Fast inversion of Hessenberg-quasiseparable-Vandermonde matrices and resulting recurrence relations and characterizations

T.Bella, Y.Eidelman, I.Gohberg, V.Olshevsky E.Tyrtyshnikov

Abstract. Although Gaussian elimination uses $\mathcal{O}(n^3)$ operations to invert an arbitrary matrix, matrices with a special Vandermonde structure can be inverted in only $\mathcal{O}(n^2)$ operations by the *fast* Traub algorithm [T66]. It was noticed in [GO97] that with a minor modification of the Traub algorithm it can typically yield a very high accuracy. The Traub algorithm has been extended from Vandermonde matrices $V(x) = \left[\begin{array}{c} x_i^{j-1} \end{array}\right]$ involving monomials to polynomial Vandermonde matrices $V_R(x) = \left[\begin{array}{c} r_{j-1}(x_i) \end{array}\right]$ involving real orthogonal polynomials [CR93], [GO94], and the Szegö polynomials [O01].

In this paper we consider a new more general class of polynomials that we suggest to call *H-(1,1)-q.s. polynomials*. The new class is wide enough to include all of the above important special cases, e.g., monomials, real orthogonal polynomials and the Szegö polynomials. We derive a fast $\mathcal{O}(n^2)$ Traub-like algorithm to invert the associated *H-(1,1)-q.s.-Vandermonde matrices*.

In the course of considering the new Traub-like algorithm, three different sets of recurrence relations for the associated (generalized Horner) polynomials are derived to achieve the $\mathcal{O}(n^2)$ complexity. These recurrence relations apply also to the H-(1,1)-q.s. polynomials themselves; more specifically, to particular subclasses of H-(1,1)-q.s. polynomials. Thus these different sets of recurrence relations are also used to characterize some subclasses of H-(1,1)-q.s. polynomials.

The class of *quasiseparable matrices* is garnering a lot of attention recently; it has been found to be useful in designing a number of fast algorithms. The derivation of our new Traub-like algorithm is also based on exploiting the quasiseparable order (1,1) structure of the corresponding Hessenberg matrices (thus suggesting the name H-(1,1)-q.s. polynomials). The encouraging results of the first numerical experiments are presented.

1. Introduction

1.1. Inversion of polynomial-Vandermonde matrices

For a given system of polynomials $R = \{r_0(x), r_1(x), \ldots, r_{n-1}(x)\}$ and set of nodes $x = (x_1, \ldots, x_n)$, the corresponding *polynomial-Vandermonde* matrix $V_R(x) = \left[\begin{array}{c} r_{j-1}(x_i) \end{array}\right]$ is given by

$$V_R(x) = \begin{bmatrix} r_0(x_1) & r_1(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & \cdots & r_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ r_0(x_n) & r_1(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}. \tag{1.1}$$

We consider in this paper the problem of inversion of the matrix $V_R(x)$ for a given system of polynomials R in terms of the recurrence relations they satisfy.

In the simplest case where $R = \{1, x, x^2, \ldots, x^{n-1}\}$, $V_R(x)$ reduces to a classical Vandermonde matrix and the inversion algorithm is due to Traub [T66]. It was shown in [GO97] that a minor modification of the original Traub algorithm results in very good accuracy.

While the structure-ignoring approach of Gaussian elimination for inversion of $V_R(x)$ requires $O(n^3)$ operations, the special structure allows algorithms to be derived exploiting that structure, resulting in fast algorithms computing n^2 entries of the inverse in $O(n^2)$ operations. Previous work in deriving such fast algorithms for inversion of $V_R(x)$ for various special cases of the polynomial system R are listed in Table 1.

Matrix $V_R(x)$	Polynomial System R	Fast inversion algorithm
Classical Vandermonde	monomials	Traub [T66]
Chebyshev-Vandermonde	Chebyshev polynomials	Gohberg-Olshevsky [GO94]
Three-Term Vandermonde	Real orthogonal polynomials	Calvetti-Reichel [CR93]
Szegö-Vandermonde	Szegö polynomials	Olshevsky [O01]

Table 1. Fast $O(n^2)$ inversion algorithms.

1.2. A more general class of polynomials

In this paper, we consider a more general class of polynomials that contains all of those listed in Table 1 as special cases. For simplicity, we start the introduction by considering the class of polynomials that satisfy the fairly general recurrence relations[1]

$$r_k(x) = (\alpha_k x - \delta_k) \cdot r_{k-1}(x) - (\beta_k x + \gamma_k) \cdot r_{k-2}(x). \tag{1.2}$$

To show that these recurrence relations generalize those satisfied by the special cases listed in Table 1, we list each system and corresponding recurrence relations in Table 2.

Polynomial System R	Recurrence relations
monomials	$r_k(x) = x \cdot r_{k-1}(x)$
Chebyshev polynomials	$r_k(x) = 2x \cdot r_{k-1}(x) - r_{k-2}(x)$
Real orthogonal polynomials	$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k \cdot r_{k-2}(x)$
Szegö polynomials	$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k}\right) r_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x\right) r_{k-2}(x)$ Note: Typically the 2-term recurrence relations (7.10) are used for the Szegö polynomials in practice . It is convenient to use here the 3-term recurrence relations of [G48] to draw a theoretical connection to (1.2).
	$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - (\beta_k x + \gamma_k) r_{k-2}(x)$

Table 2. Systems of polynomials and corresponding recurrence relations.

[1] In fact, the algorithm we derive is valid for a slightly larger class of polynomials than those satisfying these recurrence relations. This is the class of H-$(1,1)$-q.s. polynomials, named for their relation to quasiseparable matrices; see Definition 1.3 for the formal definition, and Section 4 for more details.

1.3. Matrix interpretation of recurrence relations (1.2)

It will be shown in Theorem 4.7 below that the systems of polynomials satisfying (1.2) are related to what we suggest to call a *counterpart* matrix,

$$A = \begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2} & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right) & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right)\left(\frac{\beta_4}{\alpha_4}\right) & \cdots & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right)\left(\frac{\beta_4}{\alpha_4}\right)\cdots\left(\frac{\beta_n}{\alpha_n}\right) \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2} & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3} & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3}\left(\frac{\beta_4}{\alpha_4}\right) & \cdots & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3}\left(\frac{\beta_4}{\alpha_4}\right)\cdots\left(\frac{\beta_n}{\alpha_n}\right) \\ & \frac{1}{\alpha_2} & \frac{\delta_3}{\alpha_3}+\frac{\beta_3}{\alpha_2\alpha_3} & \frac{\left(\frac{\delta_3}{\alpha_3}+\frac{\beta_3}{\alpha_2\alpha_3}\right)\beta_4+\gamma_4}{\alpha_4} & & \vdots \\ & & \frac{1}{\alpha_3} & \frac{\delta_4}{\alpha_4}+\frac{\beta_4}{\alpha_3\alpha_4} & \ddots & \\ & & & \frac{1}{\alpha_4} & \ddots & \\ & & & \ddots & \ddots & \frac{\left(\frac{\delta_{n-1}}{\alpha_{n-1}}+\frac{\beta_{n-1}}{\alpha_{n-2}\alpha_{n-1}}\right)\beta_n+\gamma_n}{\alpha_n} \\ & & & & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n}+\frac{\beta_n}{\alpha_{n-1}\alpha_n} \end{bmatrix} \tag{1.3}$$

via

$$r_k(x) = \alpha_1 \cdots \alpha_k \det (xI - A)_{(k \times k)} . \tag{1.4}$$

That is, the polynomials $r_k(x)$ are (scaled) characteristic polynomials of the principal submatrices of the matrix A. Clearly, the matrix $A = [a_{ij}]$ is **(i)** upper Hessenberg, i.e., $i > j + 1$ implies $a_{ij} = 0$, and further **(ii)** it is irreducible, i.e., $a_{i+1,i} \neq 0$ for $i = 1, \dots, n-1$.

Two important special cases of polynomials $r_k(x)$ as in (1.4), and corresponding counterpart matrices A are presented next.

Example 1.1 (Real-orthogonal polynomials & tridiagonal matrices). The three-term recurrence relations

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k \cdot r_{k-2}(x)$$

satisfied by real-orthogonal polynomials (see Table 2) are easily seen to be a special case of (1.2) with $\beta_k = 0$. Secondly, taking $\beta_k = 0$ for each k, the matrix A of (1.3) reduces to the tridiagonal matrix

$$A = \begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} & \ddots & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & & \ddots & \frac{\delta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} \end{bmatrix}. \tag{1.5}$$

In this special tridiagonal case the relation (1.4) is classical in the theory of real orthogonal polynomials, see [SB92], theorems 3.6.3, 3.6.20.

Example 1.2 (Szegö polynomials & unitary Hessenberg matrices). Polynomials orthogonal on the unit circle, or *Szegö polynomials*, satisfy the recurrence relations[2] (cf. with Table 2)

$$r_k(x) = \left(\frac{1}{\mu_k} x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k}\right) r_{k-1}(x) - \left(\frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x\right) r_{k-2}(x), \tag{1.6}$$

see, e.g., [G48]. These recurrence relations are also a special case of (1.2) with

$$\alpha_k = \frac{1}{\mu_k}, \quad \delta_k = -\frac{1}{\mu_k} \frac{\rho_k}{\rho_{k-1}}, \quad \beta_k = \frac{\mu_{k-1}}{\mu_k} \frac{\rho_k}{\rho_{k-1}}, \quad \gamma_k = 0. \tag{1.7}$$

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[2]Here the complex numbers $\{\rho_0 := -1, \rho_1, \dots, \rho_n\}$ such that $|\rho_k| \leqslant 1$ are referred to as *reflection coefficients*, and $\mu_k := \sqrt{1 - |\rho_k|^2}$ if $|\rho_k| < 1$ and $\mu_k := 1$ if $|\rho_k| = 1$ are called *complementary parameters*.

Also, inserting the relations (1.7) into the matrix A of (1.3) results in the well-known *unitary Hessenberg matrix* of the form

$$A = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & \cdots & -\rho_0^*\mu_1\cdots\mu_{n-1}\rho_n \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\cdots\mu_{n-1}\rho_n \\ 0 & \mu_2 & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^*\rho_n \end{bmatrix} \tag{1.8}$$

Again, the relation (1.4) connecting the Szegö polynomials (1.6) and the unitary Hessenberg matrices (1.8) is well-known, see, e.g., [O01] and the references therein.

The tridiagonal (1.5), unitary Hessenberg (1.8) and counterpart (1.3) matrices are special cases of a more general class of matrices, defined next.

1.4. Main tool: quasiseparable matrces and polynomials

Definition 1.3. (Quasiseparable matrices and polynomials)

- *A matrix A is called H-$(1,n)$-quasiseparable if* **(i)** *it is upper Hessenberg, and* **(ii)** $\max(\operatorname{rank} A_{12}) = n$ *where the maximum is taken over all symmetric partitions of the form*

$$A = \left[\begin{array}{c|c} * & A_{12} \\ \hline * & * \end{array}\right]$$

- *Let $A = [a_{ij}]$ be an irreducible (i.e., $a_{i+1,i} \neq 0$), H-$(1,1)$-quasiseparable matrix with $\alpha_i = 1/a_{i+1,i}$. Then the system of polynomials related to A via*

$$r_k(x) = \alpha_1 \cdots \alpha_k \det{(xI - A)}_{(k\times k)}\,.$$

is called a system of Hessenberg-$(1,1)$-quasiseparable polynomials, or H-$(1,1)$-q.s. polynomials.

Remark 1.4. The counterpart matrix A (1.3) is irreducible H-$(1,1)$-quasiseparable. Indeed, one can see that in any partition $A = \left[\begin{array}{c|c} * & A_{12} \\ \hline * & * \end{array}\right]$ the $(k-1)$-st column $A_{12}(:,k-1)$ and the k-th column $A_{12}(:,k)$ of the matrix A_{12} are scalar multiples of each other:

$$A_{12}(:,k) = \frac{\beta_{k+2}}{\alpha_{k+2}} A_{12}(:,k-1).$$

E.g., inspect the $2\times(n-2)$ matrix

$$A_{12} = \begin{bmatrix} \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right) & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right)\left(\frac{\beta_4}{\alpha_4}\right) & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right)\left(\frac{\beta_4}{\alpha_4}\right)\left(\frac{\beta_5}{\alpha_5}\right) & \cdots \\ \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3} & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3}\left(\frac{\beta_4}{\alpha_4}\right) & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3}\left(\frac{\beta_4}{\alpha_4}\right)\left(\frac{\beta_5}{\alpha_5}\right) & \cdots \end{bmatrix}.$$

Hence we have the following remark.

Remark 1.5. Polynomials satisfying the recurrence relations (1.2) are a special case of H-$(1,1)$-q.s. polynomials.

The two matrix examples of subsection 1.3 indicate that tridiagonal matrices and unitary Hessenberg matrices are special counterpart matrices implying that the class of H-$(1,1)$-q.s. polynomials is wide enough to include the important classes of real orthogonal polynomials and Szegö polynomials as special cases.

1.5. Main problem: Inversion of H-$(1,1)$-q.s.-Vandermonde matrices

In this paper we extend the Traub algorithm to a polynomial-Vandermonde matrix $V_R(x)$ whose defining polynomial system R is a H-$(1,1)$-q.s. system. The new algorithm generalizes the corresponding algorithms for monomials, real orthogonal polynomials, and Szegö polynomials, which are themselves special cases of H-$(1,1)$-q.s. polynomials. Also, it applies to another important special case when polynomials R satisfy the general recurrence relations (1.2). The full characterization of the class of polynomials for which the new Traub-like algorithm is applicable is given in the next subsection.

1.6. Full characterization of H-$(1,1)$-q.s. matrices and polynomials via recurrence relations

It was observed in Remark 1.5 that the class of H-$(1,1)$-q.s. polynomials includes polynomials satisfying (1.2) as a special case. At the same time, in the footnote of Section 1.3 we briefly mentioned (see Section 8 for more details) that the class of polynomials (1.2) is actually slightly narrower than the class of all H-$(1,1)$-q.s. polynomials. Hence the latter cannot be characterized via three-term recurrence relations (1.2).

As it will be shown in Section 8, the class of H-$(1,1)$-q.s. polynomials $\{r_k(x)\}$ can be fully characterized by the two-term recurrence relations of the form

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{1.9}$$

where $\{G_k(x)\}$ are some auxiliary polynomials. In this case the relation (1.4) is true for $\{r_k(x)\}$ and the following H-$(1,1)$-q.s. matrix

$$A = \begin{bmatrix} -\frac{\theta_1}{\delta_1} & -\beta_1(\frac{\gamma_2}{\delta_2}) & -\beta_1\alpha_2(\frac{\gamma_3}{\delta_3}) & -\beta_1\alpha_2\alpha_3(\frac{\gamma_4}{\delta_4}) & \cdots & -\beta_1\alpha_2\alpha_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ \frac{1}{\delta_1} & -\frac{\theta_2}{\delta_2} & -\beta_2(\frac{\gamma_3}{\delta_3}) & -\beta_2\alpha_3(\frac{\gamma_4}{\delta_4}) & \cdots & -\beta_2\alpha_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & \frac{1}{\delta_2} & -\frac{\theta_3}{\delta_3} & -\beta_3(\frac{\gamma_4}{\delta_4}) & \ddots & -\beta_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & 0 & \frac{1}{\delta_3} & -\frac{\theta_4}{\delta_4} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\beta_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & \cdots & 0 & 0 & \frac{1}{\delta_{n-1}} & -\frac{\theta_n}{\delta_n} \end{bmatrix}. \tag{1.10}$$

Two remarks are due.

Remark 1.6. First observe that a choice of coefficients including $\alpha_k = 0$ in (1.9) converts (1.10) into a tridiagonal matrix as in (1.5). Secondly, the choices

$$\alpha_k = \mu_k, \quad \beta_k = \rho_{k-1}^*\mu_k, \quad \gamma_k = \frac{\rho_k}{\mu_k}, \quad \delta_k = \frac{1}{\mu_k}, \quad \theta_k = \frac{\rho_{k-1}^*\rho_k}{\mu_k}$$

for the coefficients of (1.9) convert (1.10) into a unitary Hessenberg matrix (1.8). It is another indication that the class of polynomials $\{r_k(x)\}$ in (1.9) included the real orthogonal polynomials and the Szegö polynomials as important special cases.

