Trajectory Planning

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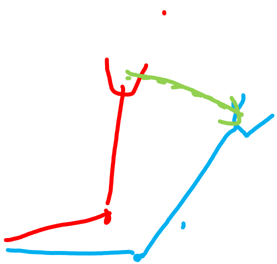
Path vs. Trajectory

<u>Path</u> an ordered locus of points in space, which the manipulator should follow. Pure geometric description of motion

<u>Trajectory</u> a path on which timing law is specified.(eg: velocities and accelerations at each point



Robot Motion Planing

<u>Path Planing</u>

Geometric path:

issue is finding optimal path

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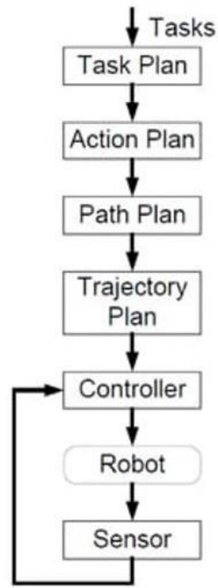
<u>Trajectory</u>

Approximate the desired path by class of polynomial functions.
Generate time based "control set points" for the manipulator from initial position to the destination



Robot Motion Planing

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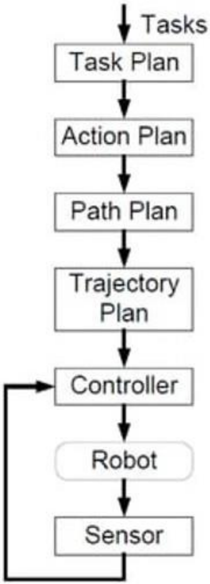
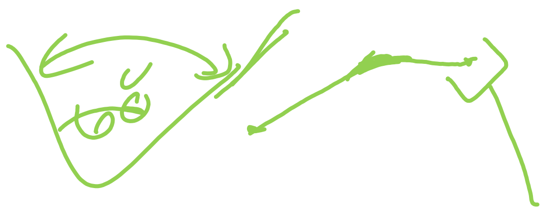


Trajectory Planning

- Methods of computing a smooth trajectory that describes the desired motion of a manipulator in multidimensional space
- Trajectory refers to a time history of position, velocity, and acceleration for each degree of freedom
- Should not violate saturation limits of joint drives



Joint Space vs. Operational Space

Cartesian Space

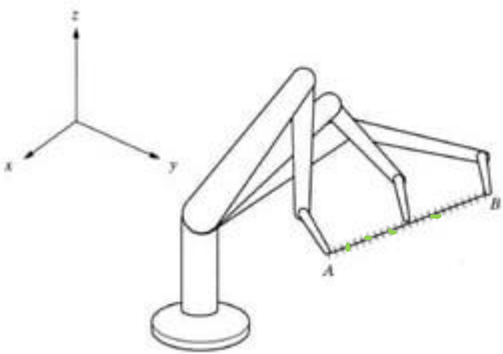
Operational Space Description:

- Description of the motion to be made by robot by its joint values
- The motion between two points is unpredictable

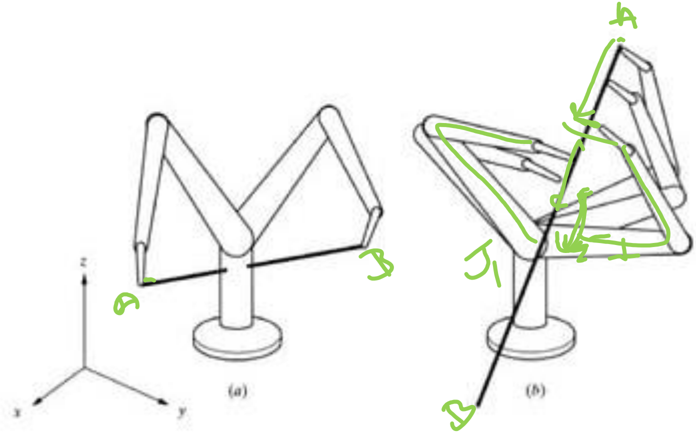
Joint Space Description:

- Motion between the two points is known at all times and controllable.
- Easy to visualize the trajectory, but difficult to ensure that singularity does not occur

Joint Space vs. Operational Space Example



Sequential motions of a robot to follow a straight line.



