# Cumulative Variances Proof of Method

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Using the propability density of the normal distribution, the first step of a one-dimensional random walk can be expressed as:

$$p(x_1) = \frac{1}{\sigma_1 * \sqrt{2\pi}} \exp\left(\frac{-(x_1 - \mu_1)^2}{2\sigma_1^2}\right). \tag{1}$$

For the one-dimensional random walk scenario $\mu_1$ is equal to 0. The conditional probability density of the second step, given the first step and corresponding variance $\sigma_1^2$ for one step ahead prediction, can be written as:

$$p(x_2|x_1) = \frac{1}{\sigma_1 * \sqrt{2\pi}} \exp\left(\frac{-(x_2 - x_1)^2}{2\sigma_1^2}\right). \tag{2}$$

Using the product rule for joint probability distributions and the marginalization rule leads to:

$$p(x, y) = p(x|y)p(y) = p(y|x)p(x), \tag{3}$$

$$p(x) = \int_y p(x, y)dy, \tag{4}$$

where $x, y \in (-\infty, +\infty)$. Using the combination of both rules, the probability density of the second step can be written as:

$$p(x_2) = \int_{x_1} p(x_2, x_1)dx_1 = \int p(x_2|x_1)p(x_1)dx_1, \tag{5}$$

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \int \exp\left(\frac{-(x_2 - x_1)^2}{2\sigma_1^2}\right) \exp\left(\frac{-x_1^2}{2\sigma_1^2}\right) dx_1, \tag{6}$$

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \int \exp\left(\frac{-(x_2^2 + x_1^2 - 2x_2x_1 + x_1^2)}{2\sigma_1^2}\right) dx_1, \tag{7}$$

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \int \exp\left(\frac{-\left(\sqrt{2}x_1 - \frac{1}{\sqrt{2}}x_2\right)^2}{2\sigma_1^2}\right) \exp\left(\frac{-x_2^2}{4\sigma_1^2}\right) dx_1. \tag{8}$$

This equation can be substituted by using the transformation $z = \sqrt{2}x_1 - \frac{1}{\sqrt{2}x_2}$ and $dx_1 = \frac{1}{\sqrt{2}}dz$:

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \exp\left(\frac{x_2^2}{4\sigma_1^2}\right) \int \exp\left(\frac{-z^2}{2\sigma_1^2}\right) \frac{1}{\sqrt{2}}dz. \tag{9}$$

Using an additional transformation $u = \frac{z}{\sqrt{2}\sigma}$ and $dz = \sqrt{2}\sigma du$ leads to:

$$p(x_2) = \frac{1}{\sigma_1^2 * 2\pi} \exp\left(\frac{x_2^2}{4\sigma_1^2}\right) \sigma_1 \underbrace{\int \exp\left(-u^2\right) du}$$

$\sqrt{\pi}$, (10)

$$p(x_2) = \frac{1}{\sigma_1 * 2\sqrt{\pi}} \exp\left(\frac{-x_2^2}{4\sigma_1^2}\right) = \frac{1}{\sqrt{4\sigma_1^2\pi}} \exp\left(\frac{-x_2^2}{4\sigma_1^2}\right). \tag{11}$$

Hence with $\sigma_2^2 = 2 * \sigma_1^2$ this can be written as:

$$p(x_2) = \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(\frac{-x_2^2}{2\sigma_2^2}\right). \tag{12}$$

This represents the probability density of a normally distributed variable with variance $\sigma_2^2 = 2 * \sigma_1^2$ and mean zero which completes the proof for cumulatively adding variances for $n$-step ahead predictions. Therefore:

$$\sigma_n^2 = n * \sigma_1^2, \tag{13}$$

for $n$-step ahead predictions with a constant single step variance $\sigma_1^2$.