

Stability of certain Jordan bases of real H -selfadjoint matrices

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Abstract

In this paper we prove that the new Jordan bases called γ -flipped orthogonal conjugate symmetric (γ -FOCS) are Lipschitz stable under small perturbations as well. We also establish the Lipschitz stability for the classical real canonical Jordan bases.

Keywords: Canonical Jordan bases, Indefinite inner product, Structure-preserving perturbations

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1. Introduction

The catalyst for this study was a question posed by Professor Leiba Rodman. Namely, *is it possible to have a Lipschitz stability while realizing affiliation relation between a real H -selfadjoint matrix and its real Weierstrass form via a real matrix?* That is all the matrices involved are real.

We were able to find a way to show it. However, we needed to introduce a new type of Jordan canonical bases that was Lipschitz stable. In 2008 Bella, Olshevsky and Prasad proved that the flipped orthogonal (FO) Jordan bases of H -selfadjoint matrices are Lipschitz stable under small perturbations. In 2022, Dogruer, Minenkova and Olshevsky considered the real case, and proved that for real H -selfadjoint matrices there exist a more refined bases called FOCS bases. In addition to flipped orthogonality they also possess the conjugate symmetric (CS) property. All the definitions and the structure of this paper are predated in the next chapter.

2. Preliminaries

2.1. H -selfadjoint matrices and the affiliation relation

Let us start with the definition of **the indefinite inner product**. It is a relation denoted by $[\cdot, \cdot]$ that satisfies all the axioms of the usual inner product except for positivity.

For example, every $n \times n$ invertible Hermitian matrix H determines an indefinite inner product through

$$[x, y]_H = y^* H x, \text{ for } x, y \in \mathbb{C}^n. \quad (2.1)$$

The converse is also true: for any indefinite inner product in \mathbb{C}^n we can find such invertible Hermitian H that (2.1) holds true. If H is positive-definite then $[\cdot, \cdot]_H$ turns into the classical inner product. However, in this paper we do not impose such an assumption.

H-selfadjoint matrices. A matrix A is called **H-selfadjoint** if

$$A = H^{-1} A^* H \text{ (equivalently } [Ax, y]_H = [x, Ay]_H). \quad (2.2)$$

There are many application for such matrices and they have been studied by many authors (e.g. see [4] and many references therein).

It follows from (2.2) that the eigenvalues of H -selfadjoint matrices are symmetric with respect to the real axis. In addition, the conjugate pairs of eigenvalues have the Jordan blocks of the same sizes.

Affiliation relation. For any change of basis

$$x \mapsto u = T^{-1}x, \quad y \mapsto v = T^{-1}y,$$

we have

$$[x, y]_H = y^* H x = (Tv)^* H (Tu) = v^* \overbrace{(T^* H T)}^G u = [u, v]_G.$$

That is, a new congruent matrix $G = T^* H T$ generates the same inner product in the new basis.

Two pairs (A, H) and (B, G) are called **affiliated** if

$$T^{-1} A T = B \text{ and } T^* H T = G.$$

and this equivalence relation is called the **affiliation relation**.

$$(A, H) \mapsto (B, G). \quad (2.3)$$

The affiliation relation preserves selfadjointness in the indefinite inner product, that is if A is H -selfadjoint, then B is G -selfadjoint.

For an H_0 -selfadjoint matrix there exists an invertible matrix T_0 (see e.g. [1, 4, 9] and the references therein), such that $(A_0, H_0) \xrightarrow{T_0} (J, P)$, i.e.

$$T_0^{-1}A_0T_0 = J \quad \text{and} \quad T_0^*H_0T_0 = P,$$

where J is the Jordan Form of A_0 and P has a particularly sparse simple form called *the sip matrix* in [4, Section 5.5], which is uniquely defined up to perturbation of the Jordan blocks in J . This is a very special affiliation relation. It first appeared in the works by Weierstrass [15, 16] (see also [4, Chapter 5] and [9, Chapter 3]). The columns of T_0 form a Jordan basis of A_0 , and this basis was called “canonical” (see [5]) and the pair (J, P) is sometimes called the *Weierstrass form* of A_0 . Since we introduce two more canonical bases, we suggest to call this basis *flipped-orthogonal* (or just FO) basis of (A_0, H_0) in order to distinguish them.

2.2. FO, CS, and FOCS Bases

Here is an example of an H -selfadjoint matrix.

$$A = \begin{bmatrix} 0 & 0 & 0 & -16 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 0 \end{bmatrix} = TJT^{-1}, \quad H = \frac{1}{128} \begin{bmatrix} 0 & 1 & 0 & -12 \\ 1 & 0 & -12 & 0 \\ 0 & -12 & 0 & 80 \\ -12 & 0 & 80 & 0 \end{bmatrix}, \quad (2.4)$$

where $A = H^{-1}A^*H$ and

$$J = \left[\begin{array}{cc|cc} -2i & 1 & 0 & 0 \\ 0 & -2i & 0 & 0 \\ \hline 0 & 0 & 2i & 1 \\ 0 & 0 & 0 & 2i \end{array} \right], \quad P = \left[\begin{array}{cc|cc} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{array} \right], \quad (2.5)$$

and

$$T = \begin{bmatrix} 8i & 4 & -8i & 12 \\ -4 & 4i & -4 & -8i \\ 2i & -1 & -2i & 1 \\ -1 & 0 & -1 & -i \end{bmatrix}. \quad (2.6)$$

Here, (J, P) is the FO form of (A, H) and the columns of $T = [t_i]$ satisfy

$$[t_i, t_j]_H = [e_i, e_j]_P = \delta_{i,4-j},$$

where $\delta_{i,j}$ is the Kronecker symbol. Therefore, the Jordan basis $\{t_1, \dots, t_n\}$ is called **flipped orthogonal** in agreement with its usage in [1].

Note that FO bases exist in the general complex case. However, the matrices (A, H) in our example are real, which suggests the following question.

Are there any canonical bases specific for the case of real matrices? Yes, for example a conjugate symmetric (CS) basis. Let us return to the same matrices A and H are as in (2.4). and consider another Jordan basis for A , captured by the columns of the following matrix.

$$L = \begin{bmatrix} 8i & 4 & -8i & 4 \\ -4 & 4i & -4 & -4i \\ 2i & -1 & -2i & -1 \\ -1 & 0 & -1 & 0 \end{bmatrix}. \quad (2.7)$$

The columns $L = [l_1 \ l_2 \ l_3 \ l_4]$ are conjugate of each other:

$$l_1 = \bar{l}_3 \text{ and } l_2 = \bar{l}_4.$$

Therefore, we say that L captures a **conjugate symmetric basis** of A from (2.4) and $(A, H) \xrightarrow{L} (J, G)$, where

$$G = L^* H L = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -i \\ 0 & 1 & 0 & 0 \\ 1 & i & 0 & 0 \end{bmatrix}$$

is not a sip matrix as was P in (2.6). This means L is CS but not FO. Also, if we look back at the FO basis in T from (2.5), it is not CS.

Now that we have seen the example of the FO and CS canonical bases we started to wonder: *Could we find a Jordan basis for real H -selfadjoint matrices that is flipped orthogonal and γ -conjugate symmetric at the same time?*

Here is a matrix whose columns also capture a Jordan basis of the same A as before.

$$M = \frac{1}{2} \begin{bmatrix} 16i & 16 & -16i & 16 \\ -8 & 12i & -8 & -12i \\ 4i & 0 & -4i & 0 \\ -2 & i & -2 & -i \end{bmatrix},$$

such that $(A, H) \xrightarrow{M} (J, P)$ and M is CS. So M captures a **flipped orthogonal conjugate symmetric basis**.

It is only an example.

Can one guarantee the existence of such bases? The positive answer to this question can be found in [2].

2.3. Lipschitz Stability of FO Bases

Stability of Jordan bases were studied previously, for example, in [1]. The authors proved there the Lipschitz stability of the so-called flipped orthogonal (FO) Jordan bases of *complex* H -selfadjoint matrices under small perturbations.

The following fact was presented in [1, Proposition 1.5].

Proposition 2.1. (Lipschitz stability of FO-bases) *Let $A_0 \in \mathbb{C}^{n \times n}$ be a fixed H_0 -selfadjoint matrix and let the columns of this matrix capture the FO-basis, that is,*

$$(A_0, H_0) \xrightarrow{T_0} (J, P).$$

There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds for any H -selfadjoint matrix A with A similar to A_0 and

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there exists a matrix T whose columns capture the FO-basis of A , i.e.

$$(A, H) \xrightarrow{T} (J, P)$$

and such that

$$\|T - T_0\| \leq K(\|A - A_0\| + \|H - H_0\|). \quad (2.8)$$

2.4. Main results and the structure of the paper

2.4.1. Lipschitz Stability of FOCS Bases

We are going to build on what we mentioned before and show that if the pair (A, H) is real, and if matrix T_0 captures the FO but FOCS basis of (A, H) , there is a matrix T which can be chosen to capture the γ -FOCS basis of (A, H) and still be Lipschitz stable under small perturbation preserving the Jordan structure. That is, we prove the bound (2.8), in which both matrices T_0 and T enjoy the additional CS-property.

