Cyclic dimensions, kernel multiplicities and Gohberg–Kaashoek numbers

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ABSTRACT

Two geometric characteristics, namely *cyclic dimensions* and *kernel multiplicities*, are introduced for a square matrix. The connection between these characteristics and *Gohberg–Kaashoek numbers* is studied. On this basis two simple geometric proofs are given for the theorem about the change of the Jordan structure of a given matrix under small perturbation.

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0. INTRODUCTION

Let A be a matrix in $\mathbf{C}^{n\times n}$, and $\sigma(A)$ be the set of all its eigenvalues. Let $m_1(A,\lambda_0) \geq m_2(A,\lambda_0) \geq ... \geq m_t(A,\lambda_0)$ be the sizes of all blocks corresponding to $\lambda_0 \in \sigma(A)$ in the Jordan form of A. For convenience we set $m_i(A,\lambda_0) = 0$ $(i = t+1, t+2, ..., n)$. The numbers

$$m_i(A) = \sum_{\lambda\in\sigma(A)} m_i(A,\lambda)$$

are referred to as *Gohberg-Kaashoek numbers*. They were introduced in [GK], where the problem of complete description for the Jordan structure of a matrix, which is a small perturbation of a given matrix A_0 was posed. Moreover, such a description was conjectured in [GK], and afterwards it was independently proved in [MP] and [DBT]. Before formulating their result, let us introduce the necessary notations. Let $a = \{a_i\}_1^n$, $b = \{b_i\}_1^n$ be two vectors with nonnegative integer entries, such that $a_i \geq a_{i+1}$ and $b_i \geq b_{i+1}$ $(i = 1, ..., n-1)$. We shall write $a \prec b$ if

$$\sum_{i=1}^{d} a_i \leq \sum_{i=1}^{d} b_i \qquad (d = 1, 2, ..., n) \qquad \text{and} \qquad \sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i.$$

THEOREM 0.1 ([GK], [MP], [DBT]) *Let matrix $A_0 \in \mathbf{C}^{n\times n}$ be given. Then the following statements hold.*

(i) *There exists $\varepsilon > 0$, such that any matrix $A \in \mathbf{C}^{n\times n}$ with $\|A - A_0\| < \varepsilon$ satisfies*

$$\{m_i(A_0)\}_1^n \prec \{m_i(A)\}_1^n. \tag{0.1}$$

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(ii) *The relations* (0.1) *are the only restrictions on the variation of the Jordan structure of a matrix under small perturbation.*

Note that in case where matrices A_0 and A are selfadjoint with respect to indefinite inner product, there are additional to (0.1) restrictions for the Jordan structure of a perturbation A, see, e.g. [O1], where the role played by the so-called *sign characteristic* is revealed.

Here we may also remark that the proof of the assertion (ii) of Theorem 0.1 makes no difficulties, and that it is essentially reduced to Examples 1 and 2, given in Appendix (see, e.g. [MP], [DBT]).

Both proofs in [MP] and [DBT] of the assertion (i) were purely algebraic. In the present paper two new simple proofs, which reveal the geometric aspects of the relations (0.1), are given. These proofs are obtained as a byproduct of the study of new matrix characteristics, namely of *cyclic dimensions* and *kernel multiplicities* that are introduced in the present paper.

The d-th cyclic dimension of a matrix $A \in C^{n\times n}$ is defined as the maximal dimension over all A-invariant subspaces, generated by d vectors. The behavior of cyclic dimensions under small perturbations of a matrix, and their relation to Gohberg-Kaashoek numbers is studied in Section 1.

The d-th kernel multiplicity of $A \in \mathbf{C}^{n\times n}$ is defined in Section 3 as the maximal dimension of the kernel of $f(A)$ over all polynomials $f(\lambda)$, whose degrees do not exceed d. The behavior of kernel multiplicities under small perturbations of a matrix, and their relation to Gohberg-Kaashoek numbers is studied in Section 3.

It is our pleasure to thank A.Markus for fruitful discussions, and C.Davis for suggestions that improved the exposition of the paper.

