DISPLACEMENT STRUCTURE APPROACH TO CHEBYSHEV–VANDERMONDE AND RELATED MATRICES

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In this paper we use the displacement structure concept to introduce a new class of matrices, designated as *Chebyshev–Vandermonde–like* matrices, generalizing ordinary Chebyshev–Vandermonde matrices, studied earlier by different authors. Among other results the displacement structure approach allows us to give a nice explanation for the form of the Gohberg–Olshevsky formulas for the inverses of ordinary Chebyshev–Vandermonde matrices. Furthermore, the fact that the displacement structure is inherited by Schur complements leads to a fast $O(n^2)$ implementation of Gaussian elimination with *partial pivoting* for Chebyshev–Vandermonde–like matrices.

0. MOTIVATION AND RELATED WORK

0.1. Chebyshev–Vandermonde matrices. The concept of displacement structure has been found to be useful in a large number of applications, some of which have been noted in the reviews [K], [KS2]. In this paper we explore its application to *Chebyshev–Vandermonde* matrices

$$V_T(x) = \begin{bmatrix} T_0(x_1) & T_1(x_1) & \cdots & T_{n-1}(x_1) \\ T_0(x_2) & T_1(x_2) & \cdots & T_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ T_0(x_n) & T_1(x_n) & \cdots & T_{n-1}(x_n) \end{bmatrix}; \quad V_U(x) = \begin{bmatrix} U_0(x_1) & U_1(x_1) & \cdots & U_{n-1}(x_1) \\ U_0(x_2) & U_1(x_2) & \cdots & U_{n-1}(x_2) \\ \vdots & \vdots & & \vdots \\ U_0(x_n) & U_1(x_n) & \cdots & U_{n-1}(x_n) \end{bmatrix}, \tag{0.1}$$

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where $T_0(x), T_1(x), ..., T_{n-1}(x)$ stand for Chebyshev polynomials of the first kind and $U_0(x)$, $U_1(x), ..., U_{n-1}(x)$ stand for Chebyshev polynomials of the second kind.

Chebyshev–Vandermonde matrices were studied earlier by different authors [KarSz], [G], [VS], [Hig1], [Hig2], [RO], [CR], [GO3]. Among other results, explicit formulas for the inverses and fast $O(n^2)$ algorithms for inversion ([CR], [GO3]) and for solving a linear system with Chebyshev–Vandermonde matrices ([Hig1], [Hig2], [RO]) were derived. In particular, it was shown in [GO3] that if $x_1, x_2, ..., x_n$ are n distinct points, then $V_T(x)$ and $V_U(x)$ are invertible and

$$V_T(x)^{-1} = D_0 \cdot H(d) \cdot D_0 \cdot V_T(x)^T \cdot \mathrm{diag}(c), \tag{0.2}$$

$$V_U(x)^{-1} = H(e) \cdot V_U(x)^T \cdot \mathrm{diag}(c), \tag{0.3}$$

$$V_T(x)^{-1} = 2 \cdot D_0 \cdot H(a) \cdot V_U(x)^T \cdot \mathrm{diag}(c), \tag{0.4}$$

$$V_U(x)^{-1} = 2 \cdot H(a) \cdot D_0 \cdot V_T(x)^T \cdot \mathrm{diag}(c), \tag{0.5}$$

where $H(f)$ stand for the following triangular Hankel matrix

$$H(f) = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-2} & f_{n-1} \\ f_1 & & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & 0 \\ \vdots & f_{n-2} & \cdot^{\cdot^{\cdot}} & \cdot^{\cdot^{\cdot}} & \vdots \\ f_{n-2} & f_{n-1} & \cdot^{\cdot^{\cdot}} & & \vdots \\ f_{n-1} & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad f = (f_k)_{k=0}^{n-1},$$

$D_0 = \mathrm{diag}(\frac{1}{2}, 1, ..., 1)$, and $a, c, d, e \in \mathbf{C}^n$ are certain vectors, whose exact form is not relevant at the moment.

Inversion and fast solution of a linear system are two classical applications of the concept of displacement structure. In the rest of this section we shall review briefly the basic facts and definitions of displacement structure theory. Then with this background we shall return to the study of the Chebyshev–Vandermonde matrices in (0.1), and of their natural generalizations, suggested by the concept of displacement structure, which we shall call *Chebyshev–Vandermonde–like* matrices. Among other results, the concept of displacement structure will give a nice explanation for the form of the expressions (0.2) – (0.5) for $V_T(x)^{-1}$ and $V_U(x)^{-1}$. Furthermore, using the displacement structure approach we shall design a fast $O(n^2)$ implementation of Gaussian elimination with *partial pivoting* for Chebyshev–Vandermonde–like matrices; without partial pivoting the resulting algorithms are in the family of generalized Schur algorithms derived in [KS1].

0.2. Displacement structure. Following [KKM] introduce in $\mathbf{C}^{n\times n}$ the linear operator $\nabla_{\{Z_0,Z_0^T\}}(\cdot) : \mathbf{C}^{n\times n} \to \mathbf{C}^{n\times n}$, which transforms each matrix R to its *displacement*,

$$\nabla_{\{Z_0,Z_0^T\}}(R) = R - Z_0 \cdot R \cdot Z_0^T, \tag{0.6}$$

where

$$Z_0 = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

is the lower shift matrix and the superscript ^{T} denotes transpose. The number

$$\alpha = \mathrm{rank}\nabla_{\{Z_0,Z_0^T\}}(R) = \mathrm{rank}(R - Z_0 \cdot R \cdot Z_0^T)$$

is referred to as the $\{Z_0, Z_0^T\}$-*displacement rank* of the matrix R. Let $T = \left[\ t_{i-j}\ \right]_{1\le i,j\le n}$ be an arbitrary Toeplitz matrix. Direct computation shows that only the entries in the first row and column of the displacement of T may differ from zero :

$$\nabla_{\{Z_0,Z_0^T\}}(T) = T - Z_0 \cdot T \cdot Z_0^T =$$

$$\begin{bmatrix} t_0 & t_{-1} & \cdots & t_{-n+1} \\ t_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \frac{t_0}{2} \\ t_1 \\ \vdots \\ t_{n-1} \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}^T + \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdot \begin{bmatrix} \frac{t_0}{2} \\ t_{-1} \\ \vdots \\ t_{-n+1} \end{bmatrix}^T ,$$

i.e., the $\{Z_0, Z_0^T\}$-displacement rank of T does not exceed 2. This fact suggests calling a matrix R, defined by an equation

$$\nabla_{\{Z_0,Z_0^T\}}(R) = \sum_{m=1}^{\alpha} a_m \cdot b_m^T \qquad (a_m, b_m \in \mathbf{C}^n) \tag{0.7}$$

with small α, a *Toeplitz–like* matrix (the designations *Toeplitz-type* matrix and *close-to-Toeplitz* matrix are also in use). In other words, R is referred to as a Toeplitz–like matrix if it is transformed to a matrix of low rank, say α, by the operator $\nabla_{\{Z_0,Z_0^T\}}(\cdot)$ in (0.6). It was shown in [KKM] that any matrix $R \in \mathbf{C}^{n\times n}$ is uniquely determined by its displacement, and that equality (0.7) holds if and only if

$$R = \sum_{m=1}^{\alpha} L(a_m) \cdot L(b_m)^T, \tag{0.8}$$

where $L(a)$ denotes a lower triangular Toeplitz matrix whose first column is a.

The two basic properties of displacement rank are that it is (essentially) preserved by the operations of inversion and Schur complementation.

0.3. Inversion formulas. A classical application of the first property is to the well known Gohberg–Semencul formula for the inverse of a Toeplitz matrix, which in the real symmetric case has the form

$$T^{-1} = L(a) \cdot L(a)^T - L(b) \cdot L(b)^T. \tag{0.9}$$

The exact form of the columns $a, b \in \mathbf{C}^n$ is not relevant at the moment (see e.g. [GF]). The explanation of the nice form (0.9) for T^{-1} was given in [KKM], where it was shown that for an arbitrary matrix R we have

$$\mathrm{rank}\nabla_{\{Z_0,Z_0^T\}}(R) = \mathrm{rank}\nabla_{\{Z_0,Z_0^T\}}(\tilde{I} \cdot R^{-T} \cdot \tilde{I}), \tag{0.10}$$

where

$$\tilde{I} = \text{the antidiagonal identity matrix.}$$

In other words the displacement rank of a matrix is (essentially) inherited by its inverse. Now it is easy to see that a real symmetric Toeplitz matrix T has the property $T = \tilde{I} \cdot T \cdot \tilde{I}$, and hence $T^{-1} = \tilde{I} \cdot T^{-1} \cdot \tilde{I}$. From here and from (0.10) follows that $\nabla_{\{Z_0, Z_0^T\}}(T^{-1}) \leq 2$. In fact it can be checked that $\nabla_{\{Z_0, Z_0^T\}}(T^{-1})$ has one positive and one negative eigenvalue, so there must exist $\{a, b\}$ so that

$$\nabla_{\{Z_0, Z_0^T\}}(T^{-1}) = a \cdot a^* - b \cdot b^*$$

Therefore by (0.7) – (0.8), T^{-1} must have the form shown in the Gohberg–Semencul formula (0.9); some more algebra will give formulas for $a, b \in \mathbf{C}^n$.

We shall give a similar explanation of the formulas (0.2) – (0.5), with however a slightly different definition of displacement, described next.

0.4. Generalized displacement structure and basic classes of structured matrices. The concept of displacement structure, introduced in [KKM] was progressively generalized in the work of T.Kailath and his colleagues. Through a series of studies they arrived in [KS1] at the following general definition of displacement :

$$\nabla_{\{\Omega,\Delta,F,A\}}(R) = \Omega \cdot R \cdot \Delta^* - F \cdot R \cdot A^*. \tag{0.11}$$

Among other results they showed that properties similar to those in (0.10) hold for various choices of $\{\Omega, \Delta, F, A\}$, including the two following choices, corresponding to Vandermonde and Cauchy matrices :

- *Vandermonde–like* $\Omega = \mathrm{diag}(\frac{1}{x_1}, ..., \frac{1}{x_n})$, $A = Z_0$, $\Delta = F = I$;
- *Cauchy–like* $\Omega = \mathrm{diag}(t_1, ..., t_n)$, $A = \mathrm{diag}(s_1^*, ..., s_n^*)$, $\Delta = F = I$;

These definitions are justified by the observation [HR] that with the above choices of $\{\Omega, \Delta, F, A\}$, Vandermonde matrices $V = \left[\begin{array}{c} x_i^j \end{array}\right]_{1\leq i,j\leq n}$ and Cauchy matrices $C = \left[\begin{array}{c} \frac{1}{t_i - s_j} \end{array}\right]_{1\leq i,j\leq n}$ have $\{\Omega, \Delta, F, A\}$-displacement rank 1 . Therefore *Vandermonde–like* and *Cauchy–like* matrices are defined as those that are transformed to matrices of low rank by the displacement operator $\nabla_{\{\Omega,\Delta,F,A\}}(\cdot)$, with Ω, Δ, F, A as defined above.