Cartesian-space trajectory
(a) The trajectory specified in Cartesian coordinates may force the robot to run into itself, and (b) the trajectory may requires a sudden change in the joint angles.

Trajectory in the Operational Space

- Calculate path from the initial point to the final point
- Assign a total time T_{path} to traverse the path
- Discretize the points in time and space
- Blend a continuous time function between these points
- Solve inverse kinematics at each step

<u>Advantages:</u>

- Collision free path can be obtained

<u>Disadvantages:</u>

- Computationally expensive due to inverse kinematics
- It is unknown how to set the total time T_{path}

Trajectory in the Joint Space

- Calculate inverse kinematics solution from initial point to the final point
- Assign total time T_q, using maximal velocities in joints
- Discretize the individual joint trajectories in time
- Blend a continuous function between these point

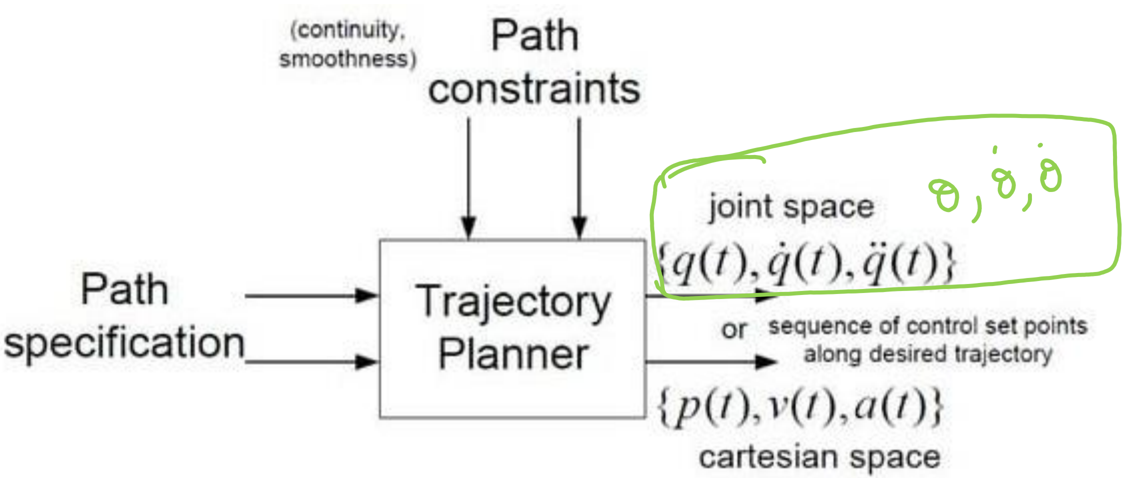
Advantages:

- Inverse kinematics is computed only once
- Can easily taken into account joint angle, velocity constraints

Disadvantages:

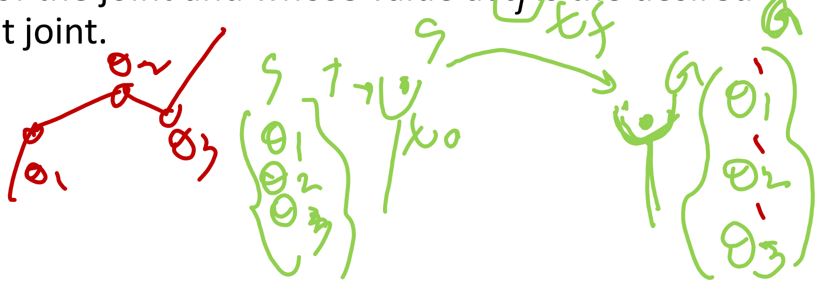
- Cannot deal with operational space obstacles

Trajectory Planning



Cubic Polynomials

- Consider the problem of moving the **tool from its initial** position to a **goal position** in a certain amount of time.
- Inverse kinematics allow the set of joint angles that correspond to the goal position and orientation to be calculated.
- The **initial position** of the manipulator is also known in the form of a set of joint angles.
- What is required is a function for **each joint** whose value at $\boldsymbol{t_0}$ is the initial position of the joint and whose value at $\boxed{\boldsymbol{t_f}}$ is the desired goal position of that joint.



