Complexity of multiplication with vectors for structured matrices

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Abstract

Fast algorithms for computing the product by a vector are presented for a number of classes of matrices whose properties relate to the properties of Toeplitz, Vandermonde or Cauchy matrices (these matrices are defined using the concept of *displacement* of a matrix) and also for their inverses. All the actions which are not dependent upon the coordinates of the input vector are singled out in a separate *preprocessing* stage. The proposed algorithms are based on new representations of these matrices, involving factor circulants.

0 Intrtoduction

In the present paper fast algorithms of computing the products of matrices by vectors are proposed for a number of classes of matrices, namely for Toeplitz-related, for Vandermonde-related, for Cauchy-related matrices and also for their inverses. The proposed algorithms are divided into two separate phases :

I. *Preprocessing* for a given matrix $A \in \mathbf{C}^{n \times n}$.

II. *Computing the product* of the matrix A by a vector $\mathbf{b} \in \mathbf{C}^n$.

All the computations which are not dependent upon the information about the vector $\mathbf{b} \in \mathbf{C}^n$ are singled out to the first phase. In the proposed algorithms the preprocessing phase involves also the actions on preparation of the given matrix, which are aimed at reduction of the complexity of the second phase of computations. The quantity of flops (i.e. operations of addition, subtraction, multiplication and division with float point) of these two phases are denoted by

$$\mathrm{Prep}(A) \qquad \text{and} \qquad \mathrm{Comp}(A, \mathbf{b}),$$

correspondingly. In what follows the limited number of well-known algorithms is often used. The notations and known estimates for their complexities are given in table 0.1. The particular implementations of these three algorithms and complexity analysis for them can be found in various sources (see, for example [1]).

Table 0.1. Algorithms

Algorithm	Notation for complexity	Estimate for complexity
The algorithm of the evaluation of $(n-1)$-th degree polynomial at n points	$\varepsilon(n)$	$O(n\log^2 n)$ flops
The algorithm of the interpolation of $(n-1)$-th degree polynomial from its values at n points	$\iota(n)$	$O(n\log^2 n)$ flops
The algorithm of Fast Fourier Transform of order n	$\phi(n)$	$O(n\log n)$ flops

In table 0.2 the data on complexity of multiplication by a vector are collected for various classes of matrices with a certain structure. Where possible, the data on complexity for the phase of preprocessing and for the phase of computating the product are given separately. The last column contains further references.

Table 0.2. Matrices

Matrix $A \in \mathbf{C}^{n\times n}$	Prep(A)	Comp$(A, \mathbf{b})$	References
Fourier matrix and its inverse	—	$\phi(n)$	Well known
Factor circulant	$\phi(n)+O(n)$	$2\phi(n)+O(n)$	Well known
Toeplitz matrix	$2\phi(n)$	$4\phi(n)+O(n)$	Well known
Vandermonde matrix	—	$\varepsilon(n)$	Well known
Inverse of Vandermonde matrix	—	$\iota(n)$	Well known
Transpose of Vandermonde matrix	$\iota(n)+\varepsilon(n)+\phi(n)+$ $+2n\log n+O(n)$	$\iota(n)+2\phi(n)+O(n)$	[11] (see also [5])
Transpose of inverse Vandermonde matrix [1]	$\iota(n)+\varepsilon(n)+\phi(n)+$ $+2n\log n+O(n)$	$\varepsilon(n)+2\phi(n)+O(n)$	[11] (see also [5])
Cauchy matrix	$\iota(n)+2\varepsilon(n)+O(n)$	$\iota(n)+\varepsilon(n)+O(n)$	[8], [11]

[1] Algorithms from [11] for transpose to Vandermonde matrix and for transpose to inverse Vandermonde matrix have the same preprocessing stage.

In the present paper fast algorithms are proposed for matrix times vector multiplication for three classes of matrices whose properties relate to the properties of Toeplitz, Vandermonde and Cauchy matrices and also for their inverses. The definitions of these classes of matrices are based on the concept of *displacement.*

Let F_f and F_b be matrices from $\mathbf{C}^{n\times n}$. Following [6] let us refer to the matrix

$$\nabla_{\{F_f,F_b\}}(A) = A - F_f A F_b^T \tag{0.1}$$

as $\{F_f, F_b\}$-*displacement* of the matrix $A \in \mathbf{C}^{n\times n}$. The number $\alpha = \operatorname{rank}\nabla_{\{F_f,F_b\}}(A)$ is called $\{F_f, F_b\}$-*displacement rank* of the matrix $A \in \mathbf{C}^{n\times n}$. Each pair of rectangular matrices G and H such that

$$\nabla_{\{F_f,F_b\}}(A) = G\cdot H^T \qquad (G,\ H \in \mathbf{C}^{n\times\beta})$$

is referred to as $\{F_f, F_b\}$-*generator of length* β $\ (\geq \alpha = \operatorname{rank}\nabla_{\{F_f,F_b\}}(A)\ )$ of the matrix A. For $F_f = F_b = Z_0$, where

$$Z_0 = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

is lower shift matrix, the concept of displacement in the form of (0.1) appeared in [14]. In [14] it was shown that every matrix $A \in \mathbf{C}^{n\times n}$ is uniquely determined by a $\{Z_0, Z_0\}$-generator $G = [\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_\beta]$, $H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_\beta]$ $\ (\mathbf{g}_m,\ \mathbf{h}_m \in \mathbf{C}^n)$ by the following formula

$$A = \sum_{m=1}^{\beta} \operatorname{Lower}(\mathbf{g}_m)\cdot \operatorname{Upper}(\mathbf{h}_m), \tag{0.2}$$

where $\operatorname{Lower}(\mathbf{r})$ is the lower triangular Toeplitz matrix with the first column $\mathbf{r} = (r_i)_{i=0}^{n-1}$ and $\operatorname{Upper}(\mathbf{s})$ is the upper triangular Toeplitz matrix with the first row $\mathbf{s}^T = ((s_i)_{i=0}^{n-1})^T$.

The matrices with low $\{Z_0, Z_0\}$-displacement rank are called close-to-Toeplitz (they are also referred to as near-to-Toeplitz, Toeplitz-like, of the Toeplitz type, etc.). The classes of close-to-Vandermonde and close-to-Cauchy matrices (the names Vandermonde-like, Cauchy-like are also used) are defined (see [13] [6]) using another displacement operator

$$\triangle_{\{F_f,F_b\}}(A) = F_f A - A F_b.$$

Namely, matrix A is called close-to-Vandermonde if it is transformed by $\triangle_{\{\operatorname{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}}(\cdot)$ in the matrix of low rank.

Furthermore, in the case when rank of the matrix $\triangle_{\{\operatorname{diag}(\mathbf{q}),\operatorname{diag}(\mathbf{t})\}}(A)$ is comparatively small, matrix A is named close-to-Cauchy matrix.

In the present paper we define three classes of matrices whose properties relate to the properties of Toeplitz, Vandermonde and Cauchy matrices and the definitions are based on the unify form (0.1) of the displacement operator. In accordance, three choices of the operators F_f and F_b for $\nabla_{\{F_f,F_b\}}(A)$ are considered :

(i) $F_f = Z_\varphi$, $F_b = Z_{\frac{1}{\varphi}}$;

(ii) $F_f = \mathrm{diag}(\mathbf{t})$ ($\mathbf{t} \in \mathbf{C}^n$), $F_b = Z_{\frac{1}{\varphi}}$;

(iii) $F_f = \mathrm{diag}(\mathbf{q})$, $F_b = \mathrm{diag}(\mathbf{t})$ ($\mathbf{q}, \mathbf{t} \in \mathbf{C}^n$).

Here diag($\mathbf{t}$) is a diagonal matrix whose entries on the main diagonal equal the coordinates of the vector $\mathbf{t} \in \mathbf{C}^n$ and

$$Z_\varphi = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & \varphi \\ 1 & 0 & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

is the φ-cyclic lower shift matrix. Matrices with low $\{F_f, F_b\}$-displacement rank will be referred to in the present paper as *Toeplitz-related* for the case (i), *Vandermonde-related* for the case (ii), and *Cauchy-related* for the case (iii). The motivation of these names will be explained separately in each case considered. Note that using the displacement operator of the form (0.1) with choices (i) - (iii), the same or almost the same classes of matrices as in [14] and [13] are defined. Nevertheless, this approach has definite advantages. Firstly, the common form (0.1) of the displacement operator enables us to reduce the problems for Vandermonde-related and for Cauchy-related matrices to the analogous problems for Toeplitz-related matrices. Secondly, the appearance in (0.1) of factor circulants Z_φ and $Z_{\frac{1}{\varphi}}$ (cases (i), (ii)) instead of the usual Z_0 enables us to receive for matrices of all three considered classes the representations involving factor circulants. That immediately leads to the low complexity algorithms for multiplication with vectors. Furthermore, in contrast with Z_0, matrices Z_φ and $Z_{\frac{1}{\varphi}}$ are invertible. This feature helps also in consideration of the problems concerning the inverses of Toeplitz-related, Vandermonde-related and Cauchy-related matrices.