1. CYCLIC DIMENSIONS

For any d vectors $f_1, f_2, ..., f_d \in \mathbf{C}^n$ let

$$S_A(f_1, ..., f_d) = \mathrm{Span}\{f_1, ..., f_d, Af_1, ..., Af_d, A^2 f_1, ..., A^2 f_d, A^{n-1} f_1, ..., A^{n-1} f_d\}$$

be the *Krylov subspace*, spanned by the vectors $f_1, ..., f_d$. In accordance with the Cayley-Hamilton theorem, the matrix A^n is a linear combination of the lower powers of the same matrix, and hence $S_A(f_1, ..., f_d)$ is the minimal A-invariant subspace, spanned by the vectors $f_1, ..., f_d$. Introduce for A the *cyclic dimensions* by

$$r_d(A) = \max_{f_1,...,f_d \in \mathbf{C}^n} \dim S_A(f_1, ..., f_d) \qquad (d = 1, 2, ..., n).$$

The following theorem shows that cyclic dimensions may only increase under small perturbations of a matrix.

THEOREM 1.1 *Let matrix $A_0 \in \mathbf{C}^{n\times n}$ be given. Then there exists $\varepsilon > 0$, such that any matrix $A \in \mathbf{C}^{n\times n}$ with $\|A - A_0\| < \varepsilon$ satisfies*

$$r_d(A_0) \le r_d(A) \qquad (d = 1, 2, ..., n) \qquad \text{and} \qquad r_n(A_0) = r_n(A). \tag{1.1}$$

Proof. Let $1 \le d \le n$, and let $f_1, ..., f_d$ be vectors satisfying

$$r_d(A_0) = \dim S_{A_0}(f_1, ..., f_d).$$

Then there are numbers α_{ijk}, such that vectors

$$g_k = \sum_{i=1}^{d} \sum_{j=0}^{n-1} \alpha_{ijk} A_0^j f_i \qquad (k = 1, ..., r_d(A_0))$$

form a basis in the subspace $S_{A_0}(f_1, ..., f_d)$. It easy to see that for A satisfying $\|A - A_0\| < \varepsilon$ with small ε, the vectors

$$g'_k = \sum_{i=1}^{d} \sum_{j=0}^{n-1} \alpha_{ijk} A^j f_i \qquad (k = 1, ..., r_d(A_0))$$

remain linearly independent. This implies the inequalities in (1.1). The last equality in (1.1) is obvious.

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In the next theorem we establish a connection between the geometric characteristics $r_d(A)$ and the Gohberg-Kaashoek numbers $m_d(A)$.

THEOREM 1.2 *Gohberg-Kaashoek numbers and cyclic dimensions of any matrix $A \in \mathbf{C}^{n\times n}$ are related by*

$$r_d(A) = \sum_{i=1}^{d} m_i(A) \qquad (d = 1, 2, ..., n). \tag{1.2}$$

Clearly, the assertion (i) of Theorem 0.1. immediately follows from Theorems 1.1 and 1.2. Moreover, relations (1.1) provide a geometric interpretation of algebraic inequalities (0.1).

The proof of Theorem 1.2 is based on three nice properties of cyclic dimensions and of the Jordan structure of a matrix. These properties will be proven in the next three Lemmas 1.3, 1.4 and 1.5. Then with this background we shall return to the proof of Theorem 1.2.

The first Lemma 1.3 claims that cyclic dimensions have the property of additivity with respect to the spectrum of a matrix. Here and henceforth by $A|_{\mathcal{M}}$ is denoted the restriction of the matrix $A \in \mathbf{C}^{n\times n}$ to A-invariant subspace $\mathcal{M} \subseteq \mathbf{C}^n$.

LEMMA 1.3 *Let $A \in \mathbf{C}^{n\times n}$ and $\mathcal{M}, \mathcal{N} \subseteq \mathbf{C}^n$ be two A-invariant subspaces, such that*

$$\sigma(A|_{\mathcal{M}}) \cap \sigma(A|_{\mathcal{N}}) = \emptyset. \tag{1.3}$$

Then for $d = 1, 2, ..., n$ the following equalities hold :

$$r_d(A|_{\mathcal{M}\dot{+}\mathcal{N}}) = r_d(A|_{\mathcal{M}}) + r_d(A|_{\mathcal{N}}). \tag{1.4}$$

