An Operator Identities Approach to Operator Bezoutians. A General Scheme and Examples*

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Abstract

In this paper we study Bezoutians using a general method known as the *method of operator identities* in the integral equations literature [S76b, S97], and under the name *displacement structure method* in the engineering [K99] and numerical literature [HR84, O03]. The latter approach allows us to introduce a generalized concept of the *operator Bezoutian* and to carry over to it the classical results of Darboux (on common roots of scalar polynomials [D1876]), and of Hermite (on polynomial stability [H1856]). Several other known results scattered in the mathematical and engineering literature (Schur-Cohn [C22], Krein [K], Sakhnovich [S76a], Anderon-Jury [AJ76], Lerer-Tysmenetsky [LT82], Lerer-Rodman [LR96a, LR96b]) are shown to appear as particular instances of our general result. The unified *operator identies (displacement structure)* approach results in a transparent concise derivation of main results allowing us to include most of known as well as new special cases in one paper.

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1 Introduction

1.1 Bezoutians. The Classical Results of Darboux and Hermite

1.2 Operator Identities (Displacement Structure) Approach to Bezoutians. A General Scheme.

Let H and G be Hilbert spaces; H being possibly infinite-dimensional, whereas G is finite dimensional: $m = \dim G$. In this paper we consider a pair of operator functions given in the form called *a realization*,

$$F(z) = I_m - zQ^*(I - Az)^{-1}\Phi, \tag{1}$$

$$G(z) = I_m - zP^*(I - Az)^{-1}\Phi, \tag{2}$$

where $A \in L(H); \Phi, P, Q \in L(G, H)$. Here $L(G, H)$ denotes the set of all linear bounded operators acting from G into H, and $L(H)$ stands for the set of all linear bounded operators acting from H into H. There are several types of realization formulas (cf. with [S99] and [BGR90]), we consider here the form shown in (1) since it is general enough to capture several special cases of our primary interest, two of them are mentioned next.

Example 1. Matrix entire functions of the exponential type. Let us specify A to be defined by

$$Af = i\int_0^x f(t)dt, \qquad f \in L_m^2(0, a) =: H.$$

It can be shown that in this case $F(z)$ and $G(z)$ are matrix entire functions of the exponential type.

Example 2. Matrix polynomials. If not only G but also H is finite dimensional as well, and if A has the Jordan canonical form then $F(z)$ and $G(z)$ are matrix polynomials.

Realization formulas are intrinsically connected with the method of operator identities [S96]. Using this method we obtain here the following results.

- A concept of common zeros of $F(z)$ and $G(z)$ for the matrix case $m > 1$ is introduced (It is trivial for the scalar case $m = 1$, but for the matrix case it is not).
- Our technique is based on introducing and analyzing the operator Bezoutian for the matrix entire functions $F(z)$ and $G(z)$.

  Note that Bezoutians of scalar polynomials are classical going back to the work of Hermite and Darboux and their connection to the root localization problems is well-understood (see, e.g., [KN36] and the references therein). Later this concept was extended to matrix polynomials [AJ76], [LT82], and to some classes of scalar entire functions [S76a, S96]. The definition of operator Bezoutian given here applies to the entire operator functions of the form (1) and (2).

1.3 Applying the General Scheme to the Classical Examples of Darboux and Hermite

1.4 Some Other Examples and The Structure of the Paper

2 A General Scheme

2.1 Definition and the Kernel Structure of the Operator Bezoutian T

In this section we introduce the operator Bezoutian T of matrix entire functions $F(z), G(z)$. To this end let us associate with the pair $F(z), G(z)$ in (1) , (2) the operator identity

$$TB - C^*T = N_2N_1^*, \qquad (T, B, C \in L(H); \quad N_1, N_2 \in L(G, H)) \tag{3}$$

where

$$B = A + \Phi Q^*, \qquad C = A + \Phi P^*, \tag{4}$$

and

$$N_1 = T^*\Phi, \tag{5}$$

We are ready to formulate the first result.

Theorem 1 *Assume that*

(i) *The relations (3), (4), and (5) hold true.*

(ii) *The two conditions (i) $M \in \mathrm{Inv}(A^*)$ and (ii) $M \subset \mathrm{Ker}\Phi^*$ imply $M = 0$ (i.e., there is no non-trivial A^*-invariant subspace annihilated by Φ^*).*

Then

$$\mathrm{Ker}T = L_1, \qquad \textit{where } L_1 \textit{ is the maximal } B\textit{-invariant subspace in } \mathrm{Ker}N_1^*. \tag{6}$$

Proof. Let us first prove that

$$\mathrm{Ker}T \subset L_1. \tag{7}$$

From the definition of N_1 in (5) it follows that for every $\varphi \in \mathrm{Ker}T$ we have $N_1^*\varphi = \Phi^*T\varphi = 0$, and hence $\mathrm{Ker}T \subset \mathrm{Ker}N_1^*$. Further, (3) implies

$$TB\varphi = 0, \tag{8}$$

and therefore $\mathrm{Ker}T \subset \mathrm{Inv}B$. Hence we have (7) by the definition of L_1.

