

# Tensor properties of multilevel Toeplitz and related matrices

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## Abstract

A general proposal is presented for fast algorithms for multilevel structured matrices. It is based on investigation of their tensor properties and develops the idea recently introduced by J. Kamm and J. G. Nagy in the block Toeplitz case. We show that tensor properties of multilevel Toeplitz matrices are related to separation of variables in the corresponding symbol, present analytical tools to study the latter, expose truncation algorithms preserving the structure, and report on some numerical results confirming advantages of the proposal.

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## 1 Introduction

Despite a remarkable progress in fast algorithms for structured matrices in the last decades, many challenging gaps remain, especially concerning multilevel structured matrices.

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Multilevel matrices frequently arise in multidimensional applications, where sizes of matrices may be very large and fast algorithms become crucial. However, most of the well-known fast algorithms for structured matrices are designed for one-level structured matrices, where request for large sizes is certainly weaker.

Unfortunately, the one-level algorithms are not easy to adapt to the multilevel case. This applies, for example, to the multilevel Toeplitz matrices: fast algorithms are well developed for the Toeplitz matrices but very thin on the ground for the two-level (multilevel) Toeplitz matrices. This is likely to reflect the fact that the fabulous Gohberg-Sementsul and related formulas for the inverse matrices are obtained only in the one-level case.

We believe that structure in the inverse matrices in the multilevel case may appear through approximation by appropriately chosen matrices of “simpler” structure. In this regard, tensor-product constructions can be attractive because of the very simple inversion formula

$$(A^1 \otimes \cdots \otimes A^p)^{-1} = (A^1)^{-1} \otimes \cdots \otimes (A^p)^{-1}.$$

The main purpose of this paper is investigation of interrelations between the multilevel structured matrices and tensor-product constructions with accent on the two-level matrices.

In Section 2 we recollect the framework for study of structures in multilevel matrices.

In Section 3 we study approximations of two-level matrices by sums of tensor products with the same structures of the factors.

In Section 4 we show that the existence problem of tensor-product approximations for multilevel Toeplitz matrices reduces to approximate separation of variables in the corresponding generating function (symbol).

In Section 5 we present useful analytical tools to study the latter separation of variables.

In Section 6 we present truncation algorithms for approximation of the inverse matrices, making a step towards better understanding of structure in the inverses to multilevel matrices.

In Section 7 we demonstrate some numerical results.

## 2 Structures in multilevel matrices

A general notion of multilevel matrix was introduced in [16]. Let  $A$  be a matrix of size  $M \times N$  with

$$M = \prod_{k=1}^p m_k, \quad N = \prod_{k=1}^p n_k.$$

Then, set

$$\mathbf{m} = (m_1, \dots, m_p), \quad \mathbf{n} = (n_1, \dots, n_p)$$

and introduce the index bijections

$$i \leftrightarrow \mathbf{i}(\mathbf{m}) = (i_1(\mathbf{m}), \dots, i_p(\mathbf{m})), \quad j \leftrightarrow \mathbf{j}(\mathbf{n}) = (j_1(\mathbf{n}), \dots, j_p(\mathbf{n}))$$

by the following rules:

$$i = \sum_{k=1}^p i_k \prod_{l=1}^{k-1} m_l, \quad j = \sum_{k=1}^p j_k \prod_{l=1}^{k-1} n_l,$$

$$0 \leq i \leq M - 1, \quad 0 \leq i_k \leq m_k - 1, \quad k = 1, \dots, p,$$

$$0 \leq j \leq N - 1, \quad 0 \leq j_k \leq n_k - 1, \quad k = 1, \dots, p.$$

Any entry  $a_{ij}$  of  $A$  can be pointed to by the index pair  $(\mathbf{i}(\mathbf{m}), \mathbf{j}(\mathbf{n}))$  revealing a certain hierarchical block structure in  $A$ . We will say that  $A$  is a  $p$ -level matrix and write

$$a_{ij} = a(\mathbf{i}, \mathbf{j}) \quad \text{or} \quad a_{ij} = a_{\mathbf{i}\mathbf{j}},$$

freely replacing  $i$  by  $\mathbf{i}$  and  $j$  by  $\mathbf{j}$ . Sometimes it might be also convenient to write

$$a_{ij} = a[i_1, j_1; \dots; i_p, j_p].$$

Introduce the truncated indices

$$\mathbf{i}_k = (i_1, \dots, i_k), \quad \mathbf{j}_k = (j_1, \dots, j_k).$$

Then  $a(\mathbf{i}_k, \mathbf{j}_k)$  will denote a block of level  $k$ . We will call  $\mathbf{m}$  and  $\mathbf{n}$  the *size-vectors* of  $A$ .

