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CHANGE OF JORDAN STRUCTURE OF G-SELFADJOINT OPERATORS AND SELFADJOINT OPERATOR-VALUED FUNCTIONS UNDER SMALL PERTURBATIONS

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ABSTRACT. The author considers the problem of the change of length of Jordan chains when passing from G_0-selfadjoint operator A_0 to G-selfadjoint operator A, provided $\|A - A_0\| + \|G - G_0\|$ is small enough. The role played by the so-called sign characteristics is clarified. The results will carry over to the case of small perturbations of holomorphic selfadjoint operator-valued functions.

Introduction

In [1]–[3] a characterization of the feasible range was obtained for the lengths of Jordan chains of linear operators and holomorphic operator-valued functions under small perturbations. In this paper the analogous problems are considered for the classes of G-selfadjoint operators and selfadjoint operator-valued functions (o.f.).

In order to describe the basic results, we introduce some notation (for the details see §§1–3). Let $\mathfrak{Z}$ be a Hilbert space and let $L(\mathfrak{Z})$ be the set of all linear operators on $\mathfrak{Z}$. If λ_0 is an isolated Fredholm eigenvalue (e.v.) of an operator A, then by $m_i(A, \lambda_0)$ $(i = 1, \ldots, r = \dim \operatorname{Ker}(A - \lambda_0 I))$ we denote the lengths of the corresponding Jordan chains (*partial multiplicities*), enumerated in nonascending order. For convenience, we put $m_i(A, \lambda_0) = 0$ for $i > r$. A bounded domain Ω $(\subset \mathbf{C})$ is said to be *normal* for an operator $A \in L(\mathfrak{Z})$ if its boundary contains no point of the spectrum of A, and the whole spectrum of A in Ω consists of a finite number of Fredholm e.v. $\{\lambda_i\}_1^n$. In that case we put

$$m_i(A, \Omega) = \sum_{j=1}^{n} m_i(A, \lambda_j).$$

If $u = \{u_i\}_1^\infty$ and $v = \{v_i\}_1^\infty$ are two nonincreasing sequences of nonnegative integers and if

$$\sum_{i=1}^{k} u_i \le \sum_{i=1}^{k} v_i, \quad k = 1, 2, \ldots, \qquad \sum_{i=1}^{\infty} u_i = \sum_{i=1}^{\infty} v_i,$$

we write $u \prec v$.

In [1]–[3] it was shown that

$$\{m_i(A_0, \Omega)\} \prec \{m_i(A, \Omega)\} \tag{0.1}$$

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for any operator A close enough to A_0. The converse was also proved: if an operator A_0 has a unique e.v. λ_0 in Ω (the general case is easily reduced to this one), and if a natural number p is given as well as nonincreasing sequences $\{m'_{ij}\}_{i=1}^{\infty}$ $(j = 1, \dots, p)$ such that

$$\{m_i(A_0, \lambda_0)\} \prec \left\{\sum_{j=1}^{p} m'_{ij}\right\},$$

then in every neighborhood of A_0 there exists an operator A having exactly p e.v. $\{\lambda_i\}_1^p$ in Ω, and in addition $m_i(A, \lambda_j) = m'_{ij}$.

We consider the problem of the change of partial multiplicities in the case when the initial operator A_0 is G_0-*selfadjoint* (i.e. $G_0 A_0 = A_0^* G_0$, where G_0 is in invertible selfadjoint operator) and the perturbed operator A is G-selfadjoint. For the solution of this problem it is highly essential that the Jordan basis of the G-selfadjoint operator A corresponding to the e.v. λ_0 can be chosen in such a way that to each chain a certain sign is assigned. In other words, to each partial multiplicity $m_i(A, \lambda_0)$ we associate a number $\varepsilon_i(A, G, \lambda_0) = \pm 1$, which we call the *sign characteristic* (the definition of the sign characteristic is given in §2; see also [4] and [5]).

In §2 the following result is proved, establishing the relationship between the change of partial multiplicities and sign characteristics under small perturbations (Theorem 2.2). Let Ω be a domain which is normal for a G-selfadjoint operator A_0. If A is a G-selfadjoint operator and $\|A - A_0\| + \|G - G_0\|$ is small enough, then

$$|\alpha_k(A, G, \Omega) - \alpha_k(A_0, G_0, \Omega)| \le \sum_{i=1}^{k}(m_i(A, \Omega) - m_i(A_0, \Omega)), \qquad k = 1, 2, \dots, \tag{0.2}$$

where $\alpha_k(A, G, \Omega)$ is the sum of sign characteristics of all Jordan chains whose length is an odd number not exceeding k, and which correspond to the real e.v. in Ω. Note that the right-hand sides of the inequalities (0.2) are nonnegative by (0.1).

In the same section some analogues of (0.2) are deduced for the chains of even length. Note that the proofs in §2 are based on the assertions obtained in §1 about the change of linear spans of the first few Jordan chains of an operator under a small perturbation (recall that the chains are enumerated in the order of nonincreasing lengths).

In §3 we obtain the analogues of the results of §2 for selfadjoint holomorphic operator-valued functions. Here partial multiplicities are understood in the sense of Keldysh [6], and sign characteristics in the sense of Kostyuchenko and Shkalikov [7].

In §§4–6 we consider the inverse problems, i.e. we search for restrictions on the numbers m'_{ij} $(i = 1, 2, \dots;\ j = 1, \dots, p)$ under which in any neighborhood of the G-selfadjoint operator A_0 there exists a G-selfadjoint operator A having p e.v. $\{\lambda_i\}_1^p$ in Ω and satisfying $m_i(A, \lambda_j) = m'_{ij}$ (in some cases this problem is modified; more will be said about this below).

The simplest case is the one considered in §4, when Ω contains no real numbers that are in the spectrum of A_0. If, moreover, it contains no pair of complex conjugate numbers that are both in the spectrum, then in our problem there are no additional restrictions in comparison with [1]–[3] (though the methods of proofs are different).

In §5 we study the case of real spectrum, but we search for the restrictions on the total values $m_i(A, \Omega)$ (*Gohberg-Kaashoek numbers*) rather than on the partial multiplicities $m_i(A, \lambda_j)$. It turns out that, again, it is not necessary to impose any other restrictions on these numbers, except for (0.1).

On the other hand, the results of §2 show that when considering our inverse problem for the partial multiplicities that correspond to real e.v., additional restrictions may arise. For example, if the sign characteristics of all chains of odd length that correspond to the real e.v. of A_0 in Ω are equal, then, by virtue of (0.2), the number of chains of odd length of the perturbed operator A that correspond to its real e.v. in Ω will be no smaller than that of A_0. Some results related to this (the most complex) case are obtained in §6. Here we assume that the operators A_0 and A have a unique e.v. λ_0 in Ω $(\lambda_0 \in \mathbf{R})$. It is shown that the inequalities

$$|\gamma_k - \gamma_k| \le \sum_{i=1}^{k}(m_i' - m_i(A_0, \lambda_0)), \ k = 1, 2 \dots, \tag{0.3}$$

where γ_k (respectively γ_k') is the number of odd numbers among $\{m_i(A_0, \lambda_0)\}_1^k$ (respectively $\{m_i(A, \lambda_0)\}_1^k$), together with condition (0.1) are sufficient in order that in any neighborhood of the G-selfadjoint operator A_0 there exists a G-selfadjoint operator A such that $m_i(A, \lambda_0) = m_i'$. In the case when the operator A_0 satisfies the so-called sign condition [8], i.e. the numbers $\varepsilon_i(A_0, G, \lambda_0)$ depend only on the evenness of the numbers $m_i(A_0, \lambda_0)$, the conditions (0.3) obviously coincide with (0.2), and hence they yield, together with (0.1), the complete solution of the problem.

All results of §§4–6 carry over to the case of selfadjoint holomorphic o.f.

For convenience, all results relating to linear operators are stated for the case of a finite-dimensional space. The general case can be reduced to this if we consider the restrictions of the operators A and A_0 to direct sums of the root subspaces, corresponding to e.v. in Ω (these direct sums have the same dimensions, and they can be identified in a suitable way).

The basic results of this paper were announced in [9].

The author takes this opportunity to express deep gratitude to A. S. Markus for posing the problem and for useful discussions.

§1. Change of the linear span of Jordan chains of an operator under a small perturbation

1. Let $\mathfrak{Z}_1$ and $\mathfrak{Z}_2$ be Hilbert spaces, and let $L(\mathfrak{Z}_1, \mathfrak{Z}_2)$ be the set of all linear operators that map $\mathfrak{Z}_1$ to $\mathfrak{Z}_2$. Instead of $L(\mathfrak{Z}, \mathfrak{Z})$ we will write $L(\mathfrak{Z})$. Throughout this paper we assume that the spaces denoted by $\mathfrak{Z}$ (with or without an index) are finite-dimensional.

Denote by $m_1(A, \lambda) \ge \cdots \ge m_r(A, \lambda)$ the orders of all Jordan blocks (*partial multiplicities*) of an operator $A \in L(\mathfrak{Z})$ that correspond to its e.v. λ. For convenience, we put $m_i(A, \lambda) = 0$ for $i > r$. Choose a Jordan basis

$$\varphi_i^j, \qquad i = 0, 1, \dots, m_j - 1, \ j = 1, \dots, r, \tag{1.1}$$

in the root subspace $\mathscr{R}(A, \lambda)$ corresponding to λ (i.e. $(A-\lambda I)\varphi_0^j = 0$, $(A-\lambda I)\varphi_i^j = \varphi_{i-1}^j$, and in addition the vectors $\{\varphi_0^j\}_1^r$ form a basis of the eigenspace $\operatorname{Ker}(A-\lambda I)$). Here $m_j = m_j(A, \lambda)$ and $r = \dim \operatorname{Ker}(A - \lambda I)$.

If the spectrum $\sigma(A)$ of an operator A in a domain $\Omega \subset \mathbf{C}$ consists of p distinct e.v. $\{\lambda_i\}_1^p$, then we put

$$\mathscr{R}(A, \Omega) = \mathscr{R}(A, \lambda_1) \dotplus \cdots \dotplus \mathscr{R}(A, \lambda_p),$$
$$\mathscr{R}^k(A, \Omega) = \mathscr{R}^k(A, \lambda_1) \dotplus \cdots \dotplus \mathscr{R}^k(A, \lambda_p),$$
$$\mathscr{R}_k(A, \Omega) = \mathscr{R}_k(A, \lambda_1) \dotplus \cdots \dotplus \mathscr{R}_k(A, \lambda_p).$$

Observe that the subspaces $\mathscr{R}^k(A, \Omega)$ and $\mathscr{R}_k(A, \Omega)$ are not uniquely determined and depend on the choice of Jordan bases of subspaces $\mathscr{R}(A, \lambda_i)$ $(i = 1, \dots, p)$. From now on, when we talk about the choice of some subspaces $\mathscr{R}^k(A, \Omega)$ or $\mathscr{R}_k(A, \Omega)$ with certain properties, we will have in mind a choice of a Jordan basis such that the corresponding subspaces $\mathscr{R}^k(A, \Omega)$ and $\mathscr{R}_k(A, \Omega)$ have these properties. It is obvious that for a fixed Jordan basis in the subspaces $\{\mathscr{R}^k(A, \Omega)\}_1^\infty$ (respectively $\{\mathscr{R}_k(A, \Omega)\}_1^\infty$) form an increasing (respectively decreasing) chain, i.e. $\mathscr{R}^k(A, \Omega) \subset \mathscr{R}^{k+1}(A, \Omega)$ $(\mathscr{R}_{k+1}(A, \Omega) \subset \mathscr{R}_k(A, \Omega))$. The numbers $m_i(A, \Omega) = \sum_{j=1}^p m_i(A, \lambda_j)$ are called the *Gohberg-Kaashoek numbers* of the operator A in the domain Ω.

A domain $\Omega \subset \mathbf{C}$ is said to be *normal for an operator* $A \in L(\mathfrak{Z})$ if its boundary $\partial\Omega$, consisting of a finite number of simple closed rectifiable curves, contains no point of the spectrum of A.

For subspaces $\mathfrak{N}$ and $\mathfrak{M}$ of $\mathfrak{Z}$, we put

$$\theta_0 = (\mathfrak{N}, \mathfrak{M}) = \max_{\substack{x \in \mathfrak{N} \\ \|x\|=1}} \min_{y \in \mathfrak{M}} \|x - y\|.$$

The symmetrized quantity $\theta(\mathfrak{N}, \mathfrak{M}) = \max\{\theta_0(\mathfrak{N}, \mathfrak{M}), \theta_0(\mathfrak{M}, \mathfrak{N})\}$ is well known and is called the *aperture* of the subspaces $\mathfrak{N}$ and $\mathfrak{M}$. It equals the norm of the difference of the orthogonal projections onto these subspaces. The quantity $\theta_0(\mathfrak{N}, \mathfrak{M})$ can be called the *semi-aperture* of the subspaces $\mathfrak{N}$ and $\mathfrak{M}$.

A subspace $\mathfrak{N} \subset \mathfrak{Z}$ is said to be *invariant* for an operator A (*A-invariant*) if $A\mathfrak{N} \subset \mathfrak{N}$. We denote by $A|\mathfrak{N}$ the restriction of A to the A-invariant subspace $\mathfrak{N}$.

If the space $\mathfrak{Z}$ is represented as a direct sum of its subspaces $\mathfrak{Z}_1, \dots, \mathfrak{Z}_m$, then it is convenient to write an operator $A \in L(\mathfrak{Z})$ as an operator matrix $A = (A_{ij})_{i,j=1}^m$, where $A_{ij} \in L(\mathfrak{Z}_i, \mathfrak{Z}_j)$.