Let us now prove

$$L_1 \subset \mathrm{Ker}T. \tag{9}$$

To this end let us prove that the subspace $H_1 := \overline{TL_1}$ is zero. The operator identity (3) implies that H_1 is C^*-invariant. Since $L_1 \subset \mathrm{Ker}N_1^*$ by the definition, the operator identity (3) implies $H_1 \subset \mathrm{Ker}\Phi^*$. It means that $C^*|_{H_1} = A^*|_{H_1}$, i.e., the operators C^* and A^* coincide on H_1. Hence The condition (ii) of the theorem yields $H_1 = 0$ implying (9).

The inclusions (7) and (9) finally imply (6).

■

2.2 Properties of the Operator B. Auxiliary Results

In this section we prove several auxiliary results.

Proposition 2 *If z is a regular point of $(I - Az)^{-1}$ and $F(z)^{-1}$ then z is a regular point $(I - Bz)^{-1}$ as well, and moreover*

$$(I - Bz)^{-1} = (I - Az)^{-1} + z(I - Az)^{-1}\Phi F(z)^{-1}Q^*(I - Az)^{-1}. \tag{10}$$

Proof. Denote

$$f = (I - Bz)^{-1}g. \tag{11}$$

Then by (4) we have

$$g = (I - Az)f - z\Phi Q^* f,$$

and hence

$$f = (I - Az)^{-1}g + z(I - Az)^{-1}\Phi Q^* f. \tag{12}$$

Since

$$Q^* f = Q^*(I - Az)^{-1}g + zQ^*(I - Az)^{-1}\Phi Q^* f.$$

we obtain that

$$Q^* f = F(z)^{-1}Q^*(I - Az)^{-1}g. \tag{13}$$

The desired relation (10) follows directly from equating (11) and (12), and substituting (13).

■

Remark 3 *In view of (1) and (10) we have*

$$(I - Bz)^{-1}\Phi = (I - Az)^{-1}\Phi F(z)^{-1}. \tag{14}$$

Proposition 4 *The following relation*

$$(B - zI)^{p+1} = \sum_{s=0}^{p}(A - zI)^{p-s}\Phi Q^*(B - zI)^s + (A - zI)^{p+1}. \tag{15}$$

holds true.

Proof. We prove (15) by induction. The case $p = 0$,

$$(B - zI) = \Phi Q^* + (A - zI), \tag{16}$$

is clearly valid by the definition of B in (4). Now, assuming

$$(B - zI)^p = \sum_{s=0}^{p-1} (A - zI)^{p-1-s} \Phi Q^* (B - zI)^s + (A - zI)^p,$$

and multiplying it by (16) we obtain

$$(B - zI)^{p+1} = \sum_{s=0}^{p-1} (A - zI)^{p-s} \Phi Q^* (B - zI)^s + (A - zI)^{p+1} + \Phi Q^* (B - zI)^{p+1}$$

which yields the (15).

■

Let λ_j be an eigenvalue of the operator B and let f_{p_j} be a corresponding $(p_j + 1)$-st generalized eigenvector, i.e.,

$$(B - \lambda_j)^{p_j+1} f_{p_j} = 0, \qquad (B - \lambda_j)^{p_j} f_{p_j} \neq 0 \tag{17}$$

The equation (15) then implies

$$f_{p_j} = - \sum_{s=0}^{p_j} (A - \lambda_j I)^{-s-1} \Phi Q^* (B - \lambda_j I)^s f_{p_j} = \sum_{s=0}^{p_j} (A - \lambda_j I)^{-s-1} h_s, \tag{18}$$

where

$$h_s = - \Phi Q^* (B - \lambda_j I)^s f_{p_j}. \tag{19}$$

Let us now consider the whole chain of the generalized eigenvectors

$$f_{p_j - k} = (B - \lambda_j I)^k f_{p_j}, \qquad 0 \le k \le p_j \tag{20}$$

Proposition 5 *The following relations*

$$f_{p_j - k} = \sum_{s=0}^{p_j - k} (A - \lambda_j I)^{-s-1} h_{s+k}, \qquad 0 \le k \le p_j \tag{21}$$

hold true.

