A QUASISEPARABLE APPROACH TO FIVE–DIAGONAL CMV AND COMPANION MATRICES

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Abstract. Recent work in the characterization of structured matrices in terms of the systems of polynomials (and specifically the recurrence relations they satisfy) related as characteristic polynomials of principal submatrices is furthered in this paper. It has been shown previously that quasiseparable structure (essentially low–rank blocks up to but excluding the main diagonal) is particularly useful in providing such recurrence relations characterizations. Some classical classes of matrices with quasiseparable structure include tridiagonal (related to real orthogonal polynomials) and banded matrices, unitary Hessenberg matrices (related to Szegö polynomials), and semiseparable matrices, as well as others. Hence working with the class of quasiseparable matrices provides new results which generalize and unify known results.

Previous work has focused on characterizing $(H,1)$–quasiseparable matrices, matrices with order–one quasiseparable structure that are also upper Hessenberg. This restriction permits a useful bijection between the sets of matrices and systems of polynomials, and thus the results derived are complete characterizations. In this paper, the authors introduce the concept of a twist transformation, and use them to explain the relationship between $(H,1)$–quasiseparable matrices and the subclass of $(1,1)$–quasiseparable matrices (without the upper Hessenberg restriction) which are related to the same systems of polynomials. These results explain the discoveries of Cantero, Fiedler, Kimura, Moral and Velázquez of five–diagonal matrices related to Horner and Szegö polynomials in the context of quasiseparable matrices.

Key words. quasiseparable matrices, semiseparable matrices, CMV matrices, Kimura, unitary Hessenberg matrices, Fiedler matrices, banded matrices, five–diagonal matrices, companion matrices, well–free matrices, orthogonal polynomials, Szegö polynomials, twist transformation.

AMS subject classifications.

1. Introduction.

Various polynomial systems $\{r_k(x)\}_{k=0}^n$ are often associated with Hessenberg matrices $H=[m_{ij}]_{ij=1}^n$ as scaled characteristic polynomials of principal submatrices of H; that is, via the relation

$$r_0(x)=\lambda_0, \quad r_k(x)=\lambda_0\lambda_1\dots\lambda_k\det(xI-H_{k\times k}), \quad k=1,\dots,n. \tag{1.1}$$

Moreover, the relation (1.1) establishes a bijection [5] if $\lambda_k=\frac{1}{m_{k,k}}$ and λ_0,λ_n are two parameters, so

$$\{r_k(x)\}_{k=0}^n \longleftrightarrow \{H,\lambda_0,\lambda_n\}. \tag{1.2}$$

1.1. From Hessenberg to five–diagonal matrices. Two examples.

It is widely known that Szegö polynomials $\{\phi_k^{\#}(x)\}_{k=0}^n$ orthogonal on the unit circle are connected via (1.1) with a certain (almost[1]) unitary Hessenberg matrix

$$M=\begin{bmatrix} -\rho_0^*\rho_1 & -\rho_0^*\mu_1\rho_2 & -\rho_0^*\mu_1\mu_2\rho_3 & \cdots & -\rho_0^*\mu_1\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ \mu_1 & -\rho_1^*\rho_2 & -\rho_1^*\mu_2\rho_3 & \cdots & -\rho_1^*\mu_2\mu_3\cdots\mu_{n-1}\rho_n \\ 0 & \mu_2 & -\rho_2^*\rho_3 & \cdots & -\rho_2^*\mu_3\cdots\mu_{n-1}\rho_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \mu_{n-1} & -\rho_{n-1}^*\rho_n \end{bmatrix}, \tag{1.3}$$

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[1] Throughout the paper, matrices referred to as unitary Hessenberg are almost unitary, differing from unitary in the last column. Specifically, $M=UD$ for a unitary matrix U and diagonal matrix $D=\text{diag}\{1,\dots,1,\rho_n\}$.

where ρ_k are *reflection coefficients*[2] and μ_k are *complementary parameters*. The details on this relation can be found in [26, 29, 4, 39, 2, 11, 35, 33, 34]. The matrix M has a rather dense structure in comparison with the tridiagonal Jacobi matrix [1, 15, 25] for orthogonal polynomials on the real line. However, the bijection (1.2) implies that for a given system of Szegö polynomials the only Hessenberg matrix related to that system via (1.1) is M. The situation is much different if we do not restrict the matrix to the class of strictly upper Hessenberg matrices.

It was found first by Kimura [28] and independently by Cantero, Moral and Velázquez [16, 17, 18] that Szego polynomials are also related via (1.1) (with $\lambda_k = \frac{1}{\mu_k}$) to the following five–diagonal matrix:

$$\mathcal{K} = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{1.4}$$

which has been called a CMV matrix since the paper [16] triggered deep interest in the orthogonal polynomials community. It is reputed that CMV matrices are better than unitary Hessenberg matrices in studying of properties of polynomials orthogonal on the unit circle (mostly because of its banded structure).

Shortly after the discovery of CMV matrices, other non–Hessenberg matrices related to important systems of polynomials via (1.1) were discovered. Consider the well known companion matrix

$$C = \begin{bmatrix} -m_1 & -m_2 & \cdots & -m_{n-1} & -m_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{1.5}$$

The characteristic polynomials $\{p_k(x)\}_{k=0}^n$ of its leading submatrices are the so–called Horner polynomials. It was shown by Fiedler [40] that the five–diagonal matrix

$$F = \begin{bmatrix} -m_1 & -m_2 & 1 & & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & -m_3 & 0 & -m_4 & 1 & & & \\ & 1 & 0 & 0 & 0 & 0 & & \\ & & 0 & -m_5 & 0 & -m_6 & 1 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{1.6}$$

is also related to the same set of Horner polynomials.

1.2. Quasiseparable approach. Twist transformation. What do CMV matrices (1.4) and Fiedler matrices (1.6) have in common? It turns out that both of them belong to the class of $(1,1)$–quasiseparable matrices defined next.

Definition 1.1 (Rank definition of $(1,1)$–quasiseparable matrices). *A matrix A is called $(1,1)$–quasiseparable (i.e., order one quasiseparable) if*

$$\max_{1\leqslant i\leqslant n-1} \operatorname{rank} A(1:i, i+1:n) = \max_{1\leqslant i\leqslant n-1} \operatorname{rank} A(i+1:n, 1:i) = 1.$$

[2] Reflection coefficients are also known in various contexts as Schur parameters [36], Verblunsky coefficients [37].

Indeed, every submatrix $A(1:i,i+1:n)$ or $A(i+1:n,1:i)$ of CMV and Fiedler matrices consists of at most two nonzero elements in the same row or column and, therefore, $\operatorname{rank} A(1:i,i+1:n) = \operatorname{rank} A(i+1:n,1:i) = 1$ for all $i = 1,\dots n-1$. It can be also shown that unitary Hessenberg matrices (1.3) and companion matrices (1.5) are also $(1,1)$–quasiseparable, we refer to [9] for details.

Eidelman and Gohberg in [19] gave an alternative definition of $(1,1)$–quasiseparable matrices in terms of small number of parameters they are described by. Such sparse representations are often at the heart of fast algorithms involving this and similar classes of structured matrices.

Definition 1.2 (Generator definition of (1,1)-qs matrices). *A matrix A is called $(1,1)$–quasiseparable if it can be represent in the form*

$$\begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \cdots & \cdots & g_1b_2\dots b_{n-1}h_n \\ p_2q_1 & d_2 & g_2h_3 & \cdots & \cdots & g_2b_3\dots b_{n-1}h_n \\ p_3a_2q_1 & p_3q_2 & d_3 & \cdots & \cdots & g_3b_4\dots b_{n-1}h_n \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & d_{n-1} & g_{n-1}h_n \\ p_na_{n-1}\dots a_2q_1 & p_na_{n-1}\dots a_3q_2 & p_na_{n-1}\dots a_4q_3 & \cdots & p_nq_{n-1} & d_n \end{bmatrix},$$

where the parameters $\{q_k, a_k, p_k, d_k, g_k, b_k, h_k\}$, all scalars, are called generators *of A.*

One of many useful properties of $(1,1)$–quasiseparable matrices is the existence of two–term recurrence relations for polynomials related to them via (1.1).

Theorem 1.3. *[23] Let $\{r_k(x)\}_{k=0}^n$ be a system of polynomials related to a $(1,1)$–quasiseparable matrix A via (1.1). Then they satisfy two–term recurrence relations*

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_0 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \lambda_k \begin{bmatrix} a_kb_kx - c_k & -q_kg_k \\ p_kh_k & x - d_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}, \tag{1.7}$$

where $c_k = d_ka_kb_k - q_kp_kb_k - g_kh_ka_k$.

Remark 1.4. *The choice of generators of an $(1,1)$–quasiseparable matrix is not unique.*

What one can get immediately from this theorem is that the interchange of lower and upper generators as in

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k \tag{1.8}$$

for some k does not change the recurrence relations (1.7) and, hence, does not change polynomials $\{r_k(x)\}_{k=0}^n$. We propose to call an operation described by (1.8) a *twist–transformation.*

We next show that both CMV and Fiedler matrices can be obtained via twist–transformations from unitary Hessenberg and companion matrices, respectively.

