VARIATION OF THE JORDAN STRUCTURE OF G-SELF-ADJOINT OPERATORS AND SELF-ADJOINT OPERATOR FUNCTIONS UNDER SMALL PERTURBATIONS

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1. In [1-3] one has obtained a description of the possible domain of variation of the lengths of the Jordan chains of linear operators and holomorphic operator-functions under small perturbations. In this note similar problems are extended to the classes of G-self-adjoint operators and self-adjoint operator functions. We elucidate the role played by the so-called sign characteristic [4-6].

2. Let $\mathfrak{H}$ be a Hilbert space and let $L(\mathfrak{H})$ be the set of all linear bounded operators in $\mathfrak{H}$. If λ_0 is an isolated Fredholm eigenvalue of the operator $A \in L(\mathfrak{H})$, then by $m_i(A, \lambda_0)$ ($i = 1,\ldots, r = \dim\operatorname{Ker}(A - \lambda_0 I)$) we denote the lengths of the corresponding Jordan chains, numbered in nonincreasing order. For the sake of convenience we set $m_i(A, \lambda_0) = 0$ for $i > r$. A bounded domain $\Omega\ (\subset \mathbb{C})$ is said to be normal for the operator $A \in L(\mathfrak{H})$, if on its boundary there are no points of spectrum of A, while the whole spectrum of A in Ω consists of a finite number of Fredholm eigenvalues $\{\lambda_i\}_1^n$. We set $m_i(A, \Omega) = m_i(A, \lambda_1) + \ldots + m_i(A, \lambda_n)$.

If $\alpha = \{\alpha_i\}_1^\infty$, $\beta = \{\beta_i\}_1^\infty$ are two nonincreasing finite sequences of nonnegative integers and the relations $\sum_{i=1}^k \alpha_i \leqslant \sum_{i=1}^k \beta_i\ (k = 1, 2, \ldots)$, $\sum_{i=1}^\infty \alpha_i = \sum_{i=1}^\infty \beta_i$ hold, then we shall write $\alpha \prec \beta$.

In [1-3] it has been proved that $\{m_i(A, \Omega)\} \prec \{m_i(A', \Omega)\}$ for any operator $A' \in L(\mathfrak{H})$, sufficiently close to A. The converse theorem has been also proved: if the operator A has in Ω a unique eigenvalue λ_0 (the general case reduces easily to this) and if there are given a natural number p and a nonincreasing sequence $\{m_{ij}\}_{i=1}^\infty\ (j = 1, \ldots, p)$ such that $\{m_i(A, \lambda_0)\} \prec \{m_i'\}$, where $m_i' = m_{i1} + \ldots + m_{ip}$, then in any neighborhood of the operator A there exists an operator A' which has in Ω exactly p eigenvalues $\{\lambda_i\}_1^p$ and, moreover, $m_i(A', \lambda_j) = m_{ij}$.

Assume that in $\mathfrak{H}$ there is defined also an indefinite inner product $[f, g] = (Gf, g)\ (f, g \in \mathfrak{H})$, where $G \in L(\mathfrak{H})$ is an invertible self-adjoint operator. An operator A is said to be G-self-adjoint if $[Af, g] = [f, Ag]\ (f, g \in \mathfrak{H})$. It is easy to show that the above formulated inverse theorem from [1-3] for $\lambda_0 \notin \mathbb{R}$ remains valid for the case when the initial operator A and its perturbation A' are G-self-adjoint. The following theorem shows that in this case also for $\lambda_0 \in \mathbb{R}$ for the numbers $m_i(A', \Omega)$ additional restrictions do not arise [in Sec. 3 we show that such restrictions occur for the numbers $m_i(A', \lambda_j)$].

<u>THEOREM 1.</u> Suppose that Ω is a normal domain for a G-self-adjoint operator $A \in L(\mathfrak{H})$, containing only one of its eigenvalues $\lambda_0 \in \mathbb{R}$, and assume that condition $\{m_i(A, \lambda_0)\} \prec \{m_i'\}$ is satisfied. Then in each neighborhood of the operator A there exists a G-self-adjoint operator $A' \in L(\mathfrak{H})$ such that $\sigma(A') \cap \Omega \subset \mathbb{R}$ and $m_i(A', \Omega) = m_i'$.