Cubic Polynomials

$\theta_1, \theta_2, \theta_3$

- In making a single smooth motion, at least four constraints on $\boldsymbol{\theta(t)}$ are evident.

$$\theta(0) = \theta_0$$
$$\theta(t_f) = \theta_f$$

- the function be continuous in **velocity**, which in this case means that the **initial** and **final** velocity are **zero**:

$$\dot{\theta}(0) = 0,$$
$$\dot{\theta}(t_f) = 0.$$

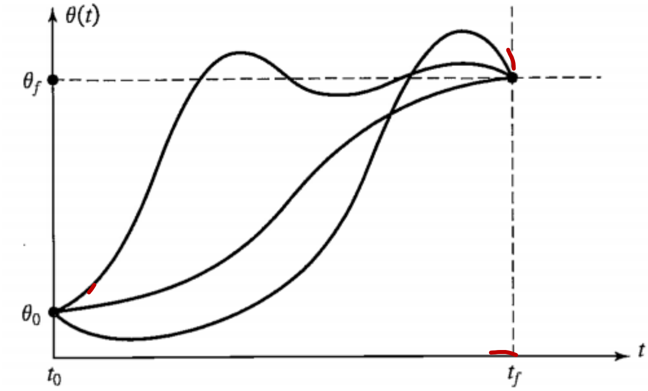


FIGURE 7.2: Several possible path shapes for a single joint.

Cubic Polynomials

- These four constraints can be satisfied by a polynomial of third order: $$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 \quad (1)$$

$\frac{d\theta(t)}{dt} = \dot{\theta}(t)$

- Joint velocity and acceleration along this path are

$$\dot{\theta}(t) = a_1 + 2a_2 t + 3a_3 t^2 \quad (2)$$

$\frac{d\dot{\theta}(t)}{dt} = \ddot{\theta}(t)$

$t = 0 \qquad \theta(0) = a_0$

$$\ddot{\theta}(t) = 2a_2 + 6a_3 t \quad (3)$$

- Applying the four constraints gives four equations for the unknown a

$t = t_f \rightarrow (1), (2)$

$$\theta_0 = a_0$$
$$0 = a_1$$

$\dot{\theta}(0) = a_1$

$$\theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \quad (A)$$
$$\dot{\theta}(t_f) = 0 = a_1 + 2a_2 t_f + 3a_3 t_f^2 \quad (B)$$

- The solution:

$$a_0 = \theta_0, \qquad a_1 = 0, \qquad a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0), \qquad a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0)$$

(B) $2a_2 \cancel{t_f} = -3a_3 t_f^{\cancel{2}}$ $\qquad a_2 = -\frac{3}{2} a_3 t_f$

Cubic Polynomials

- The solution:

$$a_0 = \theta_0, \qquad a_1 = 0, \qquad a_2 = \frac{3}{{t_f}^2}(\theta_f - \theta_0), \qquad a_3 = -\frac{2}{{t_f}^3}(\theta_f - \theta_0)$$

- This cubic polynomial can be used to connect any initial joint-angle position with any desired final position.

- This solution is valid only for the case when the joint starts and finishes at zero velocity.

- The single Cubic Polynomial equation that satisfies these conditions is:

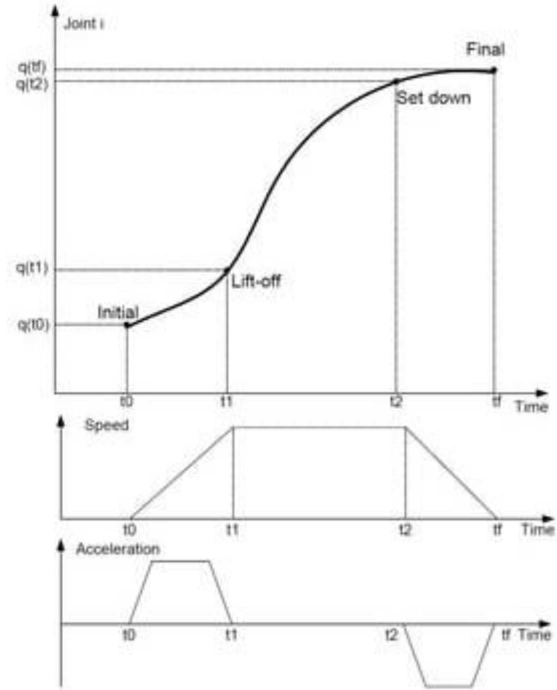
$$\theta(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

$$\boxed{\theta(t) = \theta_0 + \frac{3}{{t_f}^2}(\theta_f - \theta_0)t^2 - \frac{2}{{t_f}^3}(\theta_f - \theta_0)t^3}$$

