission line having characteristic impedance Z_0 arrives at a termination located at $z=0$. The impedance looking into the termination is Z_L, which may be real-, imaginary-, or complex-valued. The questions are: Under what circumstances is a reflection – i.e., a leftward traveling wave – expected, and what precisely is that wave?

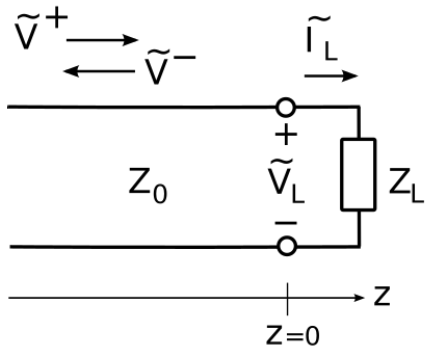


Figure 3.12.1: A wave arriving from the left incident on a termination located at $z=0$.

The potential and current of the incident wave are related by the constant value of Z_0. Similarly, the potential and current of the reflected wave are related by Z_0. Therefore, it suffices to consider *either* potential or current. Choosing potential, we may express the incident wave as

$$\widetilde{V}^{+}(z)=V_0^{+}e^{-j\beta z}$$

where V_0^+ is determined by the source of the wave, and so is effectively a "given." Any reflected wave must have the form

$$\widetilde{V}^{-}(z)=V_0^{-}e^{+j\beta z}$$

Therefore, the problem is solved by determining the value of V_0^- given V_0^+, Z_0, and Z_L.

Considering the situation at $z=0$, note that by definition we have

$$Z_L \triangleq \frac{\widetilde{V}_L}{\tilde{I}_L} \tag{3.12.1}$$

where $\widetilde{V}_L$ and $\tilde{I}_L$ are the potential across and current through the termination, respectively. Also, the potential and current on either side of the $z=0$ interface must be equal. Thus,

$$\widetilde{V}^{+}(0)+\widetilde{V}^{-}(0)=\widetilde{V}_L \tag{3.12.2}$$

$$\tilde{I}^{+}(0)+\tilde{I}^{-}(0)=\tilde{I}_L \tag{3.12.3}$$

where $\tilde{I}^{+}(z)$ and $\tilde{I}^{-}(z)$ are the currents associated with $\widetilde{V}^{+}(z)$ and $\widetilde{V}^{-}(z)$, respectively. Since the voltage and current are related by Z_0, Equation 3.12.3 may be rewritten as follows:

$$\frac{\widetilde{V}^{+}(0)}{Z_0}-\frac{\widetilde{V}^{-}(0)}{Z_0}=\tilde{I}_L \tag{3.12.4}$$

Evaluating the left sides of Equations 3.12.2 and 3.12.4 at $z=0$, we find:

$$\begin{aligned} V_0^{+}+V_0^{-} &= \widetilde{V}_L \\ \frac{V_0^{+}}{Z_0}-\frac{V_0^{-}}{Z_0} &= \tilde{I}_L \end{aligned}$$

Substituting these expressions into Equation 3.12.1 we obtain:

$$Z_L=\frac{V_0^{+}+V_0^{-}}{V_0^{+}/Z_0-V_0^{-}/Z_0}$$

Solving for V_0^- we obtain

$$V_0^- = \frac{Z_L - Z_0}{Z_L + Z_0} V_0^+$$

Thus, the answer to the question posed earlier is that

$$V_0^- = \Gamma V_0^+ \text{ , where}$$

$$\boxed{\Gamma \triangleq \frac{Z_L - Z_0}{Z_L + Z_0}} \tag{3.12.5}$$

The quantity Γ is known as the *voltage reflection coefficient*. Note that when $Z_L = Z_0$, $\Gamma = 0$ and therefore $V_0^- = 0$. In other words,

> If the terminating impedance is equal to the characteristic impedance of the transmission line, then there is no reflection.

If, on the other hand, $Z_L \neq Z_0$, then $|\Gamma| > 0$, $V_0^- = \Gamma V_0^+$, and a leftward-traveling reflected wave exists.

Since Z_L may be real-, imaginary-, or complex-valued, Γ too may be real-, imaginary-, or complex-valued. Therefore, V_0^- may be different from V_0^+ in magnitude, sign, or phase.

Note also that Γ is *not* the ratio of I_0^- to I_0^+. The ratio of the *current* coefficients is actually $-\Gamma$. It is quite simple to show this with a simple modification to the above procedure and is left as an exercise for the student.

Summarizing:

> The voltage reflection coefficient Γ, given by Equation 3.12.5, determines the magnitude and phase of the reflected wave given the incident wave, the characteristic impedance of the transmission line, and the terminating impedance.

We now consider values Γ that arise for commonly-encountered terminations.

Matched Load. ($Z_L = Z_0$). In this case, the termination may be a device with impedance Z_0, or the termination may be another transmission line having the same characteristic impedance. When $Z_L = Z_0$, $\Gamma = 0$ and there is no reflection.

Open Circuit. An "open circuit" is the absence of a termination. This condition implies $Z_L \to \infty$, and subsequently $\Gamma \to +1$. Since the *current* reflection coefficient is $-\Gamma$, the reflected current wave is 180° out of phase with the incident current wave, making the total current at the open circuit equal to zero, as expected.

Short Circuit. "Short circuit" means $Z_L = 0$, and subsequently $\Gamma = -1$. In this case, the phase of Γ is 180°, and therefore, the potential of the reflected wave cancels the potential of the incident wave at the open circuit, making the total potential equal to zero, as it must be. Since the *current* reflection coefficient is $-\Gamma = +1$ in this case, the reflected current wave is in phase with the incident current wave, and the magnitude of the total current at the short circuit non-zero as expected.

Purely Reactive Load. A purely reactive load, including that presented by a capacitor or inductor, has $Z_L = jX$ where X is reactance. In particular, an inductor is represented by $X > 0$ and a capacitor is represented by $X < 0$. We find

$$\Gamma = \frac{-Z_0 + jX}{+Z_0 + jX}$$

The numerator and denominator have the same magnitude, so $|\Gamma| = 1$. Let ϕ be the phase of the denominator ($+Z_0 + jX$). Then, the phase of the numerator is $\pi - \phi$. Subsequently, the phase of Γ is $(\pi - \phi) - \phi = \pi - 2\phi$. Thus, we see that the phase of Γ is no longer limited to be 0° or 180°, but can be any value in between. The phase of reflected wave is subsequently shifted by this amount.

Other Terminations. Any other termination, including series and parallel combinations of any number of devices, can be expressed as a value of Z_L which is, in general, complex-valued. The associated value of $|\Gamma|$ is limited to the range 0 to 1. To see this, note:

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{Z_L/Z_0 - 1}{Z_L/Z_0 + 1}$$

Note that the smallest possible value of $|\Gamma|$ occurs when the numerator is zero; i.e., when $Z_L = Z_0$. Therefore, the smallest value of $|\Gamma|$ is zero. The largest possible value of $|\Gamma|$ occurs when $Z_L/Z_0 \to \infty$ (i.e., an open circuit) or when $Z_L/Z_0 = 0$ (a short circuit); the result in either case is $|\Gamma| = 1$. Thus,

$$\boxed{0 \leq |\Gamma| \leq 1}$$

3.13: Standing Waves

> A *standing wave* consists of waves moving in opposite directions. These waves add to make a distinct magnitude variation as a function of distance that does not vary in time.

