Matched Filtering for Generalized Stationary Processes

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***Abstract*— The methods for solving optimal filtering problems in the case of the classical stationary processes are well-known since late forties. Practice often gives rise to what is not a classical stationary process but a generalized one, white noise is one simplest example. Hence it is of interest to describe the system action on the generalized stationary processes, and then to carry over filtering methods to them. For the most general stochastic processes this seems to be a challenging problem. In this correspondence we *identify* a rather general class of S_J-*generalized stationary* processes for which the desired extension can be done for matched filters. The considered class can be considered as a model of colored noise, and it is wide enough to include white noise, positive frequencies white noise, as well as certain generalized processes occurring in practice, namely when the smoothing effect gives rise the situation in which the distribution of probabilities may not exist at some time instances.**

I. Motivation

1.1. Classical Stationary Processes. A stochastic process $x(t)$ is called *stationary in the wide sense* if its expectation is a constant,

$$E[x(t)] = const,$$

and the correlation function depends only on the difference $(t - s)$, i.e.,

$$E[x(t)\overline{x(s)}] = K_x(t - s).$$

We assume that $E[|x(t)|^2] < \infty$.

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Let us consider a finite memory system that maps the input stochastic process $x(t)$ into the output stochastic process $y(t)$ in accordance with the following rule:

$$y(t) = \int_{t-\omega}^{\tau} x(s)h(t-s)ds. \tag{1}$$

Wiener's seminal monograph [W49] described the construction of the optimal filter in the case $\omega = \infty$. His results were extended to the case $\omega < \infty$ in [ZR50].

1.2. Wiener filters and matched filters. Let the system receives a signal $a(t)$ corrupted by noise $x(t)$, which we assume, for a moment, to be a zero-mean stochastic process *stationary in the wide sense* (the more natural *generalized stationary* case is discussed below). In accordance with (1) the system outputs

$$a_o(t)+y(t) = \int_0^T h(\tau)[a(t-\tau)+x(t-\tau)]d\tau, \tag{2}$$

as shown in the following picture

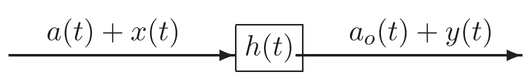

Figure 1. Classical Optimal Filter.

The objective is to find the form of $h(t)$ that makes the system optimal in the sense of a certain criterion. Two particular criteria are recalled next.

Wiener filter. In the case of random signals the typical criterion is to minimize the mean-square value of the difference between the actual system output $a_o(t) + y(t)$ and the actual value $a(t)$ being observed. In this case $h(t)$ is called the *Wiener filter* [H00].

Matched filter. Here we consider a different approach mostly used when the signal is deterministic. In this case the standard criterion is to maximize the signal-to-noise ratio

$$S/N = \frac{a_o^2(t_0)}{\sigma^2},$$

where

$$\sigma^2 = E[|y(t)|^2] = \int_0^T \int_0^T h(u)B(u-v)h(v)dvdu,$$

such filters are referred to as *matched filters* [H00]. Here $B(u)$ is the correlation function of the process $x(t)$. Matched filters are heavily used, e.g., in radar systems [LM04]. Radar systems operate by periodically transmitting very short bursts of radio-frequency energy. The received signal is simply one or more replicas of the transmitted signal that are created by being reflected from any objects that the transmitted signal illuminates. Thus, the form of the received signals is known exactly. The things that are not known about received signals are the number of reflections, the time delay between the transmitted and received signals, the amplitude, and even whether there is a received signal or not. It can be shown that the probability of detecting a weak radar signal in the presence of noise is greatest when the signal-to-noise ratio is the greatest.

A similar situation arises in digital communication systems [H00]. In such a system the message to be transmitted is converted to a sequence of binary signals, say 0 and 1. Each of them is represented as a time function having a specified form. For example, a negative rectangular pulse can represent 0 while a positive rectangular pulse can represent 1. Again, in the presence of noise the probability to recognize the right pulse at the receiver is greatest when the signal-to-noise ratio is greatest.