Remark 1.7. The reader might think that the two-term recurrence relations (1.9) generalize the classical Szegö formula

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k}\begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix}\begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix} \tag{1.11}$$

for the Szegö polynomials $\{\phi_k^{\#}(x)\}$. Apparently, this is not the case. Indeed,

- In section 8.2 below we will obtain a true generalization

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix} \tag{1.12}$$

  of the Szegö formula (1.11).
- The difference between the general (1.9) and the Szegö-type (1.12) two-term relations is as follows. It is easy to see that in the Szegö-type relations (1.12) the polynomials have the same degree: $\deg G_k(x) = \deg r_k(x)$, while in the general relations (1.9) the degrees are different: $\deg G_k(x) = \deg r_k(x) - 1$.
- Although the latter fact might seem to be quite minor at first glance, it indicates the difference between (1.12) and (1.9) which is actually more substantial. In fact, the class of polynomials satisfying (1.12) is only a proper subset of H-$(1,1)$-q.s. polynomials, so (1.12) cannot be used to characterize the latter. It is the most general recurrence relations (1.9) that provide a full characterization of the class of H-$(1,1)$-q.s. polynomials.

1.7. Structure of the paper

In Section 2 an inversion formula valid for a general system of polynomials (although expensive in general) is presented. The formula presented there reduces the problem of inversion of $V_R(x)$ to that of evaluating the so-called associated polynomials $\widehat{R}$ corresponding to the polynomial system R. In Section 3 a relation between the polynomial systems R and $\widehat{R}$ is presented in terms of their confederate matrices. This relation suggests a procedure for evaluating the associated polynomials $\widehat{R}$. In Section 4 quasiseparable matrices are defined, and a conversion from the polynomial language (i.e. polynomials satisfying (1.2)) to the matrix language (i.e. *generators* of a quasiseparable matrix) is given. In Section 5, perturbed recurrence relations are presented for the associated polynomials $\widehat{R}$. It is these recurrence relations that allow the computational speedup that results in a fast $O(n^2)$ algorithm, presented formally in Section 7. Three different sets of recurrence relations are given, two generalizing known formulas for real orthogonal polynomials and Szegö polynomials, and a third that produces new formulas for these cases. In Section 6, a fast algorithm for computing the coefficients of the master polynomial is presented. This is required in step 2 of Algorithm 3.2. In Section 7 the algorithm is presented in full detail. The reduction in the special cases of monomials, real orthogonal polynomials, and Szegö polynomials is examined in detail as well. In Section 8, several theorems are presented that completely characterize systems of H-$(1,1)$-q.s. polynomials in terms of the Hessenberg, order $(1,1)$ quasiseparable matrices they correspond to, and vice versa. Section 9 consists of some results of numerical experiments with the proposed algorithm, and conclusions are offered in the final section. Some proofs have additionally been postponed to the appendix, which follows the conclusions.

2. Inversion formula

In this section we present a formula and overview of the suggested algorithm used to invert a polynomial-Vandermonde matrix as in (1.1). Such a matrix is completely determined by n polynomials $R = \{r_0(x), \dots, r_{n-1}(x)\}$ and n nodes $x = (x_1, \dots, x_n)$. The desired inverse $V_R(x)^{-1}$ is given by the formula

$$V_R(x)^{-1} = \tilde{I} \cdot V_{\widehat{R}}^T(x) \cdot \operatorname{diag}(c_1, \dots, c_n), \tag{2.1}$$

(see [O98], [O01]) where

$$c_i = \prod_{\substack{k=1 \\ k\neq i}}^{n} (x_k - x_i)^{-1}, \tag{2.2}$$

$\tilde{I}$ is the antidiagonal matrix

$$\tilde{I} = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \iddots & 1 & 0 \\ 0 & \iddots & \iddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}, \tag{2.3}$$

and $\widehat{R}$ is the system of *associated (generalized Horner) polynomials*, defined as follows: if we define the *master polynomial* $P(x)$ by $P(x) = (x-x_1)\cdots(x-x_n)$, then for the polynomial system $R = \{r_0(x), \dots, r_{n-1}(x), P(x)\}$, the associated polynomials $\widehat{R} = \{\widehat{r}_0(x), \dots, \widehat{r}_{n-1}(x), P(x)\}$ are those satisfying the relations

$$\frac{P(x) - P(y)}{x - y} = \sum_{k=0}^{n-1} r_k(x) \cdot \widehat{r}_{n-k-1}(y), \tag{2.4}$$

see [KO97]. A discussion showing the existence of polynomials satisfying these relations (2.4) for any polynomial system R is given in [BEGKO07]. This definition can be seen as a generalization of the Horner polynomials associated with the monomials, cf. with the discussion in section 3.3 below.

This discussion gives a relation between the inverse $V_R(x)^{-1}$ and the polynomial-Vandermonde matrix $V_{\widehat{R}}(x)$, where $\widehat{R}$ is the system of polynomials associated with R. To use this in order to invert $V_R(x)$, one needs to evaluate the polynomials $\widehat{R}$ at the nodes x to form $V_{\widehat{R}}^T(x)$, and a procedure for doing this is suggested by their confederate matrices defined next.

3. Confederate matrix interpretation of associated polynomials $\widehat{R}$ involved in the inversion formula (2.1)

3.1. Definition of the confederate matrix

We next give the definition of the confederate matrix of a polynomial with respect to a given system of polynomials. Let polynomials $R = \{r_0(x),\ r_1(x)\ , \dots,\ r_n(x)\}$ with $\deg(r_k) = k$ be specified by the general recurrence relations

$$r_k(x) = (\alpha_k x - a_{k-1,k}) \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \ldots - a_{0,k} \cdot r_0(x) \tag{3.1}$$

for $k = 1, \ldots, n$. Following [MB79], define for the polynomial

$$P(x) = P_0 \cdot r_0(x) + P_1 \cdot r_1(x) + \ldots + P_{n-1} \cdot r_{n-1}(x) + P_n \cdot r_n(x) \tag{3.2}$$

its *confederate* matrix

$$C_R(P) = \underbrace{\begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \frac{a_{0,k}}{\alpha_k} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \frac{a_{1,k}}{\alpha_k} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \vdots & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} \\ 0 & 0 & \frac{1}{\alpha_3} & \ddots & \frac{a_{k-2,k}}{\alpha_k} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{a_{k-1,k}}{\alpha_k} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{\alpha_k} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} \end{bmatrix}}_{C_R(r_n)} - \begin{bmatrix} P_0 \\ P_1 \\ P_2 \\ \vdots \\ \vdots \\ P_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & \frac{1}{\alpha_n P_n} \end{bmatrix} \tag{3.3}$$

with respect to the polynomial system R. Notice that the coefficients of the recurrence relations for the k^{th} polynomial $r_k(x)$ from (3.1) are contained in the k^{th} column of $C_R(r_n)$, as the highlighted column shows. Notice also that in the special case $P(x) = r_n(x)$, the second matrix in (3.3) vanishes. We refer to [MB79] for many useful properties of the confederate matrix and only recall here that $\det(xI - C_R(P)) = P(x)/(\alpha_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_n)$, and that similarly, the characteristic polynomial of the $k \times k$ leading submatrix of $C_R(P)$ is equal to $r_k(x)/\alpha_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_k$.

Example 3.1. It will be convenient, for future reference, to collect in the next table several special cases of conferederate matrices.

Recurrence Relations of R	Confederate matrix $C_R(r_n)$
$r_k(x) = x r_{k-1}(x)$	$\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$
Monomials	Companion matrix

TABLE 3. Polynomial systems and corresponding confederate matrices.

Recurrence Relations of R	Confederate matrix $C_R(r_n)$
$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k r_{k-2}(x)$ Real orthogonal polynomials	$\begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} & \ddots & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & & \ddots & \frac{\delta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} \end{bmatrix}$ Tridiagonal matrix
$r_k(x) = [\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k}] r_{k-1}(x)$ $- \frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x r_{k-2}(x)$ Szegö polynomials	$\begin{bmatrix} -\rho_0^* \rho_1 & -\rho_0^* \mu_1 \rho_2 & -\rho_0^* \mu_1 \mu_2 \rho_3 & \cdots & -\rho_0^* \mu_1 \cdots \mu_{n-1} \rho_n \\ \mu_1 & -\rho_1^* \rho_2 & -\rho_1^* \mu_2 \rho_3 & \cdots & -\rho_1^* \mu_2 \cdots \mu_{n-1} \rho_n \\ 0 & \mu_2 & -\rho_2^* \rho_3 & \cdots & -\rho_2^* \mu_3 \cdots \mu_{n-1} \rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^* \rho_n \end{bmatrix}$ Unitary Hessenberg matrix
$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x)$ $-(\beta_k x + \gamma_k) r_{k-2}(x)$ polynomials satisfying (1.2)	$\begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2} & \cdots & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2} \left(\frac{\beta_3}{\alpha_3}\right) \left(\frac{\beta_4}{\alpha_4}\right) \cdots \left(\frac{\beta_n}{\alpha_n}\right) \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} + \frac{\beta_2}{\alpha_1 \alpha_2} & \cdots & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3} \left(\frac{\beta_4}{\alpha_4}\right) \cdots \left(\frac{\beta_n}{\alpha_n}\right) \\ & & \ddots & \\ & & \ddots & \frac{\left(\frac{\delta_{n-1}}{\alpha_{n-1}}+\frac{\beta_{n-1}}{\alpha_{n-2}\alpha_{n-1}}\right)\beta_n+\gamma_n}{\alpha_n} \\ & & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} + \frac{\beta_n}{\alpha_{n-1}\alpha_n} \end{bmatrix}$ counterpart matrix

TABLE 3. Polynomial systems and corresponding confederate matrices.

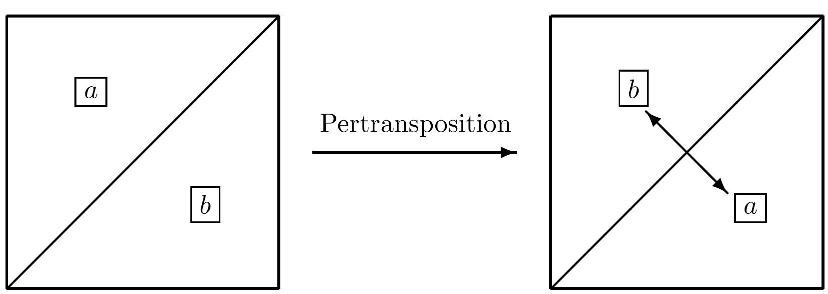
3.2. The relationship between $C_R(P)$ (defining $V_R(x)$) & $C_{\widehat{R}}(P)$ (defining $V_{\widehat{R}}(x)$ used in the inversion formula)

The motivation for considering confederate matrices is that they will allow the computation of the polynomials associated with the given system of polynomials. The confederate matrices of R and $\widehat{R}$ are related by

$$C_{\widehat{R}}(P) = \tilde{I} \cdot C_R(P)^T \cdot \tilde{I}. \tag{3.4}$$

(see [O98], [O01]). The passage from $C_R(P)$ to $C_{\widehat{R}}(P)$ in (3.4) can be seen as a transposition across the antidiagonal, or a pertransposition, or more visually,

$C_R(P) =$



$= C_{\widehat{R}}(P)$

As an example, consider for a moment the 5×5 Hessenberg confederate matrix

$$C_R(P) = \begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \frac{a_{04}}{\alpha_4} & \frac{a_{05}}{\alpha_5} - \frac{P_0}{P_5 \alpha_5} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \frac{a_{14}}{\alpha_4} & \frac{a_{15}}{\alpha_5} - \frac{P_1}{P_5 \alpha_5} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \frac{a_{24}}{\alpha_4} & \frac{a_{25}}{\alpha_5} - \frac{P_2}{P_5 \alpha_5} \\ 0 & 0 & \frac{1}{\alpha_3} & \frac{a_{34}}{\alpha_4} & \frac{a_{35}}{\alpha_5} - \frac{P_3}{P_5 \alpha_5} \\ 0 & 0 & 0 & \frac{1}{\alpha_4} & \frac{a_{45}}{\alpha_5} - \frac{P_4}{P_5 \alpha_5} \end{bmatrix} \tag{3.5}$$

corresponding to a system R of polynomials satisfying (3.1). From (3.4) above, the system of polynomials associated with R, denoted by $\widehat{R} = \{\widehat{r}_0(x), \widehat{r}_1(x), \dots, \widehat{r}_5(x)\}$, has confederate matrix

$$C_{\widehat{R}}(P) = \begin{bmatrix} \frac{a_{45}}{\alpha_5} - \frac{P_4}{P_5\alpha_5} & \frac{a_{35}}{\alpha_5} - \frac{P_3}{P_5\alpha_5} & \frac{a_{25}}{\alpha_5} - \frac{P_2}{P_5\alpha_5} & \frac{a_{15}}{\alpha_5} - \frac{P_1}{P_5\alpha_5} & \frac{a_{05}}{\alpha_5} - \frac{P_0}{P_5\alpha_5} \\ \frac{1}{\alpha_4} & \frac{a_{34}}{\alpha_4} & \frac{a_{24}}{\alpha_4} & \frac{a_{14}}{\alpha_4} & \frac{a_{04}}{\alpha_4} \\ 0 & \frac{1}{\alpha_3} & \frac{a_{23}}{\alpha_3} & \frac{a_{13}}{\alpha_3} & \frac{a_{03}}{\alpha_3} \\ 0 & 0 & \frac{1}{\alpha_2} & \frac{a_{12}}{\alpha_2} & \frac{a_{02}}{\alpha_2} \\ 0 & 0 & 0 & \frac{1}{\alpha_1} & \frac{a_{01}}{\alpha_1} \end{bmatrix}. \tag{3.6}$$

In accordance with (2.1), the main computational burden is to compute $V_{\widehat{R}}$, i.e. to evaluate n polynomials $\{\hat{r}_k(x)\}_0^{n-1}$ at n points $\{x_k\}_1^n$. Using (3.6) to accomplish this is expensive in the general case, since it leads to the full n-term recurrence relations, for instance the highlighted column in (3.6) implies that the recurrence relations for $\widehat{r}_4(x)$ are given by

$$\widehat{r}_4(x) = \alpha_1 \cdot x \cdot \widehat{r}_3(x) - \frac{\alpha_1}{\alpha_2} a_{1,2} \cdot \widehat{r}_3(x) - \frac{\alpha_1}{\alpha_3} a_{1,3} \cdot \widehat{r}_2(x) - \frac{\alpha_1}{\alpha_4} a_{1,4} \cdot \widehat{r}_1(x) - \frac{\alpha_1}{\alpha_5}(a_{1,5} + \frac{P_1}{P_5}) \cdot \widehat{r}_0(x) \tag{3.7}$$

Additionally, the coefficients P_i of the master polynomial $P(x)$ decomposed into the R basis must be computed; that is, $\{P_0, \dots, P_n\}$ such that

$$\prod_{k=1}^{n}(x - x_k) = P_0 r_0(x) + P_1 r_1(x) + \cdots + P_{n-1} r_{n-1}(x) + P_n r_n(x). \tag{3.8}$$

as they are present in the recurrence relations defining $\widehat{r}_k(x)$.

All of this allows us to present a sketch of the Traub-like inversion algorithm next. The detailed algorithm will be provided in Section 7 below after deriving next several formulas that will be required to implement its steps 2 and 3.

Algorithm 3.2. *[A sketch of the Traub-like inversion algorithm]*

1. *Compute the entries of diag*$(c_1, \dots, c_n)$ *via* (2.2)*.*
2. *Compute the coefficients* $\{P_0, P_1, \dots, P_{n-1}\}$ *of the master polynomial* $P(x)$ *as in* (3.8)*.*
3. *Evaluate the* n *polynomials of* $\widehat{R}$ *with confederate matrix specified via* (3.4) *at the* n *nodes* x_k *to form* $V_{\widehat{R}}(x)$*.*
4. *Compute the inverse* $V_R(x)^{-1}$ *via* (2.1)*.*

We next present the simplest special case, when the polynomial system R is a system of monomials, and how the algorithm resulting is the classical Traub algorithm with complexity $O(n^2)$.