Proof. Let $\xi_i = \varphi_i + \psi_i$, where $\varphi_i \in \mathcal{M}$, $\psi_i \in \mathcal{N}$ $(i = 1, 2, ..., d)$. Let us first show that

$$S_A(\varphi_1, ..., \varphi_d) \subseteq S_A(\xi_1, ..., \xi_d). \tag{1.5}$$

To this end choose numbers α_{ijk}, such that vectors

$$g_k = \sum_{i=1}^{d} \sum_{j=1}^{n-1} \alpha_{ijk} A^j \varphi_i \qquad (k = 1, ..., \dim S_A(\varphi_1, ..., \varphi_d))$$

form a basis in $S_A(\varphi_1, ..., \varphi_d)$. Furthermore, let $f(\lambda)$ be the minimal polynomial of the matrix $A|_{\mathcal{N}}$. From (1.3) it follows that the matrix $f(A|_{\mathcal{M}})$ is invertible. Hence the vectors $f(A)g_k$ $(k = 1, ..., \dim S_A(\varphi_1, ..., \varphi_d))$ also form a basis in the A-invariant subspace $S_A(\varphi_1, ..., \varphi_d) \subseteq \mathcal{M}$. Furthermore, since $f(A)\psi_i = 0$ for $i = 1, ..., d$, hence

$$f(A)g_k = \sum_{i=1}^{d} \sum_{j=1}^{n-1} \alpha_{ijk} A^j f(A) \varphi_i = \sum_{i=1}^{d} \sum_{j=1}^{n-1} \alpha_{ijk} A^j f(A) \xi_i,$$

and (1.5) follows. The inclusion

$$S_A(\psi_1, ..., \psi_d) \subseteq S_A(\xi_1, ..., \xi_d) \tag{1.6}$$

is deduced with exactly the same arguments. Furthermore, from (1.5), (1.6) and the obvious inclusion

$$S_A(\xi_1, ..., \xi_d) \subseteq S_A(\varphi_1, ..., \varphi_d) \dot{+} S_A(\psi_1, ..., \psi_d)$$

it follows that

$$S_A(\xi_1, ..., \xi_d) = S_A(\varphi_1, ..., \varphi_d) \dot{+} S_A(\psi_1, ..., \psi_d). \tag{1.7}$$

The latter equality implies (1.4). The lemma is proved.

■

The next lemma asserts that when one passes from an arbitrary matrix $A \in \mathbf{C}^{n\times n}$ to its restriction $A|_{\mathcal{M}}$ to an A-invariant subspace $\mathcal{M} \in \mathbf{C}^{n\times n}$, the sizes of the corresponding Jordan blocks may only decrease. The proof for this statement can be found in [GLR, Theorem 4.1.4], but we shall give here another short proof.

Lemma 1.4 *Let $A \in \mathbf{C}^{n\times n}$ and $\mathcal{M} \subseteq \mathbf{C}^n$ be an A-invariant subspace. Then for each $\lambda_0 \in \sigma(A)$ the following inequalities hold*

$$m_i(A|_{\mathcal{M}}, \lambda_0) \leq m_i(A, \lambda_0) \qquad (i = 1, 2, ..., n).$$

Proof. Let us observe that the number

$$\dim \operatorname{Ker}(A - \lambda_0 I)^i - \dim \operatorname{Ker}(A - \lambda_0 I)^{i-1}$$

is equal to the number of the blocks with the sizes at least i, corresponding to the eigenvalue λ_0 in the Jordan form of A. In other words,

$$\max\{l : m_l(A, \lambda_0) \geq i\} = \dim \operatorname{Ker}(A - \lambda_0 I)^i - \dim \operatorname{Ker}(A - \lambda_0 I)^{i-1}.$$