2.4.2. Lipschitz stability of RC bases

We refer the reader to [4] for an introduction to real canonical bases.

Let us come back to our example from (2.4). Here is the real Jordan form J_R of A and its real canonical basis such that $R^*HR = P$.

$$J_R = R^{-1}AR = \left[\begin{array}{cc|cc} 0 & -2 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 \end{array} \right] \quad \text{and} \quad R = \begin{bmatrix} -8 & 8 & 8 & 8 \\ -4 & -4 & -6 & 6 \\ -2 & 2 & 0 & 0 \\ -1 & -1 & -0.5 & 0.5 \end{bmatrix}.$$

In this paper, via FOCS bases we were able to obtain results on stability of *real Jordan canonical forms*.

We prove that there exists an invertible matrix S_0 such that

$$S_0^{-1}A_0S_0 = J_R \quad \text{and} \quad S_0^*H_0S_0 = P,$$

where J_R is a real Jordan form of A_0 (see Theorem 6.1 for details) and the columns of S_0 capture a *Real Canonical (RC) basis*. Moreover, if $A \in \mathbb{R}^{n \times n}$ is a perturbation of A_0 , preserving the Jordan structure, that is H -selfadjoint for some self-adjoint real matrix H , then there exists an invertible real matrix S such that

$$S^{-1}AS = J_R \quad \text{and} \quad S^*HS = P,$$

where the columns of S capture RC basis of A and this basis is Lipschitz stable, that is

$$\|S - S_0\| \leq K(\|A - A_0\| + \|H - H_0\|).$$

2.4.3. Weak Affiliation

Now let us clarify what the difference is between *affiliation* and *weak affiliation*.

Definition 2.2. (Weak Affiliation Relation) *Two pairs (A, H) and (B, G) are called **weakly affiliated** if for some invertible matrix T matrices $T^{-1}AT$ and B have the same Jordan basis, and $T^*HT = G$. The relation $(A, H) \xrightarrow{T}_w (B, G)$ is called the **weak affiliation relation**.*

In [1, Theorem 1.8] it was showed that if a small perturbation A of A_0 has the same Jordan structure as A_0 , H is a small perturbation of H_0 , and as before A_0 is H_0 -selfadjoint and A is H -selfadjoint, then an affiliation matrix

S exists, and it can be chosen to be a small perturbation of the identity matrix, and a Lipschitz-type bound

$$\|I - S\| \leq K(\|A - A_0\| + \|H - H_0\|)$$

holds for some positive K depending on A_0 only. More details on the weak affiliation case are presented in Section 6.

2.4.4. *The Structure of the Paper*

For the sake of brevity we used examples of FO and FOCS bases to present our main result. In Section 2 we introduce these concepts rigorously. In [2] we established the existence of FOCS bases. That construction does not seem to be useful to derive our main result, Lipschitz stability. Hence, in Section 3 we present a different proof for the existence of FOCS bases. The new construction allows us to establish the Lipschitz stability in Section 4. In Section 5 the Lipschitz stability of real canonical forms is established. Finally, we slightly generalize the results of Sections 4 and 5 to the weak affiliation case in Section 6.

3. Canonical Bases and Forms

Let us introduce all the characters involved in this story. We have already presented examples of several canonical bases in Section 1. In this section, we formally define FO and CS bases.

3.1. *Flipped-Orthogonal Basis*

This type of bases was presented in [1]. As before, let $J_A = T^{-1}AT$ denote the Jordan canonical form of A , that is

$$J_A = \begin{bmatrix} J(\lambda_1) & & 0 \\ & \ddots & \\ 0 & & J(\lambda_m) \end{bmatrix},$$

where each

$$J(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_k \end{bmatrix}$$

is a single Jordan block. If we partition T correspondingly

$$T = [T_1 \mid \dots \mid T_m];$$

then the columns of T_k form a Jordan chain of A corresponding to $\lambda_k \in \sigma(A)$, and the columns of T form a Jordan basis of A .

Definition 3.1. (Affiliation Relation) *Two pairs (A, H) and (B, G) are called **affiliated** if $T^{-1}AT = B$ and $T^*HT = G$. The relation $(A, H) \xrightarrow{T} (B, G)$ is called the **affiliation relation**.*

Even if there exists an invertible matrix T such that $(A, H) \xrightarrow{T} (B, G)$, and the matrix $B = T^{-1}AT$ is the Jordan form of A , the matrix $G = T^*HT$ is not necessarily simple.

As before we consider H -selfadjoint matrices A . From the definition of H -selfadjoint matrices we know that A is similar to A^* , meaning that nonreal eigenvalues of A come in conjugate pairs. Moreover, the sizes of Jordan blocks of A corresponding to such λ and to $\bar{\lambda}$ are identical. Let us denote by $\lambda_1, \dots, \lambda_\alpha$ all real eigenvalues of H -selfadjoint matrix A , and by $\lambda_{\alpha+1}, \dots, \lambda_\beta$ all eigenvalues of A from the upper half plane. Then the Jordan canonical form of A has the form

$$T^{-1}AT = J_A = \begin{bmatrix} J_1 & & & & & \\ & \ddots & & & & \\ & & J_\alpha & & & \\ & & & \hat{J}_{\alpha+1} & & \\ & & & & \ddots & \\ & & & & & \hat{J}_\beta \end{bmatrix}$$

where J_k is a single Jordan block corresponding to real λ_k as before (for $k = 1, \dots, \alpha$), the matrices

$$\hat{J}_k = \begin{bmatrix} J(\lambda_k) & 0 \\ 0 & J(\bar{\lambda}_k) \end{bmatrix}$$

for $k = \alpha + 1, \dots, \beta$ are direct sums of two Jordan blocks of the same size corresponding to λ_k and $\bar{\lambda}_k$, respectively.

The following fact (see [4, 9]) shows that there is such a matrix T that makes T^*HT sparse.

Proposition 3.2. *Let $A \in \mathbb{C}^{n \times n}$ be a fixed H -selfadjoint matrix for some self-adjoint matrix H and J be its Jordan form. Then there exists an invertible matrix T such that*

$$(A, H) \xrightarrow{T} (J, P) \quad (3.9)$$

where

$$J = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus \hat{J}(\lambda_{\alpha+1}) \oplus \cdots \oplus \hat{J}(\lambda_\beta) \quad (3.10)$$

and $J(\lambda_i)$ is a Jordan block for real eigenvalues $\lambda_1, \dots, \lambda_\alpha$ and

$$\hat{J}(\lambda_k) = \begin{bmatrix} J(\lambda_k) & 0 \\ 0 & J(\bar{\lambda}_k) \end{bmatrix} \quad (3.11)$$

is a direct sum of two Jordan blocks of the same size corresponding to λ_k and $\bar{\lambda}_k$,

$$P = P_1 \oplus \cdots \oplus P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P_\beta \quad (3.12)$$

where P_k is a sip matrix $\epsilon_k \tilde{I}_k$ (i.e. $\begin{bmatrix} 0 & \cdots & 0 & \epsilon_k \\ \vdots & \ddots & \epsilon_k & 0 \\ 0 & \ddots & \ddots & \vdots \\ \epsilon_k & 0 & \cdots & 0 \end{bmatrix}$) of the same size as $J(\lambda_k)$

and $\epsilon_k = \pm 1$ for $k = 1, \dots, \alpha$ and a sip matrix \tilde{I}_k (i.e. $\begin{bmatrix} 0 & \cdots & 0 & 1 \\ \vdots & \ddots & 1 & 0 \\ 0 & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$) of the

same size as $\hat{J}(\lambda_k)$ for $k = \alpha + 1, \dots, \beta$.

This shows us the existence of a canonical basis for which P has a very particular form.

Think back to the example of A and H from (2.4). An illustration to Proposition 3.2 would be matrix T in (4). The following is a formal definition of the basis captured by columns of T .