Proof.
In view of (18) and (21) we have

$$f_{p_j-1} = \sum_{s=0}^{p_j} (A - \lambda_j I)^{-s} h_s + \Phi Q^* f_{p_j} = \sum_{s=0}^{p_j-1} (A - \lambda_j I)^{-s-1} h_{s+1} + \Phi Q^* f_{p_j} + h_0 = \sum_{s=0}^{p_j-1} (A - \lambda_j I)^{-s-1} h_{s+1}.$$

Repeating the latter argument we obtain the equation (21).

■

It follows from (21) that

$$f_0 = (A - \lambda_j I)^{-1} h_{p_j}, \tag{22}$$

where

$$h_{p_j} = - \Phi Q^* (B - \lambda_j I)^{p_j} f_{p_j} = - \Phi Q^* f_0. \tag{23}$$

According to (22) and (23) we have

$$Q^* f_0 = - Q^* (A - \lambda_j I)^{-1} \Phi Q^* f_0,$$

i.e.,

$$F(z_j) Q^* f_0 = 0, \qquad \text{where} \qquad z_j = \frac{1}{\lambda_j}. \tag{24}$$

We introduce the matrix function

$$F_1(\lambda) = F(\frac{1}{\lambda}) = I_m + Q^*(A - \lambda I)^{-1}\Phi. \tag{25}$$

It is easy to see that

$$F_1^{(k)}(\lambda) = k!Q^*(A - \lambda I)^{-k-1}\Phi, \qquad k = 1, 2, \dots \tag{26}$$

Hence we have

$$Q^*(A - \lambda I)^{-k-1}\Phi = \frac{1}{k!}F_1^{(k)}(\lambda) \tag{27}$$

Using (19) - (21) we obtain

$$Q^* f_{p_j - k} = -\sum_{s=0}^{p_j - k} Q^*(A - \lambda_j I)^{-s-1}\Phi Q^* f_{p_j - k - s}, \tag{28}$$

where $0 \leq k \leq p_j$. Due to (26) and (28) we deduce that

$$\sum_{s=0}^{p_j - k} \frac{1}{s!}F_1^{(k)}(\lambda_j)Q^* f_{p_j - k - s} = 0, \qquad 0 \leq k \leq p_j. \tag{29}$$

Proposition 6 *If the operators A and B do not have common eigenvalues than*

$$Q^* f_0 \neq 0. \tag{30}$$

Proof. It follows from (19) that $(B - \lambda_j I)f_0 = 0$. If $Q^* f_0 = 0$ then λ_j is the common eigenvalue of A and B. Hence $Q^* f_0 \neq 0$.

■

Remark 7 *If $m = 1$ and if the inequality (30) is valid then (29) is equivalent to*

$$F_1^{(s)}(\lambda_j) = 0, \qquad 0 \leq s \leq p_j. \tag{31}$$

2.3 Common zeros of the matrix entire functions $F(z)$ and $G(z)$. A generalization of the Darboux result

Similarly to (1) and (25) we use (2) introduce

$$G_1(\lambda) = G(\frac{1}{\lambda}) = I_m + P^*(A - \lambda I)^{-1}\Phi. \tag{32}$$

Let λ_j be an eigenvalue if the operator C and let g_{q_j} be the corresponding generalized eigenvector of the order q_j. With these notations the relation (29) takes the form

$$\sum_{s=0}^{q_j - k} \frac{1}{s!}G_1^{(s)}(\lambda_j)P^* g_{q_j - k - s} = 0, \qquad 0 \leq k \leq q_j \tag{33}$$

The following statement follows immediately from proposition 6

Proposition 8 *If the operators A and C do not have common eigenvalues then*

$$P^* g_0 \neq 0. \tag{34}$$

Theorem 9 *Let the assumptions of theorem 1 and the proposition 8 are fulfilled, and let $m = 1$, i.e. $F(z)$ and $G(z)$ are scalar functions. If*

$$Q - P = N_1\Sigma, \qquad \Sigma \in L(G), \tag{35}$$

then the number of common zeros of the functions $F(z)$ and $G(z)$ coincides with $\dim \mathrm{Ker}T$.

Proof. Let

$$F^{(s)}(z_j) = G^{(s)}(z_j) = 0, \qquad 0 \leq s \leq \rho_j. \tag{36}$$

By using (27), (35) and 36) we deduce that

$$N_1^*(A - \lambda_j I)^{-s-1}\Phi = 0, \qquad 0 \leq s \leq \rho_j, \tag{37}$$

where $\lambda_j = \frac{1}{z_j}$. The relations (18) and (37) imply that the generalized eigenvectors $f_s(0 \leq s \leq \rho_j)$ belong to the subspace L_1. Now, let

$$F^{(s)}(z_j) = 0, \qquad 0 \leq s \leq \rho_j, \tag{38}$$

and the corresponding generalized eigenvectors $f_s(0 \leq s \leq \rho_j)$ belong to L_1. Now, (21) and (30) imply (37). In view of (37) and (38) the relations (36) are valid. Hence the number of common zeros of the functions $F(z)$ and $G(z)$ id equal to $\dim L_1$. The assertion of the theorem now follows directly from the theorem 1.