By definition,  $A$  itself is a single block of level 0. It consists of  $m_1 \times n_1$  blocks  $a(\mathbf{i}_1, \mathbf{j}_1)$ , these blocks being said to belong to the 1st level of  $A$ . At the same time,  $A$  consists of  $(m_1 m_2) \times (n_1 n_2)$  blocks  $a(\mathbf{i}_2, \mathbf{j}_2)$  of the 2nd level of  $A$ , and so on. It is important to note that each block of level  $k < p$  consists of  $m_{k+1} \times n_{k+1}$  blocks of level  $k + 1$ . Further on we chiefly assume that  $M = N$  and  $\mathbf{m} = \mathbf{n}$ .

Multilevel block partitionings are of interest only if the blocks of the levels exhibit some structure. For example,  $A$  is a  $p$ -level Toeplitz matrix if every block of level  $0 \leq k < p$  is a block Toeplitz matrix with the blocks of the next level. An equivalent

definition is to say that  $a(\mathbf{i}, \mathbf{j})$  depends actually only on  $\mathbf{i} - \mathbf{j}$ . Thus, in the case of a  $p$ -level Toeplitz matrix we may write

$$A = [a(\mathbf{i} - \mathbf{j})].$$

A  $p$ -level matrix  $C$  is called a  $p$ -level circulant if every block of level  $0 \leq k < p$  is a block circulant matrix with the blocks of level  $k + 1$ . Equivalently,  $a(\mathbf{i}, \mathbf{j})$  depends only on

$$(\mathbf{i} - \mathbf{j})(\bmod \mathbf{n}) \equiv ((i_1 - j_1)(\bmod n_1), \dots, (i_p - j_p)(\bmod n_p)),$$

and one may write

$$C = [c((\mathbf{i} - \mathbf{j})(\bmod \mathbf{n}))].$$

Below we illustrate the structure of  $A$  and  $C$  in the case  $p = 2$  and  $\mathbf{n} = (3, 2)$ :

$$A = \left[ \begin{array}{cc|cc|cc} a(0,0) & a(0,-1) & a(-1,0) & a(-1,-1) & a(-2,0) & a(-2,-1) \\ a(0,1) & a(0,0) & a(-1,1) & a(-1,0) & a(-2,1) & a(-2,0) \\ \hline a(1,0) & a(1,-1) & a(0,0) & a(0,-1) & a(-1,0) & a(-1,-1) \\ a(1,1) & a(1,0) & a(0,1) & a(0,0) & a(-1,1) & a(-1,0) \\ \hline a(2,0) & a(2,-1) & a(1,0) & a(1,-1) & a(0,0) & a(0,-1) \\ a(2,1) & a(2,0) & a(1,1) & a(1,0) & a(0,1) & a(0,0) \end{array} \right],$$

$$C = \left[ \begin{array}{cc|cc|cc} c(0,0) & c(0,1) & c(2,0) & c(2,1) & c(1,0) & c(1,1) \\ c(0,1) & c(0,0) & c(2,1) & c(2,0) & c(1,1) & c(1,0) \\ \hline c(1,0) & c(1,1) & c(0,0) & c(0,1) & c(2,0) & c(2,1) \\ c(1,1) & c(1,0) & c(0,1) & c(0,0) & c(2,1) & c(2,0) \\ \hline c(2,0) & c(2,1) & c(1,0) & c(1,1) & c(0,0) & c(0,1) \\ c(2,1) & c(2,0) & c(1,1) & c(1,0) & c(0,1) & c(0,0) \end{array} \right].$$

A general description of structure in multilevel matrices can be introduced in the following way [17]. Denote by  $\mathbf{S}_\alpha$  a sequence of linear subspaces  $\mathbf{S}_\alpha^1, \mathbf{S}_\alpha^2, \dots$  with  $\mathbf{S}_\alpha^n$  being a subspace in the space of all  $n \times n$  matrices. Obviously,  $\mathbf{S}_\alpha^n$  can be considered as a class of structured matrices of order  $n$ , and, if structures for individual  $n$  are worthy to consider as “traces” of a common structure, then  $\mathbf{S}_\alpha$  is a reference to this common structure. Let us write  $A \in \mathbf{S}_\alpha$  if there exists  $n$  such that  $A \in \mathbf{S}_\alpha^n$ . Sometimes  $\mathbf{S}_\alpha$  will be referred to as *structured classes*.

It is easy to see that Toeplitz or circulant matrices can be described exactly in this way. Moreover, diagonal, three-diagonal, banded matrices as well as matrices with a prescribed pattern of sparsity are all examples of the same description style.