Example 1.5 (Unitary Hessenberg and CMV matrices). *By comparing $(1,1)$–-quasiseparable generators of unitary Hessenberg (Table 1.1) and CMV (Table 1.2) matrices we conclude that the second is obtained from the first via twist transformations for even indices.*

Example 1.6 (Companion and Fiedler matrices). *Similarly, comparing Tables 1.3 and 1.4, one can see that the Fiedler matrix is obtained from the companion matrix via twist transformations for odd indices $k > 1$.*

Table 1.1
Generators of unitary Hessenberg matrix

d_k	a_k	b_k	q_k	g_k	p_k	h_k
$-\rho_{k-1}^*\rho_k$	0	μ_k	μ_k	$-\rho_{k-1}^*\mu_k$	1	ρ_k

Table 1.2
Generators of CMV matrix

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
odd	$-\rho_{k-1}^*\rho_k$	0	μ_k	μ_k	$-\rho_{k-1}^*\mu_k$	1	ρ_k
even	$-\rho_{k-1}^*\rho_k$	μ_k	0	$-\rho_{k-1}^*\mu_k$	μ_k	ρ_k	1

The invariance of systems of polynomials under twist transformation together with Examples 1.5 and 1.6 explains why unitary Hessenberg and CMV as well as companion and Fiedler matrices share the same systems of characteristic polynomials.

1.3. Main results and structure of the paper. As we have mentioned already, unitary Hessenberg and companion matrices are both strictly upper Hessenberg and $(1,1)$–quasiseparable. Such matrices have often been called $(H,1)$–quasiseparable (see Definition 2.5). In Section 2 of the present paper we study matrices obtained from $(H,1)$–quasiseparable matrices via twist transformation (which we call *twisted* $(H,1)$–quasiseparable matrices). We also show that every $(H,1)$–quasiseparable matrix can be transformed to a certain five–diagonal matrix (one of twisted $(H,1)$–quasiseparable matrices) via twist transformations.

The next part of the paper is devoted to the study of recurrence relations for (scaled) characteristic polynomials of principal submatrices of five–diagonal twisted $(H,1)$–quasiseparable matrices. In the recent paper [9] authors derived specific recurrence relations for various subclasses of $(H,1)$–quasiseparable matrices. Moreover, because of the bijection (1.2), they have obtained a full characterization of subclasses of $(H,1)$–quasiseparable matrices via recurrence relations satisfied by polynomials related to them via (1.1). We give a brief survey of the results of [9] in Section 3 in order to exploit them in Section 4 in connection with five–diagonal matrices.

In particular, we derive the class of five–diagonal matrices which are connected to polynomials satisfying three–term recurrence relations

$$\begin{gathered} r_0(x) = \alpha_0, \quad r_1(x) = (\alpha_1 x + \beta_1)\cdot r_0(x), \\ r_k(x) = (\alpha_k x + \beta_k)\cdot r_{k-1}(x) + (\gamma_k x + \delta_k)\cdot r_{k-2}(x), \quad \alpha_k \neq 0, \quad k = 2,\dots,n. \end{gathered} \tag{1.9}$$

It was shown by Geronimus [24] that, under the additional restriction of $\rho_k \neq 0$ for every k, the corresponding Szegö polynomials $\{\phi_k^{\#}(x)\}_{k=0}^n$ satisfy three–term recurrence relations

$$\begin{gathered} \phi_0^{\#}(x) = \frac{1}{\mu_0}, \quad \phi_1^{\#}(x) = \frac{1}{\mu_1}(x\cdot\phi_0^{\#}(x) + \rho_1\rho_0^*\cdot\phi_0^{\#}(x)), \\ \phi_k^{\#}(x) = \left[\frac{1}{\mu_k}\cdot x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}\right]\phi_{k-1}^{\#}(x) - \frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k}\cdot x\cdot\phi_{k-2}^{\#}(x), \end{gathered} \tag{1.10}$$

which are of type (1.9). One can also check that the Horner polynomials, related to the Fiedler matrix (1.6), satisfy the recurrence relations

$$p_k(x) = \left(x + \frac{m_k}{m_{k-1}}\right)p_{k-1}(x) - \frac{m_k}{m_{k-1}}x\cdot p_{k-2} \tag{1.11}$$

TABLE 1.3
Generators of companion matrix

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
$k=1$	$-m_1$	–	–	1	1	–	–
$k \neq 1$	0	0	1	1	0	1	$-m_k$

TABLE 1.4
Generators of Feidler matrix

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
$k=1$	$-m_1$	–	–	1	1	–	–
$k>1$ odd	0	1	0	0	1	$-m_k$	1
even	0	0	1	1	0	1	$-m_k$

under the restriction $m_k \neq 0$ for every k. Recurrence relations (1.11) are of type (1.9) as well. Hence, both five diagonal CMV matrices (1.4) and Fiedler matrices (1.6) belong to the new class of five–diagonal matrices related to polynomials satisfying three–term recurrence relations. This result as well as other results on polynomials related to five–diagonal matrices are presented in Section 4.

Finally, in Section 5 we derive a nested decomposition of twisted $(H,1)$–quasiseparable matrices which can also be used to obtain them from the original $(H,1)$–quasiseparable matrices.

2. Twist transformations and twisted $(H,1)$–quasiseparable matrices.

2.1. Twist transformations. A system of polynomials can be related to many distinct $(1,1)$–quasiseparable matrices (Definition (1.2)) via (1.1). For instance, a nonsymmetric $(1,1)$–quasiseparable matrix and its transpose share the same system of polynomials. In this section we show how for a given $(1,1)$–quasiseparable matrix one can obtain another $(1,1)$–quasiseparable matrices related to the same system of polynomials as the original one.

DEFINITION 2.1 (Twist transformation). *We say that an $n \times n$ $(1,1)$–quasiseparable matrix $\widetilde{A}$ having generators $\{\widetilde{p}_k, \widetilde{q}_k, \widetilde{a}_k, \widetilde{g}_k, \widetilde{h}_k, \widetilde{b}_k, \widetilde{d}_k\}$ is obtained via twist transformation from another $n \times n$ $(1,1)$–quasiseparable matrix A with generators $\{p_k, q_k, a_k, g_k, h_k, b_k, d_k\}$ if there exists a set $K \subset \{1, 2, \dots, n\}$ such that*

$$\begin{cases} \widetilde{q}_1 = g_1, \quad \widetilde{g}_1 = q_1, \quad \widetilde{d}_1 = d_1 & \text{if } 1 \in K, \\ \widetilde{p}_n = h_n, \quad \widetilde{h}_n = p_n, \quad \widetilde{d}_n = d_n & \text{if } n \in K, \\ \widetilde{p}_k = h_k, \quad \widetilde{q}_k = g_k, \quad \widetilde{a}_k = b_k, & \\ \widetilde{h}_k = p_k, \quad \widetilde{g}_k = q_k, \quad \widetilde{b}_k = a_k, \quad \widetilde{d}_k = d_k & \text{if } k \in K, \end{cases} \tag{2.1}$$

and all other generators of $\widetilde{A}$ and A are equal. Additionally, if K contains a single index, we call the transformation an elementary twist transformation.

In other words, $\widetilde{A}$ is obtained from A via the interchange of lower and upper generators

$$a_k \longleftrightarrow b_k, \quad p_k \longleftrightarrow h_k, \quad q_k \longleftrightarrow g_k$$

for some subset of indices k. This is why we propose to call the operations of (2.1) *twist–transformations.*

The significant feature of the twist transformation is that it transforms one $(1,1)$–quasiseparable matrix into another while preserving the coefficients of the recurrence relations (1.7) and, thus, also preserving the characteristic polynomials of all of their submatrices. The next theorem exploits this fact.

THEOREM 2.2. *The system of polynomials related to a* $(1,1)$*–quasiseparable matrix* A *is invariant under twist transformations.*

Proof. It suffices to prove the theorem for an elementary twist transformation with $K=\{k\}$. Let $\widetilde{A}$ be the matrix obtained from A via (2.1) and $\{r_k(x)\}_{k=0}^n$ and $\{\widetilde{r}_k(x)\}_{k=0}^n$ be the system of polynomials related to A and $\widetilde{A}$, respectively. Considering the recurrence relations (1.7) for polynomials related to $(1,1)$–quasiseparable matrices and noticing that

$$\widetilde{a}_k\widetilde{b}_k = a_kb_k, \quad \widetilde{p}_k\widetilde{h}_k = p_kh_k, \quad \widetilde{d}_k = d_k,$$
$$\widetilde{d}_k\widetilde{a}_k\widetilde{b}_k - \widetilde{q}_k\widetilde{p}_k\widetilde{b}_k - \widetilde{g}_k\widetilde{h}_k\widetilde{a}_k = d_ka_kb_k - q_kp_kb_k - g_kh_ka_k.$$

we conclude that both systems of polynomials $\{r_k(x)\}_{k=0}^n$ and $\{\widetilde{r}_k(x)\}_{k=0}^n$ satisfy the same recurrence relations and, hence, coincide. □

COROLLARY 2.3. *Examples 1.5 and 1.6 show that CMV matrices* (1.4) *and Fiedler matrices* (1.6) *are obtained via twist transformations from unitary Hessenberg matrices* (1.3) *and companion matrices* (1.5)*, respectively. Hence, unitary Hessenberg matrices and CMV matrices share the same systems of characteristic polynomials, as do companion matrices and Fiedler matrices,*

COROLLARY 2.4. *For an arbitrary* $(1,1)$*–quasiseparable matrix* A *of size* n*, there are* 2^n *(possibly not distinct) matrices obtained from* A *via twist–transformations related to the same system of polynomials as* A*.*

2.2. Twisted $(H,1)$–quasiseparable matrices. Following [9], we define the class of matrices which are both strictly[3] upper Hessenberg and $(1,1)$–quasiseparable. The definition, like Definition 1.2 above, is given in terms of generators, see [9] for an equivalent definition in terms of ranks.

