Eigenstructure of Order-One-Quasiseparable Matrices. Three-term and Two-term Recurrence Relations

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Abstract

This paper presents explicit formulas and algorithms to compute the eigenvalues and eigenvectors of *order-one-quasiseparable matrices*. Various recursive relations for characteristic polynomials of their principal submatrices are derived. The cost of evaluating the characteristic polynomial of an $N \times N$ matrix and its derivative is only $O(N)$. This leads immediately to several versions of a fast *quasiseparable Newton iteration* algorithm. In the Hermitian case we extend the Sturm property to the characteristic polynomials of order-one-quasiseparable matrices which yields to several versions of a fast *quasiseparable bisection algorithm*.

Conditions guaranteeing that an eigenvalue of a order-one-quasiseparable matrix is simple are obtained, and an explicit formula for the corresponding eigenvector is derived. The method is further extended to the case when these conditions are not fulfilled. Several particular examples with tridiagonal, (almost) unitary Hessenberg, and Toeplitz matrices are considered.

The algorithms are based on new three-term and two-term recurrence relations for the characteristic polynomials of principal submatrices of *order-one-quasiseparable matrices* R. It turns out that the latter *new class of polynomials* generalizes and includes two classical families: (i) polynomials orthogonal on the real line (that play a crucial role in a number of classical algorithms in numerical linear algebra), and (ii) the Szego polynomials (that play a significant role in signal processing). Moreover, new formulas can be seen as generalizations of the classical three-term recurrence relations for the real orthogonal polynomials and of the two-term recurrence relations for the Szego polynomials.

1 Introduction

1.1. Quasiseparable and diagonal-plus-semiseparable matrices. Following [6] we refer to R as *quasiseparable* order (r_L, r_U) matrix if

$$r_L = \max \operatorname{rank} R_{21}, \qquad r_U = \max \operatorname{rank} R_{12}, \tag{1.1}$$

where the maximum is taken over all *symmetric* partitions of the form $R = \left[\begin{array}{c|c} * & R_{12} \\ \hline R_{21} & * \end{array}\right]$. In case $r_U = r_L = r$ one refers to R as an order-r-quasiseparable matrix. Quasiseparable matrices generalize *diagonal-plus-semiseparable matrices* [15], i.e., those of the form

$$R = diag(d) + tril(R_L) + triu(R_U), \qquad \text{where} \qquad \operatorname{rank} R_L = r_L, \qquad \operatorname{rank} R_U = r_U. \tag{1.2}$$

with some vector d , and with some matrices R_L, R_U. Here $tril(R)$ and $triu(R)$ are the standard MATLAB notations standing for the strictly lower and stryctly upper triangular parts of R, resp. Clearly, diagonal-plus-semiseparable matrices are quasiseparable, but not vice versa, e.g., the tridiagonal matrix in (1.5) is not diagonal-plus-semiseparable. Quasiseparable matrices are also known under different names such as matrices with a low Hankel rank [5], low mosaic rank matrices or weekly semiseparable matrices [20]. In the context of some applications, e.g., to integral equations, electromagnetics, boundary value problems, the less general diagonal-plus-semiseparable matrices occur more often. Moreover, the orders higher than $(1,1)$ have attracted more attention perhaps since they exhibit somewhat more computational challenges.

The next section suggests that even in the simplest case of the order $(1,1)$ the more general quasiseparable matrices are by no means less interesting. Moreover, their connection to the theories of orthogonal polynomials on the line and on the unit circle (the Szego polynomials) makes them a useful tool for solving several problems in numerical analysis which is a topic of this and the forthcoming publications.

1.2. Order one matrices. Two motivating examples.

Example 1.1 Tridiagonal matrices and real orthogonal polynomials

- *It is well known that the polynomials* $\{\widetilde{\gamma}_k(x)\}$ orthogonal *with respect to the inner product of the form*

$$< p(x), q(x) >= \int_a^b p(x)q(x)w^2(x)dx$$

satisfy three-term recurrence relations*:*

$$\widetilde{\gamma}_k(x) = (\alpha_k \cdot x - \beta_k) \cdot \widetilde{\gamma}_{k-1}(x) - \gamma_k \cdot \widetilde{\gamma}_{k-2}(x), \tag{1.3}$$

where $\alpha_k, \beta_k, \gamma_k$ *depend on the weight function* $w^2(x)$*.*

- *The relations (1.3) translate into the matrix form*

$$\widetilde{\gamma}_k(x) = (\alpha_0 \cdot \ldots \cdot \alpha_k) \cdot \det(xI - R_{k\times k}) \qquad (1 \le k \le N) \tag{1.4}$$

where

$$R = \begin{bmatrix} \frac{\beta_1}{\alpha_1} & \frac{\gamma_2}{\alpha_2} & 0 & \cdots & 0 & 0 \\ \frac{1}{\alpha_1} & \frac{\beta_2}{\alpha_2} & \frac{\gamma_3}{\alpha_3} & \ddots & \vdots & 0 \\ 0 & \frac{1}{\alpha_2} & \frac{\beta_3}{\alpha_3} & \ddots & 0 & \vdots \\ \vdots & 0 & \frac{1}{\alpha_3} & \ddots & \frac{\gamma_{n-1}}{\alpha_{n-1}} & 0 \\ \vdots & \vdots & \ddots & \ddots & \frac{\beta_{n-1}}{\alpha_{n-1}} & \frac{\gamma_n}{\alpha_n} \\ 0 & 0 & \cdots & 0 & \frac{1}{\alpha_{n-1}} & \frac{\beta_n}{\alpha_n} \end{bmatrix} \tag{1.5}$$

Example 1.2 (Almost) unitary Hessenberg matrices and polynomials orthogonal on the unit circle (Szego polynomials)

- *It is well known that the polynomials* $\{\widetilde{\gamma}_k(x)\}$ orthogonal *with respect to the inner product of the form*

$$< p(e^{iz}), q(e^{iz}) >= \frac{1}{2\pi} \int_{-\pi}^{\pi} p(e^{iz}) \cdot [q(e^{iz})]^* w^2(z)dz \tag{1.6}$$

satisfy two-term recurrence relations *(involving also another set of polynomials* $\{G_k(x)\}$*):*

$$\begin{bmatrix} G_0(x) \\ \widetilde{\gamma}_0(x) \end{bmatrix} = \frac{1}{\mu_0} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad \begin{bmatrix} G_{k+1}(x) \\ \widetilde{\gamma}_{k+1}(x) \end{bmatrix} = \frac{1}{\mu_{k+1}} \begin{bmatrix} 1 & -\rho_{k+1}^* \\ -\rho_{k+1} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & x \end{bmatrix} \begin{bmatrix} G_k(x) \\ \widetilde{\gamma}_k(x) \end{bmatrix}. \tag{1.7}$$

where $\mu_k = \sqrt{1 - |\rho_k|^2}$ *with (*$\mu_k := 1$ *if* $\rho_k = 1, \rho_0 = -1$*). Polynomials* $\{\widetilde{\gamma}_k(x)\}$ *are defined by* N *parameters* $\{\rho_k\}$ *that are called the* reflection coefficients *and that are determined by the wight function* $w^2(x)$*.*

- *The relations (1.7) translate into the matrix form*

$$\widetilde{\gamma}_k(x) = \frac{\det(xI - R_{k\times k})}{\mu_0 \cdot \ldots \cdot \mu_k} \qquad (1 \le k \le N) \tag{1.8}$$

where

$$R = \begin{bmatrix} -\rho_1\rho_0^* & -\rho_2\mu_1\rho_0^* & -\rho_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_{n-1}\mu_{n-2}\ldots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\ldots\mu_1\rho_0^* \\ \mu_1 & -\rho_2\rho_1^* & -\rho_3\mu_2\rho_1^* & \cdots & -\rho_{n-1}\mu_{n-2}\ldots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\ldots\mu_2\rho_1^* \\ 0 & \mu_2 & -\rho_3\rho_2^* & \cdots & -\rho_{n-1}\mu_{n-2}\ldots\mu_3\rho_2^* & -\rho_n\mu_{n-1}\ldots\mu_3\rho_2^* \\ \vdots & \ddots & \mu_3 & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{n-1}\rho_{n-2}^* & -\rho_n\mu_{n-1}\rho_{n-2}^* \\ 0 & \cdots & \cdots & 0 & \mu_{n-1} & -\rho_n\rho_{n-1}^* \end{bmatrix} \tag{1.9}$$

It is known that $R \cdot \operatorname{diag}(1, \ldots, 1, \frac{1}{\rho_n})$ *is unitary.*

- *Along with the two-term recurrence relations (1.8) translates into three-term recurrence relations*

$$\widetilde{\gamma}_0(x) = \frac{1}{\mu_0}, \qquad \widetilde{\gamma}_1(x) = \frac{1}{\mu_1}(x \cdot \widetilde{\gamma}_0(x) + \rho_1\rho_0^* \cdot \widetilde{\gamma}_0(x)),$$

$$\widetilde{\gamma}_k(x) = [\frac{1}{\mu_k} \cdot x + \frac{\rho_k}{\rho_{k-1}}\frac{1}{\mu_k}] \cdot \widetilde{\gamma}_{k-1}(x) - \frac{\rho_k}{\rho_{k-1}}\frac{\mu_{k-1}}{\mu_k} \cdot x \cdot \widetilde{\gamma}_{k-2}(x). \tag{1.10}$$

Observation. A closer look at the submatrices R_{12} of the two matrices R in (1.5) and (1.9),

$$\underbrace{\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \frac{\gamma_4}{\alpha_4} & 0 & \cdots & 0 & 0 \end{bmatrix}}_{R_{12} \text{ for } (1.5)}, \underbrace{\begin{bmatrix} -\rho_4\mu_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_{n-1}\mu_{n-2}\ldots\mu_1\rho_0^* & -\rho_n\mu_{n-1}\ldots\mu_1\rho_0^* \\ -\rho_4\mu_3\mu_2\rho_1^* & \cdots & -\rho_{n-1}\mu_{n-2}\ldots\mu_2\rho_1^* & -\rho_n\mu_{n-1}\ldots\mu_2\rho_1^* \\ -\rho_4\mu_3\rho_2^* & \cdots & -\rho_{n-1}\mu_{n-2}\ldots\mu_3\rho_2^* & -\rho_n\mu_{n-1}\ldots\mu_3\rho_2^* \end{bmatrix}}_{R_{12} \text{ for } (1.9)}$$

reveals that these R_{12} are both rank-one, so that in both examples R is a order-one quasiseparable matrix.