The remarks of the referees were important and constructive. In particular under the influence of these remarks we added the section 4.3. It is our pleasure to thank the referees for their help as well as for bringing the papers [15] and [4] to our attention.

1 Toeplitz-related matrices

Here we set $F_f = Z_\varphi$, $F_b = Z_{\frac{1}{\varphi}}$. In this case matrix

$$\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(A) = A - Z_\varphi A Z_{\frac{1}{\varphi}}^T \tag{1.1}$$

is called *φ-cyclic displacement* of $A \in \mathbf{C}^{n\times n}$, the number $\alpha = \mathrm{rank}\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(A)$ is named *φ-cyclic displacement rank* of A, and $\{Z_\varphi, Z_{\frac{1}{\varphi}}\}$-generator of A is referred to as *φ-cyclic generator*. Straightforward computation shows, that φ-cyclic displacement rank of any Toeplitz matrix $A = (a_{i-j})_{i,j=0}^{n-1}$ does not exceed two. Indeed,

$$\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(A) = \left[\begin{array}{c|ccc} 0 & a_{-1} - \varphi a_{n-1} & \cdots & a_{-n+1} - \varphi a_1 \\ \hline a_1 - \frac{1}{\varphi} a_{-n+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n-1} - \frac{1}{\varphi} a_{-1} & 0 & \cdots & 0 \end{array}\right] =$$

$$= \begin{bmatrix} 1 & \beta \\ 0 & a_1 - \frac{1}{\varphi} a_{-n+1} \\ \vdots & \vdots \\ 0 & a_{n-1} - \frac{1}{\varphi} a_{-1} \end{bmatrix} \cdot \begin{bmatrix} -\beta & 1 \\ a_{-1} - \varphi a_{n-1} & 0 \\ \vdots & \vdots \\ a_{-n+1} - \varphi a_1 & 0 \end{bmatrix}^T \tag{1.2}$$

where β may be an arbitrary complex number. By analogy with (1.2), if

$$\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(A) = G \cdot H^T \qquad (G,\ H \in \mathbf{C}^{n \times \alpha})$$

and the number α is small in comparison with the dimension of the space, then we will refer to $A \in \mathbf{C}^{n \times n}$ as a *Toeplitz-related matrix.*

In [10] it was shown how to reconstruct an arbitrary matrix $A \in \mathbf{C}^{n \times n}$ from a φ-cyclic generator and any one of its rows or columns. Moreover, the corresponding formulae express A in the form of the sum of the products of factor circulants, i.e. matrices of the form

$$\mathrm{Circ}_\varphi(\mathbf{r}) = \begin{bmatrix} r_0 & \varphi r_{n-1} & \cdots & \cdots & \varphi r_1 \\ r_1 & r_0 & \varphi r_{n-1} & & \vdots \\ \vdots & r_1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \varphi r_{n-1} \\ r_{n-1} & \cdots & \cdots & r_1 & r_0 \end{bmatrix} \qquad (\mathbf{r} = (r_i)_{i=0}^{n-1}).$$

Let us formulate the result.

Theorem 1.1 ([10]). *Let $\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(\cdot)$ be the linear operator in $\mathbf{C}^{n \times n}$ defined by* (1.1). *Then the following statements hold:*

(i). *The equality $\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(A) = 0$ holds if and only if A is a φ-circulant.*

(ii). *If the equation*

$$\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(X) = G \cdot H^T, \tag{1.3}$$

with

$$G = [\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_\alpha],\ \ H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_\alpha],$$

where $\mathbf{g}_m, \mathbf{h}_m$ $(m = 1, 2, ..., \alpha)$ are given vectors, is solvable with respect to $X \in \mathbf{C}^{n \times n}$, then

$$\sum_{m=1}^{\alpha} \mathrm{Circ}_\varphi(\mathbf{g}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{h}_m)^T = 0. \tag{1.4}$$

(iii). *If* 2α *vectors* $\mathbf{g}_m$ *and* $\mathbf{h}_m$ $(m = 1, 2, ..., \alpha)$ *satisfy the condition* (1.4)*, then the equation* (1.3) *has the solution*

$$X = \mathrm{Circ}_{lr} + \frac{\varphi}{\varphi - \psi} \sum_{m=1}^{\alpha} \mathrm{Circ}_{\psi}(\mathbf{g}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{h}_m)^T. \tag{1.5}$$

Here ψ $(\neq \varphi)$ *is an arbitrary complex number and* Circ_{lr} *is any* φ*-circulant. The last rows of the matrices* X *and* Circ_{lr} *are the same.*

(iv). *Under the conditions of the assertion* (iii) *the solution* X *of the equation* (1.3) *may also be written in the form*

$$X = \mathrm{Circ}_{lc} + \frac{\psi}{\psi - \varphi} \sum_{m=1}^{\alpha} \mathrm{Circ}_{\varphi}(\mathbf{g}_m) \cdot \mathrm{Circ}_{\frac{1}{\psi}}(\mathbf{h}_m)^T. \tag{1.6}$$

Here $\psi(\neq \varphi)$ *is an arbitrary complex number and* Circ_{lc} *is any* φ*-circulant. The last columns of the matrices* X *and* Circ_{lc} *are the same.*

The formulae (1.5) and (1.6) enable the proposition of the algorithms for fast multiplication with a vector for Toeplitz-related matrices. Before stating the result, let us introduce the necessary notations. Let ξ be arbitrary complex number met the condition $\xi^n = \varphi$. Denote by

$$D_{\varphi} = \mathrm{diag}((\xi^i)_{i=0}^{n-1}) \qquad \text{and} \qquad \Lambda_{\varphi}(\mathbf{r}) = \mathrm{diag}(\mathcal{F} D_{\varphi} \mathbf{r}) \quad (\mathbf{r} \in \mathbf{C}^n). \tag{1.7}$$

Here by $\mathrm{diag}(\mathbf{r})$ is denoted the diagonal matrix whose entries on the main diagonal equal the coordinates of the vector $\mathbf{r} \in \mathbf{C}^n$. It is well known (see [7]), that for matrix $\mathrm{Circ}_{\varphi}(\mathbf{r})$ the following representation holds

$$\mathrm{Circ}_{\varphi}(\mathbf{r}) = D_{\varphi}^{-1} \cdot \mathcal{F}^* \cdot \Lambda_{\varphi}(\mathbf{r}) \cdot \mathcal{F} \cdot D_{\varphi}, \tag{1.8}$$

where $\mathcal{F} = \frac{1}{\sqrt{n}}(\omega^{ij})_{i,j=0}^{n-1}$ is the Fourier matrix and superscript * means conjugate transpose. Here $\omega = e^{\frac{2\pi i}{n}}$ is the primitive n-th root from unity.

Theorem 1.2 *Let the matrix* $A \in \mathbf{C}^{n \times n}$ *be given by a* φ*-cyclic generator of length* α *and by its last column (or last row). Then*

$$\mathrm{Comp}(A, \mathbf{b}) \leq (2\alpha + 3) \cdot \phi(n) + O(n)$$

and

$$\mathrm{Prep}(A) \leq (2\alpha + 1) \cdot \phi(n) + O(n).$$

Proof. Let

$$G = [\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_{\alpha}], \; H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_{\alpha}] \qquad (\mathbf{g}_m, \mathbf{h}_m \in \mathbf{C}^n)$$

be the φ-cyclic generator of A and $\mathbf{s} = (s_i)_{i=0}^{n-1}$ be last column of A. Then for A the representation (1.6) holds. Let us represent all the factor circulants from the right-hand side of (1.6) in form of (1.8). This procedure can be accomplished in $(2\alpha+1)\cdot\phi(n)+O(n)$ flops, and then (1.6) takes the shape

$$A = D_{\varphi}^{-1}\cdot\mathcal{F}^*\cdot\Big(\Lambda_{lc}\cdot\mathcal{F}\cdot D_{\varphi}+\frac{\psi}{\psi-\varphi}\cdot\big(\sum_{m=1}^{\alpha}\Lambda_{\varphi}(\mathbf{g}_m)\cdot\mathcal{F}\cdot D_{\varphi}\cdot D_{\psi}^{-1}\cdot\mathcal{F}^*\cdot\Lambda_{\psi}(\mathbf{h}_m)\big)\cdot\mathcal{F}D_{\psi}\Big). \quad (1.9)$$

Here $\Lambda_{lc} = \mathrm{diag}(\mathcal{F}D_{\varphi}\begin{bmatrix} s_{n-1} \\ s_{n-2} \\ \vdots \\ s_0 \end{bmatrix})$ is the central factor of the representation (1.8), corresponding to φ-circulant Circ_{lc}.