According to the latter equality, it is sufficient to prove the following inequalities

$$\dim \operatorname{Ker}(A|_{\mathcal{M}} - \lambda_0 I)^i - \dim \operatorname{Ker}(A|_{\mathcal{M}} - \lambda_0 I)^{i-1} \leq \dim \operatorname{Ker}(A - \lambda_0 I)^i - \dim \operatorname{Ker}(A - \lambda_0 I)^{i-1} \tag{1.8}$$

for $i = 2, ..., n$. Denote by $l \in \mathbf{N}$ the left hand side of (1.8), and let $g_1, ..., g_l \in \operatorname{Ker}(A|_{\mathcal{M}} - \lambda_0 I)^i$ be l vectors, which are linearly independent modulo subspace $\operatorname{Ker}(A|_{\mathcal{M}} - \lambda_0 I)^{i-1}$. In this case the vectors $g_1, ..., g_l \in \operatorname{Ker}(A - \lambda_0 I)^i$ are linearly independent modulo subspace $\operatorname{Ker}(A - \lambda_0 I)^{i-1}$. This fact implies that the right hand side of (1.8) is at least l. The lemma is proved.

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Finally, Lemma 1.5 asserts that an A-invariant subspace, generated by d vectors cannot contain more than d eigenvectors, corresponding to the same eigenvalue.

Lemma 1.5 *Given matrix $A \in \mathbf{C}^{n \times n}$ and vectors $f_1, ..., f_d \in \mathbf{C}^n$. Then for each $\lambda_0 \in \sigma(A)$*

$$\dim \operatorname{Ker}(A|_{S_A(f_1,...,f_d)} - \lambda_0 I) \leq d. \tag{1.9}$$

Proof. Obviously,

$$S_A(f_1, ..., f_d) = \operatorname{Span}(f_1, ..., f_d, (A|_{S_A(f_1,...,f_d)} - \lambda_0 I)f_1, ..., (A|_{S_A(f_1,...,f_d)} - \lambda_0 I)f_d, ...$$
$$..., (A|_{S_A(f_1,...,f_d)} - \lambda_0 I)^{n-1} f_1, ..., (A|_{S_A(f_1,...,f_d)} - \lambda_0 I)^{n-1} f_d).$$

According to the latter equality $S_A(f_1, ..., f_d)$ is a linear span of dn vectors, only first d of which may not belong to $\operatorname{Im}(A|_{S_A(f_1,...,f_d)} - \lambda_0 I)$. Therefore, the codimension of the subspace $\operatorname{Im}(A|_{S_A(f_1,...,f_d)} - \lambda_0 I)$ in $S_A(f_1, ..., f_d)$ does not exceed d, and (1.9) follows. The lemma is proved.

■

Now we are able to prove Theorem 1.1.

Proof of Theorem 1.1. In accordance with Lemma 1.3 it is sufficient to prove the equalities (1.2) for the simplest case, when matrix A has only one eigenvalue λ_0. In the latter situation the equalities (1.2) are reduced to the following form

$$r_d(A) = \sum_{i=1}^{d} m_i(A, \lambda_0) \qquad (d = 1, 2, ..., n). \tag{1.10}$$

Let the vectors

$$\begin{cases} \varphi_{1,1}, & \cdots & , \varphi_{1,m_1(A,\lambda_0)} \\ \varphi_{2,1} & \cdots & , \varphi_{2,m_2(A,\lambda_0)} \\ \vdots & & \vdots \\ \varphi_{t,1}, & \cdots & , \varphi_{t,m_t(A,\lambda_0)} \end{cases}$$

be a Jordan basis of the matrix A. Then for $d = 1, 2, ..., n$

$$\dim S_A(\varphi_{1,m_1(A,\lambda_0)}, ..., \varphi_{d,m_d(A,\lambda_o)}) = \sum_{i=1}^{d} m_i(A, \lambda_0), \tag{1.11}$$

which implies

$$r_d(A) \geq \sum_{i=1}^{d} m_i(A, \lambda_0) \qquad (d = 1, 2, ..., n). \tag{1.12}$$

Now let us show that the converses of inequalities (1.12) hold. To this end let for some $1 \leq d \leq n$ the vectors $f_1, ... f_d \in \mathbf{C}^n$ satisfy $r_d(A) = \dim S_A(f_1, ..., f_d)$. Obviously, in this case $r_d(A) = \sum_{i=1}^{n} m_i(A|_{S_A(f_1,...,f_d)}, \lambda_0)$. From Lemma 1.5 it follows that $m_i(A|_{S_A(f_1,...,f_d)}, \lambda_0) = 0$ for $i = d+1, ..., n)$; and hence $r_d(A) = \sum_{i=1}^{d} m_i(A|_{S_A(f_1,...,f_d)}, \lambda_0)$. This equality and Lemma 1.4 yield

$$r_d(A) \leq \sum_{i=1}^{d} m_i(A, \lambda_0) \qquad (d = 1, 2, ..., n). \tag{1.13}$$

Inequalities (1.12) and (1.13) imply (1.10), and Theorem 1.2 follows.

