A fast Björck-Pereyra-like algorithm for solving quasiseparable-Hessenberg-Vandermonde systems

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Abstract. In this paper we derive a fast $O(n^2)$ algorithm for solving linear systems where the coefficient matrix is a polynomial-Vandermonde $V_R(x) = [r_{j-1}(x_i)]$ matrix with polynomials R related to a quasiseparable matrix. The result is a generalization of the well-known Björck-Pereyra algorithm for classical Vandermonde systems. As will be shown, many important systems of polynomials are related to quasiseparable matrices in this way, and thus this result also generalizes the algorithms derived in [RO91] for Chebyshev polynomials, [H90] for real orthogonal polynomials, and [BEGKO07] for Szegö polynomials. Numerical experiments are presented comparing the algorithm to standard structure-ignoring methods.

1. Introduction

The fact that the n^2 entries of Vandermonde matrices $V(x) = \left[\ x_i^{j-1}\ \right]$ are determined by only n parameters allows the design of fast algorithms, typically of an order of magnitude reduction in complexity over standard algorithms. Specifically, given a linear system $V(x)a = f$, an algorithm due to Björck and Pereyra [BP70] can solve the system in $O(n^2)$ operations by a factorization of $V(x)^{-1}$ as a product of bidiagonal factors. This is as opposed to the $O(n^3)$ operations required by Gaussian elimination.

More generally, for a given system of polynomials $R = \{r_0(x), r_1(x), \ldots, r_{n-1}(x)\}$, we can consider the corresponding *polynomial-Vandermonde* matrix $V_R(x) = \left[\ r_{j-1}(x_i)\ \right]$, given by

$$V_R(x) = \begin{bmatrix} r_0(x_1) & r_1(x_1) & \cdots & r_{n-1}(x_1) \\ r_0(x_2) & r_1(x_2) & \cdots & r_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ r_0(x_n) & r_1(x_n) & \cdots & r_{n-1}(x_n) \end{bmatrix}. \tag{1.1}$$

The reduction in complexity of fast algorithms for Vandermonde matrices attracted much attention in numerical linear algebra literature, and the Björck-Pereyra algorithm has been generalized for several important classes of polynomials $\{r_k(x)\}$, notably Chebyshev polynomials (resulting in Chebyshev-Vandermonde matrices) in [RO91], polynomials orthogonal on a real interval (resulting in three-term Vandermonde matrices) in [H90], and Szegö polynomials (resulting in Szegö-Vandermonde matrices) in [BEGKO07].

We next define the confederate matrix of a polynomial with respect to a given system of polynomials. Let polynomials $R = \{r_0(x),\ r_1(x)\ ,\ldots,\ r_n(x)\}$ be specified by the general recurrence relations

$$r_k(x) = (\alpha_k x - a_{k-1,k}) \cdot r_{k-1}(x) - a_{k-2,k} \cdot r_{k-2}(x) - \ldots - a_{0,k} \cdot r_0(x). \tag{1.2}$$

Following [MB79], define for the polynomial

$$\beta(x) = \beta_0 \cdot r_0(x) + \beta_1 \cdot r_1(x) + \ldots + \beta_{n-1} \cdot r_{n-1}(x) + r_n(x) \tag{1.3}$$

its *confederate* matrix

$$C_R(\beta) = \underbrace{\begin{bmatrix} \frac{a_{01}}{\alpha_1} & \frac{a_{02}}{\alpha_2} & \frac{a_{03}}{\alpha_3} & \cdots & \frac{a_{0,k}}{\alpha_k} & \cdots & \cdots & \frac{a_{0,n}}{\alpha_n} \\ \frac{1}{\alpha_1} & \frac{a_{12}}{\alpha_2} & \frac{a_{13}}{\alpha_3} & \cdots & \frac{a_{1,k}}{\alpha_k} & \cdots & \cdots & \frac{a_{1,n}}{\alpha_n} \\ 0 & \frac{1}{\alpha_2} & \frac{a_{23}}{\alpha_3} & \cdots & \vdots & \cdots & \cdots & \frac{a_{2,n}}{\alpha_n} \\ 0 & 0 & \frac{1}{\alpha_3} & \ddots & \frac{a_{k-2,k}}{\alpha_k} & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{a_{k-1,k}}{\alpha_k} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \frac{1}{\alpha_k} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} \end{bmatrix}}_{C_R(r_n)} - \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \\ \vdots \\ \vdots \\ \beta_{n-1} \end{bmatrix} \begin{bmatrix} 0 & 0 & \cdots & 0 & 1/\alpha_n \end{bmatrix} \quad (1.4)$$

with respect to the polynomial system R. Notice that the coefficients of the recurrence relations for the k^{th} polynomial $r_k(x)$ from (1.2) are contained in the k^{th} column of $C_R(r_n)$, as the highlighted column shows.

We refer to [MB79] for many useful properties of the confederate matrix and only recall here that $\det(xI - C_R(\beta)) = \beta(x)/(\alpha_0 \cdot \alpha_1 \cdots \alpha_n)$, and that similarly, the characteristic polynomial of the $k \times k$ leading submatrix of $C_R(\beta)$ is equal to $r_k(x)/\alpha_0 \cdot \alpha_1 \cdot \ldots \cdot \alpha_k$. The motivation for introducing confederate matrices can be seen in Table 1, where detailed examples of confederate matrices corresponding to several important special cases of systems of polynomials are presented.

A matrix A is called upper quasiseparable of rank n_U if $\max(\text{rank} A_{12}) = n_U$ where the maximum is taken over all symmetric partitions of the form

$$A = \left[\begin{array}{c|c} * & A_{12} \\ \hline * & * \end{array}\right] \quad (1.5)$$

Recall that a matrix $A = [a_{ij}]$ is called *upper Hessenberg* if all entries below the first subdiagonal are zeros; that is, $a_{ij} = 0$ if $i > j + 1$. Polynomials corresponding to rank n_U upper quasiseparable Hessenberg matrices are the Hessenberg quasiseparable polynomials, or H-q/s polynomials. To emphasize the order of quasiseparability of the corresponding quasiseparable matrix, we write H-$(1, n_U)$-q/s polynomials. The 1 refers to the lower order of quasiseparability, which is always one in the Hessenberg case.