- Path Profile

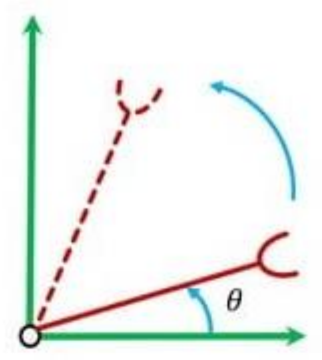
- Velocity Profile

- Acceleration Profile



Example

- A single-link manipulator with a revolt joint stopping at θ = 15 degrees. It's desired to move the joint in a smooth manner to θ = 75 degrees in 3 seconds. Find the coefficients of a cubic that accomplishes this motion and brings the manipulator to rest at the goal.



Example

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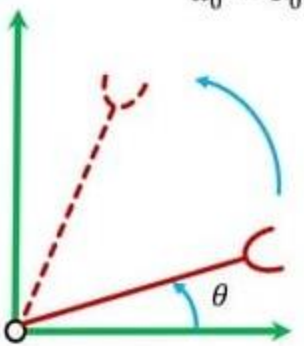
Solution:

$$a_0 = \theta_0, \qquad a_1 = 0, \qquad a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0), \qquad a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0)$$



$$a_0 = 15,$$
$$a_1 = 0,$$
$$a_2 = \frac{3}{9}(75 - 15) = 20,$$
$$a_3 = -\frac{2}{27}(75 - 15) = -4.44$$

Joint Position

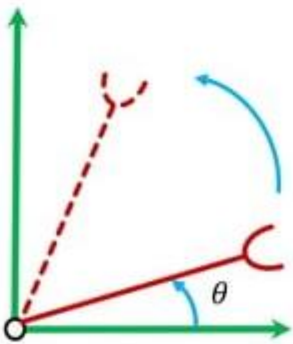
$$\theta(t) = 15 + 20\,t^2 - 4.44\,t^3$$

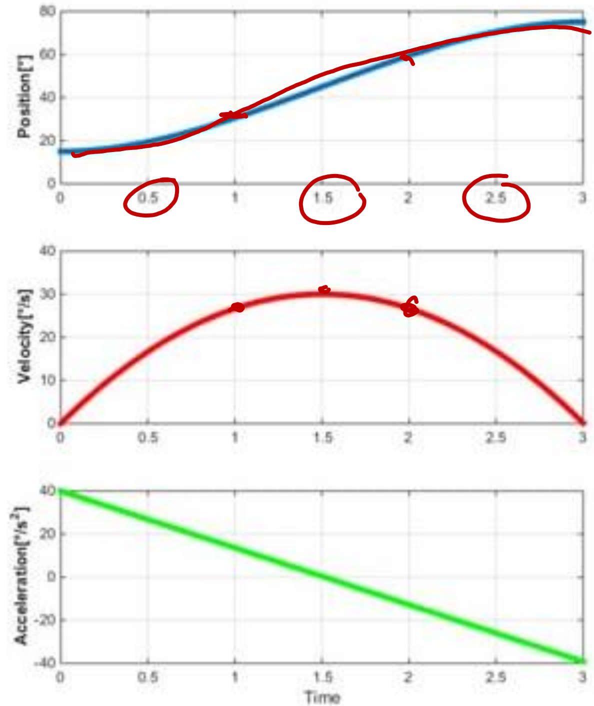
Joint velocity along this path,

$$\dot{\theta}(t) = 40\,t - 13.32\,t^2$$

Joint acceleration along this path,

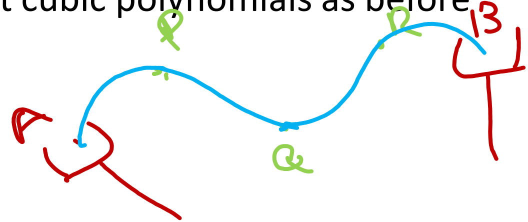
$$\ddot{\theta}(t) = 40 - 26.4\,t$$





Cubic Polynomials with Via Points

- If the desired velocities of the joints at the via points have non-zero values, then we can construct cubic polynomials as before with considering new constraints:
  - Initial value $\theta(0) = \theta_0$
  - Final value $\theta(t_f) = \theta_f$
  - Initial velocity $\dot{\theta}(0) = \dot{\theta}_0$
  - Final velocity $\dot{\theta}(t_f) = \dot{\theta}_f$