■

In the next proposition we observe that it is possible to chose the sequence of vectors $f_1, f_2, ...,$ so that for each $d = 1, 2, ...,$ the first d vectors $f_1, f_2, ..., f_d$ generate the d-th cyclic dimension $r_d(A)$.

Proposition 1.6 *Let $A \in \mathbf{C}^{n\times n}$ be arbitrary. There exists a set of vectors $f_1, f_2, ..., f_t$, satisfying*

$$\dim S_A(f_1, ..., f_d) = r_d(A) \qquad (d = 1, 2, ..., t), \tag{1.14}$$

and $r_t(A) = n$.

Proof. Assume that A has exactly l eigenvalues, i.e. $\sigma(A) = \{\lambda_1, ..., \lambda_l\}$, where eigenvalues $\lambda_1, ..., \lambda_l$ of A are ordered so that $t_j = \dim \mathrm{Ker}(A - \lambda_j I) \geq t_{j+1} = \dim \mathrm{Ker}(A - \lambda_{j+1} I)$. Let the vectors

$$\left\{ \begin{array}{lll} \varphi_{1,1}^{(j)}, & \cdots & , \varphi_{1,m_1(A,\lambda_j)}^{(j)} \\ \varphi_{2,1}^{(j)} & \cdots & , \varphi_{2,m_2(A,\lambda_j)}^{(j)} \\ \vdots & & \vdots \\ \varphi_{t_j,1}^{(j)}, & \cdots & , \varphi_{t_j,m_{t_j}(A,\lambda_j)}^{(j)} \end{array} \right. \qquad (j = 1, ..., l) \tag{1.15}$$

form a Jordan basis of A. Then the vectors

$$f_i = \sum_{j=1}^{q_i} \varphi_{i,m_i(A,\lambda_j)}^{(j)} \qquad (i = 1, 2, ..., t_1)$$

with $q_i = \max\{j \mid t_j \geq i\}$, satisfy the condition (1.14). Indeed, in accordance with (1.7) for $d = 1, 2, ..., t_1$ the following decomposition holds :

$$S_A(f_1, ..., f_d) = S_A(\varphi_{1,m_1(A,\lambda_1)}^{(1)}, ..., \varphi_{u_1,m_{u_1}(A,\lambda_1)}^{(1)}) \dot{+} ... \dot{+} S_A(\varphi_{1,m_1(A,\lambda_l)}^{(l)}, ..., \varphi_{u_l,m_{u_l}(A,\lambda_l)}^{(l)}),$$

where $u_i = \min\{d, t_i\}$. From here, (1.11) and Theorem 1.1 the equalities (1.14) follow.

■

We conclude this section with a remark concerning a more general situation, where A is a linear operator acting in infinite dimensional space H. In this case cyclic dimensions can be defined by

$$r_d(A) = \max_{f_1,...,f_d \in H} \dim \mathrm{Span}\{A^j f_i \mid i = 1, ..., d; \; j = 1, 2, ...\}.$$

Furthermore, if all the values $r_d(A)$ $(d = 1, 2, ...)$ are finite, then all the results of this section on cyclic dimensions (i.e. Theorem 1.1 and Lemma 1.2) remain valid. Moreover, their proofs simply repeat the arguments given above for the finite dimensional case. Therefore Theorem 1.1 can be regarded as an infinite dimensional generalization of Theorem 0.1.