It is important to note that this subclass contains all of the previous classes listed in Table 1 as special cases. This can be seen by presenting a typical submatrix A_{12} from the partition described in (1.5) for each confederate matrix. For instance, if A is tridiagonal, then the submatrix A_{12} has the form

$$A_{12} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \frac{\delta_k}{\alpha_k} & 0 & \cdots & 0 \end{bmatrix}$$

which can easily be observed to have rank one. If the matrix A is $(1, j)$-banded, that is, has only one nonzero subdiagonal and j nonzero superdiagonals, then a typical submatrix A_{12} has the form

$$A_{12} = \begin{bmatrix} 0 & & \cdots & \cdots & 0 \\ \vdots & & & & \\ 0 & & & & \vdots \\ \frac{a_{k-j-1,k}}{\alpha_k} & \ddots & & & \vdots \\ \vdots & \ddots & \ddots & & \\ \frac{a_{k-2,k}}{\alpha_k} & \cdots & \frac{a_{k-2,k+j-1}}{\alpha_{k+j-1}} & 0 & \cdots & 0 \end{bmatrix}$$

where the "height" and "width" of the triangular nonzero pattern are both j. Thus it is easy to see that the matrix A_{12} has rank j. Finally, if A corresponds to the Szegö polynomials, then the corresponding submatrix

Recurrence Relations of R	Confederate matrix $C_R(r_n)$
$r_k(x) = x \cdot r_{k-1}(x)$ Monomials	$\begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$ Companion matrix
$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k r_{k-2}(x)$ Real orthogonal polynomials	$\begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} & \ddots & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & \ddots & \ddots & \frac{\delta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} \end{bmatrix}$ Tridiagonal matrix
$\begin{aligned} r_k(x) = (\alpha_k x - a_{k-1,k}) \cdot r_{k-1}(x) \\ -a_{k-2,k} \cdot r_{k-2}(x) \\ \vdots \\ -a_{k-j,k} \cdot r_{k-j}(x) \end{aligned}$ $j+1$-recurrent polynomials	$\begin{bmatrix} \frac{a_{0,1}}{\alpha_1} & \cdots & \frac{a_{0,j}}{\alpha_j} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{a_{1,2}}{\alpha_2} & \cdots & \frac{a_{1,j+1}}{\alpha_{j+1}} & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & & & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \frac{a_{n-j,n}}{\alpha_n} \\ \vdots & & \ddots & \frac{1}{\alpha_{n-2}} & & \vdots \\ 0 & \cdots & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{a_{n-1,n}}{\alpha_n} \end{bmatrix}$ $(1,j)$-banded matrix
$\begin{aligned} r_k(x) = [\tfrac{1}{\mu_k} \cdot x + \tfrac{\rho_k}{\rho_{k-1}} \tfrac{1}{\mu_k}] r_{k-1}(x) \\ - \tfrac{\rho_k}{\rho_{k-1}} \tfrac{\mu_{k-1}}{\mu_k} \cdot x \cdot r_{k-2}(x) \end{aligned}$ Szegö polynomials	$\begin{bmatrix} -\rho_1\rho_0^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\cdots\mu_1\rho_0^* \\ \mu_1 & \ddots & -\rho_{n-1}\mu_{n-2}\cdots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\cdots\mu_2\rho_1^* \\ 0 & \ddots & \vdots & \vdots \\ \vdots & & -\rho_{n-1}\rho_{n-2}^* & -\rho_n\mu_{n-1}\rho_{n-2}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n\rho_{n-1}^* \end{bmatrix}$ Unitary Hessenberg matrix

TABLE 1. Polynomial systems and corresponding confederate matrices.

A_{12} has the form

$$A_{12} = \begin{bmatrix} -\rho_k\mu_{k-1}\cdots\mu_3\mu_2\mu_1\rho_0^* & -\rho_{k-1}\mu_{k-2}\cdots\mu_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\mu_2\mu_1\rho_0^* \\ -\rho_k\mu_{k-1}\cdots\mu_3\mu_2\rho_1^* & -\rho_{k-1}\mu_{k-2}\cdots\mu_3\mu_2\rho_1^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\mu_2\rho_1^* \\ -\rho_k\mu_{k-1}\cdots\mu_3\rho_2^* & -\rho_{k-1}\mu_{k-2}\cdots\mu_3\rho_2^* & \cdots & -\rho_n\mu_{n-1}\cdots\mu_3\rho_2^* \end{bmatrix},$$

which is also rank 1.

These three special cases of quasiseparable matrices imply that the corresponding systems of polynomials are special cases of H-q/s polynomials. That is, real orthogonal polynomials, $j+1$-recurrent polynomials, and Szegö polynomials are all special cases of H-q/s polynomials.

Furthermore, tridiagonal matrices and unitary Hessenberg matrices are quasiseparable of order one, so both are special cases of the more restrictive class of H-$(1,1)$-q/s polynomials. The $(1,j)$-banded matrices are quasiseparable of order j, and the higher ranks involved in this special case mean the polynomials are not H-$(1,1)$-q/s polynomials, but instead are H-$(1,j)$-q/s polynomials.

Table 2 contains a listing of previous work in fast $O(n^2)$ algorithms for polynomial-Vandermonde matrices, including generalizations of the Björck-Pereyra algorithm for solving linear systems and generalizations of the Traub algorithm [T66] for inversion.

Matrix	Polynomials	Confederate matrix	Fast system solver	Fast inversion alg.
Classical Vandermonde	monomials	companion matrix	Björck-Pereyra [BP70]	Traub [T66]
Chebyshev-Vandermonde	Chebyshev polynomials	tridiagonal matrix	Reichel-Opfer [RO91]	Gohberg-Olshevsky [GO94]
Three-Term Vandermonde	Real orthogonal polynomials	tridiagonal matrix	Higham [H90]	Calvetti-Reichel [CR93]
Szegö-Vandermonde	Szegö polynomials	unitary Hessenberg matrix	[BEGKO07]	Olshevsky [O01]

TABLE 2. Fast $O(n^2)$ algorithms for polynomial-Vandermonde matrices & systems.

The problem considered in this paper is, given a polynomial-Vandermonde matrix $V_R(x)$ corresponding to a system of H-q/s polynomials R and a right-hand-side vector f, to solve the corresponding linear system $V_R(x)a = f$. In the next section the classical Björck-Pereyra algorithm is presented. We begin by recalling the classical Björck-Pereyra algorithm in the next section.

2. The classical Björck-Pereyra algorithm

In [BP70], the authors derive a representation for the inverse $V(x)^{-1}$ of an $n \times n$ Vandermonde matrix as the product of bidiagonal matrices, that is,

$$V(x)^{-1} = U_1 \cdots U_{n-1} \cdot \tilde{L}_{n-1} \cdots \tilde{L}_1$$

and used this result to solve the linear system $V(x)a = f$ by computing the solution vector

$$a = U_1 \cdots U_{n-1} \cdot \tilde{L}_{n-1} \cdots \tilde{L}_1 f$$

which solves the linear system in $\frac{5}{2}n^2$ operations. This is an order of magnitude improvement over Gaussian elimination, which is well known to require $O(n^3)$ operations in general. This favorable complexity results from the fact that the matrices U_k and L_k are sparse. More specifically, the factors U_k and L_k are given by

$$U_k = \left[\begin{array}{c|ccccc} I_{k-1} & & & & & \\ \hline & 1 & -x_k & & & \\ & & 1 & \ddots & & \\ & & & \ddots & -x_k & \\ & & & & 1 & \end{array}\right], \tag{2.1}$$

$$\tilde{L}_k = \left[\begin{array}{c|ccc} I_k & & & \\ \hline & \frac{1}{x_{k+1}-x_1} & & \\ & & \ddots & \\ & & & \frac{1}{x_n - x_{n-k}} \end{array}\right] \cdot \left[\begin{array}{c|cccc} I_{k-1} & & & & \\ \hline & 1 & & & \\ & -1 & 1 & & \\ & & \ddots & \ddots & \\ & & & -1 & 1 \end{array}\right]. \tag{2.2}$$

In the next section we will present the new Björck-Pereyra-like algorithm for the most general case considered in this paper: the general Hessenberg case.