1.3. An extension to generalized stationary processes. Motivation. The *white noise* $x(t)$ (having equal intensity at all frequencies within a broad band) is not a stochastic process in the *classical sense* as defined above. In fact, white noise can be thought of as the *derivative of a Brownian motion*, which is a continuous stationary stochastic process $B(t)$. It can be shown that $B(t)$ is nowhere differentiable, a fact explaining the highly irregular motions that Robert Brown observed. This means that white noise $\frac{dB(t)}{dt}$ does not exist in the ordinary sense. In fact, white noise is a *generalized stochastic process* (defined, e.g., in the main text below), cf. with [VG61].

Generally, any receiving device has a certain "inertia" and hence instead of actually measuring the classical stochastic process $\xi(t)$ it measures its averaged value

$$\Phi(\varphi) = \int \varphi(t)\xi(t)dt, \qquad (3)$$

where $\varphi(t)$ is a certain function characterizing the device. Since small changes in φ yield small changes in $\Phi(\varphi)$ (small changes in the receiving devices yield closed measurements), hence Φ is a continuous linear functional (see (3)), i.e., a generalized stochastic process [VG61].

Hence it is very natural and important to solve the optimal filtering problem in the case of generalized stochastic processes.

1.4. The main result and the structure of the correspondence. In Sec. II we recall the definition of generalized stationary processes, and consider a special class of them that contains white noise as well as certain other naturally occurring subclasses.

A disadvantage of the formulas (1) and (2) is that they are valid for the classical stationary processes only. In Sec. III we specify one way to extend these formulas to describe the system action on the generalized stationary processes, e.g., on white noise.

In Sec. IV we introduce the the concept of *generalized optimal filters* that operate on the generalized stationary processes. We prove that in the simplest case of classical stationary processes our definition coincided with the standard one. We then formulate the problem of designing the *generalized matched filter*.

In Sec. V we consider a special class of the so called S_J-generalized processes Φ, i.e., such that

$$E[\Phi(\varphi)\Phi(\psi)] = (S_J\varphi, \psi)_{L^2},$$

for the subclass of $\varphi(t), \psi(t)$ that vanish on $J = [a,]$. Here

$$S_J\varphi = \frac{d}{dt}\int_a^b \varphi(u)s(t-u)du \in L^2(a,b).$$

We provide formulas that completely solve the problem of designing S_J-generalized matched filters. We show that if the equation

$$S_J f_0 = g_0, \qquad \text{with} \qquad g_0(t) = a(t_0 - t).$$

is solvable then the matched filter is given by

$$h_{opt} = \frac{f_0}{(g_0, f_0)_{L^2}}.$$

Finally, in Sec. VI we provide several examples that show the simplicity of the suggested approach. In general, the operator S_J can be considered as a model to study colored noise, and the suggested technique is especially beneficial when S_J is invertible, then the solution for the matching filtering problem is unique. For example, white noise corresponds to the simplest case $S_J = I$, and the solution reduces to the known simplest form

$$h_{opt} = \frac{a(t_0 - t)}{\int_0^T |a(t_0 - t)|^2 dt}.$$

We also provide the solutions to several "colored cases," e.g., to the positive frequencies white nose and some others.

II. Generalized Stationary Processes

Let $\mathcal{K}$ be the set of all infinitely differentiable finite functions. A stochastic functional Φ assigns to any $\varphi(t) \in \mathcal{K}$ a stochastic value $\Phi(\varphi)$. A stochastic functional Φ is called linear if

$$\Phi(\alpha\varphi + \beta\psi) = \alpha\Phi(\varphi) + \beta\Phi(\psi).$$

Let us now assume that all the stochastic values $\Phi(\varphi)$ have the expectations $m(\varphi)$ that depend continuously on φ as

$$m(\varphi) = E[\Phi(\varphi)] = \int_{-\infty}^{\infty} x dF(x),$$

where

$$F(x) = P[\Phi(\varphi) \leq x].$$

The functional $m(\varphi)$ is a linear one in the space $\mathcal{K}$. The bilinear functional

$$B(\varphi, \psi) = E[\Phi(\varphi)\Phi(\psi)]$$

is a correlation functional of a stochastic process. It is supposed that that $B(\varphi, \psi)$ is continuously dependent on each of the arguments.