DEFINITION 2.5 (Generator definition of $(H,1)$–quasiseparable matrices). *A matrix* A *is called* $(H,1)$*–quasiseparable if it can be represented in the form*

$$A = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \cdots & \cdots & g_1b_2\dots b_{n-1}h_n \\ q_1 & d_2 & g_2h_3 & \cdots & \cdots & g_2b_3\dots b_{n-1}h_n \\ 0 & q_2 & d_3 & \cdots & \cdots & g_3b_4\dots b_{n-1}h_n \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & q_{n-2} & d_{n-1} & g_{n-1}h_n \\ 0 & \cdots & \cdots & 0 & q_{n-1} & d_n \end{bmatrix}, \tag{2.2}$$

where the parameters $\{q_k \neq 0, d_k, g_k, b_k, h_k\}$ *are called* generators *of* A*.*

REMARK 2.6. *Comparing Definitions 1.2 and 2.5 one can easily see that an* $(1,1)$*–quasiseparable matrix is* $(H,1)$*–quasiseparable if and only if it has a choice of generators such that* $a_k = 0$*,* $p_k = 1$*,* $q_k \neq 0$*.*

It is easy to check that both unitary Hessenberg matrices (1.3) and companion matrices (1.5) are $(H,1)$–quasiseparable (in fact, the generators listed for them in Tables 1.1 and 1.3 demonstrate this fact). As we have seen, CMV matrices (1.4) and Fiedler matrices (1.6) can be obtained from them via twist transformations. In order

[3] i.e. having nonzero subdiagonal elements.

to generalize these results, we define next the entire class of matrices which can be obtained from $(H,1)$–quasiseparable matrices via twist–transformations.

Definition 2.7 (Twisted $(H,1)$–quasiseparable matrices). *A $(1,1)$–quasiseparable matrix A is called twisted $(H,1)$–quasiseparable if it can be obtained from an $(H,1)$–quasiseparable matrix via twist transformations.*

Performing the twist transformation of the matrix (2.2) explicitly one can give the following alternative definition of twisted $(H,1)$–quasiseparable matrices in terms of their generators.

Definition 2.8 (Generator definition of twisted $(H,1)$–quasiseparable matrices). *A $(1,1)$–quasiseparable matrix A is twisted $(H,1)$–quasiseparable if and only if it has a choice of generators $\{p_k,\, q_k,\, a_k,\, g_k,\, h_k,\, b_k,\, d_k\}$ such that*

$$\begin{cases} q_1 \neq 0 \quad or \quad g_1 \neq 0, \\ a_k = 0, q_k \neq 0, p_k = 1 \quad or \quad b_k = 0, g_k \neq 0, h_k = 1, \quad k = 2 \dots n-1, \\ p_n = 1 \quad or \quad h_n = 1. \end{cases}$$

For an arbitrary $(H,1)$–quasiseparable matrix A with given generators, according to Corollary 2.4 there are exist 2^n (possibly not distinct) twisted-$(H,1)$–quasiseparable matrices related to the same polynomial system as A. But it is feasible to distinguish them using the so-called *pattern* defined next.

Definition 2.9 (Pattern of twisted $(H,1)$–quasiseparable matrices). *For an arbitrary twisted $(H,1)$–quasiseparable matrix A we will say that a sequence of binary digits $(i_1, i_2, \dots, i_n)$ is the* pattern *of A if A can be transformed to some $(H,1)$–quasiseparable matrix H by applying the twist transformation for $k \in K$ for each $i_k = 1$. Equivalently, $(i_1, i_2, \dots, i_n)$ is the pattern of A if there exist generators of A satisfying*

$$\begin{cases} q_1 \neq 0 & if\ i_1 = 0, \\ g_1 \neq 0 & if\ i_1 = 1, \\ a_k = 0, q_k \neq 0, p_k = 1 & if\ i_k = 0, \\ b_k = 0, g_k \neq 0, h_k = 1 & if\ i_k = 1, \\ p_n = 1 & if\ i_n = 0, \\ h_n = 1 & if\ i_n = 1. \end{cases} \tag{2.3}$$

Under these conditions we will also say that $A = H(i_1, i_2, \dots, i_n)$.

Example 2.10. *In accordance to this definition any $(H,1)$–quasiseparable matrix H of size n is $H(\underbrace{0,0,\dots,0}_{n})$ and its transpose is $H(\underbrace{1,1,\dots,1}_{n})$.*

Example 2.11. *Comparing the generators of unitary Hessenberg matrices (Table 1.1) and CMV matrices (Table 1.2) it is easy to see that CMV matrices have pattern $(0,1,0,1,\dots)$. A similar observation shows that Fiedler matrices have pattern $(1,0,1,0,1,\dots)$.*

Remark 2.12. *Let H be an $(H,1)$–quasiseparable matrix specified by its generators $\{q_k,\, d_k,\, g_k,\, b_k,\, h_k\}$. Then the matrices $H(0,1,0,1,0,\dots)$ and $H(1,0,1,0,1,\dots)$*

are five–diagonal. In particular,

$$H(0,1,0,1,0,\dots) = \begin{bmatrix} d_1 & g_1 & 0 & & & & \\ q_1h_2 & d_2 & q_2h_3 & q_2b_3 & & & \\ q_1b_2 & g_2 & d_3 & g_3 & 0 & & \\ & 0 & q_3h_4 & d_4 & q_4h_5 & q_4b_5 & \\ & & q_3b_4 & g_4 & d_5 & g_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix},$$

and $H(1,0,1,0,1,\dots) = H(0,1,0,1,0,\dots)^T$. *Thus for every* $(H,1)$*–quasiseparable matrix there always exist five–diagonal twisted* $(H,1)$*–quasiseparable matrices having the same system of characteristic polynomials. More details on five–diagonal matrices will be given in Section 4.*

3. A survey of [9] results for $(H,1)$–quasiseparable polynomials. In the present section we briefly describe the main results of [9] in order to use them extensively in Section 4.

3.1. A bijection between strictly Hessenberg matrices and polynomial systems. Let $\mathbb{H}_n$ be the set of strictly upper Hessenberg $n \times n$ matrices, λ_0 and $\lambda_n\}$ be two nonzero parameters, and $\mathbb{P}_n$ be the set of polynomial systems $\{r_k\}_{k=0}^n$ with $\deg r_k = k$. We next demonstrate that there is a bijection between the triple ($\mathbb{H}_n$, λ_0, λ_n) and $\mathbb{P}_n$. Indeed, given a polynomial system $\{r_k\}_{k=0}^n$ satisfying $\deg r_k = k$, there exist unique n–term recurrence relations of the form

$$r_0(x) = \frac{1}{m_{0,0}}, \quad x \cdot r_{k-1}(x) = m_{k,k}r_k(x) - m_{k-1,k}r_{k-1}(x) - \cdots - m_{0,k}r_0(x), \\ m_{k,k} \neq 0, \quad k = 1,\dots,n. \tag{3.1}$$

This formula represents $x \cdot r_{k-1}$ in the space of all polynomials of degree at most k in terms of $\{r_j\}_{j=0}^k$, which form a basis in that space, and hence these coefficients are unique. Forming a matrix $H \in \mathbb{H}_n$ and parameters λ_0 and λ_n from these coefficients of the form

$$H = \begin{bmatrix} m_{0,1} & m_{0,2} & m_{0,3} & \cdots & m_{0,n} \\ m_{1,1} & m_{1,2} & m_{1,3} & \cdots & m_{1,n} \\ 0 & m_{2,2} & m_{2,3} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & m_{n-2,n} \\ 0 & \cdots & 0 & m_{n-1,n-1} & m_{n-1,n} \end{bmatrix}, \quad \lambda_0 = \frac{1}{m_{0,0}}, \quad \lambda_n = \frac{1}{m_{n,n}}, \tag{3.2}$$

it is clear that there is a bijection between ($\mathbb{H}_n$, λ_0, λ_n) and $\mathbb{P}_n$, as they share the same unique parameters. Furthermore, it was shown in [32] that the strictly upper Hessenberg matrix H defined in (3.2) and the polynomial system (3.1) are related via (1.1) with $\lambda_k = \frac{1}{m_{k,k}}$. This shows the desired bijection (1.2). The matrix H in (3.2) is usually called *confederate* for the system of polynomials (3.1).