■

Let us now consider the general matrix case $m > 1$, in this case one needs to explain what is meant by the number of common zeros. The standard definition is given next.

Definition 10 *Let all the generalized eigenvectors of the operator B have the order of 1. The matrix functions $F(z)$ and $G(z)$ are said to have the common zero $z_j = \frac{1}{\lambda_j}$ of the multiplicity κ_j if the following conditions are met:*

1. *There exist a $m \times \kappa_j$ matrix U_j of the full rank such that*

$$F(z_j)U_j = G(z_j)U_j = 0. \tag{39}$$

2. *The number κ_j is the maximal number for which a full-rank matrix U_j satisfying (39) can be found.*

With this definition one is able to extend the theorem 9 to the matrix case, we formulate first the simplest result when there are only eigenvectors and no generalized eigenvectors of higher oreders.

Theorem 11 *Let the conditions of the theorem 9 be fulfilled, and let all the generalized eigenvectors if the operator B have the order equal to 1. If*

$$Q - P = M_1\Sigma, \qquad \text{where} \qquad \Sigma \in L(G), \quad \det\Sigma \neq 0, \tag{40}$$

then the number of common zeros of the matrix functions $F(z)$ and $G(z)$ is equal to $dim Ker T$.

Proof. Let $F(z)$ and $G(z)$ have the common zero $z_j = \frac{1}{\lambda_j}$ of the multiplicity κ_j. In this case the matrix U_j in (39) can be be written as

$$U_j = \left[\begin{array}{cccc} u_{1j}, & u_{2j}, & \ldots, & u_{\kappa_j,j} \end{array} \right],$$

where the $m \times 1$ vectors $u_{s,j}$ ($1 \leq s \leq \kappa_j$) are all linearly independent. Let us consider the vectors

$$f_{s,j} = (A - \lambda_j)^{-1}\Phi u_{s,j}, \qquad 1 \leq s \leq \kappa_j. \tag{41}$$

Using (39) we have

$$Q^* f_{s,j} = P^* f_{s,j} = -u_{s,j}. \tag{42}$$

Now, the equations (41) and (42) imply the following assertions.

1. The vectors $f_{s,j}$ ($1 \leq s \leq \kappa_j$) are linearly independent.

2. The vectors $f_{s,j}$ are the eigenvectors of the operators B and C corresponding to the eigenvalue λ_j.

3. The relations

$$N_1^* f_{s,j} = 0, \qquad (1 \leq s \leq \kappa_j)$$

hold true, i.e., $f_{s,j} \in L_1$.

Let for some vector $f_0 \neq 0$ the relations

$$(B - \lambda_j I) f_0 = 0, \qquad f_0 \in L_1 \tag{43}$$

be fulfilled. Then

$$(C - \lambda_j I) f_0 = 0, \qquad f_0 \in L_1 \tag{44}$$

Using (22), (23) and (24) we obtain

$$F(z_j) u_0 = G(z_j) u_0 = 0, \tag{45}$$

with

$$u_o = Q^* f_0. \tag{46}$$

In view of the definition 10 the vector u_0 is a linear combination of the vectors $u_{s,j}$ ($1 \leq s \leq \kappa_j$). Hence the vector f_0 is a linear combination of the vectors $f_{s,j}$ ($1 \leq s \leq \kappa_j$). In other words, the number of common zeros of the functions $F(z)$ and $G(z)$ is equal to $\mathrm{dim} L_1 = \mathrm{dim}\mathrm{Ker} T$.

■

Let us now treat the general case.

Definition 12 *We say that the matrix functions $F(z)$ and $G(z)$ have $\mathrm{dim} L$ common zeros, where L is the maximal subspace that is simultaneously B-invariant and C-invariant.*

It follows from the proof of Theorem 11 that in the case all the generalized eigenvectors of B have the order 1 the definitions 10 and 12 coincide.

Theorem 13 *Let conditions of the theorem 9 be fulfilled. If the relation (40) is valid, and*

$$\Phi g \neq 0, \qquad (g \neq 0), \tag{47}$$

then the number of common zeros of the matrix functions $F(z)$ and $G(z)$ is equal to $\mathrm{dim}\mathrm{Ker} T$.