Definition 3.3. (Flipped-orthogonal Basis) *Let us partition T in (3.9) in the following way*

$$T = [T_1 \dots T_\alpha \mid T_{\alpha+1} \dots T_\beta].$$

Obviously, for $k = 1, \dots, \alpha$ the columns of

$$T_k = [g_{0,k}, \dots, g_{p_k-1,k}]$$

form the Jordan chain of A corresponding to λ_k . Similarly, the columns of

$$T_k = [\underbrace{g_{0,k}, \dots, g_{p_k-1,k}}_{\lambda_k} \mid \underbrace{h_{0,k}, \dots, h_{p_k-1,k}}_{\bar{\lambda}_k}]$$

for $k = \alpha + 1, \dots, \beta$ form two Jordan chains, corresponding to λ_k and $\bar{\lambda}_k$ respectively. Then the structure of the matrix P in (3.12) implies the following orthogonality relation,

$$g_{ik}^* H g_{jm} = 0, \quad h_{ik}^* H g_{jm} = 0, \quad h_{ik}^* H h_{jm} = 0, \quad \text{for } k \neq m.$$

Further, we have that

$$g_{i,k}^* H g_{j,k} = \begin{cases} \epsilon_k, & j = p_k - 1 - i \\ 0 & \text{otherwise} \end{cases} \quad \text{for } k = 1, \dots, \alpha$$

$$\begin{cases} g_{ik}^* H g_{jk} = 0, \\ h_{ik}^* H h_{jk} = 0, \\ h_{i,k}^* H g_{j,k} = \begin{cases} 1, & j = p_k - 1 - i \\ 0 & \text{otherwise} \end{cases} \end{cases} \quad \text{for } k = \alpha + 1, \dots, \beta.$$

Jordan bases, having the above property, are called *flipped-orthogonal (FO)*.

3.2. γ -Conjugate Symmetric Bases

If $A \in \mathbb{R}^{n \times n}$ and $\{q_k\}_{k=0}^{m-1}$ is its Jordan chain corresponding to a nonreal eigenvalue λ then $\{\gamma \bar{q}_k\}_{k=0}^{m-1}$ is also a Jordan chain of A but corresponding to $\bar{\lambda}$ for any non-zero $\gamma \in \mathbb{C}$. This observation leads to the following definition.

Definition 3.4. (γ -Conjugate Symmetric Basis) Suppose $A \in \mathbb{R}^{n \times n}$ and that there exists an invertible matrix N such that $A = N J N^{-1}$, where

$$J = J(\lambda_1) \oplus \dots \oplus J(\lambda_\alpha) \oplus \hat{J}(\lambda_{\alpha+1}) \oplus \dots \oplus \hat{J}(\lambda_\beta) \quad (3.13)$$

is the Jordan canonical form with $\hat{J}(\lambda_k)$ defined in (3.10) and (3.11), the matrix

$$N = [N_1 | \dots | N_\alpha | N_{\alpha+1} | \dots | N_\beta] \quad (3.14)$$

can be chosen

$$N_k = [Q_k | \gamma \bar{Q}_k] \quad (3.15)$$

for some $\gamma \neq 0$ and $k = \alpha + 1, \dots, \beta$. The columns of N capture the Jordan basis of A , and because of (3.15) we call it the γ -Conjugate Symmetric Basis.

Now that we have introduced two canonical bases, FO and γ -CS, we might ask: *Is there a basis for real H -selfadjoint matrices that is both flipped orthogonal and γ -conjugate symmetric?* The answer to this question is in our next section.

4. Existence of a γ -FOCS Basis

The FO basis was defined for pairs of matrices (A, H) , where A and H are not necessarily real. The γ -CS basis is just defined for the case of A being a real matrix.

Proposition 3.2 implies the existence of the FO basis for (A, H) , the latter, generally, does not have the CS property. Theorem 4.1 shows that if (A, H) is real, then there exists a γ -FOCS basis, that is, the one, which is simultaneously FO and γ -CS.

Theorem 4.1. (Existence of a γ -FOCS basis) *Let all the assumptions of Proposition 3.2 hold true. Assume, in addition, that both matrices A and H are real. Then N in $(A, H) \xrightarrow{N} (J, P)$ can be chosen such that if we partition it in the agreement with (3.14) then each N_k has the form*

$$N_k = [L_k | \gamma \bar{L}_k]$$

for some matrices L_k for $k = \alpha + 1, \dots, \beta$.

In other words, the columns of N form not only FO, but also a γ -CS basis. Theorem 4.1 was proven in [2]. Below we need to provide it with a different proof, since the notations and details of this new proof will be used in derivation of our main result, the Lipschitz stability.

To establish that we first need to prove the following lemma.

Lemma 4.2. *Let*

$$A = \begin{bmatrix} J_m(\lambda) & 0 \\ 0 & J_m(\bar{\lambda}) \end{bmatrix},$$

where $J_m(\lambda)$ is an $m \times m$ Jordan block with an eigenvalue $\lambda \notin \mathbb{R}$. If A is \tilde{G} -selfadjoint, then \tilde{G} has the following structure

$$\tilde{G} = \begin{bmatrix} 0 & G^* \\ G & 0 \end{bmatrix}, \quad \text{where} \quad G = \begin{bmatrix} 0 & \dots & 0 & g_0 \\ \vdots & \ddots & g_0 & g_1 \\ 0 & \ddots & \ddots & \vdots \\ g_0 & g_1 & \dots & g_{m-1} \end{bmatrix}, \quad (4.16)$$

with some coefficients g_j 's.

Proof. Recall that $\{e_k\}$ form a Jordan chain of A means

$$(A - \lambda I)e_1 = 0, \quad (A - \lambda I)e_{k+1} = e_k \quad k = 1, \dots, m-1 \quad (4.17)$$

and

$$(A - \bar{\lambda} I)e_{m+1} = 0, \quad (A - \bar{\lambda} I)e_{k+1} = e_k \quad k = m+1, \dots, 2m-1. \quad (4.18)$$

Let

$$\tilde{G} = \begin{bmatrix} X & Y \\ Z & U \end{bmatrix},$$

where $X = \{[e_i, e_j]\}_{i,j=1}^m$, $U = \{[e_i, e_j]\}_{i,j=m+1}^{2m}$, $Z = \{[e_i, e_j]\}_{i=1, j=m+1}^{m, 2m}$ and $Y = Z^*$ by conjugate symmetry of indefinite inner products.

Note that X is the zero matrix. Using (4.17), we have

$$\begin{aligned} \lambda[e_1, e_1] &= [\lambda e_1, e_1] = [Ae_1, e_1] = [e_1, Ae_1] = [e_1, \lambda e_1] \\ &= \bar{\lambda}[e_1, e_1] \Rightarrow [e_1, e_1] = 0, \end{aligned}$$

since $\lambda \neq \bar{\lambda}$. Moreover, for any $k = 1, \dots, m-1$

$$\begin{aligned} (\bar{\lambda} - \lambda)[e_{k+1}, e_1] &= [e_{k+1}, (\lambda - \bar{\lambda})e_1] = [e_{k+1}, (A - \bar{\lambda} I)e_1] \\ &= [(A - \lambda I)e_{k+1}, e_1] = [e_k, e_1] \end{aligned} \quad (4.19)$$

and

$$\begin{aligned} (\lambda - \bar{\lambda})[e_1, e_{k+1}] &= [(\lambda - \bar{\lambda})e_1, e_{k+1}] = [(A - \bar{\lambda} I)e_1, e_{k+1}] \\ &= [e_1, (A - \lambda I)e_{k+1}] = [e_1, e_k]. \end{aligned} \quad (4.20)$$

Keeping in mind that $\lambda \neq \bar{\lambda}$ and $[e_1, e_1] = 0$, we get $[e_1, e_k] = [e_k, e_1] = 0$ by induction from (4.19) and (4.20) for $k = 1, \dots, m$.

Suppose $[e_{i-1}, e_j] = 0$ and $[e_i, e_{j-1}] = 0$ for some $i, j = 2, \dots, m$. Next,

$$\begin{aligned} \lambda[e_i, e_j] &= [\lambda e_i, e_j] = [Ae_i - e_{i-1}, e_j] = [Ae_i, e_j] - [e_{i-1}, e_j] = [e_i, Ae_j] = \\ &= [e_i, \lambda e_j + e_{j-1}] = [e_i, \lambda e_j] - [e_i, e_{j-1}] = \bar{\lambda}[e_i, e_j] \quad \text{or} \quad [e_i, e_j] = 0, \end{aligned}$$

since $\lambda \neq \bar{\lambda}$. Thus, by induction on i and j we have $X = 0$.

The fact that $U = 0$ can be proved using similar argument.