- Applying the four constraints gives four equations for the unknown a

$$\begin{gathered}
\theta_0 = a_0 \\
\dot{\theta}_0 = a_1 \\
\theta_f = a_0 + a_1 t_f + a_2 t_f^2 + a_3 t_f^3 \\
\dot{\theta}_f = a_1 + 2a_2 t_f + 3a_3 t_f^2
\end{gathered}$$

Cubic Polynomials with Via Points

- Solution

$$a_0 = \theta_0$$
$$a_1 = \dot{\theta}_0$$
$$a_2 = \frac{3}{t_f^2}(\theta_f - \theta_0) - \frac{1}{t_f}(2\dot{\theta}_0 + \dot{\theta}_f)$$
$$a_3 = -\frac{2}{t_f^3}(\theta_f - \theta_0) + \frac{1}{t_f^2}(\dot{\theta}_0 + \dot{\theta}_f)$$

- Now we are able to calculate the cubic polynomial that connects any initial and final positions with any initial and final velocities

$$\theta(t) = \theta_0 + \dot{\theta}_0 t + [\ a_2\ ] t^2 + [\ a_3\ ] t^3$$

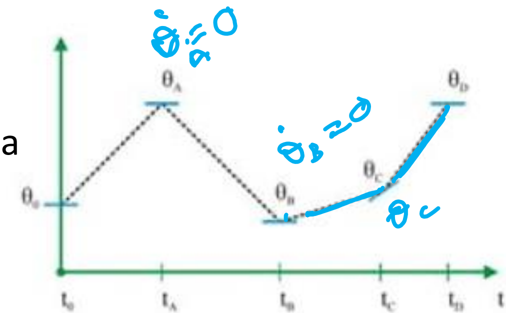
Velocities at Via Points

- There are several ways to work out the desired velocity at the via points
- The user specifies the desired velocity at each via point in terms of a Cartesian linear and angular velocity of the tool frame at that instant.
- Cartesian velocities at the via points are mapped to the desired joint velocity by using the inverse Jacobian of the manipulator at that point.

Velocities at Via Points

- The system automatically chooses the velocities at the via points.
- Desired velocities at the points are indicated with the tangents.
- The via points are connected with straight line segments.

- If the slope of these lines changes the sign at the via point, choose zero velocity (point A and B) else, choose the average of the two slopes as the via velocity (point C).



Velocities at Via Points

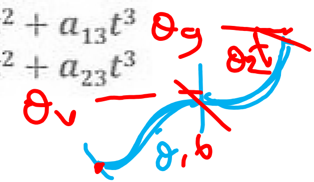
- The system automatically chooses the velocities at the via points in such a way that acceleration is continuous at the via points.
- To do this, a new approach is needed.
- We will replace the two velocity constraints at the connection of two cubics with the two constraints that velocity and acceleration be continuous.

Example

Solve for the coefficients of two cubics that are connected in a two-segment spline with continuous velocity and acceleration at the via point. The given values are:

- the initial angle θ_0
- the via point θ_v
- the goal point θ_g

$$\theta_1(t) = a_{10} + a_{11}t + a_{12}t^2 + a_{13}t^3$$
$$\theta_2(t) = a_{20} + a_{21}t + a_{22}t^2 + a_{23}t^3$$



Angular constraints for first cubic:

- Initial position $\theta_0 = a_{10}$
- Terminal position $\theta_v = a_{10} + a_{11}t_{f1} + a_{12}t_{f1}^2 + a_{13}t_{f1}^3$

Angular constraints for second cubic:

- Initial position $\theta_v = a_{20}$
- Terminal position $\theta_g = a_{20} + a_{21}t_{f2} + a_{22}t_{f2}^2 + a_{23}t_{f2}^3$

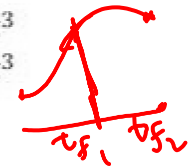
Example

Solve for the coefficients of two cubics that are connected in a two-segment spline with continuous velocity and acceleration at the via point. The given values are:

- the initial angle θ_0
- the via point θ_v
- the goal point θ_g

$a_0 \quad a_1 \quad a_2 \quad a_3$

$$\theta_1(t) = a_{10} + \underline{a_{11}}t + a_{12}t^2 + a_{13}t^3$$
$$\theta_2(t) = a_{20} + \underline{a_{21}}t + a_{22}t^2 + a_{23}t^3$$