The stochastic process Φ is called *generalized stationary in the wide sense* [S97] if for any functions $\varphi(t)$ and $\psi(t)$ from $\mathcal{K}$ and for any number h the equalities

$$m[\varphi(t)] = m[\varphi(t+h)], \tag{4}$$

$$B[\varphi(t), \psi(t)] = B[\varphi(t+h), \psi(t+h)] \tag{5}$$

hold true.

Let us denote by $\mathcal{K}_y$ the set of the functions from $\mathcal{K}$ such that $\varphi(t) = 0$ when $t \in J = [a, b]$. The correlation functional $B_y(\varphi, \psi)$ is called a *segment* of the correlation functional $B(\varphi, \psi)$ if

$$B_y(\varphi, \psi) = B(\varphi, \psi), \qquad \varphi, \psi \in \mathcal{K}_y. \tag{6}$$

In what follows we consider the generalized stationary processes of the form

$$B_y(\varphi, \psi) = (S_J \varphi, \psi)_{L^2}, \tag{7}$$

where $(\cdot, \cdot)_{L^2}$ is the inner product in the space $L^2(a, b)$, and S_J is a bounded nonnegative operator acting in $L^2(a, b)$ and having the form

$$S_J \varphi = \frac{d}{dt} \int_a^b \varphi(u) s(t-u) du. \tag{8}$$

Definition 1: Generalized stationary processes satisfying (7) and (8) are called S_J-*generalized processes*.

Example 1: **White noise.** It is well-known [L68] that the correlation functional for the Wiener stochastic process Υ (Brownian motion) is given by

$$B(\varphi, \psi) = \int_0^\infty [\widehat{\varphi}(t) - \widehat{\varphi}(\infty)][\widehat{\psi}(t) - \widehat{\psi}(\infty)] dt,$$

where

$$\widehat{\varphi}(t) = \int_0^t \varphi(t) dt, \qquad \widehat{\psi}(t) = \int_0^t \psi(t) dt.$$

White noise Φ (which is the derivative of Υ) is not a continuous stochastic process. Fortunately, it is a generalized stationary process whose correlation functional is known [L68] to be

$$B'(\varphi, \psi) = \int_0^\infty \int_0^\infty \delta(t-s)\varphi(t)\psi(s) ds$$

Thus, in this case we have $B'(\varphi, \psi) = (\varphi, \psi)_{L_2}$ and hence (7) implies that white noise Φ is a very special S_J-generalized stationary process with

$$S_J = I. \tag{9}$$

III. System action on the generalized stationary processes

Let the system receives the generalized stationary signal Φ and the deterministic signal $a(t)$. We assume that Φ is zero-mean and the correlation functional $B(\varphi, \psi)$ is known. At the output we obtain the generalized process of the form

$$a_o(t) + \Psi, \tag{10}$$

where

$$a_o(t) = \int_0^T h(\tau)a(t-\tau)d\tau. \tag{11}$$

cf. with (1) and (2).

The problem is how to describe the system action for the generalized stationary processes shown in the Figure 2.

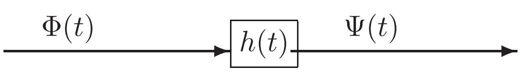

Figure 2. Mapping of the generalized stationary processes

Here we answer the latter question and define it as follows

$$\Psi(\varphi) = \Phi[\int_0^T h(\tau)\varphi(t+\tau)d\tau], \tag{12}$$

so that the overall system is described in Figure 3.