Proof. Let the operators B and C have a common chain of the generalized eigenvectors f_s ($1 \leq s \leq \rho_j$). Then the following relations

$$(A - \lambda_j) f_s + \Phi Q^* f_s = f_{s-1}, \quad f_{-1} = 0, \tag{48}$$

$$(A - \lambda_j) f_s + \Phi P^* f_s = f_{s-1}, \quad f_{-1} = 0, \tag{49}$$

hold true simultaneously. From 48 and 49 it follows that

$$\Phi(Q^* - P^*) f_s = 0, \qquad 0 \leq s \leq \rho_j. \tag{50}$$

Using (40), (47), and (50) we obtain that $N_1^* f_s = 0$, i.e.,

$$f_s \in L - 1, \qquad 0 \leq s \leq \rho. \tag{51}$$

Now, we suppose that f_s ($0 \leq s \leq \rho_j$) is a chain of the generalized eigenvectors of the operator B, and that $f_s \in L_1$. Then the relations (48) abd (50) are valid. Hence (49) is fulfilled as well, meaning that f_s ($0 \leq s \leq \rho_j$) is a chain of the generalized eigenvectors of the operator C, too. Hence $\mathrm{dim} L = \mathrm{dim} L_1 = \mathrm{dim}\mathrm{Ker} T$.

■

The following proposition is a direct consequence of (29) and (33)

Proposition 14 *If f_s ($0 \leq s \leq \rho_j$) is a common chain of the generalized eigenvectors of the operators B and C then*

$$\sum_{s=0}^{\rho_j - k} \frac{1}{s!} F_1^{(s)}(\lambda_j) Q^* f_{\rho_j - k - s} = \sum_{s=0}^{\rho_j - k} \frac{1}{s!} G_1^{(s)}(\lambda_j) P^* f_{\rho_j - k - s} = 0 \qquad 0 \leq k \leq \rho_j. \tag{52}$$

2.4 Stability. A Generalization of the Hermite result

Let us consider only one matrix function

$$F(z) = I - zQ^*(I - Az)^1\Phi. \tag{53}$$

By setting

$$C = B, \qquad \text{and} \qquad N_2 = iN_1\Sigma, \qquad \text{where} \qquad \Sigma \in L(G), \qquad \Sigma \geq 0 \tag{54}$$

the relations (3) takes the form

$$TB - B^*T = iN_1\Sigma N_1, \qquad T = T^*, \qquad N_1 = T\Phi. \tag{55}$$

We assume here that $\lambda = 0$ is not a point of the limit spectrum of the operator T, implying

$$\mathrm{dimKer}T < \infty. \tag{56}$$

It follows from (55) that $\mathrm{dimKerT}$ is simultaneously B-invariant and T-invariant. Hence H_1 is T-invariant and B^*-invariant. The operator T is bounded on H_1 together with its inverse. The spectrum of the operator

$$B_1 = P_1BP_1 \tag{57}$$

is contained in the spectrum of the operator B. Let us consider on H_1 the indefinite scalar product

$$[\varphi, \psi] = (T\varphi, \psi), \qquad \varphi, \psi \in H_1. \tag{58}$$

It follows from (55) that

$$[B_1\varphi, \psi] - [\varphi, B_1\psi] = i(\Sigma N_1^*\varphi, N_1^*\psi). \tag{59}$$

Thus, the operator B_1 is T-dissipative [K78]. Let us denote by $d_1(\lambda_j)$ the dimension of the root space of the operator B_1 with respect to the eigenvalue λ_j. We have [K78]

$$\sum_{Im\lambda_j<0} d_1(\lambda_j) \leq \kappa, \qquad \lambda_J = \frac{1}{z_j}, \tag{60}$$

where κ is the dimension of the maximal T-invariant subspace on which T is nonnegative.

Corollary 15 *(A generalization of the Hermite, Schur-Cohn and Krein theorems) If the operator A does not have the points of spectrum in the upper half plane ($Imz > 0$) and if the operator B_1 does not have real eigenvalues, then*

$$\sum_{Im\lambda_j<0} d_1(\lambda_j) = \kappa, \qquad \lambda_J = \frac{1}{z_j}, \tag{61}$$

i.e., the equality has place in (60).

Theorem 16 *(A generalization of the Hermite, Schur-Cohn and Krein theorems) Let the relations (55) be fulfilled and let*

$$T \geq \delta I, \qquad \delta 0). \tag{62}$$

Then $B = B_1$ and

$$\det F(z) \neq 0, \qquad Imz > 0. \tag{63}$$

Remark 17 *In what follows we shall need to use one more matrix function (in addition to $F(z)$ and $G(z)$ in (1) and (2)),*

$$G_2(z) = I - zP_2^*(I - Az)^{-1}\Phi,$$

for which the condition

$$Q - P_2 = N_1\Sigma_1 \qquad \Sigma_1 \in L(G), \qquad \det \Sigma_1 \neq 0$$

will hold true. Clearly, all the theorems 9, 11 and 13 will apply to the pair $F(z)$, $G_2(z)$. Moreover, in our application we shall not consider the pair $F(z)$, $G(z)$ and hence the condition (40) will not be needed.