1.3. A new class of polynomials and recurrence relations corresponding to general order-one-quasiseparable matrices. A computationally important property of $N \times N$ quasiseparable matrices R is that they can be represented by only $O((r_L + r_U)N)$ parameters via

$$R = \begin{bmatrix} d_1 & & g_i b_{ij}^{\times} h_j \\ & \ddots & \\ p_i a_{ij}^{\times} q_j & & d_N \end{bmatrix} \quad \text{where } a_{ij}^{\times} = a_{i+1} \cdot \ldots \cdot a_{j-1},\ b_{ij}^{\times} = b_{i+1} \cdot \ldots \cdot b_{j-1} \text{ with } a_{i,i+1} = b_{i,i+1} = 1. \tag{1.11}$$

We limit ourselves to the simplest case of the order one, in which all the parameters $\{d_k, p_k, q_k, g_k, h_h, a_k, b_k\}_{k=1}^N$ are just numbers. (For the higher orders k they are either vectors or matrices of the size k.)

One of the central results of this paper is the recurrence relations for the polynomials $\{\widetilde{\gamma}_k(x)\} = \det(xI - R_{k\times k})$ for a general order one quasiseparable matrix R in (1.11). The polynomials $\{\widetilde{\gamma}_k(x)\}$ form a rather narrow class though wide enough to include as special cases both families of polynomials orthogonal on the real line and on the unit circle. As could be expected, these new polynomials satisfy various recurrence relations. We have three-term recurrence relations

$$\widetilde{\gamma}_k(x) = -\psi_k(x)\widetilde{\gamma}_{k-1}(x) - \varphi_k(x)\widetilde{\gamma}_{k-2}(x) \quad \text{with} \quad \begin{bmatrix} \widetilde{\gamma}_1(x) \\ \widetilde{\gamma}_2(\lambda) \end{bmatrix} = \begin{bmatrix} d_1 - x \\ -[(d_2 - x)(d_1 - x) - p_2q_1g_1h_2] \end{bmatrix}, \tag{1.12}$$

where $\psi_k(x)$ and $\varphi_k(x)$ are given by (3.16) below, and two-term recurrence relations

$$\left[\begin{array}{c} \widetilde{f}_1(x) \\ \widetilde{\gamma}_1(x) \end{array}\right] = \left[\begin{array}{c} q_1 g_1 \\ d_1 - x \end{array}\right]; \qquad \left[\begin{array}{c} \widetilde{f}_k(x) \\ \widetilde{\gamma}_k(x) \end{array}\right] = -\left[\begin{array}{cc} c_k - a_k b_x x & q_k g_k \\ -p_k h_k & d_k - x \end{array}\right] \left[\begin{array}{c} \widetilde{f}_{k-1}(x) \\ \widetilde{\gamma}_{k-1}(x) \end{array}\right]. \tag{1.13}$$

1.4. Recurrence relations. Known and new algorithms. The classical three-term recurrence relations (1.3) lie in the heart of a number of algorithms, including, e.g., the Euclid and the Routh-Hurwitz algorithms. Moreover, several general eigenvalue algorithms first reduce a matrix to the tridiagonal form (1.5) and then find the eigenvalues of the latter. The list of related algorithms includes the tridiagonal Newton iteration, the tridiagonal Sturm sequences method, the Lanczos algorithm, and the tridiagonal QR algorithm. Hence the new generalized three-term recurrence relations (1.12) may suggest possible extensions of these methods from tridiagonal to quasiseparable matrices. This paper is more of a theoretical nature and we specify here only the quasiseparable Newton iteration and the quasiseparable Sturm sequences method. The forthcoming paper will have more computational focus, e.g., [9] discusses the numerical aspects of using the formulas of [8] to run the quasiseparable QR iteration algorithm.

Similarly, the classical unit circle recurrence relations (1.7) and (1.10) lie in the heart of a number of well-known signal processing algorithms. The algorithms of Levinson, Schur and Schur-Cohn are all based on the two-term (1.7). The split-Levinson is based on (1.10). For example, the classical Schur-Cohn algorithm for checking if a given polynomial $\widetilde{\gamma}_N(x)$ is discrete-time-stable (i.e., has all of its roots inside the unit circle) is based on setting $G_N(x) = x^n[\widetilde{\gamma}_N(\frac{1}{x^*})]^*$ and then running (1.7) backwards with wisely chosen ρ_k. Therefore, the new generalized three-term recurrence relations (1.13) can be seen as a direct generalization of the Schur-Cohn recursions.

Finally, the eigenvalue problem for semiseparable matrices is a hot topic of the concurrent research, and a number of methods are being suggested in [2, 3, 4, 11, 17, 18, 21]. The results presented here differ from their substantially. Numerical aspects based on the results of this paper will be discussed separately elsewhere.

1.5. The structure of the paper. The paper contains four sections. Section 1 is the introduction. In the nex section 2 we define the class of quasiseparable of order one matrices and formulate the basic properties of this class obtained in [7]. In Section 3 we derive recursive relations for characteristic polynomials of principal submatrices of quasiseparable matrices. In Section 4 we obtain the conditions in which an eigenvalue of a quasiseparable matrix is simple and methods of computation of the eigenvectors when these conditions are valid and non-valid.

2 Definitions and basic properties of quasiseparable matrices

2.1 Definitions

Let $\{a_k\}, k = 1, \ldots, N$ be a set of numbers. For positive integers i, j define the operation $a_{ij}^{\times}$ as follows: $a_{ij}^{\times} = a_{i-1} \cdots a_{j+1}$ for $N \geq i > j+1 \geq 2$, $a_{ij}^{\times} = a_{i+1} \cdots a_{j-1}$ for $N \geq j > i+1 \geq 2$, $a_{k+1,k}^{\times} = a_{k,k+1}^{\times} = 1$ for $1 \leq k \leq N-1$.

We consider a class of matrices R for which lower triangular and upper triangular parts have a special structure. Let R be a square matrix of size $N \times N$ with entries in the lower triangular part of the form

$$R_{ij} = p_i a_{ij}^{\times} q_j, \quad 1 \leq j < i \leq N,$$

where $p_i,\ q_j,\ a_k$ are given numbers. Then the matrix R is called *lower quasiseparable of order one* and the numbers $p_i\ (i = 2, \ldots, N),\ q_j\ (j = 1, \ldots, N-1),\ a_k\ (k = 2, \ldots, N-1)$ are called *lower generators* of the matrix R.

Let R be a square matrix of size $N \times N$ with entries in the upper triangular part of the form

$$R_{ij} = g_i b_{ij}^{\times} h_j, \quad 1 \leq i < j \leq N,$$

where $g_i,\ h_j,\ b_k$ are given numbers. Then the matrix R is called *upper quasiseparable of order one* and the numbers $g_i\ (i = 1, \ldots, N-1),\ h_j\ (j = 2, \ldots, N),\ b_k\ (k = 2, \ldots, N-1)$ are called *upper generators* of the matrix R.

If a matrix R of size $N \times N$ is lower quasiseparable of order one and upper quasiseparable of order one then it is called *quasiseparable of order one*. More precisely, quasiseparable of order one matrix is a matrix of the form

$$R_{ij} = \begin{cases} p_i a_{ij}^{\times} q_j, & 1 \le j < i \le N, \\ d_i, & 1 \le i = j \le N, \\ g_i b_{ij}^{\times} h_j, & 1 \le i < j \le N. \end{cases} \tag{2.1}$$

The elements $p_i\ (i = 2, \dots, N),\ q_j\ (j = 1, \dots, N-1),\ a_k\ (k = 2, \dots, N-1);\ g_i\ (i = 1, \dots, N-1),\ h_j\ (j = 2, \dots, N),\ b_k\ (k = 2, \dots, N-1);\ d_k\ (k = 1, \dots, N)$ are called *generators* of the matrix R.

The class of quasiseparable of order one matrices is a generalization of two well-known classes of structured matrices: tridiagonal matrices and diagonal plus semiseparable of order one matrices. If $a_k = b_k = 1,\ 2 \le k \le N-1$ then the matrix R is diagonal plus semiseparable of order one. If $a_k = b_k = 0,\ 2 \le k \le N-1$ then R is tridiagonal.

All statements below concern a quasiseparable of order one matrix R with generators denoted by $p_i\ (i = 2, \dots, N),\ q_j\ (j = 1, \dots, N-1),\ a_k\ (k = 2, \dots, N-1);\ g_i\ (i = 1, \dots, N-1),\ h_j\ (j = 2, \dots, N),\ b_k\ (k = 2, \dots, N-1);\ d_k\ (k = 1, \dots, N)$. This means that the entries of the matrix R have the form (2.1), where

$$a_{ij}^{\times} = a_{i-1} \cdots a_{j+1},\ N \ge i > j+1 \ge 2, \quad a_{j+1,j}^{\times} = 1,\ j = 1, \dots, N-1,$$

$$b_{ij}^{\times} = b_{i+1} \cdots b_{j-1},\ N \ge j > i+1 \ge 2, \quad b_{i,i+1}^{\times} = 1,\ i = 1, \dots, N-1.$$

By the generators of the matrix R we define the vectors

$$P_k = \mathrm{col}(p_i a_{i,k-1}^{\times})_{i=k}^{N},\ H_k = \mathrm{row}(b_{k-1,i}^{\times} h_i)_{i=k}^{N}, \quad k = 2, \dots, N; \tag{2.2}$$

$$Q_k = \mathrm{row}(a_{k+1,i}^{\times} q_i)_{i=1}^{k},\ G_k = \mathrm{col}(g_i b_{i,k+1}^{\times})_{i=1}^{k}, \quad k = 1, \dots, N-1. \tag{2.3}$$

which are used frequently in the paper.