3. New Björck-Pereyra-like algorithm. General Hessenberg case.

In this section we consider the linear system $V_R(x)a = f$, where $V_R(x)$ is the polynomial-Vandermonde matrix corresponding to the polynomial system R and the n distinct nodes x. No restrictions are placed on the polynomial system R at this point other than $\deg(r_k) = k$. The new algorithm is based on a decomposition of the inverse $V_R(x)^{-1}$ enabled by the following lemma.

Lemma 3.1. *Let $R = \{r_0(x), \ldots, r_n(x)\}$ be an arbitrary system of polynomials as in (1.2), and denote $R_1 = \{r_0(x), \ldots, r_{n-1}(x)\}$. Further let $x_{1:n} = (x_1, ..., x_n)$ be n distinct points. Then the inverse of $V_R(x_{1:n})$ admits a decomposition*

$$V_R(x_{1:n})^{-1} = U_1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & V_R(x_{2:n})^{-1} \end{bmatrix} L_1, \tag{3.1}$$

with

$$U_1 = \left[\begin{array}{c|ccc} \frac{1}{\alpha_0} & & & \\ 0 & & & \\ & & C_{R_1}(r_{n-1}) - x_1 I & \\ \vdots & & & \\ \hline 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} \end{array}\right], \tag{3.2}$$

$$L_1 = \begin{bmatrix} 1 & & & \\ & \frac{1}{x_2 - x_1} & & \\ & & \ddots & \\ & & & \frac{1}{x_n - x_1} \end{bmatrix} \begin{bmatrix} 1 & & & \\ -1 & 1 & & \\ \vdots & & \ddots & \\ -1 & & & 1 \end{bmatrix}. \tag{3.3}$$

The proof of Lemma 3.1 is given in Section 6, but first we present its use in solving the linear system $V_R(x)a = f$.

3.1. Solving polynomial Vandermonde systems

Like the classical Björk-Pereyra algorithm, the recursive nature of (3.1) allows a decomposition of $V_R(x)^{-1}$ into $2n - 2$ factors,

$$V_R(x)^{-1} = U_1 \cdot \left[\begin{array}{c|c} I_1 & \\ \hline & U_2 \end{array}\right] \cdots \left[\begin{array}{c|c} I_{n-2} & \\ \hline & U_{n-1} \end{array}\right] \cdot \left[\begin{array}{c|c} I_{n-2} & \\ \hline & L_{n-1} \end{array}\right] \cdots \left[\begin{array}{c|c} I_1 & \\ \hline & L_2 \end{array}\right] \cdot L_1, \tag{3.4}$$

with the lower and upper triangular factors given in (3.2), (3.3). The associated linear system can be solved by multiplying (3.4) by the right-hand side vector f.

It is emphasized that this decomposition is valid for any polynomial system R, however no computational savings are guaranteed. In order to have the desired computational savings, each multiplication of a matrix from (3.4) by a vector must be performed quickly.

The factors L_k are sparse as in the classical Björck-Pereyra algorithm, and thus multiplication by them is fast. However, unlike the classical Björck-Pereyra algorithm, the factors U_k are not sparse in general. In order to have a fast $O(n^2)$ algorithm for solving the system $V_R(x)a = f$, it is necessary to be able to multiply each matrix in (3.4) by f in $O(n)$ operations.

3.2. Differences between L_k and $\tilde{L}_k$

Note that there is a difference between the factors $\tilde{L}_k$ in (2.2) and the factors L_k in (3.3). It is easy to see that both formulas are correct because of the uniqueness of the L factor in the LU-factorization. That is, the matrices (2.2) may be used in place of the matrices (3.3).

Furthermore, our choice of L_k is not arbitrary; the presence of the factors $(x_2 - x_1)^{-1}, \ldots, (x_n - x_1)^{-1}$ suggests that the Leja ordering of the nodes may produce better numerical results ([RO91], [H90], [O03]). More details are given in Section 7.

4. Known special cases where the Björck-Pereyra-like algorithm is fast.

We next present a detailed reduction of the algorithm presented in the previous section in several important special cases, achieving the same results.

4.1. Monomials. The classical Björck-Pereyra algorithm.

Suppose the system of polynomials in the polynomial-Vandermonde matrix is simply a set of monomials; that is, $R = \{1, x, \dots, x^{n-1}, x^n\}$. Then (1.2) becomes simply

$$r_0(x) = 1, \quad r_k(x) = x r_{k-1}(x), \quad k = 1, \dots, n \tag{4.5}$$

and the corresponding confederate matrix is

$$C_R(r_n) = \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}; \tag{4.6}$$

that is, $\alpha_k = 1$ for $k = 0, \dots, n-1$. Inserting this and (4.6) into (3.2) yields (2.1), implying the factors U_k reduce to those of the classical Björck-Pereyra algorithm in this case. That is, the Vandermonde linear system $V_R(x)a = f$ can be solved via the factorization of (3.4):

$$a = V_R(x)^{-1} f = U_1 \cdot \left[\begin{array}{c|c} I_1 & \\ \hline & U_2 \end{array}\right] \cdots \left[\begin{array}{c|c} I_{n-2} & \\ \hline & U_{n-1} \end{array}\right] \cdot \left[\begin{array}{c|c} I_{n-2} & \\ \hline & L_{n-1} \end{array}\right] \cdots \left[\begin{array}{c|c} I_1 & \\ \hline & L_2 \end{array}\right] \cdot L_1 \cdot f. \tag{4.7}$$

Thus when the quasiseparable polynomials in question reduce to the monomials, the algorithm described reduces to the classical Björck-Pereyra algorithm. Due to the sparseness of the matrices involved, the overall cost of the algorithm is only $\frac{5}{2}n^2$.

4.2. Real orthogonal polynomials. The Higham algorithm.

If the polynomial system under consideration satisfies the three-term recurrence relations

$$r_k(x) = (\alpha_k x - \delta_k) r_{k-1}(x) - \gamma_k r_{k-2}(x), \tag{4.8}$$

then the resulting confederate matrix is tridiagonal

$$C_R(r_n) = \begin{bmatrix} \frac{\delta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 \\ \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} & \ddots & \ddots & \vdots \\ 0 & \frac{1}{\alpha_2} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & & \ddots & \frac{\delta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\delta_n}{\alpha_n} \end{bmatrix} \tag{4.9}$$

which leads to the matrices U_k of the form

$$U_k = \left[\begin{array}{c|cccccc} \frac{1}{\alpha_0} & \frac{\delta_1}{\alpha_1} - x_k & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & & 0 \\ 0 & \frac{1}{\alpha_1} & \frac{\delta_2}{\alpha_2} - x_k & \ddots & \ddots & & \vdots \\ & 0 & \frac{1}{\alpha_2} & \ddots & \frac{\gamma_{k-1}}{\alpha_{k-1}} & & 0 \\ \vdots & \vdots & & \ddots & \frac{\delta_{k-1}}{\alpha_{k-1}} - x_k & & \frac{\gamma_k}{\alpha_k} \\ & 0 & \cdots & 0 & \frac{1}{\alpha_{k-1}} & & \frac{\delta_k}{\alpha_k} - x_k \\ \hline 0 & & \cdots & & 0 & & \frac{1}{\alpha_{n-j}} \end{array}\right]$$

Again, the factorization (4.7) uses the above to solve the linear system. The sparseness of these matrices allows computational savings, and the overall cost of the algorithm is again $O(n^2)$.