Angular velocity constraint for first cubic: Start from rest:

$$0 = a_{11}$$

Angular velocity constraint for second cubic: End at rest:

$\frac{d}{dt}\theta_2$

$$0 = a_{21} + 2\,a_{22}t_{f2} + 3\,a_{23}t_{f2}{}^2$$

Both cubics must have the same angular velocity and acceleration at the via point:

$$a_{11} + 2\,a_{12}t_{f1} + 3\,a_{13}t_{f1}{}^2 = a_{21}$$
$$2\,a_{12} + 6\,a_{13}t_{f1} = 2\,a_{22}$$

Example

$\theta_0 = a_{10}$

$\theta_v = a_{10} + a_{11}t_{f1} + a_{12}t_{f1}^2 + a_{13}t_{f1}^3$

$\theta_v = a_{20}$

$\theta_g = a_{20} + a_{21}t_{f2} + a_{22}t_{f2}^2 + a_{23}t_{f2}^3$

$0 = a_{11}$

$0 = a_{21} + 2\,a_{22}t_{f2} + 3\,a_{23}t_{f2}^2$

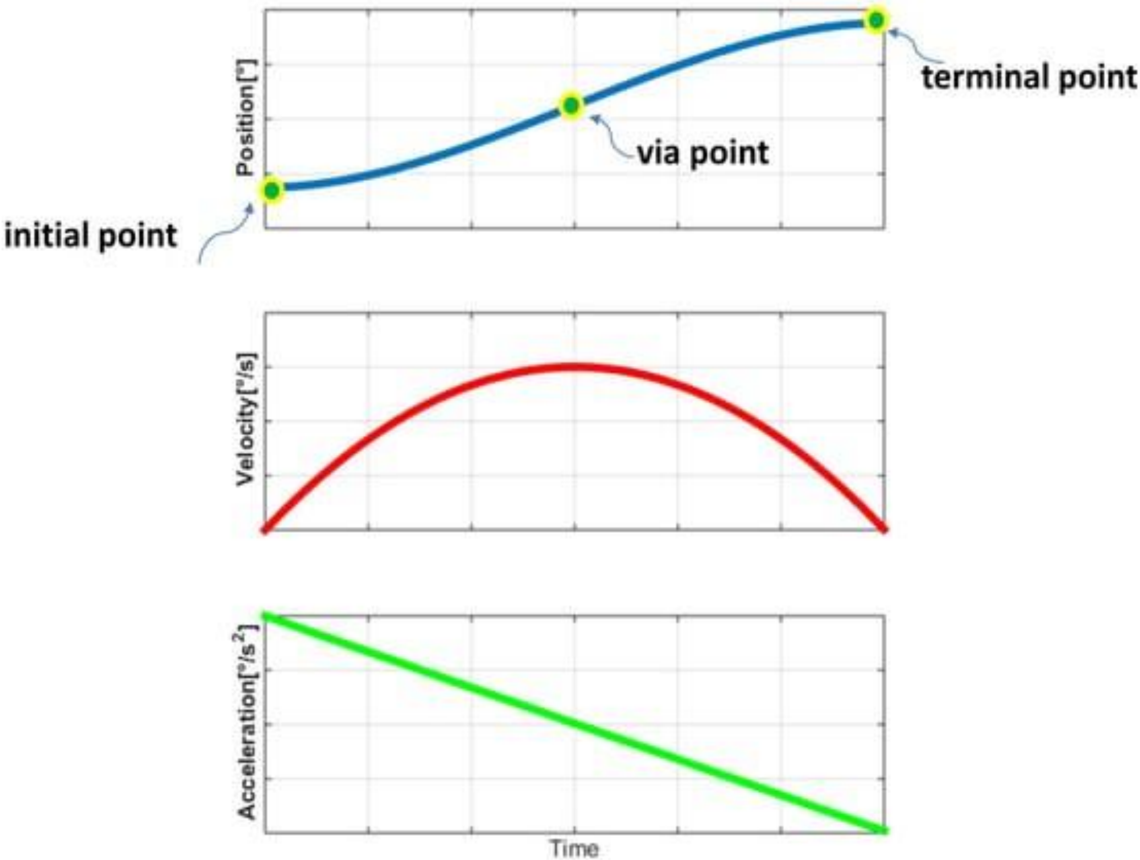
$a_{11} + 2\,a_{12}t_{f1} + 3\,a_{13}t_{f1}^2 = a_{21}$