2.2 The basic properties of quasiseparable of order one matrices

We present here some results obtained in [7] for quasiseparable of order one matrices. These results are used essentially in this paper.

Theorem 2.1 *Let R be a quasiseparable of order one matrix R with generators $p_i\ (i = 2, \dots, N),\ q_j\ (j = 1, \dots, N-1),\ a_k\ (k = 2, \dots, N-1);\ g_i\ (i = 1, \dots, N-1),\ h_j\ (j = 2, \dots, N),\ b_k\ (k = 2, \dots, N-1);\ d_k\ (k = 1, \dots, N)$. The matrix R admits the partitions*

$$R = \begin{pmatrix} A_k & G_k H_{k+1} \\ P_{k+1} Q_k & B_{k+1} \end{pmatrix}, \quad k = 1, \dots, N-1, \tag{2.4}$$

where

$$A_k = R(1:k, 1:k),\ B_k = R(k:N, k:N), \quad k = 1, \dots, N$$

and the elements P_k, Q_k, G_k, H_k are defined by (2.2), (2.3).

Next we consider some relations which follows directly from (2.4). Setting $k = N-1$ in (2.4) we obtain

$$R = \begin{pmatrix} A_{N-1} & G_{N-1} h_N \\ p_N Q_{N-1} & d_N \end{pmatrix}. \tag{2.5}$$

Applying this formula to the matrix A_k we get

$$A_k = \begin{pmatrix} A_{k-1} & G_{k-1} h_k \\ p_k Q_{k-1} & d_k \end{pmatrix}, \quad k = 1, \dots, N-1. \tag{2.6}$$

Similarly setting in (2.4) $k=1$ we obtain

$$R = \begin{pmatrix} d_1 & g_1 H_2 \\ P_2 q_1 & B_2 \end{pmatrix}. \tag{2.7}$$

Applying this formula to the matrix B_k we get

$$B_k = \begin{pmatrix} d_k & g_k H_{k+1} \\ P_{k+1} q_k & B_{k+1} \end{pmatrix}, \quad k=1,\dots,N-1. \tag{2.8}$$

Substituting (2.6) in (2.4) and using the relations

$$Q_k = \begin{pmatrix} a_k Q_{k-1} & q_k \end{pmatrix},\ G_k = \begin{pmatrix} G_{k-1} b_k \\ g_k \end{pmatrix}, \quad k=2,\dots,N-1$$

we obtain the representations

$$R = \begin{pmatrix} A_{k-1} & G_{k-1} h_k & G_{k-1} b_k H_{k+1} \\ p_k Q_{k-1} & d_k & g_k H_{k+1} \\ P_{k+1} a_k Q_{k-1} & P_{k+1} q_k & B_{k+1} \end{pmatrix}, \quad k=2,\dots,N-1. \tag{2.9}$$

Theorem 2.2 *Let $A_k = R(1:k,1:k),\ B_k = R(k:N,k:N), \quad k=1,\dots,N$ be principal submatrices of the matrix R and let*

$$\gamma_k = \det A_k, \quad \theta_k = \det B_k, \quad k=1,\dots,N$$

and

$$f_k = Q_k(\mathrm{adj} A_k) G_k,\ k=1,\dots,N-1; \quad z_k = H_k(\mathrm{adj} B_k) P_k,\ k=N,\dots,2, \tag{2.10}$$

where the elements Q_k, G_k, P_k, H_k are defined by (2.2), (2.3).

Then the following recursive relations hold:

$$\begin{gathered} \gamma_1 = d_1, \quad f_1 = q_1 g_1; \\ c_k = d_k a_k b_k - q_k p_k b_k - a_k h_k g_k, \\ f_k = c_k f_{k-1} + q_k g_k \gamma_{k-1}, \quad \gamma_k = d_k \gamma_{k-1} - p_k h_k f_{k-1}, \quad k=2,\dots,N-1; \\ \gamma_N = d_N \gamma_{N-1} - p_N h_N f_{N-1} \end{gathered} \tag{2.11}$$

and

$$\begin{gathered} \theta_N = d_N, \quad z_N = h_N p_N, \\ z_k = c_k z_{k+1} + \theta_{k+1} h_k p_k, \quad \theta_k = d_k \theta_{k+1} - g_k q_k z_{k+1}, \quad k=N-1,\dots,2; \\ \theta_1 = d_1 \theta_2 - q_1 g_1 z_2. \end{gathered} \tag{2.12}$$

In the sequel we use also the vector $V_{N-1} = \mathrm{adj} A_{N-1} G_{N-1}$, where $A_{N-1} = R(1:N-1,1:N-1)$ and G_{N-1} is defined in (2.3). This vector may be expressed explicitly via the numbers γ_k, f_k.

Theorem 2.3 *The vector $V_{N-1} = (\mathrm{adj} A_{N-1}) G_{N-1}$ is given by the formula*

$$V_{N-1} = \mathrm{col}(v_i \delta^{\times}_{i,N})_{i=1}^{N-1}, \tag{2.13}$$

where

$$v_1 = g_1, \quad v_i = g_i \gamma_{i-1} - p_i f_{i-1} b_i,\ \delta_i = d_i b_i - g_i h_i, \quad i=2,\dots,N-1. \tag{2.14}$$

The proof may be found in the proof of Theorem 4.1 of [7].

3 The characteristic polynomial

3.1 Recursive relations

In this section we derive recursive relations for characteristic polynomials $\gamma_k(\lambda) = \det(R(1:k, 1:k) - \lambda I)$, $\theta_k(\lambda) = \det(R(k:N, k:N) - \lambda I)$ $(k = 1, \ldots, N)$ of principal submatrices of a quasiseparable of order one matrix.

Theorem 3.1 *Let* $\gamma_k(\lambda) = \det(R(1:k, 1:k) - \lambda I)$, $\theta_k(\lambda) = \det(R(k:N, k:N) - \lambda I)$ $(k = 1, \ldots, N)$ *be characteristic polynomials of the principal submatrices of the matrix* R.

Then the following recursive relations hold:

$$\gamma_1(\lambda) = d_1 - \lambda, \quad f_1(\lambda) = q_1 g_1; \tag{3.1}$$

$$\gamma_k(\lambda) = (d_k - \lambda)\gamma_{k-1}(\lambda) - p_k h_k f_{k-1}(\lambda), \tag{3.2}$$

$$c_k = d_k a_k b_k - q_k p_k b_k - a_k h_k g_k, \tag{3.3}$$

$$f_k(\lambda) = (c_k - a_k b_k \lambda) f_{k-1}(\lambda) + q_k g_k \gamma_{k-1}(\lambda), \quad k = 2, \ldots, N-1; \tag{3.4}$$

$$\gamma_N(\lambda) = (d_N - \lambda)\gamma_{N-1}(\lambda) - p_N h_N f_{N-1}(\lambda) \tag{3.5}$$

and

$$\theta_N(\lambda) = d_N - \lambda, \quad z_N(\lambda) = h_N p_N, \tag{3.6}$$

$$\theta_k(\lambda) = (d_k - \lambda)\theta_{k+1}(\lambda) - g_k q_k z_{k+1}(\lambda), \tag{3.7}$$

$$z_k(\lambda) = (c_k - a_k b_k \lambda) z_{k+1}(\lambda) + \theta_{k+1}(\lambda) h_k p_k, \quad k = N-1, \ldots, 2; \tag{3.8}$$

$$\theta_1(\lambda) = (d_1 - \lambda)\theta_2(\lambda) - q_1 g_1 z_2(\lambda). \tag{3.9}$$

Here $f_k(\lambda), z_k(\lambda)$ *are auxiliary polynomials.*

Proof. The matrix $R - \lambda I$ is quasiseparable of order one with the same generators $p_i, q_j, a_k; g_i, h_j, b_k$ as the matrix R and diagonal entries $d_k - \lambda$. Substituting in (2.11) $d_k - \lambda$ instead of d_k we obtain (3.1)-(3.5). Similarly (3.6)-(3.9) follow from (2.12). □

¿From the relations (3.1)-(3.5) one can derive easily the recursions for the derivatives of the corresponding polynomials.

Corollary 3.2 *In the conditions of Theorem 3.1 the derivatives of the polynomials* $\gamma_k(\lambda)$, $f_k(\lambda)$ *satisfy the recursive relations*

$$\gamma_1'(\lambda) = -1, \quad f_1'(\lambda) = 0; \tag{3.10}$$

$$\gamma_k'(\lambda) = (d_k - \lambda)\gamma_{k-1}'(\lambda) - \gamma_{k-1}(\lambda) - p_k h_k f_{k-1}'(\lambda), \tag{3.11}$$

$$f_k'(\lambda) = (c_k - a_k b_k \lambda) f_{k-1}'(\lambda) - a_k b_k f_{k-1}(\lambda) + q_k g_k \gamma_{k-1}'(\lambda), \quad k = 2, \ldots, N-1; \tag{3.12}$$

$$\gamma_N'(\lambda) = -\gamma_{N-1}(\lambda) - (d_N - \lambda)\gamma_{N-1}'(\lambda) - p_N h_N f_{N-1}'(\lambda). \tag{3.13}$$

The relations (3.1)-(3.5) and (3.10)-(3.13) yield a $O(N)$ algorithm for evaluation of characteristic polynomials of the principal submatrices of a quasiseparable matrix and their derivatives. This algorithm may be used for a fast realization of the Newton iteration method (for the Newton method see, for instance, [19]).