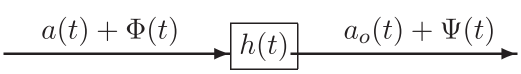

Figure 3. Generalized Matched Filter.

Proposition 1: Let $x(t)$ and $y(t)$ be the classical stationary processes. Then the formula (12) for them is equivalent to the relation

$$y(t) = \int_0^T x(t-\tau)h(\tau)d\tau. \tag{13}$$

Remark 1: The proposition 1 has the following meaning. The formula (13) describes the behavior of a a classical system, cf. with (1) and (2). It follows that the formula (12) suggested here indeed generalizes (13), and in the case of the classical stationary processes our definition coincides with the standard one.

Proof of the proposition 1. The functionals

$$\Phi(\varphi) = \int_{-\infty}^{\infty} x(t)\varphi(t)dt \tag{14}$$

and

$$\Psi(\varphi) = \int_{-\infty}^{\infty} y(t)\varphi(t)dt \tag{15}$$

are associated with the processes $x(t)$ and $y(t)$/ It follows from (13) and (14),(15) that

$$\Psi(\varphi) = \int_{-\infty}^{\infty}\int_0^T x(t-\tau)h(\tau)d\tau\varphi(t)dt =$$
$$\int_{-\infty}^{\infty} x(t)\int_0^T h(\tau)\varphi(t+\tau)d\tau dt. \tag{16}$$

Hence (13) implies (12). In the same way we can prove the converse.

■

IV. Generalized Matched Filters

The objective is to choose the function $h(\tau)$ so that it characterizes the detected signal in an optimal way. If we consider the case of the classical stationary processes $x(t)$ then the criterium of the system quality is maximizing the signal-to-noise ratio defined by the formula

$$S/N = \frac{a_o^2(t_0)}{\sigma^2} \tag{17}$$

where

$$\sigma^2 = E[|y(t)|^2] = \int_0^T\int_0^T h(u)B(u-v)h(v)dvdu \tag{18}$$

Here $B(u)$ is the correlation function of the process $x(t)$. Let us re-write the formula (18) with the help of the correlation functional $B(\varphi, \psi)$ of the generalized stationary process

$$\sigma^2 = B(h, h). \tag{19}$$

Then the formula (17) makes sense in the case of the generalized stationary process as well. We shall consider the following extremal problem.

Problem 1: Find $h(t) \in L^2(0, T)$ such that the signal-to-noise ratio S/N has the greatest value.

The problems of the above type play an important role in communication theory [H00], [S97].

V. S_J-Generalized Matched Filters

We solve the problem for the case of S_J-generalized processes where $J = [0, T]$. In this case the equality

$$\sigma^2 = (S_J h, h) \tag{20}$$

is valid. If $h(t)$ is a solution to the problem 1 then $ch(t)$ is a solution as well. Hence, without any loss of generality, we may assume that

$$\int_0^T h(t)a(t_0 - t)dt = 1. \tag{21}$$

Problem 1 is equivalent to the following problem.

Problem 2: Find the minimum of the form$(S_j h, h)$ under the constraint (21)

By solving the problem 2 we assume that there exists a function $f_0(t)$ which belongs to $L^2(0, T)$ and satisfies the relation

$$S_J f_0 = g_0, \qquad \text{where} \qquad g_0(t) = a(t_0 - t). \tag{22}$$

Remark 2: If the operator S_J is invertible then f_0 exists, and

$$f_0 = S_J^{-1} g_0. \tag{23}$$

Taking into account the equality (22) we can rewrite the condition (21) in the following form

$$(h, S_J f_0)_{L^2} = (\sqrt{S_J}, \sqrt{S_J} f_0)_{L^2} = 1. \tag{24}$$

The latter and the Schwartz inequality imply

$$1 \leq (S_J h, h)_{L^2} (S_J f_0, f_0)_{L^2},$$

i.e.,

$$(S_J h, h) \geq \frac{1}{(S_J f_0, f_0)_{L^2}} = \frac{1}{(g_0, f_0)_{L^2}} = \nu_{min}. \tag{25}$$