Next, let us see what happens to

$$Z = \begin{bmatrix} [e_1, e_{m+1}] & [e_1, e_{m+2}] & \cdots & [e_1, e_{2m}] \\ [e_2, e_{m+1}] & [e_2, e_{m+2}] & \cdots & [e_2, e_{2m}] \\ \vdots & \vdots & \ddots & \vdots \\ [e_m, e_{m+1}] & [e_m, e_{m+2}] & \cdots & [e_m, e_{2m}] \end{bmatrix}. \quad (4.21)$$

Applying (4.18) for $k = 1, 2, \dots, m-1$ we get

$$[e_k, e_{m+1}] = [(A - \lambda I)e_{k+1}, e_{m+1}] = [e_{k+1}, (A - \bar{\lambda}I)e_{m+1}] = [e_{k+1}, 0] = 0 \quad (4.22)$$

$$[e_m, e_{m+1}] = [e_m, (A - \bar{\lambda}I)e_{m+2}] = [(A - \lambda I)e_m, e_{m+2}] = [e_{m-1}, e_{m+2}]. \quad (4.23)$$

This means that matrix Z has the Hankel structure as shown in (4.16). We have

$$[e_k, e_j] = [(A - \lambda I)e_{k+1}, e_j] = [e_{k+1}, (A - \bar{\lambda}I)e_j] = [e_{k+1}, e_{j-1}] \quad (4.24)$$

for $k = 1, \dots, m-1$ and $j = m+2, \dots, 2m$. The matrix in (4.21) is Hankel and becomes G from (4.16) by using (4.22)–(4.24) and by letting $[e_k, e_{2m}] = g_{k-1}$. \square

We are at the point now to prove the main result of this section. We are going to construct the columns of matrix N in Theorem 4.1 in a way that it captures the γ -FOCS basis of (A, H) . This matrix $N = Z_1 Z_2 Z_3 Z_4$ is constructed in four steps, as described in Table 1 below.

Constructing Z_1, Z_2, Z_3, Z_4	$(A, H) \xrightarrow{Z_1} (J, \tilde{G}) \xrightarrow{Z_2} (J, \tilde{G}_1) \xrightarrow{Z_3} (J, \tilde{G}_2) \xrightarrow{Z_4} (J, P)$
Step 1 (constructing Z_1)	$(A, H) \xrightarrow{Z_1} (J, \tilde{G})$
Step 2 (constructing Z_2)	$(A, H) \mapsto \xrightarrow{Z_1} \xrightarrow{Z_2} \mapsto (J, \tilde{G}_1)$
Step 3 (constructing Z_3)	$(A, H) \mapsto \xrightarrow{Z_1} \xrightarrow{Z_2} \xrightarrow{Z_3} \mapsto (J, \tilde{G}_2)$
Step 4 (constructing Z_4)	$(A, H) \mapsto \xrightarrow{Z_1} \xrightarrow{Z_2} \xrightarrow{Z_3} \xrightarrow{Z_3=N} \mapsto (J, P)$

Table 1: Constructing N

We will construct it in a way that the columns of each of four matrices $Z_1, Z_1 Z_2, Z_1 Z_2 Z_3, Z_1 Z_2 Z_3 Z_4$ capture the γ -CS bases of A , and columns of the last matrix $N = Z_1 Z_2 Z_3 Z_4$ capture not only γ -CS, but also the desired γ -FOCS basis.

Proof of the Theorem 4.1. Let us first introduce the following notation that we will use throughout this proof. We denote by I_k (resp. \tilde{I}_k) the identity (resp. sip) matrix of the size of $J(\lambda_k)$ from (3.13).

Now let us go through the procedure step-by-step:

Step 1 (Mapping $A \rightarrow J$. Constructing Z_1) We know from Proposition 3.2 that there exists T whose columns capture the FO-basis of (A, H) , that is $(A, H) \xrightarrow{T} (J, P)$. Let us partition

$$T = [T_1 | \dots | T_\alpha | T_{\alpha+1} | \dots | T_\beta]$$

in correspondence with (3.13). The columns of T do not necessarily form a γ -CS basis yet, that is

$$T_k = [L_k | S_k] \quad \text{for} \quad k = \alpha + 1, \dots, \beta$$

with some matrices L_k and S_k possibly not related to each other. Now, define

$$Z_1 = [T_1 | \dots | T_\alpha | M_{\alpha+1} | \dots | M_\beta], \quad (4.25)$$

where

$$M_k = [L_k | \gamma \bar{L}_k] \quad \text{for} \quad k = \alpha + 1, \dots, \beta. \quad (4.26)$$

Thus, we got Z_1 for $(A, H) \xrightarrow{Z_1} (J, \tilde{G})$. Moreover, by using the same partition of \tilde{G} as in (3.12), for each block corresponding to non-real eigenvalues we observed the form as in (4.16) for $k = \alpha + 1, \dots, \alpha + \beta$

$$\tilde{G}_k = \begin{bmatrix} 0 & G_k^* \\ G_k & 0 \end{bmatrix}, \quad \text{where} \quad G_k = \begin{bmatrix} 0 & \dots & 0 & g_0^{(k)} \\ \vdots & \ddots & g_0^{(k)} & g_1^{(k)} \\ 0 & \ddots & \ddots & \vdots \\ g_0^{(k)} & g_1^{(k)} & \dots & g_{k-1}^{(k)} \end{bmatrix}$$

and \tilde{G}_k is the sip matrix $\epsilon_k \tilde{I}_k$ ($\epsilon = \pm 1$) for $k = 1, \dots, \alpha$.

By construction the columns of matrix Z_1 capture the γ -CS basis of A . However, it is not necessarily the FO basis yet.

Step 2 (Forcing the antidiagonal of \tilde{G}_1 to be real. Constructing Z_2) During this step, we use the argument and modulus of $g_0^{(k)} = e^{i\varphi_k} r_k$ for $k = \alpha + 1, \dots, \alpha + \beta$ to define

$$Z_2 = I_1 \oplus \dots \oplus I_\alpha \oplus Z_{\alpha+1}^{(2)} \oplus \dots \oplus Z_\beta^{(2)},$$

where

$$Z_k^{(2)} = \begin{bmatrix} e^{\frac{-i\varphi_k}{2}} \sqrt{\frac{s_k}{r_k}} I & 0 \\ 0 & e^{\frac{i\varphi_k}{2}} \sqrt{\frac{s_k}{r_k}} I \end{bmatrix}, \quad (4.27)$$

with $s_k = |\operatorname{Re} g_0^{(k)}|$. Then,

$$Z_2^* \widetilde{G} Z_2 = \widetilde{G}_1, \quad Z_2^{-1} J Z_2 = J,$$

and the corresponding to (3.13) partition of \widetilde{G}_1 is such that $\widetilde{G}_k^{(1)}$'s are unchanged for $k = 1, \dots, \alpha$ and $\widetilde{G}_k^{(1)} = \begin{bmatrix} 0 & (G_k^{(1)})^* \\ G_k^{(1)} & 0 \end{bmatrix}$ where $G_k^{(1)} =$

$$e^{-i\varphi_k \frac{s_k}{r_k}} G_k = \begin{bmatrix} 0 & \dots & 0 & s_k \\ \vdots & \ddots & s_k & * \\ 0 & \ddots & \ddots & \vdots \\ s_k & * & \dots & * \end{bmatrix} \text{ for } k = \alpha + 1, \dots, \alpha + \beta.$$

If we put Z_1 and Z_2 together, i.e.

$$Z_1 Z_2 = [T_1 | \dots | T_\alpha | M_{\alpha+1}^{(0)} | \dots | M_\beta^{(0)}].$$

we get that

$$M_k^{(0)} = [e^{\frac{i\varphi_k}{2}} \sqrt{\frac{s_k}{r_k}} L_k | \gamma e^{\frac{-i\varphi_k}{2}} \sqrt{\frac{s_k}{r_k}} \bar{L}_k] =: [Q_k^{(0)} | \gamma \bar{Q}_k^{(0)}] \quad \text{for } k = \alpha + 1, \dots, \beta, \quad (4.28)$$

where $Q_k^{(0)} = e^{\frac{i\varphi_k}{2}} \sqrt{\frac{s_k}{r_k}} L_k$. As we see from (4.28), the columns $Z_1 Z_2$ still capture the γ -CS basis of A .