To conclude, for an arbitrary matrix H and scaling factors $\{\lambda_k\}_{k=0}^n$ there exists a unique system of polynomials related to it via (1.1). But the converse is, of course, not true. However, a bijection does exist if we restrict our attention to strictly upper Hessenberg matrices.

3.2. Well-free polynomials and three–term recurrence relations. Consider the general three–term recurrence relations

$$\begin{gathered} r_0(x) = \alpha_0, \quad r_1(x) = (\alpha_1 x + \beta_1) \cdot r_0(x), \\ r_k(x) = (\alpha_k x + \beta_k) \cdot r_{k-1}(x) + (\gamma_k x + \delta_k) \cdot r_{k-2}(x), \quad \alpha_k \neq 0. \end{gathered} \tag{3.3}$$

These recurrence relations can be treated as the generalized version of the recurrence relations

$$r_k(x) = (\alpha_k x + \beta_k) r_{k-1}(x) + \gamma_k \cdot r_{k-2}(x), \quad \alpha_k \neq 0, \gamma_k > 0 \tag{3.4}$$

satisfied by polynomials orthogonal on the real line.

THEOREM 3.1 (General three–term recurrence relations). *A polynomial system* $\{r_k(x)\}_{k=0}^n$ *satisfies three–term recurrence relations* (3.3) *if and only if there exists an* $(H,1)$*–quasiseparable matrix* H *with the set of generators* $\{q_k, d_k, g_k, b_k, h_k \neq 0\}$ *related to it via* (1.1) *with* $\lambda_k = \frac{1}{q_k}$*. Moreover, conversion formulas between generators and recurrence relations coefficients are given in Table 3.1.*

TABLE 3.1
Conversion formulas: three–term r.r. coefficients $\Longleftrightarrow$ *quasiseparable generators.*

quasiseparable generators				
q_k	d_k	g_k	b_k	h_k
$\frac{1}{\alpha_k}$	$-\frac{\alpha_{k-1}\beta_k + \gamma_k}{\alpha_{k-1}\alpha_k}$	$-\frac{\gamma_{k+1} d_k + \delta_k}{\alpha_{k+1}}$	$-\frac{\gamma_{k+1}}{\alpha_{k+1}}$	1

three–term r.r. coefficients			
α_k	β_k	γ_k	δ_k
$\frac{1}{q_k}$	$\frac{q_{k-1} b_{k-1} - d_k}{q_k}$	$-\frac{b_{k-1}}{q_k}$	$\frac{d_k b_k - g_k}{q_{k+1}}$

REMARK 3.2. $(H,1)$*–quasiseparable matrices with the restriction* $h_k \neq 0$ *on the generators were called* well-free *in [9]. Therefore, polynomials satisfying* (3.3) *are also called* well-free *polynomials.*

3.3. Semiseparable polynomials and Szegö–type two–term recurrence relations. Another interesting family of recurrence relations for polynomials considered in [9] are the so-called *Szegö–type* two–term recurrence relations, of the form

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \delta_0 \end{bmatrix}, \quad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (x+\theta_k) \cdot r_{k-1}(x) \end{bmatrix}, \tag{3.5}$$

with $\alpha_k\delta_k - \beta_k\gamma_k \neq 0$, $\delta_k \neq 0$ and $G_k(x)$ being auxiliary polynomials.

These recurrence relations generalize those satisfied by Szegö polynomials (polynomials orthogonal on the unit circle), of the form

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix} -\rho_0^* \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix}. \tag{3.6}$$

This justifies the name Szegö–type.

THEOREM 3.3 (Szegö–type two–term recurrence relations). *A polynomial system* $\{r_k(x)\}_{k=0}^n$ *satisfies two–term recurrence relations* (3.5) *if and only if there exists*

TABLE 3.2
Conversion formulas: Szegö two–term r.r. coefficients $\Longleftrightarrow$ quasiseparable generators.

quasiseparable generators				
q_k	d_k	g_k	b_k	h_k
$\frac{1}{\delta_k}$	$\theta_k + \frac{\beta_{k-1}\gamma_k}{\delta_{k-1}\delta_k}$	$-\frac{\beta_{k-1}(\alpha_k\delta_k-\beta_k\gamma_k)}{\delta_{k-1}\delta_k}$	$(\alpha_k\delta_k-\beta_k\gamma_k)$	γ_k

Szegö–type r.r. coefficients				
α_k	β_k	γ_k	δ_k	θ_k
$\frac{b_k}{q_k} + \frac{g_{k+1}q_{k+1}}{b_{k+1}}h_k$	$\frac{g_{k+1}q_{k+1}}{b_{k+1}}q_k$	h_k	q_k	$d_k - g_k h_k$

an $(H,1)$–quasiseparable matrix H with the set of generators $\{q_k, d_k, g_k, b_k \neq 0, h_k\}$ related to it via (1.1) *with $\lambda_k = \frac{1}{q_k}$. Moreover, conversion formulas between generators and recurrence relations coefficients are given in Table 3.2.*

REMARK 3.4. $(H,1)$*–quasiseparable matrices with the restriction $b_k \neq 0$ on the generators were called* semiseparable *in [9]. Therefore, polynomials satisfying* (3.5) *are also called* semiseparable *polynomials.*

3.4. Quasiseparable polynomials and EGO–type two–term recurrence relations. The authors of [9] established that the class of polynomials related to $(H,1)$–quasiseparable matrices (Definition 2.5) are characterized as those satisfying *EGO–type* two term recurrence relations

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_0 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \beta_k & \gamma_k \\ \delta_k & \theta_k x + \varepsilon_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}, \tag{3.7}$$

with auxiliary polynomials $F_k(x)$.

THEOREM 3.5 (EGO–type two–term recurrence relations). *A polynomial system $\{r_k(x)\}_{k=0}^n$ satisfies two–term recurrence relations* (3.7) *if and only if there exists an $(H,1)$–quasiseparable matrix H with the set of generators $\{q_k, d_k, g_k, b_k, h_k\}$ related to it via* (1.1) *with $\lambda_k = \frac{1}{q_k}$. Moreover, conversion formulas between generators and recurrence relations coefficients are given in Table 3.3.*

TABLE 3.3
Conversion formulas: EGO-type r.r. coefficients $\Longleftrightarrow$ quasiseparable generators.

quasiseparable generators				
q_k	d_k	g_k	b_k	h_k
$\frac{1}{\theta_k}$	$-\frac{\varepsilon_k}{\theta_k}$	$-\gamma_k$	β_k	$\frac{\delta_k}{\theta_k}$

EGO–type r.r. coefficients				
β_k	γ_k	δ_k	θ_k	ε_k
b_k	$-g_k$	$\frac{h_k}{q_k}$	$\frac{1}{q_k}$	$-\frac{d_k}{q_k}$

REMARK 3.6. *Due to the bijection established by Theorem 3.5, it was proposed in [9] to call polynomials satisfying the recurrence relations* (3.7) $(H,1)$*–quasiseparable polynomials or, simply, quasiseparable polynomials.*

4. Five–diagonal twisted $(H,1)$–quasiseparable matrices. The results of Section 2 connect five–diagonal CMV matrices (1.4) and Fiedler matrices (1.6) to the theory of quasiseparable matrices (through the concept of twist transformations). In fact, the quasiseparable approach (Section 3) leads to several new results on five–diagonal matrices.

In this section we investigate recurrence relations satisfied by polynomials $\{r_k(x)\}_{k=0}^n$ related to five–diagonal twisted $(H,1)$–quasiseparable matrices via

$$r_0(x) = \lambda_0, \quad r_k(x) = \lambda_0\lambda_1 \ldots \lambda_k \det(xI - A_{k\times k}), \quad k = 1,\ldots,n, \tag{4.1}$$

with $\lambda_k \neq 0$ being additional parameters. It turns out that in contrast to the case of Hessenberg matrices there are no bijections like (1.2) between five–diagonal twisted $(H,1)$–quasiseparable matrix and polynomial systems. The details will be given in Section 4.2.

We start by deriving an entrywise description of five–diagonal twisted $(H,1)$–quasiseparable matrices in order to distinguish them from general five–diagonal matrices.