2. LEMMA 1.1. *Let λ_0 be an e.v. of an operator $A \in L(\mathfrak{Z})$. For arbitrary numbers $\varepsilon > 0$ and $q \in \mathbf{N}$ and any q vectors $\{\psi_k\}_1^q$ in $\mathscr{R}(A, \lambda_0)$, it is possible to select a chain of subspaces $\{\mathscr{R}^k(A, \lambda_0)\}_1^\infty$ such that there exist vectors $\varphi_k \in \mathscr{R}^k(A, \lambda_0)$ $(k = 1, \dots, q)$ satisfying*

$$\|\varphi_k - \psi_k\| < \varepsilon, \qquad k = 1, \dots, q.$$

PROOF. Obviously it suffices to conduct the proof for the case when $q < r = \dim \operatorname{Ker}(A - \lambda_0 I)$.

We construct the required Jordan basis in q steps. Let $\{f_{ik}^j\}$ $(i = 0, 1, \dots, m_j - 1;\ j = 1, \dots, r;\ k = 0, \dots, q)$ be Jordan bases of $\mathscr{R}(A, \lambda_0)$, where the first one $\{f_{i0}^j\}$ is arbitrary, and each of the others is related to the one preceeding it in the following way. For a natural $s \in [1, q]$ consider the subspaces $\mathscr{R}^{s-1}(A, \lambda_0)$ and $\mathscr{R}_{s-1}(A, \lambda_0)$ corresponding to the Jordan basis $\{f_{i,s-1}^j\}$. Let $\psi_s = \psi_s^1 + \psi_s^2$ be the decomposition of ψ_s relative to the decomposition of $\mathscr{R}(A, \lambda_0)$ as the direct sum of subspaces $\mathscr{R}^{s-1}(A, \lambda_0)$ and $\mathscr{R}_{s-1}(A, \lambda_0)$. The basis $\{f^j_{is}\}$ is obtained from $\{f^j_{i,s-1}\}$ by replacing the Jordan chain $\{f^j_{i,s-1}\}_{i=1}^{m_s}$ of the operator A by the chain

$$f^s_{is} = \delta_s f^s_{i,s-1} + (A - \lambda_0 I)^{m_s - i - 1}\psi_s^2, \qquad i = 0, \dots, m_s - 1. \tag{1.2}$$

In addition, the positive number δ_s, satisfying the condition

$$\delta_s < \varepsilon \|f^s_{m_s, -1, s-1}\|^{-1}, \tag{1.3}$$

is selected so that the first vector f^s_{0s} in the chain (1.2) is not contained in the linear span of the eigenvectors $f^r_{0,s-1}$ $(r \neq s)$ from the remaining chains. Since $\psi_s^2 \in \mathscr{R}_{s-1}(A, \lambda_0)$, it follows that the system of vectors $\{f^j_{is}\}$ is a Jordan basis of $\mathscr{R}(A, \lambda_0)$. From (1.3) it follows that the vectors $\varphi_k = \psi^1_k + f^k_{m_k - 1, q}$ $(k = 1, \dots, q;$ $m_k = m_k(A, \lambda_0))$ satisfy the assertions of the lemma.

THEOREM 1.2. *Let Ω be a domain, normal for an operator $A_0 \in L(\mathfrak{Z})$, and let a chain of subspaces $\{\mathscr{R}^k(A_0, \Omega)\}_1^\infty$ be chosen. There exists a number $K > 0$ such that for every operator $A \in L(\mathfrak{Z})$ it is possible to select a chain of subspaces $\{\mathscr{R}^k(A, \Omega)\}_1^\infty$ satisfying*

$$\theta_0(\mathscr{R}^k(A_0, \Omega), \mathscr{R}^k(A, \Omega)) \leq K\|A - A_0\|, \qquad k = 1, 2, \dots \tag{1.4}$$

PROOF. Suppose first that $\Omega = \mathbf{C}$ and that the operator A_0 has only one e.v. λ_0; in addition, without loss of generality, we can assume that $\lambda_0 = 0$. Let the subspaces $\{\mathscr{R}^k(A_0, 0)\}_1^\infty$ correspond to a Jordan basis f^j_i $(i = 0, \dots, m_j - 1;$ $j = 1, \dots, r)$ of A_0, and let $\{\lambda_l\}_1^p$ be all distinct e.v. of the operator A. Let $f^j_{m_j - 1} = \psi_{j1} + \dots + \psi_{jp}$ $(j = 1, \dots, r)$ be the decomposition of the vector $f^j_{m_j-1}$ relative to the direct sum of subspaces $\mathscr{R}(A, \lambda_1) \dotplus \dots \dotplus \mathscr{R}(A, \lambda_p)$. According to Lemma 1.1, for every $\delta > 0$ we can select a chain of subspaces $\{\mathscr{R}^j(A, \lambda_l)\}_{j=1}^\infty$ such that there exist vectors $\psi_{jl} \in \mathscr{R}^j(A, \lambda_l)$ $(j = 1, \dots, r;\ l = 1, \dots, p)$ satisfying

$$\|\varphi_{jl} - \psi_{jl}\| < \delta. \tag{1.5}$$

Put $\mathscr{R}^j(A, \mathbf{C}) = \mathscr{R}^j(A, \lambda_1) \dotplus \dots \dotplus \mathscr{R}^j(A, \lambda_p)$ $(j = 1, 2, \dots)$. It is obvious that the vectors $h^j_i = A^{m_j - i - 1}(\sum_{l=1}^p \psi_{jl})$ $(i = 0, \dots, m_j - 1)$ belong to the subspace $\mathscr{R}^j(A, \mathbf{C})$ $(j = 1, \dots, r)$. Moreover, since the number δ in (1.5) can be assumed to be arbitrarily small, the inequalities

$$\|f^j_i - h^j_i\| \leq K_1\|A - A_0\|,$$

hold, and the number $K_1 > 0$ depends only on the initial Jordan basis $\{f^j_i\}$ of A_0. Thus, the theorem has been proved in the case when the spectrum of A_0 consists of a single point.

Now we pass to the general case. Let $\{\lambda_l\}_1^t$ be all distinct e.v. of A_0 in Ω. Choose open disks G_l $(\subset \Omega)$ centered at the points λ_l $(l = 1, \dots, t)$ with radii so small that $G_k \cap G_l = \varnothing$ for $k \neq l$. As we know, there exists a positive number ε such that, if $\|A - A_0\| < \varepsilon$, the sums of multiplicities of e.v. of A and A_0 in G_l

($l = 1, \dots, t$) coincide. It clearly suffices to establish (1.4) for operators A such that $\theta_0(\mathfrak{N}, \mathfrak{M}) \le 1$ ($\mathfrak{N}, \mathfrak{M} \subset \mathfrak{Z}$). Consider the operator

$$S = I - \sum_{l=1}^{t} (P_l(A_0) - P_l(A)) \cdot P_l(A_0),$$

where

$$P_l(A) = -\frac{1}{2\pi i} \int_{\partial G_l} (A - \lambda I)^{-1}\, d\lambda$$

is the Riesz projection onto the subspace $\mathscr{R}(A, G_l)$ (the projection $P_l(A_0)$ is defined analogously).

It is not hard to show that there exists a number $K_1 > 0$, depending only on the operator A_0 and on the disks G_l, such that

$$\|I - S\| \le K_1 \|A - A_0\|. \tag{1.6}$$

It is also easy to verify that the equality $\mathscr{R}(A_0, \lambda_l) = \mathscr{R}(S^{-1}AS, G_l)$, which was established in the first part of the proof, implies that we can select a chain of subspaces $\{\mathscr{R}^k(S^{-1}AS, G_l)\}_{k=1}^{\infty}$ in such a way that for $k = 1, 2, \dots$ and $l = 1, \dots, t$ the inequalities

$$\theta_0(\mathscr{R}^k(A_0, \lambda_l), \mathscr{R}^k(S^{-1}AS, G_l)) \le K_2 \|(S^{-1}AS - A_0)|\mathscr{R}(A_0, \lambda_l)\|, \tag{1.7}$$

hold, where $K_2 > 0$ depends only on the operator A_0. Put

$$\mathscr{R}^k(A, \Omega) = S(\mathscr{R}^k(S^{-1}AS, G_1) \dotplus \cdots \dotplus \mathscr{R}^k(S^{-1}AS, G_t)).$$

As it is easily seen,

$$\begin{aligned}
&\theta_0(\mathscr{R}^k(A_0, \Omega), \mathscr{R}^k(A, \Omega)) \\
&\quad \le \theta_0(\mathscr{R}^k(A_0, \Omega), \mathscr{R}^k(S^{-1}AS, \Omega)) + \theta_0(\mathscr{R}^k(S^{-1}AS, \Omega), \mathscr{R}^k(A, \Omega)).
\end{aligned}$$

From this and from equations (1.7) and (1.6), taking into account the obvious relation $\theta_0(\mathfrak{N}, S\mathfrak{N}) \le \|I - S\|$, it follows that the subspaces $\{\mathscr{R}^k(A, \Omega)\}_1^{\infty}$ satisfy (1.4). The theorem is proved.

3. The following theorem is a generalization of the well-known (and also a corollary of (1.6)) bound for the quantity $\theta(\mathscr{R}(A_0, \Omega), \mathscr{R}(A, \Omega))$ in terms of $\|A - A_0\|$.

THEOREM 1.3. *If in Theorem* 1.2 *the operator* $A \in L(\mathfrak{Z})$ *satisfies*

$$\sum_{i=1}^{k} m_i(A, \Omega) = \sum_{i=1}^{k} m_i(A_0, \Omega)$$

for some $k \in \mathbf{N}$, *then* θ_0 *can be replaced by* θ *in the* kth *inequality in* (1.4). *In that case, it can also be assumed that the subspace* $\mathscr{R}^k(A, \Omega)$ *satisfies*

$$\theta(\mathscr{R}_k(A_0, \Omega), \mathscr{R}_k(A, \Omega)) \le K\|A - A_0\|. \tag{1.8}$$

PROOF. The first assertion is obvious, since $\theta_0(\mathfrak{N}, \mathfrak{M}) = \theta(\mathfrak{N}, \mathfrak{M})$ when $\dim \mathfrak{M} = \dim \mathfrak{N}$. Furthermore, by Theorem 1.2 we can assume that the subspaces $\mathscr{R}^k(A, \Omega)$ and $\mathscr{R}(A, \Omega)$ satisfy

$$\begin{aligned}
\theta(\mathscr{R}^k(A_0, \Omega), \mathscr{R}^k(A, \Omega)) &\le K_1\|A - A_0\|, \\
\theta(\mathscr{R}(A_0, \Omega), \mathscr{R}(A, \Omega)) &\le K_1\|A - A_0\|.
\end{aligned}$$

It is not hard to select an invertible operator $T \in L(\mathfrak{Z})$ mapping the subspaces $\mathscr{R}^k(A_0, \Omega)$ and $\mathscr{R}(A_0, \Omega)$ into the subspaces $\mathscr{R}^k(A, \Omega)$ and $\mathscr{R}(A, \Omega)$, respectively, and satisfying the inequality

$$\|I - T\| \leq K_2 \|A - A_0\|, \tag{1.9}$$

where the number $K_2 > 0$ depends only on A_0. It is easy to verify that the operator $A_1 = T^{-1}AT$ satisfies the equalities $\mathscr{R}(A_0, \Omega) = \mathscr{R}(A_1, \Omega)$ and $\mathscr{R}^k(A_0, \Omega) = \mathscr{R}^k(A_1, \Omega)$. Let $\widetilde{A}_0$ and $\widetilde{A}_1$ be the restrictions of A_0 and A_1 to the subspace $\mathscr{R}(A_0, \Omega)$. Write down the matrices of $\widetilde{A}_0$ and $\widetilde{A}_1$ relative to the decomposition of $\mathscr{R}(A_0, \Omega)$ as the direct sum of $\mathscr{R}^k(A_0, \Omega)$ and $\mathscr{R}_k(A_0, \Omega)$:

$$\widetilde{A}_0 = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}, \qquad \widetilde{A}_1 = \begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{pmatrix}.$$

Since the subspace $\mathscr{R}^k(\widetilde{A}_1, \Omega)$ $(= \mathscr{R}^k(A_0, \Omega))$ possesses an $\widetilde{A}_1$-invariant complement, the operator $\widetilde{A}_1$ can be reduced to the diagonal form. It is easy to see that this can be done with the help of the similarity operator

$$S = \begin{pmatrix} I & X \\ 0 & I \end{pmatrix}, \tag{1.10}$$

where X is a solution of the equation $B_{11}X - XB_{22} = B_{12}$. Consider the transformations $Q(X) = A_{11}X - XA_{22}$ and $R(X) = B_{11}X - XB_{22}$, acting on the space $L(\mathscr{R}_k(A_0, \Omega), \mathscr{R}^k(A_0, \Omega))$. From [10] (Chapter VIII, §2, Theorem 1) it follows that the dimensions of the kernels of these transformations coincide and are equal to $k\sum_{k+1}^{\infty} m_j(A_0, \Omega)$. Therefore, for all A close enough to A_0 (and it suffices to prove the theorem only for such A), the equation $R(Z) = Y$ (if it is solvable) has a solution Z such that $\|Z\| \leq K_3\|Y\|$, where $K_3 > 0$ depends only on A_0. In particular, we can assume that

$$\|X\| \leq K_3 \|B_{12}\|. \tag{1.11}$$

Extend the operator S to the whole space $\mathfrak{Z}$ by setting it equal to the identity operator on the subspace $\mathscr{R}(A_0, \mathbf{C}\backslash\overline{\Omega})$. By (1.10) and (1.11), S satisfies

$$\|I - S\| \leq K_4 \|A - A_0\|. \tag{1.12}$$

Since $\|A - A_0\|$ is small enough, we can assume, by virtue of (1.9) and (1.12), that the operators T and S are invertible. In addition, it is easy to see that

$$\mathscr{R}^k(S^{-1}T^{-1}ATS, \Omega) = \mathscr{R}^k(A_0, \Omega),$$
$$\mathscr{R}_k(S^{-1}T^{-1}ATS, \Omega) = \mathscr{R}_k(A_0, \Omega).$$

Putting $\mathscr{R}^k(A, \Omega) = TS(\mathscr{R}^k(A_0, \Omega))$ and $\mathscr{R}_k(A, \Omega) = TS(\mathscr{R}_k(A_0, \Omega))$, with the help of (1.9) and (1.12) we see that (1.8) is valid. The theorem is proved.