3.3. Example: monomial bases & the Horner recursion, and the classical Traub algorithm

Monomials $R = \{1, x, x^2, \dots, x^{n-1}\}$ satisfy the obvious recurrence relations $x^k = x \cdot x^{k-1}$ and hence the confederate matrix (3.3) becomes

$$C_R(P) = \begin{bmatrix} 0 & 0 & \cdots & 0 & -P_0 \\ 1 & 0 & \cdots & 0 & -P_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -P_{n-1} \end{bmatrix} \tag{3.9}$$

which is the well-known companion matrix. Using the pertransposition rule (3.4) of section 3.2, we obtain the confederate matrix

$$C_{\widehat{R}}(\widehat{r}_n) = \begin{bmatrix} -P_{n-1} & -P_{n-2} & \cdots & -P_1 & -P_0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & 0 \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}. \tag{3.10}$$

for the associated polynomials $\widehat{r}_k(x)$. Using the formula (3.1), we read from the matrix (3.10) the familiar Horner recurrence relations

$$\widehat{r}_0(x) = 1, \quad \widehat{r}_k(x) = x\widehat{r}_{k-1}(x) + P_{n-k}. \tag{3.11}$$

To sum up, in the case of the monomials our approach yields the classical $O(n^2)$ Traub algorithm that will be generalized in the rest of the paper. The generalization will be using the concept of a quasiseparable matrix defined next.

4. Main tool: Quasiseparable matrices and polynomials

4.1. Quasiseparable matrices and polynomials.

4.1.1. Quasiseparable matrices. We begin with the *rank definition* of quasiseparability; an equivalent definition will be given afterwards.

Definition 4.1 (Rank definition for H-$(1,1)$-quasiseparable matrices). *A matrix A is called H-$(1,n)$-quasiseparable if* **(i)** *it is Hessenberg, and* **(ii)** $\max(\operatorname{rank} A_{12}) = n$ *and* $\max(\operatorname{rank} A_{21}) = 1$, *where the maximum is taken over all symmetric partitions of the form*

$$A = \left[\begin{array}{c|c} * & A_{12} \\ \hline A_{21} & * \end{array}\right] \tag{4.1}$$

Example 4.2. Each of the confederate matrices in Table 3 is H-$(1,n)$-quasiseparable.

Tridiaginal matrix: Indeed, if A is tridiagonal as in (1.5), then the submatrix A_{12} has the form

$$A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{\delta_k}{\alpha_k} & 0 & \cdots & 0 \end{bmatrix}$$

which can easily be observed to have rank one.

Unitary Hessenberg matrix: Also, if A is a unitary Hessenberg matrix as in (1.8), then for instance the corresponding submatrix A_{12} of size $3 \times (n-4)$

$$A_{12} = \begin{bmatrix} -\rho_4\mu_3\mu_2\mu_1\rho_0^* & -\rho_5\mu_4\mu_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\mu_2\mu_1\rho_0^* \\ -\rho_4\mu_3\mu_2\rho_1^* & -\rho_5\mu_4\mu_3\mu_2\rho_1^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\mu_2\rho_1^* \\ -\rho_4\mu_3\rho_2^* & -\rho_5\mu_4\mu_3\rho_2^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \end{bmatrix},$$

clearly has rank one. Thus, since tridiagonal matrices and unitary Hessenberg matrices have this special structure, they are quasiseparable of order one.

Counterpart matrix: The counterpart matrix A of (1.3) is irreducible H-$(1,1)$-quasiseparable. Indeed, one can see that in any partition $A = \left[\begin{array}{c|c} * & A_{12} \\ \hline * & * \end{array}\right]$ the $(k-1)$-st column $A_{12}(:,k)$ and the k-th column $A_{12}(:,k)$ of the matrix A_{12} are scalar multiples of each other:

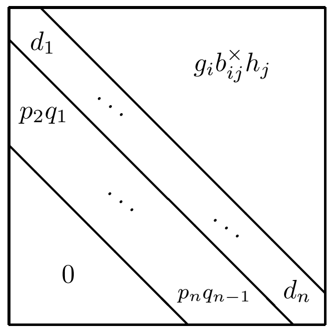
$$A_{12}(:,k) = \frac{\beta_{k+2}}{\alpha_{k+2}} A_{12}(:,k-1).$$

E.g., inspect r the $2 \times (n-2)$ matrix

$$A_{12} = \begin{bmatrix} \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right) & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right)\left(\frac{\beta_4}{\alpha_4}\right) & \frac{\frac{\delta_1}{\alpha_1}\beta_2+\gamma_2}{\alpha_2}\left(\frac{\beta_3}{\alpha_3}\right)\left(\frac{\beta_4}{\alpha_4}\right)\left(\frac{\beta_5}{\alpha_5}\right) & \cdots \\ \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3} & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3}\left(\frac{\beta_4}{\alpha_4}\right) & \frac{\left(\frac{\delta_2}{\alpha_2}+\frac{\beta_2}{\alpha_1\alpha_2}\right)\beta_3+\gamma_3}{\alpha_3}\left(\frac{\beta_4}{\alpha_4}\right)\left(\frac{\beta_5}{\alpha_5}\right) & \cdots \end{bmatrix}.$$

We present next an alternate definition of quasiseparable matrices that is equivalent to Definition 4.1.

Definition 4.3 (Generator definition for H-$(1,n)$-quasiseparable matrices). *A matrix A is called H-$(1,n)$-quasiseparable if* **(i)** *it is Hessenberg, and* **(ii)** *it can be represented in the form*

$A =$ 

where $b_{ij}^{\times} = b_{i+1} \cdots b_{j-1}$ for $j > i+1$ and $b_{ij}^{\times} = 1$ for $j = i+1$. The elements

$$\{p_k, q_k, d_k, g_k, b_k, h_k\},$$

called the generators of the matrix A, are matrices of sizes

	p_k	q_k	d_k	g_k	b_k	h_k
sizes	1×1	1×1	1×1	$1 \times r''_k$	$r''_{k-1} \times r''_k$	$r''_{k-1} \times 1$
range	$k \in [2,n]$	$k \in [1,n-1]$	$k \in [1,n]$	$k \in [1,n-1]$	$k \in [2,n-1]$	$k \in [2,n]$

subject to $\max_k r''_k = n$.

4.1.2. Quasiseparable polynomials.

Definition 4.4 (H-$(1,1)$-Quasiseparable polynomials). *Let $A = [a_{ij}]$ be an irreducible (i.e., $a_{i+1,i} \neq 0$), H-$(1,1)$-quasiseparable matrix with $\alpha_i = 1/a_{i+1,i}$. Then the system of polynomials related to A via*

$$r_k(x) = \alpha_1 \cdots \alpha_k \det (xI - A)_{(k \times k)} .$$

is called a system of Hessenberg-$(1,1)$-quasiseparable polynomials, or H-$(1,1)$-q.s. polynomials.

Briefly, H-$(1,1)$-q.s. polynomial systems are systems of polynomials whose confederate matrices are H-$(1,1)$-q.s.

Example 4.5. Example 4.2 means that the class of H-$(1,1)$-q.s. polynomials includes as special cases the important classical polynomial classes of **(i)** real orthogonal polynomials; **(ii)** Szegö polynomials; as well as **(iii)** a fairly general class of polynomials satisfying (1.2).

Remark 4.6. It is useful to note that Definition 4.3 and formula (3.1) imply that H-$(1,1)$-quasiseparable polynomials satisfy n-term recurrence relations

$$r_k(x) = \frac{1}{p_{k+1} q_k} \left[(x - d_k) r_{k-1}(x) - \sum_{j=0}^{k-2} \left(g_{j+1} b_{j+1,k}^{\times} h_k r_j(x) \right) \right] \tag{4.2}$$

that use only $O(n)$ parameters, i.e., generators. The formula (4.2) is not sparse and hence expensive. We postpone presenting the more computationally efficient formulas for H-$(1,1)$-quasiseparable polynomials until section 8 since they are not needed for the derivation of the Traub-like algorithm.

4.1.3. Conversion of recurrence relation coefficients into quasiseparable generators. In the rest of the paper we will be deriving the algorithm for inversion of H-$(1,1)$-q.s.-Vandermonde matrices using generators. However, in many examples, see, e.g., in Table 3, the polynomials are often defined not by the generators but via recurrence relations (1.2). The next theorem shows how to convert the latter into the former. The formal proof is postponed to the appendix (see page 28).

Theorem 4.7 (Recurrence relation coefficients ⇒ quasiseparable generators). *Let $R = \{r_0(x), \ldots, r_n(x)\}$ be a system of polynomials s.t.* $\deg(r_k) = k$ *and denote by $C_R(r_n)$ the confederate matrix of $r_n(x)$ with respect to R. Suppose R satisfies the recurrence relations (*1.2*). Then $C_R(r_n)$ is an irreducible Hessenberg-*$(1,1)$*-quasiseparable matrix with generators*

$$d_1 = \frac{\delta_1}{\alpha_1}, \quad d_k = \frac{\delta_k}{\alpha_k} + \frac{\beta_k}{\alpha_{k-1}\alpha_k}, \quad k = 2, \ldots, n \tag{4.3}$$

$$p_{k+1}q_k = \frac{1}{\alpha_k}, \quad g_k = \frac{d_k\beta_{k+1} + \gamma_{k+1}}{\alpha_{k+1}}, \quad k = 1, \ldots, n-1 \tag{4.4}$$

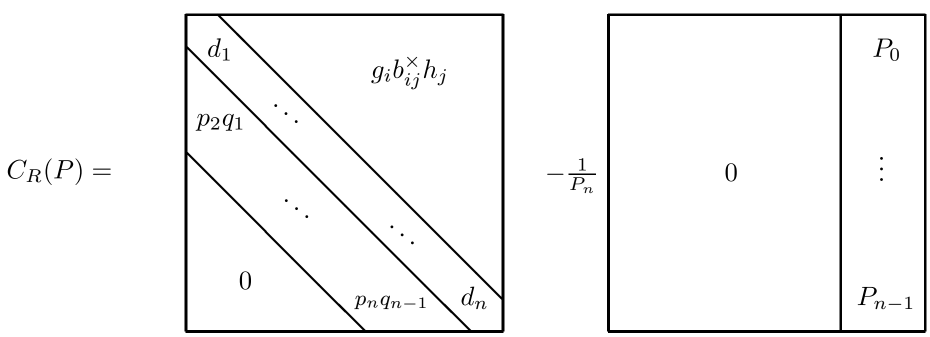
$$b_k = \frac{\beta_{k+1}}{\alpha_{k+1}}, \quad k = 2, \ldots, n-1 \qquad h_k = 1, \quad k = 2, \ldots, n \tag{4.5}$$

Remark 4.8. As one can see, in this case the matrix $C_R(r_n)$ has the form displayed in (1.3).

We are ready to start the derivation of the main algorithm.

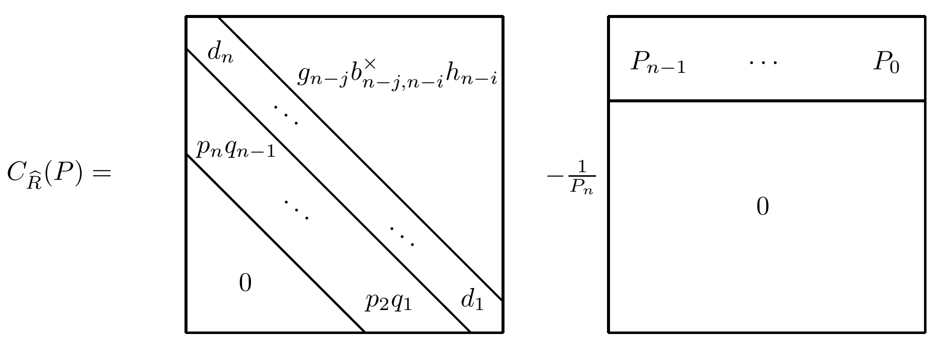
4.2. Inversion problem: a quasiseparable conferderate matrix interpretation

4.2.1. Computing the associated polynomials. In this section we specify the general procedure of Section 3.2 to quasiseparable confederate matrices. Given a polynomial system satisfying recurrence relations (1.2), the coefficients $\{\alpha_k, \delta_k, \beta_k, \gamma_k\}$ can be used in Theorem 4.7 to generate an irreducible Hessenberg $(1,1)$-quasiseparable matrix of the form shown in Definition 4.3. Next, considering the confederate matrix with respect to the master polynomial $P(x) = \prod(x - x_k)$ defined by the nodes $x_k, k = 1, \ldots, n$, we have



(4.6)

Applying (3.4) gives us the confederate matrix for the associated polynomials as



(4.7)

¿From this last equation we can see that the n-term recurrence relations satisfied by the associated polynomials $\widehat{R}$ are given by

$$\widehat{r}_k(x) = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left[(x - \widehat{d}_k)\widehat{r}_{k-1}(x) - \sum_{j=0}^{k-2}\left(\widehat{g}_{j+1}\widehat{b}^{\times}_{j+1,k}\widehat{h}_k\widehat{r}_j(x)\right) - \frac{P_{n-k}}{P_n}\widehat{r}_0(x)\right] \tag{4.8}$$

where, in order to simplify the formulas, we introduce the notation

$$\widehat{p}_k = q_{n-k+1}, \quad \widehat{q}_k = p_{n-k+1}, \quad \widehat{d}_k = d_{n-k+1}, \quad \widehat{g}_k = h_{n-k+1}, \quad \widehat{b}_k = b_{n-k+1}, \quad \widehat{h}_k = g_{n-k+1}. \tag{4.9}$$

Having found explicit n-term recurrence relations for the system of polynomials associated with the given system satisfying (1.2), the next goal is to find *sparse* recurrence relations. The motivation is that the n-term recurrence relations are slow; they lead to $O(n^3)$ algorithms, while two- and three-term recurrence relations lead to $O(n^2)$ algorithms.

5. Perturbed recurrence relations (to be used in step 3 of Algorithm 3.2)

In this section we consider the case where R is a system of H-$(1,1)$-q.s. polynomials, and we derive sparse recurrence relations for the associated system of polynomials. For quasiseparable-$(1,1)$-polynomials themselves, such recurrence relations are derived in [EGO05]. Obtaining these formulas for the leading minors of $C_{\widehat{R}}(P)$ of the form shown in (4.7) is not immediate, as the second term now affects each column, and the result is that the leading submatrices are now of order $(1,2)$, as opposed to those of the form shown in (4.6), which are all of order $(1,1)$.

We begin with recurrence relations generalizing classical two- and three-term recurrence relations; that is, the two-term recurrence relations satisfied by the Szegö polynomials, and the three-term recurrence relations satisfied by both the Szegö polynomials and real orthogonal polynomials. Special conditions on the generators must be satisfied for these two results to be applicable in more general cases.

Theorem 5.1 (Three-term recurrence relations). *Let $R = \{r_0(x), \ldots, r_{n-1}(x), P(x)\}$ be a system of H-$(1,1)$-q.s. polynomials corresponding to an irreducible Hessenberg-$(1,1)$-quasiseparable matrix of size $n \times n$ with generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ as in Definition 4.3, with the convention that $g_n = 1, b_n = 0$. Suppose further that $g_k \neq 0$ for $k = 1, \ldots, n-1$. Then the system of polynomials $\widehat{R}$ associated with R satisfies the recurrence relations below.*

Limitation: $g_k \neq 0, k = 1, \ldots, n-1$

$$\widehat{r}_0(x) = P_n, \qquad \widehat{r}_1(x) = \frac{1}{\widehat{p}_2\widehat{q}_1}\left(x - \widehat{d}_1\right)\widehat{r}_0(x) + \frac{1}{\widehat{p}_2\widehat{q}_1}P_{n-1}$$

$$\widehat{r}_k(x) = (\widehat{\alpha}_k x - \widehat{\delta}_k)\cdot\widehat{r}_{k-1}(x) - (\widehat{\beta}_k x + \widehat{\gamma}_k)\cdot\widehat{r}_{k-2}(x) + \widehat{\alpha}_k P_{n-k} - \widehat{\beta}_k P_{n-k+1}, \quad k = 2, \ldots, n-1 \tag{5.1}$$

where

$$\widehat{\alpha}_k = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}, \quad \widehat{\delta}_k = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left(\widehat{d}_k - \frac{\widehat{p}_k\widehat{q}_{k-1}\widehat{h}_k\widehat{b}_{k-1}}{\widehat{h}_{k-1}}\right) \tag{5.2}$$

$$\widehat{\beta}_k = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\frac{\widehat{h}_k\widehat{b}_{k-1}}{\widehat{h}_{k-1}}, \quad \widehat{\gamma}_k = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\frac{\widehat{h}_k}{\widehat{h}_{k-1}}\left(\widehat{h}_{k-1}\widehat{g}_{k-1} - \widehat{d}_{k-1}\widehat{b}_{k-1}\right), \tag{5.3}$$

and the coefficients $P_k, k = 0, \ldots, n$ are as defined in (3.8).