Step 3 (Making G_1 the unit-antidiagonal. Constructing Z_3) This time we just scale all the columns. We use the result from Step 2 to define

$$Z_3 = I_1 \oplus \dots \oplus I_\alpha \oplus Z_{\alpha+1}^{(3)} \oplus \dots \oplus Z_\beta^{(3)},$$

where $Z_k^{(3)} = \frac{1}{\sqrt{s_k}} I$ for $k = \alpha + 1, \dots, \alpha + \beta$. That is we simply scaling \widetilde{G}_1 . Then

$$Z_3^* \widetilde{G}_1 Z_3 = \widetilde{G}_2, \quad Z_3^{-1} J Z_3 = J,$$

where as before we have the block structure of \widetilde{G}_2 such that $\widetilde{G}_k^{(2)}$ being a sip matrix of the same size as $J(\lambda_k)$ for $k = 1, \dots, \alpha$ and $\widetilde{G}_k^{(2)} =$

$$\begin{bmatrix} 0 & (G_k^{(2)})^* \\ G_k^{(2)} & 0 \end{bmatrix} \text{ with } G_k^{(2)} = \begin{bmatrix} 0 & \dots & 0 & 1 \\ \vdots & \ddots & 1 & * \\ 0 & \ddots & \ddots & \vdots \\ 1 & * & \dots & * \end{bmatrix} \text{ for } k = \alpha + 1, \dots, \alpha + \beta.$$

We constructed such matrix Z_3 in order to have

$$Z_1 Z_2 Z_3 = [T_1 | \dots | T_\alpha | M_{\alpha+1}^{(1)} | \dots | M_\beta^{(1)}].$$

where we get the following form for $M_k^{(1)}$'s

$$M_k^{(1)} = [\frac{1}{\sqrt{s}} Q_k^{(0)} | \gamma \frac{1}{\sqrt{s}} \overline{Q}_k^{(0)}] =: [Q_k^{(1)} | \gamma \overline{Q}_k^{(1)}],$$

by letting $Q_k^{(1)} = \frac{e^{\frac{i\varphi_k}{2}}}{\sqrt{r_k}} L_k$. Then, we have $(J, \widetilde{G}_1) \xrightarrow{Z_3} (J, \widetilde{G}_2)$. Moreover, $(A, H) \xrightarrow{Z_1 Z_2 Z_3} (J, \widetilde{G}_2)$. Therefore, the columns of $Z_1 Z_2 Z_3$ capture the γ -CS basis of A but as before we cannot guarantee it to be FO yet. The flipped orthogonality property is still missing.

Step 4 (Zeroing sub-antidiagonal entires of G_2). Constructing Z_4

Now, we look for a matrix Z_4 such that $N = Z_1 Z_2 Z_3 Z_4$ has all the properties formulated in the theorem. Let us define Z_4 in the following way

$$Z_4 = I_1 \oplus \dots \oplus I_\alpha \oplus Z_{\alpha+1}^{(4)} \oplus \dots \oplus Z_{\alpha+\beta}^{(4)},$$

where $Z_k^{(4)} = \begin{bmatrix} F_k & 0 \\ 0 & \overline{F}_k \end{bmatrix}$ for $k = \alpha + 1, \dots, \alpha + \beta$. The exact formulas for F_k 's will be found later. Then, once again, we get that

$$Z_1 Z_2 Z_3 Z_4 = [T_1 | \dots | T_\alpha | M_{\alpha+1}^{(2)} | \dots | M_\beta^{(2)}],$$

where

$$M_k^{(2)} = [Q_k^{(1)} | \gamma \overline{Q}_k^{(1)}] \begin{bmatrix} F_k & 0 \\ 0 & \overline{F}_k \end{bmatrix} = [Q_k^{(1)} F_k | \gamma \overline{Q}_k^{(1)} \overline{F}_k] = [Q_k^{(2)} | \gamma \overline{Q}_k^{(2)}] \quad (4.29)$$

with $Q_k^{(2)} = e^{\frac{i\varphi_k}{2}} \frac{1}{\sqrt{r_k}} L_k F_k$ will have the desired γ -CS form with any invertible F_k . In addition, we want F_k to satisfy

$$(Z_k^{(4)})^{-1} \begin{bmatrix} J(\lambda_k) & 0 \\ 0 & J(\overline{\lambda}_k) \end{bmatrix} Z_k^{(4)} = \begin{bmatrix} J(\lambda_k) & 0 \\ 0 & J(\overline{\lambda}_k) \end{bmatrix}. \quad (4.30)$$

The relation in (4.30) is satisfied if F_k is an upper triangular Toeplitz matrix that commutes with $J(\lambda_k)$. Furthermore, an immediate computation shows that

$$\begin{aligned} Z_k^{(4)*} \widetilde{G}_k^{(2)} Z_k^{(4)} &= \begin{bmatrix} F_k^* & 0 \\ 0 & F_k^T \end{bmatrix} \begin{bmatrix} 0 & (G_k^{(2)})^* \\ G_k^{(2)} & 0 \end{bmatrix} \begin{bmatrix} F_k & 0 \\ 0 & \overline{F}_k \end{bmatrix} \\ &= \begin{bmatrix} 0 & F_k^* G_k^{(2)*} \overline{F}_k \\ F_k^T G_k^{(2)} F_k & 0 \end{bmatrix} = \begin{bmatrix} 0 & (F_k^T G_k^{(2)} F_k)^* \\ F_k^T G_k^{(2)} F_k & 0 \end{bmatrix}. \end{aligned}$$

Hence, if F_k satisfies

$$F_k^T G_k^{(2)} F_k = \widetilde{I}_k \quad (4.31)$$

for $k = \alpha + 1, \dots, \alpha + \beta$, then

$$Z_4^* \widetilde{G}_2 Z_4 = P$$

and we are done.

Therefore, it remains to solve (4.31) for an upper triangular Toeplitz matrix F_k , and the proof will be complete. Since $\widetilde{I}^2 = I$, we have

$$F_k^T G_k^{(2)} \widetilde{I}_k \widetilde{I}_k F_k \widetilde{I}_k = \widetilde{I}_k \widetilde{I}_k = I_k, \quad \text{then} \quad \underbrace{F_k^T}_F \underbrace{G_k^{(2)} \widetilde{I}_k \widetilde{I}_k}_{G_3} \underbrace{F_k \widetilde{I}_k}_F = I_k.$$

Since the lower triangular Toeplitz matrices F and G_3 commute, we have

$$F^2 G_3 = I_k.$$

We see that F^2 is the inverse of the unit lower triangular Toeplitz matrix G_3 . Hence, F can be chosen to be a unit lower triangular Toeplitz matrix. Thus, $F_k = F^\top$ is a unit upper triangular Toeplitz matrix and, hence, it commutes with $J(\lambda_k)$.

Step 5 (Combining the affiliation matrices Z_1, Z_2, Z_3 , and Z_4) By combining all the previous steps, we get the desired conclusion

$$(A, H) \xrightarrow{N} (J, P)$$

where $N = Z_1 Z_2 Z_3 Z_4$. In other words, the columns of N form a flipped-orthogonal Jordan basis. Furthermore, (4.29) shows that this Jordan basis is also γ -conjugate-symmetric. This concludes the proof of Theorem 4.1. \square

5. Stability of γ -FOCS Canonical Forms and Bases

Now, we are ready to state the main result of this section.

Theorem 5.1. (Lipschitz stability of γ -FOCS basis) *Let $A_0 \in \mathbb{R}^{n \times n}$ be a fixed H_0 -selfadjoint matrix, where $H_0 \in \mathbb{R}^{n \times n}$, and let the columns of this matrix capture the FOCS-basis, that is,*

$$(A_0, H_0) \xrightarrow{T_0} (J, P).$$

There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds for any H -selfadjoint matrix A with the pair (A, H) real and A invertible and similar to A_0 , and

$$\|A - A_0\| + \|H - H_0\| < \delta, \quad (5.32)$$

there is an invertible matrix $N \in \mathbb{R}^{n \times n}$, whose columns capture the γ -FOCS basis of A , i.e.

$$(A, H) \xrightarrow{N} (J, P),$$

and such that

$$\|N - T_0\| \leq K(\|A - A_0\| + \|H - H_0\|). \quad (5.33)$$

This theorem says that γ -FOCS bases are Lipschitz stable under small perturbations, preserving the Jordan structure of A_0 . To prove Theorem 5.1 we need a technical result stated in Lemma 5.2 and in this proof we need to modify the notations as described next.

Θ – Calculus

In this paper we use and prove a large number of inequalities with the right-hand side

$$K(\|A - A_0\| + \|H - H_0\|)$$

with some constant $K > 0$. Each of these inequalities hold for its own K implying cumbersome notations (introducing large number of K_i 's and the necessity to explain at each step that those K_i 's are different). To avoid it we introduce the notation

$$\Theta = K(\|A - A_0\| + \|H - H_0\|),$$

where we do not keep track of the actual value of K . Though it is not quite rigorous (e.g, $3\Theta \leq 2\Theta$ is a valid inequality) but this does not lead to a confusion, and greatly simplifies the presentation. However, one has to always keep in mind that in different inequalities Θ 's may invoke different values of K .