§2. Change of sign characteristics and partial multiplicities of a G-selfadjoint operator under a small perturbation

1. In $\mathfrak{Z}$, together with the usual inner product $(\cdot, \cdot)$, we will also consider an *indefinite* inner product

$$[x, y] = (Gx, y), \qquad x, y \in \mathfrak{Z},$$

defined by means of an invertible selfadjoint operator $G\ L(\mathfrak{Z})$. The theory of spaces with an indefinite metric and of linear operators in such spaces is presented in many books (see, for example, [4], [5], and [11]). Here we only recall the fundamental definitions.

With the help of the inner product $[\cdot, \cdot]$, *G-orthogonality* of vectors and subspaces is defined in a natural way.

A subspace $\mathfrak{N} \subset \mathfrak{Z}$ is called *G-degenerate* if there exists a nonzero vector $x \in \mathfrak{N}$ that is G-orthogonal to the whole subspace $\mathfrak{N}$. A subspace $\mathfrak{N}$ containing no such vectors is said to be *G-nondegenerate.* An operator $A \in L(\mathfrak{Z})$ is called *G-selfadjoint* if $GA = A^*G$. From the last relation it follows that the spectrum of A is symmetric with respect to the real axis, and, moreover, $m_i(A, \lambda) = m_i(A, \bar{\lambda})$ $(i = 1, 2, \dots)$, where $\lambda \in \sigma(A)$.

If λ is a real e.v. of a G-selfadjoint operator $A \in L(\mathfrak{Z})$, then, in accordance with [4], §28.1 (see also [5] Theorem 1.3.3), it is possible to select a Jordan basis φ_i^j $(i = 0, \dots, m_j - 1;\ j = 1, \dots, r)$ of A in the root subspace $\mathscr{R}(A, \lambda)$ of A, corresponding to the number λ, such that for some numbers $\varepsilon_j(A, G, \lambda) = \pm 1$ $(j = 1, \dots, r)$ we have $[\varphi_i^j, \varphi_k^l] = \varepsilon_j(A, G, \lambda)$ if $j = l$ and $i + k = m_j$, $[\varphi_i^j, \varphi_k^l] = 0$ in all other cases. The numbers $\varepsilon_j(A, G, \lambda)$ are called the *sign characteristics* of the G-selfadjoint operator A corresponding to its e.v. $\lambda \in \mathbf{R}$, and they are determined up to the order of enumeration of the ones to which equal partial multiplicities $m_j(A, \lambda)$ correspond. If λ is a nonreal e.v. of A, it is possible to select Jordan bases φ_i^j and ψ_i^j $(i = 0, \dots, m_j - 1;\ j = 1, \dots, r)$ in the subspaces $\mathscr{R}(A, \lambda)$ and $\mathscr{R}(A, \bar{\lambda})$, respectively, such that $[\varphi_i^j, \varphi_k^l] = 1$ if $j = l$ and $i + k = m_j$, and $[\varphi_i^j, \varphi_k^l] = 0$ in all other cases.

Represent the space $\mathfrak{Z}$ as the direct sum

$$\mathfrak{Z} = \mathscr{R}_1 \dotplus \cdots \dotplus \mathscr{R}_s, \tag{2.1}$$

where each summand is either a root subspace of A corresponding to a real e.v., or the sum of its root subspaces corresponding to a pair of conjugate nonreal e.v. Joining the Jordan bases selected in each subspace $\mathscr{R}_i$ $(i = 1, \dots, s)$ in the way stated above, we obtain a Jordan basis of the G-selfadjoint operator A, which we call a *G-normal* basis.

2. LEMMA 2.1. *Let a G-selfadjoint operator $A \in \mathfrak{L}(\mathfrak{Z})$ be given as well as a subspace $L \subset \mathfrak{Z}$ such that both $\mathfrak{L}$ and its complement are A-invariant. Suppose also that the operator $A|\mathfrak{L}$ is similar to its adjoint. For any positive number ε, there exists a G-singular A-invariant subspace $\mathfrak{S} \subset \mathfrak{Z}$ with the property that $\theta(\mathfrak{L}, \mathfrak{S}) < \varepsilon$.*

PROOF. Consider the decomposition (2.1) of the space $\mathfrak{Z}$ and put $\mathfrak{L}_i = \mathfrak{L} \cap \mathscr{R}_i$ $(i = 1, \dots, s)$. It is easy to see that $\mathfrak{L}$ is decomposed into a direct sum of G-orthogonal A-invariant subspaces: $\mathfrak{L} = \mathfrak{L}_1 \dotplus \cdots \dotplus \mathfrak{L}_s$. Let $\mathscr{R}_i$ be the root subspace of A corresponding to its real e.v. λ_i. Denote by $\mathfrak{N}_1'$ the linear span of the vectors from one of the chains $\{g_j\}_1^t$ of any Jordan basis of $A|\mathfrak{L}_i$ and by $\mathfrak{M}_1'$ the linear span of all other vectors in that basis. If $\mathfrak{N}_1'$ is a G-nondegenerate subspace, we put $\mathfrak{N}_1 = \mathfrak{N}_1'$, otherwise we take $\mathfrak{N}_1$ to be the linear span of vectors $\{g_j + \delta f_j\}_1^t$, where $\delta > 0$ and $\{f_j\}_1^t$ is a chain of length t from a G-normal basis of $\mathscr{R}(A, \lambda_i)$. The existence of such a chain comes from the presence of an A-invariant direct complement to the subspace $\mathfrak{L}$. If the subspace $\mathfrak{N}_1$ were G-degenerate, then, by virtue of the G-selfadjointness of A, its eigenvector $g_1 + \delta f_1$ $(\in \mathfrak{N}_1)$ would be G-orthogonal to $\mathfrak{N}_1$. Since $[f_1, f_t] = \pm 1$, we have $[g_1 + \delta f_1, g_t + \delta f_t] \neq 0$ for δ small enough; hence we can assume that $\mathfrak{N}_1$ is a G-nondegenerate A-invariant subspace. Put $\mathfrak{L}_i^1 = \mathfrak{N}_1 \dotplus \mathfrak{M}_1'$. It is easy to see that, by means of the choice of δ, the quantity $\theta(\mathfrak{L}_i, \mathfrak{L}_i^1)$ can be made arbitrarily small.

Denote by $\mathfrak{B}_1$ the set of all vectors in $\mathscr{R}_i$ that are G-orthogonal to the subspace $\mathfrak{N}_1$. It is well known that when passing from $\mathfrak{N}_1$ to $\mathfrak{B}_1$ the properties of being a G-nondegenerate or A-invariant subspace are preserved. Put $\mathfrak{M}_1 = \mathfrak{L}_i^1 \cap \mathfrak{B}_1$. Next we act in the subspace $\mathfrak{M}_1$ in exactly the same way as we acted in $\mathfrak{L}_i$: we choose a chain from an arbitrary Jordan basis of the operator $A|\mathfrak{M}_1$, we consider, if necessary, its perturbation by a chain of the same length from a G-normal basis of the operator $A|\mathfrak{B}_1$, and we continue this process until we construct a G-nondegenerate A-invariant subspace $\mathfrak{S}_i = \mathfrak{L}_i^q$ $(q = \dim(\operatorname{Ker}(A - \lambda_i I) \cap \mathfrak{L}_i))$, where the number $\theta(\mathfrak{L}_i, \mathfrak{S}_i)$ can be assumed to be as small as needed.

In a similar way, the existence of such a subspace $\mathfrak{S}_i$ is proved in the case when $\mathscr{R}_i$ is a direct sum of two root subspaces, corresponding to some complex conjugate e.v. Put $\mathfrak{S} = \mathfrak{S}_1 \dotplus \cdots \dotplus \mathfrak{S}_s$. Obviously, the subspace $\mathfrak{S}$ has all required properties. The lemma is proved.

3. Let λ be a real e.v. of a G-selfadjoint operator $A \in L(\mathfrak{Z})$. Denote by $\alpha_k(A, G, \lambda)$ (respectively $\beta_k(A, G, \lambda)$) the sum of the $\varepsilon_i(A, G, \lambda)$ that correspond to the odd (respectively even and nonzero) numbers among $\{m_i(A, \lambda)\}_1^k$. Obviously, $\alpha_k(A, G, \lambda)$ and $\beta_k(A, G, \lambda)$ depend on the indexing of the $\varepsilon_i(A, G, \lambda)$; hence they are not uniquely determined. Nevertheless, if $m_{k+1}(A, \lambda) > m_k(A, \lambda)$ for some k, then $\alpha_k(A, G, \lambda)$ and $\beta_k(A, G, \lambda)$ are uniquely determined. If among the e.v. of A in a domain $\Omega \subset \mathbf{C}$ there are n real e.v. $\{\lambda_i\}_1^n$, we put

$$\alpha_k(A, G, \Omega) = \sum_{j=1}^{n} \alpha_k(A, G, \lambda_j).$$

Finally, if Ω contains no real e.v. of A, we assume $\alpha_k(A, G, \Omega) = 0$.

THEOREM 2.2. *Let Ω be a normal domain for a G_0-selfadjoint operator $A_0 \in L(\mathfrak{Z})$. Also, suppose that an indexing of sign characteristics of real e.v. of A_0 in Ω is fixed. Then there exists a number $\delta > 0$ such that for any selfadjoint operator $G \in L(\mathfrak{Z})$ and for any G-selfadjoint operator $A \in L(\mathfrak{Z})$ satisfying $\|A - A_0\| + \|G - G_0\| < \delta$, the indexing of sign characteristics of real e.v. of A in Ω can be done in such a way that*

$$|\alpha_k(A, G, \Omega) - \alpha_k(A_0, G_0, \Omega)| \leq \sum_{j=1}^{k} (m_j(A, \Omega) - m_j(A_0, \Omega)), \qquad k = 1, 2, \ldots. \tag{2.2}$$

PROOF. Obviously, it suffices to prove the theorem in the case when Ω contains no nonreal points of the spectrum of A_0. Let the chain of subspaces $\{\mathscr{R}^k(A_0, \Omega)\}_1^\infty$ correspond to a G-normal basis in $\mathscr{R}(A_0, \Omega)$. According to Theorem 1.2, for any $\varepsilon > 0$ there exists $\delta > 0$ such that it is possible to select a chain of subspaces $\{\mathscr{R}^k(A, \Omega)\}_1^\infty$ having the property

$$\theta_0(\mathscr{R}^k(A_0, \Omega), \mathscr{R}^k(A, \Omega)) < \varepsilon, \qquad k = 1, 2, \ldots. \tag{2.3}$$

In addition, it follows from Lemma 2.1 that the subspaces $\mathscr{R}^k(A, \Omega)$ $(k = 1, 2, \dots)$ can be assumed G-nondegenerate, or, in other words, that the chain $\{\mathscr{R}^k(A, \Omega)\}_1^\infty$ corresponds to a G-normal basis in $\mathscr{R}(A, \Omega)$. Now, by [4], Russian p. 323,

$$\begin{aligned} \alpha_k(A_0, G_0, \Omega) &= \operatorname{sig}[P_k(A_0)G_0|\mathscr{R}^k(A_0, \Omega)], \\ \alpha_k(A, G, \Omega) &= \operatorname{sig}[P_k(A)G|\mathscr{R}^k(A, \Omega)], \end{aligned} \tag{2.4}$$

where $P_k(A_0)$ (respectively $P_k(A)$) is the orthogonal projection onto the subspace $\mathscr{R}^k(A_0, \Omega)$ (respectively $\mathscr{R}^k(A, \Omega)$), and $\operatorname{sig} H$ denotes the difference between the numbers of positive and negative e.v. of a selfadjoint operator H (*signature of* H). It follows from (2.3) that there exist subspaces $\mathscr{R}^k \subset \mathscr{R}^k(A, \Omega)$ $(k = 1, 2, \dots)$ such that $\theta(\mathscr{R}^k(A_0, \Omega), \mathscr{R}^k) < \varepsilon$ $(k = 1, 2, \dots)$, and therefore $\dim \mathscr{R}^k = \dim \mathscr{R}^k(A_0, \Omega)$. The number δ can be taken so small that

$$\operatorname{sig}[P_k(A_0)G_0|\mathscr{R}^k(A_0, \Omega)] = \operatorname{sig}[P_k G|\mathscr{R}^k], \; k = 1, 2, \dots, \tag{2.5}$$

where P_k is the orthogonal projection onto $\mathscr{R}^k$. The operator $P_k(A)G|\mathscr{R}^k(A, \Omega)$ does not have fewer positive e.v. (counting multiplicities) than the operator $P_k G|\mathscr{R}^k$ (see, for example, [12] Chapter III, Problem 238). The analogous assertion is valid for negative e.v. of these operators. Therefore

$$|\operatorname{sig}[P_k(A)G|\mathscr{R}^k(A, \Omega)] - \operatorname{sig}[P_k G|\mathscr{R}^k]| \le \dim \mathscr{R}^k(A, \Omega) - \dim \mathscr{R}^k, \quad k = 1, 2, \dots$$

Hence, taking into account (2.4), (2.5), and the fact that

$$\dim \mathscr{R}^k(A, \Omega) = \sum_{j=1}^k m_j(A, \Omega),$$

$$\dim \mathscr{R}^k = \dim \mathscr{R}^k(A_0, \Omega) = \sum_{j=1}^k m_j(A_0, \Omega),$$

we get the theorem.