Denote by  $\mathbf{S}_\alpha^n \circ \mathbf{S}_\beta^m$  a subspace in the space of all two-level matrices with size-vector  $(\mathbf{n}, \mathbf{m})$  and define it by the claim that

$$A = [a_{(i_1, i_2)(j_1, j_2)}] \in \mathbf{S}_\alpha^n \circ \mathbf{S}_\beta^m$$

if and only if

$$A_{i_2 j_2}^2 \equiv [a_{(i_1, i_2)(j_1, j_2)}]_{i_1 j_1=0}^{n-1} \in \mathbf{S}_\alpha^n \quad \forall 0 \leq i_2, j_2 \leq m-1,$$

and

$$A_{i_1 j_1}^1 \equiv [a_{(i_1, i_2)(j_1, j_2)}]_{i_2 j_2=0}^{m-1} \in \mathbf{S}_\beta^m \quad \forall 0 \leq i_1, j_1 \leq n-1.$$

By  $\mathbf{S}_\alpha \circ \mathbf{S}_\beta$  we mean a sequence  $\mathbf{S}_\alpha^n \circ \mathbf{S}_\beta^m$  with the two indices  $n, m = 1, 2, \dots$ . We call  $\mathbf{S}_\alpha \circ \mathbf{S}_\beta$  the *level product* of  $\mathbf{S}_\alpha$  and  $\mathbf{S}_\beta$ .

Let  $\mathbf{G} = \mathbf{S}_0$  denote a special *no-structure* sequence with  $\mathbf{G}^n$  being full spaces of  $n \times n$  matrices. Then, obviously,

$$\mathbf{S}_\alpha \circ \mathbf{G} = \mathbf{G} \circ \mathbf{S}_\alpha = \mathbf{S}_\alpha.$$

An important and natural generalization of the above-considered construction comes with the assumption that  $\mathbf{S}_\alpha$  is a sequence of subspaces  $\mathbf{S}_\alpha^n$  of multilevel matrices with size-vector  $\mathbf{n}$ . Then,  $\mathbf{S}_\alpha \circ \mathbf{S}_\beta$  means a sequence of subspaces  $\mathbf{S}_\alpha^n \circ \mathbf{S}_\beta^m$  of multilevel matrices with size-vector  $(\mathbf{n}, \mathbf{m})$ . The definition for  $\mathbf{S}_\alpha^n \circ \mathbf{S}_\beta^m$  mimics the above definition with minor changes in the following way:

$$A = [a_{\mathbf{i}_1 \mathbf{i}_2}(\mathbf{j}_1, \mathbf{j}_2)] \in \mathbf{S}_\alpha^n \circ \mathbf{S}_\beta^m$$

if and only if

$$A_{\mathbf{i}_2 \mathbf{j}_2}^2 \equiv [a_{(\mathbf{i}_1, \mathbf{i}_2)(\mathbf{j}_1, \mathbf{j}_2)}]_{\mathbf{i}_1, \mathbf{j}_1 \in \mathcal{I}_\mathbf{n}} \in \mathbf{S}_\alpha^n \quad \forall \mathbf{i}_2, \mathbf{j}_2 \in \mathcal{I}_\mathbf{m},$$

and

$$A_{\mathbf{i}_1 \mathbf{j}_1}^1 \equiv [a_{(\mathbf{i}_1, \mathbf{i}_2)(\mathbf{j}_1, \mathbf{j}_2)}]_{\mathbf{i}_2, \mathbf{j}_2 \in \mathcal{I}_\mathbf{m}} \in \mathbf{S}_\beta^m \quad \forall \mathbf{i}_1, \mathbf{j}_1 \in \mathcal{I}_\mathbf{n},$$

where  $\mathcal{I}_\mathbf{n}$  reads

$$\mathcal{I}_\mathbf{n} = \{(i_1, \dots, i_p) : 0 \leq i_k \leq n_k - 1, k = 1, \dots, p\}$$

provided that  $\mathbf{n} = (n_1, \dots, n_p)$  with  $p = p(\mathbf{n})$ .