The amount of operations for computing using (1.9) the product of the Toeplitz-related matrix by a vector equals

$$(2\alpha+3)\cdot\phi(n)+O(n)$$

flops.$\square$

Corollary 1.3 *Let the matrix $A \in \mathbf{C}^{n\times n}$ be given by a φ-cyclic generator of length α and by its last column (or last row). Then the product of A with an arbitrary matrix from $\mathbf{C}^{n\times n}$ can be computed in*

$$(2\alpha+3)\cdot\phi(n)\cdot n+O(n^2)$$

flops.

2 Vandermonde-related matrices

Here we consider the choice $F_f = \mathrm{diag}(\mathbf{t}),\ F_b = Z_{\frac{1}{\varphi}}$, where $\mathbf{t} = (t_i)_{i=0}^{n-1} \in \mathbf{C}^n$ and the number $\varphi \in \mathbf{C}$ is chosen such that

$$t_i^n \neq \varphi \qquad (i = 0, 1, ..., n-1). \quad (2.1)$$

By $V(\mathbf{t})$ is denoted the Vandermonde matrix of the form

$$V(\mathbf{t}) = \begin{bmatrix} 1 & t_0 & \cdots & t_0^{n-1} \\ 1 & t_1 & \cdots & t_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & t_{n-1} & \cdots & t_{n-1}^{n-1} \end{bmatrix}.$$

It is not difficult to realize that

$$\nabla_{\{\mathrm{diag}(\mathbf{t}),Z_{\frac{1}{\varphi}}\}}(V(\mathbf{t})) = \begin{bmatrix} 1-\frac{1}{\varphi}t_0^n \\ 1-\frac{1}{\varphi}t_1^n \\ \vdots \\ 1-\frac{1}{\varphi}t_{n-1}^n \end{bmatrix}\cdot\mathbf{e}_0^T, \quad (2.2)$$

where as above by $Z_{\frac{1}{\varphi}}$ is denoted the lower $\frac{1}{\varphi}$-cyclic shift matrix.

If $\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}$-displacement rank of $A \in \mathbf{C}^{n\times n}$ is small in comparison with $n \in \mathbf{N}$, then, by analogy with (2.2), A is called *Vandermonde-related matrix.* The following theorem shows how any matrix $A \in \mathbf{C}^{n\times n}$ may be reconstructed from a $\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}$-generator.

Theorem 2.1 *Let $G = [\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_\alpha]$ and $H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_\alpha]$ be arbitrary matrices from $\mathbf{C}^{n\times\alpha}$ $(\mathbf{g}_m, \mathbf{h}_m \in \mathbf{C}^n)$. Then the unique solution $X \in \mathbf{C}^{n\times n}$ of the equation*

$$\nabla_{\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}}(X) = G \cdot H^T \tag{2.3}$$

is given by

$$X = \mathrm{diag}(\Big(\frac{1}{1-\frac{1}{\varphi}t_i^n}\Big)_{i=0}^{n-1}) \cdot \sum_{m=1}^{\alpha} \mathrm{diag}(\mathbf{g}_m) \cdot V(\mathbf{t}) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{h}_m)^T. \tag{2.4}$$

Proof. The uniqueness of the solution follows from the condition (2.1) (see, for example, p. 411 in [16]). Let us check that the matrix X defined by (2.4) solves the equation (2.3). Indeed, since matrices $Z_{\frac{1}{\varphi}}^T$ and $\mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{h}_m)^T$ are commuting as φ-circulants, hence

$$\nabla_{\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}}(X) = \mathrm{diag}(\Big(\frac{1}{1-\frac{1}{\varphi}t_i^n}\Big)_{i=0}^{n-1}) \cdot \sum_{m=1}^{\alpha} \mathrm{diag}(\mathbf{g}_m) \cdot \nabla_{\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}}\big(V(\mathbf{t})\big) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{h}_m)^T.$$

Furthermore, using (2.2) we have

$$\nabla_{\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}}(X) = \sum_{m=1}^{\alpha} \mathrm{diag}(\mathbf{g}_m) \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \mathbf{e}_0^T \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{h}_m)^T = \sum_{m=1}^{\alpha} \mathbf{g}_m \cdot \mathbf{h}_m^T = G \cdot H^T. \square$$

If $\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}$-displacement rank of the matrix $A \in \mathbf{C}^{n\times n}$ equals $\alpha \in \mathbf{N}$, then the formula (2.4) gives the following estimates for the complexity of multiplication with a vector :

$$\mathrm{Comp}(A, \mathbf{b}) \le \alpha \cdot \varepsilon(n) + (\alpha + 1) \cdot \phi(n) + O(n) \qquad \text{and} \qquad \mathrm{Prep}(A) = \alpha \cdot \phi(n) + O(n).$$

Below we show how, using (2.4) at the stage of preprocessing, one may reduce the complexity of the second stage of computing the product.

Theorem 2.2 *Let the matrix $A \in \mathbf{C}^{n\times n}$ be given by a $\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}$-generator of A of length α. Then*

$$\mathrm{Comp}(A, \mathbf{b}) \le \varepsilon(n) + (2\alpha + 5) \cdot \phi(n) + O(n)$$

and

$$\mathrm{Prep}(A) \le (2\alpha + 2) \cdot \iota(n) + 2 \cdot \varepsilon(n) + (6\alpha + 7) \cdot \phi(n) + 2n \log n + O(n).$$

Proof. Set

$$p(\lambda) = \prod_{i=0}^{n-1}(\lambda - t_i) = \lambda^n + \sum_{i=0}^{n-1} r_i \cdot \lambda^i, \qquad \mathbf{r} = (r_i)_{i=0}^{n-1} \in \mathbf{C}^n.$$

Then the following equality holds

$$V(\mathbf{t}) \cdot C_{\mathbf{r}} = \mathrm{diag}(\mathbf{t}) \cdot V(\mathbf{t}) \tag{2.5}$$

where

$$C_{\mathbf{r}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -r_0 \\ 1 & 0 & \cdots & 0 & -r_1 \\ 0 & 1 & \cdots & 0 & -r_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -r_{n-1} \end{bmatrix}$$

is the *companion matrix* of the polynomial $p(\lambda)$ with coefficients r_i $(i = 0, 1, ..., n-1)$. From (2.5) and the equality

$$\nabla_{\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}}(A) = G \cdot H^T,$$

where

$$G = [\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_\alpha],\ \ H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_\alpha] \qquad (\mathbf{g}_m, \mathbf{h}_m \in \mathbf{C}^n)$$

is the $\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}$-generator of A it follows that

$$\nabla_{\{C\mathrm{r}, Z_{\frac{1}{\varphi}}\}}(\tilde{A}) = G_1 \cdot H^T,$$

with $\tilde{A} = V(\mathbf{t})^{-1} \cdot A$ and $G_1 = V(\mathbf{t})^{-1} \cdot G$. Furthermore, taking into account the equality $C_{\mathbf{r}} = Z_\varphi - (\mathbf{r} + \varphi \mathbf{e}_0) \cdot \mathbf{e}_{n-1}^T$, we have

$$\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(\tilde{A}) = \tilde{G} \cdot \tilde{H}^T,$$

where $\tilde{G} = [\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, , ..., \tilde{\mathbf{g}}_{\alpha+1}], \tilde{H} = [\mathbf{h}_1, \mathbf{h}_2, , ..., \mathbf{h}_{\alpha+1}]$ with

$$\tilde{\mathbf{g}}_m = V(\mathbf{t})^{-1}\mathbf{g}_m \quad (m = 1, 2, ..., \alpha), \quad \tilde{\mathbf{g}}_{\alpha+1} = -\mathbf{r} - \varphi \mathbf{e}_0 \quad \text{and} \quad \mathbf{h}_{\alpha+1} = Z_{\frac{1}{\varphi}} A^T V(\mathbf{t})^{-T} \mathbf{e}_{n-1}.$$

Thus, the matrix A can be represented in the form

$$A = V(\mathbf{t}) \cdot \tilde{A}, \tag{2.6}$$

where matrix $\tilde{A}$ has φ-cyclic displacement rank at most $\alpha + 1$. The latter equality and theorem 1.2 imply that

$$\mathrm{Comp}(A, \mathbf{b}) \leq \varepsilon(n) + (2\alpha + 5) \cdot \phi(n) + O(n).$$

Corresponding estimate for the value $\mathrm{Prep}(A)$ follows from the detailed description of the phase of preprocessing, which is given below.□

Preprocessing for matrix $A \in \mathbf{C}^{n \times n}$ given by a $\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}$-generator

$$G = [\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_\alpha], H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_\alpha] \qquad (\mathbf{g}_m, \mathbf{h}_m \in \mathbf{C}^n).$$

1. Compute $\{C_{\mathbf{r}}, Z_{\frac{1}{\varphi}}\}$-generator $G_1 = [\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, ..., \tilde{\mathbf{g}}_\alpha], H$ of matrix $\tilde{A} = V(\mathbf{t})^{-1} \cdot A$. To do it, compute in $\alpha \cdot \iota(n)$ flops vectors $\tilde{\mathbf{g}}_m = V(\mathbf{t})^{-1}\mathbf{g}_m \quad (m = 1, 2, ..., \alpha)$.