Lemma 5.2. *Let T_0 and N be given matrices such that $N = Z_1 Z_2 Z_3 Z_4$ with some Z_1, Z_2, Z_3, Z_4 . For any H -selfadjoint matrix A , having the same Jordan structure as A_0 , if*

$$\|Z_1 - T_0\| < \Theta, \quad (5.34)$$

$$\|Z_k - I\| < \Theta \quad \text{for } k = 2, 3, 4, \quad (5.35)$$

then we have

$$\|N - T_0\| < \Theta.$$

Proof. Suppose that (5.34) and (5.35) are true. Obviously, the chain of inequalities

$$\|Z_1\| = \|Z_1 - T_0 + T_0\| \leq \|Z_1 - T_0\| + \|T_0\| < \Theta + \|T_0\|$$

holds true due to (5.34). Now, using inequalities (5.34) and (5.35) again, we get

$$\begin{aligned} \|N - T_0\| &= \|Z_1 Z_2 Z_3 Z_4 - T_0\| = \|Z_1 Z_2 Z_3 Z_4 - Z_1 Z_2 Z_3 + Z_1 Z_2 Z_3 - Z_1 Z_2 + Z_1 Z_2 \\ &\quad - Z_1 + Z_1 - T_0\| = \|Z_1 Z_2 Z_3(Z_4 - I)\| + \|Z_1 Z_2(Z_3 - I)\| + \|Z_1(Z_2 - I)\| \\ &\quad + \|Z_1 - T_0\| \leq \|Z_1 Z_2 Z_3\| \|Z_4 - I\| + \|Z_1 Z_2\| \|Z_3 - I\| + \|Z_1\| \|Z_2 - I\| \\ &\quad + \|Z_1 - T_0\| = \|Z_1 Z_2 Z_3 - Z_1 Z_2 + Z_1 Z_2 - Z_1 + Z_1\| \|Z_4 - I\| + \|Z_1 Z_2 - Z_1 \\ &\quad + Z_1\| \|Z_3 - I\| + \|Z_1\| \|Z_2 - I\| + \|Z_1 - T_0\| \leq (\|Z_1 Z_2(Z_3 - I)\| \\ &\quad + \|Z_1(Z_2 - I)\| + \|Z_1\|) \|Z_4 - I\| + (\|Z_1(Z_2 - I)\| + \|Z_1\|) \|Z_3 - I\| \\ &\quad + \|Z_1\| \|Z_2 - I\| + \|Z_1 - T_0\| < (\|Z_1 Z_2\| \|Z_3 - I\| + \|Z_1\| \|Z_2 - I\| \\ &\quad + \|Z_1\|) \Theta + (\|Z_1\| \|Z_2 - I\| + \|Z_1\|) \Theta + \|Z_1\| \Theta + \Theta \\ &< (\|Z_1 Z_2 - Z_1 + Z_1\| \Theta + \|Z_1\| \Theta + \|Z_1\|) \Theta + (\|Z_1\| \Theta + \|Z_1\|) \Theta \\ &\quad + \|Z_1\| \Theta + \Theta = (\|Z_1(Z_2 - I) + Z_1\| \Theta + \|Z_1\| \Theta + \|Z_1\|) \Theta \\ &\quad + (\|Z_1\| \Theta + \|Z_1\|) \Theta + \|Z_1\| \Theta + \Theta < ((\|Z_1\| \Theta + \|Z_1\|) \Theta + \|Z_1\| \Theta \end{aligned}$$

$$\begin{aligned}
& + \|Z_1\|) \Theta + (\|Z_1\| \Theta + \|Z_1\|) \Theta + \|Z_1\| \Theta + \Theta \\
& < \Theta (\|T_0\| + \Theta) \cdot (\Theta^2 + 3\Theta + 1) + 1 \leq \Theta.
\end{aligned}$$

□

Now we are in a position to prove Theorem 5.1.

Plan of the proof of Theorem 5.1: In fact, the desired matrix N satisfying (5.33) has already been designed in the proof of Theorem 4.1. It was constructed there in four steps with matrix Z_k chosen in the k th step (see Table 1). N was given by $N = Z_1 Z_2 Z_3 Z_4$. Here, we repeat these steps, preserving all the notations from the proof of Theorem 4.1, and prove the inequalities (5.34) and (5.35) for each step. In view of Lemma 5.2 this will completely prove Theorem 5.1.

Proof of Theorem 5.1. We are ready to proceed with the 1st step of this plan.

Stability of Step 1. Again, let us consider $(A, H) \xrightarrow{T} (J, P)$ and $(A_0, H_0) \xrightarrow{T_0} (J, P)$. Proposition 2.1 yields that T can be chosen such that

$$\|T - T_0\| < \Theta. \quad (5.36)$$

Let us partition T_0 and T in correspondence with (3.13)

$$\begin{aligned}
T &= [T_1 | \dots | T_\alpha | T_{\alpha+1} | \dots | T_\beta], \\
T_0 &= [T_1^{(0)} | \dots | T_\alpha^{(0)} | T_{\alpha+1}^{(0)} | \dots | T_\beta^{(0)}].
\end{aligned}$$

Recall that T_0 captures the γ -FOCS basis of A_0 , while T , according to Proposition 2.1, captures only the FO-basis not necessarily γ -CS. Hence,

$$T_k^{(0)} = [L_k^{(0)} | \gamma \overline{L_k}^{(0)}], \quad \text{while} \quad T_k = [L_k | S_k] \quad \text{for} \quad k = \alpha + 1, \dots, \beta$$

with some possibly not related to each other matrices L_k and S_k .

According to Proposition 2.1

$$\|L_k - L_k^{(0)}\| \leq \|T - T_0\| \quad \text{and} \quad \|S_k - \gamma \overline{L_k}^{(0)}\| \leq \|T - T_0\|.$$

The former and (5.36) imply that the matrix Z_1 defined in (4.25) satisfies

$$\|Z_1 - T_0\| \leq \Theta. \quad (5.37)$$

Note that

$$\begin{aligned}\|\tilde{G} - P\| &= \|Z_1^* H Z_1 - T_0^* H_0 T_0\| \leq \|Z_1 - T_0\| \|H\| \|Z_1\| + \|T_0\| \|H - H_0\| \|Z_1\| + \\ &\quad + \|T_0\| \|H_0\| \|Z_1 - T_0\| \leq \Theta.\end{aligned}\tag{5.38}$$

The first inequality in Lemma 5.2 which is (5.34) holds true. Let us check the rest.

Stability of Step 2. Our goal is to prove the following inequality for Z_2 defined in (4.27)

$$\|I - Z_2\| \leq \Theta.\tag{5.39}$$

From the way we defined Z_2 , we got

$$\|I - Z_2\| \leq \max_k \left\{ \left| 1 - e^{-\frac{i\varphi_k}{2}} \sqrt{\frac{s_k}{r_k}} \right|, \left| 1 - e^{\frac{i\varphi_k}{2}} \sqrt{\frac{s_k}{r_k}} \right| \right\}.$$

The next part is analogous to the argument in [1, Lemma 4.5]. We have

$$\begin{aligned}\left| 1 - e^{-\frac{i\varphi_k}{2}} \sqrt{\frac{s_k}{r_k}} \right| &= \left| \frac{\sqrt{r_k} e^{i\varphi_k} - \sqrt{s_k}}{\sqrt{r_k} e^{i\varphi_k}} \right| = \left| \frac{\sqrt{g_0^{(k)}} - \sqrt{s_k}}{\sqrt{g_0^{(k)}}} \right| = \left| \frac{g_0^{(k)} - s_k}{g_0^{(k)} + \sqrt{g_0^{(k)}} s_k} \right| \\ &\leq \frac{|g_0^{(k)} - 1|}{|g_0^{(k)}|} \leq \frac{1}{|g_0^{(k)}|} \|\tilde{G} - P\| \leq \Theta.\end{aligned}\tag{5.40}$$

The last inequality in (5.40) follows from (5.38). Therefore, we see that the desired inequality for Z_2 (5.39) is now proven.

Stability of Step 3. During this step, we need to prove the following inequality

$$\|I - Z_3\| \leq \Theta.\tag{5.41}$$

We constructed Z_3 so

$$\|I - Z_3\| \leq \max_k \left\| I_k - \frac{1}{\sqrt{s_k}} I_k \right\| \leq \max_k \left| 1 - \frac{1}{\sqrt{s_k}} \right|.$$

Note that s_k is a real number. Hence we get

$$\left| 1 - \frac{1}{\sqrt{s_k}} \right| = \left| \frac{\sqrt{s_k} - 1}{\sqrt{s_k}} \right| = \left| \frac{s_k - 1}{\sqrt{s_k}(\sqrt{s_k} + 1)} \right| = \left| \frac{s_k - 1}{s_k + \sqrt{s_k}} \right| \leq \frac{|s_k - 1|}{|s_k|}$$

$$\leq \frac{|g_0^{(k)} - 1|}{|s_k|} \leq \frac{\|\tilde{G} - P\|}{|s_k|} \leq \Theta.$$

Thus, (5.41) holds true.