For k large enough, the right-hand side of (2.2) vanishes, and so (2.2) becomes the equality $\alpha_k(A, G, \Omega) = \alpha_k(A_0, G_0, \Omega)$. This equality is contained in Theorem III.1.1 from [5], where the question of the possible values of the difference between the numbers of real e.v. of A and A_0 in Ω is considered too.

4. Theorem 2.3. *Let the hypotheses of Theorem 2.2 be satisfied. Suppose also that the real spectrum of the operator A_0 (respectively A) in Ω consists of exactly one point λ_0 (respectively λ_1). Then the indexing of the numbers $\varepsilon_i(A, G, \lambda_1)$ can be done in such a way that, besides the inequalities (2.2), the inequalities*

$$|\beta_k(A, G, \lambda_1) - \beta_k(A_0, G_0, \lambda_0)| \le \sum_{j=1}^k (m_j(A, \Omega) - m_j(A_0, \Omega)) + C_k, \qquad k = 1, 2, \dots, \tag{2.6}$$

also hold, where $C_k = \max\{0, k - \dim \operatorname{Ker}(A - \lambda_1 I)\}$.

In the simplest case when $m_j(A, \lambda_1) = m_j(A_0, \lambda_0)$, Theorem 2.2 and Theorem 2.3 imply the equalities $\varepsilon_j(A, G, \lambda_1) = \varepsilon_j(A_0, G_0, \lambda_0)$ $(j = 1, \dots, r)$, where $r = \dim \operatorname{Ker}(A_0 - \lambda_0 I)$. This result has been known ([5], p. 283).

THEOREM 2.4. *Let the hypotheses of Theorem 2.2 be satisfied. Suppose also that the real spectrum of the operator A_0 in Ω consists of a single point λ_0, and that the operator A has no real e.v. in Ω. Then the inequalities* (2.2), *hold, and also*

$$|\beta_k(A_0, G_0, \lambda_0)| \le \sum_{j=1}^{k}(m_j(A, \Omega) - m_j(A_0, \Omega)) + k, \qquad k = 1, 2, \dots \tag{2.7}$$

Theorems 2.3 and 2.4 will be proved in §5, since for the proof we will need some information from the theory of holomorphic selfadjoint matrix-valued functions.

5. A G-selfadjoint operator $A \in L(\mathfrak{Z})$ is said [8] to *satisfy the sign condition* at the point $\lambda_0 \in \mathbf{R}$ if the sign characteristics $\varepsilon_i(A, G, \lambda_0)$ corresponding to an e.v. λ_0 of A depend only on the evenness of the numbers $m_i(A, \lambda_0)$. If $\{\lambda_j\}_1^s$ is a complete system of real e.v. of A, in a domain Ω normal for it, and if the operator A satisfies the sign condition at each of the points λ_j $(j = 1, \dots, s)$, we will say that the operator A *satisfies the sign condition* in Ω. In the case when Ω contains no real e.v. of A, we will also say that A satisfies the sign condition in Ω.

In [8] and [13] it was established that the sign conditions play an important role in the study of the stability of some special invariant subspaces of A, as well as in the study of the stability of some special solutions of the matrix Riccati equations. An example is given there for which the sign condition ceases to be satisfied when passing from a G_0-selfadjoint operator A_0 to a G-selfadjoint operator A, where $\|A - A_0\| + \|G - G_0\|$ can be taken arbitrarily small. From that example it follows that for an operator the property of satisfying the sign condition is, in general, not stable. The next two theorems show that this property is stable in some classes of G-selfadjoint operators A.

THEOREM 2.5. *Let Ω be a normal domain for a G_0-selfadjoint operator $A_0 \in L(\mathfrak{Z})$, containing exactly one point λ_0 of the real spectrum of A_0, and in addition let A_0 satisfy the sign condition at λ_0. Then there exists a number $\delta > 0$ such that for any selfadjoint operator $G \in L(\mathfrak{Z})$ and any G-selfadjoint operator $A \in L(\mathfrak{Z})$ having in Ω only one real e.v. (say, λ_1), with the condition $\|A - A_0\| + \|G - G_0\| < \delta$, the following assertions are true*:

(a) *The operator A satisfies the sign condition at the point λ_1 and, furthermore, the sign characteristics $\varepsilon_i(A, G, \lambda_1)$ and $\varepsilon_i(A_0, G_0, \lambda_0)$ corresponding to the odd (even) numbers $m_i(A, \lambda_1)$ and $m_i(A_0, \lambda_0)$ are equal.*

(b) *The number of odd numbers in the sequences $\{m_i(A, \lambda_1)\}_1^\infty$ and $\{m_i(A_0, \lambda_0)\}_1^\infty$ is the same.*

PROOF. Put $r = \dim \operatorname{Ker}(A_0 - \lambda_0 I)$ and $r_1 = \dim \operatorname{Ker}(A - \lambda_1 I)$. By (2.2) and (2.6), for $k = r$,

$$\alpha_r(A, G, \lambda_1) = \alpha_r(A_0, G_0, \lambda_0), \tag{2.8}$$

$$|\beta_r(A, G, \lambda_1) - \beta_r(A_0, G_0, \lambda_0)| \le r - r_1. \tag{2.9}$$

From (2.8) it follows that the number of odd numbers among $\{m_i(A, \lambda_1)\}_1^r$ in any case is not less than $|\alpha_r(A_0, G_0, \lambda_0)|$. On the other hand, from (2.9) it follows that this quantity cannot be strictly larger than $|\alpha_r(A_0, G_0, \lambda_0)|$. Thus property (b) is proved. Property (a) comes directly from (b) and relations (2.8) and (2.9). The theorem is proved.

COROLLARY 2.6. *Let Ω be a normal domain for a G_0-selfadjoint operator $A_0 \in L(\mathfrak{Z})$, and let the operator A_0 satisfy the sign condition in Ω. Then there exists a* *number $\delta > 0$ such that a G-selfadjoint operator $A \in L(\mathfrak{Z})$ satisfies the sign condition in Ω, provided $\|A - A_0\| + \|G - G_0\| < \delta$, and the number of real e.v. of A in Ω is not larger than the corresponding number for A_0.*

§3. Selfadjoint operator-valued functions and their perturbations

1. Let Ω be a domain in $\mathbf{C}$, let $\mathfrak{G}$ be a complex Hilbert space, and let $W(\lambda)$ be a holomorphic operator-valued function (o.f.) in Ω with values in $L(\mathfrak{G})$. The set of all points $\lambda_0 \in \Omega$ for which the operator $W(\lambda)$ is not invertible will be called the *spectrum* of the o.f. $W(\lambda)$ (in the domain Ω); if the equation $W(\lambda_0)\varphi_0 = 0$ has a nonzero solution φ_0, then λ_0 is said to be an *eigenvalue* (e.v.) of $W(\lambda)$, and φ_0 is said to be an *eigenvector* of $W(\lambda)$ corresponding to an e.v. λ_0. Following Keldysh [6], we introduce the concept of the multiplicity of an e.v. of a holomorphic o.f. (as well as of the partial multiplicity). The vectors $\varphi_1, \dots, \varphi_m$ are said to be *associated* to an eigenvector φ_0, if

$$\sum_{s=0}^{t} \frac{1}{s!} W^{(s)}(\lambda)\varphi_{t-s} = 0, \qquad t = 1, \dots, m.$$

The number $m+1$ is called the *length* of the chain $\varphi_0, \dots, \varphi_m$ that consists of an eigenvector and the associated vectors. If the lengths of all chains corresponding to an eigenvector φ_0 are bounded, the largest one is called the *multiplicity* of φ_0. If these lengths are unbounded, we take the multiplicity of φ_0 to be infinite.

If the eigenspace $\operatorname{Ker} W(\lambda_0)$ is finite-dimensional and the multiplicity of each vector $\varphi \in \operatorname{Ker} W(\lambda_0)$ $(\varphi \neq 0)$ is finite, then the system

$$\varphi_i^j,\ i = 0, \dots, m_j - 1;\ j = 1, \dots, r = \dim \operatorname{Ker} W(\lambda_0),$$

is called the *canonical system* of eigenvectors and associated vectors of the o.f. $W(\lambda)$, corresponding to λ_0, where φ_0^j is an eigenvector of multiplicity m_j, $\varphi_1^j, \dots, \varphi_{m_j-1}^j$ is a chain of vectors associated to it, and in addition m_1 is the largest among multiplicities of all eigenvectors corresponding to λ_0; m_j $(j = 2, \dots, r)$ is the largest among multiplicities of all eigenvectors not belonging to the subspace spanned by the vectors $\{\varphi_0^k\}_{k=1}^{j-1}$. The numbers m_j (*partial multiplicities*) will be denoted by $m_j(W(\lambda), \lambda_0)$ $(j = 1, \dots, r)$. For convenience, put $m_j(W(\lambda), \lambda_0) = 0$ for $j > r$. The number

$$m(W(\lambda), \lambda_0) = \sum_{j=1}^{\infty} m_j(W(\lambda), \lambda_0)$$

is called the *multiplicity* of the e.v. λ_0. If $\dim \operatorname{Ker} W(\lambda_0) = \infty$ or if the multiplicity of at least one vector in $\operatorname{Ker} W(\lambda_0)$ is infinite, we put $m(W(\lambda), \lambda_0) = \infty$.

A bounded domain $\Omega \subset \mathbf{C}$ whose boundary consists of a finite number of simple closed rectifiable curves is said to be *normal* for an o.f. $W(\lambda)$ holomorphic in Ω and continuous in $\overline{\Omega}$, if the operator $W(\lambda)$ is invertible for all $\lambda \in \partial\Omega$ and if it is a Fredholm operator (i.e. $\dim \operatorname{Ker} W(\lambda) < \infty$ and $\dim \mathfrak{G} / \operatorname{Ran} W(\lambda) < \infty$) for all $\lambda \in \Omega$. It is well known that the whole spectrum of an o.f. $W(\lambda)$ in a domain Ω normal for it consists of a finite number of e.v. $\{\lambda_j\}_1^s$, each of them having a finite multiplicity [14]. Therefore, for such a domain, it is possible to define the sums

$$m_i(W(\lambda), \Omega) = \sum_{j=1}^{s} m_i(W(\lambda), \lambda_j), \qquad i = 1, 2, \dots,$$

called the *Gohberg-Kaashoek numbers* of an o.f. $W(\lambda)$ in the domain Ω.

It is known [6] that the principal part of the o.f. $W^{-1}(\lambda)$ in the neighborhood of its pole $\lambda = \lambda_0$ is of the form

$$\sum_{j=1}^{r}\left[\frac{(\cdot\,,\,\psi_0^j)\varphi_0^j}{(\lambda-\lambda_0)^{m_j}}+\cdots+\frac{(\cdot\,,\,\psi_0^j)\varphi_{m_j-1}^j+\cdots+(\cdot\,,\,\psi_{m_j-1}^j)\varphi_0^j}{\lambda-\lambda_0}\right], \tag{3.1}$$

where

$$\varphi_i^j, \qquad i=0,\dots,m_j-1;\; j=1,\dots,r, \tag{3.2}$$

is the canonical system of eigenvectors and associated vectors of the holomorphic o.f. $W(\lambda)$ corresponding to the number λ_0, and

$$\psi_i^j, \qquad i=0,\dots,m_j-1,\; j=1,\dots,r, \tag{3.3}$$

is the canonical system of eigenvectors and associated vectors of the holomorphic o.f. $(W(\overline{\lambda}))^*$ corresponding to the number $\overline{\lambda}_0$, and it is uniquely determined provided the system (3.2) has already been selected. In [7] (Lemma 1.1) it was shown that if λ_0 is a real isolated Fredholm e.v. of a selfadjoint holomorphic o.f. $W(\lambda)$ (i.e. $W(\lambda) = (W(\overline{\lambda}))^*$), then the system (3.2) in the expansion (3.1) can be selected in such a way that the vectors (3.3) are determined by the relations

$$\psi_i^j = \varepsilon_j(W(\lambda),\lambda_0)\varphi_i^j, \qquad i=0,\dots,m_j-1,\; j=1,\dots,r, \tag{3.4}$$

where $\varepsilon_j(W(\lambda),\lambda_0) = \pm 1$ (this result was stated in [7] for a quadratic pencil, but the proof remains valid in the case of a holomorphic o.f., too). The numbers $\varepsilon_j(W(\lambda),\lambda_0)$ will be called the *sign characteristics* of the o.f. $W(\lambda)$, corresponding to the e.v. λ_0. These numbers are determined up to the order of enumeration of those to which equal numbers $m_j(W(\lambda),\lambda_0)$ correspond.

2. In what follows we will need the concept of a spectral node of a holomorphic o.f. We will observe the definition in [15] (there are other alternatives).

Let $\mathfrak{G}$ be a Hilbert space and let $W(\lambda)$ be a holomorphic o.f. in a domain Ω, with values in $L(\mathfrak{G})$. Suppose also that the spectrum of $W(\lambda)$ in Ω is compact. The quintet $\theta = (A, B, C, \mathfrak{S}, \mathfrak{G})$ is called the *spectral node* of the o.f. $W(\lambda)$ in the domain Ω if $\mathfrak{G}$ is a Hilbert space and the operators $A \in L(\mathfrak{S})$, $B \in L(\mathfrak{G}, \mathfrak{S})$, and $C \in L(\mathfrak{S}, \mathfrak{G})$ satisfy the following conditions:

1°. $\sigma(A) \subset \Omega$.