To see how this can happen, first consider that an incident wave $V_0^+ e^{-j\beta z}$, which is traveling in the $+z$ axis along a lossless transmission line. Associated with this wave is a reflected wave $V_0^- e^{+j\beta z} = \Gamma V_0^+ e^{+j\beta z}$, where Γ is the voltage reflection coefficient. These waves add to make the total potential

$$\begin{aligned}\widetilde{V}(z) &= V_0^+ e^{-j\beta z} + \Gamma V_0^+ e^{+j\beta z} \\ &= V_0^+ \left(e^{-j\beta z} + \Gamma e^{+j\beta z}\right)\end{aligned}$$

The magnitude of $\widetilde{V}(z)$ is most easily found by first finding $|\widetilde{V}(z)|^2$, which is:

$$\begin{aligned}&\tilde{V}(z)\tilde{V}^*(z) \\ &= |V_0^+|^2 \left(e^{-j\beta z} + \Gamma e^{+j\beta z}\right)\left(e^{-j\beta z} + \Gamma e^{+j\beta z}\right)^* \\ &= |V_0^+|^2 \left(e^{-j\beta z} + \Gamma e^{+j\beta z}\right)\left(e^{+j\beta z} + \Gamma^* e^{-j\beta z}\right) \\ &= |V_0^+|^2 \left(1 + |\Gamma|^2 + \Gamma e^{+j2\beta z} + \Gamma^* e^{-j2\beta z}\right)\end{aligned}$$

Let ϕ be the phase of Γ; i.e.,

$$\Gamma = |\Gamma| e^{j\phi}$$

Then, continuing from the previous expression:

$$\begin{aligned}&|V_0^+|^2 \left(1 + |\Gamma|^2 + |\Gamma| e^{+j(2\beta z + \phi)} + |\Gamma| e^{-j(2\beta z + \phi)}\right) \\ &= |V_0^+|^2 \left(1 + |\Gamma|^2 + |\Gamma| \left[e^{+j(2\beta z + \phi)} + e^{-j(2\beta z + \phi)}\right]\right)\end{aligned}$$

The quantity in square brackets can be reduced to a cosine function using the identity

$$\cos\theta = \frac{1}{2}\left[e^{j\theta} + e^{-j\theta}\right]$$

yielding:

$$|V_0^+|^2 \left[1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \phi)\right]$$

Recall that this is $|\widetilde{V}(z)|^2$. $|\widetilde{V}(z)|$ is therefore the square root of the above expression:

$$\left|\widetilde{V}(z)\right| = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2|\Gamma|\cos(2\beta z + \phi)}$$

Thus, we have found that the magnitude of the resulting total potential varies sinusoidally along the line. This is referred to as a standing wave because the variation of the magnitude of the phasor resulting from the interference between the incident and reflected waves does not vary with time.

We may perform a similar analysis of the current, leading to:

$$\left|\tilde{I}(z)\right| = \frac{|V_0^+|}{Z_0} \sqrt{1 + |\Gamma|^2 - 2|\Gamma|\cos(2\beta z + \phi)}$$

Again we find the result is a standing wave.

Now let us consider the outcome for a few special cases.

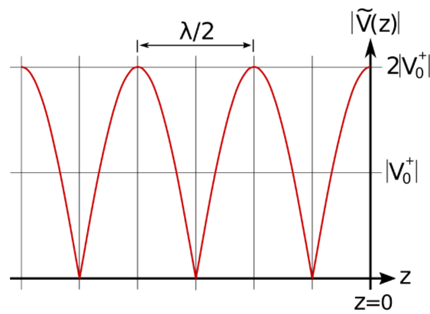
Matched load. When the impedance of the termination of the transmission line, Z_L, is equal to the characteristic impedance of the transmission line, Z_0, $\Gamma = 0$ and there is no reflection. In this case, the above expressions reduce to $|\widetilde{V}(z)| = |V_0^+|$ and $|\tilde{I}(z)| = |V_0^+|/Z_0$, as expected.

Open or Short-Circuit. In this case, $\Gamma = \pm 1$ and we find:

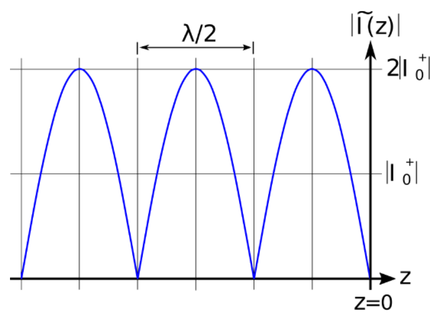
$$\left|\widetilde{V}(z)\right| = |V_0^+| \sqrt{2 + 2\cos(2\beta z + \phi)}$$

$$\left|\tilde{I}(z)\right| = \frac{|V_0^+|}{Z_0} \sqrt{2 - 2\cos(2\beta z + \phi)}$$

where $\phi = 0$ for an open circuit and $\phi = \pi$ for a short circuit. The result for an open circuit termination is shown in Figure $3.13.1$(a) (potential) and $3.13.1$(b) (current). The result for a short circuit termination is identical except the roles of potential and current are reversed. In either case, note that voltage maxima correspond to current minima, and vice versa.



(a) Potential.



(b) Current.

Figure $3.13.1$: Standing wave associated with an opencircuit termination at $z = 0$ (incident wave arrives from left).

Also note:

> The period of the standing wave is $\lambda/2$; i.e., one-half of a wavelength.

This can be confirmed as follows. First, note that the frequency argument of the cosine function of the standing wave is $2\beta z$. This can be rewritten as $2\pi(\beta/\pi)z$, so the frequency of variation is β/π and the period of the variation is π/β. Since $\beta = 2\pi/\lambda$, we see that the period of the variation is $\lambda/2$. Furthermore, this is true regardless of the value of Γ.

Mismatched loads. A common situation is that the termination is neither perfectly-matched ($\Gamma = 0$) nor an open/short circuit ($|\Gamma| = 1$). Examples of the resulting standing waves are shown in Figure $3.13.2$

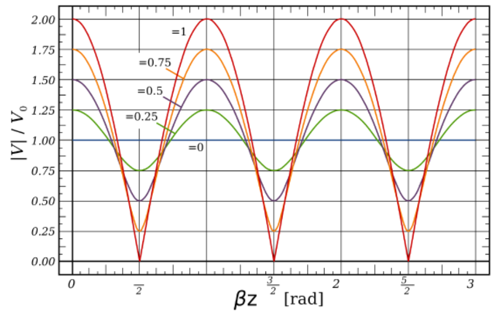


Figure 3.13.2: Standing waves associated with loads exhibiting various reflection coefficients. In this figure the incident wave arrives from the right.

3.14: Standing Wave Ratio

Precise matching of transmission lines to terminations is often not practical or possible. Whenever a significant mismatch exists, a standing wave (Section 3.13) is apparent. The quality of the match is commonly expressed in terms of the *standing wave ratio* (SWR) of this standing wave.

> *Standing wave ratio* (SWR) is defined as the ratio of the maximum magnitude of the standing wave to minimum magnitude of the standing wave.

In terms of the potential:

$$\boxed{\text{SWR} \triangleq \frac{\text{maximum } |\widetilde{V}|}{\text{minimum } |\widetilde{V}|}}$$

SWR can be calculated using a simple expression, which we shall now derive. In Section 3.13, we found that:

$$\left|\widetilde{V}(z)\right| = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2\,|\Gamma| \cos(2\beta z + \phi)}$$

The maximum value occurs when the cosine factor is equal to $+1$, yielding:

$$\max \left|\widetilde{V}\right| = |V_0^+| \sqrt{1 + |\Gamma|^2 + 2\,|\Gamma|}$$

Note that the argument of the square root operator is equal to $(1 + |\Gamma|)^2$; therefore:

$$\max \left|\widetilde{V}\right| = |V_0^+|\,(1 + |\Gamma|)$$

Similarly, the minimum value is achieved when the cosine factor is equal to -1, yielding:

$$\min \left|\widetilde{V}\right| = |V_0^+| \sqrt{1 + |\Gamma|^2 - 2\,|\Gamma|}$$

So:

$$\min \left|\widetilde{V}\right| = |V_0^+|\,(1 - |\Gamma|)$$

Therefore:

$$\boxed{\text{SWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|}} \tag{3.14.1}$$

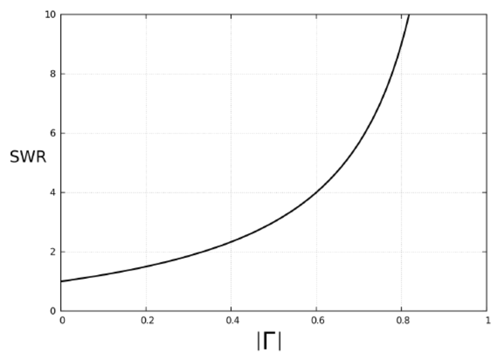


Figure 3.14.1: Relationship between SWR and $|\Gamma|$.

This relationship is shown graphically in Figure $3.14.1$. Note that SWR ranges from 1 for perfectly-matched terminations ($\Gamma = 0$) to infinity for open- and short-circuit terminations ($|\Gamma| = 1$).

It is sometimes of interest to find the magnitude of the reflection coefficient given SWR. Solving Equation 3.14.1 for $|\Gamma|$ we find:

$$|\Gamma| = \frac{\mathrm{SWR}-1}{\mathrm{SWR}+1} \tag{3.14.2}$$

SWR is often referred to as the *voltage standing wave ratio* (VSWR), although repeating the analysis above for the current reveals that the current SWR is equal to potential SWR, so the term "SWR" suffices.