3. Let $\lambda_0 \in \mathbb{R}$ be an isolated Fredholm eigenvalue of a G-self-adjoint operator $A \in L(\mathfrak{H})$. In the corresponding root subspace one can select a Jordan basis $\varphi_{ij}\ (j = 1, \ldots, m_i(A, \lambda_0);\ i = 1, \ldots, r = \dim\operatorname{Ker}(A - \lambda_0 I))$ of the operator A such that for some $\varepsilon_i(A, \lambda_0) = \pm 1$ ($i = 1,\ldots,r$) we should have $[\varphi_{ij}, \varphi_{kl}] = \varepsilon_i(A, \lambda_0)$, if $k = i$ and $j + l = m_i(A, \lambda_0) + 1$, while $[\varphi_{ij}, \varphi_{kl}] = 0$ in the remaining cases. The numbers $\varepsilon_i(A, \lambda_0)$ are called the sign characteristics of the corresponding chains [numbers $m_i(A, \lambda_0)$] (for more details see [4, 5]). We denote by $\alpha_k(A, \lambda_0)$ ($\beta_k(A, \lambda_0)$) the sum of the numbers $\varepsilon_i(A, \lambda_0)$, corresponding to an odd (even) number among $m_i(A, \lambda_0)$ ($i = 1,\ldots,k$). If among the eigenvalues of the operator A in the domain Ω one has n real eigenvalues $\{\lambda_i\}^n$, then we set $\alpha_k(A, \Omega) = \alpha_k(A, \lambda_1) + \ldots + \alpha_k(A, \lambda_n)$.

<u>THEOREM 2.</u> Let Ω be a normal domain for a G-self-adjoint operator $A \in L(\mathfrak{H})$. There exists $\delta > 0$ such that for any self-adjoint operator G' and any G'-self-adjoint operator A' satisfying the condition $\|A' - A\| + \|G' - G\| < \delta$ we have the inequalities

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$$|\alpha_k(A',\Omega)-\alpha_k(A,\Omega)|\leqslant\sum_{i=1}^{k}(m_i(A',\Omega)-m_i(A,\Omega))\quad(k=1,2,\ldots).$$

We mention that the question of the possible values of the multiplicities of the eigenvalues, leaving the real axis in the case of perturbations, has been considered in [5, Subsec. III.1.1].

COROLLARY 1. If under the assumptions of Theorem 2 the operator A (resp., A') has in Ω a unique eigenvalue $\lambda_0\in\mathbf{R}$ (resp., $\lambda_1\in\mathbf{R}$), then for $k = 1,\ldots,\dim\operatorname{Ker}(A-\lambda_0 I)$ we have

$$|\beta_k(A',\lambda_1)-\beta_k(A,\lambda_0)|\leqslant\sum_{i=1}^{k}(m_i(A',\lambda_1)-m_i(A,\lambda_0))+\max(0,k-\dim\operatorname{Ker}(A'-\lambda_1 I)).$$

In the simplest case $m_i(A',\lambda_1)=m_i(A,\lambda_0)$ $(i=1,2,\ldots)$. from Theorem 2 and Corollary 1 there follows that $\varepsilon_i(A',\lambda_1)=\varepsilon_i(A,\lambda_0)$ $(i=1,\ldots,\dim\operatorname{Ker}(A-\lambda_0 I))$. This result is known [5, p. 283].

4. In the case when both the initial and the perturbed operators have in Ω only one eigenvalue, then one can indicate the following sufficient condition for the existence of a perturbed operator A' with prescribed numbers $m_i(A',\lambda_0)$:

THEOREM 3. Suppose that the conditions of Theorem 1 hold. We denote by γ_k (resp., γ_k') the number of odd numbers among $m_i(A,\lambda_0)$ (resp., m_i') (i = 1, 2,...). If

$$|\gamma_k'-\gamma_k|\leqslant\sum_{i=1}^{k}(m_i'-m_i(A,\lambda_0))\qquad(k=1,2,\ldots),\tag{1}$$

then in any neighborhood of the operator A there exists a G-self-adjoint operator $A'\in L(\mathfrak{H})$, having in Ω a unique eigenvalue λ_0 and, moreover, $m_i(A',\lambda_0)=m_i'$ (i = 1, 2,...).

In certain cases, Theorems 2 and 3 allow us to obtain a complete description of the possible domain of variation of the lengths of the Jordan chains of a perturbed G-self-adjoint operator.

THEOREM 4. Let Ω be a normal domain for a G-self-adjoint operator $A\in L(\mathfrak{H})$, containing only one of its eigenvalues $\lambda_0\in\mathbf{R}$. We also assume that the sign characteristics $\varepsilon_i(A,\lambda_0)$ depend only on the parity of the numbers $m_i(A,\lambda_0)$. The following statements are equivalent: a) in any neighborhood of the operator A there exists a G-self-adjoint operator $A'\in L(\mathfrak{H})$, which has in Ω a unique eigenvalue λ_0 and, moreover, $m_i(A',\lambda_0)=m_i'$ (i = 1, 2;...); b) $\{m_i(A,\lambda_0)\}\prec\{m_i'\}$ and the inequalities (1) are satified.

5. All the results of Secs. 2-4 can be carried over to holomorphic self-adjoint perturbations of holomorphic self-adjoint operator functions (compare with [2, 3], where one has obtained similar generalizations but without the assumptions on self-adjointness). Moreover, instead of the lengths of the Jordan chains one considers the partial multiplicities in the sense of Keldysh [7], while the sign characteristics are understood in the sense of Kostyuchenko and Shkalikov [6].

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