Using the recursions (3.1)-(3.5) one can derive easily a $O(N^2)$ algorithm for the computation of the coefficients of the characteristic polynomial of the matrix R.

In assumption that some generators of the matrix R are zeros or non-zeros the recursive relations (3.1)-(3.5) may be given in a more convenient for our next purposes form.

In the sequel to simplify the exposition we use the auxiliary linear in λ expressions

$$d_k(\lambda) = d_k - \lambda, \quad k = 1, \dots, N;$$
$$c_k(\lambda) = (d_k - \lambda)a_k b_k - q_k p_k b_k - h_k g_k a_k,$$
$$l_k(\lambda) = (d_k - \lambda)a_k - q_k p_k, \quad \delta_k(\lambda) = (d_k - \lambda)b_k - h_k g_k, \quad k = 2, \dots, N-1.$$

The expressions $l_k(\lambda), \delta_k(\lambda)$ play essential role in the main theorem below. One can check easily that

$$d_k(\lambda)c_k(\lambda) = l_k(\lambda)\delta_k(\lambda) - q_k p_k h_k g_k. \tag{3.14}$$

Theorem 3.3 *Let for some $k \in \{3, \dots, N-1\}$ the inequality $p_{k-1}h_{k-1} \neq 0$ holds. Then the polynomials $\gamma_j(\lambda) = \det(R(1:j, 1:j) - \lambda I)$ $j = k-2, k-1, k$ satisfy the three-term recurrence relation:*

$$\gamma_k(\lambda) = \psi_k(\lambda)\gamma_{k-1}(\lambda) - \varphi_k(\lambda)\gamma_{k-2}(\lambda), \tag{3.15}$$

where

$$\psi_k(\lambda) = d_k - \lambda + \frac{p_k h_k}{p_{k-1}h_{k-1}} c_{k-1}(\lambda),$$

$$\varphi_k(\lambda) = \frac{p_k h_k}{p_{k-1}h_{k-1}} l_{k-1}(\lambda)\delta_{k-1}(\lambda). \tag{3.16}$$

Proof. Substituting $k-1$ instead of k in (3.4) and (3.2) we obtain

$$f_{k-1}(\lambda) = c_{k-1}(\lambda)f_{k-2}(\lambda) + q_{k-1}g_{k-1}\gamma_{k-2}(\lambda), \tag{3.17}$$

$$\gamma_{k-1}(\lambda) = d_{k-1}(\lambda)\gamma_{k-2}(\lambda) - p_{k-1}h_{k-1}f_{k-2}(\lambda). \tag{3.18}$$

The relation (3.18) implies

$$f_{k-2}(\lambda) = \frac{1}{p_{k-1}h_{k-1}}[d_{k-1}(\lambda)\gamma_{k-2}(\lambda) - \gamma_{k-1}(\lambda)]. \tag{3.19}$$

Substituting (3.19) in (3.17) and taking into account (3.14) we obtain

$$\begin{aligned}
f_{k-1}(\lambda) &= \frac{c_{k-1}(\lambda)}{p_{k-1}h_{k-1}}[d_{k-1}(\lambda)\gamma_{k-2}(\lambda) - \gamma_{k-1}(\lambda)]q_{k-1}g_{k-1}\gamma_{k-2}(\lambda) \\
&= -\frac{c_{k-1}(\lambda)}{p_{k-1}h_{k-1}}\gamma_{k-1}(\lambda) + \left[\frac{c_{k-1}(\lambda)}{p_{k-1}h_{k-1}}d_{k-1}(\lambda) + g_{k-1}h_{k-1}\right]\gamma_{k-2}(\lambda) \\
&= -\frac{c_{k-1}(\lambda)}{p_{k-1}h_{k-1}}\gamma_{k-1}(\lambda) + \frac{1}{p_{k-1}h_{k-1}} l_{k-1}(\lambda)\delta_{k-1}(\lambda)\gamma_{k-2}(\lambda).
\end{aligned}$$

Substitution of the last expression in (3.2) yields (3.15). □

Corollary 3.4 *Let $p_i h_i \neq 0$ $(i = 2, \dots, N-1)$. Then the polynomials $\gamma_k(\lambda) = \det(R(1:k, 1:k) - \lambda I)$ $(k = 1, \dots, N)$ satisfy the three-term recurrence relations:*

$$\gamma_1(\lambda) = d_1 - \lambda, \quad \gamma_2(\lambda) = (d_2 - \lambda)(d_1 - \lambda) - p_2 q_1 g_1 h_2; \tag{3.20}$$

$$\gamma_k(\lambda) = \psi_k(\lambda)\gamma_{k-1}(\lambda) - \varphi_k(\lambda)\gamma_{k-2}(\lambda), \quad k = 3, \dots, N, \tag{3.21}$$

where

$$\psi_k(\lambda) = d_k - \lambda + \frac{p_k h_k}{p_{k-1}h_{k-1}} c_{k-1}(\lambda), \quad \varphi_k(\lambda) = \frac{p_k h_k}{p_{k-1}h_{k-1}} l_{k-1}(\lambda)\delta_{k-1}(\lambda).$$

Proof. The relations (3.20) directly follow from (3.1) and from (3.2) with $k = 2$. For $k = 3, \dots, N$ the relations (3.21) follow from Theorem 3.3. $\square$

This corollary generalizes the well-known results for tridiagonal matrices and the corresponding result of the paper [10] obtained for symmetric semiseparable matrices represented as a sum of a diagonal matrix and a matrix with a tridiagonal inverse.

Next we combine the cases $p_k h_k = 0$ and $p_k h_k \neq 0$. We use here the following auxiliary relations.

Lemma 3.5 *For any $2 \leq s \leq k \leq N-1$ the polynomials $f_k(\lambda)$ defined in (3.1)-(3.4) satisfy the relations*

$$f_k(\lambda) = (c_{k+1,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s}^{k} (c_{k+1,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda). \tag{3.22}$$

Proof. The proof is by induction on k. For $k = s$ by virtue of (3.4) we have

$$f_k(\lambda) = f_s(\lambda) = c_s(\lambda) f_{s-1}(\lambda) + q_s g_s \gamma_{s-1}(\lambda) = (c_{s+1,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + (c_{s+1,j}(\lambda))^{\times} q_s g_s \gamma_{s-1}(\lambda)$$

which implies (3.22). Assume that (3.22) is valid for some $k \geq s$. By virtue of (3.4) and the equality

$$c_{k+1}(\lambda)(c_{k+1,s-1}(\lambda))^{\times} = (c_{k+2,s-1}(\lambda))^{\times}$$

we have

$$\begin{aligned} f_{k+1}(\lambda) &= c_{k+1}(\lambda) f_k(\lambda) + q_{k+1} g_{k+1}(\lambda) \gamma_k(\lambda) \\ &= c_{k+1}(\lambda)[(c_{k+1,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s}^{k} (c_{k+1,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda)] + q_{k+1} g_{k+1} \gamma_k(\lambda) \\ &= (c_{k+2,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s}^{k} (c_{k+2,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda) + (c_{k+2,k+1}(\lambda))^{\times} q_{k+1} g_{k+1} \gamma_k(\lambda) \\ &= (c_{k+2,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s}^{k+1} (c_{k+2,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda) \end{aligned}$$

which completes the proof of the lemma. $\square$

Theorem 3.6 *Assume that for some $2 \leq s < k-1 \leq N-1$, $p_s h_s \neq 0$, $p_{s+1} h_{s+1} = \cdots = p_{k-1} h_{k-1} = 0$. Then*

$$\gamma_k(\lambda) = \psi_{k,s}(\lambda) \gamma_s(\lambda) - \varphi_{k,s}(\lambda) \gamma_{s-1}(\lambda), \tag{3.23}$$

where

$$\psi_{k,s}(\lambda) = (d_{k+1,s}(\lambda))^{\times} + \frac{p_k h_k}{p_s h_s} (c_{k,s-1}(\lambda))^{\times} - p_k h_k \sum_{j=s+1}^{k-1} (c_{k,j}(\lambda))^{\times} q_j g_j (d_{j,s}(\lambda))^{\times},$$

$$\varphi_{k,s}(\lambda) = \frac{p_k h_k}{p_s h_s} (c_{k,s}(\lambda))^{\times} l_s(\lambda) \delta_s(\lambda).$$

Proof. By virtue of Lemma 3.5 we have

$$f_k(\lambda) = (c_{k,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s+1}^{k-1} (c_{k,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda) + (c_{k,s}(\lambda))^{\times} q_s g_s \gamma_{s-1}(\lambda). \tag{3.24}$$

As a direct consequence of (3.2) we get that if for some $1 \leq i < j \leq N$, $p_{i+1} h_{i+1} = \cdots = p_j h_j = 0$ then

$$\gamma_j(\lambda) = (d_j - \lambda) \cdots (d_{i+1} - \lambda) \gamma_i(\lambda). \tag{3.25}$$

Hence it follows that

$$\gamma_{j-1}(\lambda)=\gamma_s(\lambda)(d_{j,s}(\lambda))^{\times},\quad j=s+1,\dots,k. \tag{3.26}$$

Next by virtue of (3.2) we have

$$f_{s-1}(\lambda)=\frac{1}{p_sh_s}[(d_s-\lambda)\gamma_{s-1}(\lambda)-\gamma_s(\lambda)]. \tag{3.27}$$

Substituting (3.27) and (3.26) in (3.24) we obtain

$$f_{k-1}(\lambda)=\frac{(c_{k,s-1}(\lambda))^{\times}}{p_sh_s}(d_s-\lambda)\gamma_{s-1}(\lambda)-\frac{(c_{k,s-1}(\lambda))^{\times}}{p_sh_s}\gamma_s(\lambda)$$
$$+\sum_{j=s+1}^{k-1}(c_{k,j}(\lambda))^{\times}q_jg_j(d_{j,s}(\lambda))^{\times}\gamma_s(\lambda)+(c_{k,s}(\lambda))^{\times}q_sg_s\gamma_{s-1}(\lambda).$$