The proof is given in the appendix, see page 29. The following notations will be used to present the two-term recurrence relations.

$$u_k(x) = (x - \widehat{d}_k) + \frac{\widehat{g}_k\widehat{h}_k}{\widehat{b}_k}, \qquad v_k = \widehat{p}_{k+1}\widehat{b}_{k+1}\widehat{q}_k - \frac{\widehat{g}_{k+1}\widehat{h}_k}{\widehat{b}_k}. \tag{5.4}$$

Theorem 5.2 (Szegö-type recurrence relations). *Let $R = \{r_0(x), \ldots, r_{n-1}(x), P(x)\}$ be a system of H-$(1,1)$-q.s. polynomials corresponding to an irreducible Hessenberg-$(1,1)$-quasiseparable matrix of size $n \times n$ with generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ as in Definition 4.3, with the convention that $g_n = 0, b_n = 1$. Suppose further that $b_k \neq 0$ for $k = 2, \ldots, n-1$. Then the system of polynomials $\widehat{R}$ associated with R satisfy the recurrence relations below.*

Limitation: $b_k \neq 0, k = 2, \ldots, n-1$

$$\begin{bmatrix} G_0(x) \\ \widehat{r}_0(x) \end{bmatrix} = \begin{bmatrix} -\widehat{g}_1 P_n \\ P_n \end{bmatrix}, \tag{5.5}$$

$$\begin{bmatrix} G_k(x) \\ \widehat{r}_k(x) \end{bmatrix} = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k} \begin{bmatrix} v_k & -\widehat{g}_{k+1} \\ \widehat{h}_k/\widehat{b}_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ u_k(x)\widehat{r}_{k-1}(x) + P_{n-k} \end{bmatrix}, \quad k = 1, \ldots, n-1 \tag{5.6}$$

with auxiliary polynomials $G_k(x)$, and the coefficients $P_k, k = 0, \ldots, n$ are as defined in (3.8).

See the appendix, page 29, for the proof.

The formulas of the previous two theorems generalize the classical formulas for monomials, real-orthogonal polynomials, and Szegö polynomials (demonstrated below). We emphasize at this point that these formulas have limitations in the general case: Theorem 5.1 requires nonzero g_k for each k, and Theorem 5.2 requires nonzero b_k for each k. The next theorem is more general, and does not have any such limitations.

Theorem 5.3 ([EGO05]-type recurrence relations). *Let $R = \{r_0(x), \ldots, r_{n-1}(x), P(x)\}$ be a system of H-$(1,1)$-q.s. polynomials corresponding to an irreducible Hessenberg-$(1,1)$-quasiseparable matrix of size $n \times n$ with generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ as in Definition 4.3, with the convention that $q_n = 0, b_n = 0$. Then the system of polynomials $\widehat{R}$ associated with R satisfy the recurrence relations below.*

Limitation: none.

$$\begin{bmatrix} F_0(x) \\ \widehat{r}_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ P_n \end{bmatrix}, \tag{5.7}$$

$$\begin{bmatrix} \widehat{F}_k(x) \\ \widehat{r}_k(x) \end{bmatrix} = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k} \begin{bmatrix} \widehat{q}_k\widehat{p}_k\widehat{b}_k & -\widehat{q}_k\widehat{g}_k \\ \widehat{p}_k\widehat{h}_k & x - \widehat{d}_k \end{bmatrix} \begin{bmatrix} \widehat{F}_{k-1}(x) \\ \widehat{r}_{k-1}(x) \end{bmatrix} + \frac{1}{\widehat{p}_{k+1}\widehat{q}_k} \begin{bmatrix} 0 \\ P_{n-k} \end{bmatrix} \tag{5.8}$$

with auxiliary polynomials $\widehat{F}_k(x)$, and the coefficients $P_k, k = 0, \ldots, n$ are as defined in (3.8).

The proof is given in the appendix, page 30.

The recursions of this last theorem are the most applicable and will be used below to formulate a characterization of the class of H-$(1,1)$-q.s. polynomials.

6. Computing the coefficients of the master polynomial (to be used in step 2 of Algorithm 3.2)

Note that in order to use the recurrence relations of the previous section it is necessary to decompose the master polynomial $P(x)$ into the R basis; that is, the coefficients P_k as in (3.8) must be computed. To this end, an efficient method of calculating these coefficients follows.

It is easily seen that the last polynomial $r_n(x)$ in the system R does not affect the resulting confederate matrix $C_R(P)$. Thus, if $\bar{R} = \{r_0(x), \ldots, r_{n-1}(x), xr_{n-1}(x)\}$, we have $C_R(P) = C_{\bar{R}}(P)$. Decomposing the polynomial $P(x)$ into the $\bar{R}$ basis can be done recursively by setting $r_n^{(0)}(x) = 1$ and then for $k = 0, \ldots, n-1$ updating $r_n^{(k+1)}(x) = (x - x_{k+1}) \cdot r_n^{(k)}(x)$.

Lemma 6.1. *Let $R = \{r_0(x), \ldots, r_n(x)\}$ be given by (3.1), and $f(x) = \sum_{i=1}^{k} a_i \cdot r_i(x)$, where $k < n-1$. Then the coefficients of $x \cdot f(x) = \sum_{i=1}^{k+1} b_i \cdot r_i(x)$ can be computed by*

$$\begin{bmatrix} b_0 \\ \vdots \\ b_k \\ b_{k+1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{C_R(r_n)} & 0 \\ \hline 0 & \cdots & 0 & \frac{1}{\alpha_n} & 0 \end{array} \right] \begin{bmatrix} a_0 \\ \vdots \\ a_k \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{6.1}$$

Proof. It can be easily checked that

$$x \cdot \begin{bmatrix} r_0(x) & r_1(x) & \cdots & r_n(x) \end{bmatrix} - \begin{bmatrix} r_0(x) & r_1(x) & \cdots & r_n(x) \end{bmatrix} \cdot \left[\begin{array}{cccc|c} \multicolumn{4}{c|}{C_R(r_n)} & 0 \\ \hline 0 & \cdots & 0 & \frac{1}{\alpha_n} & 0 \end{array}\right]$$

$$= \begin{bmatrix} 0 & \cdots & 0 & x \cdot r_n(x) \end{bmatrix}.$$

Multiplying the latter equation by the column of the coefficients we obtain (6.1). □

This lemma suggests the following algorithm for computing coefficients $\{P_0, P_1, \ldots, P_{n-1}, P_n\}$ in (3.8) of the master polynomial.

Algorithm 6.2. *[Coefficients of the master polynomial in the R basis]*
Cost: *$O(n \times m(n))$, where $m(n)$ is the cost of multiplication of an $n \times n$ quasiseparable matrix by a vector.*
Input: *A quasiseparable confederate matrix $C_R(r_n)$ and n nodes $x = (x_1, x_2, \ldots, x_n)$.*

1. *Set* $\begin{bmatrix} P_0^{(0)} & \cdots & P_{n-1}^{(0)} & P_n^{(0)} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$
2. *For* $k = 1 : n$,

$$\begin{bmatrix} P_0^{(k)} \\ \vdots \\ P_{n-1}^{(k)} \\ P_n^{(k)} \end{bmatrix} = \left(\left[\begin{array}{ccc|c} \multicolumn{3}{c|}{C_{\bar{R}}(x \cdot r_{n-1}(x))} & 0 \\ \hline 0 & \cdots \; 0 & 1 & 0 \end{array}\right] - x_k \cdot I \right) \cdot \begin{bmatrix} P_0^{(k-1)} \\ \vdots \\ P_{n-1}^{(k-1)} \\ P_n^{(k-1)} \end{bmatrix}$$

   where $\bar{R} = \{r_0(x), \ldots, r_{n-1}(x), x r_{n-1}(x)\}$.
3. *Take* $\begin{bmatrix} P_0 & \cdots & P_{n-1} & P_n \end{bmatrix} = \begin{bmatrix} P_0^{(n)} & \cdots & P_{n-1}^{(n)} & P_n^{(n)} \end{bmatrix}$

Output: *Coefficients $\{P_0, P_1, \ldots, P_{n-1}, P_n\}$ such that (3.8) is satisfied.*

It is clear that the computational burden in implementing this algorithm is in multiplication of the matrix $C_{\bar{R}}(r_n)$ by the vector of coefficients. The cost of each such step is $O(m(n))$, where $m(n)$ is the cost of multiplication of an $n \times n$ quasiseparable matrix by a vector, thus the cost of computing the n coefficients is $O(n \times m(n))$. Using a fast $O(n)$ algorithm for multiplication of a quasiseparable matrix by a vector from [EG992], the cost of this algorithm is $O(n^2)$.

7. The overall Traub-like algorithm

7.1. General case. H-$(1,1)$-q.s. polynomials and general Traub-like algorithms.

In this section we specify in detail the process of computing the inverse of a Hessenberg-$(1,1)$-quasiseparable-Vandermonde matrix via the Traub-like algorithm.

The algorithm takes as input the generators of the Hessenberg, $(1,1)$-quasiseparable confederate matrix corresponding to the system of polynomials R. In the case where the recurrence relations of the form (1.2) are known, the following algorithm can be used to compute these generators.

Algorithm 7.1. *[Preprocessing step]*
Input: *The set $\{\alpha_k, \delta_k, \beta_k, \gamma_k\}$ such that a polynomial system R satisfies (1.2).*

1. *Compute the generators d_k via*

$$d_1 = \frac{\delta_1}{\alpha_1}, \quad d_k = \frac{\delta_k}{\alpha_k} + \frac{\beta_k}{\alpha_{k-1}\alpha_k}, \quad k = 2, \ldots, n$$

2. *Compute the generators p_k, q_k, g_k, b_k, h_k via*

$$p_{k+1} q_k = \frac{1}{\alpha_k}, \quad g_k = \frac{d_k \beta_{k+1} + \gamma_{k+1}}{\alpha_{k+1}}, \quad k = 1, \ldots, n-1$$

$$b_k = \frac{\beta_{k+1}}{\alpha_{k+1}}, \quad k = 2, \ldots, n-1 \qquad h_k = 1, \quad k = 2, \ldots, n$$

Output: *Generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of a quasiseparable confederate matrix corresponding to a system of polynomials R*

The next algorithm applies to any set of generators, and thus those arising from the recurrence relations (1.2) via the previous algorithm form a subset of all possible inputs. In this algorithm we will make use of MATLAB notations; for instance $V_{\widehat{R}}(i:j,k:l)$ will refer to the block of $V_{\widehat{R}}(x)$ consisting of rows i through j and columns k through l.

Algorithm 7.2. *[Traub-like inversion algorithm]*
Input: *Generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of a quasiseparable confederate matrix corresponding to a system of polynomials R and n nodes $x = (x_1, x_2, \ldots, x_n)$.*

1. *Compute the entries of diag$(c_1, \ldots, c_n)$ via (2.2): $c_i = \prod_{\substack{k=1\\k\neq i}}^{n}(x_k - x_i)^{-1}$.*
2. *Compute the coefficients $\{P_0, \ldots, P_n\}$ of the master polynomial $P(x)$ as in (3.8) via Algorithm 6.2.*
3. *Evaluate the n polynomials of $\widehat{R}$ with specified via (3.4) at the n nodes x_k to form $V_{\widehat{R}}(x)$. Theorems 5.1-5.3 each provide an algorithm for this, choose ONE of the following steps:*
   - **Theorem 5.1 - Three-term recurrence relations**. *(Limitation: $g_k \neq 0$.)*
     (a) *Set* $V_{\widehat{R}}(:,1) = P_n \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad V_{\widehat{R}}(:,2) = \frac{1}{\widehat{p}_2\widehat{q}_1}\left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \widehat{d}_1 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right) V_{\widehat{R}}(:,1) + \frac{1}{\widehat{p}_2\widehat{q}_1} P_{n-1} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$
     (b) *For $k = 2 : n-1$, compute $\widehat{\alpha}_k, \widehat{\delta}_k, \widehat{\beta}_k, \widehat{\gamma}_k$ via (5.2)-(5.3), and*
$$V_{\widehat{R}}(:,k+1) = \left(\widehat{\alpha}_k \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \widehat{\delta}_k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right) V_{\widehat{R}}(:,k) - \left(\widehat{\beta}_k \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \widehat{\gamma}_k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right) V_{\widehat{R}}(:,k-1) + (\widehat{\alpha}_k P_{n-k} - \widehat{\beta}_k P_{n-k+1}) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$
     Note: The product of two column vectors is understood to be componentwise.
   - **Theorem 5.2 - Szegö-like recurrence relations**. *(Limitation: $b_k \neq 0$.)*
     (a) *Set* $V_{\widehat{R}}(:,1) = P_n \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad G_{\widehat{R}}(:,1) = -\widehat{g}_1 P_n \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}.$
     (b) *For $k = 1 : n-1$, compute v_k via (5.4),*
$$u_k(x) = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \left(\frac{\widehat{g}_k\widehat{h}_k}{\widehat{b}_k} - \widehat{d}_k\right) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix},$$
$$V_{\widehat{R}}(:,k+1) = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left(\frac{\widehat{h}_k}{\widehat{b}_k} G_{\widehat{R}}(:,k) + u_k(x) V_{\widehat{R}}(:,k) + P_{n-k} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right)$$
     and
$$G_{\widehat{R}}(:,k+1) = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left(v_k G_{\widehat{R}}(:,k) - \widehat{g}_{k+1} u_k(x) V_{\widehat{R}}(:,k) - \widehat{g}_{k+1} P_{n-k} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right)$$
     Note: The product of two column vectors is understood to be componentwise.
   - **Theorem 5.3 - [EGO05]-like recurrence relations**. *(Limitation: NONE.)*

(a) *Set* $V_{\widehat{R}}(:,1) = P_n \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \qquad F_{\widehat{R}}(:,1) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$

(b) *For* $k = 1 : n-1$, *compute*

$$V_{\widehat{R}}(:,k+1) = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left(\widehat{p}_k\widehat{h}_k F_{\widehat{R}}(:,k) + \left(\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \widehat{d}_k \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right) V_{\widehat{R}}(:,k) + P_{n-k}\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}\right)$$

and

$$F_{\widehat{R}}(:,k+1) = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left(\widehat{q}_k\widehat{p}_k\widehat{b}_k F_{\widehat{R}}(:,k) - \widehat{q}_k\widehat{g}_k V_{\widehat{R}}(:,k)\right)$$

Note: The product of two column vectors is understood to be componentwise.

4. *Compute the inverse* $V_R(x)^{-1}$ *via (2.1):*

$$V_R(x)^{-1} = \tilde{I} \cdot V_{\widehat{R}}^T(x) \cdot \operatorname{diag}(c_1, \ldots, c_n)$$

Output: *Entries of* $V_R(x)^{-1}$, *the inverse of the polynomial-Vandermonde matrix.*

In what follows we show how this algorithm generalizes the previous work in the important special cases of monomials, real orthogonal polynomials, and Szegö polynomials. The reductions in all three special cases are summarized in Table 4.

Special Case	R.R. Type	Resulting R.R.
Monomials	Theorem 5.1 - 3-term r.r.	(7.1)
	Theorem 5.2 - Szegö-type r.r.	(7.1)
	Theorem 5.3 - [EGO05]-type r.r.	(7.1)
Real orthogonal polynomials	Theorem 5.1 - 3-term r.r.	(7.7)
	Theorem 5.2 - Szegö-type r.r.	N/A, $b_k = 0$.
	Theorem 5.3 - [EGO05]-type r.r.	(7.7)
Szegö polynomials	Theorem 5.1 - 3-term r.r.	(7.15)
	Theorem 5.2 - Szegö-type r.r.	(7.13)
	Theorem 5.3 - [EGO05]-type r.r.	(7.14)

TABLE 4. Reduction of derived recurrence relations in special cases.