As it is well-known, the equality in (25) takes place if and only if

$$\sqrt{S_J} h = \beta \sqrt{S_J} f_0. \tag{26}$$

Hence we obtain the minimal value

$$\nu_{min} = \frac{1}{(g_0, f_0)_{L^2}} \tag{27}$$

when

$$h = \beta f_0. \tag{28}$$

Now, in view of the condition (21) we have

$$\beta = \frac{1}{(g_0, f_0)_{L^2}}. \tag{29}$$

Thus, the solution to the problem 1 has the form

$$h_{opt} = \frac{f_0}{(g_0, f_0)_{L^2}}, \qquad \mu_{max} = S/N = (g_0, f_0)_{L^2}. \tag{30}$$

where g_0 and f_0 are defined by the formulas (22) and (23), respectively.

Proposition 2: If the operator S_J is invertible then there always exist a unique solution of the form (30) with $f_0 = S_J^{-1} g_0$, i.e.,

$$h_{opt} = \frac{S_J^{-1} g_0}{(g_0, S_J^{-1} g_0)_{L^2}}, \tag{31}$$

$$\mu_{max} = S/N = (g_0, S_J^{-1} g_0)_{L^2}. \tag{32}$$

Proposition 3: If $(S_J h, h) > 0$ when $h \neq 0$ then the problem 1 has at most one solution.

VI. Some examples

In general, the above approach can be considered as a way of modelling colored noise via (7) and (8). The above technique is beneficial when the operator S_J is invertible, then the problem can be solved either explicitly or numerically. We provide several such (known and new) examples next.

Example 2: **White noise.** Recall that in white noise (9) case $S_J = I$ and hence the solution

$$h_{opt} = \frac{a(t_0 - t)}{\int_0^T |a(t_0 - t)|^2 dt},$$

and

$$\mu_{max} = S/N = \int_0^T |a(t_0 - t)|^2 dt$$

is unique.

Example 3: **Positive frequencies white noise (PF-white noise).** Here we consider

$$S_J f = Df + \frac{j}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{x - t} dt \tag{33}$$

Here the first term generalizes white noise (white noise corresponds to $D = 1$), and the form of the second term is patterned on the Hilbert transform. If $D \geq 1$ then $S_J \geq 0$. If $D > 1$ then $S_J \geq (D - 1)I$ and so is invertible. Hence, in this important case, the solution h_{opt} and μ_{max} to the problem 1 can be found numerically, by solving (30) and (23).

In fact, the operator S_J in (33) corresponds to the noise having equal intensity at all *positive frequencies* and the zero intensity at the negative frequencies. Indeed, as it was noted in the example 1 the white nose case corresponds to $S_J = I$, and for the latter the kernel $s(t)$ in (8) is the delta function which is the Fourier transform of $f(z) = 1$. The kernel of the operator S_J in (33) is a Fourier transform of $f(z) = \begin{cases} 1 & z \geq 0 \\ 0 & z < 0 \end{cases}$, so the name PF-white noise.

Example 4: **Difference kernel.** Let us consider the case when $J = [a, b], \quad s(+0) - s(-0) = 1$, $s'(u) = K(u)$. In this case the operator S_J takes the form

$$S_J\varphi = \varphi(t) + \int_0^T K(t-u)\varphi(u)du. \qquad (34)$$

If

$$K(t) = \sum_{j=1}^{N} e^{-\alpha_j|t|}\beta_j, \qquad \alpha_j > 0, \qquad \beta_j > 0$$

then the authors have a procedure for constructing S_J^{-1}. Hence, in this important case, the solution h_{opt} and μ_{max} to the problem 1 can be found explicitly as well, see (30) and (23).

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