¿From here using the equalities

$$(c_{k,s-1}(\lambda))^{\times}=(c_{k,s}(\lambda))^{\times}c_s(\lambda),\quad \frac{c_s(\lambda)}{p_sh_s}(d_s-\lambda)+q_sg_s=\frac{l_s(\lambda)\delta_s(\lambda)}{p_sh_s}$$

we obtain

$$f_{k-1}(\lambda)=\Big[-\frac{(c_{k,s-1}(\lambda))^{\times}}{p_sh_s}+\sum_{j=s+1}^{k-1}(c_{k,j}(\lambda))^{\times}q_jg_j(d_{j,s}(\lambda))^{\times}\Big]\gamma_s(\lambda)+\frac{(c_{k,s}(\lambda))^{\times}}{p_sh_s}l_s(\lambda)\delta_s(\lambda)\gamma_{s-1}(\lambda). \tag{3.28}$$

Next using (3.26) we get

$$(d_k-\lambda)\gamma_{k-1}(\lambda)=(d_k-\lambda)(d_{k,s}(\lambda))^{\times}\gamma_s(\lambda)=(d_{k+1,s}(\lambda))^{\times}\gamma_s(\lambda). \tag{3.29}$$

Substituting (3.28) and (3.29) in (3.2) we obtain (3.23). □

3.2 The Sturm property

In this subsection we establish for characteristic polynomials of principal leading submatrices of a Hermitian quasiseparable of order one matrix an analog of the well-known Sturm property. This property yields an information on the location of the eigenvalues of quasiseparable matrices and is a basis for the bisection method (for the bisection method see, for instance, [19, p. 301]).

Theorem 3.7 *Let R be a Hermitian quasiseparable of order one matrix. Let $\gamma_0(\lambda)\equiv 1$ and $\gamma_k(\lambda)=\det(R(1:k,1:k)-\lambda I)$, $k=1,\dots,N$ be the characteristic polynomials of the principal leading submatrices of the matrix R. Let $\nu(\lambda)$ be the number of sign changes in the sequence*

$$\gamma_N(\lambda),\gamma_{N-1}(\lambda),\dots,\gamma_1(\lambda),\gamma_0(\lambda) \tag{3.30}$$

and let (α,β) be an interval on the real axis such that $\gamma_N(\alpha)\neq 0$, $\gamma_N(\beta)\neq 0$. Then:

1)$\nu(\beta)\geq\nu(\alpha)$;

2)the difference $\nu(\beta)-\nu(\alpha)$ equals the number of eigenvalues of the matrix R in the interval (α,β) counting each eigenvalue in accordance with its multiplicity.

Proof. Let us show that if $\gamma_N(\lambda_0)\neq 0$ and $\gamma_i(\lambda_0)=0$ for some i, $1\leq i\leq N-1$ then $\gamma_{i+1}(\lambda_0)=0$. If this is not the case then also $\gamma_{i+2}(\lambda_0)=0$, in the case $p_{i+1}h_{i+1}\neq 0$ it follows from Theorem 3.3 and for $p_{i+1}h_{i+1}\neq 0$ from the equality (3.2). Hence we may apply the Gundelfinger theorem (see, for instance, [16, p. 298]) to the matrix $R-\beta I$ and conclude that the number of negative eigenvalue of this matrix or equivalently the number of eigenvalues of the matrix R which are less than β, is equal to $\nu(\beta)$. Similarly we obtain that the number of eigenvalues of the matrix R which are less than α is equal to $\nu(\alpha)$ which completes the proof of the theorem. □

4 Eigenvectors and multiplicities

In this section we study eigenvalues and eigenvectors of quasiseparable of order one matrices. We obtain the conditions in which an eigenvalue of a matrix is simple and in which its multiplicity is greater than one. In both cases we derive explicit formulas for the corresponding eigenvectors.

4.1 Simple eigenvalues

Theorem 4.1 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$. Assume that $\lambda = \lambda_0$ is an eigenvalue of the matrix R and*

$$|d_1 - \lambda_0| + |q_1| > 0, \quad |d_1 - \lambda_0| + |g_1| > 0; \tag{4.1}$$

$$l_k(\lambda_0) = (d_k - \lambda_0)a_k - p_k q_k \neq 0, \quad \delta_k(\lambda_0) = (d_k - \lambda_0)b_k - g_k h_k \neq 0, \quad k = 2, \dots, N-1; \tag{4.2}$$

$$|d_N - \lambda_0| + |p_N| > 0, \quad |d_N - \lambda_0| + |h_N| > 0. \tag{4.3}$$

Then the eigenvalue λ_0 has the geometric multiplicity one. Moreover the corresponding to λ_0 eigenvector v may be determined as follows:

1)if $p_2 h_2 = \dots = p_N h_N = 0$ then

$$v = \begin{pmatrix} 1 \\ B_2(\lambda_0)^{-1} y_2 \end{pmatrix}, \tag{4.4}$$

where $B_2(\lambda_0) = R(2:N, 2:N) - \lambda_0 I$, $y_2 = -R(1, 2:N)$ and $\theta_2(\lambda_0) = \det B_2(\lambda_0) \neq 0$;

2)if for some m, $2 \leq m \leq N-1$ we have $p_m h_m \neq 0$ and $p_{m+1} h_{m+1} = \dots = p_N h_N = 0$ then

$$v = \begin{pmatrix} A_{m-1}(\lambda_0)^{-1} x_m \\ 1 \\ B_{m+1}(\lambda_0)^{-1} y_m \end{pmatrix}, \tag{4.5}$$

where $A_{m-1}(\lambda_0) = R(1:m-1, 1:m-1) - \lambda_0 I$, $x_m = -R(1:m-1, m)$, $B_{m+1}(\lambda_0) = R(m+1:N, m+1:N) - \lambda_0 I$, $y_m = P_{m+1}(a_m \frac{f_{m-1}(\lambda_0)}{\gamma_{m-1}(\lambda_0)} h_m - q_m)$, $P_{m+1} = \mathrm{col}(p_k a^{\times}_{km})_{k=m+1}^{N}$ and

$$\gamma_{m-1}(\lambda_0) = \det A_{m-1}(\lambda_0) \neq 0, \quad \theta_{m+1} = \det B_{m+1}(\lambda_0) \neq 0;$$

3)if $p_N h_N \neq 0$ then

$$v = \begin{pmatrix} A_{N-1}(\lambda_0)^{-1} x_N \\ 1 \end{pmatrix}, \tag{4.6}$$

where $A_{N-1}(\lambda_0) = R(1:N-1, 1:N-1) - \lambda_0 I$, $x_N = -R(1:N-1, N)$ and $\gamma_{N-1}(\lambda_0) = \det A_{N-1}(\lambda_0) \neq 0$. Furthermore in this case the eigenvector may be also determined by the values of the polynomials $\gamma_k(\lambda)$, $f_k(\lambda)$ at the point λ_0 as follows:

$$v = \begin{pmatrix} -V_{N-1}(\lambda_0) h_N \\ \gamma_{N-1}(\lambda_0) \end{pmatrix}, \tag{4.7}$$

where $V_{N-1}(\lambda_0) = \mathrm{col}(v_i(\lambda_0)\delta^{\times}_{iN}(\lambda_0))_{i=1}^{N-1}$,

$$v_1(\lambda_0) = g_1, \quad v_i(\lambda_0) = g_i \gamma_{i-1}(\lambda_0) - p_i f_{i-1}(\lambda_0) b_i, \ i = 1, \dots, N-1.$$

Proof. We start proving the part 1). At first let us show that $\theta_2(\lambda_0) \neq 0$. By virtue of (3.6) we have $z_N(\lambda_0) = 0$ and next by virtue of (3.8) we obtain $z_{N-1}(\lambda_0) = \dots = z_2(\lambda_0) = 0$ and using (3.6)and (3.7) we obtain

$$\theta_2(\lambda_0) = (d_N - \lambda_0) \dots (d_2 - \lambda_0).$$

By virtue of conditions (4.2), (4.3) and equalities $p_2 h_2 = \dots = p_N h_N = 0$ we conclude that $d_j - \lambda_0 \neq 0$, $j = 2, \dots, N$ and hence $\theta_2(\lambda_0) \neq 0$.

Consider the partition of the matrix $R-\lambda_0 I$ in the form

$$R-\lambda_0 I=\left(\begin{array}{cc} d_1-\lambda_0 I & r_2' \\ r_2 & B_2(\lambda_0)\end{array}\right), \tag{4.8}$$

where $B_2(\lambda_0)=R(2:N,2:N)-\lambda_0 I,\ r_2=R(2:N,1),\ r_2'=R(1,2:N)$. Let v be a corresponding to λ_0 eigenvector. We represent this vector in the form $v=\left(\begin{array}{c}\alpha\\ v'\end{array}\right)$ where v' is a $N-1$-dimensional vector and α is a scalar. From the equality $(R-\lambda_0)v=0$ using the representation (4.8) we obtain $B_2(\lambda_0)v'+\alpha r_2=0$. This means $v'=-B_2^{-1}(\lambda_0)\alpha r_2$, i.e. $v=\alpha v_0$, where $v_0=\alpha\left(\begin{array}{c}1\\ B_2^{-1}(\lambda_0)y_2\end{array}\right)$. This implies that the eigenvalue λ_0 has the geometric multiplicity one. Taking $\alpha=1$ we obtain (4.4).

2)In the same way as in the proof of the part 1) we obtain

$$z_N(\lambda_0)=z_{N-1}(\lambda_0)=\cdots=z_m(\lambda_0)=0 \tag{4.9}$$

and $\theta_m(\lambda_0)\neq 0$.