In this case, the entire algorithm presented reduces to Algorithm 2.1 in [H90]. In particular, the multiplication of a vector by the matrices specified in (4.7) involving L_k and U_k can be seen as Stage I and Stage II in that algorithm, respectively.

4.3. Szegö polynomials. The [BEGKO07] algorithm.

If the system of quasiseparable polynomials are the Szegö polynomials $\Phi^{\#} = \{\phi_0^{\#}(x), \ldots, \phi_n^{\#}(x)\}$ represented by the reflection coefficients ρ_k and complimentary parameters μ_k (see [BC92]), then they satisfy the recurrence relations

$$\phi_0^{\#}(x) = 1, \quad \phi_1^{\#}(x) = \frac{1}{\mu_1} \cdot x\phi_0^{\#}(x) - \frac{\rho_1}{\mu_1}\phi_0^{\#}(x)$$

$$\phi_k^{\#}(x) = \left[\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}\right]\phi_{k-1}^{\#}(x) - \frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k} \cdot x \cdot \phi_{k-2}^{\#}(x) \tag{4.10}$$

which are known to be associated to the almost unitary Hessenberg matrix

$$C_{\Phi^{\#}}(\phi_n^{\#}) = \begin{bmatrix} -\rho_1\rho_0^* & \cdots & -\rho_{n-1}\mu_{n-2}\cdots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\cdots\mu_1\rho_0^* \\ \mu_1 & \ddots & -\rho_{n-1}\mu_{n-2}\cdots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\cdots\mu_2\rho_1^* \\ 0 & \ddots & \vdots & \vdots \\ \vdots & & -\rho_{n-1}\rho_{n-2}^* & -\rho_n\mu_{n-1}\rho_{n-2}^* \\ 0 & \cdots & \mu_{n-1} & -\rho_n\rho_{n-1}^* \end{bmatrix} \tag{4.11}$$

In particular, if (4.11) is inserted into the factors (3.2) in (4.7), then the result is exactly that derived in [BEGKO07, (3.10) and (3.15)], where the nice properties of the matrix $C_{\Phi^{\#}}(\phi_n^{\#})$ were used to provide a computational speedup. Specifically, the algorithm is made fast by the factorization

$$C_{\Phi^{\#}}(\phi_n^{\#}) = G(\rho_1) \cdot G(\rho_2) \cdot \ldots \cdot G(\rho_{n-1}) \cdot \tilde{G}(\rho_n), \tag{4.12}$$

where

$$G(\rho_j) = \operatorname{diag}\{I_{j-1}, \begin{bmatrix} \rho_j & \mu_j \\ \mu_j & -\rho_j^* \end{bmatrix}, I_{n-j-1}\}, \quad j = 1, 2, \ldots, n-1$$

and

$$\tilde{G}(\rho_n) = \operatorname{diag}\{I_{n-1}, \rho_n\}$$

see, for instance, [G82], [BC92], or [R95]. This gives an overall computational cost of $O(n^2)$.

In the next section, we present a new special case which contains all previous special cases.

5. A new special case. Hessenberg-quasiseparable polynomials.

5.1. Rank definition.

As stated in the introduction, a matrix A is called upper quasiseparable of order n_U if $\max(\operatorname{rank} A_{12}) = n_U$ where the maximum is taken over all symmetric partitions of the form

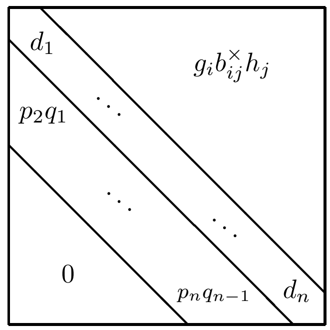
$$A = \left[\begin{array}{c|c} * & A_{12} \\ \hline * & * \end{array}\right]$$

Also, a matrix $A = [a_{ij}]$ is called *upper Hessenberg* if all entries below the first subdiagonal are zeros; that is, $a_{ij} = 0$ if $i > j + 1$. Polynomials corresponding to order n_U upper quasiseparable Hessenberg matrices are H-$(1, n_U)$-q/s polynomials, or simply H-q/s polynomials.

5.2. Generator definition.

We next present an equivalent definition of an order n_U upper quasiseparable Hessenberg matrix in terms of its *generators*. The equivalence of these two definitions is shown in [EG991]. An $n \times n$ matrix $C_R(r_n)$ is called order n_U upper quasiseparable Hessenberg if it is of the form

$C_R(r_n) =$

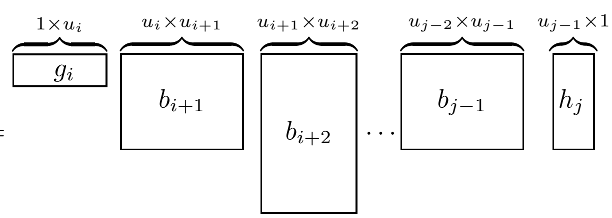


with

$$b_{ij}^{\times} = (b_{i+1}) \cdots (b_{j-1}), \quad b_{i,i+1} = 1 \tag{5.13}$$

Here p_k, q_k, d_k are scalars, the elements g_k are row vectors of maximal size n_U, h_k are column vectors of maximal size n_U, and b_k are matrices of maximal size $n_U \times n_U$ such that all products make sense. The elements in the upper part of the matrix $g_i b_{ij}^{\times} h_j$ are products of a row vector, a (possibly empty) sequence of matrices possibly of different sizes, and finally a column vector, as depicted here:

$g_i b_{ij}^{\times} h_j =$ (5.14)



with $u_k \leq n_U$ for each $k = 1, \ldots, n-1$.

5.3. Special choices of generators.

We next provide a detailed reduction of this new class to those special cases listed above in terms of generators. The choices of generators and the corresponding matrices and polynomial systems are listed in Table 3.

Polynomials	Matrix	p_k	q_k	d_k	g_k	b_k	h_k
Monomials (4.5)	Lower bidiagonal (4.6)	1	1	0	0	0	1
Real orthogonal (4.8)	Tridiagonal (4.9)	1	$1/\alpha_k$	δ_k/α_k	γ_k/α_k	0	1
Szegö polynomials (4.10)	Unitary Hessenberg (4.11)	1	μ_k	$-\rho_k \rho_{k-1}^*$	ρ_{k-1}^*	μ_{k-1}	$-\mu_{k-1}\rho_k$

Table 3. Specific choices of generators resulting in various special cases.