4.1. Full description of five–diagonal twisted $(H,1)$–quasiseparable matrices. Consider a 6×6 five–diagonal matrix

$$A = \begin{bmatrix} \star & \star & \star & 0 & 0 & 0 \\ \star & \star & \star & \star & 0 & 0 \\ \star & \star & \star & \star & \star & 0 \\ 0 & \star & \star & \star & \star & \star \\ 0 & 0 & \star & \star & \star & \star \\ 0 & 0 & 0 & \star & \star & \star \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & 0 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & m_{24} & 0 & 0 \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & 0 \\ 0 & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ 0 & 0 & m_{53} & m_{54} & m_{55} & m_{56} \\ 0 & 0 & 0 & m_{64} & m_{65} & m_{66} \end{bmatrix}. \tag{4.2}$$

It is $(1,1)$–quasiseparable (Definition 1.1) if and only if all its submatrices $A(1:i, i+1, n)$ and $A(i+1:n, 1:i)$ for $i = 2,\ldots n-1$ are of rank one. For instance, the submatrix $A(1:2, 3:6)$ highlighted in (4.2) is of rank one if and only if $m_{13}\cdot m_{24} = 0$. This observation leads to the following simple theorem.

Theorem 4.1. *[Characterizations of five–diagonal $(1,1)$–quasiseparable matrices]* ***Entrywise characterization.*** *A five–diagonal matrix $A = [m_{ij}]_{i,j=1}^n$ is $(1,1)$–quasiseparable if and only if*

$$m_{i,i+2}\cdot m_{i+1,i+3} = m_{i+2,i}\cdot m_{i+3,i+1} = 0, \quad i = 1,\ldots,n-3. \tag{4.3}$$

Generator characterization. *An $(1,1)$–quasiseparable matrix A is five–diagonal if and only if it has a choice of generators such that*

$$a_k \cdot a_{k+1} = b_k \cdot b_{k+1} = 0, \quad k = 2,\ldots,n-2. \tag{4.4}$$

The conditions (4.3) and (4.4) actually imply that in a five–diagonal $(1,1)$–quasiseparable matrix every nonzero entry on second sub(super)diagonal is surrounded by two zero entries on that sub(super)diagonal.

We give next necessary and sufficient conditions for a five–diagonal matrix to be twisted $(H,1)$–quasiseparable.

Theorem 4.2. *A five–diagonal matrix $A = [m_{ij}]_{i,j=1}^n$ is twisted $(H,1)$–quasiseparable if and only if*

$$\begin{aligned} m_{i,i+2}\cdot m_{i+1,i+3} = m_{i+2,i}\cdot m_{i+3,i+1} &= 0, \quad i = 1,\ldots,n-3, \\ m_{i,i+2}\cdot m_{i+2,i} &= 0, \quad i = 1,\ldots,n-2. \end{aligned} \tag{4.5}$$

The proof is given in the appendix.

The Venn diagram of subclasses of five–diagonal matrices is given in Figure 4.1 and is a consequence of Theorems 4.1 and 4.2.

$(1,1)$–quasiseparable matrices, condition (4.4)

five–diagonal matrices

twisted $(H,1)$–quasiseparable matrices, condition (4.5)

FIG. 4.1. *Subclasses of five–diagonal matrices*

4.2. Non–uniqueness of five–diagonal twisted $(H,1)$–quasiseparable matrices. As we have seen in Section 3.1 there is a bijection between $(H,1)$–quasiseparable matrices together with two nonzero parameters q_0 and q_n and systems of polynomials related to them via (4.1) with $\lambda_k = \frac{1}{q_k}$, where q_k, $k = 1, \dots, n-1$ are generators as in Definition 2.5. In contrast to this bijection for a given system of polynomials (related to some $(H,1)$–quasiseparable matrix) there are infinitely many five–diagonal twisted $(H,1)$–quasiseparable matrices related to it via (4.1). We describe next two reasons for this non–uniqueness.

Reason 1: Non–uniqueness of patterns. Let H be an $(H,1)$–quasiseparable matrix with generators $\{q_k, d_k, g_k, b_k, h_k\}$. Then the twisted $(H,1)$–quasiseparable matrices with patterns $(\star, 1, 0, 1, 0, \dots)$ and $(\star, 0, 1, 0, 1, \dots)$ obtained from H via twist transformations are five–diagonal. For example,

$$H(0,1,0,1,0,\dots) = \begin{bmatrix} d_1 & g_1 & 0 & & & & \\ q_1h_2 & d_2 & q_2h_3 & q_2b_3 & & & \\ q_1b_2 & g_2 & d_3 & g_3 & 0 & & \\ & 0 & q_3h_4 & d_4 & q_4h_5 & q_4b_5 & \\ & & q_3b_4 & g_4 & d_5 & g_5 & 0 \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.6}$$

and $H(1,0,1,0,1,\dots) = H(0,1,0,1,0,\dots)^T$. Moreover, all five–diagonal twisted $(H,1)$–quasiseparable matrices obtained from H share the same system of polynomials (Theorem 2.2).

REMARK 4.3. *Five–diagonal matrices of the following zero patterns*

$$\begin{bmatrix} \star & \star & 0 & & & & & \\ \star & \star & \star & \star & & & & \\ \star & \star & \star & \star & 0 & & & \\ & 0 & \star & \star & \star & \star & & \\ & & \star & \star & \star & \star & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \quad \begin{bmatrix} \star & \star & \star & & & & & \\ \star & \star & \star & 0 & & & & \\ 0 & \star & \star & \star & \star & & & \\ & \star & \star & \star & \star & 0 & & \\ & & 0 & \star & \star & \star & \star & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.7}$$

always satisfy restrictions (4.5) *and, hence, are always twisted $(H,1)$–quasiseparable. In addition, any matrices of patterns $(\star,1,0,1,0,\dots)$ and $(\star,0,1,0,1,\dots)$ have the first and second of zero patterns of* (4.7)*, respectively.*

Another fact which is worth mentioning is that if an $(H,1)$–quasiseparable matrix H has generators such that $b_k = 0$ for some k then there are two (possibly distinct) five–diagonal twisted $(H,1)$–quasiseparable matrices of patterns

$$H(i_1,\dots,i_{k-1},0,i_{k+1},\dots,i_n) \quad \text{and} \quad H(i_1,\dots,i_{k-1},1,i_{k+1},\dots,i_n).$$

Reason 2: Non–uniqueness of quasiseparable generators. Let H be an $(H,1)$–quasiseparable matrix with generators. It is known that the choice of upper quasiseparable generators $\{g_k, b_k, h_k\}$ of H is not unique. On the other hand five–diagonal twisted $(H,1)$–quasiseparable matrices of fixed patterns obtained from H are defined via these generators in a unique way. For example, one can see that different choices of $\{g_k, b_k, h_k\}$ lead to different five–diagonal matrices (4.6).

EXAMPLE 4.4. *Quasiseparable generators (Definition 1.2) of the companion matrix*

$$C = \begin{bmatrix} -m_1 & -m_2 & \cdots & -m_{n-1} & -m_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{4.8}$$

were given in Table 1.3. However, it can be easily checked that the generators given in Table 4.1 also correspond to C.

TABLE 4.1
Alternative generators of companion matrix

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
$k=1$	$-m_1$	$-$	$-$	1	1	$-$	$-$
$k=2$	0	0	$-m_3$	1	0	1	$-m_2$
$k>2$	0	0	$-m_{k+1}$	1	0	1	1

If we then apply the twist transformation for odd indices to these alternate generators, we arrive at the following five–diagonal matrix

$$\widehat{F} = \begin{bmatrix} -m_1 & -m_2 & -m_3 & & & & & \\ 1 & 0 & 0 & 0 & & & & \\ 0 & 1 & 0 & 1 & -m_5 & & & \\ & -m_4 & 0 & 0 & 0 & 0 & & \\ & & 0 & 1 & 0 & 1 & -m_7 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.9}$$

of pattern $(1,0,1,0,\dots)$ having $(1,1)$–quasiseparable generators listed in Table 4.2.

Let us note that this new five–diagonal companion matrix (4.9) *differs from the matrix* (1.6) *derived by Fiedler [40], although it also has the property for which the Fiedler matrix was recognized: the same system of polynomials related via* (1.1).

4.3. General three–term recurrence relations. Let us recall that under the additional restriction of $\rho_k \neq 0$ for each k, the corresponding Szegö polynomials satisfy three–term recurrence relations

$$\begin{gathered} \phi_0^{\#}(x) = \frac{1}{\mu_0}, \quad \phi_1^{\#}(x) = \frac{1}{\mu_1}(x \cdot \phi_0^{\#}(x) + \rho_1 \rho_0^* \cdot \phi_0^{\#}(x)), \\ \phi_k^{\#}(x) = \left[\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}} \frac{1}{\mu_k}\right] \phi_{k-1}^{\#}(x) - \frac{\rho_k}{\rho_{k-1}} \frac{\mu_{k-1}}{\mu_k} \cdot x \cdot \phi_{k-2}^{\#}(x), \quad k = 2,\dots,n. \end{gathered} \tag{4.10}$$

TABLE 4.2
Quasiseparable generators of matrix (4.9)

k	d_k	a_k	b_k	q_k	g_k	p_k	h_k
$k=1$	$-m_1$	$-$	$-$	1	1	$-$	$-$
$k=2$	0	0	$-m_3$	1	0	1	$-m_2$
$k>2$, odd	0	$-m_{k+1}$	0	0	1	1	1
$k>2$, even	0	0	$-m_{k+1}$	1	0	1	1

This system of polynomials corresponds to a five–diagonal twisted $(H,1)$–quasiseparable matrix (in fact, the CMV matrix). The theorem below gives necessary and sufficient conditions for the existence of general three–term recurrence relations for polynomials in terms of five–diagonal matrices they are related to via (4.1).