Here we may also note that displacement operators of the form

$$\nabla_{\{\Omega,A\}}(R) = \Omega \cdot R - R \cdot A^* \tag{0.12}$$

were first closely studied by Heinig and Rost [HR], who used them to derive fast algorithms for determining R^{-1}.

0.5. Triangular factorization. In the approach of Kailath and his colleagues, the problems of inversion, solution of (least-squares) linear equations, etc, have generally been attacked by using the property that displacement structure is inherited by the Schur complements, so that an appropriate variant of a so-called generalized Schur algorithm for triangular factorization of R can be applied (see, e.g. [CK], [KS1], [KS2]). The key result is that the Gaussian elimination procedure for LU factorization of R can be speeded up by exploiting the displacement structure, requiring only $O(n^2)$ operations against $O(n^3)$ required for an arbitrary matrix. Such speed up was perhaps first shown by Schur [S] for Toeplitz matrices and for certain generalizations thereof, which were called quasi-Toeplitz matrices in [LAK]. In the work of T.Kailath and his colleagues this result was extended to more general classes of matrices. The main focus of their work was the derivation of fast algorithms for computing the successive Schur complements of R, or equivalently for speeding up Gaussian elimination for such matrices (see e.g. the reviews [K], [KS2]). These algorithms were called *generalized Schur algorithms*, because they reduce to the classical Schur algorithm in the special case $\Omega = \Delta = I$, $F = A = Z_0$.

To be a little bit more specific let us assume that $\operatorname{rank}\nabla_{\{\Omega,\Delta,F,A\}}(R) = \alpha$. Then we can factor (non-uniquely)

$$\nabla_{\{\Omega,\Delta,F,A\}}(R) = G \cdot B^*,$$

where $\{G, B\}$ have α columns. The number α is called a $\{\Omega, \Delta, F, A\}$-*displacement rank* of R, and the matrices $\{G, B\}$ are called an $\{\Omega, \Delta, F, A\}$-*generator* of R. For many matrices R encountered in applications, the matrices $\Omega, \Delta, F, A \in \mathbf{C}^{n\times n}$ are simple in form (e.g., Jordan or band matrices), and the matrices G, B have only $2\alpha n$ entries (compared with n^2 in R), and α is often less than n. Translation of the Gaussian elimination procedure for R into appropriate operations on its generator $\{G, B\}$ gives an $O(n^2)$ generalized Schur algorithm.

0.6. Pivoting. The standard way to cope with error accumulation in Gaussian elimination is to apply the partial pivoting technique, i.e. to maximize the (1,1) entry of a matrix by means of row permutations, and then to repeat this procedure for each of the successive Schur complements. Of course a row permutation can destroy the structure of a matrix, as is certainly true for Toeplitz–like matrices. In a recent paper [GKO] it was observed that partial pivoting can be incorporated into fast Gaussian elimination algorithms for Vandermonde–like matrices and for Cauchy–like matrices, and moreover for any matrix R satisfying

$$\nabla_{\{\Omega,A\}}(R) = \Omega \cdot R - R \cdot A^* = G \cdot B^*, \qquad (G, B \in \mathbf{C}^{n\times\alpha}), \tag{0.13}$$

where Ω is a diagonal matrix : $\Omega = \operatorname{diag}(t_1, t_2, ..., t_n)$. Indeed, interchange of the 1-st and k-th rows of R is equivalent to multiplication by a corresponding permutation matrix P. It is easy to see that after a row permutation, the new matrix $P \cdot R$ satisfies the displacement equation (0.13) with the diagonal matrix Ω replaced by the diagonal matrix $P \cdot \Omega \cdot P^T$ and with G replaced by $P \cdot G$. This means that a row interchange does not destroy the displacement structure of Vandermonde–like or Cauchy–like matrix. In fact for these two structured classes it allows us to incorporate partial pivoting into the fast Gaussian elimination algorithm.

0.7. Transformations of structured matrices. Let $R \in \mathbf{C}^{n\times n}$ satisfy the displacement equation (0.13) and $T_1, T_2 \in \mathbf{C}^{n\times n}$ be two invertible matrices. It is straightforward to see that the matrix $\hat{R} = T_1^{-1} \cdot R \cdot T_2$ satisfies

$$\nabla_{\{\hat{\Omega},\hat{A}\}}(\hat{R}) = \hat{\Omega} \cdot \hat{R} - \hat{R} \cdot \hat{A} = \hat{G} \cdot \hat{B},$$

in which $\hat{\Omega} = T_1^{-1} \cdot \Omega \cdot T_1$, $\hat{A} = T_2^{-1} \cdot A \cdot T_2$, $\hat{G} = T_1^{-1} \cdot G$, $\hat{B} = B \cdot T_2$. This enables one to change the form of the matrices Ω and A in the displacement equation, and therefore to transform a structured matrix from one class to another. Ideas of this kind have been utilized earlier by various authors. In [P], the translation of a matrix from one structured class to another was discussed in the context of the extension of known structured algorithms to other basic structured classes. In [GO2] this technique was utilized for transformation of Vandermonde–like matrices and Cauchy–like matrices into Toeplitz–like matrices, thereby allowing use of the FFT for reducing the complexity of computing matrix–vector products for matrices from all basic structured classes.

An important application of the transformation of Toeplitz–like matrices to Cauchy–like matrices was suggested in [H]. The point is that Cauchy-like matrices admits fast implementation of Gaussian elimination with partial pivoting. To be more specific let us note that all basic classes of structured matrices can be defined via several possible displacement operators with diagonalizable Ω and F (see, e.g. [KKM], [AG], [GO1], [GO2], [H], [GKO]); the corresponding displacement ranks may differ by some small integer. For example, Toeplitz–like matrices can be defined not only via (0.6), but also as matrices with low $\{Z_1, Z_{-1}\}$-displacement rank α :

$$\nabla_{\{Z_1,Z_{-1}\}}(R) = Z_1 \cdot R - R \cdot Z_{-1} = G \cdot B^*, \qquad (G, B \in \mathbf{C}^{n\times\alpha}), \tag{0.14}$$

where we denote

$$Z_\phi = \begin{bmatrix} 0 & 0 & \cdots & 0 & \phi \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

Since Z_1 is a rank-one perturbation of Z_0, the displacement ranks defined by (0.6) and (0.14) may differ at most by 2.

As is well known,

$$Z_1 = \mathcal{F}^* \cdot D_1 \cdot \mathcal{F}, \qquad Z_{-1} = D_0^{-1} \cdot \mathcal{F}^* \cdot D_{-1} \cdot \mathcal{F} \cdot D_0,$$

where $\mathcal{F}$ is the (normalized) Discrete Transform matrix and D_{-1}, D_0, D_1 are certain unitary diagonal matrices. From this and (0.14) it follows that $\mathcal{F} \cdot R \cdot D_0^{-1} \cdot \mathcal{F}^*$ is a Cauchy–like matrix with $\{D_1, D_{-1}\}$-generator $\{\mathcal{F} \cdot G,\ \ B \cdot D_0^{-1} \cdot \mathcal{F}^*\}$. Thus the transformation of a Toeplitz–like matrix into a Cauchy–like matrix can essentially be achieved via 2α Fast Fourier Transforms on the columns of its generator $\{G, B\}$. Using these arguments new fast Levinson-type and Schur-type Toeplitz solvers were designed in [H] and [GKO], respectively. Moreover a large set of numerical experiments showed that the algorithm from [GKO] compares favorably with other fast Toeplitz solvers, especially for ill-conditioned positive definite, indefinite and non symmetric Toeplitz matrices.

We may also remark that Toeplitz–plus–Hankel–like matrices and Vandermonde–like matrices can be transformed into Cauchy–like matrices by multiplication by Discrete Fourier, Cosine or Sine transform matrices. This means that a structured matrix $R : \mathbf{C}^{n\times n} \to \mathbf{C}^{n\times n}$ can gain Toeplitz–like pattern, Vandermonde–like pattern, Cauchy–like pattern, by choosing an appropriate bases in the domain and range spaces. This suggests solving linear system with any structured matrix using the $O(n^2)$ algorithm derived in [GKO] for Cauchy–like matrices.

1. MAIN RESULTS

Returning to the results described in subsection 0.1, one can observe that the present situation with Chebyshev–Vandermonde matrices is somewhat similar to the situation with Toeplitz matrices before the first "displacement" paper [KKM]. The main goal of the present paper is to apply displacement structure methods to the investigation of Chebyshev–Vandermonde matrices.

Chebyshev–Vandermonde–like matrices. In the next section we first observe that Chebyshev–Vandermonde matrices are transformed by the displacement operator on the left hand side of (0.13), with

$$\Omega = \operatorname{diag}(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}), \qquad A = 2 \cdot \sum_{i=1}^{[\frac{n}{2}]} (-1)^{i-1} \cdot Z_0^{2i-1}, \tag{1.1}$$

to matrices of rank one. This fact suggests the introduction of a more general class of matrices, which we may call *Chebyshev–Vandermonde–like* matrices. Moreover, the displacement structure approach allows us to extend to this new class all the results described in subsection 0.1.

Inversion formulas. Similar to the results mentioned in subsection 0.3, we shall show that inverses of Chebyshev–Vandermonde–like matrices also possess a similar displacement structure; we also present an explicit formula that recovers inverse of a Chebyshev–Vandermonde–like matrix from its generator. This will give a nice explanation and generalization for the fact that all the four formulas (0.2) – (0.5) share the same form.

