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3D Computer Vision
Exercise Sheet 1

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Exercise 1 (Properties of Rotation Matrices).
Rotation matrices are orthogonal matrices with determinant 1.

1. Show that $U^T = U^{-1}$ holds for a *general* rotation matrix $U \in \mathbb{R}^{3\times3}$. Note that by definition $UU^T = I \Leftrightarrow U^{-1} = U^T$.

2. What is the geometric interpretation of the determinatent of a square 3×3 matrix A? Why does a determination matrix have determinant 1?. Hint: Compare the determinants of base transformations(e.g. a rotation, translation,...)

Solution (to Exercise 1.1).
For every orthogonal matrix $U \in \mathbb{R}^{n\times n}$ it holds

$$UU^T = I$$

by definition(see [1, Def. 22.1b)]). Now, with the remark in Exercise 1 that rotation matrices are orthogonal matrices with determinant 1, the statement apparently holds true.

Solution (to Exercise 1.2).
Let $A \in \mathbb{R}^{3\times3}$ be an arbitrary square matrix. We denote A by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \tag{1}$$

Then we use the definition of a 3×3 determinant.

$$det(A) = a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{33} - a_{11}a_{32}a_{23}$$

Furthermore, we have the *triple product* V defined by three vectors

$$\begin{aligned}
V(A) &= \left(\begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} \times \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \end{pmatrix} \right) \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \\
&= \begin{pmatrix} a_{21}a_{32} - a_{31}a_{22} \\ a_{31}a_{12} - a_{32}a_{11} \\ a_{11}a_{22} - a_{21}a_{12} \end{pmatrix} \cdot \begin{pmatrix} a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} \\
&= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13} - a_{32}a_{23}a_{33} - a_{11}a_{32}a_{23} \\
&= det(A)
\end{aligned}$$

We see that the determinant of a 3×3 matrix is equal to the triple product. The triple product equals the volume of the parallelepiped spanned by the three columns of the matrix. Rotation matrices do not change the Volume of the parallelepiped. This leads to them having a determinant equal to 1.

References

[1] Gathmann, Andreas. Grundlagen der Mathmatik, Vorlesungsskript TU Kaiserslautern, 2016.