2. DUAL TO GOHBERG-KAASHOEK NUMBERS

Let $\{m_i\}_1^n$ be a vector with nonnegative integer entries, satisfying $m_i \geq m_{i+1}$ $(i = 1, 2, ...$ $..., n-1)$. The vector $k = \{k_i\}_1^n$ is referred to as *dual* to $\{m_i\}_1^n$ if it satisfies

$$k_i = \max_{1 \leq l \leq n} \{l : m_l \geq i\}.$$

Following [MO, 7.B] introduce for $\{m_i\}_1^n$ the *incidence* matrix $B \in \mathbf{C}^{n \times m_1}$, so that the first m_i entries in the i-th row of B are ones, and the other entries of the i-th row are zeros $(i = 1, ..., n)$. It is easy to see that the sum of the entries of the i-th row of B is equal to m_i $(i = 1, ..., n)$, and the sum of the entries in the i-th column of B is equal to k_i $(i = 1, ..., m_1)$. Obviously, $k_i = 0$ for $i = m_1 + 1, ..., n$.

Example. Let $\{m_i\}_1^5 = \begin{bmatrix} 4 & 3 & 1 & 0 & 0 \end{bmatrix}^T$, then the corresponding incidence matrix has the form $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Counting in this matrix the the column sums, one obtains that the dual vector is given by $\{k_i\}_1^5 = \begin{bmatrix} 3 & 2 & 2 & 1 & 0 \end{bmatrix}^T$.

The following lemma is combined from the statements 7.B.2 and 7.B.5 in [MO].

Lemma 2.1 *Let $\{k_i\}_1^n$ and $\{k'_i\}_1^n$ be the vectors, dual to $\{m_i\}_1^n$ and $\{m'_i\}_1^n$, respectively. Then the following assertions are equivalent :*

(i) $\{m_i\}_1^n \prec \{m'_i\}_1^n$;

(ii) $\{k'_i\}_1^n \prec \{k_i\}_1^n$.

Let $A \in \mathbf{C}^{n \times n}$ be given and $\sigma(A) = \{\lambda_1, ..., \lambda_l\}$. Denote by $\{k_i(A, \lambda_j)\}_{i=1}^n$ the vectors, dual to the vectors$\{m_i(A, \lambda_j)\}_{i=1}^n$ $(j = 1, ..., l)$, respectively. Let $\{k_i(A)\}_{i=1}^n$ be the vector, dual to the vector $\{m_i(A)\}_{i=1}^n$, whose entries are the Gohberg-Kaashoek numbers of A.

In accordance with Lemma 2.1, Theorem 0.1 is equivalent to the following theorem.

Theorem 2.2 *Given $A_0 \in \mathbf{C}^{n \times n}$, there exists $\varepsilon > 0$, such that any matrix $A \in \mathbf{C}^{n \times n}$ with $\|A - A_0\| < \varepsilon$ satisfies*

$$\{k_i(A)\}_1^n \prec \{k_i(A_0)\}_1^n. \tag{2.1}$$

The direct simple proof of Theorem 2.2 will be given in the next section.

3. KERNEL MULTIPLICITIES

Introduce for $A \in \mathbf{C}^{n\times n}$ the *kernel multiplicities* by

$$p_d(A) = \max_{f(\lambda)\in\mathbf{C}_d[\lambda]} \dim \mathrm{Ker} f(A) \qquad (d = 1, ..., m_1(A)).$$

Here $\mathbf{C}_d[\lambda]$ stand for the set of all complex polynomials in λ whose degrees do not exceed d. Remark that Theorem 0.1 was proved in [MP] by making use of the numbers

$$\theta_d(A) = \min_{f(\lambda)\in\mathbf{C}_d[\lambda]} \mathrm{rank} f(A) \qquad (d = 1, ..., m_1(A)).$$

Clearly, algebraic characteristics $\theta_d(A)$ and kernel multiplicities $p_d(A)$ are related as follows : $\theta_d(A) + p_d(A) = n$ $(d = 1, ..., m_1(A))$.

The following theorem shows that kernel multiplicities may only decrease under small perturbations of a matrix.