Fast Gaussian elimination with partial pivoting. Furthermore, we shall design a generalized Schur algorithm for the LU factorization of Chebyshev–Vandermonde–like matrices. As noted in subsection 0.6, we shall show how this generalized Schur algorithm can be modified to accommodate partial pivoting, giving the $O(n^2)$ algorithm which we shall call a fast Gaussian elimination with partial pivoting (fast GEPP) algorithm for Chebyshev–Vandermonde–like matrices. This algorithm allows us to solve a linear system of equations with Chebyshev–Vandermonde-like coefficient matrices in $O(\alpha n^2)$ operations. Using this result we further show that computing all n^2 entries of the inverse matrix has the same complexity of $O(\alpha n^2)$ operations.

Transformation into Cauchy–like matrices. Chebyshev–Vandermonde–like matrices can be defined via several possible choices of Ω and A for displacement operator on the left hand side of (0.13). Here is one alternative to the choice in (1.1) :

$$\Omega = 2 \cdot \operatorname{diag}(x_1, x_2, ..., x_n), \qquad A^* = \mathcal{C} \cdot D_C \cdot \mathcal{C}^T, \tag{1.2}$$

where $\mathcal{C}$ is a Discrete Cosine-II matrix and D_C is a certain real diagonal matrix, which will be specified in the main text below. In accordance with the arguments in subsection 0.7, the equation (1.2) implies that a Chebyshev–Vandermonde–like matrix can be transformed to a Cauchy–like matrix by multiplication by a Discrete Cosine-II transform matrix. This enables solving linear systems with Chebyshev–Vandermonde–like matrices in $O(n^2)$ operations using the fast GEPP algorithm derived in [GKO] for Cauchy–like matrices.

2. CHEBYSHEV–VANDERMONDE–LIKE MATRICES

Let $V_T(x)$ and $V_U(x)$ be Chebyshev–Vandermonde matrices as in (0.1), where $x_1, x_2, ..., x_n$ are assumed to be nonzero. Using the three-term recurrence relations

$$T_0(x) = 1, \qquad T_1(x) = x, \qquad T_n(x) = 2x \cdot T_{n-1}(x) - T_{n-2}(x), \tag{2.1}$$

$$U_0(x) = 1, \qquad U_1(x) = 2x, \qquad U_n(x) = 2x \cdot U_{n-1}(x) - U_{n-2}(x), \tag{2.2}$$

it can be easily seen that Chebyshev–Vandermonde matrices possess displacement structure with respect to the displacement operator

$$\nabla_{\{F,A\}}(R) = F \cdot R - R \cdot A, \tag{2.3}$$

where

$$F = D_{\frac{1}{x}} = \mathrm{diag}(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n})$$

and

$$A = W = \begin{bmatrix} 0 & 2 & 0 & -2 & 0 & \cdots & & & \\ 0 & 0 & 2 & 0 & -2 & \cdots & & & \\ & & \ddots & \ddots & \ddots & \ddots & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \vdots & \\ \vdots & & & & \ddots & \ddots & \ddots & -2 \\ & & & & & \ddots & \ddots & 0 \\ & & & & & & \ddots & 2 \\ 0 & & & \cdots & & & & 0 \end{bmatrix} = 2 \cdot \sum_{i=1}^{[\frac{n}{2}]} (-1)^{i-1} \cdot (Z_0^T)^{2i-1}.$$

Here Z_0 stand as above for the lower shift matrix. More precisely, the following statement holds.

LEMMA 2.1 *Let $\nabla_{\{D_{\frac{1}{x}},W\}}(\cdot) : \mathbf{C}^{n\times n} \to \mathbf{C}^{n\times n}$ be the displacement operator given by* (2.3) *and $D_0 = \mathrm{diag}(\frac{1}{2}, 1, ..., 1)$. Then the Chebyshev-Vandermonde matrices satisfy*

$$\nabla_{\{D_{\frac{1}{x}},W\}}(V_T(x) \cdot D_0) = \begin{bmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \\ \vdots \\ \frac{1}{x_n} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \end{bmatrix}, \tag{2.4}$$

$$\nabla_{\{D_{\frac{1}{x}},W\}}(V_U(x)) = \begin{bmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \\ \vdots \\ \frac{1}{x_n} \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & -1 & 0 & 1 & 0 & -1 & \cdots \end{bmatrix}. \tag{2.5}$$

PROOF. Substituting in the last equality in (2.1) the same expression written for $T_{n-2}(x)$, and proceeding similarly, one finally obtains the equalities

$$\frac{1}{x}T_n(x) - 2\sum_{k=0}^{\frac{n}{2}-1}(-1)^k T_{n-1-2k}(x) = \frac{(-1)^{\frac{n}{2}}}{x} \qquad (n = 2, 4, 6, ...),$$

$$\frac{1}{x}T_n(x) - 2\sum_{k=0}^{\frac{n-1}{2}-1}(-1)^k T_{n-1-2k}(x) - (-1)^{\frac{n-1}{2}} = 0 \qquad (n = 1, 3, 5, ...).$$

These equalities are equivalent to (2.4). Formula (2.5) is similarly deduced from the recurrence relations (2.2).

$\square$

By analogy with (2.4) and (2.5) we shall refer to a matrix R of the form

$$\nabla_{\{D_{\frac{1}{x}},W\}}(R) = G \cdot B^T, \qquad G, B \in \mathbf{C}^{n \times \alpha}$$

(with small α) as a *Chebyshev–Vandermonde–like* matrix. The matrices $\{G, B\}$ are called a $\{D_{\frac{1}{x}}, W\}$-*generator* of R, and the smallest number α of columns over all possible generators is called the $\{D_{\frac{1}{x}}, W\}$-*displacement rank* of R. The following proposition shows how any square matrix can be recovered from its $\{D_{\frac{1}{x}}, W\}$-generator. Here, and henceforth, by $U(a)$ is denoted an upper triangular Toeplitz matrix with first row $a \in \mathbf{C}^n$.

THEOREM 2.2 *Let $\nabla_{\{D_{\frac{1}{x}},W\}}(\cdot) : \mathbf{C}^{n\times n} \to \mathbf{C}^{n\times n}$ stand for the displacement operator in* (2.3), *and let $\varphi_i = \left[\ \varphi_{i,k}\ \right]_{1\le k\le\alpha} \in \mathbf{C}^{1\times\alpha}$, $\psi_i = \left[\ \psi_{i,k}\ \right]_{1\le k\le\alpha} \in \mathbf{C}^{\alpha\times 1}$ $(i = 1, 2, ..., n)$. Then the unique solution R of the equation*

$$\nabla_{\{D_{\frac{1}{x}},W\}}(R) = D_{\frac{1}{x}} \cdot R - R \cdot W = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} \cdot \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix} \tag{2.6}$$

is given by

$$R = \sum_{i=1}^{\alpha} \mathrm{diag}(c_i) \cdot V_T(x) \cdot D_0 \cdot U(d_i), \tag{2.7}$$

where $D_0 = \mathrm{diag}(\frac{1}{2}, 1, \cdots, 1)$, and

$$c_i = \left[\ x_i \cdot \varphi_{k,i}\ \right]_{1\le k\le n}, \qquad d_i = \left[\ 2\psi_{k,i} + 4 \cdot \textstyle\sum_{s=1}^{[\frac{k-1}{2}]} \psi_{k-2s,i}\ \right]_{1\le k\le n}.$$

The matrix R also can be represented as

$$R = \sum_{i=1}^{\alpha} \mathrm{diag}(c_i) \cdot V_U(x) \cdot U(a_i), \tag{2.8}$$

where $a_i = \left[\ a_{k,i}\ \right]_{1\le k\le n}$ with $a_{1,i} = \psi_{1,i}$, $a_{2,i} = \psi_{2,i}$, and $a_{k,i} = \psi_{k,i} + \psi_{k-2,i}$ for $(k = 3, 4, ..., n)$.

PROOF. Since the spectra of matrices $D_{\frac{1}{x}}$ and W in (2.3) have no intersection, there is only one matrix, satisfying (2.6) (see, for example [LT, page 411]).

Let R be given by (2.7). Since $D_{\frac{1}{x}}$ and $\mathrm{diag}(c_i)$ obviously commute, and W and $U(d_i)$ commute, being upper triangular Toeplitz matrices, we can write

$$\nabla_{\{D_{\frac{1}{x}},W\}}(R) = \sum_{i=1}^{\alpha} \mathrm{diag}(c_i) \cdot \nabla_{\{D_{\frac{1}{x}},W\}}(V_T(x) \cdot D_0) \cdot U(d_i) =$$

$$= \sum_{i=1}^{\alpha} \operatorname{diag}(\varphi_{1,i}x_1, \varphi_{2,i}x_2, ..., \varphi_{n,i}x_n) \cdot \begin{bmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \\ \vdots \\ \frac{1}{x_n} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{2} & 0 & -1 & 0 & 1 & \cdots \end{bmatrix} \cdot U(d_{1,i}, d_{2,i}, ..., d_{n,i}) =$$

$$= \sum_{i=1}^{\alpha} \begin{bmatrix} \varphi_{1,i} \\ \varphi_{2,i} \\ \vdots \\ \varphi_{n,i} \end{bmatrix} \cdot \begin{bmatrix} \psi_{1,i} & \psi_{2,i} & \cdots & \psi_{n,i} \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} \cdot \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix},$$

and (2.7) follows.

Formula (2.8) can be verified similarly.

□

In section 4 we shall design a fast $O(n^2)$ algorithm for triangular factorization of Chebyshev–Vandermonde–like matrices. To this end we shall need the following corollary of Theorem 2.2, which allows to recover with $O(\alpha n)$ operations the entries of the first row and column of a matrix from its $\{D_{\frac{1}{x}}, W\}$-generator.