5.4. Fast multiplication using the quasiseparable structure

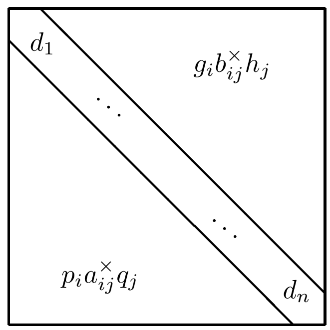
In order to have a fast $O(n^2)$ algorithm for solving systems in this special case, in view of (4.7) it suffices to have an algorithm for $O(n)$ multiplication of a quasiseparable matrix by a vector since each matrix U_k contains a quasiseparable matrix as in (3.2). With such an algorithm, each multiplication in (3.4) could be implemented in $O(n)$ operations, hence the total cost of computing the solution a would be $O(n^2)$. We first give a slightly more general definition.

A matrix A is called (n_L, n_U)-quasiseparable if $\max(\text{rank} A_{12}) = n_U$ and $\max(\text{rank} A_{21}) = n_L$, where the maximum is taken over all symmetric partitions of the form

$$A = \left[\begin{array}{c|c} * & A_{12} \\ \hline A_{21} & * \end{array}\right] \tag{5.15}$$

Similar to the generator representation for upper quasiseparable matrices given above, arbitrary order quasiseparable matrices can be expressed in terms of generators as

$C_R(r_n) =$



with $a_{ij}^{\times} = (a_{i-1}) \cdots (a_{j+1})$, $a_{i+1,i} = 1$, b_{ij} is as defined in (5.13). Now d_k are scalars, the elements p_k, g_k are row vectors of maximal size n_L and n_U respectively, q_k, h_k are column vectors of maximal size n_L and n_U respectively, and a_k, b_k are matrices of maximal size $n_L \times n_L$ and $n_U \times n_U$ respectively, such that all products make sense. The entries in the lower triangular part are defined by the product $p_i a_{ij}^{\times} q_j$, which has a similar form to that shown in (5.14).

Such a fast algorithm for multiplying a quasiseparable matrix by a vector is suggested by the following decomposition, valid for any (n_L, n_U)-quasiseparable matrix.

Proposition 5.1. *Let $C_R(r_n)$ be an $n \times n$ (n_L, n_U)-quasiseparable matrix specified by its generators as in Section 5.2. Then $C_R(r_n)$ admits the decomposition*

$$C_R(r_n) = L + D + U$$

where

$$L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & p_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & p_n \end{bmatrix} \left[\begin{array}{c|c} \begin{matrix} 0 & \cdots & 0 \end{matrix} & 0 \\ \hline \widetilde{A}^{-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{array}\right] \begin{bmatrix} q_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & q_{n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & d_n \end{bmatrix}$$

$$U = \begin{bmatrix} g_1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & g_{n-1} & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & \widetilde{B}^{-1} \\ \hline 0 & \begin{matrix} 0 & \cdots & 0 \end{matrix} \end{array}\right] \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & h_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & h_n \end{bmatrix} \tag{5.16}$$

with

$$\widetilde{A} = \begin{bmatrix} I & & & \\ -a_2 & \ddots & & \\ & \ddots & I & \\ & & -a_{n-1} & I \end{bmatrix}, \qquad \widetilde{B} = \begin{bmatrix} I & -b_2 & & \\ & \ddots & \ddots & \\ & & I & -b_{n-1} \\ & & & I \end{bmatrix}$$

and I represents the identity matrix of appropriate size.

We emphasize that the "diagonal" matrices in Proposition 5.1 are only diagonal matrices in the $(1,1)$-quasiseparable case, as otherwise some of the generators p_k, q_k, g_k, h_k are row/column vectors. To emphasize this, the product forming the matrix U has the following form:

$$U = \begin{bmatrix} \boxed{g_1} & & & \\ & \boxed{g_2} & & \\ & & \ddots & \\ & & & \boxed{g_{n-1}} & \\ & & & & 1 \end{bmatrix} \begin{bmatrix} 0 & \begin{bmatrix} \boxed{I} & \boxed{-b_2} & & & \\ & \boxed{I} & \boxed{-b_3} & & \\ & & \ddots & \ddots & \\ & & & \boxed{I} & \boxed{-b_{n-1}} \\ & & & & \boxed{I} \end{bmatrix}^{-1} \\ \vdots & \\ 0 & \\ 0 & \cdots \quad 0 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & \boxed{h_2} & & \\ & & \boxed{h_3} & \\ & & & \ddots & \\ & & & & \boxed{h_n} \end{bmatrix}$$

In light of this decomposition, we see that the matrix-vector product is reduced to five diagonal scalings and two back-substitutions with bidiagonal matrices. The justification of the proposition follows from the next simple lemma, which can be seen by direct confirmation.

Lemma 5.2. *Let $b_k, k = 1, \ldots, m-1$ be matrices of sizes $u_k \times u_{k+1}$ (with u_1, u_m arbitrary) then*

$$\begin{bmatrix} I_{n_1} & -b_1 & & & \\ & I_{n_2} & -b_2 & & \\ & & \ddots & \ddots & \\ & & & I_{n_{m-1}} & -b_{m-1} \\ & & & & I_{n_m} \end{bmatrix}^{-1} = \begin{bmatrix} I_{n_1} & b_1 & b_1b_2 & b_1b_2b_3 & \cdots & b_1b_2b_3\cdots b_{m-1} \\ & I_{n_2} & b_2 & b_2b_3 & \cdots & b_2b_3\cdots b_{m-1} \\ & & \ddots & \ddots & & \vdots \\ & & & I_{n_{m-2}} & b_{m-2} & b_{m-2}b_{m-1} \\ & & & & I_{n_{m-1}} & b_{m-1} \\ & & & & & I_{n_m} \end{bmatrix}$$

We next consider the computational cost of the multiplication algorithm suggested by this proposition. Let $C_R(r_n)$ be a quasiseparable matrix of order (n_L, n_U) whose generators are of the following sizes:

Generator	p_k	a_k	q_k	d_k	g_k	b_k	h_k
Size	$1 \times l_{k-1}$	$l_k \times l_{k-1}$	$l_k \times 1$	1×1	$1 \times u_k$	$u_{k-1} \times u_k$	$u_{k-1} \times 1$

for numbers $l_k, u_k, k = 1, \ldots, n$. Define also $l_0 = u_0 = 0$. Then it can be seen that the computational cost is

$$c = 3n + 2\sum_{k=1}^{n-1}(u_k + l_k + u_k u_{k-1} + l_k l_{k-1})$$

flops (additions plus multiplications). Since $l_k \leq n_L$ and $u_k \leq n_U$ for $k = 1, \ldots, n$, we also have

$$c \leq 3n + 2(n-1)(n_U + n_L + n_U^2 + n_L^2).$$

Thus, for values of n_L and n_U much less than n, the quasiseparability is useful as the multiplication can be carried out in $O(n)$ arithmetic operations.