Thus, having defined some classes of structured matrices  $\mathbf{S}_{\alpha_1}, \dots, \mathbf{S}_{\alpha_p}$  we can easily introduce a new class

$$\mathbf{S}_\gamma = \mathbf{S}_{\alpha_1} \circ \dots \circ \mathbf{S}_{\alpha_p}$$

of multilevel structured matrices. The number of levels for  $\mathbf{S}_\gamma$  is the sum of the numbers of levels for the classes involved. In line with these definitions, if  $\mathbf{T}$  stands

for the Toeplitz matrices then  $\mathbf{T} \circ \mathbf{T}$  means two-level Toeplitz matrices and, in the general case,

$$\mathbf{T}^p = \mathbf{T} \circ \dots \circ \mathbf{T} \quad (\mathbf{T} \text{ is repeated } p \text{ times})$$

means  $p$ -level Toeplitz matrices. Similarly, if  $\mathbf{C}$  stands for circulants then  $\mathbf{C}^p$  denotes  $p$ -level circulant matrices.

Also, we can easily describe a mixture of Toeplitz and circulant structures on different levels: for example,  $\mathbf{T} \circ \mathbf{C}$  identifies block Toeplitz matrices with circulant blocks while  $\mathbf{C} \circ \mathbf{T}$  designates block circulant matrices with Toeplitz blocks.

Another approach to construction of multilevel structured matrices exploits the notion of Kronecker (tensor) product. Consider matrices

$$A^k = [a_{i_k j_k}^k], \quad 0 \leq i_k, j_k \leq n_k - 1, \quad k = 1, \dots, p,$$

and define  $A = [a_{\mathbf{ij}}]$  as a  $p$ -level matrix of size-vector  $\mathbf{n} = (n_1, \dots, n_p)$  with the entries

$$a_{\mathbf{ij}} = a_{i_1 j_1}^1 a_{i_2 j_2}^2 \dots a_{i_p j_p}^p, \quad \mathbf{i} = (i_1, \dots, i_p), \mathbf{j} = (j_1, \dots, j_p).$$

This matrix  $A$  is called the Kronecker (tensor) product of matrices  $A^1, \dots, A^p$  and denoted by

$$A = A^1 \otimes \dots \otimes A^p.$$

The level-product and tensor-product approaches to building up multilevel matrices are naturally related by the following elementary

**Proposition.** *If  $A^k \in \mathbf{S}_{\alpha_k}$ ,  $k = 1, \dots, p$ , then*

$$A^1 \otimes \dots \otimes A^p \in \mathbf{S}_{\alpha_1} \circ \dots \circ \mathbf{S}_{\alpha_p}.$$

### 3 Optimal Kronecker approximations

Suppose that  $A$  is a two-level matrix of size-vector  $\mathbf{n} = (n_1, n_2)$  and try to approximate it by a sum of Kronecker products of the form

$$A_r = \sum_{k=1}^r A_k^1 \otimes A_k^2,$$

where the sizes of  $A_k^1$  and  $A_k^2$  are  $n_1 \times n_1$  and  $n_2 \times n_2$ , respectively. If  $A = A_r$  and  $r$  is the least possible number of the Kronecker-product terms whose sum is  $A$  then  $r$  is called the tensor rank of  $A$ .

Optimal approximations minimizing  $\|A - A_r\|_F$  can be obtained via the SVD algorithm due to the following observation [18]. Denote by

$$\mathcal{V}_n(A) = [b_{(i_1, j_1)(i_2, j_2)}]$$

a two-level matrix with size-vectors  $(n_1, n_1)$  and  $(n_2, n_2)$  defined by the rule

$$b_{(i_1, j_1)(i_2, j_2)} = a_{(i_1, i_2)(j_1, j_2)}.$$

Then, as is readily seen, the tensor rank of  $A$  is equal to the rank of  $\mathcal{V}_n(A)$ . Moreover,

$$\|A - A_r\|_F = \|\mathcal{V}_n(A) - \mathcal{V}_n(A_r)\|_F,$$

which reduces the problem of optimal tensor approximation to the problem of optimal lower-rank approximation.

In practice we are interested only in the cases when  $r \ll n_1, n_2$ , so low-rank approximations being exactly what we need to find for  $\mathcal{V}_n(A)$  and then convert to low-tensor-rank approximations for  $A$  via  $\mathcal{V}_n^{-1}$ . Computational vehicles can be the SVD or Lanzos bidiagonalization algorithm. The latter should be preferred if  $\mathcal{V}_n(A)$  admits a fast matrix-by-vector multiplication procedure. However, a drawback of both vehicles in this direct approach is that the Kronecker factors may lose any structure, or, in other words,  $A_r$  may show up “lesser structure” than  $A$ .