COROLLARY 2.3 *Let* $R = \begin{bmatrix} r_{ij} \end{bmatrix}_{1 \le i,j \le n}$ *be a Chebyshev–Vandermonde–like matrix, satisfying* (2.6) *with rows* $\varphi_i = \begin{bmatrix} \varphi_{i,k} \end{bmatrix}_{1 \le k \le \alpha} \in \mathbf{C}^{1 \times \alpha}$*, and columns* $\psi_i = \begin{bmatrix} \psi_{i,k} \end{bmatrix}_{1 \le k \le \alpha} \in \mathbf{C}^{\alpha \times 1}$*. Then*

$$\begin{bmatrix} r_{11} \\ r_{21} \\ \vdots \\ r_{n1} \end{bmatrix} = \sum_{k=1}^{\alpha} \begin{bmatrix} \varphi_{1,k} \cdot x_1 \\ \varphi_{2,k} \cdot x_2 \\ \vdots \\ \varphi_{n,k} \cdot x_n \end{bmatrix} \cdot \psi_{1k}, \tag{2.9}$$

and for $k = 1, 2, ..., \alpha$

$$\begin{aligned} r_{11} &= x_1 \cdot \sum_{k=1}^{\alpha} \varphi_{1k} \cdot \psi_{1k}, \\ r_{12} &= x_1 \cdot \sum_{k=1}^{\alpha} \varphi_{1k} \cdot (\psi_{2k} + 2\psi_{1k}), \\ r_{1,s} &= x_1 \cdot \sum_{k=1}^{\alpha} \varphi_{1k} \cdot (\psi_{sk} + \psi_{s-2,k}) + 2r_{1,s-1} - r_{1,s-2} \qquad (s = 3, 4, ...n). \end{aligned} \tag{2.10}$$

PROOF. Writing for R the expression (2.8), and calculating the entries in its first column, one immediately obtains (2.9). The equalities in (2.10) also follow from (2.8) via the following arguments. Let $a = \begin{bmatrix} a_i \end{bmatrix}_{1 \le i \le n} \in C^n$ be arbitrary and set

$$\begin{bmatrix} h_1 & h_2 & \cdots & h_n \end{bmatrix} = \begin{bmatrix} U_0(x_1) & U_1(x_1) & \cdots & U_{n-1}(x-1) \end{bmatrix} \cdot \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_2 \\ 0 & \cdots & 0 & a_1 \end{bmatrix}.$$

Then

$$h_1 = U_0(x_1) \cdot a_1, \qquad h_2 = U_0(x_1) \cdot a_2 + U_1(x_1) \cdot a_1,$$

and from the recurrence relations (2.2) it follows that

$$h_s = U_0(x_1) \cdot a_s + 2x_1 \cdot h_{s-1} - h_{s-2} \qquad (s = 3, 4, ..., n). \tag{2.11}$$

Using the rule (2.11) for calculating the entries of the first row of expression (2.8), written for R, one obtains (2.10).

□

3. INVERSION OF CHEBYSHEV–VANDERMONDE–LIKE MATRICES

As mentioned in the introduction, inverses of Toeplitz–like, Vandermonde–like or Cauchy–like matrices also possess a similar displacement structure [KKM], [HR], [CK], [GO2], [KS2]. In this section it will be verified that the same is true for Chebyshev–Vandermonde–like matrices. This fact will give an explanation of the form of the formulas (0.2) – (0.5) for inverses of ordinary Chebyshev–Vandermonde matrices. Note that the formulas (0.2) – (0.5) were derived in [GO3] by using certain generalization of Bezoutian matrices. In order to give an explanation based on displacement structure arguments, we need to prove the following result.

THEOREM 3.1 *Let R be an invertible matrix. Then*

$$\text{rank}\nabla_{\{D_{\frac{1}{x}},W\}}(R) = \text{rank}\nabla_{\{D_{\frac{1}{x}},W\}}(R^{-T} \cdot \tilde{I}), \tag{3.1}$$

where $\tilde{I}$ is antidiagonal identity matrix.

PROOF. Let $\text{rank}\nabla_{\{D_{\frac{1}{x}},W\}}(R) = \alpha$, then one can factor (non-uniquely)

$$\nabla_{\{D_{\frac{1}{x}},W\}}(R) = D_{\frac{1}{x}} \cdot R - R \cdot W = G \cdot B^T, \qquad G, B \in \mathbf{C}^{n\times\alpha}. \tag{3.2}$$

Multiplying the equality (3.2) by R^{-1} from both sides and then taking transposes, one obtains

$$D_{\frac{1}{x}} \cdot R^{-T} - R^{-T} \cdot W^T = (R^{-T} \cdot B) \cdot (G^T \cdot R^{-T}). \tag{3.3}$$

Furthermore, since W is a real Toeplitz matrix, $W^T = \tilde{I} \cdot W \cdot \tilde{I}$. This fact and (3.3) imply

$$\nabla_{\{D_{\frac{1}{x}},W\}}(R^{-T} \cdot \tilde{I}) = D_{\frac{1}{x}} \cdot (R^{-T} \cdot \tilde{I}) - (R^{-T} \cdot \tilde{I}) \cdot W = (R^{-T} \cdot B) \cdot (G^T \cdot R^{-T} \cdot \tilde{I}),$$

and (3.1) follows.

□

Combining Theorems 2.2 and 3.1, one obtains the following result. Recall that $H(f)$ denotes the following triangular Hankel matrix

$$H(f) = \begin{bmatrix} f_0 & f_1 & \cdots & f_{n-2} & f_{n-1} \\ f_1 & & \iddots & \iddots & 0 \\ \vdots & f_{n-2} & \iddots & \iddots & \vdots \\ f_{n-2} & f_{n-1} & \iddots & & \vdots \\ f_{n-1} & 0 & \cdots & \cdots & 0 \end{bmatrix} \qquad f = (f_k)_{k=0}^{n-1}.$$

THEOREM 3.2 *Let R be an invertible matrix satisfying*

$$\nabla_{\{D_{\frac{1}{x}},W\}}(R) = D_{\frac{1}{x}} \cdot R - R \cdot W = G \cdot B^T, \qquad G, B \in \mathbf{C}^{n\times\alpha}.$$

Then

$$R^{-1} = \sum_{i=1}^{\alpha} H(d_i) \cdot D_0 \cdot V_T^T(x) \cdot \operatorname{diag}(c_i), \tag{3.4}$$

where $D_0 = \operatorname{diag}(\frac{1}{2}, 1, ..., 1)$,

$$c_i = \left[\ \varphi_{k,i} x_1 \ \right]_{1\le k\le n}, \qquad d_i = \left[\ 2\psi_{n+1-k,i} + 4 \cdot \sum_{s=1}^{[\frac{n-k}{2}]} \psi_{n+1-k-2s,i} \ \right]_{1\le k\le n},$$

and the rows $\varphi_i = \left[\ \varphi_{i,k} \ \right]_{1\le k\le\alpha} \in \mathbf{C}^{1\times\alpha}$ *and columns* $\psi_i = \left[\ \psi_{i,k} \ \right]_{1\le k\le n} \in \mathbf{C}^{\alpha\times 1}$ *are determined by the 2α linear systems of equations :*

$$R \cdot \begin{bmatrix} \psi_n^T \\ \psi_{n-1}^T \\ \vdots \\ \psi_1^T \end{bmatrix} = G, \qquad R^T \cdot \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} = B. \tag{3.5}$$

The matrix R^{-1} can also be represented as

$$R^{-1} = \sum_{i=1}^{\alpha} H(a_i) \cdot V_U^T(x) \cdot \operatorname{diag}(c_i), \tag{3.6}$$

where $a_i = \left[\ a_{k,i} \ \right]_{1\le k\le n,}$ *with* $a_{k,i} = \psi_{n+1-k,i} + \psi_{n-1-k,i}$ *for* $k = 1, 2, ..., n-2$ *and* $a_{n-1,i} = \psi_{2,i}$, $a_{n,i} = \psi_{1,i}$.

PROOF. Repeating the same arguments as in the proof of Theorem 3.1, one obtains

$$\nabla_{\{D_{\frac{1}{x}},W\}}(R^{-T} \cdot \tilde{I}) = D_{\frac{1}{x}} \cdot (R^{-T} \cdot \tilde{I}) - (R^{-T} \cdot \tilde{I}) \cdot W = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} \cdot \left[\ \psi_1 \quad \psi_1 \quad \cdots \quad \psi_n \ \right].$$

Furthermore, the formulas (2.7) and (2.8) written for $R^{-T} \cdot \tilde{I}$ imply (3.4) and (3.6), respectively.

$\square$

Now it is clear that the form (3.4), (3.6) of the Gohberg–Olshevsky formulas (0.2) – (0.5) is not a mere coincidence, but a reflection of the fact that the inverses of Chebyshev–Vandermonde matrices also possess a similar displacement structure. Indeed, using equality (2.4), one can realize that matrix $V_T(x)^{-1}$ in fact has the form (0.2) with

$$c = \left[\; x_k \cdot \varphi_k \; \right]_{1 \le k \le n}, \quad d = \left[\; 2\psi_{n+1-k} + 4 \cdot \sum_{s=1}^{[\frac{n-k}{2}]} \psi_{n+1-k-2s} \; \right]_{1 \le k \le n}, \tag{3.7}$$

where the numbers φ_i and ψ_i are determined via the solutions of two linear systems

$$V_T^T(x) \cdot \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \\ \vdots \end{bmatrix}, \qquad V_T(x) \cdot \begin{bmatrix} \psi_n \\ \psi_{n-1} \\ \vdots \\ \psi_1 \end{bmatrix} = \begin{bmatrix} \frac{1}{x_1} \\ \frac{1}{x_2} \\ \vdots \\ \frac{1}{x_n} \end{bmatrix}.$$

In [GO3], an alternative to the description (3.7) for the entries of the matrices in (0.2) – (0.5) was obtained :

$$a = (a_i)_{i=1}^n \qquad \text{with} \qquad \prod_{k=1}^n (x - x_k) = \sum_{k=0}^n a_k \cdot T_k(x),$$

$$c = \Big(\frac{1}{\prod_{\substack{i=1 \\ i \ne k}}^n (x_k - x_i)}\Big)_{i=1}^n, \qquad d = \Big(4 \cdot \sum_{i=0}^{[\frac{n-k-1}{2}]} a_{k+2i}\Big)_{k=1}^n,$$

$$e = (e_k)_{k=1}^n, \qquad \text{where} \qquad e_n = a_n, \qquad e_{n-1} = a_{n-1}, \qquad e_k = a_k - a_{k+2} \qquad (k = 1, 2, ..., n-2).$$

Fast and accurate algorithms that compute all the n^2 entries of $V_T(x)^{-1}$ and $V_U(x)^{-1}$ in $7n^2$ operations were designed in [GO3] on the basis of the formulas (0.2) – (0.5). In the next section we shall extend this result and show that all n^2 entries of the inverse of a Chebyshev–Vandermonde–like matrix can be computed in $O(\alpha n^2)$ operations, where α is the $\{D_{\frac{1}{x}}, W\}$-displacement rank of a matrix.