SWR < 2 or so is usually considered a "good match," although some applications require SWR < 1.1 or better, and other applications are tolerant to SWR of 3 or greater.

Example 3.14.1: Reflection Coefficient for Various Values of SWR

What is the reflection coefficient for the above-cited values of SWR? Using Equation 3.14.2, we find:

- SWR = 1.1 corresponds to $|\Gamma| = 0.0476$.
- SWR = 2.0 corresponds to $|\Gamma| = 1/3$.
- SWR = 3.0 corresponds to $|\Gamma| = 1/2$.

3.15: Input Impedance of a Terminated Lossless Transmission Line

Consider Figure $3.15.1$, which shows a lossless transmission line being driven from the left and which is terminated by an impedance Z_L on the right. If Z_L is equal to the characteristic impedance Z_0 of the transmission line, then the input impedance Z_{in} will be equal to Z_L. Otherwise Z_{in} depends on both Z_L and the characteristics of the transmission line. In this section, we determine a general expression for Z_{in} in terms of Z_L, Z_0, the phase propagation constant β, and the length l of the line.

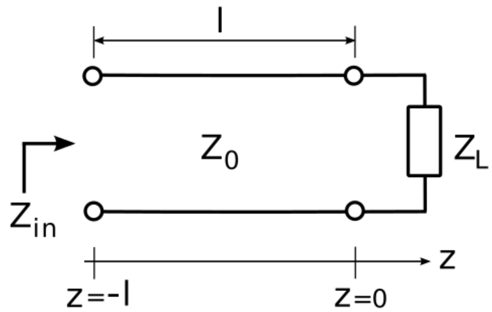


Figure 3.15.1: A transmission line driven by a source on the left and terminated by an impedance Z_L at $z=0$ on the right

Using the coordinate system indicated in Figure $3.15.1$, the interface between source and transmission line is located at $z=-l$. Impedance is defined at the ratio of potential to current, so:

$$Z_{in}(l) \triangleq \frac{\widetilde{V}(z=-l)}{\tilde{I}\,(z=-l)}$$

Now employing expressions for $\widetilde{V}(z)$ and $\tilde{I}\,(z)$ from Section 3.13 with $z=-l$, we find:

$$\begin{aligned} Z_{in}(l) &= \frac{V_0^+\left(e^{+j\beta l}+\Gamma e^{-j\beta l}\right)}{V_0^+\left(e^{+j\beta l}-\Gamma e^{-j\beta l}\right)/Z_0} \\ &= Z_0\frac{e^{+j\beta l}+\Gamma e^{-j\beta l}}{e^{+j\beta l}-\Gamma e^{-j\beta l}} \end{aligned}$$

Multiplying both numerator and denominator by $e^{-j\beta l}$:

$$\boxed{Z_{in}(l) = Z_0\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}} \tag{3.15.1}$$

Recall that Γ in the above expression is:

$$\Gamma = \frac{Z_L-Z_0}{Z_L+Z_0} \tag{3.15.2}$$

Summarizing:

> Equation 3.15.1 is the input impedance of a lossless transmission line having characteristic impedance Z_0 and which is terminated into a load Z_L. The result also depends on the length and phase propagation constant of the line.

Note that $Z_{in}(l)$ is periodic in l. Since the argument of the complex exponential factors is $2\beta l$, the frequency at which $Z_{in}(l)$ varies is β/π, and since $\beta=2\pi/\lambda$, the associated period is $\lambda/2$. This is very useful to keep in mind because it means that all possible values of $Z_{in}(l)$ are achieved by varying l over $\lambda/2$. In other words, changing l by more than $\lambda/2$ results in an impedance which could have been obtained by a smaller change in l. Summarizing to underscore this important idea:

> The input impedance of a terminated lossless transmission line is periodic in the length of the transmission line, with period $\lambda/2$.

Not surprisingly, $\lambda/2$ is also the period of the standing wave (Section 3.13). This is because – once again – the variation with length is due to the interference of incident and reflected waves.

Also worth noting is that Equation 3.15.1 can be written entirely in terms of Z_L and Z_0, since Γ depends only on these two parameters. Here's that version of the expression:

$$Z_{in}(l) = Z_0 \left[\frac{Z_L + jZ_0 \tan \beta l}{Z_0 + jZ_L \tan \beta l} \right] \tag{3.15.3}$$

This expression can be derived by substituting Equation 3.15.2 into Equation 3.15.1 and is left as an exercise for the student.

Finally, note that the argument βl appearing Equations 3.15.1 and 3.15.3 has units of radians and is referred to as *electrical length*. Electrical length can be interpreted as physical length expressed with respect to wavelength and has the advantage that analysis can be made independent of frequency.

3.16: Input Impedance for Open- and Short-Circuit Terminations

Let us now consider the input impedance of a transmission line that is terminated in an open- or short-circuit. Such a transmission line is sometimes referred to as a *stub*. First, why consider such a thing? From a "lumped element" circuit theory perspective, this would not seem to have any particular application. However, the fact that this structure exhibits an input impedance that depends on length (Section 3.15) enables some very useful applications.

First, let us consider the question at hand: What is the input impedance when the transmission line is open- or short-circuited? For a short circuit, $Z_L = 0$, $\Gamma = -1$, so we find

$$\begin{aligned} Z_{in}(l) &= Z_0 \frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}} \\ &= Z_0 \frac{1-e^{-j2\beta l}}{1+e^{-j2\beta l}} \end{aligned}$$

Multiplying numerator and denominator by $e^{+j\beta l}$ we obtain

$$Z_{in}(l) = Z_0 \frac{e^{+j\beta l} - e^{-j\beta l}}{e^{+j\beta l} + e^{-j\beta l}}$$

Now we invoke the following trigonometric identities:

$$\begin{aligned} \cos\theta &= \tfrac{1}{2}\left[e^{+j\theta} + e^{-j\theta}\right] \\ \sin\theta &= \tfrac{1}{j2}\left[e^{+j\theta} - e^{-j\theta}\right] \end{aligned} \tag{3.16.1}$$

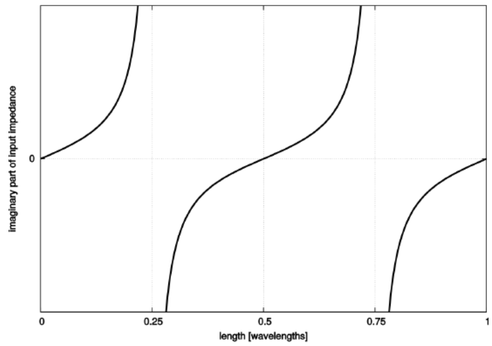
Employing these identities, we obtain:

$$Z_{in}(l) = Z_0 \frac{j2\left(\sin\beta l\right)}{2\left(\cos\beta l\right)}$$

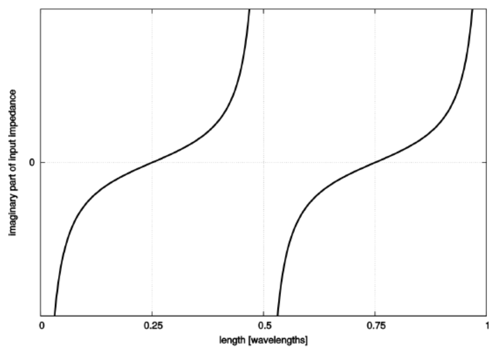
and finally:

$$\boxed{Z_{in}(l) = +jZ_0 \tan\beta l} \tag{3.16.2}$$

Figure 3.16.1(a) shows what's going on. As expected, $Z_{in} = 0$ when $l = 0$, since this amounts to a short circuit with no transmission line. Also, Z_{in} varies periodically with increasing length, with period $\lambda/2$. This is precisely as expected from standing wave theory (Section 3.13). What is of particular interest now is that as $l \rightarrow \lambda/4$, we see $Z_{in} \rightarrow \infty$. Remarkably, the transmission line has essentially transformed the short circuit termination into an open circuit!



(a) Short-ciruit termination ($Z_L = 0$.



(b) Open-circuit termination ($Z_L \to \infty$).

Figure 3.16.1: Input reactance ($\text{Im}\{Z_{in}\}$) of a stub. $\text{Re}\{Z_{in}\}$ is always zero.