2. Compute the last row $\mathbf{s}^T = \mathbf{e}_{n-1}^T V(\mathbf{t})^{-1} A$ of the matrix $\tilde{A} = V(\mathbf{t})^{-1} \cdot A$. Using for A the representation (2.4) and then factoring each factor circulant as in (1.8), we have

$$\mathbf{s} = D_{\frac{1}{\varphi}}^{-1} \cdot \mathcal{F}^* \cdot \Big(\sum_{m=1}^{\alpha} \Lambda_{\frac{1}{\varphi}}(\mathbf{h}_m) \cdot \mathcal{F} \cdot D_{\frac{1}{\varphi}} \cdot V(\mathbf{t})^T \cdot \mathrm{diag}(\mathbf{g}_m) \Big) \cdot \mathrm{diag}(\Big(\frac{1}{1 - \frac{1}{\varphi} t_i^n} \Big)_{i=0}^{n-1}) \cdot V(\mathbf{t})^{-T} \cdot \mathbf{e}_{n-1}$$

According to the data in the table 0.2, vector $\mathbf{s}$ can be computed in $(\alpha+1) \cdot \iota(n) + 2 \cdot \varepsilon(n) + (4\alpha + 4) \cdot \phi(n) + 2n \log n + O(n)$ flops.

3. Compute φ-cyclic generator $\tilde{G} = [\tilde{\mathbf{g}}_1, \tilde{\mathbf{g}}_2, , ..., \tilde{\mathbf{g}}_{\alpha+1}]$, $\tilde{H} = [\mathbf{h}_1, \mathbf{h}_2, , ..., \mathbf{h}_{\alpha+1}]$ of the matrix $\tilde{A}$. To do it :

   3.1. Compute in $\iota(n)$ flops the coefficients of the polynomial $p(\lambda) = \prod_{i=0}^{n-1}(\lambda - t_i) = \lambda^n + \sum_{i=0}^{n-1} \lambda^i \cdot r_i$.

   3.2. Compute in one flop the coordinates of the vector $\tilde{\mathbf{g}}_{\alpha+1} = -\mathbf{r} - \varphi \mathbf{e}_0$, where $\mathbf{r} = (r_i)_{i=0}^{n-1}$.

   3.3. Compute in one flop the coordinates of the vector $\mathbf{h}_{\alpha+1} = Z_{\frac{1}{\varphi}} A^T V(\mathbf{t})^{-T} \mathbf{e}_{n-1} = Z_{\frac{1}{\varphi}} \mathbf{s}$, where $\mathbf{s}$ is the vector, computed at step 2.

4. On the basis of the information on its φ-cyclic generator and the last row write down for $\tilde{A}$ the decomposition (1.5), then represent in $(2\alpha + 3) \cdot \phi(n) + O(n)$ flops all the factor circulants in the right-hand side in the form of (1.8) and substitute the derived expression in (2.6).

Theorem 2.2 yields the following result.

Corollary 2.3 *Let matrix $A \in \mathbf{C}^{n \times n}$ be given by a $\{\mathrm{diag}(\mathbf{t}), Z_{\frac{1}{\varphi}}\}$-generator of A of length α. Then the product of A by an arbitrary matrix from $\mathbf{C}^{n \times n}$ can be computed in*

$$\varepsilon(n) \cdot n + (2\alpha + 5) \cdot \phi(n) \cdot n + O(n^2)$$

flops.

3 Cauchy-related matrices

Let s_i, t_i $(i = 0, 1, ..., n-1)$ be $2n$ different complex numbers and consider the choice $F_f = \mathrm{diag}(\mathbf{s})$, $F_b = \mathrm{diag}(\mathbf{t})$ with $\mathbf{s} = (s_i)_{i=0}^{n-1}$ and $\mathbf{t} = (t_i)_{i=0}^{n-1}$. Without loss of generality we can assume that $s_i \neq 0$ $(i = 0, 1, ..., n-1)$. It is easy to realize that for Cauchy matrix

$$C(\mathbf{s}, \mathbf{t}) = \Big(\frac{1}{s_i - t_j} \Big)_{i=0}^{n-1}$$

the following equality holds

$$\nabla_{\{\mathrm{diag}(\mathbf{q}), \mathrm{diag}(\mathbf{t})\}}(C(\mathbf{s}, \mathbf{t})) = \mathbf{q} \cdot \left[\begin{array}{cccc} 1 & 1 & \cdots & 1 \end{array} \right], \tag{3.1}$$

where $\mathbf{q} = (\frac{1}{s_i})_{i=0}^{n-1}$.

If matrix $A \in \mathbf{C}^{n\times n}$ satisfies the condition

$$\nabla_{\{\mathrm{diag}(\mathbf{q}),\mathrm{diag}(\mathbf{t})\}}(A) = G \cdot H^T$$

with $G,\ H \in \mathbf{C}^{n\times\alpha}$ and the number $\alpha \in \mathbf{N}$ is small in comparison with the size of the matrix, then by analogy with (3.1), we shall refer to A as a *Cauchy-related matrix.*

The following theorem shows that any matrix is uniquely determined by a $\{\mathrm{diag}(\mathbf{q}), \mathrm{diag}(\mathbf{t})\}$-generator.

Theorem 3.1 *Let* $G = [\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_\alpha]$ and $H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_\alpha]$ *be arbitrary matrices from* $\mathbf{C}^{n\times n}$ $(\mathbf{g}_m, \mathbf{h}_m \in \mathbf{C}^n)$ *and* $\mathbf{s} = (s_i)_{i=0}^{n-1}$, $\mathbf{t} = (t_i)_{i=0}^{n-1}$, *where* $s_i\ (\neq 0), t_i\ (i = 0, 1, ..., n-1)$ *are* $2n$ *different numbers. Then the unique solution* $X \in \mathbf{C}^{n\times n}$ *of the equation*

$$\nabla_{\{\mathrm{diag}(\mathbf{q}),\mathrm{diag}(\mathbf{t})\}}(X) = G \cdot H^T \tag{3.2}$$

with $\mathbf{q} = (\frac{1}{s_i})_{i=0}^{n-1}$ *is given by*

$$X = \mathrm{diag}(\mathbf{s}) \cdot \sum_{m=1}^{\alpha} \mathrm{diag}(\mathbf{g}_m) \cdot C(\mathbf{s}, \mathbf{t}) \cdot \mathrm{diag}(\mathbf{h}_m). \tag{3.3}$$

Proof. The uniqueness of the solution of the equation (3.2) follows from the fact that the numbers s_i, t_i are pairwise different (see, for example, p. 411 in [16]). Furthermore, let matrix X be given by (3.3). Then

$$\nabla_{\{\mathrm{diag}(\mathbf{q}),\mathrm{diag}(\mathbf{t})\}}(X) = \mathrm{diag}(\mathbf{s}) \cdot \sum_{m=1}^{\alpha} \mathrm{diag}(\mathbf{g}_m) \cdot \nabla_{\{\mathrm{diag}(\mathbf{q}),\mathrm{diag}(\mathbf{t})\}}(C(\mathbf{s}, \mathbf{t})) \cdot \mathrm{diag}(\mathbf{h}_m) =$$

$$= \sum_{m=1}^{\alpha} \mathrm{diag}(\mathbf{g}_m) \cdot \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \cdot \mathrm{diag}(\mathbf{h}_m) = \sum_{m=1}^{\alpha} \mathbf{g}_m \cdot \mathbf{h}_m^T = G \cdot H^T. \square$$

Formula (3.3) implies that

$$\mathrm{Comp}(A, \mathbf{b}) \leq \alpha \cdot \iota(n) + \alpha \cdot \varepsilon(n) + O(n)$$

with $\iota(n) + 2\varepsilon(n) + O(n)$ flops preprocessing. Below we show how this estimate can be reduced.