2°. The o.f. $W^{-1}(\lambda) - C(\lambda I - A)^{-1}B$ has an analytic continuation on the whole domain Ω.

3°. The o.f. $W(\lambda)C(\lambda I - A)^{-1}$ has an analytic continuation on the whole domain Ω.

4°. $\bigcap_0^\infty CA^j = \{0\}$.

Various properties of spectral nodes can be found in [15]–[17] (see the bibliography there, too). Here we mention only some of them. For any o.f. $W(\lambda)$ with spectrum compact in Ω, a spectral node always exists and it is determined up to a similarity, i.e. if $\theta_1 = (A_1, B_1, C_1, \mathfrak{S}_1, \mathfrak{G})$ and $\theta_2 = (A_2, B_2, C_2, \mathfrak{S}_2, \mathfrak{G})$ are spectral nodes of an o.f. $W(\lambda)$ holomorphic in Ω, then $A_1 = S^{-1}A_2S$, $C_1 = C_2S$, and $B_1 = S^{-1}B_2$ for an invertible selfadjoint operator $S \in L(\mathfrak{S}_1, \mathfrak{S}_2)$ which is uniquely determined if the nodes θ_1 and θ_2 have already been selected. One knows that the operator A

(which is said to be the *main operator* of the node θ) is the *spectral linearization* of the o.f. $W(\lambda)$ in Ω, i.e. together with 1° it satisfies, for every $\lambda \in \Omega$, the equality

$$\begin{pmatrix} W(\lambda) & 0 \\ 0 & I_{\mathfrak{G}} \end{pmatrix} = E(\lambda) \begin{pmatrix} I_{\mathfrak{G}} & 0 \\ 0 & \lambda I_{\mathfrak{G}} - A \end{pmatrix} F(\lambda),$$

where $I_{\mathfrak{S}}$ and $I_{\mathfrak{G}}$ are the identity operators on the spaces $\mathfrak{S}$ and $\mathfrak{G}$, and $E(\lambda)$ and $F(\lambda)$ are holomorphic o.f., invertible everywhere in Ω, with values in $L(\mathfrak{G} \dotplus \mathfrak{S})$. Obviously, the spectrum and the e.v. of A coincide with the spectrum and the e.v. of $W(\lambda)$, and the multiplicities of e.v. as well as the partial multiplicities are also equal.

3. Using the concept of a spectral node, one can give another characterization of the sign characteristics of a selfadjoint o.f. corresponding to its real e.v. Let $\theta = (A, B, C, \mathfrak{S}, \mathfrak{G})$ be a spectral node of a holomorphic selfadjoint o.f. $W(\lambda)$ in Ω. According to [15], Theorem 2.8, the quintet $\theta^* = (A^*, C^*, B^*, \mathfrak{S}, \mathfrak{G})$ is also a spectral node of $W(\lambda)$ in Ω, and in addition the operator $S \in L(\mathfrak{S})$ realizing the similarity between θ and θ^* is selfadjoint (see [15], equality (1.3a)). If the domain Ω is normal for the selfadjoint holomorphic o.f. $W(\lambda)$ (in this case the space $\mathfrak{G}$ is necessarily finite-dimensional), it follows from what has been said that the operator A is S-selfadjoint, and that the sign characteristics of real e.v. of A do not depend on the choice of a particular node θ. We will show that these sign characteristics coincide with the sign characteristics of the same e.v. of $W(\lambda)$. It suffices to consider the case when the domain Ω contains only one e.v. $\lambda_0 \in \mathbf{R}$ or $W(\lambda)$. Let $\mathfrak{Z}$ be the Hilbert space of dimension $m(W(\lambda), \lambda_0)$ with an orthonormal basis e_i^j $(i = 0, \dots, m_j - 1;\ j = 1, \dots, r)$, where $r = \dim \operatorname{Ker} W(\lambda_0)$ and $m_j = m_j(W(\lambda), \lambda_0)$. Define the operators $A, S \in L(\mathfrak{Z})$, $C \in L(\mathfrak{Z}, \mathfrak{G})$, and $B \in L(\mathfrak{G}, \mathfrak{Z})$ by the equalities $Ae_0^j = 0$, $Ae_i^j = e_{i-1}^j$ $(i = 1, \dots, m_j - 1)$, $Se_i^j = \varepsilon_j(W(\lambda), \lambda_0) e_{m_j - i - 1}^j$, $Ce_i^j = \varphi_i^j$ $(i = 0, \dots, m_j - 1;\ j = 1, \dots, r)$, and $B = S^{-1}C^*$, where φ_i^j $(i = 0, \dots, m_j - 1;\ j = 1, \dots, r)$ is the canonical system of eigenvectors and associated vectors of the holomorphic selfadjoint o.f. $W(\lambda)$, such that the system ψ_i^j from (3.1) satisfies the conditions (3.4). A direct verification shows that $\theta = (A, B, C, \mathfrak{Z}, \mathfrak{G})$ (hence θ^*, too) is a spectral node of $W(\lambda)$ in Ω. It is also easy to verify that the similarity between θ and θ^* is realized by the selfadjoint operator S and that the sign characteristics of the S-selfadjoint operator A coincide with the sign characteristics of $W(\lambda)$:

$$\varepsilon_i(A, S, \lambda_0) = \varepsilon_i(W(\lambda), \lambda_0), \qquad i = 1, \dots, r.$$

4. Denote by $\alpha_k(W(\lambda), \lambda_0)$ (respectively $\beta_k(W(\lambda), \lambda_0)$) the sum of the $\varepsilon_i(W(\lambda), \lambda_0)$ corresponding to the odd (respectively, even and positive) numbers among $\{m_i(W(\lambda), \lambda_0)\}_1^k$. Obviously, the numbers $\alpha_k(W(\lambda), \lambda_0)$ and $\beta_k(W(\lambda), \lambda_0)$ are not uniquely determined and depend on the order of numeration of the numbers $\varepsilon_i(W(\lambda), \lambda_0)$ corresponding to the same partial multiplicities $m_i(W(\lambda), \lambda_0)$.

In §2 (Theorems 2.2–2.4) restrictions were determined upon the range of the sign characteristics as well as of the partial multiplicities of operators selfadjoint with respect to an indefinite metric, under small perturbations. The following theorems show that the similar restrictions apply in the case of small perturbations of a holomorphic selfadjoint o.f.

THEOREM 3.1. *Let $\mathfrak{G}$ be a Hilbert space and let Ω be a normal domain for a selfadjoint o.f. $W(\lambda)$ holomorphic in Ω and continuous in $\overline{\Omega}$, with values in $L(\mathfrak{G})$.*

Suppose also that the order of enumeration of sign characteristics of real e.v. of $W_0(\lambda)$ *in* Ω *is fixed. Then there exists a number* $\delta > 0$ *such that* Ω *is a normal domain for any selfadjoint o.f.* $W(\lambda)$ *with values in* $L(\mathfrak{G})$, *holomorphic in* Ω, *continuous in* $\overline{\Omega}$, *and satisfying the condition*

$$\|W(\lambda) - W_0(\lambda)\| < \delta, \qquad \lambda \in \partial\Omega.$$

Moreover the indexing of sign characteristics of real e.v. of $W(\lambda)$ *in* Ω *can be realized in such a way that for* $k = 1, 2, \dots$,

$$|\alpha_k(W(\lambda), \Omega) - \alpha_k(W_0(\lambda), \Omega)| \le \sum_{j=1}^{k}(m_j(W(\lambda), \Omega) - m_j(W_0(\lambda), \Omega)). \tag{3.5}$$

THEOREM 3.2. *Let the hypotheses of Theorem* 3.1 *be satisfied. Suppose also that the real spectrum of the o.f.* $W_0(\lambda)$ *(respectively* $W(\lambda)$*) in the domain* Ω *consists of exactly one point* λ_0 *(respectively* λ_1*). Then the enumeration of the sign characteristics for* $W(\lambda)$ *can be realized in such a way that the inequalities* (3.5) *hold, and also*

$$|\beta_k(W(\lambda), \lambda_1) - \beta_k(W_0(\lambda), \lambda_0)| \le \sum_{j=1}^{k}(m_j(W(\lambda), \Omega) - m_j(W_0(\lambda), \Omega)) + C_k$$

$(k = 1, 2, \dots)$, *where* $C_k = \max\{0, k - \dim \operatorname{Ker} W(\lambda_1)\}$.

THEOREM 3.3. *Let the hypotheses of Theorem* 3.1 *be satisfied. Suppose also that the real spectrum of the o.f.* $W_0(\lambda)$ *in* Ω *consists of a single point* λ_0 *and the o.f.* $W(\lambda)$ *has no points of the real spectrum in* Ω. *Then, the inequalities* (3.5) *hold, and also*

$$|\beta_k(W_0(\lambda), \lambda_0)| \le \sum_{j=1}^{k}(m_j(W(\lambda), \Omega) - m_j(W_0(\lambda), \Omega)) + k, \qquad k = 1, 2, \dots.$$

PROOF OF THEOREMS 3.1–3.3. Let $\theta_0 = (A_0, B_0, C_0, \mathfrak{Z}, \mathfrak{G})$ be a spectral node of the holomorphic selfadjoint o.f. $W_0(\lambda)$ in Ω, and let $G_0 \in L(\mathfrak{Z})$ be the selfadjoint operator realizing the similarity between θ_0 and θ_0^*. In [16], Theorem 4.5, it was shown that for δ small enough the o.f. $W(\lambda)$ has a spectral node $\theta = (A, B, C, \mathfrak{Z}, \mathfrak{G})$ in Ω such that $\|A - A_0\| + \|B - B_0\| + \|C - C_0\|$ can be taken as small as needed. By virtue of [15] (equality (1.3a)) the quantity $\|G - G_0\|$ is going to be as small as needed too, where G is the selfadjoint operator in $L(\mathfrak{Z})$ realizing the similarity between the spectral nodes θ and θ^* of the holomorphic selfadjoint o.f. $W(\lambda)$ in Ω. Since the real e.v. that belong to Ω, as well as their partial multiplicities and sign characteristics for the o.f. $W(\lambda)$ and $W_0(\lambda)$ the operators A and A_0, respectively, coincide, in order to complete the proof it suffices to quote Theorems 2.2–2.4.

§4. The inverse problem in the case of the nonreal spectrum

1. Let F be the set of all nonincreasing finite sequences of nonnegative integers, except for the zero sequence. If $u = \{u_i\}_1^\infty$ and $V = \{v_i\}_1^\infty$ are sequences from F, we will write $u \prec v$ if the following relations are satisfied:

$$\sum_{i=1}^{k} u_i \le \sum_{i=1}^{k} v_i, \quad k = 1, 2, \dots, \qquad \sum_{i=1}^{\infty} u_i = \sum_{i=1}^{\infty} v_i.$$

In [1]–[3] the following result was obtained. Let an operator $A_0 \in L(\mathfrak{Z})$ be given together with a domain $\Omega \subset \mathbf{C}$ normal for A_0. Then there exists a number $\delta > 0$ such that

$$\{m_i(A_0, \Omega)\} \prec \{m_i(A, \Omega)\} \tag{4.1}$$

for any operator $A \in L(\mathfrak{Z})$ with $\|A - A_0\| < \delta$.

Note that this assertion can be obtained as a consequence of Theorem 1.2. For that we need only use the obvious equalities

$$\dim \mathscr{R}^k(A_0, \Omega) = \sum_{i=1}^{k} m_i(A_0, \Omega),$$

$$\dim \mathscr{R}^k(A, \Omega) = \sum_{i=1}^{k} m_i(A, \Omega), \qquad k = 1, 2, \dots,$$

and the fact that the inequality $\theta_0(\mathfrak{N}, \mathfrak{M}) < 1$ (where $\mathfrak{N}, \mathfrak{M} \subset \mathfrak{Z}$ are arbitrary subspaces) implies $\dim \mathfrak{N} < \dim \mathfrak{M}$.

In [1]–[3] the converse was also proved: if an operator A_0 has a unique e.v. λ_0 in a domain Ω normal for it (the general case is easily reduced to this one), and if a natural number p and sequences $\{m_{ij}\}_{i=1}^{\infty} \in F$ $(j = 1, \dots, p)$ are given, satisfying

$$\{m_i(A_0, \lambda_0)\} \prec \left\{ \sum_{j=1}^{p} m_{ij} \right\},$$

then in any neighborhood of A_0 there exists an operator A having exactly p e.v. $\{\lambda_j\}_1^p$ in Ω, and in addition $m_i(A, \lambda_j) = m_{ij}$.

This result indicates that the relations (4.1) are the only restrictions on the possible change of partial multiplicities of linear operators under small perturbations. The next theorem shows that for $\lambda_0 \notin \mathbf{R}$ this assertion remains valid in the case when the initial operator A_0 and its small perturbations are G-selfadjoint.

THEOREM 4.1. *Let Ω be a normal domain for a G-selfadjoint operator $A_0 \in L(\mathfrak{Z})$ containing only one e.v. λ_0 of A_0, with $\lambda_0 \notin \mathbf{R}$. Suppose also that a natural number p and the sequences $\{m_{ij}\}_{i=1}^{\infty} \in F$ $(j = 1, \dots, p)$ are given so that*

$$\{m_i(A_0, \lambda_0)\} \prec \left\{ \sum_{j=1}^{p} m_{ij} \right\}. \tag{4.2}$$

Then in any neighborhood of A_0 there exists a G-selfadjoint operator $A \in L(\mathfrak{Z})$ having exactly p e.v. $\{\lambda_j\}_1^p$ in Ω and such that $m_i(A, \lambda_j) = m_{ij}$ $(i = 1, 2, \dots;$ $j = 1, \dots, p)$.