For an open circuit termination, $Z_L \to \infty$, $\Gamma = +1$, and we find

$$\begin{aligned} Z_{in}(l) &= Z_0 \frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}} \\ &= Z_0 \frac{1+e^{-j2\beta l}}{1-e^{-j2\beta l}} \end{aligned}$$

Following the same procedure detailed above for the short-circuit case, we find

$$\boxed{Z_{in}(l) = -jZ_0 \cot \beta l} \tag{3.16.3}$$

Figure 3.16.1(b) shows the result for open-circuit termination. As expected, $Z_{in} \to \infty$ for $l = 0$, and the same $\lambda/2$ periodicity is observed. What is of particular interest now is that at $l = \lambda/4$ we see $Z_{in} = 0$. In this case, the transmission line has transformed the *open* circuit termination into a *short* circuit.

Now taking stock of what we have determined:

> The input impedance of a short- or open-circuited lossless transmission line is completely imaginary-valued and is given by Equations 3.16.2 and 3.16.3, respectively.

> The input impedance of a short- or open-circuited lossless transmission line alternates between open- ($Z_{in} \to \infty$) and short-circuit ($Z_{in} = 0$) conditions with each $\lambda/4$-increase in length.

3.17: Applications of Open- and Short-Circuited Transmission Line Stubs

The theory of open- and short-circuited transmission lines – often referred to as *stubs* – was addressed in Section 3.16. These structures have important and wide-ranging applications.

In particular, these structures can be used to replace discrete inductors and capacitors in certain applications. To see this, consider the short-circuited line (Figure 3.16.1(a) of Section 3.16). Note that each value of l that is less than $\lambda/4$ corresponds to a particular positive reactance; i.e., the transmission line "looks" like an inductor. Also note that lengths between $\lambda/4$ and $\lambda/2$ result in reactances that are negative; i.e., the transmission line "looks" like a capacitor. Thus, it is possible to replace an inductor or capacitor with a short-circuited transmission line of the appropriate length. The input impedance of such a transmission line is identical to that of the inductor or capacitor at the design frequency. The variation of reactance with respect to frequency will *not* be identical, which may or may not be a concern depending on the bandwidth and frequency response requirements of the application. Open-circuited lines may be used in a similar way.

This property of open- and short-circuited transmission lines makes it possible to implement impedance matching circuits (see Section 3.16 a), filters, and other devices entirely from transmission lines, with fewer or no discrete inductors or capacitors required. Transmission lines do not suffer the performance limitations of discrete devices at high frequencies and are less expensive. A drawback of transmission line stubs in this application is that the lines are typically much larger than the discrete devices they are intended to replace.

Example 3.17.1: Emitter Induction Using Short-Circuited Line

In the design of low-noise amplifiers using bipolar transistors in common-emitter configuration, it is often useful to introduce a little inductance between the emitter and ground. This is known as "inductive degeneration," "emitter induction," or sometimes by other names. It can be difficult to find suitable inductors, especially for operation in the UHF band and higher. However, a microstrip line can be used to achieve the desired inductive impedance. Determine the length of a stub that implements a 2.2 nH inductance at 6 GHz using microstrip line with characteristic impedance $50\ \Omega$ and phase velocity $0.6c$.

Solution

At the design frequency, the impedance looking into this section of line from the emitter should be equal to that of a 2.2 nH inductor, which is $+j\omega L = +j2\pi f L = +j82.9\ \Omega$. The input impedance of a short-circuited stub of length l which is grounded (thus, short-circuited) at the opposite end is $+jZ_0 \tan\beta l$ (Section 3.16). Setting this equal to $+j82.9\ \Omega$ and noting that $Z_0 = 50\ \Omega$, we find that $\beta l \cong 1.028$ rad. The phase propagation constant is (Section 3.8):

$$\beta = \frac{\omega}{v_p} = \frac{2\pi f}{0.6c} \cong 209.4\ \text{rad/m}$$

Therefore, the length of the microstrip line is $l = (\beta l)/\beta \cong 4.9$ mm.

3.18: Measurement of Transmission Line Characteristics

This section presents a simple technique for measuring the characteristic impedance Z_0, electrical length βl, and phase velocity v_p of a lossless transmission line. This technique requires two measurements: the input impedance Z_{in} when the transmission line is short-circuited and Z_{in} when the transmission line is open-circuited.

In Section 3.16, it is shown that the input impedance Z_{in} of a short-circuited transmission line is

$$Z_{in}^{(SC)} = +jZ_0 \tan \beta l$$

and when a transmission line is terminated in an open circuit, the input impedance is

$$Z_{in}^{(OC)} = -jZ_0 \cot \beta l$$

Observe what happens when we multiply these results together:

$$Z_{in}^{(SC)} \cdot Z_{in}^{(OC)} = Z_0^2$$

that is, the product of the measurements $Z_{in}^{(OC)}$ and $Z_{in}^{(SC)}$ is simply the square of the characteristic impedance. Therefore

$$Z_0 = \sqrt{Z_{in}^{(SC)} \cdot Z_{in}^{(OC)}}$$

If we instead divide these measurements, we find

$$\frac{Z_{in}^{(SC)}}{Z_{in}^{(OC)}} = -\tan^2 \beta l$$

Therefore:

$$\tan \beta l = \left[-\frac{Z_{in}^{(SC)}}{Z_{in}^{(OC)}} \right]^{1/2}$$

If l is known in advance to be less than $\lambda/2$, then the electrical length βl can be determined by taking the inverse tangent. If l is of unknown length and longer than $\lambda/2$, one must take care to account for the periodicity of tangent function; in this case, it may not be possible to unambiguously determine βl. Although we shall not present the method here, it is possible to resolve this ambiguity by making multiple measurements over a range of frequencies.

Once βl is determined, it is simple to determine l given β, β given l, and then v_p. For example, the phase velocity may be determined by first finding βl for a known length using the above procedure, calculating $\beta = (\beta l)/l$, and then $v_p = \omega/\beta$.

3.19: Quarter-Wavelength Transmission Line

Quarter-wavelength sections of transmission line play an important role in many systems at radio and optical frequencies. The remarkable properties of open- and short-circuited quarter-wave line are presented in Section 3.16 and should be reviewed before reading further. In this section, we perform a more general analysis, considering not just open- and short-circuit terminations but any terminating impedance, and then we address some applications.

The general expression for the input impedance of a lossless transmission line is (Section 3.15):

$$Z_{in}(l) = Z_0 \frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}} \tag{3.19.1}$$

Note that when $l = \lambda/4$:

$$2\beta l = 2 \cdot \frac{2\pi}{\lambda} \cdot \frac{\lambda}{4} = \pi$$

Subsequently:

$$\begin{aligned} Z_{in}(\lambda/4) &= Z_0 \frac{1+\Gamma e^{-j\pi}}{1-\Gamma e^{-j\pi}} \\ &= Z_0 \frac{1-\Gamma}{1+\Gamma} \end{aligned}$$

Recall that (Section 3.15):

$$\Gamma = \frac{Z_L - Z_0}{Z_L + Z_0}$$

Substituting this expression and then multiplying numerator and denominator by $Z_L + Z_0$, one obtains

$$\begin{aligned} Z_{in}(\lambda/4) &= Z_0 \frac{(Z_L+Z_0)-(Z_L-Z_0)}{(Z_L+Z_0)+(Z_L-Z_0)} \\ &= Z_0 \frac{2Z_0}{2Z_L} \end{aligned}$$

Thus,

$$\boxed{Z_{in}(\lambda/4) = \frac{Z_0^2}{Z_L}} \tag{3.19.2}$$

Note that the input impedance is inversely proportional to the load impedance. For this reason, a transmission line of length $\lambda/4$ is sometimes referred to as a *quarter-wave inverter* or simply as a *impedance inverter*.

Quarter-wave lines play a very important role in RF engineering. As impedance inverters, they have the useful attribute of transforming small impedances into large impedances, and vice-versa – we'll come back to this idea later in this section. First, let's consider how quarter-wave lines are used for impedance matching. Look what happens when we solve Equation 3.19.2 for Z_0:

$$Z_0 = \sqrt{Z_{in}(\lambda/4) \cdot Z_L} \tag{3.19.3}$$

This equation indicates that we may match the load Z_L to a source impedance (represented by $Z_{in}(\lambda/4)$) simply by making the characteristic impedance equal to the value given by the above expression and setting the length to $\lambda/4$. The scheme is shown in Figure $3.19.1$.

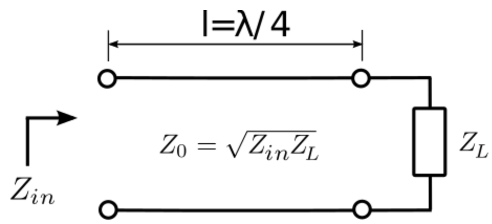


Figure 3.19.1: Impedance-matching using a quarter-wavelength transmission line.