4. FAST GAUSSIAN ELIMINATION WITH PARTIAL PIVOTING FOR CHEBYSHEV–VANDERMONDE–LIKE MATRICES

Gaussian elimination for an arbitrary matrix R_1 is based on recursive Schur complementation as in

$$R_1 = \begin{bmatrix} d_1 & u_1 \\ l_1 & R_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{d_1} l_1 & I \end{bmatrix} \cdot \begin{bmatrix} d_1 & u_1 \\ 0 & R_2 \end{bmatrix}, \tag{4.1}$$

where $R_2 = R_{22}^{(1)} - \frac{1}{d_1} l_l u_1$ is the Schur complement of the (1,1) entry d_1 in the matrix R_1. Step (4.1) gives the first column $\begin{bmatrix} 1 \\ \frac{1}{d_1} \end{bmatrix}$ of L, and the first row $\begin{bmatrix} d_1 & u_1 \end{bmatrix}$ of U, in the LU factorization of R_1. In order to compute the entire LU factorization for R_1, one has to write down for the Schur complement R_2 a factorization similar to the one in (4.1), and to proceed further recursively.

It is a well known fact in displacement structure theory that the Schur complement of a structured matrix remains in the same structured class. This result was proved in [M] for Toeplitz–like matrices, i.e., for the displacement structure as in the Stein type equation (0.6). Later this statement was generalized by different authors. In particular it was extended in [CK] to the case of displacement structures as in the Sylvester type equation (2.3) with lower triangular F and upper triangular A. In the most general form this statement can be found in [KS1], where it appeared for the case of generalized displacement structure as in (0.11) with lower triangular matrices Ω, Δ, F, A. Moreover, a generalized Schur algorithm designed there, provides this result with a constructive proof. Below we shall use a variant of the generalized Schur algorithm specified for a matrix R_1 satisfying the Sylvester type displacement equation

$$\nabla_{\{F_1,A_1\}}(R_1) = \begin{bmatrix} f_1 & 0 \\ * & F_2 \end{bmatrix} \cdot R_1 - R_1 \cdot \begin{bmatrix} a_1 & * \\ 0 & A_2 \end{bmatrix} = \begin{bmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \\ \vdots \\ \varphi_n^{(1)} \end{bmatrix} \cdot \begin{bmatrix} \psi_1^{(1)} & \psi_2^{(1)} & \cdots & \psi_n^{(1)} \end{bmatrix}, \tag{4.2}$$

where $\varphi_i^{(1)} \in \mathbf{C}^{1\times\alpha}$, $\psi_i^{(1)} \in \mathbf{C}^{\alpha\times 1}$ $(i = 1, 2, ..., n)$.

Assume that the (1,1) entry d_1 of $R_1 = \begin{bmatrix} d_1 & u_1 \\ l_1 & R_{22}^{(1)} \end{bmatrix}$ is nonzero. In accordance with the result of [CK], mentioned above, the Schur complement $R_2 = R_{22}^{(1)} - \frac{1}{d_1} l_1 u_1$ inherits the displacement structure of R_1, i.e. it satisfies the Sylvester type displacement equation

$$F_2 \cdot R_2 - R_2 \cdot A_2 = \begin{bmatrix} \varphi_2^{(2)} \\ \varphi_3^{(2)} \\ \vdots \\ \varphi_n^{(2)} \end{bmatrix} \cdot \begin{bmatrix} \psi_2^{(2)} & \psi_3^{(2)} & \cdots & \psi_n^{(2)} \end{bmatrix}, \tag{4.3}$$

with some $\varphi_i^{(2)} \in \mathbf{C}^{1\times\alpha}$ and $\psi_i^{(2)} \in \mathbf{C}^{\alpha\times 1}$. In order to obtain a generator recursion, let us multiply the equality (4.2) by $\begin{bmatrix} 1 & 0 \\ -\frac{1}{d_1} \cdot l_1 & I \end{bmatrix}$ from the left, and by $\begin{bmatrix} 1 & -\frac{1}{d_1} \cdot u_1 \\ 0 & I \end{bmatrix}$ from the right. Then using the standard Schur complementation formula

$$R_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{d_1} \cdot l_1 & I \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 \\ 0 & R_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{d_1} \cdot u_1 \\ 0 & I \end{bmatrix},$$

one obtains

$$\begin{bmatrix} f_1 & 0 \\ * & F_2 \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 \\ 0 & R_2 \end{bmatrix} - \begin{bmatrix} d_1 & 0 \\ 0 & R_2 \end{bmatrix} \cdot \begin{bmatrix} a_1 & * \\ 0 & A_2 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ -\frac{1}{d_1} \cdot l_1 & I \end{bmatrix} \cdot \begin{bmatrix} \varphi_1^{(2)} \\ \varphi_2^{(2)} \\ \vdots \\ \varphi_n^{(2)} \end{bmatrix} \cdot \begin{bmatrix} \psi_1^{(2)} & \psi_2^{(2)} & \cdots & \psi_n^{(2)} \end{bmatrix} \cdot \begin{bmatrix} 1 & -\frac{1}{d_1} \cdot u_1 \\ 0 & I \end{bmatrix}.$$

Equating the (2,2) block entries, we have

$$\begin{bmatrix} \varphi_2^{(2)} \\ \varphi_3^{(2)} \\ \vdots \\ \varphi_n^{(2)} \end{bmatrix} = \begin{bmatrix} \varphi_2^{(1)} \\ \varphi_3^{(1)} \\ \vdots \\ \varphi_n^{(1)} \end{bmatrix} - \frac{1}{d_1} l_1 \varphi_1^{(1)}, \tag{4.4}$$

and

$$\begin{bmatrix} \psi_2^{(2)} & \psi_3^{(2)} & \cdots & \psi_n^{(2)} \end{bmatrix} = \begin{bmatrix} \psi_2^{(1)} & \psi_3^{(1)} & \cdots & \psi_n^{(1)} \end{bmatrix} - \frac{1}{d_1} \psi_1^{(1)} u_1. \tag{4.5}$$

As noted above, in the most general form the recursions corresponding to displacement structure as in (0.11), appeared in [KS1]. The specific form (4.4) and (4.5) of the recursion, corresponding to a Sylvester type displacement equation (4.2), was derived in [GO4], [GKO]. From it, a fast $O(n^2)$ algorithm incorporating partial pivoting was designed in [GKO] for Vandermonde–like and Cauchy–like matrices. In the rest of this section we derive such a fast GEPP algorithm for Chebyshev–Vandermonde–like matrices.

Now assume that the choice of the matrices F_1 and A_1 in (4.2) corresponds to that for the Chebyshev–Vandermonde matrices (see, e.g. (2.3)) :

$$F_1 = D_{\frac{1}{x}} = \operatorname{diag}(\frac{1}{x_1}, \frac{1}{x_2}, ..., \frac{1}{x_n}), \qquad A_1 = W. \tag{4.6}$$

Let R_1 be a Chebyshev–Vandermonde matrix given by its generator

$$G = \begin{bmatrix} \varphi_1^{(1)} \\ \vdots \\ \varphi_k^{(1)} \\ \vdots \\ \varphi_n^{(1)} \end{bmatrix}, \qquad B = \begin{bmatrix} \psi_1^{(1)} & \psi_2^{(1)} & \cdots \psi_n^{(1)} \end{bmatrix}. \tag{4.7}$$

Then Gaussian elimination with partial pivoting for R_1 can be translated to the language of operations with the entries of its generator as follows.

- First one has to recover via (2.9) and (2.10) the first column and the first row of R_1 from its generator.

- Next to determine the position, say (k,1), of the entry with maximal magnitude in the first column. Applying the partial pivoting technique means the interchange of the 1-st and the k-th rows of R_1 (or equivalently multiplication by a corresponding permutation matrix P_1), and then performing elimination with the permuted version of R_1 :

$$P_1 \cdot R_1 = \begin{bmatrix} 1 & 0 \\ \frac{1}{d_1} l_1 & I \end{bmatrix} \cdot \begin{bmatrix} d_1 & u_1 \\ 0 & R_2 \end{bmatrix}. \tag{4.8}$$

  The above permutation is then expressed in terms of the following manipulations with the generator of a matrix : exchange the 1-st and the k-th diagonal entries t_1 and t_k of F_1 in (4.6); and exchange the 1-st and k-th rows $\varphi_1^{(1)}$ and $\varphi_k^{(1)}$ in the matrix G in (4.7).

- Now one has the first column $\begin{bmatrix} 1 \\ \frac{1}{d_1} l_1 \end{bmatrix}$ of L and the first row $\begin{bmatrix} d_1 & u_1 \end{bmatrix}$ of U in LU factorization of the permuted matrix $P_1 \cdot R_1$ as in (4.8).
- Next compute by (4.4), (4.5) the generator for the Schur complement R_2 of the permuted matrix $P_1 \cdot R_1$ in (4.8).

Proceeding recursively, one finally obtains the factorization $R_1 = P \cdot L \cdot U$, where $P = P_1 \cdot P_2 \cdot ... \cdot P_{n-1}$ and P_k is the permutation used at the k-th step of the recursion.

Here is the fast algorithm based on the above arguments.

ALGORITHM 4.1 Fast GEPP for Chebyshev–Vandermonde–like matrix.