We propose an alternative approach that allows us to preserve structure in the Kronecker factors. To introduce it, recall and adapt the proposal of [10] in the case  $A \in \mathbf{T} \circ \mathbf{T}$ . In this case  $\mathcal{V}_n(A) = [a_{(i_1 - j_1)(i_2 - j_2)}]$  has coinciding elements whenever  $i_1 - j_1 = \mu$  and  $i_2 - j_2 = \nu$ ,  $1 - n_1 \leq \mu \leq n_1 - 1$ ,  $1 - n_2 \leq \nu \leq n_2 - 1$ . It suggests to consider only independent free-parameter elements and take up a smaller matrix

$$W(A) = [a_{\mu\nu}], \quad 1 - n_1 \leq \mu \leq n_1 - 1, \quad 1 - n_2 \leq \nu \leq n_2 - 1. \quad (1)$$

Let us find a low-rank approximation

$$W(A) \approx W(A_r) \equiv \sum_{k=1}^r u^k (v^k)^\top,$$

$$u^k = [u_\mu^k], \quad 1 - n_1 \leq \mu \leq n_1 - 1, \quad v^k = [v_\nu^k], \quad 1 - n_2 \leq \nu \leq n_2 - 1,$$

then set

$$U^k = [u_{i_1 - j_1}^k], \quad 0 \leq i_1, j_1 \leq n_1 - 1,$$

$$V^k = [v_{i_2 - j_2}^k], \quad 0 \leq i_2, j_2 \leq n_2 - 1,$$

and consider the tensor approximation

$$A \approx A_r = \sum_{k=1}^r U^k \otimes V^k. \quad (2)$$

This approximation remains optimal in the subspace of interest and in appropriately chosen norm. If  $A \in \mathbf{T} \circ \mathbf{T}$  then set

$$\|A\|_{\mathbf{T} \circ \mathbf{T}} \equiv \|W(A)\|_F,$$

where  $W(A)$  is defined by (1). It is easy to see that  $A_r$  in (2) belongs to  $\mathbf{T} \circ \mathbf{T}$  and

$$\|A - A_r\|_{\mathbf{T} \circ \mathbf{T}} = \|W(A) - W(A_r)\|_F.$$

We can develop the above into quite a general construction. Let  $A \in \mathbf{S}_1 \circ \mathbf{S}_2$  for some structure classes  $\mathbf{S}_1$  and  $\mathbf{S}_2$ . If  $A$  is of size-vector  $\mathbf{n} = (n_1, n_2)$  then  $A \in \mathbf{S}_1^{n_1} \circ \mathbf{S}_2^{n_2}$ . According to the definition (see Section 1),  $\mathbf{S}^n$  is a linear subspace in the space of all matrices of order  $n$ . Hence, any matrix  $P \in \mathbf{S}^n$  can be uniquely defined by some its entries which can be referred to as *free-parameter entries*. The number of free-parameter entries is equal to  $\dim \mathbf{S}^n$ . The free-parameter entries can be chosen, as a rule, by many ways, but they occupy the same positions for all matrices  $P \in \mathbf{S}^n$ . Let us fix some stencil for the free-parameter entries and denote by  $W(P)$  a vector-column of the free-parameter entries of  $P$ . By the construction,  $P \leftrightarrow W(P)$  is a bijection, and we may write  $P = W^{-1}(W(P))$ .

Now, let  $W_1$  and  $W_2$  denote the free-parameter bijections for  $\mathbf{S}_1$  and  $\mathbf{S}_2$ , respectively. The stencils for  $W_1$  and  $W_2$  naturally generate the stencil of free-parameters for  $A \in \mathbf{S}_1^{n_1} \circ \mathbf{S}_2^{n_2}$ . For convenience of further use, we define  $W(A)$  as a matrix of size  $(\dim \mathbf{S}_1^{n_1}) \times (\dim \mathbf{S}_2^{n_2})$  as follows:

$$W(A) = [w_{\mu\nu}], \quad w_{\mu\nu} = a_{i_1, j_1}(i_2, j_2), \quad (3)$$

where  $\mu$  runs over the free-parameter positions  $(i_1, j_1)$  of  $\mathbf{S}_1$  while  $\nu$  runs over the free-parameter positions  $(i_2, j_2)$  of  $\mathbf{S}_2$ . Let us write  $W = W_1 \circ W_2$ .