Note additionally that the implementation of the algorithm suggested by the above decomposition coincides with the algorithm derived in [EG992] for the same purpose.

6. Derivation of the new Björck-Pereyra-like algorithm

In this section the algorithm presented is derived and the main enabling lemma is proved. The section begins with some background material.

6.1. Associated (generalized Horner) polynomials

Following [KO97] define the *associated polynomials* $\widehat{R} = \{\widehat{r}_0(x), \dots, \widehat{r}_n(x)\}$ for a given system of polynomials $R = \{r_0(x), \dots, r_n(x)\}$ via the relation

$$\frac{r_n(x) - r_n(y)}{x - y} = \begin{bmatrix} r_0(x) & r_1(x) & r_2(x) & \cdots & r_{n-1}(x) \end{bmatrix} \cdot \begin{bmatrix} \hat{r}_{n-1}(y) \\ \hat{r}_{n-2}(y) \\ \vdots \\ \hat{r}_1(y) \\ \hat{r}_0(y) \end{bmatrix}, \tag{6.17}$$

with additionally $\widehat{r}_n(x) = r_n(x)$.

However, before proceeding we first clarify the existence of such polynomials. Firstly, the polynomials associated with the monomials exist. Indeed, if P is the system of $n+1$ polynomials $P = \{1, x, x^2, ..., x^{n-1}, r_n(x)\}$, then

$$\frac{r_n(x) - r_n(y)}{x - y} = \begin{bmatrix} 1 & x & x^2 & \cdots & x^{n-1} \end{bmatrix} \cdot \begin{bmatrix} \hat{p}_{n-1}(y) \\ \hat{p}_{n-2}(y) \\ \vdots \\ \hat{p}_1(y) \\ \hat{p}_0(y) \end{bmatrix} = \sum_{i=0}^{n-1} x^i \cdot \hat{p}_{n-1-i}(y), \tag{6.18}$$

and in this case the associated polynomials $\widehat{P}$ can be seen to be the classical Horner polynomials (see, e.g., [KO97, Section 3.]).

Secondly, given a system of polynomials $R = \{r_0(x), r_1(x), \dots, r_{n-1}(x), r_n(x)\}$, there is a corresponding system of polynomials $\hat{R} = \{\hat{r}_0(x), \hat{r}_1(x), \dots, \hat{r}_{n-1}(x), \hat{r}_n(x)\}$ (with $\hat{r}_n(x) = r_n(x)$) satisfying (6.17). One can see that, given a polynomial system R with $\deg(r_k) = k$, the polynomials in R can be obtained from the monomial basis by

$$\begin{bmatrix} 1 & x & x^2 & \cdots & x_{n-1} \end{bmatrix} S = \begin{bmatrix} r_0(x) & r_1(x) & r_2(x) & \cdots & r_{n-1}(x) \end{bmatrix} \tag{6.19}$$

where S is an $n \times n$ upper triangular invertible matrix capturing the recurrence relations of the polynomial system R. Inserting SS^{-1} into (6.18) between the row and column vectors and using (6.19), we see that the polynomials associated with R are

$$\begin{bmatrix} \hat{r}_{n-1}(y) \\ \hat{r}_{n-2}(y) \\ \vdots \\ \hat{r}_1(y) \\ \hat{r}_0(y) \end{bmatrix} = S^{-1} \begin{bmatrix} \hat{p}_{n-1}(y) \\ \hat{p}_{n-2}(y) \\ \vdots \\ \hat{p}_1(y) \\ \hat{p}_0(y) \end{bmatrix} \tag{6.20}$$

where $\widehat{P} = \{\widehat{p}_0(x), \dots, \widehat{p}_{n-1}(x)\}$ are the classical Horner polynomials and S is from (6.19).

The following lemma will be needed in the proof presented below.

Lemma 6.1. *Let $R = \{r_0(x), \dots, r_{n-1}(x)\}$ be a system of quasiseparable polynomials satisfying (1.2), and for $k = 1, 2, \dots, n-1$ denote by $R^{(k)}$ the system of polynomials $R^{(k)} = \{\hat{r}_0^{(k)}(x), \dots, \hat{r}_k^{(k)}(x)\}$ associated with the truncated system $\{r_0(x), \dots, r_k(x)\}$. Then*

$$\begin{bmatrix} \hat{r}_0^{(1)}(x) & \hat{r}_1^{(2)}(x) & \cdots & \hat{r}_{n-2}^{(n-1)}(x) \\ & \hat{r}_0^{(2)}(x) & \cdots & \hat{r}_{n-3}^{(n-1)}(x) \\ & & \ddots & \vdots \\ & & & \hat{r}_0^{(n-1)}(x) \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{\alpha_1} & -x + \frac{a_{1,2}}{\alpha_2} & \cdots & \frac{1}{\alpha_{n-1}} a_{1,n-1} \\ & \frac{1}{\alpha_2} & \cdots & \frac{1}{\alpha_{n-1}} a_{2,n-1} \\ & & \ddots & \vdots \\ & & & -x + \frac{a_{n-2,n-1}}{\alpha_{n-1}} \\ & & & \frac{1}{\alpha_{n-1}} \end{bmatrix} \tag{6.21}$$

Proof. From [KO97] we have the formula

$$C_{\hat{R}}(\hat{r}_n) = \tilde{I} \cdot C_R(r_n)^T \cdot \tilde{I}, \qquad (\text{with} \quad \hat{r}_n(x) = r_n(x)), \tag{6.22}$$

where $\tilde{I}$ is the antidiagonal matrix, which provides a relation between the confederate matrix of a polynomial system R and that of the polynomials associated with R. From this we have the following n-term recurrence relations for the truncated associated polynomials:

$$\hat{r}_m^{(k)}(x) = \alpha_m \left[\left(x - \frac{a_{m,m+1}}{\alpha_{m+1}} \right) \hat{r}_{m-1}^{(k)} - \frac{a_{m,m+2}}{\alpha_{m+2}} \hat{r}_{m-2}^{(k)} - \cdots - \frac{a_{m,k}}{\alpha_k} \hat{r}_0^{(k)} \right], \quad m = 1, \ldots, k-1, \tag{6.23}$$

with

$$\hat{r}_0^{(k)} = 1/\alpha_k. \tag{6.24}$$

Now consider the product

$$\begin{bmatrix} \frac{1}{\alpha_1} & -x + \frac{a_{1,2}}{\alpha_2} & \cdots & \cdots & \frac{a_{1,n-1}}{\alpha_{n-1}} \\ & \ddots & \ddots & & \vdots \\ & & \frac{1}{\alpha_i} & -x + \frac{a_{i,i+1}}{\alpha_i+1} \quad \cdots & \frac{a_{i,n-1}}{\alpha_{n-1}} \\ & & & \ddots & -x + \frac{a_{n-2,n-1}}{\alpha_{n-1}} \\ & & & & \frac{1}{\alpha_{n-1}} \end{bmatrix} \begin{bmatrix} \hat{r}_0^{(1)}(x) & \cdots & \hat{r}_{j-1}^{(j)}(x) & \cdots & \hat{r}_{n-2}^{(n-1)}(x) \\ & & \hat{r}_{j-2}^{(j)}(x) & & \hat{r}_{n-3}^{(n-1)}(x) \\ & \ddots & \vdots & & \vdots \\ & & \hat{r}_0^{(j)}(x) & & \hat{r}_1^{(n-1)}(x) \\ & & & \ddots & \hat{r}_0^{(n-1)}(x) \end{bmatrix}. \tag{6.25}$$

The (i,j) entry of this product defined by the highlighted row and column can be seen as (6.23) with $k = j, m = j - i$ if $i \neq j$ and (6.24) with $k = i, m = 0$ if $i = j$. Thus this product is the identity, implying (6.21). □

With this completed, next is the proof of Lemma 3.1 from Section 3.