2.5 Some further results

We return to the consideration of the operator identity

$$TA - BT = L_1 M_1^* + L_2 M_2^*, \tag{64}$$

where $A, B, T \in L(H)$ and $L_k, M_k \in L(G, H)$. We suppose that there exists an operator $\Phi \in L(G, H)$ such that

$$T\Phi = L_2, \qquad \Phi^* T = M_1^*. \tag{65}$$

We further introduce the following operator function

$$\rho(\lambda, \mu) = \Phi^*(B - \mu I)^{-1} T (A - \lambda I)^{-1} \Phi. \tag{66}$$

It plays an important role in a number of applications (cf. with **[?]** **[?]**) including Bezoutians.

Theorem 18 *Let the relations (64) and (65) be fulfilled. Then the function $\rho(\lambda, \mu)$ defined in (66) admits the following representation*

$$\rho(\lambda, \mu) = \frac{d(\lambda)b(\lambda) + c(\lambda)a(\lambda)}{\mu - \lambda} \tag{67}$$

where

$$\begin{aligned} a(\lambda) = &\quad - \quad M_1^*(A - \lambda I)^{-1}\Phi, & (68)\\ b(\lambda) = &\quad I- \quad M_2^*(A - \lambda I)^{-1}\Phi, & (69)\\ c(\lambda) = &\quad I+ \quad \Phi^*(B - \lambda I)^{-1} L_1, & (70)\\ d(\lambda) = &\quad \quad \Phi^*(B - \lambda I)^{-1} L_2. & (71) \end{aligned}$$

Proof. It follows from (64) that

$$T(A - \lambda I) - (B - \mu I)T = L_1 M_1^* + L_2 M_2^* + (\mu - \lambda)T, \tag{72}$$

and hence

$$(B - \mu I)^{-1} T - T(A - \lambda I)^{-1} = \tag{73}$$
$$\left[(B - \mu I)^{-1} L_1\right] \cdot \left[M_1^*(A - \lambda I)^{-1}\right] + \left[(B - \mu I)^{-1} L_2\right] \cdot \left[M_2^*(A - \lambda I)^{-1}\right] + (\mu - \lambda)(B - \mu I)^{-1} T (A - \lambda I)^{-1}$$

Using (65) and (73) we have

$$\Phi^*(B - \mu I)^{-1} L_2 - M_1^*(A - \lambda I)^{-1}\Phi = \tag{74}$$
$$\left[\Phi^*(B - \mu I)^{-1} L_1\right] \cdot \left[M_1^*(A - \lambda I)^{-1}\Phi\right] + \left[\Phi^*(B - \mu I)^{-1} L_2\right] \cdot \left[M_2^*(A - \lambda I)^{-1}\Phi\right] + (\mu - \lambda)\rho(\lambda, \mu).$$

The assertion of the theorem now follows directly from the latter equation.

■

Having proved theorem 18 we notice that (67) implies the following statement.

Corollary 19 *Let the conditions of theorem 18 be fulfilled. Then the relation*

$$d(\lambda)b(\lambda) + c(\lambda)a(\lambda) = 0 \tag{75}$$

holds true.

Proposition 20 *Let the relations (64) and (75) be fulfilled. Then*

$$(\Phi^* T - M_1^*)(A - \lambda I)^{-1}\Phi = \Phi^*(B - \lambda I)^{-1}(T\Phi - L_2) \tag{76}$$

Proof. When $\lambda = \mu$ we deduce from (73) that

$$\Phi^*(B - \lambda I)^{-1} T\Phi - \Phi^* T(A - \lambda I)^{-1})\Phi = \tag{77}$$
$$\left[\Phi^*(B - \lambda I)^{-1} L_1\right] \cdot \left[M_1^*(A - \lambda I)^{-1}\Phi\right] + \left[\Phi^*(B - \lambda I)^{-1} L_2\right] \cdot \left[M_1 2^*(A - \lambda I)^{-1}\Phi\right].$$

Using (68) - (71) and (75) we obtain from (77)

$$\Phi^*(B - \lambda I)^{-1} T\Phi - \Phi^* T(A - \lambda I)^{-1})\Phi = a(\lambda) + d(\lambda). \tag{78}$$

Using (68), (71) together with (78) we obtain the desired (76).