Let us define the norm

$$\|A\|_{\mathbf{S}} \equiv \|W(A)\|_F, \quad A \in \mathbf{S}_1^{n_1} \circ \mathbf{S}_2^{n_2}. \quad (4)$$

**Theorem 3.1** *Let  $A \in \mathbf{S}_1^{n_1} \circ \mathbf{S}_2^{n_2}$ ,  $W_1$  and  $W_2$  be the free-parameter bijections for  $\mathbf{S}_1^{n_1}$  and  $\mathbf{S}_2^{n_2}$ , and  $W = W_1 \circ W_2$  be defined by (3). Assume that  $W(A)$  admits a dyadic (skeleton) approximation*

$$D_r = \sum_{k=1}^r u^k (v^k)^\top, \quad u^k = [u_\mu^k], \quad v^k = [v_\nu^k], \quad (5)$$

where  $\mu$  and  $\nu$  run over the free-parameter stencils for  $W_1$  and  $W_2$ , respectively. Then  $A$  admits a tensor approximation

$$A_r = \sum_{k=1}^r U^k \otimes V^k, \quad U_k = W_1^{-1}(u^k), \quad V_k = W_2^{-1}(v^k). \quad (6)$$



Then  $D_r = W(A_r)$  and

$$\|A - A_r\|_S = \|W(A) - W(A_r)\|_F. \quad (7)$$

The proof is actually given by the preceeding arguments.

An important consequence of Theorem 3.1 is the optimality of approximation of tensor rank  $r$  in the  $S$ -norm, which follows from the well-known optimality properties of low-rank approximations in the Frobenius norm.

Theorem 3.1 obviously generalizes the corresponding result for the  $\mathbf{T} \circ \mathbf{T}$  (doubly Toeplitz) matrices [10]. It is easy to apply this theorem to many classes of structured matrices. For illustration, consider block Toeplitz matrices with tridiagonal blocks. If  $\mathbf{B}$  denotes the tridiagonal matrices, then this class can be designated by  $\mathbf{T} \circ \mathbf{B}$ . Let  $A \in \mathbf{T}^{n_1} \circ \mathbf{B}^{n_2}$ . In this case  $W(A)$  is obviously a matrix of size  $(2n_1 - 1) \times (3n_2 - 2)$ . Thus, application of the SVD (or, even better, the Lanczos bidiagonalization) is feasible, and Theorem 3.1 states that optimal tensor approximations exist with the first Kronecker factors being Toeplitz and second ones being tridiagonal.

Our arguments can be extended over to the matrices with the number of levels greater than two. However, in this case such a powerful tool as the theory and algorithms for the singular value decomposition is not available (cf [2,4,9]). Thus, in practice we are interesting to exploit the case of two levels as far as possible (cf [5,8,14]). Moreover, accurate low-rank approximation can be often obtained from picking up only a relatively small number of entries, which leads to very efficient practical algorithms (cf [6,7,15]).

#### 4 Tensor properties and separability of symbols

Consider a family of multilevel Toeplitz matrices associated with the Fourier expansion of a generating function (symbol)  $F$ . In the case of  $p$  levels,  $F$  is a  $p$ -variate function

$$F(x_1, \dots, x_p) = \sum_{k_1=-\infty}^{\infty} \cdots \sum_{k_p=-\infty}^{\infty} f_{k_1, \dots, k_p} \exp(i(k_1 x_1 + \dots + k_p x_p)) \quad (8)$$

and the entries of  $A \in \mathbf{T}^p$  are given by

$$a_{(i_1, \dots, i_p)(j_1, \dots, j_p)} = f_{i_1-j_1, \dots, i_p-j_p}. \quad (9)$$

By  $\|A\|_C$  we mean the maximal in modulus entry of  $A$ , and by  $\|A\|_{(1)}$  the 1-norm of Schatten (the sum of all singular values of  $A$ ).

**Theorem 4.1** Assume that  $A \in \mathbf{T}^p$  is generated by

$$F(x_1, \dots, x_p) \in L_1(\Pi), \quad \Pi = [-\pi, \pi]^p,$$

according to (8), (9). Then, a separable approximation

$$F_r(x_1, \dots, x_p) = \sum_{k=1}^r \phi_k^1(x_1) \dots \phi_k^p(x_p)$$

of the symbol  $F$  implies that  $A$  admits a tensor approximation

$$A_r = \sum_{k=1}^r A_k^1 \otimes \dots \otimes A_k^p$$

with the entriwise error estimate

$$\|A - A_r\|_C \leq \frac{1}{(2\pi)^p} \|F - F_r\|_{L_1(\Pi)}$$

and the Schatten 1-norm estimate

$$\frac{1}{N} \|A - A_r\|_{(1)} \leq \frac{2}{(2\pi)^p} \|F - F_r\|_{L_1(\Pi)},$$

where  $N$  is the order of  $A$ .