Theorem 3.2 *Let matrix* $A \in \mathbf{C}^{n\times n}$ *be given by a* $\{\mathrm{diag}(\mathbf{q}), \mathrm{diag}(\mathbf{t})\}$*-generator. Then*

$$\mathrm{Comp}(A, \mathbf{b}) \leq \iota(n) + \varepsilon(n) + (2\alpha + 9) \cdot \phi(n) + O(n)$$

and

$$\mathrm{Prep}(A) \leq (4\alpha + 6) \cdot \iota(n) + (2\alpha + 8) \cdot \varepsilon(n) + (2\alpha + 11)\phi(n) + 4n \log n + O(n). \tag{3.4}$$

Proof. Set

$$p(\lambda)=\prod_{i=0}^{n-1}(\lambda-t_i)=\lambda^n+\sum_{i=0}^{n-1}r_i\cdot\lambda^i,\quad \mathbf{r}=(r_i)_{i=0}^{n-1}\in\mathbf{C}^n,$$

$$u(\lambda)=\prod_{i=0}^{n-1}(\lambda-\frac{1}{s_i})=\lambda^n+\sum_{i=0}^{n-1}v_i\cdot\lambda^i,\quad \mathbf{v}=(v_i)_{i=0}^{n-1}\in\mathbf{C}^n.$$

According to (2.5) :

$$\operatorname{diag}(\mathbf{t})=V(\mathbf{t})^{-T}\cdot C_{\mathbf{r}}^T\cdot V(\mathbf{t})^T,$$
$$\operatorname{diag}(\mathbf{q})=V(\mathbf{q})\cdot C_{\mathbf{v}}\cdot V(\mathbf{q})^{-1}.$$

From the last equalities it follows that if

$$G=[\mathbf{g}_1,\mathbf{g}_2,...,\mathbf{g}_\alpha]\quad\text{and}\quad H=[\mathbf{h}_1,\mathbf{h}_2,...,\mathbf{h}_\alpha]\quad(\mathbf{g}_m,\ \mathbf{h}_m\in\mathbf{C}^n)$$

is the $\{\operatorname{diag}(\mathbf{q}),\operatorname{diag}(\mathbf{t})\}$-generator of A, then

$$\nabla_{\{C\mathrm{v},C\mathrm{r}\}}(\tilde{A})=G_1\cdot H_1^T, \tag{3.5}$$

where $\tilde{A}=V(\mathbf{q})^{-1}\cdot A\cdot V(\mathbf{t})^{-T}$, $G_1=V(\mathbf{q})^{-1}\cdot G$ and $H_1=V(\mathbf{t})^{-1}\cdot H$. Furthermore, substituting the expressions $C_{\mathbf{v}}=Z_\varphi-(\mathbf{v}+\varphi\mathbf{e}_0)\cdot\mathbf{e}_{n-1}^T$ and $C_{\mathbf{r}}=Z_{\frac{1}{\varphi}}-(\mathbf{r}+\frac{1}{\varphi}\mathbf{e}_0)\cdot\mathbf{e}_{n-1}^T$ in (3.5), we get

$$\nabla_{\{Z_\varphi,Z_{\frac{1}{\varphi}}\}}(\tilde{A})=\tilde{G}\cdot\tilde{H}^T,$$

where

$$\tilde{G}=[\tilde{\mathbf{g}}_1,\tilde{\mathbf{g}}_2,...,\tilde{\mathbf{g}}_{\alpha+2}]\quad\text{and}\quad\tilde{G}=[\tilde{\mathbf{h}}_1,\tilde{\mathbf{h}}_2,...,\tilde{\mathbf{h}}_{\alpha+2}]$$

with

$$\tilde{\mathbf{g}}_m=V(\mathbf{q})^{-1}\mathbf{g}_m,\quad \tilde{\mathbf{h}}_m=V(\mathbf{t})^{-1}\mathbf{h}_m\quad(m=1,2,...,\alpha),$$
$$\tilde{\mathbf{g}}_{\alpha+1}=-\mathbf{v}-\varphi\mathbf{e}_0,\quad \tilde{\mathbf{h}}_{\alpha+1}=C_{\mathbf{r}}\cdot V(\mathbf{t})^{-1}A^T\cdot V(\mathbf{q})^{-T}\mathbf{e}_{n-1},$$
$$\tilde{\mathbf{g}}_{\alpha+2}=Z_\varphi\cdot V(\mathbf{q})^{-1}\cdot A\cdot V(\mathbf{t})^{-T}\mathbf{e}_{n-1},\quad \tilde{\mathbf{h}}_{\alpha+2}=-\mathbf{r}-\frac{1}{\varphi}\mathbf{e}_0.$$

Thus, A can be represented in the form

$$A=V(\mathbf{q})\cdot\tilde{A}\cdot V(\mathbf{t})^T, \tag{3.6}$$

where the φ-cyclic displacement of $\tilde{A}$ does not exceed $\alpha+2$. From this representation (3.6) and theorem 1.2 it follows that

$$\operatorname{Comp}(A,\mathbf{b})\le\iota(n)+\varepsilon(n)+(2\alpha+9)\cdot\phi(n)+O(n).$$

The corresponding estimate for the value $\operatorname{Prep}(A)$ follows from the detailed description of the preprocessing, which is given below.□

Preprocessing for matrix A given by $\{\operatorname{diag}(\mathbf{q}),\operatorname{diag}(\mathbf{t})\}$-generator

$$G=[\mathbf{g}_1,\mathbf{g}_2,...,\mathbf{g}_\alpha]\quad\text{and}\quad H=[\mathbf{h}_1,\mathbf{h}_2,...,\mathbf{h}_\alpha]\quad(\mathbf{g}_m,\mathbf{h}_m\in\mathbf{C}^n).$$

1. Compute, using (3.3) in $(\alpha+3)\cdot\iota(n)+(\alpha+4)\cdot\varepsilon(n)+3\cdot\phi(n)+2n\log n+O(n)$ flops the last column $\mathbf{f}=V(\mathbf{q})^{-1}\cdot A\cdot V(\mathbf{t})^{-T}\mathbf{e}_{n-1}$ of the matrix $\tilde{A}=V(\mathbf{q})^{-1}AV(\mathbf{t})^{-T}$.

2. Compute $\{C_{\mathbf{v}},C_{\mathbf{r}}\}$-generator

$$G_1=[\tilde{\mathbf{g}}_1,\tilde{\mathbf{g}}_2,...,\tilde{\mathbf{g}}_\alpha],\qquad H_1=[\tilde{\mathbf{h}}_1,\tilde{\mathbf{h}}_2,...,\tilde{\mathbf{h}}_\alpha]$$

of matrix $\tilde{A}$. To do this compute in $2\alpha\cdot\iota(n)$ flops the vectors

$$\tilde{\mathbf{g}}_m=V(\mathbf{q})^{-1}\mathbf{g}_m,\qquad \tilde{\mathbf{h}}_m=V(\mathbf{t})^{-1}\mathbf{h}_m\qquad (m=1,2,...,\alpha).$$

3. Compute the φ-cyclic generator

$$\tilde{G}=[\tilde{\mathbf{g}}_1,\tilde{\mathbf{g}}_2,...,\tilde{\mathbf{g}}_{\alpha+2}],\qquad \tilde{G}=[\tilde{\mathbf{h}}_1,\tilde{\mathbf{h}}_2,...,\tilde{\mathbf{h}}_{\alpha+2}]$$

of matrix $\tilde{A}$. To do this :

3.1. Compute using formula (3.3) in $(\alpha+3)\cdot\iota(n)+(\alpha+4)\cdot\varepsilon(n)+3\cdot\phi(n)+2n\log n+O(n)$ flops the vector $\tilde{\mathbf{h}}_{\alpha+1}=C_{\mathbf{r}}\cdot V(\mathbf{t})^{-1}A^T\cdot V(\mathbf{q})^{-T}\mathbf{e}_{n-1}$.

3.2. Set $\tilde{\mathbf{g}}_{\alpha+1}=-\mathbf{v}-\varphi\mathbf{e}_0$.

3.3. Compute in $O(n)$ flops the vector $\tilde{\mathbf{g}}_{\alpha+2}=-Z_\varphi\mathbf{f}$, where $\mathbf{f}$ is the last column of $\tilde{A}=V(\mathbf{q})^{-1}AV(\mathbf{t})^{-T}$ computed at the step 1.

3.4. Set $\tilde{\mathbf{h}}_{\alpha+2}=-\mathbf{r}-\frac{1}{\varphi}\mathbf{e}_0$.

4. On the basis of the information on φ-cyclic generator and last column $\tilde{A}$, compute in $(2\alpha+5)\cdot\phi(n)$ flops the parameters of representation $\tilde{A}$ in the form (1.8) and substitute it in (3.6).