By virtue of (3.25) and the fact that λ_0 is an eigenvalue of the matrix R we have

$$(d_N-\lambda_0)\cdots(d_{m+1}-\lambda_0)\gamma_m(\lambda_0)=\gamma_N(\lambda_0)=0. \tag{4.10}$$

The equalities $p_{m+1}h_{m+1}=\cdots=p_Nh_N$ and the conditions (4.2), (4.3) imply $d_j-\lambda_0\neq 0,\ j=m+1,\dots,N$ and therefore from (4.10) we conclude that $\gamma_m(\lambda_0)=0$. Let us prove that $\gamma_{m-1}(\lambda_0)\neq 0$. Assume that it is not true. We will show that in this case the equalities

$$\gamma_1(\lambda_0)=\gamma_2(\lambda_0)=\cdots=\gamma_{m-1}(\lambda_0)=\gamma_m(\lambda_0)=0 \tag{4.11}$$

hold. We have the following cases:

(a)$m=2$. By the assumption we have $\gamma_1(\lambda_0)=\gamma_2(\lambda_0)=0$ and hence (4.11) follows.

(b)$p_2h_2=\cdots=p_{m-1}h_{m-1}=0$. By virtue of (3.25) we have

$$\gamma_{j+1}(\lambda)=(d_{j+1}-\lambda)\gamma_j(\lambda),\quad j=m-2,\dots,1$$

and since by virtue of the conditions (4.2) the inequalities $d_{j+1}-\lambda_0\neq 0,\ j=m-2,\dots,1$ hold, we obtain (4.11).

(c)$p_{m-1}h_{m-1}\neq 0$. Applying Theorem 3.3 we obtain

$$\gamma_m(\lambda)=\psi_m(\lambda)\gamma_{m-1}(\lambda)-\varphi_m(\lambda)\gamma_{m-2}(\lambda), \tag{4.12}$$

where

$$\varphi_m(\lambda)=\frac{p_mh_m}{p_{m-1}h_{m-1}}l_{m-1}(\lambda)\delta_{m-1}(\lambda).$$

Here $p_mh_m=0$ by the condition, $l_{m-1}(\lambda_0)\delta_{m-1}(\lambda_0)\neq 0$ by virtue of (4.11) and hence $\varphi_m(\lambda_0)\neq 0$. Since $\gamma_m(\lambda_0)=\gamma_{m-1}(\lambda_0)=0$ the equality (4.12) implies $\gamma_{m-2}(\lambda_0)=0$.

(d)$p_sh_s\neq 0,\ p_{s+1}h_{s+1}=\cdots=p_{m-1}h_{m-1}$ for some s such that $2\le s<m-1$. By virtue of (3.25) we have

$$\gamma_{j+1}(\lambda)=(d_{j+1}-\lambda)\gamma_j(\lambda),\quad j=s,\dots,m-2$$

and since by virtue of the conditions (4.2), the inequalities $d_{j+1}-\lambda_0\neq 0,\ j=m-2,\dots,1$ hold, we obtain

$$\gamma_s(\lambda_0)=\gamma_{s+1}(\lambda_0)=\cdots=\gamma_{m-1}(\lambda_0)=0.$$

Next apply Theorem 3.6. We have

$$\gamma_m(\lambda)=\psi_{m,s}(\lambda)\gamma_s(\lambda)-\varphi_{m,s}(\lambda)\gamma_{s-1}(\lambda), \tag{4.13}$$

where

$$\varphi_{m,s}(\lambda) = \frac{p_m h_m}{p_s h_s} c_{m-1}(\lambda) \cdots c_{s+1}(\lambda) l_s(\lambda) \delta_s(\lambda).$$

Here $p_m h_m \neq 0$ by the condition, $l_s(\lambda_0)\delta_s(\lambda_0) \neq 0$ by virtue of (4.2) and $c_j(\lambda_0) \neq 0$ by virtue of (4.2) and the equality (3.14). Thus $\varphi_{m,s}(\lambda_0) \neq 0$ and since $\gamma_m(\lambda_0) = 0,\ \gamma_s(\lambda_0) = 0$ from (4.13) we conclude that $\gamma_{s-1}(\lambda_0) = 0$.

In the cases (a) and (b) the proof of (4.11) is finished. In the case (c) we apply the method described above to the polynomials $\gamma_{m-1}(\lambda), \gamma_{m-2}(\lambda)$ and in the case (d) we apply this method to the polynomials $\gamma_s(\lambda), \gamma_{s-1}(\lambda)$. We go on in this way and finally obtain (4.11).

Thus we have the equalities (4.11). Hence in particular follows that $\gamma_1(\lambda_0) = d_1 - \lambda_0 = 0$ and by virtue of the condition (4.1) and the second equality in (3.1) we have $f_1(\lambda_0) = q_1 g_1 \neq 0$. Let t be the minimal index such that $p_t h_t \neq 0$. It is clear that $t \leq k$. Using (3.2) and (4.11) we obtain

$$\gamma_t(\lambda_0) = (d_t - \lambda_0)\gamma_{t-1}(\lambda_0) - p_t h_t f_{t-1}(\lambda_0) = -p_t h_t f_{t-1}(\lambda_0). \tag{4.14}$$

By virtue of Lemma 3.5 and (4.11) we have

$$f_{t-1}(\lambda_0) = (c_{t,1}(\lambda_0))^{\times} f_1(\lambda_0) + \sum_{j=2}^{t-1} (c_{t,1}(\lambda_0))^{\times} q_j g_j \gamma_{j-1}(\lambda_0) = c_{t-1}(\lambda_0) \cdots c_2(\lambda_0) f_1(\lambda_0).$$

¿From the conditions (4.2) and the equalities (3.14) we obtain $c_j(\lambda_0) \neq 0,\ j = 2, \dots, t-1$ and hence $f_{t-1}(\lambda_0) \neq 0$. But from (4.14) we conclude that $\gamma_t(\lambda_0) \neq 0$ which is a contradiction. Thus we have proved that $\gamma_{m-1}(\lambda_0) \neq 0$.

Applying the partition (2.9) with $k = m$ to the matrix $R - \lambda_0 I$ we get

$$R - \lambda_0 I = \begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1} h_m & G_{m-1} b_m H_{m+1} \\ p_m Q_{m-1} & d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1} a_m Q_{m-1} & P_{m+1} q_m & B_{m+1}(\lambda_0) \end{pmatrix}, \tag{4.15}$$

where $A_{m-1}(\lambda_0) = A_{m-1} - \lambda_0 I,\ B_{m+1}(\lambda_0) = B_{m+1} - \lambda_0 I$ and the elements Q_m, G_m, P_m, H_m are defined by (2.2), (2.3). Moreover from (2.10) we obtain

$$Q_{m-1} A_{m-1}^{-1}(\lambda_0) G_{m-1} = \frac{f_{m-1}(\lambda_0)}{\gamma_{m-1}(\lambda_0)}, \quad H_{m+1} B_{m+1}^{-1}(\lambda_0) P_{m+1} = \frac{z_{m+1}(\lambda_0)}{\theta_{m+1}(\lambda_0)}. \tag{4.16}$$

Let v be a corresponding to λ_0 eigenvector. We represent this vector in the form $v = \begin{pmatrix} v' \\ \alpha \\ v'' \end{pmatrix}$, where v', v'' are $m-1$ and $N-m$-dimensional vectors and α is a scalar. From the equality $(R - \lambda_0 I)v = 0$ using the representation (4.15) we obtain

$$\begin{aligned} &A_{m-1}(\lambda_0) v' + G_{m-1}(h_m \alpha + b_m H_{m+1} v'') = 0, \\ &P_{m+1}(a_m Q_{m-1} v' + q_m \alpha) + B_{m+1}(\lambda_0) v'' = 0. \end{aligned} \tag{4.17}$$

By virtue of invertibility of the matrices $A_{m-1}(\lambda_0), B_{m+1}(\lambda_0)$ we may rewrite (4.17) in the form

$$v' = -A_{m-1}^{-1}(\lambda_0) G_{m-1}(h_m \alpha + b_m H_{m+1} v''), \tag{4.18}$$

$$v'' = -B_{m+1}^{-1}(\lambda_0) P_{m+1}(a_m Q_{m-1} v' + q_m \alpha). \tag{4.19}$$

Multiplying (4.19) by H_{m+1} and using the second equality from (4.16) and the equality $z_m = 0$ from (4.9) we obtain $H_{m+1} v'' = 0$. Substituting this in (4.18) we obtain

$$v' = -A_{m-1}^{-1}(\lambda_0) G_{m-1} h_m \alpha = A_{m-1}^{-1}(\lambda_0) x_m \alpha. \tag{4.20}$$

Substituting (4.20) in (4.19) we obtain

$$v'' = B_{m+1}^{-1}(\lambda_0)P_{m+1}(a_m Q_{m-1}A_{m-1}^{-1}(\lambda_0)G_{m-1}h_m - q_m)\alpha$$

and using the first equality from (4.16) we conclude that

$$v'' = B_{m+1}^{-1}(\lambda_0)P_{m+1}(a_m \frac{f_{m-1}(\lambda_0)}{\gamma_{m-1}(\lambda_0)}h_m - q_m)\alpha = B_{m+1}^{-1}(\lambda_0)y_m\alpha. \tag{4.21}$$

Thus from (4.20), (4.21) we conclude that $v = \alpha v_0$, where $v_0 = \alpha \begin{pmatrix} A_{m-1}^{-1}(\lambda_0)x_m \\ 1 \\ B_{m+1}^{-1}(\lambda_0)y_m \end{pmatrix}$. This implies that the eigenvalue λ_0 has the geometric multiplicity one. Taking $\alpha = 1$ we obtain (4.5).