PROOF. Consider the selfadjoint operator bundle $L(\lambda) = GA_0 - \lambda G$. Theorem 7.1 of [17] implies that there exist an operator pencil $R(\lambda)$ with coefficients in $L(\mathfrak{Z})$ and with only one point λ_0 of its spectrum in Ω, and an entire (i.e. holomorphic in the whole plain) o.f. $Q(\lambda)$ such that

$$L(\lambda) = Q(\lambda)R(\lambda) = (R(\bar{\lambda}))^*(Q(\bar{\lambda}))^*, \qquad \lambda \in \mathbf{C},$$

with the operator $Q(\lambda_0)$ invertible. According to [16] there exists an entire o.f. $F(\lambda)$ such that the operator $F(\lambda_0)$ is invertible and $(Q(\bar{\lambda}))^* = F(\lambda)R(\lambda)$ for any $\lambda \in \mathbf{C}$. Thus, the selfadjoint pencil $L(\lambda)$ admits the factorization

$$L(\lambda) = (R(\bar{\lambda}))^* F(\lambda) R(\lambda), \qquad \lambda \in \mathbf{C}. \tag{4.3}$$

Moreover, (4.3) implies that the o.f. $F(\lambda)$ is selfadjoint and partial multiplicities corresponding to λ_0 for the pencils $L(\lambda)$ and $R(\lambda)$ are equal.

Denote by Ω_1 an open disk centered at the point $\lambda = 0$ and with radius large enough so that it contains the domain Ω as well as the whole spectrum of the operator A_0 (or, which is the same, the spectrum of $L(\lambda)$). From the results of [1]–[3] and Theorem 4.3 from [16] it follows that there exists an o.f. $P(\lambda)$, holomorphic in Ω_1 and continuous in $\overline{\Omega}_1$, having exactly p e.v. $\{\lambda_j\}_1^p$ in Ω, with the partial multiplicities $m_i(P(\lambda), \lambda_j) = m_{ij}$, and moreover, the quantity

$$\max_{\lambda \in \partial\Omega_1} \|P(\lambda) - R(\lambda)\| \tag{4.4}$$

can be taken to be as small as desired. Consider the selfadjoint o.f.

$$W(\lambda) = (P(\overline{\lambda}))^* F(\lambda) P(\lambda), \qquad \lambda \in \Omega_1,$$

holomorphic in Ω_1. Obviously, the spectrum of $W(\lambda)$ in Ω consists of p e.v. $\{\lambda_j\}_1^p$, $m_i(W(\lambda), \lambda_j) = m_{ij}$ $(i = 1, 2, \ldots;\ j = 1, \ldots, p)$, and, in addition,

$$\|W(\lambda) - L(\lambda)\| \le K_1 \max_{\lambda \in \partial\Omega_1} \|P(\lambda) - R(\lambda)\|, \tag{4.5}$$

where $K_1 > 0$ depends only on the initial factorization (4.3). It is not hard to show that the quintet $\theta = (A_0, G^{-1}, I, 3, 3)$ is a spectral node of the selfadjoint pencil $L(\lambda)$ in Ω_1 and that the similarity between the spectral nodes θ and θ^* is realized by the operator G. By Theorem 4.5 of [16], for the quantity (4.4) small enough, there exists a spectral node $\theta_1 = (A_1, B_1, C_1, 3, 3)$ for $W(\lambda)$ satisfying

$$\|A_1 - A_0\| + \|B_1 - G^{-1}\| + \|C_1 - I\| \le K_2 \max_{\lambda \in \partial\Omega_1} \|W(\lambda) - L(\lambda)\|. \tag{4.6}$$

From this, according to formula (1.3a) from [15], it follows that the selfadjoint operator $G_1 \in L(3)$ realizing the similarity between the spectral nodes θ_1 and θ_1^* of $W(\lambda)$ satisfies

$$\|G_1 - G\| \le K_3 \max_{\lambda \in \partial\Omega_1} \|W(\lambda) - L(\lambda)\|. \tag{4.7}$$

Note that the positive constants K_2 and K_3 depend only on the pencil $L(\lambda)$ and its spectral node θ.

Thus we have constructed a G_1-selfadjoint operator $A_1 \in L(3)$, having exactly p e.v. $\{\lambda_j\}_1^p$ in Ω, with partial multiplicities $m_i(A_1, \lambda_j) = m_{ij}$. Moreover, from (4.5)–(4.7), by virtue of the arbitrary smallness of the quantity (4.4), it follows that the number $\|A_1 - A_0\| + \|G_1 - G\|$ can be taken as small as desired. To complete the proof, we need to use the following lemma.

LEMMA 4.2. *Let a G-selfadjoint operator $A_0 \in L(3)$ be given. For any $\varepsilon > 0$ there exists $\delta > 0$ such that if a selfadjoint operator $G_1 \in L(3)$ and a G_1-selfadjoint operator $A_1 \in L(3)$ satisfy the inequality $\|A_1 - A_0\| + \|G_1 - G\| < \delta$, then there exists a G-selfadjoint operator $A \in L(3)$ with $\|A - A_0\| < \delta$, and moreover the e.v. and their partial multiplicities as well as the sign characteristics of real e.v. of A and A_1 are equal.*

PROOF. First note that for δ small enough the selfadjoint operators G_1 and G have the same number of positive e.v. (counting multiplicities). Therefore the operators G_1 and G are congruent, i.e. $G = T^* G_1 T$ for an invertible operator $T \in L(3)$. In [18] it was shown that the operator T can be chosen in such a way that

$$\|I - T\| \le K_1 \|G_1 - G\|, \tag{4.8}$$

where the number K_1 depends only on the operator G. It is not hard to verify that the operator $A = T^{-1}A_1T$ is G-selfadjoint. Moreover, from (4.8) it follows that

$$\|A - A_0\| \le K_2(\|A_1 - A_0\| + \|G_1 - G\|),$$

where the number K_2 depends only on the initial operators A_0 and G. It is easy to see that the e.v. and their multiplicities and partial multiplicities for the operators A and A_1 coincide. By Theorem 1.3.6 of [5], the sign characteristics of real e.v. for these operators are also equal. The lemma is proved.

With the help of the results of this section it is easy to examine the case when the domain Ω contains several nonreal points of the spectrum of A_0, no pair being complex conjugates. And if there are conjugate points of the spectrum of A_0 in Ω, then the decomposition of such e.v. under a perturbation takes place in exactly the same way.

§5. The inverse problem for Gohberg-Kaashoek numbers in the case of the real spectrum

1. We will need the following characterization of the sign characteristics of real eigenvalues of selfadjoint matrix-valued functions (m.f.) (see, for example, [5], Theorem 11.3.3). Let a selfadjoint m.f. $W(\lambda)$, holomorphic in $\mathbf{C}$ and with values in $L(\mathfrak{Z})$, be given. Then there exists a holomorphic m.f. $U(\lambda)$ of the real argument, whose values are unitary matrices, such that for $\lambda \in \mathbf{R}$

$$W(\lambda) = U^*(\lambda)\,\mathrm{diag}(\mu_1(\lambda), \dots, \mu_n(\lambda))U(\lambda), \tag{5.1}$$

where $n = \dim \mathfrak{Z}$ and $\mu_1(\lambda), \dots, \mu_n(\lambda)$ are holomorphic functions of a real argument. The corresponding diagonal matrix is denoted by $\mathrm{diag}(\mu_1, \dots, \mu_n)$. If λ_0 is a real e.v. of the m.f. $W(\lambda)$, then the equalities $\mu_j(\lambda) = (\lambda - \lambda_0)^{m_j}\nu_j(\lambda)$ hold, where $m_j = m_j(W(\lambda), \lambda_0)$, and for $m_j \ne 0$

$$\operatorname{sgn}\nu_j(\lambda_0) = \varepsilon_j(W(\lambda), \lambda_0).$$

Using this characterization of sign characteristics for m.f., we can prove Theorems 2.3 and 2.4.

PROOF OF THEOREM 2.3. Together with the operators A_0, G_0, A, and G, consider the selfadjoint pencils $W_0(\lambda) = G_0A_0 - \lambda G_0$ and $W(\lambda) = GA - \lambda G$. As we have already seen, $\theta_0 = (A_0, G_0^{-1}, I, \mathfrak{Z}, \mathfrak{Z})$ and $\theta = (A, G^{-1}, I, \mathfrak{Z}, \mathfrak{Z})$ are spectral nodes of the m.f. $W_0(\lambda)$ and $W(\lambda)$ in Ω, and therefore the partial multiplicities and the sign characteristics corresponding to λ_0 (respectively λ_1) for A_0 and $W_0(\lambda)$ (respectively for A and $W(\lambda)$) coincide. From (5.1) it follows that when passing from $W_0(\lambda)$ and $W(\lambda)$ to $(\lambda - \lambda_0)W_0(\lambda)$ and $(\lambda - \lambda_1)W(\lambda)$ respectively, the first n $(= \dim \mathfrak{Z})$ partial multiplicities of λ_0 and λ_1 are increased by 1, and the sign characteristics remain unchanged. These considerations and condition (3.5), written down for $(\lambda - \lambda_1)W(\lambda)$ and $(\lambda - \lambda_0)W_0(\lambda)$, now imply (2.6). The theorem is proved.

Theorem 2.4 can be proved in a similar way.

2. Theorems 2.3 and 2.4 show that, in the case $\lambda_0 \in \mathbf{R}$, Theorem 4.1 is no longer true, i.e. additional restrictions (besides (4.2)) on the range of possible values for the partial multiplicities of the perturbed operator appear. Nevertheless, even in that case no restrictions except (4.1) on the Gohberg-Kaashoek numbers appear. The next theorem shows this.

THEOREM 5.1. *Let Ω be a normal domain for a G-selfadjoint operator $A_0 \in L(\mathfrak{Z})$ containing only one e.v. λ_0 of A_0, where $\lambda_0 \in \mathbf{R}$, and let the sequence $\{m_i'\}_1^\infty$ in F*

be given so that $\{m_i(A_0, \lambda_0)\} \prec \{m'_i\}$. *Then in every neighborhood of* A_0 *there exists a G-selfadjoint operator* $A \in L(\mathfrak{Z})$ *such that* $\sigma(A) \cap \Omega \subset \mathbf{R}$ *and* $m_i(A, \Omega) = m'_i$.

PROOF. Consider a selfadjoint matrix pencil

$$W_0(\lambda) = \operatorname{diag}(\varepsilon_i(\lambda - \lambda_0)^{m_i})_{i=1}^r,$$

where $r = \dim \operatorname{Ker}(A_0 - \lambda_0 I)$, $\varepsilon_i = \varepsilon_i(A_0, G, \lambda_0)$, and $m_i = m_i(A_0, \lambda_0)$, and fix an open disk Ω centered at λ_0. With the help of m'_1 steps, we construct a selfadjoint matrix pencil $W_{m'_1}(\lambda)$ such that $m_i(W_{m'_1}(\lambda), \Omega) = m'_i$, where the coefficients of $W_{m'_1}(\lambda)$ can be assumed to be as close as needed to the corresponding coefficients of $W_0(\lambda)$.

Denote by s_1 the largest number i for which $m'_i \geq 1$. Let the integers $k \geq 0$ and $l > 0$ be such that $m_i = m_{s_1}$ if and only if $k < i \leq k + l$. Denote by M the set consisting of the natural numbers $1, \dots, k$ (under the condition that $k \geq 1$) and $(k + l) + (s_1 - k) + 1, \dots, k + l$. Consider the matrix pencils

$$V_1(\lambda) = \operatorname{diag}(v_i(\lambda))_1^r, \qquad U_1(\lambda) = \operatorname{diag}(u_i(\lambda))_1^r,$$

where $v_i(\lambda) = (\lambda - \lambda_1)$ and $u_i(\lambda) = \varepsilon_i(\lambda - \lambda_0)^{m_i - 1}$ for $i \in M$, and $v_i(\lambda) = 1$ and $u_i(\lambda) = \varepsilon_i(\lambda - \lambda_0)^{m_i}$ for any other i. It is not hard to check that the coefficients of the selfadjoint pencil $W_1(\lambda) = V_1(\lambda)U_1(\lambda)$ can be made as close to the corresponding coefficients of $W_0(\lambda)$ as desired if the real number λ_1 is chosen close enough to λ_0. It is also obvious that $m_i(U_1(\lambda), \lambda_0) = m_i(W_0(\lambda), \lambda_0) - 1$ for $i \in M$ and $m_i(U_1(\lambda), \lambda_0) = m_i(W_0(\lambda), \lambda_0)$ for $i \notin M$. From these relations it follows that $\{m_i(U_1(\lambda), \lambda_0)\} \prec \{m''_i\}$, where $m''_i = m'_i - 1$ for $i = 1, \dots, s_1$ and $m''_i = m'_i$ $(= 0)$ for any other i.

In the second step we deal with the matrix pencil $U_1(\lambda)$ and the sequence $\{m''_i\}_1^\infty$ in exactly the same way as we dealt with the pencil $W_0(\lambda)$ and the sequence $\{m'_i\}_1^\infty$ in the first step. As a result, we obtain the number s_2, the sequence $\{m'''_i\}_1^\infty$ and the pencils $V_2(\lambda)$ and $U_2(\lambda)$. Continuing this process, after m'_1 steps we obtain a pencil $U_{m'_1}(\lambda)$ which no longer depends on λ: $U_{m'_1}(\lambda) = \operatorname{diag}(\varepsilon_i)_1^r$. As for $V_i(\lambda)$, it is of diagonal form, with s_i of its diagonal entries equal to $(\lambda - \lambda_i)$, and the remaining equal to 1 $(i = 1, \dots, m'_1)$. Selecting the real numbers λ_i $(i = 1, \dots, m'_1)$ close enough to λ_0, we can assume that the coefficients of the selfadjoint pencil $W_{m'_1}(\lambda) = V_1(\lambda) \cdots V_{m'_1}(\lambda)$ are as close as desired to the corresponding coefficients of $W_0(\lambda)$. If all the numbers λ_i $(i = 1, \dots, m'_1)$ are distinct, it is not hard to obtain from the relations $s_i = \max\{j : m'_j \geq i\}$ that $m_i(W_{m'_1}(\lambda), \Omega) = m'_i$ $(i = 1, 2, \dots)$. In order to complete the proof it remains to use the following lemma.