<u>Input</u> *Nonzero points* $x_1, x_2, ..., x_n$*;*
Row vectors $\varphi_1, ..., \varphi_n \in \mathbf{C}^{1\times\alpha}$*;*
Column vectors $\psi_1, ..., \psi_n \in \mathbf{C}^{\alpha\times 1}$*;*

<u>Output</u> *The factorization* $R_1 = P^T \cdot L \cdot U$ *of permuted version of a Chebyshev–Vandermonde–like matrix* R_1*, satisfying* (4.2), (4.6), *where* P *is permutation, matrix* $L = \begin{bmatrix} l_{ij} \end{bmatrix}_{i,j=1}^m$ *is unit lower triangular, matrix* $U = \begin{bmatrix} u_{ij} \end{bmatrix}_{i,j=1}^m$ *is upper triangular.*

<u>Initialization</u> *Set* $L = \begin{bmatrix} l_{ij} \end{bmatrix}_{i,j=1}^m$, $U = \begin{bmatrix} u_{ij} \end{bmatrix}_{i,j=1}^m$ *to be zero matrices, and* P *to be identity matrix.*

for $i = 1 : n$
  for $k = i : n$
    $l_{ki} = \sum_{m=1}^{\alpha} \varphi_{ki} x_i \psi_{i1}$
  end
  find $i \le q \le m$ so that $|l_{qi}| = \max_{i\le k\le m} |l_{ki}|$
  $u_{ii} = l_{qi}$
  swap x_i and x_q
  swap φ_i and φ_q
  swap i-th and q-th rows in L
  swap i-th and q-th rows in P
  $u_{i,i+1} = x_i \cdot \sum_{k=1}^{\alpha} (\varphi_{i,k} \cdot \psi_{i+1,k} + 2u_{ii})$
  for $k = i + 2 : m$
    $u_{ik} = x_i \cdot (\sum_{m=1}^{\alpha} \varphi_{im} \cdot (\psi_{km} + \psi_{k-2,m}) + 2u_{i,k-1} - u_{i,k-2}$
  end
  $l_{ii} = 1$
  for $k = i + 1 : m$
    $l_{k,i} = l_{k,i} \cdot \frac{1}{u_{ii}}$
    $\psi_k = \psi_k - \psi_i \cdot u_{ik} \cdot \frac{1}{u_{ii}}$
    $\varphi_k = \varphi_k - \varphi_i \cdot l_{ki}$
  end
end

□

Once the LU factorization of a Chebyshev–Vandermonde–like matrix is computed in $O(\alpha n^2)$ operations by Algorithm 4.1, then a linear system with R_1 can be solved in $O(n^2)$ operations via forward and back substitution [GL].

As mentioned above, it was shown in [GO3] that all n^2 entries of matrices $V_T(x)^{-1}$ and $V_U^{-1}(x)$ can be computed in $7n^2$ operations. The results of sections 3 and 4 allow to extend this result to the more general class of Chebyshev–Vandermonde–like matrices. Moreover the following statement holds.

PROPOSITION 4.2 *Let R be any invertible matrix, specified by its $\nabla_{\{D_{\frac{1}{x}},W\}}$-generator. Then all n^2 entries of R^{-1} can be computed using $O(\alpha n^2)$ operations, where α is a $\{D_{\frac{1}{x}},W\}$-displacement rank of R.*

PROOF. Recall that in accordance with Theorem 3.1, matrix R^{-1} can be represented in the form

$$R^{-1} = \sum_{i=1}^{\alpha} H(a_i) \cdot V_U(x)^T(x) \cdot \mathrm{diag}(c_i). \tag{4.9}$$

where the entries of the vectors a_i and c_i $(i = 1, 2, ..., \alpha)$ in (4.9) are described in terms of the solutions of 2α linear systems (3.3) with matrix R. Therefore the entries of the factors on the right hand side of (4.9) can be computed in $O(\alpha n^2)$ operations, via algorithm 4.1 and then forward and back substitution. Having these entries computed, one can multiply the factors on the right hand side of (4.9) in $O(\alpha n^2)$ operations using the rule (2.11) for multiplication of a Hankel matrix and a transposed Chebyshev–Vandermonde matrix.

□

5. ALTERNATIVE DISPLACEMENT OPERATORS FOR CHEBYSHEV–VANDERMONDE–LIKE MATRICES

Recall that Chebyshev–Vandermonde–like matrices were introduced in Section 2 via a displacement operator of the form

$$\nabla_{\{F,A\}}(R) = F \cdot R - R \cdot A, \tag{5.1}$$

where

$$F = D_{\frac{1}{x}}, \qquad A = W$$

were diagonal and upper triangular matrices, respectively (see e.g. (2.3)). However Chebyshev–Vandermonde–like matrices have low displacement rank for several other choices of matrices F and A as well. For our purposes in the rest of this paper, the following choices will be useful :

$$F = 2D_x = \mathrm{diag}(2x_1, 2x_2, ..., 2x_n), \qquad A = Y_{\gamma,\delta} = \begin{bmatrix} \gamma & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 & 1 \\ 0 & \cdots & 0 & 1 & \delta \end{bmatrix}. \tag{5.2}$$

or

$$F = 2D_x = \mathrm{diag}(2x_1, 2x_2, ..., 2x_n), \qquad A = Z_1 + Z_1^T, \tag{5.3}$$

where Z_1 stands for the circulant lower shift matrix,

$$Z_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & & & \vdots \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}.$$

LEMMA 5.1 *Let* $\nabla_{\{2D_x, Y_{\gamma,\delta}\}}(\cdot) : \mathbf{C}^{n\times n} \to \mathbf{C}^{n\times n}$ *be the displacement operator given by* (5.1), (5.2), *and* $\nabla_{\{2D_x, Z_1+Z_1^T\}}(\cdot) : \mathbf{C}^{n\times n} \to \mathbf{C}^{n\times n}$ *be the displacement operator given by* (5.1), (5.3). *Then Chebyshev–Vandermonde matrices satisfy*

$$\nabla_{\{2D_x, Y_{\gamma,\delta}\}}(V_T(x)) = \begin{bmatrix} x_1 - \gamma \\ x_2 - \gamma \\ \vdots \\ x_n - \gamma \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} T_n(x_1) - \delta T_{n-1}(x_1) \\ T_n(x_2) - \delta T_{n-1}(x_2) \\ \vdots \\ T_n(x_n) - \delta T_{n-1}(x_n) \end{bmatrix} \cdot \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{5.4}$$

$$\nabla_{\{2D_x, Y_{\gamma,\delta}\}}(V_U(x)) = \begin{bmatrix} -\gamma \\ -\gamma \\ \vdots \\ -\gamma \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} U_n(x_1) - \delta U_{n-1}(x_1) \\ U_n(x_2) - \delta U_{n-1}(x_2) \\ \vdots \\ U_n(x_n) - \delta U_{n-1}(x_n) \end{bmatrix} \cdot \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{5.5}$$

$$\nabla_{\{2D_x, Z_1+Z_1^T\}}(V_T(x)) = \begin{bmatrix} x_1 - T_{n-1}(x_1) \\ x_2 - T_{n-1}(x_2) \\ \vdots \\ x_n - T_{n-1}(x_n) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} T_n(x_1) - 1 \\ T_n(x_2) - 1 \\ \vdots \\ T_n(x_n) - 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}, \tag{5.6}$$

$$\nabla_{\{2D_x, Z_1+Z_1^T\}}(V_U(x)) = \begin{bmatrix} -U_{n-1}(x_1) \\ -U_{n-1}(x_2) \\ \vdots \\ -U_{n-1}(x_n) \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} U_n(x_1) - 1 \\ U_n(x_2) - 1 \\ \vdots \\ U_n(x_n) - 1 \end{bmatrix} \cdot \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}. \tag{5.7}$$

PROOF. From the recurrence relations (2.1) and (2.2) it immediately follows that only the entries in the first and the last columns of the matrices on the left hand sides of (5.4) – (5.7) may differ from zero. Calculating these entries, one obtains the assertions of the lemma.

□

Lemma 2.1 and Lemma 5.1 suggest three alternatives for displacement operators that can define Chebyshev–Vandermonde–like matrices. The following statement shows that no matter how one introduces Chebyshev–Vandermonde–like matrices, all the definitions, based on Lemma 2.1 or on Lemma 5.1, describe in fact the same class of matrices.

LEMMA 5.2 *Let the matrix $R \in \mathbf{C}^{n\times n}$ be arbitrary. Then*

$$|\mathrm{rank}\nabla_{\{2D_x,Y_{\gamma,\delta}\}}(R) - \mathrm{rank}\nabla_{\{2D_x,Z_1+Z_1^T\}}(R)| \leq 2. \tag{5.8}$$

$$|\mathrm{rank}\nabla_{\{D_{\frac{1}{x}},W\}}(R) - \mathrm{rank}\nabla_{\{2D_x,Y_{\gamma,\delta}\}}(R)| \leq 3. \tag{5.9}$$

PROOF. Inequality (5.8) immediately follows from the fact that matrix $Y_{\gamma,\delta}$ is a rank-two perturbation of $Z_1 + Z_1^T$.

Now let us prove (5.9). Since the matrix

$$W_1 = W + 2e_n \cdot e_1^T$$

is a rank one perturbation of W, the rank of the matrix

$$\nabla_{\{D_{\frac{1}{x}},W_1\}}(R) = D_{\frac{1}{x}} \cdot R - R \cdot W_1 \tag{5.10}$$

may differ from the $\nabla_{\{D_{\frac{1}{x}},W_1\}}$-displacement rank of R by no more than by 1. Hence multiplying (5.10) by $2D_x = 2D_{\frac{1}{x}}^{-1}$ from the left and by $2W_1^{-1}$ from the right, one obtains the following inequality

$$|\mathrm{rank}\nabla_{\{D_{\frac{1}{x}},W\}}(R) - \mathrm{rank}\nabla_{\{2D_x,2W_1^{-1}\}}(R)| \leq 1. \tag{5.11}$$

It is easy to see that the matrix

$$2W_1^{-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & 1 & \ddots & & & 0 \\ 0 & 1 & 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 & 1 & 0 \\ \vdots & & & \ddots & 1 & 0 & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 & 0 \end{bmatrix}$$

is a rank two perturbation of the matrix $Y_{\gamma,\delta}$ for any $\gamma, \delta \in \mathbf{C}$. From this and (5.11), the inequality (5.9) follows.

□

In the rest of this paper we shall make use of the fact that matrices $Y_{\gamma,\delta}$ with $\gamma, \delta \in \{1, -1\}$ or $\gamma = \delta = 0$ can be diagonalized by Fast Trigonometric Transform matrices. In particular, the following statement holds (see for example, [BDF]).