**Proof.** It suffices to take into account the following:

$$\begin{aligned} (A_r)_{i_1-j_1, \dots, i_p-j_p} = \\ \frac{1}{(2\pi)^p} \int_{-\pi}^{\pi} \dots \int_{-\pi}^{\pi} F_r(x_1, \dots, x_p) \exp(-i(i_1-j_1)x_1 + \dots + (i_p-j_p)x_p) dx_1 \dots dx_p = \\ \frac{1}{(2\pi)^p} \sum_{k=1}^p \prod_{l=1}^p \left( \int_{-\pi}^{\pi} \phi_k^l(x_l) \exp(-i(i_l-j_l)x_l) dx_l \right). \quad \square \end{aligned}$$

## 5 Analytical tools for approximate separability

Separation of variables is a topic of permanent interest in approximation theory (cf [2]). The purpose here is to relate the number of separable terms to the corresponding approximation accuracy. Obviously, the results depend on the smoothness properties of functions under query. In applications, a closer attention is obviously paid to functions with certain types of singularities.

Let us consider bivariate symbols  $F(x_1, x_2)$  on  $\Pi = [-\pi, \pi]^2$ . Then, one can apply general results for the so-called *asymptotically smooth* functions [13,14].  $F$  is

called asymptotically smooth if it attains a finite value at any point except for  $(0, 0)$  and all its mixed derivatives satisfy the inequality

$$\left| \frac{\partial^{k_1} \partial^{k_2}}{(\partial x_1)^{k_1} (\partial x_2)^{k_2}} F(x_1, x_2) \right| \leq c d^{k_1+k_2} (k_1 + k_2)! (x_1^2 + x_2^2)^{(g-k_1-k_2)/2},$$

$$(x_1, x_2) \neq (0, 0),$$

for all sufficiently large nonnegative  $k_1, k_2$  with constants  $c, d > 0$  and a real-valued constant  $g$  independent of  $k_1$  and  $k_2$ . In our case it is sufficient to consider  $F$  only for  $(x, x_2) \in \Pi$ .

**Theorem 5.1** [14] *Assume that  $F$  is asymptotically smooth and arbitrary values  $0 < h, q < 1$  be chosen. Then for any  $m = 1, 2, \dots$  there exists a separable function  $F_r(x_1, x_2)$  with  $r$  terms such that*

$$r \leq (c_0 + c_1 \log h^{-1})m,$$

$$|F(x_1, x_2) - F_r(x_1, x_2)| \leq c_2 q^m (x_1^2 + x_2^2)^{g/2}, \quad (x_1, x_2) \notin [-h, h]^2,$$

where  $c_0, c_1, c_2$  are constants depending on  $q$  but not on  $m$ .

A direct corollary of this is the following

**Theorem 5.2** *Let  $A^n$  be a two-level Toeplitz matrix of size-vector  $\mathbf{n}$ , generated by asymptotically smooth symbol  $F$  such that*

$$\int_{-h}^h \int_{-h}^h |F(x_1, x_2)| dx_1 dx_2 = O(h^\tau), \quad \tau > 0,$$

and assume additionally that  $g > -4$ . Then, for any  $\varepsilon > 0$  there exists a tensor approximation  $A_r^n \in \mathbf{T} \circ \mathbf{T}$  with  $r$  terms such that

$$r \leq C_1 \log^2 \varepsilon^{-1},$$

$$\|A^n - A_r^n\|_C \leq C_2 \varepsilon,$$

where  $C_1$  and  $C_2$  do not depend on  $\mathbf{n}$ .

**Proof.** Given  $\varepsilon > 0$ , choose  $h$  so that  $h^\tau \sim \varepsilon$  and chose  $m$  so that  $q^m \sim \varepsilon$ . Also, take into account that the function  $(x_1^2 + x_2^2)^{g/2}$  will be  $L_1$ -integrable on  $[-\pi, \pi]^2$  for  $g > -4$ . It remains to have recourse to Theorems 5.1 and 4.1.  $\square$

Applications also give rise to functions like, for instance,

$$F(x_1, x_2) = \frac{\sin(7(x_1^2 + x_2^2)^{1/2})}{(x_1^2 + x_2^2)^{1/2}} \quad (10)$$

that are not asymptotically smooth. Note, by the way, that the derivatives of this  $F$  are not bounded. Acquisition of separable approximations in such cases requires

some special tools. An excellent vehicle for many practical cases can be developed on the base of E. T. Whittaker's cardinal function ("a function of royal blood", by his words) and Sinc-functions [11]. This vehicle works good also for many asymptotically smooth functions.