THEOREM 4.5. *A system of polynomials* $R = \{r_k(x)\}_{k=0}^n$ *satisfies three–term recurrence relations*

$$\begin{gathered} r_0(x) = \alpha_0, \quad r_1(x) = (\alpha_1 x + \beta_1) \cdot r_0(x), \\ r_k(x) = (\alpha_k x + \beta_k) \cdot r_{k-1}(x) + (\gamma_k x + \delta_k) \cdot r_{k-2}(x), \quad \alpha_k \neq 0. \end{gathered} \tag{4.11}$$

if and only if it is related to a matrix A *of the following zero pattern*

$$\begin{bmatrix} \star & \star & 0 & & & & & \\ \star & \star & \star & \star & & & & \\ \star & \star & \star & \star & 0 & & & \\ & 0 & \star & \star & \star & \star & & \\ & & \star & \star & \star & \star & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.12}$$

with nonzero highlighted entries via (4.1) *with* $\lambda_k = \alpha_k$.

Proof.

[**Necessity**] Obviously, the matrix A is twisted $(H,1)$–quasiseparable (see Remark 4.3) and its general representation is

$$A = \begin{bmatrix} d_1 & g_1 & 0 & & & & & \\ q_1 h_2 & d_2 & q_2 h_3 & q_2 b_3 & & & & \\ q_1 b_2 & g_2 & d_3 & g_3 & 0 & & & \\ & 0 & q_3 h_4 & d_4 & q_4 h_5 & q_4 b_5 & & \\ & & q_3 b_4 & g_4 & d_5 & g_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.13}$$

Let $q_k = \frac{1}{\lambda_k}$, then all other its generators $\{d_k, g_k, b_k, h_k\}$ are defined uniquely. Moreover, the generators h_k are all nonzero. Hence, there exists a unique $(H,1)$–quasiseparable matrix H having generators $\{q_k, d_k, g_k, b_k, h_k \neq 0\}$ and according to the Theorem 2.2 it is related to the system of polynomials R via (4.1). It was proved in Theorem 3.1 that polynomials related to $(H,1)$–quasiseparable matrices with $h_k \neq 0$ satisfy the recurrence relations (4.13) and, hence, so do $\{r_k(x)\}_{k=0}^n$.

[**Sufficiency**] Let R satisfy three–term recurrence relations (4.11), it follows from Theorem 3.1 that there exists a unique $(H,1)$–quasiseparable matrix H with $q_k = \frac{1}{\alpha_k}$ and $h_k \neq 0$ such that it is related to R via (4.1) with $\lambda_k = \alpha_k$. Let $A = H(0,1,0,1,\dots)$ be a five–diagonal twisted $(H,1)$–quasiseparable matrix obtained from H via twist transformations, then it has the zero pattern (4.12) and by the Theorem 2.2 is related to the system of polynomials R via (4.1) with $\lambda_k = \alpha_k$. $\square$

The following corollary follows directly from Theorem 3.1, Remark 4.3, and Theorem 4.5.

COROLLARY 4.6. *A system of polynomials satisfies three–term recurrence relations* (4.11) *if and only if it is related to a twisted* $(H, 1)$*–quasiseparable matrix* A. *As such, the matrix* A *is twist equivalent to some (not twisted)* $(H, 1)$*–quasiseparable matrix* B *via some pattern* $(i_k)_{k=1}^n$. *Table 3.1 gives conversion formulas between the three–term recurrence relation coefficients and the generators of* B. *We next present Table 4.3, which gives a conversion from three–term recurrence relation coefficients to the generators of the matrix* A *itself.*

TABLE 4.3
Conversion formulas for the generators of a twisted $(H, 1)$*–quasiseparable matrix* A *in terms of the corresponding three–term recurrence relation coefficients.*

	g_k	b_k	h_k	d_k
if $i_k = 0$	$-\frac{\gamma_{k+1}d_k+\delta_k}{\alpha_{k+1}}$	$-\frac{\gamma_{k+1}}{\alpha_{k+1}}$	1	$-\frac{\alpha_{k-1}\beta_k+\gamma_k}{\alpha_{k-1}\alpha_k}$
if $i_k = 1$	$\frac{1}{\alpha_k}$	0	1	$-\frac{\alpha_{k-1}\beta_k+\gamma_k}{\alpha_{k-1}\alpha_k}$

	p_k	a_k	q_k
if $i_k = 0$	1	0	$\frac{1}{\alpha_k}$
if $i_k = 1$	1	$-\frac{\gamma_{k+1}}{\alpha_{k+1}}$	$-\frac{\gamma_{k+1}d_k+\delta_k}{\alpha_{k+1}}$

EXAMPLE 4.7. *Observing a CMV matrix*

$$K = \begin{bmatrix} -\rho_0^*\rho_1 & \rho_0^*\mu_1 & 0 & & & & & \\ -\mu_1\rho_2 & -\rho_1^*\rho_2 & -\mu_2\rho_3 & \mu_2\mu_3 & & & & \\ \mu_1\mu_2 & \rho_1^*\mu_2 & -\rho_2^*\rho_3 & \rho_2^*\mu_3 & 0 & & & \\ & 0 & -\mu_3\rho_4 & -\rho_3^*\rho_4 & -\mu_4\rho_5 & \mu_4\mu_5 & & \\ & & \mu_3\mu_4 & \rho_3^*\mu_4 & -\rho_4^*\rho_5 & \rho_4^*\mu_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.14}$$

it is easy to see that the additional restriction $\rho_k \neq 0$ *implies that all highlighted entries in* (4.14) *are not zeros*[4] *and, hence, the matrix* K *satisfies conditions of Theorem 4.5. This proves the existence of recurrence relations* (4.10) *for polynomials related to* K.

EXAMPLE 4.8. *Applying Theorem 4.5 to the transpose of a Fiedler matrix*

$$F^T = \begin{bmatrix} -m_1 & 1 & 0 & & & & & \\ -m_2 & 0 & -m_3 & 1 & & & & \\ 1 & 0 & 0 & 0 & 0 & & & \\ & 0 & -m_4 & 0 & -m_5 & 1 & & \\ & & 1 & 0 & 0 & 0 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{4.15}$$

[4] The definition of the complementary parameters μ_k is $\mu_k = \begin{cases} \sqrt{1-|\rho_k|^2} & |\rho_k| < 1 \\ 1 & |\rho_k| = 1 \end{cases}$, which are thus always nonzero.

we conclude that three–term recurrence relations (4.11) *must exist for Horner polynomials under the condition* $m_k \neq 0$ *for* $k = 2, \ldots, n$. *Indeed, from the recurrence relations* $p_k(x) = x \cdot p_{k-1}(x) + m_k$ *we can get that*

$$1 = \frac{p_{k-1} - x \cdot p_{k-2}}{m_{k-1}},$$

and, hence

$$p_k(x) = x \cdot p_{k-1}(x) + m_k \cdot 1 = \left(x + \frac{m_k}{m_{k-1}} \right) p_{k-1}(x) - \frac{m_k}{m_{k-1}} x \cdot p_{k-2}.$$

4.4. Szegö–type two–term recurrence relations. It is well-known that Szegö polynomials related to CMV matrices satisfy two–term recurrence relations

$$\begin{bmatrix} \phi_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix} -\rho_0^* \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \phi_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \frac{1}{\mu_k} \begin{bmatrix} 1 & -\rho_k^* \\ -\rho_k & 1 \end{bmatrix} \begin{bmatrix} \phi_{k-1}(x) \\ x\phi_{k-1}^{\#}(x) \end{bmatrix}. \tag{4.16}$$

In this section we consider the general form of recurrence relations (4.16) (which were called Szegö–type in [9]) and derive the class of five–diagonal twisted $(H,1)$–quasiseparable matrices related to polynomials satisfying them.

Theorem 4.9. *A system of polynomials* $R = \{r_k(x)\}_{k=0}^n$ *satisfies Szegö–type two–term recurrence relations*

$$\begin{bmatrix} G_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \delta_0 \end{bmatrix}, \quad \begin{bmatrix} G_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \alpha_k & \beta_k \\ \gamma_k & \delta_k \end{bmatrix} \begin{bmatrix} G_{k-1}(x) \\ (x + \theta_k) \cdot r_{k-1}(x) \end{bmatrix} \tag{4.17}$$

with $\alpha_k\delta_k - \beta_k\gamma_k \neq 0$, $\delta_k \neq 0$ *if and only if it is related to a matrix* A *of the following zero pattern*

$$\begin{bmatrix} \star & \star & 0 & & & & & \\ \star & \star & \star & \star & & & & \\ \star & \star & \star & \star & 0 & & & \\ & 0 & \star & \star & \star & \star & & \\ & & \star & \star & \star & \star & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix} \tag{4.18}$$

with nonzero highlighted entries via (4.1) *with* $\lambda_k = \delta_k$.