7.2. First special case. Monomials and the classical Traub algorithm.

As shown in the previous section, the well known companion matrix (3.9) results when the polynomial system R is simply a system of monomials. By choosing the generators $p_k = 1, q_k = 1, d_k = 0, g_k = 1, b_k = 1$, and $h_k = 0$, the matrix (4.6) reduces to (3.9), and also (4.7) reduces to the confederate matrix for the Horner polynomials (3.10). In this special case, the perturbed three-term recurrence relations of Theorem 5.1 become

$$\widehat{r}_0(x) = P_n, \quad \widehat{r}_k(x) = x\widehat{r}_{k-1}(x) + P_{n-k}, \tag{7.1}$$

coinciding with the known recurrence relations for the Horner polynomials, used in the evaluation of the polynomial

$$P(x) = P_0 + P_1 x + \cdots + P_{n-1}x^{n-1} + P_n x^n. \tag{7.2}$$

In fact, after eliminating the auxiliary polynomials present in Theorems 5.2 and 5.3, these recurrence relations also reduce to (7.1). Thus all of the presented recurrence relations generalize those used in the classical Traub algorithm.

7.3. Second special case. Real orthogonal polynomials and the Calvetti-Reichel algorithm.

Consider the almost tridiagonal confederate matrix

$$C_R(P) = \begin{bmatrix} d_1 & h_2 & 0 & \cdots & 0 & -P_0/P_n \\ q_1 & d_2 & h_3 & \ddots & \vdots & -P_1/P_n \\ 0 & q_2 & d_3 & h_4 & 0 & \vdots \\ 0 & 0 & q_3 & d_4 & \ddots & -P_{n-3}/P_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & h_n - P_{n-2}/P_n \\ 0 & \cdots & 0 & 0 & q_{n-1} & d_n - P_{n-1}/P_n \end{bmatrix}. \tag{7.3}$$

The corresponding system of polynomials R satisfy three-term recurrence relations; for instance, the highlighted column implies

$$r_3(x) = \frac{1}{q_3}(x - d_3)r_2(x) - \frac{h_3}{q_3}r_1(x) \tag{7.4}$$

by the definition of the confederate matrix. Thus, confederate matrices of this form correspond to systems of polynomials satisfying three-term recurrence relations, or systems of polynomials orthogonal on a real interval, and the polynomial $P(x)$. Such confederate matrices can be seen as special cases of our general class by choosing the generators $p_k = 1, b_k = 0$, and $g_k = 1$, and in this case the matrix (4.6) reduces to (7.3).

To invert the corresponding polynomial-Vandermonde matrix by our algorithm, we first find the confederate matrix $C_{\widehat{R}}(P)$ of the polynomial system $\widehat{R}$ associated with R. That is, we must evaluate the polynomials corresponding to the confederate matrix

$$C_R(P) = \begin{bmatrix} d_n - P_{n-1}/P_n & h_n - P_{n-2}/P_n & -P_{n-3}/P_n & \cdots & -P_1/P_n & -P_0/P_n \\ q_{n-1} & d_{n-1} & h_{n-1} & \ddots & \vdots & \\ 0 & q_{n-2} & d_{n-2} & h_{n-2} & 0 & \vdots \\ 0 & 0 & q_{n-3} & d_{n-3} & \ddots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & h_2 \\ 0 & \cdots & 0 & 0 & q_1 & d_1 \end{bmatrix}. \tag{7.5}$$

Note that the highlighted column corresponds to the full recurrence relation

$$\widehat{r}_3(x) = \frac{1}{q_{n-3}}(x - d_{n-2})\widehat{r}_2(x) - \frac{h_{n-1}}{q_{n-3}}\widehat{r}_1(x) + \frac{1}{q_{n-3}}\frac{P_{n-3}}{P_n}\widehat{r}_0(x) \tag{7.6}$$

In this case the perturbed three-term recurrence relations from Theorem 5.1 as well as the two-term recurrence relations from Theorem 5.3 both become

$$\widehat{r}_k(x) = \frac{1}{q_{n-k}}(x - d_{n-k})\widehat{r}_{k-1}(x) - \frac{q_{n-k+1}}{q_{n-k}}h_{n-k+1}\widehat{r}_{k-2}(x) + \frac{1}{q_{n-k}}P_{n-k} \tag{7.7}$$

which coincides with the Clenshaw rule for evaluating

$$P(x) = P_0 r_0(x) + P_1 r_1(x) + \cdots + P_{n-1}r_{n-1}(x) + P_n r_n(x). \tag{7.8}$$

Thus our formula generalizes both the Clenshaw rule and the algorithms designed for inversion of three-term-Vandermonde matrices in [CR93] and [GO94].

Notice that the Szegö-like two-term recurrence relations of Theorem 5.2 are inapplicable as $b_k = 0$ is a necessary choice of generators.

7.4. Third special case. Szegö polynomials and the [O98] algorithm.

Next consider the important special case of the almost unitary Hessenberg matrix

$$C_R(P) = \begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & \cdots & -\rho_0^*\mu_1\cdots\mu_{n-1}\rho_n \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\cdots\mu_{n-1}\rho_n \\ 0 & \mu_2 & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^*\rho_n \end{bmatrix} \tag{7.9}$$

that corresponds to the Szegö polynomials (represented by the reflection coefficients ρ_k and complimentary parameters μ_k) as in Section 1, and polynomial $P(x)$. The Szegö polynomials are known to satisfy the two-term recurrence relations

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^\#(x) \end{bmatrix} = \frac{1}{\mu_0}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^\#(x) \end{bmatrix} = \frac{1}{\mu_k}\begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix}\begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^\#(x) \end{bmatrix} \tag{7.10}$$

as well as the three-term recurrence relations

$$\phi_0^\#(x) = 1, \quad \phi_1^\#(x) = \frac{1}{\mu_1}\cdot x\phi_0^\#(x) - \frac{\rho_1}{\mu_1}\phi_0^\#(x)$$

$$\phi_k^\#(x) = \left[\frac{1}{\mu_k}\cdot x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}\right]\phi_{k-1}^\#(x) - \frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k}\cdot x\cdot\phi_{k-2}^\#(x) \tag{7.11}$$

(see [GS58], [G48]). As above, the polynomials associated with the system of Szegö polynomials are determined by the confederate matrix

$$C_{\widehat{R}}(P) = \begin{bmatrix} -\rho_n\rho_{n-1}^* - P_{n-1}/P_n & -\rho_n\mu_{n-1}\rho_{n-2}^* - P_{n-2}/P_n & \cdots & -\rho_n\mu_{n-1}\dots\mu_1\rho_0^* - P_0/P_n \\ \mu_{n-1} & -\rho_{n-1}\rho_{n-2}^* & \cdots & -\rho_{n-1}\mu_{n-2}\dots\mu_1\rho_0^* \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & \mu_1 & -\rho_1\rho_0^* \end{bmatrix}. \tag{7.12}$$

For this special case, let $p_k = 1$, $q_k = \mu_k$, $d_k = -\rho_k\rho_{k-1}^*$, $g_k = \rho_{k-1}^*$, $b_k = \mu_{k-1}$, and $h_k = -\mu_{k-1}\rho_k$ (alternatively $g_k = \rho_{k-1}^*\mu_k$, $b_k = \mu_k$, $h_k = -\rho_k$). This choice of generators reduces (4.6) to the matrix (7.9) as well as (4.7) to (7.12), and in this case the perturbed two-term recurrence relations of Theorem 5.2 become

$$\begin{bmatrix} \widehat{\phi}_0(x) \\ \widehat{\phi}_0^\#(x) \end{bmatrix} = \frac{1}{\mu_n}\begin{bmatrix} -\rho_n \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \widehat{\phi}_k(x) \\ \widehat{\phi}_k^\#(x) \end{bmatrix} = \frac{1}{\mu_{n-k}}\begin{bmatrix} 1 & -\rho_{n-k}^* \\ -\rho_{n-k} & 1 \end{bmatrix}\begin{bmatrix} \widehat{\phi}_{k-1}(x) \\ x\widehat{\phi}_{k-1}^\#(x) + P_{n-k} \end{bmatrix}, \tag{7.13}$$

coinciding with those recurrence relations derived in [O98]. The recurrence relations from Theorem 5.3 reduce to new two-term recurrence relations; that is, relations that do not generalize those derived in [O98]. They become

$$\begin{bmatrix} \widehat{F}_k(x) \\ \widehat{\phi}_k^\#(x) \end{bmatrix} = \frac{1}{\mu_{n-k}}\begin{bmatrix} \mu_{n-k}\mu_{n-k+1} & -\mu_{n-k}\rho_{n-k+1}^* \\ -\mu_{n-k+1}\rho_{n-k} & x+\rho_{n-k}\rho_{n-k+1}^* \end{bmatrix}\begin{bmatrix} \widehat{F}_{k-1}(x) \\ \widehat{\phi}_{k-1}^\#(x) \end{bmatrix} + \begin{bmatrix} 0 \\ P_{n-k} \end{bmatrix}, \tag{7.14}$$

Also, the perturbed three-term recurrence relations of Theorem 5.1 reduce to

$$\widehat{\phi}_0(x) = \frac{1}{\mu_n}, \quad \widehat{\phi}_1(x) = \left\{\frac{1}{\mu_{n-1}}\cdot x\widehat{\phi}_0(x) - \frac{\rho_{n-1}\rho_n^*}{\mu_{n-1}}\phi_0(x)\right\} + \frac{P_{n-1}}{\mu_{n-1}}.$$

$$\begin{aligned} \widehat{\phi}_k(x) &= \left[\frac{1}{\mu_{n-k}}\cdot x + \frac{\rho_{n-k}}{\rho_{n-k+1}}\frac{1}{\mu_{n-k}}\right]\widehat{\phi}_{k-1}(x) - \frac{\rho_{n-k}}{\rho_{n-k+1}}\frac{\mu_{n-k+1}}{\mu_{n-k}}\cdot x\cdot\widehat{\phi}_{k-2}(x) \\ &\quad + \frac{P_{n-k} - P_{n-k+1}\mu_{n-k+1}\frac{\rho_{n-k}}{\rho_{n-k+1}}}{\mu_{n-k}}. \end{aligned} \tag{7.15}$$

in this case, also coinciding with the perturbed three-term recurrence relations in [O98]. Thus both of these theorems generalize the recurrence relations derived in [O98] as well.

8. Full characterizations of H-$(1,1)$-q.s. polynomials via recurrence relations

One reason that Algorithm 7.2 is fast is that it converts the generators of the quasiseparable confederate matrix into coefficients of recurrence relations for the associated system of polynomials $\widehat{R}$ involved in the inversion formula (2.1) for $V_R(x)^{-1}$. In fact, we have presented three such conversion methods, summarized along with their cases of applicability in Table 5.

R.R. type	Recurrence relations for $\widehat{R}$	Restrictions
3-term r.r.	Theorem 5.1	$g_k \neq 0$
Szegö-type r.r.	Theorem 5.2	$b_k \neq 0$
[EGO05]-type r.r.	Theorem 5.3	none

Table 5. Conversion formulas for the system of associated polynomials $\widehat{R}$.

With these conversions, the derivation of the Traub-like algorithm is complete. However, there are still two questions left to answer. Both questions are related not to the associated polynomials $\widehat{R}$ (for which the results are listed in Table 5) but are related to the original polynomial system R involved in the original polynomial-Vandermonde matrix $V_R(x)$.

First, in many applications polynomials are given in terms of their recurrence relations, and in order to run Algorithm 7.2 one needs to include a preprocessing step for this conversion, such as that specified in Theorem 4.7 and Algorithm 7.1. The results of this section can be used to answer this question as in Table 6. A shaded element indicates the result has not yet been stated.

R.R. type	R.R. Coefficients ⇒ Generators	Generators ⇒ R.R. Coefficients
3-term r.r.	Theorem 4.7	Corollary 8.3 restrictions: $h_k \neq 0$.
Szegö-type r.r.	Theorem 8.6	Corollary 8.10 restrictions: $b_k \neq 0$.
[EGO05]-type r.r.	Theorem 8.12	Corollary 8.15 restrictions: none.

Table 6. Conversion formulas for the system of polynomials R.

Secondly, the input of Algorithm 7.2 is the set of generators of a Hessenberg-$(1,1)$-quasiseparable matrix $C_R(r_n)$, and the algorithm applies to the class of H-$(1,1)$-q.s. polynomials. Hence one may want to give a complete characterization of this class in terms of various recurrence relations satisfied.

The results of this section in terms of this characterization can be easily summarized in Figure 1, where each class of matrix is shown, and the recurrence relations are listed with the corresponding class of matrix.

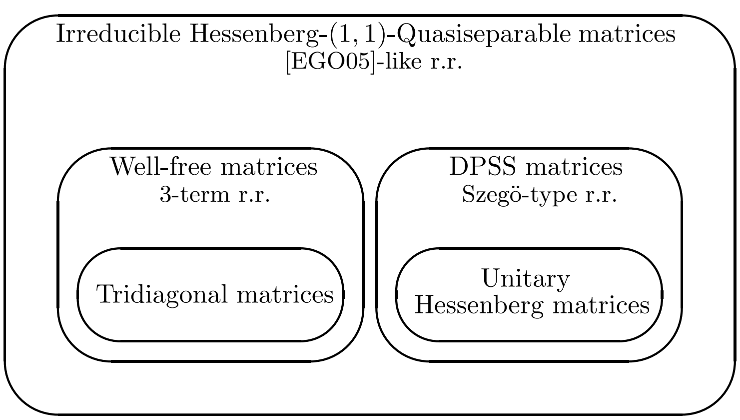


Figure 1. Classes of matrices.

8.1. Well-free matrices & three-term recurrence relations

In this section we give a definition of a new class of matrices, and show that the class is exactly that whose corresponding polynomials satisfy an analog of the 3-term recurrence relations derived above for the associated polynomials.

Definition 8.1 (Well-free matrices).

- *An $n \times n$ matrix $A = (A_{i,j})$ is said to have a* **well** *in column k if $A_{i,k} = 0$ for $1 \leqslant i < k$ and there exists a pair (i, j) with $1 \leqslant i < k$ and $k < j \leqslant n$ such that $A_{i,j} \neq 0$.*
- *An irreducible H-$(1,1)$-q.s. matrix is said to be* **well-free** *if none of its columns contain wells.*

In words, a matrix has a well in column k if all entries above the main diagonal in the k-th column are zero, **except** if all entries in the upper-right block to the right of these zeros are also zeros. For instance, the matrices

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

have wells in the 4th, 2nd, and 3rd columns, respectively, but the matrices

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

are all well-free.

Notice that the class of tridiagonal matrices is a subclass of well-free matrices, as tridiagonal matrices are both H-$(1,1)$-quasiseparable and contain no wells under any circumstances.

Lemma 8.2. *An irreducible H-$(1,1)$-q.s. matrix is well-free if and only if there exists a choice of generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of the matrix such that $h_k \neq 0$ for $k = 2, \ldots, n$.*

The proof is found in the appendix, page 31.

The result in Theorem 4.7 tells us that if polynomials r_k satisfy the recurrence relations (1.2) then the corresponding confederate matrix A has a nice structure. Precisely, A is a Hessenberg-$(1,1)$-quasiseparable matrix given by the generators (4.3)-(4.5). Notice that it is guaranteed from (4.5) that $h_k = 1$, $k = 2, \ldots, n$. The next statement shows that, provided $h_k \neq 0$, the converse is also true. Although the condition $h_k \neq 0$ may look unnecessary when one starts with the matrices, when beginning with the polynomials satisfying (1.2) and converting to matrices, the condition $h_k = 1$ is automatically satisfied as shown in (4.5).

Corollary 8.3 (Quasiseparable generators $\Rightarrow$ recurrence relations coefficients). *Let $C_R(r_n)$ be an irreducible Hessenberg-$(1,1)$-quasiseparable matrix specified by the generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ with $\mathbf{h_k \neq 0}$ for $k = 2, \ldots, n$; that is, $C_R(r_n)$ is* **well-free**. *Then the polynomial system R with confederate matrix $C_R(r_n)$ satisfies (1.2) with*

$$\alpha_k = \frac{1}{p_{k+1}q_k}, \quad \delta_k = \frac{1}{p_{k+1}q_k}\left(d_k - \frac{p_k q_{k-1} b_{k-1} h_k}{h_{k-1}}\right) \tag{8.1}$$

$$\beta_k = \frac{1}{p_{k+1}q_k}\frac{h_k b_{k-1}}{h_{k-1}}, \quad \gamma_k = \frac{1}{p_{k+1}q_k}\frac{h_k}{h_{k-1}}\left(h_{k-1}g_{k-1} - d_{k-1}b_{k-1}\right) \tag{8.2}$$

This relation is in fact a corollary of Theorem 5.1. Indeed, the corresponding matrix of the generators presented in Theorem 5.1 is of the form (4.7), and that of Corollary 8.3 is of the form (4.6). Hence we can see that the recurrence relations (8.1), (8.2) are specifications of (5.2), (5.3) with $P_i = 0$.