Let us remark that step 1 above contains preprocessing of the matrix $C(\mathbf{s},\mathbf{t})$, while step 3.1 contains preprocessing of the matrix $C(\mathbf{s},\mathbf{t})^T$. Using the results from [11], it is possible to compute the product of the matrix $C(\mathbf{s},\mathbf{t})^T$ by a vector in $\iota(n)+\varepsilon(n)+4\cdot\phi(n)+O(n)$ flops on the basis of the results of preprocessing of the matrix $C(\mathbf{s},\mathbf{t})$. Therefore the estimate (3.4) in the theorem 3.2 can be replaced by the following estimate

$$\text{Prep}(A)\le(4\alpha+5)\cdot\iota(n)+(2\alpha+6)\cdot\varepsilon(n)+(6\alpha+11)\phi(n)+4n\log n+O(n).$$

Corollary 3.3 *Let matrix $A\in\mathbf{C}^{n\times n}$ be given by a $\{\text{diag}(\mathbf{q}),\text{diag}(\mathbf{t})\}$-generator. Then the product of A with an arbitrary matrix from $\mathbf{C}^{n\times n}$ can be computed in $\iota(n)\cdot n+\varepsilon(n)\cdot n+(2\alpha+7)\cdot\phi(n)\cdot n+O(n^2)$ flops.*

4 Inverses of Toeplitz-related , Vandermonde-related and Cauchy-related matrices

4.1 Inverses of Toeplitz-related matrices

It is easy to see that if matrices F_f and F_b are invertible, then $\{F_f,F_b\}$-displacement of matrix A and $\{F_b^{-T},F_f^{-T}\}$-displacement of matrix A^{-1} are related by

$$\nabla_{\{F_b^{-T},F_f^{-T}\}}(A^{-1})=-A^{-1}\cdot\nabla_{\{F_f,F_b\}}(A)\cdot F_b^{-T}\cdot A^{-1}\cdot F_f^{-1}.$$

For $F_f = Z_\varphi$ and $F_b = Z_{\frac{1}{\varphi}}$ (this choice corresponds to Toeplitz matrices) the last equality takes the shape

$$\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(A^{-1}) = -A^{-1} \cdot \nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(A) \cdot Z_\varphi A^{-1} Z_{\frac{1}{\varphi}}^T. \tag{4.1}$$

This equality implies that φ-cyclic displacement rank is invariant under the matrix inversion and therefore the inverses of Toeplitz-related matrices are themselves Toeplitz-related. Moreover, the equality (4.1) yields that if

$$G = [\mathbf{g}_1, \mathbf{g}_2, ..., \mathbf{g}_\alpha], \qquad H = [\mathbf{h}_1, \mathbf{h}_2, ..., \mathbf{h}_\alpha] \qquad (\mathbf{g}_m, \mathbf{h}_m \in \mathbf{C}^n) \tag{4.2}$$

is φ-cyclic generator of matrix $A \in \mathbf{C}^{n \times n}$, then computing of φ-cyclic generator

$$R = [\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_\alpha], \quad S = [\mathbf{s}_1, \mathbf{s}_2, ..., \mathbf{s}_\alpha] \quad (\mathbf{r}_m, \mathbf{s}_m \in \mathbf{C}^n)$$

of the inverse matrix A^{-1} is reduced to the solving of the following 2α equations :

$$A\mathbf{r}_m = \mathbf{g}_m \qquad (m = 1, 2, ..., \alpha), \tag{4.3}$$

$$\mathbf{s}_m^T \cdot Z_\varphi A Z_{\frac{1}{\varphi}}^T = -\mathbf{h}_m^T \qquad (m = 1, 2, ..., \alpha) \tag{4.4}$$

Furthermore, the last row of A^{-1} is recovered from one more equation

$$\mathbf{y}^T A = \mathbf{e}_{n-1}^T. \tag{4.5}$$

On the basis of the information on φ-cyclic generator and the last row of A^{-1} we can write down for it the formula (1.5) :

$$A^{-1} = \mathrm{Circ}_{lr} + \frac{\varphi}{\varphi - \psi} \cdot \sum_{m=1}^{\alpha} \mathrm{Circ}_\psi(\mathbf{r}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{s}_m)^T,$$

where ψ ($\neq \varphi$) is arbitrary, the matrix Circ_{lr} is the φ-circulant with the last row $\mathbf{y}^T$ fulfilling the equation (4.5) and vectors $\mathbf{r}_m$, $\mathbf{s}_m$ $(m = 1, 2, ..., \alpha)$ are the solutions of the equations (4.3) and (4.4), correspondingly. Proposing the above scheme, we suppose the matrix A to be invertible. Note that this assumption is unnecessary, since the solvability of the equations (4.4) and (4.5) automatically yields the invertibility of A. Indeed, assume that we succeeded to solve the equations (4.4) and (4.5), but there exists nonzero vector $\mathbf{p} \in \mathbf{C}^n$ such that $A\mathbf{p} = 0$. Then (4.4) imply $\nabla_{\{Z_\varphi, Z_{\frac{1}{\varphi}}\}}(A) Z_\varphi \mathbf{p} = 0$ and hence $A Z_\varphi \mathbf{p} = 0$. Using the same arguments, we get the equalities $A Z_\varphi^k \mathbf{p} = 0$ $(k = 0, 1, ..., n-1)$. Premultiplying the last equalities by the solution $\mathbf{y}^T$ of the equation (4.5), we can conclude that all the coordinates of the vector $\mathbf{p}$ are identically equal to zero. Thus, the invertibility of A is now proved.

In the next subsection the above scheme will be embodied for the inverses of Toeplitz matrices. Here we show that divide-and-conquer implementation from [6] of generalized Schur algorithm is useful for computing the φ-cyclic displacement of strongly regular Toeplitz-related matrices (a strongly regular matrix is, by definition, a matrix whose all leading minors are nonzero).

Theorem 4.1 *Let strongly regular matrix* $A \in \mathbf{C}^{n\times n}$ *be given by a* φ*-cyclic generator and the last row (or last column). Then*

$$\mathrm{Comp}(A^{-1}, \mathbf{b}) \le (2\alpha + 3) \cdot \phi(n) + O(n) \tag{4.6}$$

and

$$\mathrm{Prep}(A^{-1}) \le O(\alpha^3 n \log^2 n). \tag{4.7}$$

Proof. As was mentioned, the φ-cyclic displacement rank is inherited under the matrix inversion. Therefore the estimate (4.6) immediately follows from theorem 1.2. The estimate (4.7) follows from the detailed description of the preprocessing for matrix A^{-1}.□

Preprocessing for the inverse of matrix A with given φ-cyclic generator of length $\alpha \in \mathbf{N}$ and the last row.

1. Compute in $O(\alpha n \log n)$ flops for matrix A its $\{Z_0, Z_0\}$-generator of length $\alpha+2$, using equalities $Z_\varphi = Z_0 + \varphi \mathbf{e}_0 \cdot \mathbf{e}_{n-1}^T$ and $Z_{\frac{1}{\varphi}}^T = Z_0 + \frac{1}{\varphi}\mathbf{e}_{n-1} \cdot \mathbf{e}_0^T$.

2. Using the algorithm from [6] compute in $O(\alpha^3 n \log^2 n)$ flops for matrix A^{-1} its $\{Z_0, Z_0\}$-generator of length $\alpha + 4$.

3. Using representation (0.2) corresponding to the computed above $\{Z_0, Z_0\}$-generator of A^{-1}, compute in $O(\alpha n \log n)$ flops the φ-cyclic generator of A^{-1} (that is the vectors $\mathbf{r}_m$, $\mathbf{s}_m$ in (4.2) and (4.3)) and the last row $\mathbf{y}^T = \mathbf{e}_{n-1}^T A^{-1}$.

4. On the basis of φ-cyclic generator of A^{-1} and its last row, compute in $O(\alpha n \log n)$ flops the parameters of the representation A^{-1} in the form (1.8) as is prescribed in the section 1.

4.2 Inverses of Toeplitz matrices

If a matrix is the inverse of the given Toeplitz matrix, then its product by a vector can be computed using the Gohberg-Semencul formula. In [12] (see also [9]) it was proved, that if for Toeplitz matrix $A = (a_{i-j})_{i,j=0}^{n-1}$, the equations

$$A\mathbf{x} = \mathbf{e}_0, \quad A\mathbf{y} = \mathbf{e}_{n-1} \tag{4.8}$$

have solutions $\mathbf{x} = (x_i)_{i=0}^{n-1}$, $\mathbf{y} = (y_i)_{i=0}^{n-1}$ and $x_0 \neq 0$, then A is invertible and

$$T^{-1} = \frac{1}{x_0}\left(\begin{bmatrix} x_0 & 0 & \cdots & \cdots & 0 \\ x_1 & x_0 & & & \vdots \\ \vdots & x_1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ x_{n-1} & \cdots & \cdots & x_1 & x_0 \end{bmatrix} \cdot \begin{bmatrix} y_{n-1} & y_{n-2} & \cdots & \cdots & y_0 \\ 0 & y_{n-1} & y_{n-2} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & y_{n-2} \\ 0 & \cdots & \cdots & 0 & y_{n-1} \end{bmatrix} - \right.$$

$$
- \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ y_0 & 0 & & & \vdots \\ \vdots & y_0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ y_{n-2} & \cdots & \cdots & y_0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & x_{n-1} & \cdots & \cdots & x_1 \\ 0 & 0 & x_{n-1} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & x_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix}). \tag{4.9}
$$

On the basis of this formula a number of fast algorithms for solving Toeplitz systems of equations was elaborated. These algorithms were divided into two separate stages: first solving the equations (4.8) and second, application of the formula (4.9). Let us denote by $\tau(n)$ the complexity of solving one equation of the type (4.8). According to [3]

$$
\tau(n) \leq O(n \log^2 n)
$$

flops. Thus the Gohberg-Semencul formula enables one to compute the product of the inverse of Toeplitz matrix by a vector in $16 \cdot \phi(n) + O(n)$ flops after $2 \cdot \tau(n) + 8 \cdot \phi(n) + O(n)$ flops preprocessing. The implementation of the scheme from the first part of subsection 4.1 leads to the reduction of this estimate.

