

## Cumulative Variances Proof of Method

Using the propability density of the normal distribution, the first step of a one-dimensional random walk can be expressed as Probability:2013:

$$p(x_1) = \frac{1}{\sigma_1 * \sqrt{2\pi}} \exp\left(\frac{-(x_1 - \mu_1)^2}{2\sigma_1^2}\right). \quad (1)$$

For the one-dimensional random walk scenario  $\mu_1$  is equal to 0. The conditional probability density of the second step, given the first step and corresponding variance  $\sigma_1^2$  for one step ahead prediction, can be written as Probability:2013:

$$p(x_2|x_1) = \frac{1}{\sigma_1 * \sqrt{2\pi}} \exp\left(\frac{-(x_2 - x_1)^2}{2\sigma_1^2}\right). \quad (2)$$

Using the product rule for joint probability distributions and the marginalization rule leads to Probability:2013:

$$p(x, y) = p(x|y)p(y) = p(y|x)p(x), \quad (3)$$

$$p(x) = \int_y p(x, y) dy, \quad (4)$$

where  $x, y \in (-\infty, +\infty)$ . Using the combination of both rules, the probability density of the second step can be written as:

$$p(x_2) = \int_{x_1} p(x_2, x_1) dx_1 = \int p(x_2|x_1)p(x_1) dx_1, \quad (5)$$

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \int \exp\left(\frac{-(x_2 - x_1)^2}{2\sigma_1^2}\right) \exp\left(\frac{-x_1^2}{2\sigma_1^2}\right) dx_1, \quad (6)$$

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \int \exp\left(\frac{-(x_2^2 + x_1^2 - 2x_2x_1 + x_1^2)}{2\sigma_1^2}\right) dx_1, \quad (7)$$

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \int \exp\left(\frac{-\left(\sqrt{2}x_1 - \frac{1}{\sqrt{2}}x_2\right)^2}{2\sigma_1^2}\right) \exp\left(\frac{-x_2^2}{4\sigma_1^2}\right) dx_1. \quad (8)$$

This equation can be substituted by using the transformation  $z = \sqrt{2}x_1 - \frac{1}{\sqrt{2}}x_2$  and  $dx_1 = \frac{1}{\sqrt{2}}dz$ :

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \exp\left(\frac{-x_2^2}{4\sigma_1^2}\right) \int \exp\left(\frac{-z^2}{2\sigma_1^2}\right) \frac{1}{\sqrt{2}} dz. \quad (9)$$

Using an additional transformation  $u = \frac{z}{\sqrt{2}\sigma}$  and  $dz = \sqrt{2}\sigma du$  leads to:

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \exp\left(\frac{x_2^2}{4\sigma_1^2}\right) \sigma_1 \underbrace{\int \exp(-u^2) du}_{\sqrt{\pi}}, \quad (10)$$

$$p(x_2) = \frac{1}{\sigma_1 * 2\sqrt{\pi}} \exp\left(\frac{-x_2^2}{4\sigma_1^2}\right) = \frac{1}{\sqrt{4\sigma_1^2\pi}} \exp\left(\frac{-x_2^2}{4\sigma_1^2}\right). \quad (11)$$

Hence with  $\sigma_2^2 = 2 * \sigma_1^2$  this can be written as:

$$p(x_2) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(\frac{-x_2^2}{2\sigma_2^2}\right). \quad (12)$$

This represents the probability density of a normally distributed variable with variance  $\sigma_2^2 = 2 * \sigma_1^2$  and mean zero which completes the proof for cumulatively adding variances for  $n$ -step ahead predictions. Therefore:

$$\sigma_n^2 = n * \sigma_1^2, \quad (13)$$

for  $n$ -step ahead predictions with a constant single step variance  $\sigma_1^2$ .