Theorem 4.7 and Corollary 8.3 give not only a method of computing generators from parameters of the recurrence relations and vice versa, but they also together give a characterization of polynomials satisfying (1.2) in terms of their quasiseparable matrices, stated as the following corollary.

Corollary 8.4 (Full characterization of well-free matrices via 3-term recurrence relations). *Let $R = \{r_0(x), \dots, r_n(x)\}$ be a polynomial system s.t.* $\deg(r_k) = k$ *and denote by $C_R(r_n)$ the irreducible upper Hessenberg confederate matrix of $r_n(x)$ with respect to R. Then the system R satisfies the recurrence relations (1.2) if and only if $C_R(r_n)$ is* **well-free***; that is, $(1,1)$-quasiseparable with $\mathbf{h_k} \neq \mathbf{0}$ for $k = 2, \dots, n$.*

Remark 8.5. The reader may have noticed the difference in the requirement of $h_k \neq 0$ for the polynomials R in this section and $g_k \neq 0$ for the associated polynomials $\widehat{R}$ in Theorem 5.1. The class of matrices for which Theorem 5.1 applies is the class of **tunnel-free** matrices. Following Definition 8.1, a matrix A is called tunnel-free if its pertranspose $\tilde{I} \cdot A^T \cdot \tilde{I}$ is well-free.

8.2. Diagonal-plus-semiseparable matrices & Szegö-type recurrence relations

We begin this section with an analog of Theorem 4.7 for when polynomials satisfying Szegö-like two-term recurrence relations are known.

Theorem 8.6 (Recurrence relation coefficients ⇒ quasiseparable generators). *Let $R = \{r_0(x), \dots, r_n(x)\}$ be a system of polynomials s.t.* $\deg(r_k) = k$ *and denote by $C_R(r_n)$ the confederate matrix of $r_n(x)$ with respect to R. Suppose R satisfies the recurrence relations*

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (\delta_k x + \theta_k) r_{k-1}(x) \end{bmatrix}. \tag{8.3}$$

Then $C_R(r_n)$ is an irreducible Hessenberg-$(1,1)$-quasiseparable matrix of the form

$$\begin{bmatrix}
-\frac{\theta_1+\gamma_1}{\delta_1} & -(\alpha_1-\beta_1\gamma_1)\frac{\gamma_2}{\delta_2} & -(\alpha_1-\beta_1\gamma_1)(\alpha_2-\beta_2\gamma_2)\frac{\gamma_3}{\delta_3} & \cdots & -(\alpha_1-\beta_1\gamma_1)\cdots(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\frac{\gamma_n}{\delta_n} \\
\frac{1}{\delta_1} & -\frac{\theta_2+\gamma_2\beta_1}{\delta_2} & -\beta_1(\alpha_2-\beta_2\gamma_2)\frac{\gamma_3}{\delta_3} & \cdots & -\beta_1(\alpha_2-\beta_2\gamma_2)\cdots(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\frac{\gamma_n}{\delta_n} \\
0 & \frac{1}{\delta_2} & -\frac{\theta_3+\gamma_3\beta_2}{\delta_3} & \ddots & -\beta_2(\alpha_3-\beta_3\gamma_3)\cdots(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\frac{\gamma_n}{\delta_n} \\
\vdots & \ddots & \ddots \quad \ddots & \ddots & \vdots \\
\vdots & & & \ddots & -\beta_{n-1}(\alpha_{n-1}-\beta_{n-1}\gamma_{n-1})\frac{\gamma_n}{\delta_n} \\
0 & \cdots & 0 & \frac{1}{\delta_{n-1}} & -\frac{\theta_n+\gamma_n\beta_{n-1}}{\delta_n}
\end{bmatrix} \tag{8.4}$$

that is, the generators of $C_R(r_n)$ are given by

$$d_1 = -\frac{\theta_1+\gamma_1}{\delta_1}, \qquad d_k = -\frac{\theta_k+\gamma_k\beta_{k-1}}{\delta_k}, \quad k = 2, \dots, n$$
$$p_k = 1, \quad k = 2, \dots, n$$
$$q_k = \frac{1}{\delta_k}, \quad k = 1, \dots, n-1$$
$$g_1 = 1, \qquad g_k = \beta_{k-1}, \quad k = 2, \dots, n-1$$
$$b_k = \alpha_{k-1} - \beta_{k-1}\gamma_{k-1}, \quad k = 2, \dots, n-1$$
$$h_k = -\frac{\gamma_k}{\delta_k}(\alpha_{k-1} - \beta_{k-1}\gamma_{k-1}), \quad k = 2, \dots, n$$

For the proof, see the appendix. We next give a detailed example of the specification of this result to the Szegö case.

Example 8.7 (Szegö polynomials). With the choices

$$\alpha_k = \frac{1}{\mu_k}, \quad \beta_k = -\rho_k^*, \quad \gamma_k = -\frac{\rho_k}{\mu_k}, \quad \delta_k = \frac{1}{\mu_k}, \quad \theta_k = 0$$

from which it follows that

$$\alpha_k - \beta_k\gamma_k = \frac{1-|\rho_k|^2}{\mu_k} = \mu_k, \quad \frac{\gamma_k}{\delta_k} = -\rho_k$$

the two-term recurrence relations (8.3) become

$$\begin{bmatrix} \phi_k(x) \\ \phi_k^\#(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^\#(x) \end{bmatrix} \tag{8.5}$$

and the matrix (8.4) reduces to the matrix

$$\begin{bmatrix} \rho_1 & \mu_1\rho_2 & \mu_1\mu_2\rho_3 & \cdots & \mu_1\cdots\mu_{n-1}\rho_n \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\cdots\mu_{n-1}\rho_n \\ 0 & \mu_2 & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^*\rho_n \end{bmatrix}.$$

Using the convention that $\rho_0 := -1$ to insert $1 = -\rho_0^*$ throughout the first row, this matrix becomes exactly the unitary Hessenberg matrix displayed in (1.8). This demonstrates that the Szegö polynomials are a special case of polynomials satisfying (8.3), and likewise the unitary Hessenberg matrix is a special case of those of the form (8.4).

We also note that the condition $b_k \neq 0$ is *not* satisfied by the real orthogonal polynomials, and hence the form (8.4) cannot be used for them.

Next, we define the class of Hessenberg diagonal-plus-upper-semiseparable matrices, and show they are the class of matrices whose polynomials satisfy a version of the Szegö-type recurrence relations of Theorem 5.2. In this definition, we use the standard MATLAB notation triu(A) to denote the strictly upper triangular portion of the matrix A.

Definition 8.8 (Diagonal-plus-upper-semiseparable (DPSS) matrices). *An irreducible H-$(1,1)$-q.s. matrix A is called diagonal-plus-upper-semiseparable (DPSS) if it is of the form*

$$A = B + \mathrm{triu}(A_U)$$

for a rank-one matrix A_U and a lower bidiagonal matrix B.

Lemma 8.9. *An irreducible H-$(1,1)$-q.s. matrix is DPSS if and only if there exists a choice of generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ of the matrix such that $b_k \neq 0$ for $k = 2, \ldots, n-1$.*

The proof is given in the appendix, page 32.

Corollary 8.10 (Quasiseparable generators ⇒ recurrence relations coefficients). *Let $C_R(r_n)$ be an irreducible Hessenberg-$(1,1)$-quasiseparable matrix specified by the generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ with $\mathbf{b_k} \neq \mathbf{0}$ for $k = 2, \ldots, n-1$; that is, $C_R(r_n)$ is* **DPSS**. *Then the polynomial system R with confederate matrix $C_R(r_n)$ satisfies*

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} -g_1 \\ 1 \end{bmatrix}, \tag{8.6}$$

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{p_{k+1}q_k} \begin{bmatrix} v_k & -g_{k+1} \\ h_k/b_k & 1 \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ u_k(x)r_{k-1}(x) \end{bmatrix} \tag{8.7}$$

with

$$u_k(x) = (x - d_k) + \frac{g_k h_k}{b_k}, \qquad v_k = p_{k+1}b_{k+1}q_k - \frac{g_{k+1}h_k}{b_k}. \tag{8.8}$$

As in the case of the three-term recurrence relations, Theorem 8.6 and Corollary 8.10 together give a characterization of polynomials satisfying two-term recurrence relations of the form (8.7) in terms of their quasiseparable matrices, stated as the following corollary.

Corollary 8.11 (Full characterization of DPSS matrices via Szegö-type recurrence relations). *Let $R = \{r_0(x), \ldots, r_n(x)\}$ be a polynomial system s.t. $\deg(r_k) = k$ and denote by $C_R(r_n)$ the irreducible upper Hessenberg confederate matrix of $r_n(x)$ with respect to R. Then the system R satisfies the recurrence relations (8.7) if and only if $C_R(r_n)$ is* **DPSS**; *that is, $(1,1)$-quasiseparable with $\mathbf{b_k} \neq \mathbf{0}$ for $k = 2, \ldots, n-1$.*

8.3. [EGO05]-type recurrence relations

Theorem 8.12 (Recurrence relation coefficients ⇒ quasiseparable generators). *Let $R = \{r_0(x), \ldots, r_n(x)\}$ be a system of polynomials s.t. $\deg(r_k) = k$ and denote by $C_R(r_n)$ the confederate matrix of $r_n(x)$ with respect to R. Suppose R satisfies the recurrence relations*

$$\begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k x + \theta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix} \tag{8.9}$$

Then $C_R(r_n)$ is an irreducible Hessenberg-$(1,1)$-quasiseparable matrix of the form

$$\begin{bmatrix} -\frac{\theta_1}{\delta_1} & -\beta_1(\frac{\gamma_2}{\delta_2}) & -\beta_1\alpha_2(\frac{\gamma_3}{\delta_3}) & -\beta_1\alpha_2\alpha_3(\frac{\gamma_4}{\delta_4}) & \cdots & -\beta_1\alpha_2\alpha_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ \frac{1}{\delta_1} & -\frac{\theta_2}{\delta_2} & -\beta_2(\frac{\gamma_3}{\delta_3}) & -\beta_2\alpha_3(\frac{\gamma_4}{\delta_4}) & \cdots & -\beta_2\alpha_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & \frac{1}{\delta_2} & -\frac{\theta_3}{\delta_3} & -\beta_3(\frac{\gamma_4}{\delta_4}) & \ddots & -\beta_3\alpha_4\cdots\alpha_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & 0 & \frac{1}{\delta_3} & -\frac{\theta_4}{\delta_4} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & -\beta_{n-1}(\frac{\gamma_n}{\delta_n}) \\ 0 & \cdots & 0 & 0 & \frac{1}{\delta_{n-1}} & -\frac{\theta_n}{\delta_n} \end{bmatrix} \tag{8.10}$$

that is, the generators of $C_R(r_n)$ are

$$d_k = -\frac{\theta_k}{\delta_k}, \quad k = 1, \dots, n$$
$$p_k = 1, \quad k = 2, \dots, n$$
$$q_k = \frac{1}{\delta_k}, \quad k = 1, \dots, n-1$$
$$g_k = \beta_k, \quad k = 1, \dots, n-1$$
$$b_k = \alpha_k, \quad k = 2, \dots, n-1$$
$$h_k = -\frac{\gamma_k}{\delta_k}, \quad k = 2, \dots, n$$

For the proof of this result, see the appendix.

Example 8.13 (Szegö polynomials). If we choose

$$\alpha_k = \mu_k, \quad \beta_k = \rho^*_{k-1}\mu_k, \quad \gamma_k = \frac{\rho_k}{\mu_k}, \quad \delta_k = \frac{1}{\mu_k}, \quad \theta_k = \frac{\rho^*_{k-1}\rho_k}{\mu_k}$$

the two-term recurrence relations (8.9) do *not* reduce to the known two-term recurrence relations for the Szegö polynomials (7.10), but become instead the new relations

$$\begin{bmatrix} \phi_k(x) \\ \phi^\#_k(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & \rho^*_{k-1} \\ \rho_k & x + \rho^*_{k-1}\rho_k \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ \phi^\#_{k-1}(x) \end{bmatrix}. \tag{8.11}$$

The matrix (8.10) does in fact reduce to the matrix the unitary Hessenberg matrix displayed in (1.8).

Both the classical Szegö formula (8.5) and the new formula (8.11) describe, of course, the same Szegö polynomials $\{\phi^\#_k(x)\}$. However, the auxilary polynomials $\{\phi_k(x)\}$ used in the two above formulas are different. Indeed, it is well-known that the classical Szegö formula (8.5) we have

$$\phi_k(x) = x^n \cdot [\phi^\#_k(\frac{1}{x^*})]^*,$$

and in particular, $\deg \phi_k(x) = \deg \phi^\#_k(x)$. At the same time, it is easy to see that the auxiliary polynomials $\{\phi_k(x)\}$ of the new formula (8.11) are different; in particular $\deg \phi_k(x) = \deg \phi^\#_k(x) - 1$.

Example 8.14 (Real orthogonal polynomials). For systems with $\alpha = 0$, the matrix (8.10) becomes tridiagonal, and the corresponding system of polynomials are orthogonal on a real interval. The relations (8.9) becomes just the familiar three-term recurrence relations for real orthogonal polynomials, as $\alpha_k = 0$ in (8.9) implies that the auxiliary polynomial G_{k-1} is just a scaled version of r_{k-2}, and hence r_k is formed from r_{k-1} and r_{k-2} as in the usual three-term recurrence relations.

Corollary 8.15 (Quasiseparable generators ⇒ recurrence relations coefficients). *Let $C_R(r_n)$ be an irreducible Hessenberg-$(1,1)$-quasiseparable matrix specified by the generators $\{p_k, q_k, d_k, g_k, b_k, h_k\}$. Then the polynomial system R with confederate matrix $C_R(r_n)$ satisfies*

$$\begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \frac{1}{p_{k+1}q_k} \begin{bmatrix} q_k p_k b_k & -q_k g_k \\ p_k h_k & x - d_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}. \tag{8.12}$$

There are no restrictions required for these recurrence relations to be applicable as in the recurrence relations above which generalize classical relations. As a result, these recurrence relations completely characterize H-$(1,1)$-q.s. polynomials:

Corollary 8.16 (Full characterization of H-(1,1)-q.s. matrices via [EGO05]-type recurrence relations). *Let $R = \{r_0(x), \ldots, r_n(x)\}$ be a polynomial system s.t. $\deg(r_k) = k$ and denote by $C_R(r_n)$ the irreducible upper Hessenberg confederate matrix of $r_n(x)$ with respect to R. Then the system R satisfies the recurrence relations (8.12) if and only if $C_R(r_n)$ is $(1,1)$-quasiseparable.*

9. Numerical Experiments

The numerical properties of the Traub algorithm and its generalizations (that are the special cases of our general algorithm) were studied by different authors. It was noticed in [GO97] that a version of the Traub algorithm can yield high accuracy in certain cases if the algorithm is preceded with the *Leja ordering* of the nodes; that is, ordering such that

$$|x_1| = \max_{1\leqslant i\leqslant n} |x_i|, \qquad \prod_{j=1}^{k-1} |x_k - x_j| = \max_{k\leqslant i\leqslant n} \prod_{j=1}^{k-1} |x_i - x_j|, \quad k = 2, \ldots, n-1$$

(see [RO91], [H90], [O03]) It was noticed in [GO97] that the same is true for Chebyshev-Vandermonde matrices.

No error analysis was done, but the conclusions of the above authors was that in many cases the Traub algorithm and its extensions can yield much better accuracy than Gaussian elimination, even for very ill-conditioned matrices.