Consider the case

$$F(x_1, x_2) = \mathcal{F}(\xi, \theta), \quad \xi = \left(\frac{x_1}{\pi}\right)^2, \quad \theta = \left(\frac{x_2}{\pi}\right)^2,$$

then  $0 \leq \xi \leq 1$  and  $0 \leq \theta \leq 1$ . The goal is approximate separation of variables  $\xi$  and  $\theta$ . The approach of [11] capitalizes on outstanding properties of functions of complex variable  $z$  analytic in a strip  $|\operatorname{Im} z| \leq d$ . Thus, the enterprise must begin with finding a way to make the initial problem fit into that framework. A useful possibility is the change of variable

$$\xi = \frac{1}{\cosh u}, \quad \cosh u = \frac{\exp(u) + \exp(-u)}{2}, \quad -\infty \leq u \leq +\infty,$$

with coming back to  $\xi$  in the end using the formula

$$u = \log(\xi^{-1}(1 + \sqrt{1 - \xi^2}))$$

or, alternatively,

$$u = \log(\xi^{-1}(1 - \sqrt{1 - \xi^2})).$$

In order to separate  $u$  and  $\theta$  we make use of the assumption that

$$g(u, \theta) \equiv \mathcal{F}\left(\frac{1}{\cosh u}, \theta\right)$$

can be considered as the trace of a function

$$g(z, \theta) \equiv \mathcal{F}\left(\frac{1}{\cosh z}, \theta\right)$$

that is analytic with respect to  $z$  in the strip  $|\operatorname{Im} z| \leq d$ . Moreover, the constructions of [11] require that  $g(z, \theta)$  enjoys as well the following properties:

$$\mathcal{J}(g, d, \theta) \equiv \int_{-\infty}^{\infty} (|g(u + id, \theta)| + |g(u - id, \theta)|) du < +\infty, \quad (11)$$

$$\lim_{u \rightarrow \infty} \int_{-d}^d (|g(u + iv, \theta)| + |g(-u + iv, \theta)|) dv = 0, \quad (12)$$

$$|g(u, \theta)| \leq c \exp(-p|u|), \quad c, p > 0. \quad (13)$$

Then  $g(u, \theta)$  can be approximated by

$$g_n(u, \theta) \equiv \sum_{k=-n}^n g(kh, \theta) S_{kh}(u), \quad (14)$$

where

$$S_{kh}(u) = \frac{\sin\left(\frac{\pi}{h}(u - kh)\right)}{\left(\frac{\pi}{h}(u - kh)\right)}, \quad (15)$$

and  $h$  can be chosen so that

$$|g(u, \theta) - g_n(u, \theta)| \leq P \exp(-Q\sqrt{n}), \quad P, Q > 0. \quad (16)$$

One can see that the interpolation formula (14) makes the wanted job of separation of  $u$  and  $\theta$ . All the same, one should be careful with the above construction because  $P$  and  $Q$  in the error estimate (16) *may depend* on  $\theta$ . Moreover, even  $d$  might appear to depend on  $\theta$ . Note also that the properties (11), (12), (13) are not taken for granted, they must be verified and are likely not to hold initially but appear only after some suitable transformation of the problem. Nevertheless, the approach can be adapted to successfully treat, for example, the function (10).

Let us give more details pertinent to functions similar to (10). Instead of (10), however, consider a more general function of the form

$$\Phi(\xi, \theta) = \frac{\exp(i\kappa(\xi + \theta)^\nu)}{(\xi + \theta)^\nu}, \quad 0 < \nu < 2, \quad \kappa \geq 0.$$

Take some  $\mu > 0$  and set up

$$\mathcal{F}(\xi, \theta) = \xi^\mu \Phi(\xi, \theta).$$

Then, consider

$$g(z, \theta) = \frac{1}{(\cosh z)^\mu} \Phi\left(\frac{1}{\cosh z}, \theta\right).$$

First of all, note that  $g(z, \theta)$  is analytic at any  $z$  such that

$$\cosh z \neq 0, \quad \frac{1}{\cosh z} + \theta \neq 0.$$

It is not difficult to see that  $g(z, \theta)$  is analytic in any strip  $|\operatorname{Im} z| \leq d < \pi/2$ . Verification of (11) results in the observation that

$$\mathcal{J}(g, d, \theta) = O\left(\frac{1}{\theta^\nu}\right).$$

Condition (12) is evidently fulfilled. Concerning (13), we find that it holds true with

$$c = O\left(\frac{1}{\theta^\nu}\right), \quad p = -\mu.$$

Consequently, the estimate (16) is valid. Some further details of theory in [11] can lead to the assertion that

$$P = O\left(\frac{1}{\theta^\nu}\right)$$

while  $Q$  is greater than  $\sqrt{\mu}$ .

## 6 Truncation algorithms for the inverse matrices

## 7 Numerical results

### References

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