LEMMA 5.2. *Let* $A_0 \in L(\mathfrak{Z})$ *be a G-selfadjoint operator, let a domain* Ω *be normal for it, and let* Ω *contain only one e.v.* λ_0 *of* A_0, *with* $\lambda_0 \in \mathbf{R}$. *Put* $W_0(\lambda) = \operatorname{diag}(\varepsilon_i(\lambda - \lambda_0)^{m_i})_{i=1}^r$, *where* $r = \dim \operatorname{Ker}(A_0 - \lambda_0 I)$, $m_i = m_i(A_0, \lambda_0)$, *and* $\varepsilon_i = \varepsilon_i(A_0, G, \lambda_0)$. *Let the numbers* m'_i $(i = 1, 2, \dots)$ *and* ε'_i $(i = 1, \dots, r')$ *be such that for every* $\delta > 0$ *a selfadjoint matrix pencil* $W(\lambda)$ *with the following properties can be singled out*:

1) $\|W(\lambda) - W_0(\lambda)\| < \delta$ $(\lambda \in \partial\Omega)$.
2) λ_0 *is the unique e.v. of* $W(\lambda)$ *in* Ω.
3) $m_i(W(\lambda), \lambda_0) = m'_i$ *and* $\varepsilon_i(W(\lambda), \lambda_0) = \varepsilon'_i$.

Then in every neighborhood of A_0 there exists a G-selfadjoint operator $A \in L(\mathfrak{Z})$ such that λ_0 is the unique e.v. of A in Ω, and $m_i(A, \lambda_0) = m_i'$ and $\varepsilon_i(A, G, \lambda_0) = \varepsilon_i'$.

PROOF. Put $\widetilde{A}_0 = A_0|\mathscr{R}(A_0, \lambda_0)$ and $\widetilde{G}_0 = PG|\mathscr{R}(A_0, \lambda_0)$, where P is the orthogonal projection onto the subspace $\mathscr{R}(A_0, \lambda_0)$. Let $\theta = (\widetilde{A}, \widetilde{B}, \widetilde{C}, \mathscr{R}(A_0, \lambda_0), \mathfrak{G})$ be a spectral node of $W_0(\lambda)$ in Ω, and let $\widetilde{G} \in L(\mathscr{R}(A_0, \lambda_0))$ be the selfadjoint operator realizing the similarity between θ and θ^*. By Theorem 1.3.6 of [5], the equalities $m_i(\widetilde{A}_0, \lambda_0) = m_i(\widetilde{A}, \lambda_0)$ and $\varepsilon_i(\widetilde{A}_0, \widetilde{G}_0, \lambda_0) = \varepsilon_i(\widetilde{A}, \widetilde{G}, \lambda_0)$ imply that there exists an invertible operator $T \in L(\mathscr{R}(A_0, \lambda_0))$ such that $\widetilde{A}_0 = T^{-1}\widetilde{A}T$ and $\widetilde{G}_0 = T^*\widetilde{G}T$. Put $\widetilde{B}_0 = T^{-1}\widetilde{B}$ and $\widetilde{C}_0 = \widetilde{C}T$. It is easy to see that the quintet $\theta_0 = (\widetilde{A}_0, \widetilde{B}_0, \widetilde{C}_0, \mathscr{R}(A_0, \lambda_0), \mathfrak{G})$ is a spectral node of the selfadjoint pencil $W_0(\lambda)$ in Ω, and the similarity between θ_0 and θ_0^* is realized by the selfadjoint operator $\widetilde{G}_0$. The subsequent arguments are presented briefly, because they are analogous to those in the proof of Theorem 4.1. For the pencil $W(\lambda)$ there exists a spectral node $\theta_1 = (\widetilde{A}_1, \widetilde{B}_1, \widetilde{C}_1, \mathscr{R}(A_0, \lambda_0), \mathfrak{G})$ such that

$$\|\widetilde{A}_1 - \widetilde{A}_0\| + \|\widetilde{B}_1 - \widetilde{B}_0\| + \|\widetilde{C}_1 - \widetilde{C}_0\| \le K_1 \max_{\lambda\in\partial\Omega} \|W(\lambda) - W_0(\lambda)\|,$$
$$\|\widetilde{G}_1 - \widetilde{G}_0\| \le K_2 \max_{\lambda\in\partial\Omega} \|W(\lambda) - W_0(\lambda)\|,$$

where $\widetilde{G}_1$ is the selfadjoint operator realizing the similarity between θ_1 and θ_1^*, and the positive constants K_1 and K_2 depend only on $W_0(\lambda)$ and θ_0 ([16], Theorem 4.3). From this and condition 1) it follows that $\|\widetilde{A}_1 - \widetilde{A}_0\| + \|\widetilde{G}_1 - \widetilde{G}_0\|$ can be assumed to be as small as desired. Therefore, by virtue of Lemma 4.2, in every neighborhood of $\widetilde{A}_0$ there exists a $\widetilde{G}_0$-selfadjoint operator $\widetilde{A}_2 \in L(\mathscr{R}(A_0, \lambda_0))$ such that $m_i(\widetilde{A}_2, \lambda_0) = m_i'$ and $\varepsilon_i(\widetilde{A}_2, \widetilde{G}_0, \lambda_0) = \varepsilon_i'$. It is easy to see that the operator A which acts as $\widetilde{A}_2$ on the subspace $\mathscr{R}(A_0, \lambda_0)$ and as A_0 on the subspace $\mathscr{R}(A_0, \mathbf{C}\backslash\lambda_0)$, possesses all the required properties. Lemma 5.2, and with it Theorem 5.1, are proved.

§6. The inverse problem in the case of a unique real eigenvalue

1. In this section we consider the case when both the initial G-selfadjoint operator $A_0 \in L(\mathfrak{Z})$ and the G-selfadjoint operator A that is close to it have in a domain Ω only one e.v., which is real.

Consider first the model examples of perturbations for the pencil of matrices of second order

$$W_0(\lambda) = \operatorname{diag}(\varepsilon_i(\lambda - \lambda_0)^{m_i})_1^2,$$

where $\varepsilon_i = \pm 1$, $m_i \in \mathbf{N}$, $m_1 \ge m_2$, and $\lambda_0 \in \mathbf{R}$. Obviously (see (5.1)), $\varepsilon_i(W_0(\lambda), \lambda_0) = \varepsilon_i$ and $m_i(W_0(\lambda), \lambda_0) = m_i$. We analyze various situations that may arise under a perturbation of $W_0(\lambda)$.

EXAMPLE 6.1. If the number $m_1 + m_2$ is odd, we consider the selfadjoint matrix pencil

$$W(\lambda) = W_0(\lambda) + \begin{pmatrix} 0 & \delta(\lambda - \lambda_0)^s \\ \delta(\lambda - \lambda_0)^s & \delta^2\varepsilon_1(\lambda - \lambda_0)^{m_2 - 1} \end{pmatrix},$$

where $s = (m_1 + m_2 - 1)/2$ and $\delta \in \mathbf{R}$. It is not hard to verify that the spectrum of $W(\lambda)$ consists of a single point λ_0. We find that partial multiplicities and the sign characteristics corresponding to the e.v. of λ_0 of $W(\lambda)$. From the Smith canonical form for the m.f. $W(\lambda)$ (see, for example, [10], Chapter VI, §3.1) it follows that $m_2(W(\lambda), \lambda_0)$ is the same as the multiplicity of the root λ_0 of the greatest common divisor of all entries of the matrix $W(\lambda)$, i.e. $m_2(W(\lambda), \lambda_0) = m_2 - 1$. Since the multiplicity of the e.v. λ_0 is $m_1 + m_2$, it follows that $m_1(W(\lambda), \lambda_0) = m_1 + 1$. If the number $\delta > 0$ is small enough, then, by Theorem 3.1, $\alpha_2(W(\lambda), \lambda_0) = \alpha_2(W_0(\lambda), \lambda_0)$ and, therefore, the same sign corresponds to any chain of odd length associated to $W(\lambda)$ as to any chain of odd length associated to $W_0(\lambda)$. Theorem 3.2 implies the same assertion for the chains of even lengths. Consequently, for $\delta > 0$ small enough, the equalities $\varepsilon_1(W(\lambda), \lambda_0) = \varepsilon_2$ and $\varepsilon_2(W(\lambda), \lambda_0) = \varepsilon_1$ are true (if $m_2 - 1 \neq 0$).

EXAMPLE 6.2. If the number $m_1 + m_2$ is even, $m_2 \geq 2$, and $\varepsilon_1 = \varepsilon_2$, we put

$$W(\lambda) = W_0(\lambda) + \begin{pmatrix} 0 & \delta(\lambda - \lambda_0)^s \\ \delta(\lambda - \lambda_0)^s & \delta^2 \varepsilon_1 (\lambda - \lambda_0)^{m_2 - 2} \end{pmatrix},$$

where $s = (m_1 + m_2 - 2)/2$ and $\delta \in \mathbf{R}$. By analogy with Example 6.1, it is possible to verify that the spectrum of $W(\lambda)$ consists of a single point λ_0, and $m_1(W(\lambda), \lambda_0) = m_1 + 2$ and $m_2(W(\lambda), \lambda_0) = m_2 - 2$; besides that, for $\delta > 0$ small enough, $\varepsilon_1(W(\lambda), \lambda_0) = \varepsilon_1$ and $\varepsilon_2(W(\lambda), \lambda_0) = \varepsilon_2$ (if $m_2 > 2$).

EXAMPLE 6.3. If the number $m_1 + m_2$ is even, but $\varepsilon_1 = -\varepsilon_2$, then we put

$$W(\lambda) = W_0(\lambda) + \delta \begin{pmatrix} (\lambda - \lambda_0)^{m_1 - 1} & (\lambda - \lambda_0)^s \\ (\lambda - \lambda_0)^s & (\lambda - \lambda_0)^{m_2 - 1} \end{pmatrix}, \tag{6.1}$$

where $s = (m_1 + m_2 - 2)/2$ and $\delta \in \mathbf{R}$. In exactly the same way as in the previous examples, one verifies that λ_0 is a unique e.v. of the selfadjoint pencil $W(\lambda)$, and that $m_1(W(\lambda), \lambda_0) = m_1 + 1$ and $m_2(W(\lambda), \lambda_0) = m_2 - 1$. In order to find the sign characteristics of $W(\lambda)$, we compute the diagonal entries of the middle factor on the right-hand side of (5.1), when $W(\lambda)$ on the left-hand side of that formula is of the form (6.1). It is easy to see that these holomorphic functions $\mu_1(\lambda)$ and $\mu_2(\lambda)$ of a real argument are roots of the equation $\det(\mu(\lambda)I - W(\lambda)) = 0$. It is not hard to verify, by writing these roots explicitly, that $\nu_2(\lambda) = \mu_2(\lambda)/(\lambda - \lambda_0)^{m_2 - 1}$ has at λ_0 the same sign $\varepsilon_2(W(\lambda), \lambda_0)$ as the number $\delta \in \mathbf{R}$. It follows from Theorems 3.1 and 3.2 that $\varepsilon_1(W(\lambda), \lambda_0) = -\operatorname{sgn}\delta$ and $\varepsilon_2(W(\lambda), \lambda_0) = \operatorname{sgn}\delta$, for $|\delta| \neq 0$ small enough.

2. LEMMA 6.4. *Let λ_0 ($\in \mathbf{R}$) be a unique e.v. of a G-selfadjoint operator $A_0 \in L(\mathfrak{Z})$ that belongs to a domain Ω normal for A_0, and let the natural numbers s and t be given ($s < t \leq \dim \operatorname{Ker}(A_0 - \lambda_0 I)$). In any neighborhood of A_0 there exists a G-selfadjoint operator A having in Ω only one e.v. λ_0, and such that $m_i(A, \lambda_0) = m_i(A_0, \lambda_0)$ and $\varepsilon_i(A, G, \lambda_0) = \varepsilon_i(A_0, G, \lambda_0)$ for $i \neq s, t$, whereas for $i = s, t$ these quantities take the following values:*

(a) *If the number $m_s(A_0, \lambda_0) + m_t(A_0, \lambda_0)$ is odd, then $m_s(A, \lambda_0) = m_s(A_0, \lambda_0) + 1$, $m_t(A, \lambda_0) = m_t(A_0, \lambda_0) - 1$, $\varepsilon_s(A, G, \lambda_0) = \varepsilon_t(A_0, G, \lambda_0)$, and $\varepsilon_t(A, G, \lambda_0) = \varepsilon_s(A_0, G, \lambda_0)$.*

(b) *If the number $m_s(A_0, \lambda_0) + m_t(A_0, \lambda_0)$ is even and $\varepsilon_s(A_0, G, \lambda_0) = \varepsilon_t(A_0, G, \lambda_0)$, then $m_s(A, \lambda_0) = m_s(A_0, \lambda_0) + 2$, $m_t(A, \lambda_0) = m_t(A_0, \lambda_0) - 2$, and $\varepsilon_s(A, G, \lambda_0) = \varepsilon_t(A, G, \lambda_0) = \varepsilon_s(A_0, G, \lambda_0)$.*

(c) *If, $m_s(A_0, G, \lambda_0) + m_t(A_0, G, \lambda_0)$ is even but $\varepsilon_s(A_0, G, \lambda_0) = -\varepsilon_t(A_0, G, \lambda_0)$, then $m_s(A, \lambda_0) = m_s(A_0, \lambda_0) + 1$, $m_t(A, \lambda_0) = m_t(A_0, \lambda_0) - 1$, $\varepsilon_s(A, G, \lambda_0) = -\varepsilon_t(A, G, \lambda_0)$.*

PROOF. It suffices to establish the lemma for the restriction of A_0 to the linear span of all vectors from the sth and tth chains of any G-normal basis in $\mathscr{R}(A_0, \lambda_0)$. For that we need to use one of Examples 6.1–6.3 and quote Lemma 5.2.