We made our preliminary experiments with the general algorithm, and our conclusions are consistent with the experience of our colleagues. In all cases we studied the proposed algorithm yields better accuracy than Gaussian elimination, e.g., in the new special cases of Szegö-Vandermonde and Hessenberg-$(1,1)$-quasiseparable-Vandermonde matrices. However, our experiments need to be done for different special cases and also the numerical properties of different recurrence relations are worth analyzing.

The algorithm has been implemented in MATLAB version 7. The results of the algorithm using standard MATLAB code, and hence double precision arithmetic, were compared with exact solutions calculated using the MATLAB Symbolic Toolbox command `vpa()`, which allows software-implemented precision of arbitrary numbers of digits. The number of digits was set to 64, however in cases where the condition number of the coefficient matrix exceeded 10^{30}, this was raised to 100 digits to maintain accuracy.

We compare the forward accuracy of the inverse computed by the algorithm with respect to the inverse computed in high precision, defined by

$$e = \frac{\|V_R(x)^{-1} - \widehat{V_R}(x)^{-1}\|_2}{\|V_R(x)^{-1}\|_2} \tag{9.1}$$

where $\widehat{V_R}(x)^{-1}$ is the solution computed by each algorithm in MATLAB in double precision, and $V_R(x)^{-1}$ is the exact solution. In the tables, TraubQS denotes the proposed Traub-like algorithm, and `inv()` indicates MATLAB's inversion command. Finally, $\mathrm{cond}(V)$ denotes the condition number of the matrix V computed via the MATLAB command `cond()`.

Experiment 1. In this experiment, the problem was chosen by choosing the generators that define the recurrence relations of the polynomial system randomly in $(-1,1)$, and the nodes x_k were selected equidistant on $(-1,1)$ via the formula

$$x_k = -1 + 2\left(\frac{k}{n-1}\right), \quad k = 0, 1, \ldots, n-1$$

We test the accuracy of the inversion algorithm for various sizes n of matrices generated in this way. Some results are tabulated in Table 7, and shown graphically in Figure 2.

n	`cond(V)`	`inv()`	TraubQS
10	4.2e04	4.1e-14	3.4e-15
	2.2e05	2.5e-14	6.3e-15
	3.7e08	1.0e-13	8.9e-14
15	1.1e10	3.5e-11	3.5e-11
	1.1e11	1.5e-12	4.8e-13
	4.7e11	1.3e-13	7.7e-14
20	7.6e14	1.1e-10	3.4e-12
	1.2e15	4.2e-11	1.1e-11
	7.8e17	1.2e-09	1.7e-15
25	4.8e19	1.2e-09	1.7e-13
	1.1e24	5.9e-07	1.3e-11
	1.5e27	8.4e-08	2.4e-09
30	3.3e24	7.2e-07	1.1e-13
	5.0e27	2.8e-06	1.7e-11
	1.8e30	1.3e-03	9.5e-10
35	7.3e23	2.4e-04	6.9e-10
	8.3e26	2.6e-03	1.2e-06
	1.4e27	2.9e-05	1.4e-08
40	1.1e31	8.2e-02	2.4e-13
	2.4e32	3.4e+00	9.9e-12
	1.7e33	1.2e-01	1.0e-08
45	4.3e30	1.7e-01	1.7e-05
	1.7e31	5.9e-01	1.0e-08
	3.9e35	1.0e-02	2.4e-08
50	2.1e42	1.0e+00	4.7e-06
	3.9e44	1.0e+00	7.0e-06
	6.6e45	1.0e+00	6.3e-06

Table 7. Equidistant nodes on $(-1, 1)$.

Notice that the performance of the proposed inversion algorithm is an improvement over that of MATLAB's standard inversion command `inv()` in this specific case.

Experiment 2. Next, the values for the generators and the nodes were chosen randomly on the unit disc. We test the accuracy for various 30×30 matrices generated in this way, and present some results in Table 8 and Figure 3.

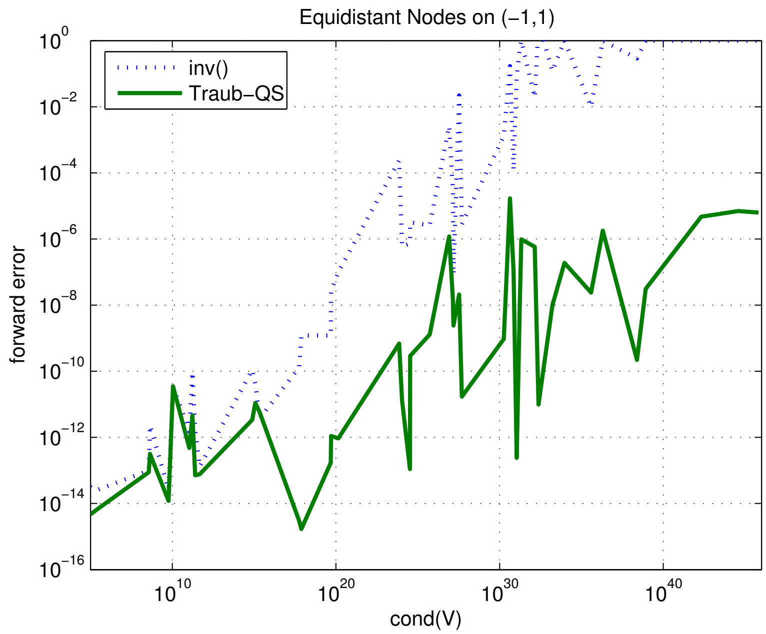


FIGURE 2. Equidistant nodes on $(-1, 1)$.

cond(V)	inv()	TraubQS
1.7e21	1.3e-07	3.5e-14
3.9e23	1.2e-05	9.6e-15
4.3e23	2.7e-03	1.2e-14
2.8e24	2.4e-05	8.4e-13
2.9e24	3.9e-03	4.3e-12
1.8e25	6.8e-07	2.6e-12
2.2e25	8.9e-03	3.4e-14
3.1e25	1.3e-03	3.6e-14
3.5e25	2.9e-03	7.9e-14
6.8e25	1.0e+00	2.2e-11
2.2e27	1.0e-02	2.9e-11
4.9e27	3.6e+00	2.3e-13
6.6e27	9.9e+00	7.6e-13
7.6e27	4.6e-04	2.0e-12
2.0e28	1.9e-03	5.7e-14
2.4e28	6.9e-04	9.6e-15
2.6e28	2.5e-02	1.2e-13
5.2e28	2.4e-05	1.7e-12
6.9e30	1.2e-03	2.5e-14
1.4e33	1.0e+00	2.9e-13

Table 8. Random parameters on the unit disc.

10. Conclusions

In this paper we used properties of confederate matrices to extend the classical Traub algorithm for inversion of Vandermonde matrices to the general polynomial-Vandermonde case. The relation between polynomial

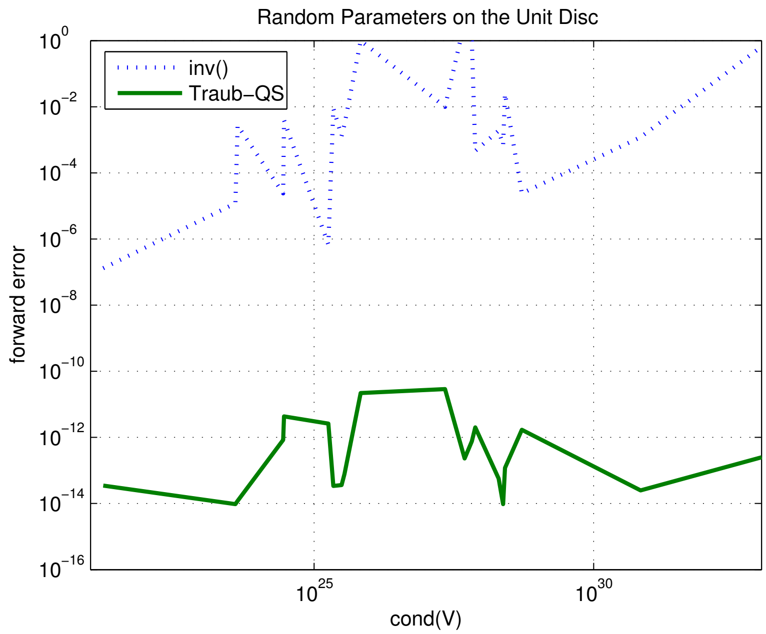


FIGURE 3. Random parameters on the unit disc.

systems satisfying some recurrence relations and quasiseparable matrices allowed an order of magnitude computational savings in this case, resulting in an $\mathcal{O}(n^2)$ algorithm as opposed to Gaussian elimination, which requires $\mathcal{O}(n^3)$ operations. The connection between the various sparse recurrence relations derived for this purpose was investigated, and the result is a complete characterization of the system of polynomials studied in terms of the related quasiseparable matrix. Finally, some numerical experiments were presented that indicate that, under some circumstances, the resulting algorithm can give better performance than Gaussian elimination.

Appendix

Proof of Theorem 4.7. We prove by induction on k that the choice of generators (4.3)-(4.5) results in the n-term recurrence relations (4.2) corresponding to an irreducible Hessenberg-$(1,1)$-quasiseparable matrix.

For $k=1$, the recurrence relations (1.2) result in

$$r_1(x) = (\alpha_1 x - \delta_1) r_0(x), \tag{A.1}$$

and using (4.3)-(4.5) we arrive at (4.2) for $k=1$. For $k=2$, inserting the relation

$$x r_0(x) = \frac{1}{\alpha_1}(r_1(x) + \delta_1 r_0(x))$$

from (A.1) into

$$r_2(x) = (\alpha_2 x - \delta_2) r_1(x) - (\beta_2 x + \gamma_2) r_0(x)$$

results in

$$r_2(x) = \left(\alpha_2 x - \left(\delta_2 + \frac{\beta_2}{\alpha_1}\right)\right) r_1(x) - \left(\beta_2 \frac{\delta_1}{\alpha_1} + \gamma_2\right) r_0(x),$$

and again using (4.3)-(4.5) this becomes

$$r_2(x) = \frac{1}{p_3 q_2}\left[(x - d_2) r_1(x) - g_1 r_0(x)\right]$$

which is (4.2) for $k=2$.

Next suppose the choice of generators (4.3)-(4.5) results in (4.2) for some $k-1$ with $k \geqslant 3$. By adding and subtracting the quantity $\frac{\beta_k}{\alpha_{k-1}} r_{k-1}(x)$ to (1.2), we have

$$r_k(x) = \alpha_k \left[\left(x - \left(\frac{\delta_k}{\alpha_k} + \frac{\beta_k}{\alpha_k \alpha_{k-1}} \right) \right) r_{k-1}(x) + \left(\frac{\beta_k}{\alpha_k \alpha_{k-1}} \right) r_{k-1}(x) - \left(\frac{\beta_k}{\alpha_k} x + \frac{\gamma_k}{\alpha_k} \right) r_{k-2}(x) \right]. \tag{A.2}$$

By the inductive hypothesis,

$$\left(\frac{\beta_k}{\alpha_k \alpha_{k-1}} \right) r_{k-1}(x) - \left(\frac{\beta_k}{\alpha_k} x + \frac{\gamma_k}{\alpha_k} \right) r_{k-2}(x) = -\left(\frac{d_{k-1}\beta_k + \gamma_k}{\alpha_k} \right) r_{k-2}(x) - \sum_{j=0}^{k-3} g_{j+1} b^{\times}_{j+1,k-1} \frac{\beta_k}{\alpha_k} r_j(x),$$

and using (4.3)-(4.5), this is furthermore equal to

$$-g_{k-1} r_{k-2}(x) - \sum_{j=0}^{k-3} g_{j+1} b^{\times}_{j+1,k-1} b_{k-1} r_j(x) = \sum_{j=0}^{k-2} g_{j+1} b^{\times}_{j+1,k} r_j(x) \tag{A.3}$$

using the identities $b^{\times}_{k-1,k} = 1$ and $b^{\times}_{j+1,k-1} b_{k-1} = b^{\times}_{j+1,k}$. Inserting (A.3) into (A.2) and using (4.3) once more gives (4.2) for k. This completes the proof. □

Proof of Theorem 5.1. Let $S = \{s_0(x), s_1(x), \ldots, s_{n-1}(x)\}$ be the system of polynomials corresponding to the Hessenberg order $(1,2)$-quasiseparable matrix $C_{\widehat{R}}(P)$ of the form in (4.7). Then from (3.1) and (4.7), we have for $k = 1, 2, \ldots, n-1$

$$\begin{aligned} s_k(x) &= \frac{1}{\widehat{p}_{k+1}\widehat{q}_k} \Big[(x - \widehat{d}_k) s_{k-1}(x) - \widehat{g}_{k-1}\widehat{h}_k s_{k-2}(x) - \widehat{g}_{k-2}\widehat{b}_{k-1}\widehat{h}_k s_{k-3}(x) \\ & \quad - \cdots - \widehat{g}_2 \widehat{b}_3 \cdots \widehat{b}_{k-1}\widehat{h}_k s_1(x) - \widehat{g}_1 \widehat{b}_2 \cdots \widehat{b}_{k-1}\widehat{h}_k s_0(x) + P_{n-k} \Big]. \end{aligned} \tag{A.4}$$

It suffices to show that the system of polynomials $\{\widehat{r}_0(x), \widehat{r}_1(x), \ldots, \widehat{r}_{n-1}(x)\}$ defined by the recurrence relations in (5.1)-(5.3) coincide with the system S; that is, that $\widehat{r}_k(x) = s_k(x)$ for each k. We present this proof by induction. By direct confirmation, it is seen that $\widehat{r}_0(x) = s_0(x)$ and $\widehat{r}_1(x) = s_1(x)$.

Next suppose that the conclusion is true for each index less than or equal to $k-1$ for some $2 \leqslant k \leqslant n-1$; that is,

$$\widehat{r}_i(x) = s_i(x), \quad i = 0, 1, \ldots, k-1. \tag{A.5}$$

Then using (A.4) for $k-1$ and (A.5), we have

$$\begin{aligned} x\widehat{r}_{k-2}(x) &= \widehat{p}_k \widehat{q}_{k-1} \widehat{r}_{k-1}(x) + \widehat{d}_{k-1}\widehat{r}_{k-2}(x) \\ & \quad + \widehat{g}_{k-2}\widehat{h}_{k-1}\widehat{r}_{k-3}(x) + \widehat{g}_{k-3}\widehat{b}_{k-2}\widehat{h}_{k-1}\widehat{r}_{k-4}(x) \\ & \quad + \cdots + \widehat{g}_2 \widehat{b}_3 \cdots \widehat{b}_{k-2}\widehat{h}_{k-1}\widehat{r}_1(x) \\ & \quad + \widehat{g}_1 \widehat{b}_2 \cdots \widehat{b}_{k-2}\widehat{h}_{k-1}\widehat{r}_0(x) - P_{n-k+1} \end{aligned} \tag{A.6}$$

Next, the polynomial $\widehat{r}_k(x)$ satisfies the recurrence relations (5.1), noting that by hypothesis, $\widehat{h}_k = g_{n-k+1} \neq 0$ for each k. Inserting (A.6) into (5.1) and using (A.5), we arrive at

$$\begin{aligned} \widehat{r}_k(x) &= \frac{1}{\widehat{p}_{k+1}\widehat{q}_k} \Big[(x - \widehat{d}_k) s_{k-1}(x) - \widehat{g}_{k-1}\widehat{h}_k s_{k-2}(x) - \widehat{g}_{k-2}\widehat{b}_{k-1}\widehat{h}_k s_{k-3}(x) \\ & \quad - \cdots - \widehat{g}_2 \widehat{b}_3 \cdots \widehat{b}_{k-1}\widehat{h}_k s_1(x) - \widehat{g}_1 \widehat{b}_2 \cdots \widehat{b}_{k-1}\widehat{h}_k s_0(x) + P_{n-k} \Big]. \end{aligned}$$

and hence in light of (A.4), we have $\widehat{r}_k(x) = s_k(x)$. This completes the proof. □

Proof of Theorem 5.2. Suppose first that the generators are such that $g_k \neq 0$ for each k. The proof in this case will be given by showing that the system of polynomials generated by the perturbed two-term recurrence relations (5.5)-(5.6) coincide with those given by Theorem 5.1. From (5.6) and the relationship

$$\left(v_k + \frac{\hat{g}_{k+1}\hat{h}_k}{\hat{b}_k} \right) \begin{bmatrix} v_k & -\hat{g}_{k+1} \\ \hat{h}_k/\hat{b}_k & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \hat{g}_{k+1} \\ -\hat{h}_k/\hat{b}_k & v_k \end{bmatrix} \tag{A.7}$$

we obtain

$$\left(v_k + \frac{\hat{g}_{k+1}\hat{h}_k}{\hat{b}_k} \right) \begin{bmatrix} G_{k-1}(x) \\ u_k(x)\hat{r}_{k-1}(x) + P_{n-k} \end{bmatrix} = \widehat{p}_{k+1}\widehat{q}_k \begin{bmatrix} 1 & \hat{g}_{k+1} \\ -\hat{h}_k/\hat{b}_k & v_k \end{bmatrix} \begin{bmatrix} G_k(x) \\ \hat{r}_k(x) \end{bmatrix} \tag{A.8}$$

Thus, we have the following expression for $u_k(x)\hat{r}_{k-1}(x) + P_{n-k}$,

$$\left(v_k + \frac{\hat{g}_{k+1}\hat{h}_k}{\hat{b}_k}\right)(u_k(x)\hat{r}_{k-1}(x) + P_{n-k}) = \widehat{p}_{k+1}\widehat{q}_k\left(\frac{\hat{h}_k}{\hat{b}_k}G_k(x) + v_k\hat{r}_k(x)\right) \tag{A.9}$$

Using (5.6) for $k+1$, we have that

$$\widehat{p}_{k+2}\widehat{q}_{k+1}\hat{r}_{k+1}(x) = \frac{\hat{h}_{k+1}}{\hat{b}_{k+1}}G_k(x) + u_{k+1}\hat{r}_k(x) + P_{n-k-1} \tag{A.10}$$

which gives that $G_k(x)$ is given by

$$G_k(x) = \left(\frac{\hat{b}_{k+1}}{\hat{h}_{k+1}}\right)(\widehat{p}_{k+2}\widehat{q}_{k+1}\hat{r}_{k+1}(x) - u_{k+1}\hat{r}_k(x) - P_{n-k-1}) \tag{A.11}$$

Inserting (A.11) into (A.9) and shifting from $k+1$ to k, we arrive at (5.1) as desired. Using the assumptions and $g_k \neq 0$ for each k, Theorem 5.1 implies the result.