Example 3.19.1: 300-to-50 Ω match using an quarter-wave section of line

Design a transmission line segment that matches $300\ \Omega$ to $50\ \Omega$ at 10 GHz using a quarter-wave match. Assume microstrip line for which propagation occurs with wavelength 60% that of free space.

Solution

The line is completely specified given its characteristic impedance Z_0 and length l. The length should be one-quarter wavelength with respect to the signal propagating in the line. The free-space wavelength $\lambda_0 = c/f$ at 10 GHz is $\cong 3$ cm. Therefore, the wavelength of the signal in the line is $\lambda = 0.6\lambda_0 \cong 1.8$ cm, and the length of the line should be $l = \lambda/4 \cong 4.5$ mm.

The characteristic impedance is given by Equation ref{m0091_eQWZ0}:

$$Z_0 = \sqrt{300\ \Omega \cdot 50\ \Omega} \cong 122.5\ \Omega$$

This value would be used to determine the width of the microstrip line, as discussed in Section 3.11.

It should be noted that for this scheme to yield a real-valued characteristic impedance, the product of the source and load impedances must be a real-valued number. In particular, this method is *not* suitable if Z_L has a significant imaginary-valued component and matching to a real-valued source impedance is desired. One possible workaround in this case is the two-stage strategy shown in Figure 3.19.2 In this scheme, the load impedance is first transformed to a real-valued impedance using a length l_1 of transmission line. This is accomplished using Equation 3.19.1 (quite simple using a numerical search) or using the Smith chart (see “Additional Reading” at the end of this section). The characteristic impedance Z_{01} of this transmission line is not critical and can be selected for convenience. Normally, the smallest value of l_1 is desired. This value will always be less than $\lambda/4$ since $Z_{in}(l_1)$ is periodic in l_1 with period $\lambda/2$; i.e., there are two changes in the sign of the imaginary component of $Z_{in}(l_1)$ as l_1 is increased from zero to $\lambda/2$. After eliminating the imaginary component of Z_L in this manner, the real component of the resulting impedance may then be transformed using the quarter-wave matching technique described earlier in this section.

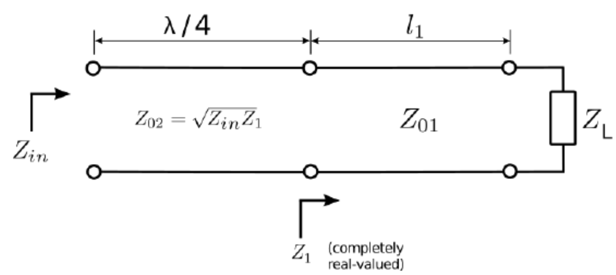


Figure 3.19.2: Impedance-matching a complex-valued load impedance using quarter-wavelength transmission line.

Example 3.19.2: Matching a patch Antenna to 50 Ω

A particular patch antenna exhibits a source impedance of $Z_A = 35 + j35\ \Omega$. (See “Microstrip antenna” in “Additional Reading” at the end of this section for some optional reading on patch antennas.) Interface this antenna to $50\ \Omega$ using the technique described above. For the section of transmission line adjacent to the patch antenna, use characteristic impedance $Z_{01} = 50\ \Omega$. Determine the lengths l_1 and l_2 of the two segments of transmission line, and the characteristic impedance Z_{02} of the second (quarter-wave) segment.

Solution

The length of the first section of the transmission line (adjacent to the antenna) is determined using Equation 3.19.1:

$$Z_1(l_1) = Z_{01} \frac{1 + \Gamma e^{-j2\beta_1 l_1}}{1 - \Gamma e^{-j2\beta_1 l_1}}$$

where β_1 is the phase propagation constant for this section of transmission line and

$$\Gamma \triangleq \frac{Z_A - Z_{01}}{Z_A + Z_{01}} \cong -0.0059 + j0.4142$$

We seek the value of smallest positive value of $\beta_1 l_1$ for which the imaginary part of $Z_1(l_1)$ is zero. This can determined using a Smith chart (see "Additional Reading" at the end of this section) or simply by a few iterations of trial-and-error. Either way we find $Z_1(\beta_1 l_1 = 0.793 \text{ rad}) \cong 120.719 - j0.111\ \Omega$, which we deem to be close enough to be acceptable. Note that $\beta_1 = 2\pi/\lambda$, where λ is the wavelength of the signal in the transmission line. Therefore

$$l_1 = \frac{\beta_1 l_1}{\beta_1} = \frac{\beta_1 l_1}{2\pi}\lambda \cong 0.126\lambda$$

The length of the second section of the transmission line, being a quarter-wavelength transformer, should be $l_2 = 0.25\lambda$. Using Equation 3.19.3, the characteristic impedance Z_{02} of this section of line should be

$$Z_{02} \cong \sqrt{(120.719\ \Omega)(50\ \Omega)} \cong 77.7\ \Omega$$

Discussion. The total length of the matching structure is $l_1 + l_2 \cong 0.376\lambda$. A patch antenna would typically have sides of length about $\lambda/2 = 0.5\lambda$, so the matching structure is nearly as big as the antenna itself. At frequencies where patch antennas are commonly used, and especially at frequencies in the UHF (300–3000 MHz) band, patch antennas are often comparable to the size of the system, so it is not attractive to have the matching structure also require a similar amount of space. Thus, we would be motivated to find a smaller matching structure.

Although quarter-wave matching techniques are generally effective and commonly used, they have one important contraindication, noted above – They often result in structures that are large. That is, any structure which employs a quarter-wave match will be at least $\lambda/4$ long, and $\lambda/4$ is typically large compared to the associated electronics. Other transmission line matching techniques – and in particular, single stub matching (Section 3.23) – typically result in structures which are significantly smaller.

The impedance inversion property of quarter-wavelength lines has applications beyond impedance matching. The following example demonstrates one such application:

Example 3.19.3: RF/DC decoupling in transistor amplifiers

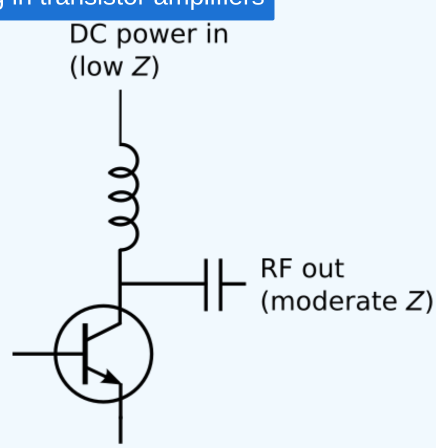


Figure 3.19.3: Use an inductor to decouple the DC input power from the RF output signal at the output of a common-emitter RF amplifier

Transistor amplifiers for RF applications often receive DC current at the same terminal which delivers the amplified RF signal, as shown in Figure 3.19.3. The power supply typically has a low output impedance. If the power supply is directly connected to the transistor, then the RF will flow predominantly in the direction of the power supply as opposed to following the desired path, which exhibits a higher impedance. This can be addressed using an inductor in series with the power supply output. This works because the inductor exhibits low impedance at DC and high impedance at RF. Unfortunately, discrete inductors are often not practical at high RF frequencies. This is because practical inductors also exhibit parallel capacitance, which tends to *decrease* impedance.

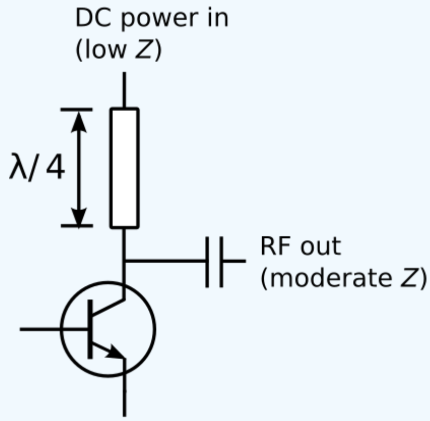


Figure 3.19.4: Decoupling of DC input power and RF output signal at the output of a common-emitter RF amplifier, using a quarter-wavelength transmission line.

A solution is to replace the inductor with a transmission line having length $\lambda/4$ as shown in Figure 3.19.4 A wavelength at DC is infinite, so the transmission line is essentially transparent to the power supply. At radio frequencies, the line transforms the low impedance of the power supply to an impedance that is very large relative to the impedance of the desired RF path. Furthermore, transmission lines on printed circuit boards are much cheaper than discrete inductors (and are always in stock!).