LEMMA 5.3 *Let $Y_{\gamma,\delta}$ be defined as in* (5.2). *Then*

$$Y_{11} = \mathcal{C} \cdot D_C \cdot \mathcal{C}^T, \qquad Y_{00} = \mathcal{S} \cdot D_S \cdot \mathcal{S}, \tag{5.12}$$

where

$$\mathcal{C} = \left[\ \sqrt{\tfrac{2}{n}}(q_j \cos \tfrac{(2k-1)\cdot(j-1)\pi}{2n})\ \right]_{1\leq k,j\leq n}, \qquad (q_1 = \frac{1}{\sqrt{2}}, q_2 = ... = q_n = 1),$$

is the (normalized) Discrete Cosine Transform–II matrix,

$$\mathcal{S} = \left[\ \sqrt{\tfrac{2}{n+1}} \sin(\tfrac{kj\pi}{n+1}) \ \right]_{1 \le k,j \le n}$$

is the (normalized) Discrete Sine-I Transform matrix, and

$$D_C = \operatorname{diag}(2, 2\cos(\frac{\pi}{n}), 2\cos(\frac{2\pi}{n}), ..., 2\cos(\frac{(n-1)\pi}{n})),$$

$$D_S = \operatorname{diag}(2\cos(\frac{\pi}{n+1}), 2\cos(\frac{2\pi}{n+1}), ..., 2\cos(\frac{n\pi}{n+1})).$$

The following statement shows how to recover any matrix from its $\{2D_x, Y_{00}\}$-generator.

THEOREM 5.4 *Let* $\nabla_{\{2D_x, Y_{00}\}}(\cdot) : \mathbf{C}^{n\times n} \to \mathbf{C}^{n\times n}$ *denote the displacement operator in* (5.1), (5.2), *and assume that*

$$\{x_1, x_2, ..., x_n\} \cap \{\cos(\frac{\pi}{n+1}), \cos(\frac{2\pi}{n+1}), ..., \cos(\frac{n\pi}{n+1}))\} = \emptyset. \tag{5.13}$$

Then the unique solution R of the equation

$$\nabla_{\{2D_x, Y_{00}\}}(R) = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} \cdot \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix}, \tag{5.14}$$

with rows $\varphi_i = \left[\ \varphi_{i,k} \ \right]_{1 \le k \le \alpha} \in \mathbf{C}^{1\times\alpha}$, *and columns* $\psi_i = \left[\ \psi_{i,k} \ \right]_{1 \le k \le \alpha} \in \mathbf{C}^{\alpha\times 1}$, *is given by*

$$R = \sum_{k=1}^{\alpha} \operatorname{diag}(c_k) \cdot V_U(x) \cdot \mathcal{S} \cdot D(d_k) \cdot \mathcal{S}, \tag{5.15}$$

where, as in (5.12), $\mathcal{S}$ *stands for the DST-I matrix,*

$$c_k = \left[\ \tfrac{\varphi_{ik}}{U_n(x_i)} \ \right]_{1 \le i \le n}, \qquad d_k = \left[\ \sqrt{\tfrac{n+1}{2}} \tfrac{\omega_{ik}}{\sin(\frac{in}{n+1}\pi)} \ \right]_{1 \le i \le n},$$

and where $\omega_i = \left[\ \omega_{i,k} \ \right]_{1 \le k \le \alpha} \in \mathbf{C}^{1\times\alpha}$ *are determined from*

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \vdots \\ \omega_n \end{bmatrix} = \mathcal{S} \cdot \begin{bmatrix} \psi_1^T \\ \psi_2^T \\ \vdots \\ \psi_n^T \end{bmatrix}.$$

PROOF. In view of the assumption in (5.13), the spectra of the matrices $2D_x$ and Y_{00} have no intersection (see e.g. Lemma 5.3). Hence there is a unique solution R of (5.14) (see, for example [LT, page 411]).

Let R be given by (5.15). Note that since the points of the second set in (5.13) are zeros of $U_n(x)$, hence $U_n(x_i) \neq 0$ $(i = 1, 2, ..., n)$ and R is well-defined by (5.15). Furthermore, observe that $2D_x$ and $D(c_i)$ obviously commute, and that Y_{00} and $\mathcal{S} \cdot D(d_k) \cdot \mathcal{S}$ commute, being simultaneously diagonalizable matrices. Hence, using (5.5) with $\gamma = 0$, we have

$$\nabla_{\{2D_x, Y_{00}\}}(R) = \sum_{k=0}^{\alpha} \operatorname{diag}(c_k) \cdot \nabla_{\{2D_x, Y_{00}\}}(V_U(x)) \cdot \mathcal{S} \cdot D(d_k) \cdot \mathcal{S} =$$

$$= \sum_{k=0}^{\alpha} \operatorname{diag}(c_k) \cdot \begin{bmatrix} U_n(x_1) \\ U_n(x_2) \\ \vdots \\ U_n(x_n) \end{bmatrix} \cdot \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix} \cdot \mathcal{S} \cdot D(d_k) \cdot \mathcal{S} =$$

$$= \sum_{i=1}^{\alpha} \begin{bmatrix} \varphi_{1,i} \\ \varphi_{2,i} \\ \vdots \\ \varphi_{n,i} \end{bmatrix} \cdot \begin{bmatrix} \psi_{1,i} & \psi_{2,i} & \cdots & \psi_{n,i} \end{bmatrix} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{bmatrix} \cdot \begin{bmatrix} \psi_1 & \psi_2 & \cdots & \psi_n \end{bmatrix},$$

and (5.14) follows.

□

With the same arguments as in the proof of Theorem 3.1, one sees that for any invertible matrix R we have

$$\operatorname{rank} \nabla_{2D_x, Y_{00}}(R) = \operatorname{rank} \nabla_{2D_x, Y_{00}}(R^{-T}).$$

Combining this equality and Theorem 5.4, one immediately obtains the following result.

THEOREM 5.5 *Let R be an invertible matrix satisfying*

$$\nabla_{\{2D_x, Y_{00}\}}(R) = 2D_x \cdot R - R \cdot Y_{00} = G \cdot B^T, \qquad G, B \in \mathbf{C}^{n \times \alpha}.$$

Then

$$R^{-1} = \sum_{k=1}^{\alpha} \mathcal{S} \cdot \operatorname{diag}(d_k) \cdot \mathcal{S} \cdot V_U(x)^T \cdot \operatorname{diag}(c_k),$$

where $\mathcal{S}$ is DST-I *matrix and columns $c_k, d_k \in \mathbf{C}^n$ are determined from the 2α linear systems of equations :*

$$\begin{bmatrix} c_1 & c_2 & \cdots & c_\alpha \end{bmatrix} = \operatorname{diag}(\frac{1}{U_n(x_1)}, ..., \frac{1}{U_n(x_n)}) \cdot R^{-T} \cdot B,$$

and

$$\begin{bmatrix} d_1 & d_2 & \cdots & d_\alpha \end{bmatrix} = \sqrt{\frac{n+1}{2}} \operatorname{diag}(\frac{1}{\sin(\frac{n}{n+1}\pi)}, \frac{1}{\sin(\frac{2n}{n+1}\pi)}, ..., \frac{1}{\sin(\frac{n^2}{n+1}\pi)}) \cdot \mathcal{S} \cdot R^{-1} \cdot G.$$

6. VANDERMONDE–LIKE MATRICES

The class of *Vandermonde–like matrices* was first introduced in [HR]. It was observed there that a Vandermonde matrix

$$V(x) = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} \tag{6.1}$$

satisfies the equation

$$\nabla_{\{D_x, Z_0\}}(V(x)) = D_x \cdot V(x) - V(x) \cdot Z_0 = \begin{bmatrix} x_1^n \\ x_2^n \\ \vdots \\ x_n^n \end{bmatrix} \cdot \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix},$$

where as above $D_x = \mathrm{diag}(x_1, x_2, ..., x_{n-1})$ and Z_0 is a lower shift matrix. Therefore a Vandermonde–like matrix was defined in [HR] as a matrix with low $\{D_x, Z_0\}$-displacement rank. Later several fast algorithms for inversion and triangular factorization were proposed for Vandermonde–like matrices by different authors (see, for example, [HR], [CK], [KS1], [GO2], [GKO]).

In the next example we observe that Vandermonde matrices also possess a displacement structure corresponding to Chebyshev–Vandermonde–like matrices.

EXAMPLE. Let $x_1, x_2, ..., x_n$ be nonzero, and $V(x)$ be a Vandermonde matrix as in (6.1). Set $z_i = \frac{1+x_i^2}{2x_i}$ $(i = 1, 2, ..., n)$, $D_{\frac{1}{z}} = \mathrm{diag}(\frac{1}{z_1}, \frac{1}{z_2}, ..., \frac{1}{z_n})$. Then it is straightforward to check that

$$\nabla_{\{D_{\frac{1}{z}}, W\}}(V(x)) = D_{\frac{1}{z}} \cdot R - R \cdot W =$$

$$= \begin{bmatrix} \frac{x_1}{x_1^2+1} \\ \frac{x_2}{x_2^2+1} \\ \vdots \\ \frac{x_n}{x_n^2+1} \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & -2 & 0 & 2 & 0 & \cdots \end{bmatrix} + \begin{bmatrix} \frac{1}{x_1^2+1} \\ \frac{1}{x_2^2+1} \\ \vdots \\ \frac{1}{x_n^2+1} \end{bmatrix} \cdot \begin{bmatrix} 0 & -2 & 0 & 2 & 0 & -2 & \cdots \end{bmatrix},$$

i.e. the $\nabla_{\{D_{\frac{1}{z}}, W\}}$-displacement rank of $V(x)$ is equal to 2. Also

$$\nabla_{\{D_x + D_{\frac{1}{x}}, Y_{\gamma,\delta}\}}(V(x)) = \begin{bmatrix} \frac{1}{x_1} - \gamma \\ \frac{1}{x_2} - \gamma \\ \vdots \\ \frac{1}{x_n} - \gamma \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} + \begin{bmatrix} x_1^n - \delta x_1^{n-1} \\ x_2^n - \delta x_2^{n-1} \\ \vdots \\ x_n^n - \delta x_n^{n-2} \end{bmatrix} \cdot \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix},$$

i.e. the $\nabla_{\{D_x + D_{\frac{1}{x}}, Y_{\gamma,\delta}\}}$-displacement rank of the Vandermonde matrix is equal to 2.

$\square$

This example shows that the Vandermonde matrix $V(x)$ with the nodes $x = (x_1, x_2, ..., x_n)$ is also a Chebyshev–Vandermonde–like matrix, however with some other nodes

$z = (z_1, z_2, ..., z_n)$. It turns out that the whole class of Vandermonde–like matrices is a subclass of Chebyshev–Vandermonde–like matrices. Moreover, the following statement holds.