Stability of Step 4. To prove the bound on $\|I - Z_4\|$, we observe first that

$$\begin{aligned} \|\tilde{G}_1 - P\| &= \|Z_2\tilde{G}Z_2 - P\| \leq \|Z_2^*\tilde{G}Z_2 - Z_2^*PZ_2 + Z_2^*PZ_2 - Z_2^*P \\ &+ Z_2^*P - P\| \leq \|Z_2^*(\tilde{G} - P)Z_2\| + \|Z_2^*P(Z_2 - I)\| + \|Z_2 - I\|\|P\| \\ &\leq \|Z_2\|^2\|\tilde{G} - P\| + (\|Z_2\| + 1)\|P\|\|Z_2 - I\| \\ &< ((1 + \Theta)^2 + (2 + \Theta)^2)\Theta \leq \Theta \end{aligned} \quad (5.42)$$

follows from (5.39).

Also, note that \tilde{G}_2 is a lower anti-triangular Hankel matrix with 1's on the main antidiagonal. Thus, $\tilde{G}_2\tilde{I} = I + E$, where E is a nilpotent matrix with $E^n = 0$ and $\|E\| \leq M$ for some $n \in \mathbb{N}$ and $M > 1$. Define F as follows.

$$F = f(E) = \sum_{k=0}^{n-1} \frac{(-1)^k}{2k} \left(\prod_{j=1}^k \frac{1+2j}{2j} \right) E^k$$

where $f(E) = \sqrt{(I + E)^{-1}}$. Note that $E = \tilde{G}_2\tilde{I} - I$. So,

$$\|I - Z_4\| \leq \|I - F\| \leq \sum_{k=1}^{n-1} \|E\|^k \leq \sum_{k=1}^{n-1} (2M)^{k-1} \|E\| \leq (n-1)(2M)^{n-2} \|E\|.$$

By letting $C = (n-1)(2M)^{n-2}$ and using (5.42), we get

$$\begin{aligned} \|I - Z_4\| &\leq C\|E\| = C\|E\tilde{I}\| = C\|\tilde{G}_2 - P\| = C\|Z_3^*\tilde{G}_1Z_3 - Z_3^*PZ_3 \\ &+ Z_3^*PZ_3 - Z_3^*P + Z_3^*P - P\| \leq C(\|Z_3^*(\tilde{G}_1 - P)Z_3\| + \|Z_3^*P(Z_3 - I)\| \\ &+ \|Z_3 - I\|\|P\|) \leq C(\|Z_3\|^2\|\tilde{G}_1 - P\| + (\|Z_3\| + 1)\|P\|\|Z_3 - I\|) \\ &< ((1 + \Theta)^2 + (2 + \Theta)^2)\Theta \leq \Theta. \end{aligned}$$

Therefore, we have the last inequality we needed

$$\|I - Z_4\| \leq \Theta. \quad (5.43)$$

Step 5 (Stability of N). Finally, we arrived at the desired conclusion for

$$(A, H) \xrightarrow{N} (J, P),$$

where $N = Z_1Z_2Z_3Z_4$.

To sum up, by using Lemma 5.2, we see that $N - T_0$ is a small perturbation with $\|N - T_0\| \leq \Theta$ via (5.37), (5.39), (5.41) and (5.43). \square

6. Real Canonical Bases and Forms

In Theorem 4.1, the real matrices (A, H) were reduced to (J, P) , where, generally speaking, J is not real. Moreover, the affiliation matrix T in $(A, H) \xrightarrow{T} (J, P)$ is not necessarily real as well.

The following well-known result (see, eg., [5] and the references therein) describes a purely real relation

$$(A, H) \xrightarrow{R} (J_R, P) \quad (6.44)$$

where all five matrices are real.

Proposition 6.1. *Let $A \in \mathbb{R}^{n \times n}$ be a fixed real H -selfadjoint matrix for some self-adjoint real matrix H . Then there exists an invertible real matrix R such that $(A, H) \xrightarrow{R} (J_R, P)$. Here J_R is a real Jordan form of A , that is*

$$J_R = J(\lambda_1) \oplus \cdots \oplus J(\lambda_\alpha) \oplus \hat{J}_R(\lambda_{\alpha+1}) \oplus \cdots \oplus \hat{J}_R(\lambda_\beta), \quad (6.45)$$

where $J(\lambda_k)$ (for $k = 1, \dots, \alpha$) are real Jordan blocks as before and for $k = \alpha + 1, \dots, \beta$ we have

[illegible]

where $\lambda_k = \sigma_k + i\tau_k$. P is a sip matrix of the form

$$P = P_1 \oplus \cdots \oplus P_\alpha \oplus P_{\alpha+1} \oplus \cdots \oplus P_\beta,$$

where P_k is a sip matrix $\epsilon_k \tilde{I}$ of the same size as $J(\lambda_k)$ for $k = 1, \dots, \alpha$ and a sip matrix \tilde{I} of the same size as $\hat{J}_R(\lambda_k)$ for $k = \alpha + 1, \dots, \beta$ and $\epsilon_k = \pm 1$ for $k = 1, \dots, \alpha$.

Definition 6.2. (Real Canonical Basis (RC)) Suppose that $A \in \mathbb{R}^{n \times n}$ and J_R is the real Jordan form in (6.45). We say that columns of matrix R in (6.44) form a real canonical basis of A .

The main result of this section is the Lipschitz stability of RC bases (see Theorem 6.4). To prove it we need the following lemma that provides a simple, yet useful connection between the i -FOCS and RC bases of the real pair of matrices (A, H) . As we will see, for an arbitrary pair (A, H) , this relation is carried over with the help of the same fixed matrix S .

Lemma 6.3. Let $A, H \in \mathbb{R}^{n \times n}$, A is H -selfadjoint, and the columns of

$$T = [T_1, \dots, T_\alpha | T_{\alpha+1}, \dots, T_\beta] \text{ and } T_k = [Q_k | i\bar{Q}_k], \quad k = \alpha + 1, \dots, \beta \quad (6.46)$$

capture the i -FOCS basis of (A, H) .

Further, let $S = \text{diag}(I_1, \dots, I_\alpha | S_{\alpha+1}, \dots, S_\beta)$, where

$$S_j = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & 1 & -i & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -i & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & -i \\ -i & 1 & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & -i & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 1 & 0 & 0 \\ 0 & \cdots & \cdots & 0 & 0 & -i & 1 \end{bmatrix}.$$

Then the matrix

$$R = TS \quad (6.47)$$

is real and its columns

$$R = [T_1, \dots, T_\alpha | K_{\alpha+1}, \dots, K_\beta] \quad (6.48)$$

capture the RC basis of A . Moreover, $K_j = T_j S_j$ for $j = \alpha + 1, \dots, \beta$.

Conversely, let the columns of R capture the RC basis of A . Then the columns of $T = RS^{-1}$ capture the i -FOCS basis of A . Moreover, if we

partition T as in (6.46), then $T_j = R_j S_j^{-1}$. Finally, here is the explicit formula for S_j^{-1} :

$$S_j^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & \cdots & 0 & i & 0 & \cdots & 0 \\ i & 0 & & \vdots & 1 & 0 & & \vdots \\ 0 & 1 & \ddots & 0 & 0 & i & \ddots & 0 \\ 0 & i & \ddots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & \vdots & \ddots & \ddots & i \\ 0 & \cdots & 0 & i & 0 & \cdots & 0 & 1 \end{bmatrix}.$$

Proof. First, let us show that the columns of matrix R in (6.47) capture the RC basis of A , that is

$$(A, H) \xrightarrow{R} (J_R, P).$$

Since $(A, H) \xrightarrow{T} (J, P)$, the above relation follows from

$$(J, P) \xrightarrow{S} (J_R, P). \quad (6.49)$$

We will now prove (6.49). Without loss of generality we can consider just a single Jordan block, assuming that $\sigma(A) = \{\lambda, \bar{\lambda}\}$. The first relation $S^{-1} \hat{J} S = \hat{J}_R$ in (6.49) follows from

$$\hat{J} S = \frac{1}{\sqrt{2}} \begin{bmatrix} \sigma+i\tau & -i\sigma+\tau & 1 & -i & 0 & \cdots & 0 & 0 \\ 0 & 0 & \sigma+i\tau & -i\sigma+\tau & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -i \\ -i\sigma-\tau & \sigma-i\tau & -i & 1 & 0 & \cdots & \sigma+i\tau & -i\sigma+\tau \\ 0 & 0 & -i\sigma-\tau & \sigma-i\tau & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & -i & 1 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -i\sigma-\tau & \sigma-i\tau \end{bmatrix} = S \hat{J}_R.$$

The second relation $S^* P S = P$ in (6.49) follows from

$$S^* P S = P \Rightarrow P S^* P S = P P = I.$$

Thus, we just need to show that $PS^*PS = I$.