■

Proposition 20 implies the following theorem.

Theorem 21 *Let the following conditions be fulfilled*

(a) *The relation (75) holds true.*

(b) *The system of vectors* $(A-\lambda I)^{-1}\Phi g$ *and* $(B^*-\lambda I)^{-1}\Phi g$ *(*$g \in G$*) are complete in* H*.*

Then the following statements are true.

(i) *If the spectra* A *and* B *do not intersect then the equation (64) has one and only one bounded solution* T *and the condition (65) is fulfilled.*

(ii) *If* T *is a bounded solution of the equation (64) and one of the conditions in (65) holds true then the other condition of (65) is also valid.*

We now turn to the consideration of then symmetric case

$$TA_1 - A_1^*T = i(L_1L_2^* + L_2L_1^*) \tag{79}$$

which is an important special case of (64) with

$$M_1 = L_2, \quad M_2 = L_1, \quad A = \frac{1}{i}A_1, \quad B = \frac{1}{i}A^*. \tag{80}$$

The conditions (65) take the form

$$T\Phi = L_2, \quad \Phi^*T = L_2^*. \tag{81}$$

The relations (68) - (71) take the form

$$\begin{aligned} a(\lambda) &= \quad - \quad iL_2^*(A_1-\lambda I)^{-1}\Phi, & (82)\\ b(\lambda) &= \quad I- \quad iL_1^*(A_1-\lambda I)^{-1}\Phi, & (83)\\ c(\lambda) &= \quad I+ \quad i\Phi^*(A_1^*-\lambda I)^{-1}L_1, & (84)\\ d(\lambda) &= \quad\quad\quad i\Phi^*(A_1^*-\lambda I)^{-1}L_2. & (85)\end{aligned}$$

and finally (75) specifies to

$$d(z)b(z) + c(z)a(z) = 0, \qquad \text{with} \qquad z = i\lambda.$$

Proposition 22 *If the system (79) (81) has only one bounded solution then* T *is self-adjoint.*

We now consider the operators

$$B_1 = A + \Phi Q^* \quad C_1 = B + \Phi P^*. \tag{86}$$

It follows from (64), (65) and (86) that

$$TB_1 - C_1^*T = (L_1 - P)M_1^* + L_2(M_2 + Q^*). \tag{87}$$

If

$$P = L_1 - L_2\Sigma_1, \qquad Q = M_1\Sigma_2 - M_2$$

then the relation (87) can be rewritten

$$L_2(\Sigma_1 + \Sigma_2)M_1. \tag{88}$$

(Now we can use all the results of the section 3.)

If we put now

$$B_1 = A_1 + \Phi Q^* \quad C_1 = B_1. \tag{89}$$

then we deduce from (79), (81) and (89) that

$$TB_1 - B_1^* = iL_2(L_1^* - iQ^*) + i(L_1 + iQ)L_2^*. \tag{90}$$

If we put

$$L_1 = -iQ \tag{91}$$

then (90) takes the form

$$TB_1 - B_1T = 0 \tag{92}$$

(Now we can use all the results of section 4.)

3 Special cases

3.1 Example: Matrix Polynomials

In this section we show that the Anderson-Jury Bezoutian and Lerer-Tismenetsky results are a special case the general scheme developed in the first part of the paper.

An analogue of the Darboux result.

An analogue of the Hermite result.

3.2 Example: Rational Matrix Functions

In this section we show that the Lerer-Rodman results are a special case the general scheme developed in the first part of the paper.

An analogue of the Darboux result.

An analogue of the Hermite result.

3.3 Example: Entire Functions

3.3.1 Scalar Entire Functions. The results of [S76a]

In this section we show that the results of [S76a] follow from the general scheme in the first part of the paper. Consider entire functions of the form

$$F(z) = 1 + iz \int_0^w e^{izt} \overline{\varphi(t)} dt \tag{93}$$

$$G(z) = 1 + iz \int_0^w e^{izt} \overline{\psi(t)} dt \tag{94}$$

where $\varphi(t), \psi(t) \in L^2(0, w)$. Let us choose α and β such that $\overline{\alpha} + \beta \neq 0$, and put

$$M_1(x) = \varphi(x) - \beta M_2(x), \tag{95}$$

$$M_2(x) = [\psi(x) + \overline{\varphi(w-x)} - 1]/(\overline{\alpha} + \beta). \tag{96}$$

This case can be treated by means of the general scheme as follows. Clearly, $m = 1$. Set

$$Af = i \int_0^x f(t)dt, \qquad \Phi_1 = 1, \qquad Q_1 = i\varphi(x), \qquad P_1 = i\psi(x). \tag{97}$$