PROPOSITION 6.1 *Let $x_i \neq 0$ and $R \in \mathbf{C}^{n\times n}$ be such that* $\mathrm{rank}\nabla_{\{D_x,Z_0\}}(R) = \alpha$. *Then*

$$\mathrm{rank}\nabla_{\{2D_z,Y_{\gamma,\delta}\}}(R) \leq 2\alpha + 2, \tag{6.2}$$

where $D_z = \mathrm{diag}(z_1, z_2, ..., z_n)$ *with* $z_i = \frac{1+x_i^2}{2x_i}$.

PROOF. Observe that the class of Vandermonde–like matrices can be defined via various forms of displacement operator. Indeed, since a circulant lower shift matrix $Z_1 = Z + e_1 \cdot e_n^T$ is a rank one perturbation of Z_0, Vandermonde–like matrices can also be defined via a displacement operator of the form

$$\nabla_{\{D_x,Z_1\}}(R) = D_x \cdot R - R \cdot Z_1. \tag{6.3}$$

Multiplying the last equality by $D_x^{-1} = D_{\frac{1}{x}}$ from the left and by $Z_1^{-1} = Z_1^T$ from the right, one obtains one more displacement operator

$$\nabla_{\{D_{\frac{1}{x}},Z_1^T\}}(R) = D_{\frac{1}{x}} \cdot R - R \cdot Z_1^T, \tag{6.4}$$

associated with the class of Vandermonde–like matrices. Let R be a matrix with $\{D_x, Z_1\}$-displacement rank α. Adding (6.3) and (6.4), one can realize that the $\{D_x + D_{\frac{1}{x}}, Z_1 + Z_1^T\}$-displacement rank of R does not exceed 2α. Since the matrix $Z_1 + Z_1^T$ is a rank two perturbation of a matrix $Y_{\gamma,\delta}$ as in (5.3), the $\{D_x + D_{\frac{1}{x}}, Y_{\gamma,\delta}\}$-displacement rank of R is less than or equal to $2\alpha + 2$, and (6.2) follows.

□

7. TRANSFORMATION INTO CAUCHY–LIKE MATRICES

As was observed in [HR], an ordinary Cauchy matrix $C(x,y) = \left[\frac{1}{x_i - y_j} \right]$ satisfies the equation

$$\nabla_{\{D_x,D_y\}}(C(x,y)) = D_x \cdot C(x,y) - C(x,y) \cdot D_y = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix},$$

where $D_x = \mathrm{diag}(x_1, x_2, ..., x_n)$ and $D_y = \mathrm{diag}(y_1, y_2, ..., y_n)$ Then Cauchy–like matrices were introduced in [HR] as matrices with low $\{D_x, D_y\}$-displacement rank.

In this section we shall show that Chebyshev–Vandermonde–like matrices become Cauchy–like matrices after multiplication by Discrete Transform matrices. To this end it will be more convenient for us to adopt the definitions of Chebyshev–Vandermonde–like matrices displacement structure based on the displacement equation (5.1) with matrices F and A as in (5.2) or as in (5.3).

THEOREM 7.1 *Let R be a Chebyshev–Vandermonde–like matrix. Then the following statements hold.*

(i) *Let R be given by $\{2D_x, Y_{11}\}$-generator $\{G, B\}$:*

$$\nabla_{\{2D_x,Y_{11}\}}(R) = 2D_x \cdot R - R \cdot Y_{11} = G \cdot B^T \qquad G, B \in \mathbf{C}^{n\times\alpha}. \tag{7.1}$$

Then $R \cdot \mathcal{C}$ is a Cauchy–like matrix :

$$\nabla_{\{2D_x,D_C\}}(R \cdot \mathcal{C}) = 2D_x \cdot (R \cdot \mathcal{C}) - (R \cdot \mathcal{C}) \cdot D_C = G \cdot \hat{B}^T, \tag{7.2}$$

where $\mathcal{C}$ and D_C are as in Lemma 5.3 and

$$\hat{B} = \mathcal{C}^T \cdot B. \tag{7.3}$$

(ii) *Let R be given by $\{2D_x, Y_{00}\}$-generator :*

$$\nabla_{\{2D_x,Y_{00}\}}(R) = 2D_x \cdot R - R \cdot Y_{00} = G \cdot B^T, \qquad G, B \in \mathbf{C}^{n\times\alpha}.$$

Then $R \cdot \mathcal{S}$ is a Cauchy–like matrix :

$$\nabla_{\{2D_x,D_S\}}(R \cdot \mathcal{S}) = 2D_x \cdot (R \cdot \mathcal{S}) - (R \cdot \mathcal{S}) \cdot D_S = G \cdot \hat{B}^T, \tag{7.4}$$

where $\mathcal{S}$ and D_S are as in Lemma 5.3 and

$$\hat{B} = \mathcal{S} \cdot B.$$

(iii) *Let R be given by $\{2D_x, Z_1 + Z_1^T\}$-generator :*

$$\nabla_{\{2D_x,Z_1+Z_1^T\}}(R) = 2D_x \cdot R - R \cdot (Z_1 + Z_1^T) = G \cdot B^*, \qquad G, B \in \mathbf{C}^{n\times\alpha}$$

Then $R \cdot \mathcal{F}^*$ is a Cauchy–like matrix :

$$\nabla_{\{2D_x,D_S\}}(R \cdot \mathcal{F}^*) = 2D_x \cdot (R \cdot \mathcal{F}^*) - (R \cdot \mathcal{F}^*) \cdot D_C = G \cdot \hat{B}^*, \tag{7.5}$$

where $\mathcal{F} = \frac{1}{\sqrt{n}} \left[\, e^{\frac{2\pi i}{n}(k-1)(j-1)} \, \right]_{1\le k,j\le n}$ stands for the (normalized) Discrete Fourier matrix, $D_C = \operatorname{diag}(2, 2\cos(\frac{\pi}{n}), 2\cos(\frac{2\pi}{n}), ..., 2\cos(\frac{(n-1)\pi}{n}))$ and

$$\hat{B} = \mathcal{F} \cdot B.$$

PROOF. Recall that the matrix Y_{11} is diagonalized by the DCT-II matrix : $Y_{11} = \mathcal{C} \cdot D_C \cdot \mathcal{C}^T$, see e.g. Lemma 5.3. Substituting the latter expression into (7.1) and then multiplying it by $\mathcal{C}$ from the right, one obtains (7.2). Formula (7.4) is deduced from Lemma 5.3 by similar arguments. Analogously, formula (7.5) follows from the well known identities

$$Z_1 = \mathcal{F}^* \cdot D_1 \cdot \mathcal{F}, \qquad Z_1^T = \mathcal{F}^* \cdot D_1^* \cdot \mathcal{F},$$

where $\mathcal{F}$ is the (normalized) DFT matrix and $D_1 = \operatorname{diag}(1, exp(\frac{2\pi i}{n}), ..., exp(\frac{2\pi i}{n}(n-1)))$.

$\square$

As was mentioned in section 3, an algorithm for fast $O(n^2)$ Gaussian elimination with partial pivoting was designed for Cauchy–like matrices in [GKO]. Furthermore, it was shown there that Toeplitz–like, Toeplitz–plus–Hankel–like and Vandermonde–like matrices could be transformed to Cauchy–like matrices by multiplication with appropriate Discrete Transform matrices. This fact allowed fast $O(n^2)$ solution of linear system with these matrices by making use of the above mentioned algorithm designed in [GKO] for Cauchy–like matrices. Similarly, Theorem 7.1 suggests processing Chebyshev–Vandermonde matrices using the above mentioned algorithm for Cauchy–like matrices. To be a bit more precise, let R be a Chebyshev–Vandermonde matrix, given by a $\nabla_{\{2D_x,Y_{11}\}}$-generator $\{G,B\}$. Then a $\nabla_{\{2D_x,D_C\}}$-generator of the Cauchy–like matrix $R\cdot\mathcal{C}$ can be computed by (7.3) via α inverse FCT's, which is a fast and accurate operation. Then a fast $O(n^2)$ GEPP algorithm for a Cauchy–like matrix (Algorithm 2.1 from [GKO]) can be applied for computing the LU factorization of the permuted version of this Cauchy–like matrix, i.e.,

$$R\cdot\mathcal{C}=P\cdot L\cdot U.$$

Making use of the latter factorization one can solve linear systems with Chebyshev–Vandermonde–like matrices in $O(n^2)$ operations (via forward and back substitution and then FCT.)

Finally we remark that formulas (7.4) and (7.5) suggest two more efficient algorithms for solving linear system with Chebyshev–Vandermonde matrices via transformation into Cauchy–like matrices, by exploiting the FST and the FFT, respectively.

Acknowledgment. A referee pointed out that the Chebyshev–Vandermonde and what we call *three-term Vandermonde matrices* $V_P(x)=\left[\ P_{j-1}(x_i)\ \right]_{1\leq i,j,\leq n}$, involving orthogonal on a real interval polynomials $P_k(x)$, were also studied by Heinig, Hoppe and Rost in [HHR]. These authors observed that three-term Vandermonde matrices satisfy the displacement equation of the form (5.1) with tridiagonal matrix A, and used this fact to derive a fast Levinson-type algorithm for triangular factorization of the *inverse* of $V_P(x)$. Note that pivoting techniques are not readily used with Levinson-type algorithms. In the present paper, use of the different displacement equation (2.3), with a triangular A, allowed us to design a fast Schur-type algorithm, which computes the triangular factorization of the matrix itself. This enabled us to incorporate partial pivoting, and to design a simultaneously fast and accurate algorithms not only for Chebyshev–Vandermonde matrices, but also for the new class of Chebyshev–Vandermonde–like matrices.

Moreover, before receipt of the referee's report, we had already submitted the forthcoming paper [KO], which carries over all the results of the present paper to the more general class of polynomial Vandermonde matrices $V_P(x)$, in which the polynomials $P_k(x)$ only satisfy $\deg P_k(x)=k$. In particular one of the displacement equations in [KO] has the form (5.1), where A is a special upper Hessenberg matrix called a *confederate* matrix in [MB]. In the special case of polynomials $P_k(x)$, orthogonal on a real interval, the confederate matrix reduces to a tridiagonal form known as a *comrade* matrix, see, e.g., [MB]. Another displacement equation in [KO] is of the form (2.3) with a triangular A, and it allowed us to implement GEPP for the general class of polynomial Vandermonde-like matrices.

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