$$PS^*PS = \frac{1}{2} \begin{bmatrix} 0 & 0 & \dots & 0 & i & 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & i \\ 0 & 0 & \dots & i & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & i & \dots & 0 & 0 & 0 & 1 & \dots & 0 \\ 0 & 1 & \dots & 0 & 0 & 0 & i & \dots & 0 \\ i & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 \\ 1 & 0 & \dots & 0 & 0 & i & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 & \dots & -i & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & -i & 1 & \dots & 0 & 0 \\ -i & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 1 & -i \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 1 & -i & \dots & 0 & 0 \\ 1 & -i & 0 & 0 & \dots & 0 & 0 \end{bmatrix} = I.$$

Secondly, we show that R is real. We have

$$TS = [T_1, \dots, T_\alpha | T_{\alpha+1}S_{\alpha+1}, \dots, T_\beta S_\beta],$$

where T_k (for $k = 1, \dots, \alpha$) are real. Now, for $k = \alpha + 1, \dots, \beta$ we have

$$\begin{aligned} T_k S_k &= [Q_k | i\bar{Q}_k] S_k = [t_1, \dots, t_{p_k} | i\bar{t}_1, \dots, i\bar{t}_{p_k}] S_k = [t_1 + \bar{t}_1, i(\bar{t}_1 - t_1), t_2 \\ &\quad + \bar{t}_2, i(\bar{t}_2 - t_2), \dots, t_{p_k} + \bar{t}_{p_k}, i(\bar{t}_{p_k} - t_{p_k})], \end{aligned}$$

which is real as well. Thus, we have

$$(A, H) \xrightarrow{R} (J_R, P).$$

The converse follows from the relation $(J_R, P) \xrightarrow{S^{-1}} (J, P)$, using similar argument. □

Now we are ready to prove the main result of this section.

Theorem 6.4. (Stability of RC basis) *Let $A_0 \in \mathbb{R}^{n \times n}$ be a fixed H_0 -selfadjoint matrix where $H_0^T = H_0 \in \mathbb{R}^{n \times n}$ is an invertible matrix. Let*

$$(A_0, H_0) \xrightarrow{R_0} (J_R, P)$$

be the mapping from Proposition 6.1, that is the columns of R_0 capture the RC basis of A_0 . There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any real pair (A, H) such that A is H -selfadjoint and A is similar to A_0 and

$$\|A - A_0\| + \|H - H_0\| < \delta.$$

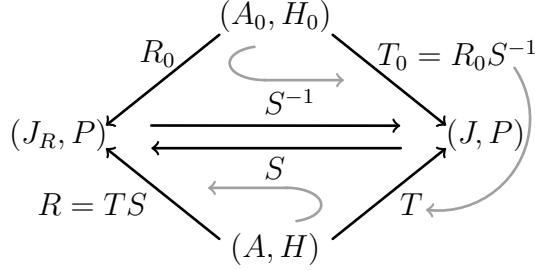
Then, there exists a matrix R such that

$$(A, H) \xrightarrow{R} (J_R, P),$$

whose columns capture the RC basis of A , and such that

$$\|R - R_0\| \leq K(\|A - A_0\| + \|H - H_0\|). \quad (6.50)$$

Proof. Here is the diagram of our proof.



Let the columns of R_0 capture the RC basis of A_0 , i.e.

$$(A_0, H_0) \xrightarrow{R_0} (J_R, P).$$

Then, by Lemma 6.3 columns of

$$T_0 = R_0 S^{-1} \tag{6.51}$$

capture the columns of i -FOCS basis of A_0 , that is $(A_0, H_0) \xrightarrow{T_0} (J, P)$.

Theorem 5.1 says that there exists a matrix T , capturing the i -FOCS basis of A , such that $(A, H) \xrightarrow{T} (J, P)$ and for some \hat{K}

$$\|T - T_0\| \leq \hat{K}(\|A - A_0\| + \|H - H_0\|). \tag{6.52}$$

Let us define $R = TS$. Then,

$$(A, H) \xrightarrow{R} (J_R, P). \tag{6.53}$$

To obtain (6.50) we observe that

$$\|R - R_0\| = \|TS - T_0S\| \leq \|S\|\|T - T_0\| \leq K(\|A - A_0\| + \|H - H_0\|)$$

follows from (6.51), (6.52) and (6.48) by denoting $K = \|S\|\hat{K}$. Thus, we complete the proof of the Theorem 6.4. \square

7. Weak Affiliation Case

In previous sections, we considered a strong affiliation relation $(A, H) \xrightarrow{T} (J, P)$ where matrices A and J were similar. Here we weaken this condition and only require A and J to have the same Jordan structure. Here is a formal definition.

Definition 7.1. (Same Jordan Structure) Denote by $\sigma(A_0)$ and $\sigma(A)$ the sets of all eigenvalues of A_0 and A , respectively. Matrices A_0 and A are said to have the **same Jordan structure** if there is a bijection $f : \sigma(A_0) \rightarrow \sigma(A)$ such that if

$$\mu = f(\lambda)$$

then $\lambda \in \sigma(A_0)$ and $\mu \in \sigma(A)$ have the same Jordan block sizes or, equivalently, if there is an invertible matrix S such that A_0 and $S^{-1}AS$ have the same Jordan bases.

We presented the weak affiliation in Section 1. Let A_0 be H_0 -selfadjoint. To remind the reader, a matrix S is called a **weak affiliation matrix** of the quadruple (A_0, H_0, A, H) if

$$(A, H) \xrightarrow{S}_w (A_0, H_0),$$

where matrices A_0 and A have the same Jordan structure. In this case (A_0, H_0) and (A, H) are called **weakly affiliated**.

In [1] authors proved not only Proposition 2.1, but also its "weak" version, which we formulate next.

Proposition 7.2. Let $A_0 \in \mathbb{C}^{n \times n}$ be a fixed H_0 -selfadjoint matrix and let columns of this matrix capture the FO-basis, that is

$$(A_0, H_0) \xrightarrow{T_0} (J, P).$$

There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any H -selfadjoint matrix A that has the same Jordan structure as A_0 and

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there exists a matrix T whose columns capture the FO-basis of A , i.e.

$$(A, H) \xrightarrow{T}_w (J, P),$$

and such that

$$\|T - T_0\| \leq K(\|A - A_0\| + \|H - H_0\|).$$

Now, we are going to introduce the "weak" versions of Theorem 5.1 and Theorem 6.4.

Theorem 7.3. (Weak Stability of γ -FOCS basis) *Let $A_0 \in \mathbb{R}^{n \times n}$ be a fixed H_0 -selfadjoint, where $H_0 \in \mathbb{R}^{n \times n}$ is a Hermitian matrix, and let the columns of this matrix capture the FOCS-basis, that is*

$$(A_0, H_0) \xrightarrow{T_0} (J, P).$$

Then, there exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any H -selfadjoint matrix A such that $A, H \in \mathbb{R}^{n \times n}$, invertible A has the same Jordan structure as A_0 , and

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there exists an invertible matrix $N \in \mathbb{R}^{n \times n}$ whose columns capture the FOCS-basis of A , that is

$$(A, H) \xrightarrow{N}_w (J, P),$$

and such that

$$\|N - T_0\| \leq K(\|A - A_0\| + \|H - H_0\|).$$

Theorem 7.4. (Weak Stability of RC basis) *Let $A_0 \in \mathbb{R}^{n \times n}$ be a fixed H_0 -selfadjoint matrix where $H_0^T = H_0 \in \mathbb{R}^{n \times n}$. Let*

$$(A_0, H_0) \xrightarrow{R_0} (J_{R_0}, P)$$

be the mapping of Theorem 6.1 that is the columns of R_0 capture the RC basis of A_0 . There exist constants $K, \delta > 0$ (depending on A_0 and H_0 only) such that the following assertion holds. For any real pair (A, H) such that A is H -selfadjoint and A has the same Jordan structure as A_0 with

$$\|A - A_0\| + \|H - H_0\| < \delta,$$

there exists a matrix R such that

$$(A, H) \xrightarrow{R} (J_R, P),$$

whose columns capture the RC basis of A , and

$$\|R - R_0\| \leq K(\|A - A_0\| + \|H - H_0\|).$$

Moreover, there exists a real weak affiliation matrix F , i.e. $(A, H) \xrightarrow{F}_w (A_0, H_0)$ such that

$$\|I - F\| \leq K(\|A - A_0\| + \|H - H_0\|).$$

The proofs of Theorem 7.3 and Theorem 7.4 follow exactly the same lines as the proofs of Theorem 5.1 and Theorem 6.4, respectively. For example, in the proof of Theorem 5.1 we used Proposition 2.1. Now, we apply the result of Proposition 7.2 instead, in order to get the proof of Theorem 7.3. All the rest remains the same.

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