Indeed, in the first section the explicit expression (1.2) for φ-cyclic displacement of Toeplitz matrix $A = (a_{i-j})_{i,j=0}^{n-1}$ was written down. The equations (4.3) in this case are of the form

$$
A\mathbf{x} = \mathbf{e}_0, \quad A\mathbf{u} = \begin{bmatrix} \beta \\ a_1 - \frac{1}{\varphi} a_{-n+1} \\ \vdots \\ a_{n-1} - \frac{1}{\varphi} a_{-1} \end{bmatrix}. \tag{4.10}
$$

The equations (4.4) take the shape

$$
\mathbf{p}^T \cdot Z_\varphi A Z_{\frac{1}{\varphi}}^T = - \begin{bmatrix} -\beta \\ a_{-1} - \varphi a_{n-1} \\ \vdots \\ a_{-n+1} - \varphi a_1 \end{bmatrix}^T, \quad \mathbf{q}^T \cdot Z_\varphi A Z_{\frac{1}{\varphi}}^T = -\mathbf{e}_0^T. \tag{4.11}
$$

Furthermore, the equality $A^T = JAJ$ with counter-identity matrix

$$
J = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix}.
$$

holds for arbitrary Toeplitz matrix $A \in \mathbf{C}^{n \times n}$. Hence (4.10) and (4.11) imply that the vectors $\mathbf{p}$ and $\mathbf{q}$ are related with the vectors $\mathbf{x} = (x_i)_{i=0}^{n-1}$ and $\mathbf{y} = (y_i)_{i=0}^{n-1}$ as follows :

$$
\mathbf{p} = \varphi Z_{\frac{1}{\varphi}} J\mathbf{u}, \quad \mathbf{q} = -\varphi Z_{\frac{1}{\varphi}} J\mathbf{x}.
$$

Therefore, in the Toeplitz case the φ-cyclic generator of A^{-1} is represented in the form :

$$R = \begin{bmatrix} x_0 & u_0 \\ x_1 & u_1 \\ x_2 & u_2 \\ \vdots & \vdots \\ x_{n-1} & u_{n-1} \end{bmatrix}, \quad S = \begin{bmatrix} u_0 & -x_0 \\ \varphi u_{n-1} & -\varphi x_{n-1} \\ \varphi u_{n-2} & -\varphi x_{n-2} \\ \vdots & \vdots \\ \varphi u_1 & -\varphi x_1 \end{bmatrix} \ (\in \mathbf{C}^{n\times 2}). \tag{4.12}$$

The factor circulant decomposition (1.5) for matrix A^{-1} corresponding to (4.12) is given by

$$A^{-1} = \frac{\varphi}{\varphi-\psi}\Big(\mathrm{Circ}_\psi(\mathbf{x})\cdot\mathrm{Circ}_\varphi(\mathbf{u}) - \mathrm{Circ}_\psi(\mathbf{u} - \frac{\varphi-\psi}{\varphi}\mathbf{e}_0)\cdot\mathrm{Circ}_\varphi(\mathbf{x})\Big). \tag{4.13}$$

The representation (4.13) is determined by the solutions of the equations (4.10). The right-hand side of the second equation in (4.10) depended upon the entries of the matrix A. Below we show how to write down for A^{-1} the factor circulant decomposition on the basis of the solutions of the equations (4.8). Such a decomposition will allow to apply at the stage of preprocessing the fast algorithm of Brent, Gustavson and Yun [3].

Lemma 4.2 *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be a Toeplitz matrix. If the equations*

$$A\mathbf{x} = \mathbf{e}_0 \quad \text{and} \quad A\mathbf{y} = \mathbf{e}_{n-1}$$

have solutions $\mathbf{x} = (x_i)_{i=0}^{n-1}$ and $\mathbf{y} = (y_i)_{i=0}^{n-1}$ and $x_0 \neq 0$ then the pair of matrices

$$R_1 = \begin{bmatrix} x_0 & 1 \\ x_1 & \frac{1}{x_0\varphi}y_0 \\ x_2 & \frac{1}{x_0\varphi}y_1 \\ \vdots & \vdots \\ x_{n-1} & \frac{1}{x_0\varphi}y_{n-2} \end{bmatrix}, \quad S_1 = \begin{bmatrix} 1 & -x_0 \\ \frac{1}{x_0}y_{n-2} & -\varphi x_{n-1} \\ \frac{1}{x_0}y_{n-3} & -\varphi x_{n-2} \\ \vdots & \vdots \\ \frac{1}{x_0}y_0 & -\varphi x_1 \end{bmatrix} \ (\in \mathbf{C}^{n\times 2}), \tag{4.14}$$

is a φ-cyclic generator of the matrix A^{-1}.

Proof. First note that to the upper left and to the lower right entries of Toeplitz matrix A there corresponds the same associated matrices. Therefore $x_0 = y_{n-1}$ since x_0 and y_{n-1} are the entries of the inverse matrix, placed in the upper left and lower right corners, correspondingly. Now let us show that the vector $\mathbf{u} = \frac{1}{\varphi x_0}Z_\varphi \mathbf{y}$ solves the second equation (4.10) for some $\beta \in \mathbf{C}^n$. Indeed,

$$A\mathbf{u} = \frac{1}{\varphi x_0}A\begin{bmatrix} \varphi y_{n-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{1}{\varphi x_0}A\begin{bmatrix} 0 \\ y_0 \\ \vdots \\ y_{n-2} \end{bmatrix} =$$

$$= \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} + \frac{1}{\varphi x_0}\begin{bmatrix} \sum_{i=1}^{n-1} a_{-i}y_{i-1} \\ -y_{n-1}a_{-n+1} \\ \vdots \\ -y_{n-1}a_{-1} \end{bmatrix} = \begin{bmatrix} \beta \\ a_1 - \frac{1}{\varphi}a_{-n+1} \\ \vdots \\ a_{n-1} - \frac{1}{\varphi}a_{-1} \end{bmatrix},$$

where $\beta = a_0 + \frac{1}{\varphi x_0}\sum_{i=1}^{n-1} a_{-i}y_{i-1}$. To prove (4.14) it remains to substitute $\mathbf{u} = \frac{1}{\varphi x_0} Z_\varphi \mathbf{y}$ in (4.12). □

The factor circulant decomposition

$$A^{-1} = \frac{1}{x_0(\varphi - \psi)}(\mathrm{Circ}_\psi(\mathbf{x}) \cdot \mathrm{Circ}_\varphi(Z_\varphi \mathbf{y}) - \mathrm{Circ}_\psi(Z_\psi \mathbf{y}) \cdot \mathrm{Circ}_\varphi(\mathbf{x}), \tag{4.15}$$

corresponds to the representation (4.14) of φ-cyclic generator of matrix A^{-1}, where $\mathbf{x}$ and $\mathbf{y}$ are the solutions of the equations (4.8) and φ, ψ ($\neq \varphi$) are arbitrary complex numbers. Formula (4.15) is a generalization of the formula of G.Ammar and P.Gader [2], where they considered the case $\varphi = 1$, $\psi = -1$ and positive definite Toeplitz matrix A. This formula allows to prove the following result.

Theorem 4.3 *Let* $A = (a_{i-j})_{i,j=0}^{n-1}$ *be invertible Toeplitz matrix with nonzero leading minor of order* $n-1$. *Then*

$$\mathrm{Comp}(A^{-1}, \mathbf{b}) \leq 6 \cdot \phi(n) + O(n) \tag{4.16}$$

and

$$\mathrm{Prep}(A^{-1}) \leq 2 \cdot \tau(n) + 4 \cdot \phi(n) + O(n). \tag{4.17}$$

Proof. The conditions of the theorem yield that the equations (4.8) are solvable and $x_0 \neq 0$. In this case the factor circulant decomposition (4.15) holds. Thus, the procedure of preprocessing consists of solving both two equations in (4.8) and then representing each factor circulant in the right hand side of (4.15) as in (1.8). The overall cost of all of these actions coincides with the right hand side of the inequality (4.17). Finally, we get

$$A^{-1} = \frac{1}{x_0(\varphi - \psi)} D_\psi^{-1} \cdot \mathcal{F}^* \cdot \Big(\Lambda_\psi(\mathbf{x}) \cdot \mathcal{F} \cdot D_\psi \cdot D_\varphi^{-1} \cdot \mathcal{F}^* \cdot \Lambda_\varphi(Z_\varphi \mathbf{y}) -$$

$$-\Lambda_\psi(Z_\psi \mathbf{y}) \cdot \mathcal{F} \cdot D_\psi \cdot D_\varphi^{-1} \cdot \mathcal{F}^* \cdot \Lambda_\varphi(\mathbf{x})\Big) \cdot \mathcal{F} \cdot D_\varphi,$$

where matrices D_φ and $\Lambda_\varphi(\cdot)$ are defined as in (1.7). From here follows the estimate (4.16).□

4.3 Improvements via Padé approximations

In the formulation of the theorem 4.3 the matrix $A = (a_{i-j})_{i,j=0}^{n-1}$ is assumed to have nonzero leading minor of order $n-1$. In this subsection it will be shown that this restriction can be removed by making use of Cabay-Choi algorithm [4] for computing the scaled Padé fractions. Before stating the result let us introduce necessary concepts.