Proof. Performing one step of Gaussian elimination on $V_R(x_{1:n})$ yields

$$V_R(x_{1:n}) = \begin{bmatrix} 1 & & & \\ 1 & 1 & & \\ \vdots & & \ddots & \\ 1 & & & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & & & \\ & x_2 - x_1 & & \\ & & \ddots & \\ & & & x_n - x_1 \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} 1 & 0 \\ 0 & \bar{R} \end{bmatrix} \cdot \left[\begin{array}{c|ccc} r_0(x_1) & r_1(x_1) & \cdots & r_{n-1}(x_1) \\ \hline 0 & & I & \end{array} \right], \tag{6.26}$$

where the matrix $\bar{R}$ has (i,m)-entry $\bar{R}_{i,m} = \frac{r_{m+1}(x_{i+1}) - r_1(x_1)}{x_{i+1} - x_1}$; that is, $\bar{R}$ consists of divided differences. By the discussion above, associated with the system R is the system $\widehat{R} = \{\hat{r}_0(x), \ldots, \hat{r}_n(x)\}$. Following the notation of Lemma 6.1, denote by $\widehat{R}^{(k)} = \{\hat{r}_0^{(k)}(x), \ldots, \hat{r}_k^{(k)}(x)\}$ the system of polynomials associated with the truncated system $\{r_0(x), ..., r_k(x)\}$. By the definition of the associated polynomials we have for $k = 1, 2, \ldots, n-1$

$$\frac{r_k(x) - r_k(y)}{x - y} = \begin{bmatrix} r_0(x) & r_1(x) & r_2(x) & \cdots & r_{k-1}(x) \end{bmatrix} \cdot \begin{bmatrix} \hat{r}_{k-1}^{(k)}(y) \\ \hat{r}_{n-2}^{(k)}(y) \\ \vdots \\ \hat{r}_1^{(k)}(y) \\ \hat{r}_0^{(k)}(y) \end{bmatrix} = \sum_{i=0}^{k-1} r_i(x) \cdot \hat{r}_{k-1-i}^{(k)}(y).$$

Finally, denoting by $\widehat{R}^{(k)} = \{\hat{r}_0^{(k)}(x), \ldots, \hat{r}_k^{(k)}(x)\}$ the system of polynomials associated with the truncated system $\{r_0(x), \ldots, r_k(x)\}$ we can further factor $\bar{R}$ as

$$\bar{R} = V_R(x_{1:n}) \cdot \begin{bmatrix} \hat{r}_0^{(1)}(x_j) & \hat{r}_1^{(2)}(x_j) & \cdots & \hat{r}_{n-2}^{(n-1)}(x_j) \\ & \hat{r}_0^{(2)}(x_j) & \cdots & \hat{r}_{n-3}^{(n-1)}(x_j) \\ & & \ddots & \vdots \\ & & & \hat{r}_0^{(n-1)}(x_j) \end{bmatrix}. \tag{6.27}$$

The last matrix on the right-hand side of (6.27) can be inverted by Lemma 6.1. Therefore, inverting (6.26) and substituting (6.21) results in (3.1) as desired. □

7. Numerical Illustrations

To check the numerical performance of the proposed algorithm, the following numerical experiments were performed. The algorithm has been implemented in MATLAB version 7, which uses double precision. These results were compared with exact solutions calculated using the MATLAB Symbolic Toolbox command `vpa()`, which allows software-implemented precision of arbitrary numbers of digits. The number of digits was set to 64, however in cases where the condition number of the coefficient matrix exceeded 10^{30}, this was raised to 100 digits to maintain accuracy.

It is known (see [RO91], [H90]) that reordering the nodes for polynomial Vandermonde matrices, which corresponds to a permutation of the rows, can affect the accuracy of related algorithms. In particular, ordering the nodes according to the *Leja ordering*

$$|x_1| = \max_{1\le i\le n} |x_i|, \qquad \prod_{j=1}^{k-1} |x_k - x_j| = \max_{k\le i\le n} \prod_{j=1}^{k-1} |t_i - t_j|, \quad k = 2, \dots, n-1$$

(see [RO91], [H90], [O03]) improves the performance of many similar algorithms. We include experiments with and without the use of Leja ordering (if the Leja ordering is not used, the nodes are ordered randomly). A counterpart of this ordering is known for Cauchy matrices, see [BKO02].

In all experiments, we compare the forward accuracy of the algorithm, defined by

$$e = \frac{\|x - \hat{x}\|_2}{\|x\|_2}$$

where $\hat{x}$ is the solution computed by each algorithm in MATLAB in double precision, and x is the exact solution. In the tables, BP-QS denotes the proposed Björck-Pereyra like algorithm with a random ordering of the nodes, and BP-QS-L denotes the same algorithm using the Leja ordering. The factors L_k from (3.3) were used. GE indicates MATLAB's Gaussian elimination. Finally, $\mathrm{cond}(V)$ denotes the condition number of the matrix V computed via the MATLAB command `cond()`.

Experiment 1. In Table 4, the values for the generators were chosen randomly on $(-1,1)$, similarly for the entries of the right hand side vector. The nodes x_k were selected equidistant on $(-1,1)$ via the formula

$$x_k = -1 + 2\left(\frac{k}{n-1}\right), \quad k = 0, 1, \ldots, n-1$$

We test the accuracy of the algorithm for various sizes n of matrices generated in this way.