$2\,a_{12} + 6\,a_{13}t_{f1} = 2\,a_{22}$

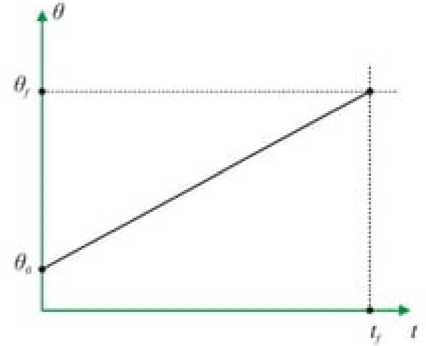
If we consider $t_f = t_{f1} = t_{f2}$, solution:

$$a_{10} = \theta_0$$
$$a_{11} = 0$$
$$a_{12} = \frac{12\theta_v - 3\theta_g - \theta_0}{4t_f^2}$$
$$a_{13} = \frac{-8\,\theta_v + 3\theta_g + 5\,\theta_0}{4t_f^3}$$

$$a_{20} = \theta_v$$
$$a_{21} = \frac{3\,\theta_g - 3\,\theta_0}{4t_f}$$
$$a_{22} = \frac{-12\theta_v + 6\theta_g + 6\theta_0}{4t_f^2}$$
$$a_{23} = \frac{8\theta_v - 5\theta_g - 3\theta_0}{4t_f^3}$$


terminal point
via point
initial point
Position[°]
Velocity[°/s]
Acceleration[°/s²]
Time

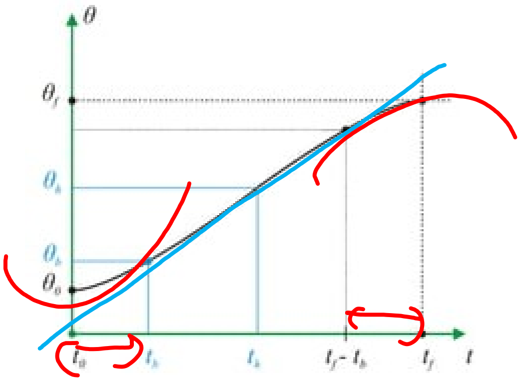
Linear Interpolation



- Another choice of joint-path shape is linear.
- That is, we simply interpolate linearly to move from the present joint position to the final position.
- Remember that, although the motion of each joint in this scheme is linear, the end-effector in general does not move in a straight line in Cartesian space.
- However, straightforward linear interpolation would cause the velocity to be discontinuous at the beginning and at the end of the motion.
- To create a smooth path with continuous position and velocity, we start with the linear function but add a parabolic blend region at each path point.

Linear Interpolation with Parabolic Blends

- During the blend part of the trajectory, constant acceleration is used to change velocity smoothly.
- The linear function and the two parabolic functions are "splined" together so that the entire path is continuous in position and velocity.



- We will assume that the parabolic blends have the same duration; therefore, the same constant acceleration is used during the blends.

Linear Interpolation with Parabolic Blends

- The velocity at the end of the blend region must equal the velocity of the linear section, and so we have:

$\dot{\theta}_{tb}$

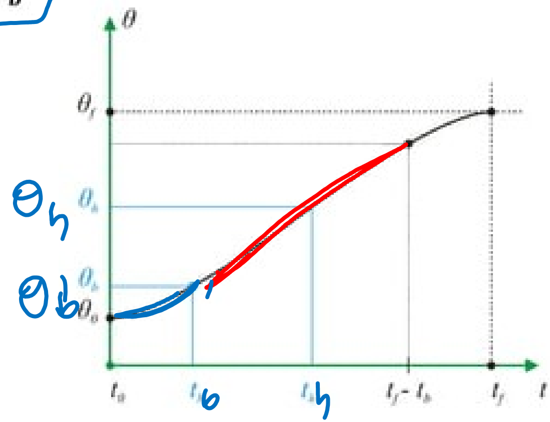
$$\ddot{\theta}t_b = \frac{\theta_h - \theta_b}{t_h - t_b}$$

- where θ_b is the joint angle at the end of the blend region, and $\ddot{\theta}$ is the joint acceleration acting during the blend region.