3)In the same way as in the proof of part 2) we show that $\gamma_{N-1}(\lambda_0) = 0$. Next for the matrix $R - \lambda_0 I$ consider the partition

$$R - \lambda_0 I = \begin{pmatrix} A_{N-1}(\lambda_0) & r_{N-1} \\ r'_{N-1} & d_N - \lambda_0 \end{pmatrix}, \tag{4.22}$$

where $A_{N-1}(\lambda_0) = R(1:N-1, 1:N-1) - \lambda_0 I,\ r_{N-1} = R(1:N-1, N),\ r'_{N-1} = R(N, 1:N-1)$. Let v be an eigenvector corresponding to the eigenvalue λ_0. We represent this vector in the form $v = \begin{pmatrix} v' \\ \alpha \end{pmatrix}$, where v' is a $N-1$-dimensional vector α is a scalar. Using (4.22) we obtain $v' = -\alpha A_{N-1}^{-1}(\lambda_0)r_{N-1}$ which implies $v = \alpha v_0$, where $v_0 = \alpha \begin{pmatrix} -A_{N-1}^{-1}(\lambda_0)r_{N-1} \\ 1 \end{pmatrix}$. This implies that the eigenvalue λ_0 has the geometric multiplicity one. Taking $\alpha = 1$ we obtain (4.6). Next by virtue of (2.5) we have $x_N = -R(1:N-1, 1:N-1) = -G_{N-1}h_N$ and moreover using the formula

$$A_{N-1}(\lambda_0)^{-1} = \frac{1}{\gamma_{N-1}(\lambda_0)}(\mathrm{adj}A_{N-1}(\lambda_0))$$

we obtain

$$v = \frac{1}{\gamma_{N-1}(\lambda_0)} \begin{pmatrix} -V_{N-1}(\lambda_0)h_N \\ \gamma_{N-1}(\lambda_0) \end{pmatrix},$$

where $V_{N-1}(\lambda_0) = (\mathrm{adj}A_{N-1}(\lambda_0))G_{N-1}$. Applying Theorem 2.3 to the matrix $R - \lambda_0 I$ and reducing the factor $\frac{1}{\gamma_{N-1}(\lambda_0)}$ we obtain the representation (4.7) for the eigenvector. $\square$

Remark. Using a fast algorithm suggested in [7] one can compute the eigenvector given by the formulas (4.4)- (4.6) in $O(N)$ arithmetic operations.

4.2 Examples

We consider some cases when the validity of the conditions of Theorem 4.1 is guaranteed for any eigenvalue.

1)The Hermitian quasiseparable of order one matrix R with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; $\overline{q_i}$ $(i = 1, \dots, N-1)$, $\overline{p_j}$ $(j = 2, \dots, N)$, a_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$ satisfying the conditions

$$\overline{p_k q_k} \neq p_k q_k,\ \overline{a_k} = a_k, \quad k = 2, \dots, N-1$$

. In this case we have

$$l_k(\lambda) = (d_k - \lambda)a_k - p_k q_k,\ \delta_k(\lambda) = (d_k - \lambda)a_k - \overline{p_k q_k}, \quad k = 2, \dots, N-1.$$

If λ is an eigenvalue of R then the number $(\lambda - d_k)a_k$ is real and the numbers $p_k q_k, \overline{p_k q_k}$ are pure imaginary. Hence it follows that the condition (4.2) holds for any eigenvalue of R.

Assume that $q_1 \neq 0,\ p_N \neq 0$. Then all the conditions of Theorem 4.1 hold and since for a Hermitian matrix the geometric multiplicity of any eigenvalue is equal to its algebraic multiplicity we conclude that all eigenvalues of the matrix R are simple.

2)The irreducible tridiagonal matrix, i.e. the tridiagonal matrix with non-zero entries on the lower and upper subdiagonals. In this case we have $a_k = b_k = 0,\ R_{k+1,k} = p_{k+1}q_k \neq 0,\ R_{k,k+1} = g_k h_{k+1} \neq 0$ ($k = 1, \dots, N-1$) and therefore

$$l_k(\lambda) = -p_k q_k \neq 0,\ \delta_k(\lambda) = -g_k h_k \neq 0, \quad k = 2, \dots, N-1.$$

and hence the conditions of Theorem 4.1 hold for any λ.

Assume additionally that the conditions

$$\overline{d_i} = d_i, \quad i = 1, \dots, N; \quad \frac{p_{i+1}q_i}{g_i h_{i+1}} > 0, \quad i = 1, \dots, N-1 \tag{4.23}$$

are valid. Let $D = \mathrm{diag}\ (\rho_i)_{i=1}^N$ be a nonsingular diagonal matrix with the entries

$$\rho_1 = 1, \quad \rho_k = \sqrt{\frac{p_k q_{k-1}}{g_{k-1}h_k}}\rho_{k-1},\ k = 2, \dots, N \tag{4.24}$$

and let $Q = D^{-1}RD$. The matrix Q is a tridiagonal Hermitian matrix. Indeed the entries of the matrix Q have the form

$$Q_{ij} = \begin{cases} \rho_{i+1}^{-1} p_{i+1} q_i \rho_i, & i = j+1, j = 1, \dots, N-1 \\ d_i, & 1 \leq i = j \leq N, \\ \rho_i^{-1} g_i h_{i+1} \rho_{i+1}, & j = i+1, i = 1, \dots, N-1 \\ 0, & |i-j| > 1. \end{cases}$$

The relations (4.24) imply

$$p_{i+1}q_i\rho_i^2 = \overline{g_i h_{i+1}}\rho_{i+1}^2, \quad i = 1, \dots, N-1 \tag{4.25}$$

which means

$$\rho_{i+1}^{-1} p_{i+1} q_i \rho_i = \overline{\rho_i^{-1} g_i h_{i+1} \rho_{i+1}}, \quad i = 1, \dots, N-1.$$

Thus from here and from Theorem 4.1 we obtain the known result that in conditions (4.23) the tridiagonal matrix R is diagonalizable and its eigenvalues are real and simple.

Now we apply Corollary 3.4 to the case of a tridiagonal matrix. We have

$$c_k(\lambda) = 0, \quad l_k(\lambda) = -p_k q_k,\ \delta_k(\lambda) = -g_k h_k,$$

and hence

$$\psi_k(\lambda) = d_k - \lambda, \quad \varphi_k(\lambda) = \frac{p_k h_k}{p_{k-1}h_{k-1}} p_{k-1}q_{k-1}g_{k-1}h_{k-1} = p_k q_{k-1} g_{k-1} h_k.$$

Hence the recursions (3.20), (3.21) one can write down in the form

$$\gamma_0(\lambda) = 1, \quad \gamma_1(\lambda) = d_1 - \lambda;$$

$$\gamma_k(\lambda) = (d_k - \lambda)\gamma_{k-1}(\lambda) - (p_k q_{k-1})(g_{k-1}h_k)\gamma_{k-2}(\lambda), \quad k = 2, \dots, N.$$

Note that it is well known (see for instance [1, p. 121]) that in the case

$$d_k = \overline{d_k},\ k = 1, \dots, N; \quad (p_k q_{k-1})(g_{k-1}h_k) > 0, \quad k = 2, \dots, N-1$$

these recursions defines polynomials orthogonal with some weight on the real line.

3)The invertible matrix with generators satisfying the conditions

$$a_k \neq 0, b_k \neq 0,\ d_k a_k - p_k q_k = 0,\ d_k b_k - g_k h_k = 0, \quad k = 2, \dots, N-1. \tag{4.26}$$

In this case we have

$$l_k(\lambda) = -\lambda a_k,\ \delta_k(\lambda) = -\lambda b_k, \quad k = 2, \ldots, N-1$$

and since by the assumption the zero is not the eigenvalue of R the condition (4.2) holds for any eigenvalue of the matrix. Suppose also that $p_{k+1}q_k \neq 0,\ g_k h_{k+1} \neq 0$ $(k = 1, \ldots, N-1)$. Then all the conditions of Theorem 4.1 hold for any eigenvalue of the matrix R.

Now as in the previous example assume that the conditions (4.23) are valid and define the nonsingular diagonal matrix $D = \text{diag}\,(\rho_i)_{i=1}^N$ via the relations (4.24). Let us show that in this case as well as in the previous example the matrix $Q = D^{-1}RD$ is a Hermitian matrix. Indeed the entries of the matrix Q have the form

$$Q_{ij} = \begin{cases} \rho_i^{-1} p_i a^{\times}_{ij} q_j \rho_j, & 1 \leq j < i \leq N, \\ d_i, & 1 \leq i = j \leq N, \\ \rho_i^{-1} g_i b^{\times}_{ij} h_j \rho_j, & 1 \leq i < j \leq N. \end{cases} \tag{4.27}$$

One should check that

$$p_i a^{\times}_{ij} q_j \rho_j^2 = \overline{g_j}\,\overline{b^{\times}_{ji}}\,\overline{h_j}\rho_i^2, \quad 1 \leq j < i \leq N. \tag{4.28}$$

¿From the relations (4.25) it follows that

$$\frac{\rho_i^2}{\rho_j^2} = \frac{\rho_i^2}{\rho_{i-1}^2} \cdots \frac{\rho_{j+1}^2}{\rho_j^2} = \frac{p_i q_{i-1}}{\overline{g_{i-1} h_i}} \frac{p_{i-1} q_{i-2}}{\overline{g_{i-2} h_{i-1}}} \cdots \frac{p_{j+1} q_j}{\overline{g_j h_{j+1}}}$$

which means

$$\frac{\rho_i^2}{\rho_j^2} = \frac{p_i}{\overline{h_i}} \frac{q_j}{\overline{g_j}} \frac{p^{\times}_{ij} q^{\times}_{ij}}{\overline{g^{\times}_{ij} h^{\times}_{ij}}}, \quad 1 \leq j < i \leq N. \tag{4.29}$$

By virtue of (4.26) and the assumption that the numbers d_k are real we have

$$d_k = \frac{p_k q_k}{a_k} = \frac{g_k h_k}{b_k} = \frac{\overline{g_k h_k}}{\overline{b_k}}, \quad k = 2, \ldots, N-1. \tag{4.30}$$

Hence it follows that

$$\frac{p^{\times}_{ij} q^{\times}_{ij}}{\overline{g^{\times}_{ij} h^{\times}_{ij}}} = \frac{a^{\times}_{ij}}{\overline{b^{\times}_{ij}}}, \quad 1 \leq j < i \leq N. \tag{4.31}$$

Comparing (4.29) and (4.31) we obtain

$$\frac{\rho_i^2}{\rho_j^2} = \frac{p_i q_j}{\overline{g_j}\,\overline{h_i}} \frac{a^{\times}_{ij}}{\overline{b^{\times}_{ij}}}, \quad 1 \leq j < i \leq N$$

which is equivalent to (4.28). Thus we conclude that in the case under consideration the matrix R is similar to a Hermitian matrix and hence R is diagonalizable and its eigenvalues are real. Moreover by Theorem 4.1 the eigenvalues of R are simple.