Theorem 3.1 *Let matrix $A_0 \in \mathbf{C}^{n\times n}$ be given. Then there exists $\varepsilon > 0$, such that any matrix $A \in \mathbf{C}^{n\times n}$ with $\|A - A_0\| < \varepsilon$ satisfies*

$$p_d(A_0) \geq p_d(A) \qquad (d = 1, 2, ..., n) \qquad \text{and} \qquad p_n(A_0) = p_n(A). \tag{3.1}$$

Proof. Suppose it were not so. Then for some d there must exist a sequence $\{A_j\}_{j=1}^{\infty}$ of matrices converging to A_0, such that $p_d(A_0) < p_d(A_j)$. By the definition of kernel multiplicities there must exist a sequence of polynomials $\{f_j(\lambda)\}_{j=1}^{\infty}$ from $\mathbf{C}_d[\lambda]$, such that

$$p_d(A_0) < \dim \mathrm{Ker} f_j(A_j). \tag{3.2}$$

Without any loss of generality we may assume that the sum of the modulii of the coefficients of each polynomial $f_j(\lambda)$ equals to 1, and then the sequence $\{f_j(\lambda)\}$ contains a subsequence $\{f_{j_t}(\lambda)\}$ converging to some polynomial $f(\lambda) \in \mathbf{C}_d[\lambda]$. Since $f_j(A_j) \to f(A_0)$, we have $\dim \mathrm{Ker} f_j(A_j) < \dim \mathrm{Ker} f(A_0)$. From here and (3.2) it follows that $p_d(A_0) < \dim \mathrm{Ker} f(A_0)$, which is impossible. This proves the theorem.

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The connection between the geometric characteristics $p_d(A)$ and the dual to Gohberg-Kaashoek numbers is given in the next theorem.

Theorem 3.2 *Given $A \in \mathbf{C}^{n\times n}$, then for $d = 1, 2, ..., n$ the following equalities hold :*

$$p_d(A) = \sum_{i=1}^{d} k_i(A). \tag{3.3}$$

Clearly, Theorems 3.1 and 3.2 immediately imply Theorem 2.2. Moreover, the relations (3.1) provide a geometric interpretation of algebraic inequalities (2.1).

The proof of Theorem 3.2 is based on the following lemma, which was stated in [O2]. We provide it here with a short proof.

Lemma 3.3 *Let $\{k_i^{(j)}\}_{i=1}^{n}$ $(j = 1, 2, ..., l)$ be l vectors, which are dual to $\{m_i^{(j)}\}_{i=1}^{n}$ $(j = 1, 2, ..., l)$, respectively. Let $\{k_i\}_1^n$ be the vector, which is dual to the vector $\{m_i\}_1^n$, where $m_i = \sum_{j=1}^{l} m_i^{(j)}$ $(i = 1, ..., n)$. Then the vector $\{k_i\}_1^n$ is obtained by arranging in a nonincreasing order the n numbers with maximal magnitude from $\{k_d^{(j)} \ : \ d = 1, ..., n;\ j = 1, ..., l\}$.*

Proof. Let $B \in \mathbf{C}^{n \times m_1}$ the incidence matrix, corresponding to the vector $\{m_i\}_1^n$. Let $B_j \in \mathbf{C}^{n \times m_1^{(j)}}$ be the incidence matrices, corresponding to the vectors $\{m_i^{(j)}\}_{i=1}^n$ $(j = 1, ..., l)$. It is easy to see that the matrix B is derived from the block matrix $\left[\begin{array}{cccc} B_1 & B_2 & \cdots & B_l \end{array} \right]$ by means of swapping the columns in the nonincreasing order. This proves the lemma.

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We are ready now to prove Theorem 3.2.

Proof of Theorem 3.2. Assume that A has exactly l distinct eigenvalues $\lambda_1, ..., \lambda_l$. The equalities

$$\dim \operatorname{Ker}(A - \lambda_j)^d = \sum_{i=1}^{d} k_i(A, \lambda_j) \qquad (j = 1, ..., l). \tag{3.4}$$

are given in [GLR, Proposition 2.2.6], and they can be easily deduced using the Jordan form of A.

Let $f(\lambda)$ be an arbitrary polynomial from $\mathbf{C}_d[\lambda]$. Represent $f(\lambda)$ as

$$f(\lambda) = (\lambda - \lambda_1)^{t_1} \cdot ... \cdot (\lambda - \lambda_l)^{t_l} \cdot g(\lambda),$$

where $g(\lambda)$ does not vanish on $\sigma(A)$. From the latter equality, formula (3.4), and the analysis of the Jordan form of A it follows that

$$\dim \operatorname{Ker} f(A) = \sum_{i=1}^{l} \dim \operatorname{Ker}(A - \lambda_i)^{t_i} = \sum_{i=1}^{l} \sum_{j=1}^{t_i} k_j(A, \lambda_i). \tag{3.5}$$

To obtain the maximum over all polynomials $f(\lambda) \in \mathbf{C}_d[\lambda]$ in (3.5), one has to choose for the right hand side d maximal numbers from the set $\{k_i(A, \lambda_j) \ : \ j = 1, ..., l; \ i = 1, ..., m_1(A, \lambda_j)\}$. From here and Lemma 3.3 the equalities (3.3) follow. The theorem is proved.