Proof.

[**Necessity**] The matrix A is twisted $(H,1)$–quasiseparable (see Remark 4.3) and its general representation is

$$A = \begin{bmatrix} d_1 & g_1 & 0 & & & & & \\ q_1h_2 & d_2 & q_2h_3 & q_2b_3 & & & & \\ q_1b_2 & g_2 & d_3 & g_3 & 0 & & & \\ & 0 & q_3h_4 & d_4 & q_4h_5 & q_4b_5 & & \\ & & q_3b_4 & g_4 & d_5 & g_5 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.19}$$

Fixing $q_k = \frac{1}{\lambda_k}$, then all other generators $\{d_k, g_k, b_k, h_k\}$ are defined uniquely. Moreover, the generators b_k are all nonzero. Hence, there exists a unique $(H,1)$–quasiseparable matrix H having generators $\{q_k, d_k, g_k, b_k \neq 0, h_k\}$ and according to Theorem

2.2 it is related to the system of polynomials R via (4.1). It is proved in Theorem 3.3 that polynomials related to $(H,1)$–quasiseparable matrices with $b_k \neq 0$ satisfy recurrence relations (4.17) and, hence, so do $\{r_k(x)\}_{k=0}^n$.

[**Sufficiency**] Let R satisfy Szegö–type recurrence relations (4.17). Then Theorem 3.3 implies that there exists a unique $(H,1)$–quasiseparable matrix H with $q_k = \frac{1}{\delta_k}$ and $b_k \neq 0$ that is related to R via (4.1) with $\lambda_k = \delta_k$. Let $A = H(0,1,0,1,\dots)$ be a five–diagonal twisted $(H,1)$–quasiseparable matrix obtained from H via twist transformations, which has the zero pattern (4.18) and by Theorem 2.2 is related to the system of polynomials R via (4.1) with $\lambda_k = \delta_k$. $\square$

The following corollary follows directly from Theorem 3.3, Remark 4.3, and Theorem 4.9.

Corollary 4.10. *A system of polynomials satisfies Szegö–type recurrence relations* (4.17) *if and only if it is related to a twisted $(H,1)$–quasiseparable matrix A. As such, the matrix A is twist equivalent to some (not twisted) $(H,1)$–quasiseparable matrix B via some pattern $(i_k)_{k=1}^n$. Table 3.2 gives conversion formulas between the Szegö–type recurrence relation coefficients and the generators of B. We next present Table 4.4, which gives a conversion from Szegö–type recurrence relation coefficients to the generators of the matrix A itself.*

Table 4.4
Conversion formulas for the generators of a twisted $(H,1)$–quasiseparable matrix A in terms of the corresponding Szegö–type recurrence relation coefficients.

	g_k	b_k	h_k	d_k
if $i_k = 0$	$-\frac{\beta_{k-1}(\alpha_k\delta_k - \beta_k\gamma_k)}{\delta_{k-1}\delta_k}$	$(\alpha_k\delta_k - \beta_k\gamma_k)$	γ_k	$\theta_k + \frac{\beta_{k-1}\gamma_k}{\delta_{k-1}\delta_k}$
if $i_k = 1$	$\frac{1}{\delta_k}$	0	1	$\theta_k + \frac{\beta_{k-1}\gamma_k}{\delta_{k-1}\delta_k}$

	p_k	a_k	q_k
if $i_k = 0$	1	0	$\frac{1}{\delta_k}$
if $i_k = 1$	γ_k	$(\alpha_k\delta_k - \beta_k\gamma_k)$	$-\frac{\beta_{k-1}(\alpha_k\delta_k - \beta_k\gamma_k)}{\delta_{k-1}\delta_k}$

Let us note also that the transpose of the Fiedler matrix (1.6) satisfies the conditions of Theorem 4.9,

$$F^T = \begin{bmatrix} -m_1 & 1 & 0 & & & & & \\ -m_2 & 0 & -m_3 & \boxed{1} & & & & \\ \boxed{1} & 0 & 0 & 0 & 0 & & & \\ & 0 & -m_4 & 0 & -m_5 & \boxed{1} & & \\ & & \boxed{1} & 0 & 0 & 0 & 0 & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{4.20}$$

as the highlighted entries in (4.20) are nonzeros. Hence, Horner polynomials satisfy Szegö–type recurrence relations (4.16). Using the generators from Table 1.3 and the conversion formulas listed in Table 3.2, we arrive at the following Szegö–type recurrence relations for Horner polynomials,

$$\begin{bmatrix} F_0(x) \\ p_0(x) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ p_k(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m_k & 1 \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ x \cdot p_{k-1}(x) \end{bmatrix}. \tag{4.21}$$

4.5. EGO–type two–term recurrence relations. It was observed in Section 4.1 that five–diagonal twisted $(H,1)$–quasiseparable matrices form a proper subclass of $(1,1)$–quasiseparable matrices. Hence, it is natural to expect that polynomials related to them via (4.1) satisfy some special recurrence relations rather than general (1.7). The next theorem shows that this is, indeed, the case.

THEOREM 4.11. *A system of polynomials $R = \{r_k(x)\}_{k=0}^n$ satisfies EGO–type two–term recurrence relations*

$$\begin{bmatrix} F_0(x) \\ r_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \theta_0 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ r_k(x) \end{bmatrix} = \begin{bmatrix} \beta_k & \gamma_k \\ \delta_k & \theta_k x + \varepsilon_k \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ r_{k-1}(x) \end{bmatrix}, \tag{4.22}$$

with $\theta_k = \lambda_k$ if and only if it is related via (4.1) *to a five–diagonal matrix $A = [m_{ij}]_{ij=1}^n$ with entries satisfying*

$$\begin{aligned} m_{i,i+2} \cdot m_{i+1,i+3} = m_{i+2,i} \cdot m_{i+3,i+1} = 0, \quad i = 1, \dots, n-3, \\ m_{i,i+2} \cdot m_{i+2,1} = 0, \quad i = 1, \dots, n-2. \end{aligned} \tag{4.23}$$

Proof.

[**Necessity**] Let the entries of A satisfy (4.23). Then according to Theorem 4.2, A is twisted $(H,1)$–quasiseparable. Hence, there exists an $(H,1)$–quasiseparable matrix related to the same system of polynomials as A by Theorem 2.2. It immediately follows from Theorem 3.5 that the related polynomials R satisfy the recurrence relations (4.22).

[**Sufficiency**] Let R satisfy EGO–type recurrence relations (4.22). Then by Theorem 3.5, there exists an $(H,1)$–quasiseparable matrix H that is related to R via (4.1). Let $A = H(0,1,0,1,\dots)$ be a twisted $(H,1)$–quasiseparable matrix obtained from H via twist transformations. Then it is five–diagonal (see Remark 4.3), satisfies the conditions (4.23) and is related to the system of polynomials R via (4.1) as desired. □

The following corollary follows directly from Theorem 3.5 and Theorem 4.11.

COROLLARY 4.12. *A system of polynomials satisfies EGO–type recurrence relations* (4.22) *if and only if it is related to a twisted $(H,1)$–quasiseparable matrix A. As such, the matrix A is twist equivalent to some (not twisted) $(H,1)$–quasiseparable matrix B via some pattern $(i_k)_{k=1}^n$. Table 3.3 gives conversion formulas between the EGO–type recurrence relation coefficients and the generators of B. We next present Table 4.5, which gives a conversion from EGO–type recurrence relation coefficients to the generators of the matrix A itself.*

TABLE 4.5

Conversion formulas for the generators of a twisted $(H,1)$–quasiseparable matrix A in terms of the corresponding EGO–type recurrence relation coefficients.

	g_k	b_k	h_k	d_k
if $i_k = 0$	$-\gamma_k$	β_k	$\frac{\delta_k}{\theta_k}$	$-\frac{\varepsilon_k}{\theta_k}$
if $i_k = 1$	$\frac{1}{\theta_k}$	0	1	$-\frac{\varepsilon_k}{\theta_k}$

	p_k	a_k	q_k
if $i_k = 0$	1	0	$\frac{1}{\theta_k}$
if $i_k = 1$	$\frac{\delta_k}{\theta_k}$	β_k	$-\gamma_k$

Let us recall that both Fiedler matrices (1.6) and CMV matrices (1.4) fulfill the conditions of Theorem 4.11. Hence, Horner and Szegö polynomials must satisfy EGO–type recurrence relations (4.22). In particular, one can easily check that Horner polynomials satisfy

$$\begin{bmatrix} F_0(x) \\ p_0(x) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} F_1(x) \\ p_1(x) \end{bmatrix} = \begin{bmatrix} 1 \\ x+m_1 \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ p_k(x) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ m_k & x \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ p_{k-1}(x) \end{bmatrix}. \tag{4.24}$$

Similarly, Szegö polynomials satisfy

$$\begin{bmatrix} F_0(x) \\ \phi_0^{\#}(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{\mu_0} \end{bmatrix}, \quad \begin{bmatrix} F_k(x) \\ \phi_k^{\#}(x) \end{bmatrix} = \begin{bmatrix} \mu_k & \rho_{k-1}^{*}\mu_k \\ \frac{\rho_k}{\mu_k} & \frac{1}{\mu_k}x - \frac{\rho_{k-1}^{*}\rho_k}{\mu_k} \end{bmatrix} \begin{bmatrix} F_{k-1}(x) \\ \phi_{k-1}^{\#}(x) \end{bmatrix}. \tag{4.25}$$

5. Nested factorization of twisted $(H,1)$–quasiseparable matrices. In this section we derive a nested factorization of twisted $(H,1)$–quasiseparable matrices which we believe might be useful in developing fast algorithms for them. The interpretation of the twist transformation (Definition 2.1) can be also given in terms of this new factorization.