n	`cond(V)`	GEPP	BP-QS	BP-QS-L
10	6.9e+06	8.7e-15	1.9e-14	1.6e-15
	3.5e+08	1.9e-14	5.3e-15	8.9e-16
	1.9e+10	7.1e-15	6.0e-16	6.4e-16
15	1.5e+10	4.4e-12	3.5e-12	6.7e-15
	7.7e+13	5.8e-12	1.4e-13	1.3e-15
	5.9e+15	3.1e-11	4.3e-13	5.7e-16
20	6.0e+17	1.0e-09	1.4e-11	4.6e-15
	2.2e+18	9.6e-14	8.5e-12	1.2e-15
	1.6e+22	6.2e-11	1.1e-11	2.3e-15
25	1.6e+20	8.0e-08	4.3e-11	4.4e-16
	1.0e+22	1.3e-08	1.1e-10	1.3e-15
	1.0e+26	8.8e-07	1.5e-10	3.2e-15
30	9.1e+18	1.2e-06	4.3e-06	1.2e-14
	8.0e+23	5.0e-08	3.3e-09	1.5e-15
	1.9e+24	5.8e-02	5.6e-10	4.4e-15
35	9.8e+23	9.3e-01	1.2e-06	2.0e-15
	7.5e+28	1.6e-03	7.1e-08	1.7e-15
	1.8e+29	1.1e-02	4.2e-06	1.7e-15
40	2.6e+25	8.6e-02	1.1e-06	8.6e-15
	2.1e+29	2.9e-02	1.4e-06	4.8e-15
	1.0e+33	1.0e+00	2.2e-05	2.4e-16
45	4.5e+31	1.0e+00	8.4e-05	2.0e-15
	9.2e+36	1.2e+00	3.2e-05	3.0e-15
	5.9e+38	1.1e+00	2.2e-04	5.2e-16
50	3.3e+37	1.0e+00	6.9e-03	1.2e-13
	2.8e+41	4.0e-01	4.8e-03	2.3e-13
	8.7e+45	1.0e+00	1.6e-02	6.3e-14

Table 4. Equidistant nodes on $(-1,1)$.

Notice that as the size of the matrices involved rises, so does the condition number of the matrices, and hence as expected, the performance of Gaussian elimination declines. The performance of the proposed algorithm with a random ordering of the nodes is an improvement over that of GE, however using the Leja ordering gives a dramatic improvement in performance in this case.

Experiment 2. In Table 5, the values for the generators and entries of the right hand side vector were chosen as in Experiment 1, and the nodes x_k were selected clustered on $(-1,1)$ via the formula

$$x_k = -1 + 2\left(\frac{k}{n-1}\right)^2, \quad k = 0, 1, \ldots, n-1$$

Again we test the accuracy for various $n \times n$ matrices generated in this way.

n	`cond(V)`	GEPP	BP-QS	BP-QS-L
10	8.6e+10	1.7e-12	9.0e-15	5.5e-16
	1.2e+11	1.9e-12	1.5e-14	1.0e-15
	3.3e+12	1.7e-13	1.6e-14	3.6e-16
15	1.3e+15	1.3e-11	3.0e-13	3.7e-15
	1.9e+16	5.1e-11	6.4e-13	3.4e-16
	1.5e+17	1.4e-11	8.6e-14	2.3e-15
20	9.7e+17	6.0e-08	1.5e-10	8.5e-14
	2.1e+18	5.4e-08	1.7e-12	8.9e-15
	1.1e+23	5.0e-08	3.1e-11	7.9e-16
25	1.8e+20	7.2e-04	6.9e-10	7.7e-14
	4.5e+20	4.8e-03	1.3e-09	5.9e-14
	4.0e+22	1.9e-03	2.9e-10	6.1e-15
30	9.1e+22	1.1e+00	6.1e-09	6.4e-15
	4.9e+24	1.0e+00	2.7e-10	2.2e-15
	1.0e+26	1.0e+00	2.9e-09	2.9e-12
35	2.1e+27	7.8e-01	2.5e-07	1.6e-14
	1.3e+28	1.0e+00	8.6e-09	2.9e-09
	4.7e+33	1.0e+00	8.3e-07	7.4e-11
40	7.5e+27	1.0e+00	2.1e-05	5.1e-08
	1.9e+33	1.0e+00	8.5e-06	2.5e-09
	3.3e+37	1.0e+00	3.8e-05	1.1e-12
45	1.8e+33	1.0e+00	6.2e-04	8.2e-04
	4.4e+34	1.0e+00	1.3e-03	1.0e-06
	1.2e+40	1.0e+00	2.0e-04	2.3e-08
50	6.5e+32	1.0e+00	7.3e-03	3.2e-06
	2.8e+36	1.0e+00	1.2e-03	5.6e-04
	1.5e+46	1.0e+00	3.1e-03	7.8e-04

Table 5. Clustered nodes on $(-1,1)$.

In this experiment, again the condition number rises with the size of the matrix, and as expected Gaussian elimination gives less accuracy as this increases. The proposed algorithm gives an improvement in this case as well, and the Leja ordering again gives an improvement.

Experiment 3. In [CF88], [BKO99] it was shown that the behavior of BKO-type algorithms can depend on the direction of the right hand side vector. We include a similar experiment here where the outcome is consistent with observations in [CF88] and [BKO99]. This is illustrated in Table 6, which shows the results for a 30×30 matrix generated by a fixed set of generators and nodes on the unit disc, and the results of applying the various algorithms to solve the system with each (left) singular vector as the right hand side.

singular vector	GEPP	BP-QS	BP-QS-L
1	4.7e-01	5.7e-02	2.7e+01
2	1.3e+00	2.8e-02	2.1e+00
3	1.6e+00	1.8e-02	1.9e+00
4	3.8e+00	3.6e-02	2.3e+00
5	5.4e+00	1.4e-02	1.6e+00
6	2.9e+00	6.3e-02	6.6e-02
7	5.3e+00	9.7e-03	2.0e-02
8	1.6e+00	2.1e-03	2.0e-03
9	5.5e-01	2.4e-04	6.6e-04
10	1.4e-01	1.4e-05	7.7e-05
11	8.1e-05	2.1e-09	3.6e-09
12	1.7e-05	5.8e-10	6.4e-10
13	7.9e-06	2.7e-10	9.4e-11
14	8.6e-06	2.7e-11	3.1e-11
15	5.8e-06	6.4e-11	6.7e-11
16	1.1e-05	1.8e-13	9.6e-13
17	1.1e-05	1.6e-15	9.6e-16
18	1.1e-05	2.5e-15	4.4e-15
19	1.1e-05	6.1e-15	2.1e-15
20	1.2e-05	8.5e-15	4.1e-14
21	1.3e-05	1.8e-15	4.4e-16
22	1.4e-05	4.7e-15	3.8e-15
23	1.3e-05	1.1e-14	1.5e-15
24	1.9e-05	2.2e-14	8.1e-15
25	5.2e-05	6.7e-14	5.0e-14
26	1.4e-05	1.9e-15	2.6e-15
27	1.3e-05	1.4e-15	2.2e-15
28	1.2e-05	2.9e-15	1.1e-15
29	1.2e-05	6.0e-15	2.5e-15
30	1.2e-05	7.3e-16	4.9e-16

Table 6. Dependence on the direction using left singular vectors.

8. Conclusion

In this paper we generalized the well known Björck-Pereyra algorithm to polynomial-Vandermonde matrices. This algorithm was derived by exploiting the properties of the corresponding confederate matrix (a Hessenberg matrix capturing the recurrence relations). In the case where the polynomial system is related to quasiseparable matrices (specifically, when the confederate matrix is quasiseparable and Hessenberg) the special properties of these matrices allowed the reduction of complexity of the overall algorithm to $O(n^2)$. This represents an order of magnitude improvement over the complexity of standard methods that ignore this structure. Initial experiments highlight some favorable numerical properties of the algorithm.

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