Additional Reading:

- “Quarter-wavelength impedance transformer” on Wikipedia.
- “Smith chart” on Wikipedia.
- “Microstrip antenna” on Wikipedia.

3.20: Power Flow on Transmission Lines

It is often important to know the power associated with a wave on a transmission line. The power of the waves incident upon, reflected by, and absorbed by a load are each of interest. In this section we shall work out expressions for these powers and consider some implications in terms of the voltage reflection coefficient (Γ) and standing wave ratio (SWR).

Let's begin by considering a lossless transmission line that is oriented along the z axis. The time-average power associated with a sinusoidal wave having potential $v(z,t)$ and current $i(z,t)$ is

$$P_{av}(z) \triangleq \frac{1}{T}\int_{t_0}^{t_0+T} v(z,t)\, i(z,t)\, dt$$

where $T \triangleq 2\pi/f$ is one period of the wave and t_0 is the start time for the integration. Since the time-average power of a sinusoidal signal does not change with time, t_0 may be set equal to zero without loss of generality.

Let us now calculate the power of a wave incident from $z<0$ on a load impedance Z_L at $z=0$. We may express the associated potential and current as follows:

$$v^+(z,t) = |V_0^+|\cos(\omega t - \beta z + \phi)$$

$$i^+(z,t) = \frac{|V_0^+|}{Z_0}\cos(\omega t - \beta z + \phi)$$

And so the associated time-average power is

$$\begin{aligned} P_{av}^+(z) &= \frac{1}{T}\int_0^T v^+(z,t)\, i^+(z,t)\, dt \\ &= \frac{|V_0^+|^2}{Z_0}\cdot\frac{1}{T}\int_0^T \cos^2(\omega t - \beta z + \phi)\, dt \end{aligned} \tag{3.20.1}$$

Employing a well-known trigonometric identity:

$$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$$

we may rewrite the integrand as follows

$$\cos^2(\omega t - \beta z + \phi) = \frac{1}{2} + \frac{1}{2}\cos(2\,[\omega t - \beta z + \phi])$$

Then integrating over both sides of this quantity

$$\int_0^T \cos^2(\omega t - \beta z + \phi)\, dt = \frac{T}{2} + 0$$

The second term of the integral is zero because it is the integral of cosine over two complete periods. Subsequently, we see that the position dependence (here, the dependence on z) is eliminated. In other words, the power associated with the incident wave is the same for all $z<0$, as expected. Substituting into Equation 3.20.1 we obtain:

$$\boxed{P_{av}^+ = \frac{|V_0^+|^2}{2Z_0}} \tag{3.20.2}$$

This is the time-average power associated with the incident wave, measured at any point $z<0$ along the line.

> Equation 3.20.2 gives the time-average power associated with a wave traveling in a single direction along a lossless transmission line.

Using precisely the same procedure, we find that the power associated with the *reflected* wave is

$$P_{av}^{-}=\frac{\left|\Gamma V_0^{+}\right|^2}{2Z_0}=|\Gamma|^2\frac{\left|V_0^{+}\right|^2}{2Z_0}$$

or simply

$$\boxed{P_{av}^{-}=|\Gamma|^2 P_{av}^{+}} \tag{3.20.3}$$

> Equation 3.20.3 gives the time-average power associated with the wave reflected from an impedance mismatch.

Now, what is the power P_L delivered to the load impedance Z_L? The simplest way to calculate this power is to use the principle of *conservation of power*. Applied to the present problem, this principle asserts that the power incident on the load must equal the power reflected plus the power absorbed; i.e.,

$$P_{av}^{+}=P_{av}^{-}+P_L$$

Applying the previous equations we obtain:

$$\boxed{P_L=\left(1-|\Gamma|^2\right)P_{av}^{+}} \tag{3.20.4}$$

> Equations 3.20.4 gives the time-average power transferred to a load impedance, and is equal to the difference between the powers of the incident and reflected waves.

Example 3.20.1: How important is it to match 50 Ω to 75 Ω?

Two impedances which commonly appear in radio engineering are $50\ \Omega$ and $75\ \Omega$. It is not uncommon to find that it is necessary to connect a transmission line having a $50\ \Omega$ characteristic impedance to a device, circuit, or system having a $75\ \Omega$ input impedance, or vice-versa. If no attempt is made to match these impedances, what fraction of the power will be delivered to the termination, and what fraction of power will be reflected? What is the SWR?

Solution

The voltage reflection coefficient going from $50\ \Omega$ transmission line to a $75\ \Omega$ load is

$$\Gamma=\frac{75-50}{75+50}=+0.2$$

The fraction of power reflected is $|\Gamma|^2=0.04$, which is 4%. The fraction of power transmitted is $1-|\Gamma|^2$, which is 96%. Going from a $50\ \Omega$ transmission line to a $75\ \Omega$ termination changes only the sign of Γ, and therefore, the fractions of reflected and transmitted power remain 4% and 96%, respectively. In either case (from Section 3.14):

$$\mathrm{SWR}=\frac{1+|\Gamma|}{1-|\Gamma|}=1.5$$

This is often acceptable, but may not be good enough in some particular applications. Suffice it to say that it is not necessarily required to use an impedance matching device to connect $50\ \Omega$ to $75\ \Omega$ devices.

3.21: Impedance Matching - General Considerations

"Impedance matching" refers to the problem of transforming a particular impedance Z_L into a modified impedance Z_{in}. The problem of impedance matching arises because it is not convenient, practical, or desirable to have all devices in a system operate at the same input and output impedances. Here are just a few of the issues:

- It is not convenient or practical to market coaxial cables having characteristic impedance equal to every terminating impedance that might be encountered.
- Different types of antennas operate at different impedances, and the impedance of most antennas vary significantly with frequency.
- Different types of amplifiers operate most effectively at different output impedances. For example, amplifiers operating as current sources operate most effectively with low output impedance, whereas amplifiers operating as voltage sources operate most effectively with high output impedances.
- Independently of the above issue, techniques for the design of transistor amplifiers rely on intentionally *mismatching* impedances; i.e., matching to an impedance different than that which maximizes power transfer or minimizes reflection. In other words, various design goals are met by applying particular impedances to the input and output ports of the transistor.[1]

For all of these reasons, electrical engineers frequently find themselves with the task of transforming a particular impedance Z_L into a modified impedance Z_{in}.

The reader is probably already familiar with many approaches to the impedance matching problem that employ discrete components and do not require knowledge of electromagnetics.[2] To list just a few of these approaches: transformers, resistive (lossy) matching, single-reactance matching, and two-reactance ("L" network) matching. However, all of these have limitations. Perhaps the most serious limitations pertain to the performance of discrete components at high frequencies. Here are just a few of the most common problems:

- Practical resistors actually behave as ideal resistors in series with ideal inductors
- Practical capacitors actually behave as ideal capacitors in series with ideal resistors
- Practical inductors behave as ideal inductors in parallel with ideal capacitors, and in series with ideal resistors.

All of this makes the use of discrete components increasingly difficult with increasing frequency.

One possible solution to these types of problems is to more precisely model each component, and then to account for the non-ideal behavior by incorporating the appropriate models in the analysis and design process. Alternatively, one may consider ways to replace particular troublesome components – or, in some cases, all discrete components – with transmission line devices. The latter approach is particularly convenient in circuits implemented on printed circuit boards at frequencies in the UHF band and higher, since the necessary transmission line structures are easy to implement as microstrip lines and are relatively compact since the wavelength is relatively small. However, applications employing transmission lines as components in impedance matching devices can be found at lower frequencies as well.

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1. For a concise introduction to this concept, see Chapter 10 of S.W. Ellingson, *Radio Systems Engineering*, Cambridge Univ. Press, 2016.↩
2. For an overview, see Chapter 9 of S.W. Ellingson, *Radio Systems Engineering*, Cambridge Univ. Press, 2016.↩

3.22: Single-Reactance Matching

An impedance matching structure can be designed using a section of transmission line combined with a discrete reactance, such as a capacitor or an inductor. In the strategy presented here, the transmission line is used to transform the real part of the load impedance or admittance to the desired value, and then the reactance is used to modify the imaginary part to the desired value. (Note the difference between this approach and the quarter-wave technique described in Section 3.19. In that approach, the first transmission line is used to zero the imaginary part.) There are two versions of this strategy, which we will now consider separately.