Let $A(z) = \sum_{i=0}^{\infty} b_i \cdot z^i$ ($b_0 \neq 0$) be formal power series and $T_{mn}(z) = \sum_{i=0}^{n} t_i \cdot z^i$ ($\not\equiv 0$) and $S_{mn}(z) = \sum_{i=0}^{m} s_i \cdot z^i$ be two polynomials. Following [4] we will refer to the rational function $\frac{S_{mn}(z)}{T_{mn}(z)}$ as *scaled Padé fraction* of type (m,n) for $A(z)$ if the following three conditions hold :

(i) $\min(m - \deg(S_{mn}(z)), n - \deg(T_{mn}(z))) = 0$,

(ii) $\mathrm{GCD}(S_{mn}(z), T_{mn}(z)) = z^{\lambda_{mn}}$ for some integer $\lambda_{mn} \geq 0$,

(iii) $A(z) \cdot T_{mn}(z) - S_{mn}(z) = z^{m+n+1} \cdot W(z)$, where $W(z)$ is a formal power series.

It is well known that the condition (iii) can be rewritten as following two systems of linear equations :

$$\begin{bmatrix} b_{m-n} & b_{m-n+1} & \cdots & b_{m-1} & b_m \\ b_{m-n-1} & b_{m-n} & \cdots & b_{m-2} & b_{m-1} \\ \vdots & \vdots & & \vdots & \vdots \\ b_{-n+1} & b_{-n+2} & \cdots & b_0 & b_1 \\ b_{-n} & b_{-n+1} & \cdots & b_{-1} & b_0 \end{bmatrix} \cdot \begin{bmatrix} t_n \\ t_{n-1} \\ \vdots \\ t_1 \\ t_0 \end{bmatrix} = \begin{bmatrix} s_m \\ s_{m-1} \\ \vdots \\ s_1 \\ s_0 \end{bmatrix}$$

and

$$\begin{bmatrix} b_m & b_{m+1} & b_{m+2} & \cdots & b_{m+n-1} & b_{m+n} \\ b_{m-1} & b_m & b_{m+1} & \cdots & b_{m+n-2} & b_{m+n-1} \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ b_{m-n+1} & b_{m-n+2} & \cdots & b_{m-1} & b_m & b_{m+1} \end{bmatrix} \cdot \begin{bmatrix} t_n \\ t_{n-1} \\ \vdots \\ t_1 \\ t_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{4.18}$$

Here $b_i = 0$ for $i < 0$.

Let $\frac{S_{mn}(z)}{T_{mn}(z)}$ be a scaled Padé fraction of type (m, n) for $A(z)$, $\mathrm{GCD}(S_{mn}(z), T_{mn}(z)) = z^{\lambda_{mn}}$ and put

$$m^* = m - \lambda_{mn} - 1, \qquad n^* = n - \lambda_{mn} - 1.$$

Following [4] we will refer to scaled Padé fraction $\frac{S_{m^*n^*}(z)}{T_{m^*n^*}(z)}$ of the type (m^*, n^*) for $A(z)$ as the *predecessor* of $\frac{S_{mn}(z)}{T_{mn}(z)}$.

For $m, n \in \mathbf{N}$ and for formal power series $A(z)$ denote by $\pi(m, n)$ the complexity of the computing for $A(z)$ the scaled Padé fraction $\frac{S_{mn}(z)}{T_{mn}(z)}$ together with its predecessor $\frac{S_{m^*n^*}(z)}{T_{m^*n^*}(z)}$. According to [4]

$$\pi(m, n) \le O((m + n) \log^2(m + n)).$$

The referee pointed out that this result enables one to drop out in theorem 4.3 the condition of invertibility of leading submatrix of order $n - 1$. In fact the following result is obtained.

Theorem 4.4 *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be invertible Toeplitz matrix. Then*

$$\mathrm{Comp}(A^{-1}, \mathbf{b}) \le 6 \cdot \phi(n) + O(n) \tag{4.19}$$

and

$$\mathrm{Prep}(A^{-1}) \le \pi(n, n) + 4 \cdot \phi(n) + O(n). \tag{4.20}$$

Proof. First of all let us show how using the Cabay-Choi algorithm [4] one can compute the solutions of the equations (4.10). Set

$$A(z) = 1 + \sum_{i=1}^{2n-1} a_{n-i} \cdot z^i$$

and let rational function $\frac{S_{nn}(z)}{T_{nn}(z)}$ be scaled Padé fraction of type (n,n) for $A(z)$. Then the equality (4.18) take the shape

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} & 0 \\ a_1 & a_0 & a_{-1} & \cdots & a_{-n+2} & a_{-n+1} \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & \cdots & a_0 & a_{-1} \end{bmatrix} \cdot \begin{bmatrix} t_n \\ t_{n-1} \\ \vdots \\ t_1 \\ t_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{4.21}$$

Since matrix $A = (a_{i-j})_{i,j=0}^{n-1}$ is invertible then from (4.21) follows that $t_0 \neq 0$ (cf. [15]). Therefore $\mathrm{GCD}(S_{nn}(z), T_{nn}(z)) = 1$ and predecessor $\frac{S_{m^*n^*}(z)}{T_{m^*n^*}(z)}$ of $\frac{S_{nn}(z)}{T_{nn}(z)}$ is scaled Padé fraction of type $(n-1, n-1)$ for $A(z)$. For polynomial $T_{n^*m^*}(z) = \sum_{i=0}^{n-1} p_i \cdot z^i$ the equality (4.18) looks as follows

$$\begin{bmatrix} a_1 & a_0 & \cdots & a_{-n+2} & a_{-n+1} \\ a_2 & a_1 & \cdots & a_{-n+3} & a_{-n+2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & \cdots & a_2 & a_1 \end{bmatrix} \cdot \begin{bmatrix} p_{n-1} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{4.22}$$

From (4.21) and (4.22) it is easy to deduce the folowing two matrix equalities :

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-n+2} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix} \cdot \begin{bmatrix} t_n \\ t_{n-1} \\ \vdots \\ t_2 \\ t_1 \end{bmatrix} = -t_0 \cdot \begin{bmatrix} 0 \\ a_{-n+1} \\ \vdots \\ a_{-2} \\ a_{-1} \end{bmatrix} \tag{4.23}$$

and

$$\begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & a_{-n+1} \\ a_1 & a_0 & a_{-1} & \cdots & a_{-n+2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & a_{-1} \\ a_{n-1} & a_{n-2} & \cdots & a_1 & a_0 \end{bmatrix} \cdot \begin{bmatrix} p_{n-1} \\ p_{n-2} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{4.24}$$

where $q = \sum_{i=0}^{n-1} a_{-i} \cdot p_{n-1-i}$ $(\neq 0)$. From equalities (4.23) and (4.24) follows that vectors

$$\mathbf{x} = \frac{1}{q} \cdot \begin{bmatrix} p_{n-1} \\ \vdots \\ p_1 \\ p_0 \end{bmatrix}, \qquad \mathbf{u} = \frac{1}{\varphi t_0} \cdot \begin{bmatrix} t_n + \varphi t_0 \\ t_{n-1} \\ \vdots \\ t_1 \end{bmatrix}$$

solve the equations (4.10).

Thus it is shown that equations (4.10) can be solved in $\pi(n,n) + O(n)$ flops. In this case the factor circulant decomposition (4.13) holds. So the procedure of preprocessing consists of solving the equations (4.10) via Cabay-Choi algorithm and then representing each factor circulant in the right hand side of (4.13) as in (1.8). The overall cost of all these actions coinsides with the right hand side if the inequality (4.20). Finally, the estimate (4.19) follows with completely the same arguments as in the proof of the theorem 4.3.□

4.4 Inverses of Vandermonde-related and Cauchy-related matrices

In the section 2 the fast algorithms for multiplication of Vandermonde-related matrix by an arbitrary vector were based on the formula (2.6). This formula represents Vandermonde-related matrix in the form of the product of a Vandermonde matrix by a Toeplitz-related matrix. Analogously, in the section 3 the representation (3.6) of Cauchy-related matrix in the form of the product of a Vandermonde matrix, Toeplitz-related and transpose to a Vandermonde matrix was used. Therefore, the derivation of the fast algorithms for the inverses of Vandermonde-related and Cauchy-related matrices is essentially reduced to the algorithms proposed in the subsection 4.1.

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