For the case of a polynomial system R where $g_j = 0$ for some j, note that the coefficients of the polynomials generated by the two-term recurrence relations (8.7) depend continuously on the entries of the 2×2 transfer matrix. Let $\{\epsilon_k\}$ be a sequence tending to zero with $\epsilon_k \neq 0$ for each k, and consider a sequence of systems of polynomials R_k with $g_j = \epsilon_k$ for each j such that $g_j = 0$ in the original polynomial system R, and all other generators the same as in R. Then the result of the theorem holds for the system R_k for every k by above, and $R_k \to R$, so by continuity, the result must hold for R as well. This completes the proof. □

Proof of Theorem 5.3. The recurrence relations (5.8) define a system of polynomials which satisfy the n-term recurrence relations

$$\widehat{r}_k(x) = (\alpha_k x - a_{k-1,k})\cdot\widehat{r}_{k-1}(x) - a_{k-2,k}\cdot\widehat{r}_{k-2}(x) - \ldots - a_{0,k}\cdot\widehat{r}_0(x) \tag{A.12}$$

for some coefficients $\alpha_k, a_{k-1,k}, \ldots, a_{0,k}$. The proof is presented by showing that these n-term recurrence relations in fact coincide exactly with (4.8), so these coefficients coincide with those of the n-term recurrence relations of the associated polynomials $\widehat{R}$; that is,

$$\alpha_k = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}, \quad a_{k-1,k} = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\widehat{d}_k, \quad a_{0,k} = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left(\widehat{g}_1\widehat{b}^{\times}_{1,k}\widehat{h}_k - \frac{P_{n-k}}{P_n}\right) \tag{A.13}$$

$$a_{j,k} = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\widehat{g}_{j+1}\widehat{b}^{\times}_{j+1,k}\widehat{h}_k, \quad j = 1,\ldots,k-2$$

Using relations for $\widehat{r}_k(x)$ and $\widehat{F}_{k-1}(x)$ from (5.8), we have

$$\widehat{r}_k(x) = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left[(x-\widehat{d}_k)\widehat{r}_{k-1}(x) - \widehat{g}_{k-1}\widehat{h}_k\widehat{r}_{k-2}(x) + \widehat{h}_k\widehat{p}_{k-1}\widehat{b}_{k-1}\widehat{F}_{k-2}(x) + \frac{P_{n-k}}{P_n}\widehat{r}_0(x)\right]. \tag{A.14}$$

Notice that again using (5.8) to eliminate $\widehat{F}_{k-2}(x)$ from the equation (A.14) will result in an expression for $\widehat{r}_k(x)$ in terms of $\widehat{r}_{k-1}(x), \widehat{r}_{k-2}(x), \widehat{r}_{k-3}(x), \widehat{F}_{k-3}(x)$, and $\widehat{r}_0(x)$ without modifying the coefficients of $\widehat{r}_{k-1}(x), \widehat{r}_{k-2}(x)$, or $\widehat{r}_0(x)$. Again applying (5.8) to eliminate $\widehat{F}_{k-3}(x)$ results in an expression in terms of $\widehat{r}_{k-1}(x), \widehat{r}_{k-2}(x), \widehat{r}_{k-3}(x), \widehat{r}_{k-4}(x), \widehat{F}_{k-4}(x)$, and $\widehat{r}_0(x)$ without modifying the coefficients of $\widehat{r}_{k-1}(x), \widehat{r}_{k-2}(x), \widehat{r}_{k-3}(x)$, or $\widehat{r}_0(x)$. Continuing in this way, the n-term recurrence relations of the form (A.12) are obtained without modifying the coefficients of the previous ones.

Suppose that for some $0 < j < k-1$ the expression for $\widehat{r}_k(x)$ is of the form

$$\begin{aligned}\widehat{r}_k(x) \quad = \quad & \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\Big[(x-\widehat{d}_k)\widehat{r}_{k-1}(x) - \widehat{g}_{k-1}\widehat{h}_k\widehat{r}_{k-2}(x) - \cdots - \widehat{g}_{j+1}\widehat{b}^{\times}_{j+1,k}\widehat{h}_k\widehat{r}_j(x) \\ & +\widehat{p}_{j+1}\widehat{b}^{\times}_{j,k}\widehat{h}_k\widehat{F}_j(x) + \frac{P_{n-k}}{P_n}\widehat{r}_0(x)\Big].\end{aligned} \tag{A.15}$$

Using (5.8) for $\widehat{F}_j(x)$ gives the relation

$$\widehat{F}_j(x) = \frac{1}{\widehat{p}_{j+1}\widehat{q}_j}\left(\widehat{q}_j\widehat{p}_j\widehat{b}_j\widehat{F}_{j-1}(x) - \widehat{q}_j\widehat{g}_j\widehat{r}_{j-1}(x)\right) \tag{A.16}$$

Inserting (A.16) into (A.15) gives

$$\begin{aligned}\widehat{r}_k(x) &= \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\Big[(x-\widehat{d}_k)\widehat{r}_{k-1}(x) - \widehat{g}_{k-1}\widehat{h}_k\widehat{r}_{k-2}(x) - \cdots - \widehat{g}_j\widehat{b}^{\times}_{j,k}\widehat{h}_k\widehat{r}_{j-1}(x) \\ &\quad + \widehat{p}_j\widehat{b}^{\times}_{j-1,k}\widehat{h}_k\widehat{F}_{j-1}(x) + \frac{P_{n-k}}{P_n}\widehat{r}_0(x)\Big].\end{aligned} \tag{A.17}$$

Therefore since (A.14) is the case of (A.15) for $j=k-2$, (A.15) is true for each $j=k-2,k-3,\ldots,0$, and for $j=0$, using the fact that $\widehat{F}_0=0$ we have

$$\widehat{r}_k(x) = \frac{1}{\widehat{p}_{k+1}\widehat{q}_k}\left[(x-\widehat{d}_k)\widehat{r}_{k-1}(x) - \widehat{g}_{k-1}\widehat{h}_k\widehat{r}_{k-2}(x) - \cdots - \widehat{g}_1\widehat{b}^{\times}_{1,k}\widehat{h}_k\widehat{r}_0(x) + \frac{P_{n-k}}{P_n}\widehat{r}_0(x)\right] \tag{A.18}$$

Since these coefficients coincide with those in (A.13) that are satisfied by the associated polynomials, the polynomials given by (5.8) must coincide with the associated polynomials. This proves the theorem. □

Proof of Lemma 8.2. Let $A=(A_{i,j})$ be a irreducible Hessenberg $(1,1)$-quasiseparable matrix with generator representation $\{p_k,q_k,d_k,g_k,b_k,h_k\}$ as in Definition 4.3, and suppose first that A is well-free. We will construct a set of generators $\{p_k,q_k,d_k,g'_k,b'_k,h'_k\}$ that generate A and $h'_k\neq 0$ for each k. Define

$$g'_k = \begin{cases} g_k & \text{if } h_{k+1}\neq 0 \\ 0 & \text{if } h_{k+1}=0\end{cases}, k=1,\ldots,n-1$$

$$b'_k = \begin{cases} b_k & \text{if } h_{k+1}\neq 0 \\ 0 & \text{if } h_{k+1}=0\end{cases}, k=2,\ldots,n-1$$

$$h'_k = \begin{cases} h_k & \text{if } h_k\neq 0 \\ 1 & \text{if } h_k=0\end{cases}, k=2,\ldots,n$$

Since A is well-free, these modifications to the elements g_k and b_k do not change the resulting matrix, and thus the set $\{p_k,q_k,d_k,g'_k,b'_k,h'_k\}$ also generates A, and all h elements are nonzero as desired.

On the other hand, suppose there exists a choice of generators such that $h_k\neq 0$ for $k\in[2,n]$. Then if an element $A_{i,j}=0$ for $1\leqslant i<j\leqslant n$, it follows that at least one element of $\{g_i,b_{i+1},b_{i+1},\ldots,b_{j-1}\}$ must equal zero. But each of these elements are also present in the expressions for $A_{i,k}$ for $k=j+1,\ldots,n$, and hence $A_{i,k}=0$ for $k=j+1,\ldots,n$ as well. Thus A cannot contain a well, and this completes the proof. □

Proof of Theorem 8.6. The specified generators in conjunction with the general n-term recurrence relations (4.2) give

$$\begin{aligned} r_k(x) &= (\delta_k x+\theta_k+\gamma_k\beta_{k-1})r_{k-1}(x) + \gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})\beta_{k-2}r_{k-2}(x) \\ &\quad +\gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})(\alpha_{k-2}-\beta_{k-2}\gamma_{k-2})\beta_{k-3}r_{k-3}(x)+\cdots+ \\ &\quad +\gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})(\alpha_{k-2}-\beta_{k-2}\gamma_{k-2})\cdots(\alpha_2-\beta_2\gamma_2)\beta_1 r_1(x)+ \\ &\quad +\gamma_k(\alpha_{k-1}-\beta_{k-1}\gamma_{k-1})(\alpha_{k-2}-\beta_{k-2}\gamma_{k-2})\cdots(\alpha_2-\beta_2\gamma_2)(\alpha_1-\beta_1\gamma_1)r_0(x) \end{aligned} \tag{A.19}$$

The proof is presented by showing that the polynomial system satisfying the two-term recurrence relations also satisfies these n-term recurrence relations. By applying the given two-term recursion, we have

$$\begin{bmatrix} G_1(x) \\ r_1(x)\end{bmatrix} = \begin{bmatrix} \beta_1 r_1(x)+\alpha_1-\beta_1\gamma_1 \\ \delta_1 x+\theta_1+\gamma_1\end{bmatrix}$$

and

$$\begin{bmatrix} G_2(x) \\ r_2(x)\end{bmatrix} = \begin{bmatrix} (\beta_2\delta_2 x+\beta_2\theta_2+\alpha_2\beta_1)r_1(x)+\alpha_2(\alpha_1-\beta_1\gamma_1) \\ (\delta_2 x+\theta_2+\gamma_2\beta_1)r_1(x)+\gamma_2(\alpha_1-\beta_1\gamma_1)\end{bmatrix} \tag{A.20}$$

giving the result for $k=1,2$. From (A.20), we have

$$\delta_2 x r_1(x) = r_2(x) - (\theta_2+\gamma_2\beta_1)r_1(x) - \gamma_2(\alpha_1-\beta_1\gamma_1)$$

and inserting this into the expression for $r_3(x)$ of the form

$$r_3(x) = \gamma_3 G_2(x) + (\delta_3 x+\theta_3)r_2(x)$$

yields (A.19) for $k=3$. Continuing in this fashion, the result follows. □

Proof of Lemma 8.9. Let $A = (A_{i,j})$ be an irreducible Hessenberg $(1,1)$-quasiseparable matrix with generator representation $\{p_k, q_k, d_k, g_k, b_k, h_k\}$ as in Definition 4.3, and suppose first that A is DPSS; in particular, $A = B + \mathrm{triu}(A_U)$ where A_U is rank-one. If $b_k = 0$ for some k, then since b_k is a factor of the elements $A_{i,j}$ for $k \in [i+1, j-1]$, we necessarily must have the upper right block $A(1:k-1, k+1:n) = 0$.[3] Since A_U is rank-one, $A(1:k-1, k+1:n) = 0$ implies that both $A(k:n, k+1:n) = 0$ and $A(1:k-1, 1:k) = 0$ as well. Hence $\mathrm{triu}(A_U) = 0$, and hence if we set $g'_k = 0$ for $k = 1, \ldots, n-1$ and $b'_k = 1$ for $k = 2, \ldots, n-1$, then the set $\{p_k, q_k, d_k, g'_k, b'_k, h_k\}$ also generates A.

Conversely, suppose the generators of A are such that $b_k \neq 0$ for $k = 2, \ldots, n-1$. Then the matrices

$$A_U = \begin{cases} g_i b^{\times}_{i,j} h_j & \text{if } 1 \leqslant i < j \leqslant n \\ g_i \dfrac{1}{b_i} h_i & \text{if } 1 < i = j < n \\ g_i \dfrac{1}{b^{\times}_{j-1,i+1}} h_j & \text{if } 1 < j < i < n \\ 0 & \text{if } j = 1 \text{ or } i = n \end{cases} \qquad B = \begin{cases} d_i & \text{if } 1 \leqslant i = j \leqslant n \\ p_i q_j & \text{if } 1 \leqslant i+1 = j \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

are well defined, $\mathrm{rank}(A_U) = 1$, B is lower bidiagonal, and $A = B + \mathrm{triu}(A_U)$. □

Proof of Theorem 8.12. Inserting the specified choice of generators into the general n-term recurrence relations (4.2), we arrive at

$$\begin{aligned} r_k(x) &= (\delta_k x + \theta_k) r_{k-1}(x) + \gamma_k \beta_{k-1} r_{k-2}(x) + \gamma_k \alpha_{k-1} \beta_{k-2} r_{k-3}(x) \\ &\quad + \gamma_k \alpha_{k-1} \alpha_{k-2} \beta_{k-3} r_{k-4}(x) + \cdots + \gamma_k \alpha_{k-1} \cdots \alpha_2 \beta_1 r_0(x) \end{aligned} \tag{A.21}$$

It suffices to show that the polynomial system satisfying the two-term recurrence relations also satisfies these n-term recurrence relations. Beginning with

$$r_k(x) = \gamma_k G_{k-1}(x) + (\delta_k x + \theta_k) r_{k-1}(x) \tag{A.22}$$

and using the relation

$$G_{k-1}(x) = \alpha_{k-1} G_{k-2}(x) + \beta_{k-1} r_{k-2}(x)$$

(A.22) becomes

$$r_k(x) = \gamma_k \alpha_{k-1} G_{k-2}(x) + \gamma_k \beta_{k-1} r_{k-2}(x) + (\delta_k x + \theta_k) r_{k-1}(x)$$

and, using a similar argument as in the proof of Theorem 5.3, we continue this procedure to obtain n-term recurrence relations. It can easily be checked that this procedure yields exactly (A.21). □

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[3]The MATLAB notation $A(i:j, k:l)$ denotes the submatrix obtained from rows $i, i+1, \ldots, j$ and columns $k, k+1, \ldots, l$.

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