$s = u \cancel{t} \frac{1}{2} at^2$

- The value of θ_b is given by $\boxed{\theta_b = \theta_0 + \frac{1}{2}\ddot{\theta}t_b^2}$
- Combining the two equations and taking into account the symmetry of the path and its duration $t_f = 2t_h$, we get:

$$\ddot{\theta}t_b(t_h - t_b) = \theta_h - \theta_0 - \frac{1}{2}\ddot{\theta}t_b^2$$



Linear Interpolation with Parabolic Blends

$$\ddot{\theta} t_b (t_h - t_b) = \theta_h - \theta_0 - \frac{1}{2} \ddot{\theta} {t_b}^2$$

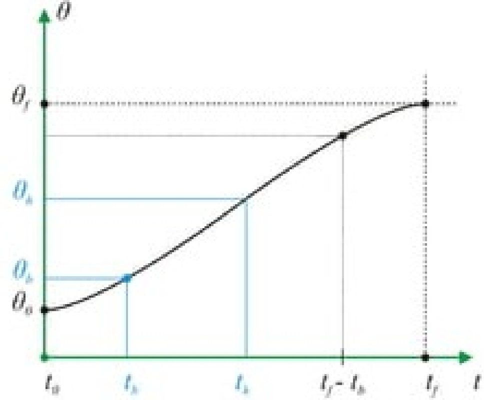
$$\Rightarrow -\ddot{\theta} {t_b}^2 + \ddot{\theta} t_b t_h = \theta_h - \theta_0 - \frac{1}{2} \ddot{\theta} {t_b}^2$$

$$\Rightarrow \frac{1}{2} \ddot{\theta} {t_b}^2 - \ddot{\theta} t_b t_h + (\theta_h - \theta_0) = 0$$

$$\Rightarrow \frac{1}{2} \ddot{\theta} {t_b}^2 - \ddot{\theta} t_b \frac{t_f}{2} + \frac{1}{2} (\theta_f - \theta_0) = 0$$

$$\Rightarrow \ddot{\theta} {t_b}^2 - \ddot{\theta} t_b t_f + (\theta_f - \theta_0) = 0$$

where $\boldsymbol{t_f}$ is the desired duration of the motion



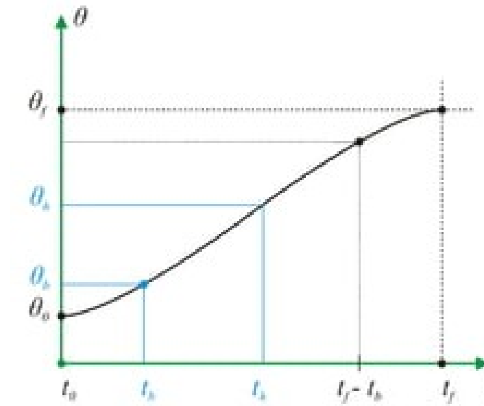
Linear Interpolation with Parabolic Blends

$$\ddot{\theta} t_b^2 - \ddot{\theta} t_b t_f + (\theta_f - \theta_0) = 0$$

- Given $\boldsymbol{\theta_f}$, $\boldsymbol{\theta_0}$ and $\boldsymbol{t_f}$ we can follow any of the path given by the choices of $\ddot{\boldsymbol{\theta}}$ and $\boldsymbol{t_b}$ that satisfy the previous equation.

- The solution of the equation for the blend duration is

$$t_b = \frac{t_f}{2} - \frac{\sqrt{\ddot{\theta}^2 t_f^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$$

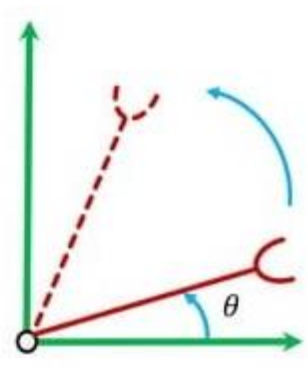


- A real solution exists if:.

$$\ddot{\theta} \geq \frac{4\,(\theta_f - \theta_0)}{t_f^2}$$

Example

- A single-link manipulator with a revolt joint stopping at θ = 15 degrees. It's desired to move the joint in a smooth manner to θ = 75 degrees in 3 seconds.
- Show two examples, one with high acceleration and one with low acceleration of a linear path with parabolic blends.



Example

- Position, velocity, and acceleration profiles for linear interpolation with parabolic blends. The set of curves on the left is based on a higher acceleration during the blends than in that on the right.