The first version is shown in Figure $3.22.1$. The purpose of the transmission line is to transform the load impedance Z_L into a new impedance Z_1 for which $\text{Re}\{Z_1\} = \text{Re}\{Z_{in}\}$. This can be done by solving the equation (from Section 3.15)

$$\text{Re}\{Z_1\} = \text{Re}\left\{Z_0 \frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}\right\} \tag{3.22.1}$$

for l, using a numerical search, or using the Smith chart.[1] The characteristic impedance Z_0 and phase propagation constant β of the transmission line are independent variables and can be selected for convenience. Normally, the smallest value of l that satisfies Equation $3.22.1$ is desired. This value will be $\leq \lambda/4$ because the real part of Z_1 spans all possible values every $\lambda/4$.

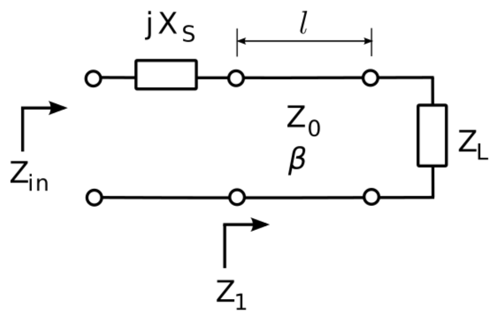

Figure 3.22.1: Single-reactance matching with a series reactance.

After matching the real component of the impedance in this manner, the imaginary component of Z_1 may then be transformed to the desired value ($\text{Im}\{Z_{in}\}$) by attaching a reactance X_s in series with the transmission line input, yielding $Z_{in} = Z_1 + jX_S$. Therefore, we choose

$$X_s = \text{Im}\{Z_{in} - Z_1\}$$

The sign of X_s determines whether this reactance is a capacitor ($X_s < 0$) or inductor ($X_s > 0$), and the value of this component is determined from X_s and the design frequency.

Example 3.22.1: Single reactance in series

Design a match consisting of a transmission line in series with a single capacitor or inductor that matches a source impedance of 50Ω to a load impedance of $33.9 + j17.6\ \Omega$ at 1.5 GHz. The characteristic impedance and phase velocity of the transmission line are 50Ω and $0.6c$ respectively.

Solution

From the problem statement: $Z_{in} \triangleq Z_S = 50\ \Omega$ and $Z_L = 33.9 + j17.6\ \Omega$ are the source and load impedances respectively at $f = 1.5$ GHz. The characteristic impedance and phase velocity of the transmission line are $Z_0 = 50\ \Omega$ and $v_p = 0.6c$ respectively.

The reflection coefficient Γ (i.e., Z_L with respect to the characteristic impedance of the transmission line) is

$$\Gamma \triangleq \frac{Z_L - Z_0}{Z_L + Z_0} \cong -0.142 + j0.239$$

The length l of the primary line (that is, the one that connects the two ports of the matching structure) is determined using the equation:

$$\mathrm{Re}\{Z_1\} = \mathrm{Re}\left\{Z_0 \frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}\right\}$$

where here $\mathrm{Re}\{Z_1\} = \mathrm{Re}\{Z_S\} = 50\ \Omega$. So a more-specific form of the equation that can be solved for βl (as a step toward finding l) is:

$$1 = \mathrm{Re}\left\{\frac{1+\Gamma e^{-j2\beta l}}{1-\Gamma e^{-j2\beta l}}\right\}$$

By trial and error (or using the Smith chart if you prefer) we find $\beta l \cong 0.408$ rad for the primary line, yielding $Z_1 \cong 50.0 + j29.0\ \Omega$ for the input impedance after attaching the primary line.

We may now solve for l as follows: Since $v_p = \omega/\beta$ (Section 3.8), we find

$$\beta = \frac{\omega}{v_p} = \frac{2\pi f}{0.6c} \cong 52.360 \text{ rad/m}$$

Therefore $l = (\beta l)/\beta \cong 7.8$ mm.

The impedance of the series reactance should be $jX_s \cong -j29.0\ \Omega$ to cancel the imaginary part of Z_1. Since the sign of this impedance is negative, it must be a capacitor. The reactance of a capacitor is $-1/\omega C$, so it must be true that

$$-\frac{1}{2\pi f C} \cong -29.0\ \Omega$$

Thus, we find the series reactance is a capacitor of value $C \cong 3.7pF$.

The second version of the single-reactance strategy is shown in Figure 3.22.2 The difference in this scheme is that the reactance is attached in parallel. In this case, it is easier to work the problem using admittance (i.e., reciprocal impedance) as opposed to impedance; this is because the admittance of parallel reactances is simply the sum of the associated admittances; i.e.,

$$Y_{in} = Y_1 + jB_p$$

where $Y_{in} = 1/Z_{in}$, $Y_1 = 1/Z_1$, and B_p is the discrete parallel susceptance; i.e., the imaginary part of the discrete parallel admittance.

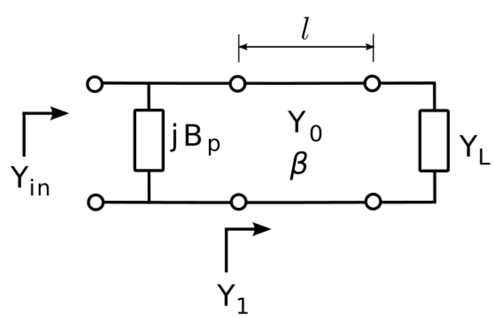

Figure 3.22.2: Single-reactance matching with a parallel reactance

So, the procedure is as follows. The transmission line is used to transform Y_L into a new admittance Y_1 for which $\mathrm{Re}\{Y_1\} = \mathrm{Re}\{Y_{in}\}$. First, we note that

$$Y_1 \triangleq \frac{1}{Z_1} = Y_0 \frac{1-\Gamma e^{-j2\beta l}}{1+\Gamma e^{-j2\beta l}}$$

where $Y_0 \triangleq 1/Z_0$ is characteristic *admittance*. Again, the characteristic impedance Z_0 and phase propagation constant β of the transmission line are independent variables and can be selected for convenience. In the present problem, we aim to solve the equation

$$\mathrm{Re}\{Y_1\} = \mathrm{Re}\left\{Y_0 \frac{1-\Gamma e^{-j2\beta l}}{1+\Gamma e^{-j2\beta l}}\right\}$$

for the smallest value of l, using a numerical search or using the Smith chart. After matching the real component of the admittances in this manner, the imaginary component of the resulting admittance may then be transformed to the desired value by attaching the susceptance B_p in parallel with the transmission line input. Since we desire jB_p in parallel with Y_1 to be Y_{in}, the desired value is

$$B_p = \text{Im}\{Y_{in} - Y_1\}$$

The sign of B_p determines whether this is a capacitor ($B_p > 0$) or inductor ($B_p < 0$), and the value of this component is determined from B_p and the design frequency.

In the following example, we address the same problem raised in Example $3.22.1$, now using the parallel reactance approach:

✓ Example $3.22.2$: Single reactance in parallel

Design a match consisting of a transmission line in parallel with a single capacitor or inductor that matches a source impedance of 50Ω to a load impedance of $33.9 + j17.6\ \Omega$ at 1.5 GHz. The characteristic impedance and phase velocity of the transmission line are 50Ω and $0.6c$ respectively.

Solution

From the problem statement: $Z_{in} \triangleq Z_S = 50\ \Omega$ and $Z_L = 33.9 + j17.6\ \Omega$ are the source and load impedances respectively at $f = 1.5$ GHz. The characteristic impedance and phase velocity of the transmission line are $Z_0 = 50\ \Omega$ and $v_p = 0.6c$ respectively.

The reflection coefficient Γ (i.e., Z_L with respect to the characteristic impedance of the transmission line) is

$$\Gamma \triangleq \frac{Z_L - Z_0}{Z_L + Z_0} \cong -0.142 + j0.239$$

The length l of the primary line (that is, the one that connects the two ports of the matching structure) is the solution to:

$$\text{Re}\{Y_1\} = \text{Re}\left\{Y_0 \frac{1 - \Gamma e^{-j2\beta l}}{1 + \Gamma e^{-j2\beta l}}\right\}$$

where here $\text{Re}\{Y_1\} = \text{Re}\{1/Z_S\} = 0.02$ mho and $Y_0 = 1/Z_0 = 0.02$ mho. So the equation to be solved for βl (as a step toward finding l) is:

$$1 = \text{Re}\left\{\frac{1 - \Gamma e^{-j2\beta l}}{1 + \Gamma e^{-j2\beta l}}\right\}$$

By trial and error (or the Smith chart) we find $\beta l \cong 0.126$ rad for the primary line, yielding $Y_1 \cong 0.0200 - j0.0116$ mho for the input admittance after attaching the primary line.