■

4. APPENDIX. TWO EXAMPLES.

Here we illustrate Theorem 0.1 with two simple examples, which are close to given in [MP], [DBT].

Example 1. Denote by $J_n(\lambda)$ a single Jordan block of the size n, corresponding to the eigenvalue λ, and let

$$A_0 = \left[\begin{array}{cc} J_k(0) & 0 \\ 0 & J_s(0) \end{array} \right], \qquad (n = k + s),$$

where we assume $k \geq s$. Clearly, 0 is the only eigenvalue of A_0 with $m_1(A_0, 0) = k$, $m_2(A_0, 0) = s$, and $m_i(A_0, 0) = 0$ for $i \geq 3$. Set

$$A_0 = \left[\begin{array}{cc} J_k(0) & D \\ 0 & J_s(0) \end{array} \right],$$

where the only nonzero entry ε of $D \in \mathbf{C}^{k\times s}$ occupies the (k,s) position. Then straightforward computation shows that $\|A - A_0\| = \varepsilon$, and that the matrix A has only eigenvalue 0 with

$$m_1(A,0) = k+1, \qquad m_2(A,0) = s-1 \qquad \text{and} \qquad m_i(A,0) = 0 \qquad \text{for} \qquad i \geq 3. \tag{4.1}$$

Indeed, let $\{e_i\}_1^n$ stands for the standard orthonormal basis in $\mathbf{C}^n$, so that $\{e_1, ..., e_k\}$ and $\{e_{k+1}, ..., e_n\}$ form two Jordan chains of the matrix A_0. It is easy to see that $\{\varepsilon e_1, \dots, \varepsilon e_{2k-n+1}, \varepsilon e_{2k-n+2} + e_{k+1}, ..., \varepsilon e_k + e_{n-1}, e_n\}$ and $\{e_{k+1}, ..., e_{n-1}\}$ form two Jordan chains for the perturbed matrix A, and (4.1) follows.

This is an example of a perturbation, where the eigenvalues of a matrix remain unchanged, with the larger of Gohberg-Kaashoek numbers increasing at the expense of the smaller.

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Example 2. Now let

$$A_0 = J_n(0) = \begin{bmatrix} J_k(0) & C \\ 0 & J_s(0) \end{bmatrix}, \qquad (n = k+s),$$

i.e. the only nonzero entry of $C \in \mathbf{C}^{k\times s}$ occupies the (k,1) position. Clearly, matrix A_0 has the only eigenvalue 0 with $m_1(A_0,0) = n$, and $m_i(A_0,0) = 0$ for $i \geq 2$. Now set

$$A = J_n(0) = \begin{bmatrix} J_k(\varepsilon) & C \\ 0 & J_s(0) \end{bmatrix}.$$

It is easy to see that the matrix A has exactly two eigenvalues ε and 0 with $m_1(A,\varepsilon) = k$ and $m_1(A,0) = s$, and $m_i(A,\varepsilon) = m_i(A,0) = 0$ for $i \geq 2$. Obviously $\|A - A_0\| = \varepsilon$ and

$$m_i(A_0) = m_i(A) \qquad (i = 1, 2, ..., n).$$

This is an example of a perturbation, where one eigenvalue of a matrix is splited into two eigenvalues, with Gohberg-Kaashoek numbers remaining unchanged.

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Let matrix $A_0 \in \mathbf{C}^{n\times n}$ be given. Following [MP], [DBT] we may remark that applying a sequence of elementary perturbations, described in the above two examples, one can easily construct the small perturbation of A_0 with any Jordan structure, obeying (0.1). This proves the assertion (ii) of Theorem 0.1.

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