We shall use the following functions

$$q(x,t) = M_2(w-t)M_1(x) + [M_1(w-t)M_2(x) \tag{98}$$

$$r(x,t) = \frac{1}{2} \int_{x+t}^{2w-|x-t|} q(\frac{s+x-t}{2}, \frac{s-x+t}{2}) ds \tag{99}$$

and the following operator

$$Rf = \int_0^w r(x,t) f(t) dt. \tag{100}$$

We will need the following result

Theorem 23 *(see [**?**]) If a bounded operator $T \in L(L^2(0,w)$ satisfies the operator equation*

$$TA - A^*T = iR \tag{101}$$

then the function $r(x,t)$ is absolutely continuous with respect to t abd

$$Tf = \frac{d}{dx} \int_0^w f(t) \frac{\partial}{\partial t} r(x,t) dt. \tag{102}$$

A direct calculation shows that

$$T1 = M_2(x), \qquad T^*1 = \overline{M_2(w-x)}. \tag{103}$$

It follows from (101) and (103) that

$$(TB - C^*T)f = -i(\overline{\alpha} + \beta)\int_0^w f(t)M_2(w-t)dtM_2(x). \tag{104}$$

Clearly, the latter equation is just a special case of (3) with

$$N_1(x) = \overline{M_2(x-w)}, \qquad N_2(x) = -i(\overline{\alpha} + \beta)M_2(x). \tag{105}$$

The relation (5) follows directly from (97), (103), (105).

In order to use the results of the section 2 we introduce the function

$$G_2(x) = e^{izw}\overline{G(\overline{z})}. \tag{106}$$

The relations (95) and (96) imply

$$\varphi(x) = 1 - \overline{M_1(w-x)} + \alpha\overline{M_2(w-x)} \tag{107}$$

and

$$\psi(x) = M_1(x) + \beta M_2(x). \tag{108}$$

With the help of (107) and (108) we rewrite the representations (93) and (94) in the forms:

$$F(z) = 1 + iz\int_0^w e^{izt}1 - [M_1(w-x) + \alpha M_2(w-x)]dt \tag{109}$$

$$G(z) = 1 + iz\int_0^w e^{izt}[\overline{M_1(t)} + \overline{\beta}\overline{M_2(t)}]dt. \tag{110}$$

Relations (106) and (110) imply

$$G_2(z) = 1 + iz\int_0^w e^{izt}[1 - M_1(w-t) - \beta M_2(w-t)]dt. \tag{111}$$

Using the notations in (97) we have

$$F(z) = 1 - zQ^*(I - Az)^{-1}\Phi, \tag{112}$$

$$G(z) = 1 - zP^*(I - Az)^{-1}\Phi, \tag{113}$$

$$G_2(z) = 1 - zP_2^*(I - Az)^{-1}\Phi, \tag{114}$$

where

$$P_21 = i\psi_2(x) = i[1\overline{M_1(w-x)} - \overline{\beta} - \overline{M_2(w-x)}]. \tag{115}$$

It is easy to see that

$$(Q - P_2)1 = i(\alpha + \overline{\beta})\overline{M_2(w-x)}. \tag{116}$$

From (105) and (116) we obtain

$$Q - P_2 = i(\alpha + \overline{\beta})N_1. \tag{117}$$

The following statement now follows from theorem 11 and remark 17.

Theorem 24 *(An analogue of the Darboux result [S76a]) If* $\mathrm{dimKer}T < \infty$ *then the number of common zeros of* $F(z)$ *and* $\overline{G(\overline{z})}$ *is equal to* $\mathrm{dimKer}T$.

Let us now turn to the important special case

$$F(z) = G(z), \quad \text{i.e.,} \quad \varphi(x) = \psi(x).$$

Choosing $\alpha = \beta$ $(Re\alpha < 0)$ we have (cf with [**?**])

$$M_1(x) + \overline{M_1(w-x)} = 1, \qquad M_2(x) = \overline{M_2(x) = \overline{M_2(w-x)}}. \tag{118}$$

Then the corresponding operator T is self-adjoint (cf. with [**?**]).

Theorems 13 and 24 imply the following theorems.

Theorem 25 *(An analogue of the Schur-Cohn and Krein theorems [S76a]) Let* $\lambda = 0$ *is not a point of the limit spectrum of the operator* T *and* $\kappa < \infty$. *Then*

$$\sum_{Imz_j > 0} k_F(z_j) - min\{k_f(Z_J), K_f(\overline{z_j})\} = \kappa.$$

3.3.2 Scalar Entire Functions. Krein's results

3.3.3 Matrix Entire Functions

3.4 More Examples

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