We may now solve for l as follows: Since $v_p = \omega/\beta$ (Section 3.8), we find

$$\beta = \frac{\omega}{v_p} = \frac{2\pi f}{0.6c} \cong 52.360 \text{ rad/m}$$

Therefore, $l = (\beta l)/\beta \cong 2.4$ mm.

The admittance of the parallel reactance should be $jB_p \cong +j0.0116$ mho to cancel the imaginary part of Y_1. The associated impedance is $1/jB_p \cong -j86.3\ \Omega$. Since the sign of this impedance is negative, it must be a capacitor. The reactance of a capacitor is $-1/\omega C$, so it must be true that

$$-\frac{1}{2\pi f C} \cong -86.3\ \Omega$$

Thus, we find the parallel reactance is a capacitor of value $C \cong 1.2pF$.

Comparing this result to the result from the series reactance method (Example $3.22.1$), we see that the necessary length of transmission line is much shorter, which is normally a compelling advantage. The tradeoff is that the parallel capacitance is much smaller and an accurate value may be more difficult to achieve.

1. For more about the Smith chart, see “Additional Reading” at the end of this section.↩

3.23: Single-Stub Matching

In Section 3.22, we considered impedance matching schemes consisting of a transmission line combined with a reactance which is placed either in series or in parallel with the transmission line. In many problems, the required discrete reactance is not practical because it is not a standard value, or because of non-ideal behavior at the desired frequency (see Section 3.21 for more about this), or because one might simply wish to avoid the cost and logistical issues associated with an additional component. Whatever the reason, a possible solution is to replace the discrete reactance with a transmission line "stub" – that is, a transmission line which has been open- or short-circuited. Section 3.16 explains how a stub can replace a discrete reactance. Figure $3.23.1$ shows a practical implementation of this idea implemented in microstrip. This section explains the theory, and we'll return to this implementation at the end of the section.

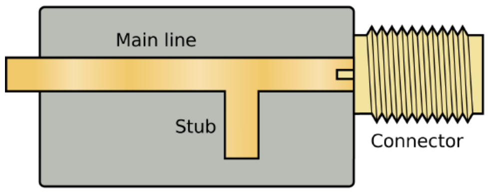

Figure $3.23.1$: A practical implementation of a singlestub impedance match using microstrip transmission line. Here, the stub is open-circuited. (CC BY SA 3.0; Spinningspark)

Figure $3.23.2$ shows the scheme. This scheme is usually implemented using the parallel reactance approach, as depicted in the figure. Although a series reactance scheme is also possible in principle, it is usually not as convenient. This is because most transmission lines use one of their two conductors as a local datum; e.g., the ground plane of a printed circuit board for microstrip line is tied to ground, and the outer conductor ("shield") of a coaxial cable is usually tied to ground. This is contrast to a discrete reactance (such as a capacitor or inductor), which does not require that either of its terminals be tied to ground. This issue is avoided in the parallel-attached stub because the parallel-attached stub and the transmission line to which it is attached *both* have one terminal at ground.

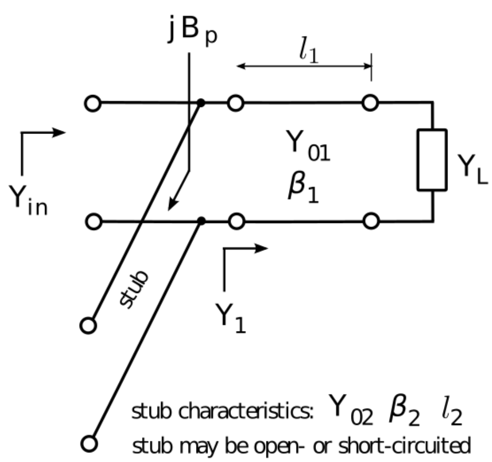

Figure $3.23.2$: Single-stub matching

The single-stub matching procedure is essentially the same as the single parallel reactance method, except the parallel reactance is implemented using a short- or open-circuited stub as opposed a discrete inductor or capacitor. Since parallel reactance matching is most easily done using admittances, [admittance] it is useful to express $Z_{in}(l) = +jZ_0 \tan \beta l$ and $Z_{in}(l) = -jZ_0 \cot \beta l$ (input impedance of an open- and short-circuited stub, respectively, from Section 3.16) in terms of susceptance:

$$B_p = -Y_{02} \cot(\beta_2 l_2) \text{ short-circuited stub}$$

$$B_p = +Y_{02} \tan(\beta_2 l_2) \text{ open-circuited stub}$$

As in the main line, the characteristic impedance $Z_{02} = 1/Y_{02}$ is an independent variable and is chosen for convenience.

A final question is when should you use a short-circuited stub, and when should you use an open-circuited stub? Given no other basis for selection, the termination that yields the shortest stub is chosen. An example of an "other basis for selection" that frequently comes up is whether DC might be present on the line. If DC is present with the signal of interest, then a short circuit termination without some kind of remediation to prevent a short circuit for DC would certainly be a bad idea.

In the following example we address the same problem raised in Section 3.22 (Examples 3.22.1 and 3.22.2), now using the single-stub approach:

Example 3.23.1: Single stub matching.

Design a single-stub match that matches a source impedance of 50Ω to a load impedance of $33.9 + j17.6\ \Omega$. Use transmission lines having characteristic impedances of 50Ω throughout, and leave your answer in terms of wavelengths.

Solution

From the problem statement: $Z_{in} \triangleq Z_S = 50\ \Omega$ and $Z_L = 33.9 + j17.6\ \Omega$ are the source and load impedances respectively. $Z_0 = 50\ \Omega$ is the characteristic impedance of the transmission lines to be used. The reflection coefficient Γ (i.e., Z_L with respect to the characteristic impedance of the transmission line) is

$$\Gamma \triangleq \frac{Z_L - Z_0}{Z_L + Z_0} \cong -0.142 + j0.239$$

The length l_1 of the primary line (that is, the one that connects the two ports of the matching structure) is the solution to the equation (from Section 3.22):

$$\mathrm{Re}\{Y_1\} = \mathrm{Re}\left\{Y_{01}\frac{1 - \Gamma e^{-j2\beta_1 l_1}}{1 + \Gamma e^{-j2\beta_1 l_1}}\right\}$$

where here $\mathrm{Re}\{Y_1\} = \mathrm{Re}\{1/Z_S\} = 0.02$mho and $Y_{01} = 1/Z_0 = 0.02$ mho. Also note

$$2\beta_1 l_1 = 2\left(\frac{2\pi}{\lambda}\right) l_1 = 4\pi\frac{l_1}{\lambda}$$

where λ is the wavelength in the transmission line. So the equation to be solved for l_1 is:

$$1 = \mathrm{Re}\left\{\frac{1 - \Gamma e^{-j4\pi l_1/\lambda}}{1 + \Gamma e^{-j4\pi l_1/\lambda}}\right\}$$

By trial and error (or using the Smith chart; see "Additional Reading" at the end of this section) we find for the primary line $l_1 \cong 0.020\lambda$, yielding $Y_1 \cong 0.0200 - j0.0116$ mho for the input admittance after attaching the primary line.

We now seek the shortest stub having an input admittance of $\cong +j0.0116$ mho to cancel the imaginary part of Y_1. For an open-circuited stub, we need

$$B_p = +Y_0 \tan 2\pi l_2/\lambda \cong +j0.0116 \text{ mho}$$

The smallest value of l_2 for which this is true is $\cong 0.084\lambda$. For a short-circuited stub, we need

$$B_p = -Y_0 \cot 2\pi l_2/\lambda \cong +j0.0116 \text{ mho}$$

The smallest positive value of l_2 for which this is true is $\cong 0.334\lambda$; i.e., much longer. Therefore, we choose the open-circuited stub with $l_2 \cong 0.084\lambda$. Note the stub is attached in parallel at the source end of the primary line.

Single-stub matching is a very common method for impedance matching using microstrip lines at frequences in the UHF band (300-3000 MHz) and above. In Figure 3.23.1, the top (visible) traces comprise one conductor, whereas the ground plane (underneath, so not visible) comprises the other conductor. The end of the stub is not connected to the ground plane, so the termination is an open circuit. A short circuit termination is accomplished by connecting the end of the stub to the ground plane using a *via*; that is, a plated-through that electrically connects the top and bottom layers.