THEOREM 5.1. *Let H be an arbitrary $(H,1)$–quasiseparable matrix specified by its generators as in Definition 2.5, and define the matrices Θ_k, Δ_k by*

$$\Theta_1 = \left[\begin{array}{cc|c} d_1 & g_1 & \\ q_1 & d_2 & \\ \hline & & I_{n-2} \end{array}\right], \quad \Delta_1 = 0_n,$$

$$\Theta_k = \left[\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & h_k & b_k & \\ & q_k & d_{k+1} & \\ \hline & & & I_{n-k-1} \end{array}\right], \Delta_k = \left[\begin{array}{c|cc|c} 0_{k-1} & & & \\ \hline & d_k - d_k h_k & g_k - d_k b_k & \\ & 0 & 0 & \\ \hline & & & 0_{n-k-1} \end{array}\right],$$

for $k = 2, \ldots, n-1$,

$$\Theta_n = \left[\begin{array}{c|c} I_{n-1} & \\ \hline & h_n \end{array}\right], \quad \Delta_n = \left[\begin{array}{c|c} 0_{n-1} & \\ \hline & d_n - d_n h_n \end{array}\right], \tag{5.1}$$

where 0_k denotes the $k \times k$ zero matrix. Then the decomposition

$$H = (\ldots((\Theta_1\Theta_2 + \Delta_2)\Theta_3 + \Delta_3)\ldots)\Theta_n + \Delta_n \tag{5.2}$$

holds.

The proof of this decomposition is deferred; it will be seen as a special case of Theorem 5.2. Equation (5.2) of this theorem can be viewed as forming the $(H,1)$–quasiseparable matrix H by the iteration

$$H_0 = I_n, \quad H_k = H_{k-1}\Theta_k + \Delta_k, \quad k = 1, \ldots, n, \quad H = H_n. \tag{5.3}$$

The next theorem gives the concept of a twist transformation in terms of this decomposition. It states that the effect of an elementary twist transformation at index k changes step k of the decomposition (5.3) via a transpose–like operation to

$$H_k = \Delta_k^T + \Theta_k^T H_{k-1};$$

that is,

$H_k = H_{k-1}\Theta_k + \Delta_k$ $\updownarrow$ $H_k = \Delta_k^T + \Theta_k^T H_{k-1}$	$\Longleftrightarrow$	$p_k \quad a_k \quad q_k$ $\updownarrow \quad \updownarrow \quad \updownarrow$ $h_k \quad b_k \quad g_k$
Twist transformation in terms of decomposition		Twist transformation in terms of generators

THEOREM 5.2. *Let H be a twisted $(H,1)$–quasiseparable matrix of pattern $(i_1, i_2, \ldots, i_n)$ with generators $\{q_k, d_k, g_k, b_k, h_k\}$. Then it can be constructed by the following procedure:*

$$H_0 = I_n, \quad H_k = \begin{cases} H_{k-1}\Theta_k + \Delta_k & \text{if } i_k = 0, \\ \Delta_k^T + \Theta_k^T H_{k-1} & \text{if } i_k = 1, \end{cases} \quad k = 1, \ldots n, \quad H = H_n. \tag{5.4}$$

The proof of this theorem is given in the appendix, and Theorem 5.1 follows as a corollary with $i_k = 0$ for $k = 1, \ldots, n$.

6. Appendix. This appendix contains the proofs omitted in the paper.

Proof. [Proof of Theorem 4.2] We first prove the "only if" implication. Let $A = [m_{ij}]$ be a five–diagonal twisted $(H,1)$–quasiseparable matrix. Since A is also $(1,1)$–quasiseparable it satisfies the conditions of Theorem 4.1 and, hence

$$m_{i,i+2} \cdot m_{i+1,i+3} = m_{i+2,i} \cdot m_{i+3,i+1} = 0, \quad i = 1, \ldots, n-2. \tag{6.1}$$

Let A be described by a set of $(1,1)$–quasiseparable generators as in the Definition 1.2. Then its generators satisfy (see Definition 2.8) $a_i \cdot b_i = 0$ which implies

$$m_{i,i+2} \cdot m_{i+2,i} = 0, \quad i = 1, \ldots, n-2.$$

Therefore, the "only if" implication is proved.

Next, let $A = [m_{ij}]$ be a five–diagonal matrix satisfying conditions (4.5). These conditions imply that A is $(1,1)$–quasiseparable (see Theorem 4.1). Hence, A has a set of generators as in the Definition 1.2. Because of the condition (6.1) and five–diagonallity we can always choose generators to be

$$a_i = 0 \quad \text{if } m_{i+2,1} = 0 \quad \text{and} \quad b_i = 0 \quad \text{if } m_{i,i+2} = 0.$$

Applying twist transformation (Definition 2.1) for corresponding indices we can convert matrix A

□

Proof. [Proof of Theorem 5.2] We will show by induction that for every $k = 2, \ldots, n$:

$$H_{k-1}(1:k, 1:k) = H(1:k, 1:k)\,|_{p_k = h_k = 1} \tag{6.2}$$

In other words, every k-th principal submatrix of H_{k-1} almost equals to the corresponding one of H and equals identically if we modify in the last generators p_k and h_k.

The basis of induction ($k = 2$) is trivial:

$$H(2:2, 2:2) = \begin{bmatrix} d_1 & g_1 h_2 \\ p_2 q_1 & d_2 \end{bmatrix}, \quad H_1(2:2, 2:2) = \begin{bmatrix} d_1 & g_1 \\ q_1 & d_2 \end{bmatrix}, \quad \text{for } i_1 = 0,$$

$$H(2:2, 2:2) = \begin{bmatrix} d_1 & q_1 h_2 \\ p_2 g_1 & d_2 \end{bmatrix}, \quad H_1(2:2, 2:2) = \begin{bmatrix} d_1 & q_1 \\ g_1 & d_2 \end{bmatrix}, \quad \text{for } i_1 = 1.$$

Assume that (6.2) holds for all indices up to k. Consider case $i_k = 0$, the remaining case $i_k = 1$ is essentially the same.

Consider matrix $H_k(1 : k+1, 1 : k+1)$:

$$\left[\begin{array}{ccc|c} & & & 0 \\ & H_{k-1}(1:k,1:k) & & \vdots \\ & & & 0 \\ \hline 0 & \cdots & 0 & 1 \end{array}\right] \begin{bmatrix} \ddots & & & \\ & 1 & & \\ & & h_k & b_k \\ & & q_k & d_{k+1} \end{bmatrix} + \begin{bmatrix} \ddots & & & \\ & 0 & & \\ & & d_k - d_k h_k & g_k - d_k b_k \\ & & 0 & 0 \end{bmatrix} \tag{6.3}$$

The last k-th column of matrix $H_{k-1}(1 : k, 1 : k)$ equals the last column of matrix $H(1 : k, 1 : k)$ if in the last we set $h_k = 1$. Therefore, performing the matrix product in (6.3) we get:

$$\left[\begin{array}{ccc|c|c} & H_{k-1}(1:k,1:k-1) & & \begin{array}{c} H_{k-1}(1:k-1,k)h_k \\ d_k h_k + (d_k - d_k h_k) \end{array} & \begin{array}{c} H_{k-1}(1:k-1,k)b_k \\ d_k b_k + (g_k - d_k b_k) \end{array} \\ \hline 0 & \cdots & 0 & q_k & d_{k+1} \end{array}\right] \tag{6.4}$$

Which is equal to $H(1 : k+1, 1 : k+1) \mid_{p_{k+1}=h_{k+1}=1}$.

By the induction we get that

$$H_{n-1} = H \mid_{p_n=h_n=1} .$$

Substituting this into recursions (5.4) we get

$$H_n = H \mid_{p_n=h_n=1} \begin{bmatrix} I_{n-1} & \\ & h_n \end{bmatrix} + \begin{bmatrix} O_{n-1} & \\ & d_n - d_n h_n \end{bmatrix} = H, \quad \text{if } i_n = 0,$$
$$H_n = \begin{bmatrix} I_{n-1} & \\ & h_n \end{bmatrix} H \mid_{p_n=h_n=1} + \begin{bmatrix} O_{n-1} & \\ & d_n - d_n h_n \end{bmatrix} = H, \quad \text{if } i_n = 1.$$

□

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