We will show that Theorems 2.2 and 2.3 give the complete characterization of the range of partial multiplicities and sign characteristics in the special case when these quantities are changed by at most two chains (and the initial e.v. has not been decomposed).

THEOREM 6.5. *Let $A_0 \in L(\mathfrak{Z})$ be a G-selfadjoint operator having in a domain Ω, normal for it, a unique e.v. λ_0 $(\in \mathbf{R})$. Let a sequence $\{m_i'\} \in F$ be given as well as a set $\{\varepsilon_i'\}_1^{r'}$, where $\varepsilon_i' = \pm 1$ and $r' = \max\{j : m_j' \neq 0\}$. Denote by α_k' (respectively β_k') the sum of the numbers ε_i' to which odd (respectively even) numbers among m_i' $(i = 1, \dots, k)$ correspond. Suppose that*

$$\sum_{i=1}^{\infty} m_i' = \sum_{i=1}^{\infty} m_i(A_0, \lambda_0)$$

and that $m_i' = m_i(A_0, \lambda_0)$ and $\varepsilon_i' = \varepsilon_i(A_0, G, \lambda_0)$ for all indices i except s and t. In order that in every neighborhood of A_0 there exist a G-selfadjoint operator A such that $\sigma(A) \cap \Omega = \{\lambda_0\}$ and $m_i(A, \lambda_0) = m_i'$ and $\varepsilon_i(A, G, \lambda_0) = \varepsilon_i'$, it is necessary and sufficient that for $k = s, t$

$$|\alpha_k' - \alpha_k(A_0, G, \lambda_0)| \leq \sum_{i=1}^{k} (m_i' - m_i(A_0, \lambda_0)),$$

$$|\beta_k' - \beta_k(A_0, G, \lambda_0)| \leq \sum_{i=1}^{k} (m_i' - m_i(A_0, \lambda_0)) + C_k,$$

where $C_k = \max\{0, k - r'\}$.

PROOF. It is easy to see that, with the help of successive applications of the perturbations determined by Lemma 6.4, one can construct a G-selfadjoint operator $A \in L(\mathfrak{Z})$ such that the whole spectrum of A in Ω consists of a single point λ_0 and $m_i(A, \lambda_0) = m_i'$ $(i = 1, 2, \dots)$. We clarify that in the case when $m_s(A_0, \lambda_0) + m_t(A_0, \lambda_0)$ is odd, we need to apply the perturbation of type (a) $m_s' - m_s(A_0, \lambda_0)$ times; in the case when $m_s(A_0, \lambda_0) + m_t(A_0, \lambda_0)$ is even and $\varepsilon_s(A_0, G, \lambda_0) = \varepsilon_t(A_0, G, \lambda_0)$ we need to use the perturbation of type (b) $(m_s' - m_s(A_0, \lambda_0))/2$ times; and in the case when $m_s(A_0, \lambda_0) + m_t(A_0, \lambda_0)$ is even but $\varepsilon_s(A_0, G, \lambda_0) = -\varepsilon_t(A_0, G, \lambda_0)$ we need to apply the perturbation of type (c) $m_s' - m_s(A_0, \lambda_0)$ times. Moreover, we can take $\|A - A_0\|$ to be as small as desired. The theorem is proved.

THEOREM 6.6. *Let the hypotheses of Theorem 5.1 be satisfied. Denote by γ_k (respectively γ_k') the number of odd numbers among $\{m_i(A_0, \lambda_0)\}_1^k$ (respectively $\{m_i'\}_1^k$). If*

$$|\gamma_k' - \gamma_k| \leq \sum_{i=1}^{k} (m_i' - m_i(A_0, \lambda_0)), \qquad k = 1, 2, \dots, \tag{6.2}$$

then in any neighborhood of A_0 there exists a G-selfadjoint operator A, having in Ω a single e.v. λ_0, and satisfying $m_i(A, \lambda_0) = m_i'$ $(i = 1, 2, \dots)$.

PROOF. Since the number of sequences $\{m_i''\}$ in F satisfying $\{m_i''\} \prec \{m_i'\}$ is finite, it suffices to prove that in every neighborhood of A_0 there exists a G-selfadjoint operator $A_1 \in L(3)$, having in Ω only one e.v. λ_0, with the sequence $\{m_i(A_1, \lambda_0)\}_1^\infty$ not coinciding with $\{m_i(A_0, \lambda_0)\}_1^\infty$ and the conditions

$$\{m_i(A_0, \lambda_0)\} \prec \{m_i(A_1, \lambda_0)\} \prec \{m_i'\}, \tag{6.3}$$

$$|\gamma_k' - \gamma_k''| \le \sum_{i=1}^{k}(m_i' - m_i(A_1, \lambda_0)), \qquad k = 1, 2, \dots, \tag{6.4}$$

being satisfied, where γ_k'' is the number of odd integers in $\{m_i(A_1, \lambda_0)\}_1^k$.

Relations (6.3) and (6.4) permit us to perform the following step in the process, which must terminate in a finite number of steps.

We pass to the proof of the theorem. Let s be the smallest index j for which $m_j(A_0, \lambda_0) < m_j'$, and let t be the smallest subsequent index j for which

$$\sum_{i=1}^{j} m_i(A_0, \lambda_0) = \sum_{i=1}^{j} m_i'.$$

The proof of the existence of the operator A_1 will be conducted separately for each of the following four alternatives concerning the values of the numbers $\{m_i(A_0, \lambda_0)\}_{i=s}^t$.

1°. Let there be no odd numbers in $\{m_i(A_0, \lambda_0)\}_{i=s}^t$. From (6.2) for $k = t$ it follows that $\gamma_t' = \gamma_t$, and so all the numbers $\{n_i'\}_s^t$ are odd. Now it follows that for $k = s, \dots, t$ the left-hand sides of (6.2) are equal to 0, and that the right-hand sides are not less than two. According to Lemma 6.4, in every neighborhood of A_0 there exists a G-selfadjoint operator A_1, having only one e.v. λ_0 in Ω, and such that only the partial multiplicities with indices s and t for the operators A_1 and A_0 differ, and

$$m_s(A_1, \lambda_0) = m_s(A_0, \lambda_0) + 2, \qquad m_t(A_1, \lambda_0) = m_t(A_0, \lambda_0) - 2. \tag{6.5}$$

In order to show this, it is necessary either to use a transformation of type (b) from Lemma 6.4, if $\varepsilon_s(A_0, G, \lambda_0) = \varepsilon_t(A_0, G, \lambda_0)$, or to apply twice a transformation of type (c) if $\varepsilon_s(A_0, G, \lambda_0) = -\varepsilon_t(A_0, G, \lambda_0)$. From (6.5) and (6.2) we get (6.4). The relations (6.3) are obvious.

2°. In the case when all the numbers $\{m_i(A_0, \lambda_0)\}_{i=s}^t$ are even, the construction of A_1 and the proof of (6.3) and (6.4) is performed analogously.

3°. Suppose now that the number $m_s(A_0, \lambda_0)$ is odd, but that there are some even numbers in $\{m_i(A_0, \lambda_0)\}_{i=s}^t$. Denote by q the largest index j such that all the numbers $\{m_i(A_0, \lambda_0)\}_{i=1}^j$ are odd. Put $p = \min\{j : m_j(A_0, \lambda_0) = m_q(A_0, \lambda_0)\}$ and $r = \max\{j : m_j(A_0, \lambda_0) = m_{q+1}(A_0, \lambda_0)\}$. Note that $r \le t$, since $m_t' < m_t(A_0, \lambda_0)$ and $m_{t+1}' \ge m_{t+1}(A_0, \lambda_0)$. In accordance with part (a) of Lemma 6.4, in any neighborhood of A_0 there exists a G-selfadjoint operator A_1 such that $m_p(A_1, \lambda_0) = m_p(A_0, \lambda_0) + 1$, $m_r(A_1, \lambda_0) = m_r(A_0, \lambda_0) - 1$, and $m_i(A_1, \lambda_0) = m_i(A_0, \lambda_0)$ for all i not equal to p or r. The relations (6.3) are obvious, and it suffices to prove (6.4) for $k = p, \dots, r-1$, since for the remaining indices k they coincide with the corresponding inequalities (6.2). Let $k \in [p, q]$. It is easy to see that when passing from the kth inequality (6.2) to the kth inequality (6.4), the expression on the right-hand side is decreased by one and the expression on the left-hand side is either decreased by one (and then (6.4) is obvious) or it is increased by one. In the latter case, as it is easily seen, $\gamma'_k = \gamma_k$, and therefore in the kth inequality (6.2) the right-hand side is no less than two, and the left-hand side equals zero. From this inequality (6.4) with the index k follows. Thus, it remains to establish (6.4) for $k \in [q+1, r-1]$ (if $q+1 \leq r-1$). For such k, consider the following alternative possibilities for the numbers $m_k(A_0, \lambda_0)$ and m'_k.

(a) If $m_k(A_0, \lambda_0) = m'_k$, the kth inequality (6.4) simply coincides with the previous one.

(b) If $m_k(A_0, \lambda_0) < m'_k$, it is not hard to verify that when passing from the $(k-1)$st to the kth inequality in (6.4), the right-hand side is increased by 1 and the left-hand side is changed by at most one. Thus, the question of validity of the kth inequality in (6.4) is reduced in this case to the question of validity of the previous inequality. Since (6.4) with index q has already been proved, it follows that all the inequalities (6.4) with the indices $k \in [q+1, r-1]$ for which $m_k(A_0, \lambda_0) < m'_k$ are satisfied.

(c) If $m_k(A_0, \lambda_0) > m'_k$, with the help of arguments analogous to those developed in the previous case we reduce the question of validity of (6.4) with index k to the question of validity of that inequality with the index $k+1$. Since (6.4) with the index r holds (it simply coincides with the rth inequality (6.2)), the inequalities (6.4) hold for all indices $k \in [q+1, r-1]$ such that $m_k(A_0, \lambda_0) > m'_k$.

4°. Finally, if $m_s(A_0, \lambda_0)$ is even in $\{m_i(A_0, \lambda_0)\}_{i=s}^{t}$ there are numbers of different parities, the proof is exactly analogous to 3°. The theorem is proved.

4. In some cases Theorems 2.2, 2.3, and 6.6 permit us to obtain the full description of the possible range of partial multiplicities of a perturbed G-selfadjoint operator.

THEOREM 6.7. *Let Ω be a normal domain for a G-selfadjoint operator $A_0 \in L(\mathfrak{Z})$, containing only one e.v. λ_0 of A_0, and let $\lambda_0 \in \mathbf{R}$. Suppose also that the operator A_0 satisfies the sign condition at λ_0. For the sequence $\{m'_i\} \in F$ the following conditions are equivalent*:

1°. *In every neighborhood of A_0 there exists a G-selfadjoint operator A having only one e.v. λ_0 in Ω, with $m_i(A, \lambda_0) = m'_i$ $(i = 1, 2, \dots)$.*

2°. *The equality*

$$\sum_{i=1}^{\infty} m_i(A_0, \lambda_0) = \sum_{i=1}^{\infty} m'_i \tag{6.6}$$

and the inequalities (6.2) *hold.*

PROOF. If condition 2° holds, 1° follows from Theorem 6.6 and the relation $\{m_i(A_0, \lambda_0)\} \prec \{m'_i\}$, which comes from 2°.

Let condition 1° be satisfied. By virtue of Theorem 2.5, the inequalities (6.2) coincide with (2.2), which were under this condition established in Theorem 2.2. The necessity of (6.6) is well known (the stability of the multiplicity). The theorem is proved.

5. All results of §§4–6 carry over to the case of holomorphic selfadjoint perturbations, small on $\partial\Omega$, of holomorphic selfadjoint o.f. We clarify this briefly. Let Ω be a normal domain for a holomorphic selfadjoint o.f. $W_0(\lambda)$ with values in $L(\mathfrak{G})$, and let $\theta_0 = (A_0, B_0, C_0, \mathfrak{Z}, \mathfrak{G})$ be a spectral node of $W_0(\lambda)$ in Ω. Let G be the selfadjoint operator realizing the similarity between the spectral nodes θ_0 and θ_0^* of $W_0(\lambda)$ in Ω. As we have already noticed, the normality of Ω implies that the space $\mathfrak{Z}$ is finite-dimensional, and that the operator A_0 is G-selfadjoint. From Theorem 4.3 of [16] it follows that there exist positive numbers δ and K with the following property: if an operator $A \in L(\mathfrak{Z})$ satisfies the condition $\|A - A_0\| < \delta$, then the quintet $\theta = (A, B_0, C_0, \mathfrak{Z}, \mathfrak{G})$ is a spectral node of a holomorphic o.f. $W(\lambda)$ in Ω, and

$$\|W(\lambda) - W_0(\lambda)\| \le K\|A - A_0\|, \qquad \lambda \in \partial\Omega.$$

From the proof of Theorem 4.3 in [16] it is obvious that the o.f. $W(\lambda)$ can be assumed to be selfadjoint if the operator A is G-selfadjoint.

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