4)The unitary Hessenberg matrix

$$R_Q = \begin{bmatrix} -\rho_1\rho_0^* & -\rho_2\mu_1\rho_0^* & -\rho_3\mu_2\mu_1\rho_0^* & \cdots & -\rho_{N-1}\mu_{N-2}\cdots\mu_1\rho_0^* & -\rho_N\mu_{N-1}\cdots\mu_1\rho_0^* \\ \mu_1 & -\rho_2\rho_1^* & -\rho_3\mu_2\rho_1^* & \cdots & -\rho_{N-1}\mu_{N-2}\cdots\mu_2\rho_1^* & -\rho_N\mu_{N-1}\cdots\mu_2\rho_1^* \\ 0 & \mu_2 & -\rho_3\rho_2^* & \cdots & -\rho_{N-1}\mu_{N-2}\cdots\mu_3\rho_2^* & -\rho_N\mu_{N-1}\cdots\mu_3\rho_2^* \\ \vdots & \ddots & \mu_3 & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{N-1}\rho_{N-2}^* & -\rho_N\mu_{N-1}\rho_{N-2}^* \\ 0 & \cdots & \cdots & 0 & \mu_{N-1} & -\rho_N\rho_{N-1}^* \end{bmatrix}, \tag{4.32}$$

where $\mu_k > 0,\ |\rho_k|^2 + \mu_k^2 = 1$ $(k = 1, \ldots, N-1),\ \rho_0 = -1,\ |\rho_N| = 1$. Such matrices were studied by various authors, see for instance [12] where it was proved that the matrix of the form (4.32) is unitary.

The matrix R_Q may be treated as a quasiseparable of order one matrix with generators $d_k = -\rho_k\rho^*_{k-1},\ a_k = 0,\ b_k = \mu_{k-1},\ p_i = 1,\ q_j = \mu_j,\ g_i = -\rho^*_{i-1},\ h_j = \mu_{j-1}\rho_j$. In this case we have $p_kq_k = \mu_k,\ g_kh_k = -\rho^*_{k-1}\mu_{k-1}\rho_k$ and next

$$l_k(\lambda) = -\mu_k, \quad \delta_k(\lambda) = -(\lambda + \rho_k\rho^*_{k-1})\mu_{k-1} + \rho^*_{k-1}\mu_{k-1}\rho_k = -\lambda\mu_{k-1}, \quad k = 2, \dots, N-1.$$

¿From here since the matrix R_Q is invertible we conclude that the condition (4.2) holds for any eigenvalue of this matrix. The other conditions of Theorem 4.1 are also valid for any eigenvalue of the matrix R_Q. Since for a unitary matrix the geometric multiplicity of any eigenvalue is equal to its algebraic multiplicity Theorem 4.1 implies that all eigenvalues of the matrix R_Q are simple.

Next consider the recursions (3.1)-(3.4). We have

$$\gamma_1(\lambda) = \rho_1 - \lambda, \quad f_1(\lambda) = \mu_1; \tag{4.33}$$

$$\gamma_k(\lambda) = (-\rho_k\rho^*_{k-1} - \lambda)\gamma_{k-1}(\lambda) - \mu_{k-1}\rho_k f_{k-1}(\lambda), \tag{4.34}$$

$$f_k(\lambda) = -\mu_{k-1}\mu_k f_{k-1}(\lambda) - \mu_k\rho^*_{k-1}\gamma_{k-1}(\lambda), \quad k = 2, \dots, N-1. \tag{4.35}$$

To compare with some known results we consider the polynomials $\widetilde{\gamma}_k(\lambda) = \det(\lambda I - R(1:k, 1:k)$ instead of $\gamma_k(\lambda) = \det(R(1:k, 1:k) - \lambda I)$, i.e. $\widetilde{\gamma}_k(\lambda) = (-1)^k\gamma_k(\lambda)$ and set $\widetilde{f}_k(\lambda) = (-1)^{k+1}f_k(\lambda)$. From the recursions (4.33)-(4.35) we obtain

$$\widetilde{\gamma}_0(\lambda) = 1, \quad \widetilde{f}_0(\lambda) = 0; \tag{4.36}$$

$$\widetilde{\gamma}_k(\lambda) = (\lambda + \rho_k\rho^*_{k-1})\widetilde{\gamma}_{k-1}(\lambda) - \mu_{k-1}\rho_k\widetilde{f}_{k-1}(\lambda), \tag{4.37}$$

$$\widetilde{f}_k(\lambda) = \mu_{k-1}\mu_k\widetilde{f}_{k-1}(\lambda) - \mu_k\rho^*_{k-1}\gamma_{k-1}(\lambda), \quad k = 1, \dots, N-1 \tag{4.38}$$

(we set here $\mu_0 = 0$). Set

$$G_k(\lambda) = -\rho^*_k\widetilde{\gamma}_k(\lambda) + \mu_k\widetilde{f}_k(\lambda), \quad k = 0, \dots, N-1. \tag{4.39}$$

Let us show that the polynomials $\widetilde{\gamma}_k(\lambda), G_k(\lambda)$ satisfy the two-terms recurrence relations

$$\begin{pmatrix} G_0(\lambda) \\ \widetilde{\gamma}_0(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{pmatrix} G_k(\lambda) \\ \widetilde{\gamma}_k(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & -\rho^*_k\lambda \\ -\rho_k & \lambda \end{pmatrix}\begin{pmatrix} G_{k-1}(\lambda) \\ \widetilde{\gamma}_{k-1}(\lambda) \end{pmatrix}, \quad k = 1, \dots, N-1. \tag{4.40}$$

These recursions define Szego polynomials orthogonal with the corresponding weight on the unit circle (see for instance [1, p. 176]) . From (4.37) and (4.39) we obtain

$$\widetilde{\gamma}_k(\lambda) = \lambda\gamma_{k-1}(\lambda) - \rho_k G_{k-1}(\lambda), \quad k = 1, 2, \dots, N.$$

Next using (4.39), (4.37), (4.38) and the equalities $\mu_k^2 + |\rho_k|^2 = 1$ we obtain

$$\begin{aligned} G_k(\lambda) &= -\rho^*_k\widetilde{\gamma}_k(\lambda) + \mu_k\widetilde{f}_k(\lambda) \\ &= -\rho^*_k(\lambda + \rho_k\rho^*_{k-1})\widetilde{\gamma}_{k-1}(\lambda) + \mu_{k-1}\rho^*_k\rho_k\widetilde{f}_{k-1}(\lambda) + \mu_k(\mu_{k-1}\mu_k\widetilde{f}_{k-1}(\lambda) - \mu_k\rho^*_{k-1}\gamma_{k-1}(\lambda)) \\ &= -\rho^*_k\lambda\widetilde{\gamma}_{k-1}(\lambda) - |\rho_k|^2\rho^*_{k-1}\gamma_{k-1}(\lambda) + \mu_{k-1}|\rho_k|^2\widetilde{f}_{k-1}(\lambda) + \mu_k^2\mu_{k-1}\widetilde{f}_{k-1}(\lambda) - \mu_k^2\rho^*_{k-1}\widetilde{f}_{k-1}(\lambda) \\ &= -\rho^*_k\lambda\widetilde{\gamma}_{k-1}(\lambda) - \rho^*_{k-1}\widetilde{\gamma}_{k-1}(\lambda) + \mu_{k-1}\widetilde{f}_{k-1}(\lambda) = -\rho^*_k\lambda\widetilde{\gamma}_{k-1}(\lambda) + G_{k-1}(\lambda), \quad k = 1, \dots, N-1 \end{aligned}$$

which completes the proof of (4.40). This result is also contained in [12].

5)The Toeplitz matrix

$$R = \begin{bmatrix} d & a & a^2 & \dots & a^{n-1} \\ b & d & a & \dots & a^{n-2} \\ b^2 & b & d & \dots & a^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \dots & d \end{bmatrix}$$

with $a \neq 0, ; b \neq 0$. This is a quasiseparable of order one matrix with generators $p_i = b^{i-1}$ ($i = 2, \dots, N$), $q_j = b^{1-j}$ ($j = 1, \dots, N-1$), $a_k = 1$ ($k = 2, \dots, N-1$); $g_i = a^{1-i}$ ($i = 1, \dots, N-1$), $h_j = a^{j-1}$ ($j = 2, \dots, N$), $b_k = 1$ ($k = 2, \dots, N-1$); $d_k = d$ ($k = 1, \dots, N$). We have

$$l_k(\lambda) = \delta_k(\lambda) = d - \lambda - 1, \quad k = 2, \dots, N-1.$$

Assume that $ab \neq 1$. Then (see [13]) the matrix

$$R - (d-1)I = \begin{bmatrix} 1 & a & a^2 & \dots & a^{n-1} \\ b & 1 & a & \dots & a^{n-2} \\ b^2 & b & 1 & \dots & a^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \dots & 1 \end{bmatrix} \tag{4.41}$$

is invertible, i.e. the number $d-1$ is not an eigenvalue of the matrix R. Thus in this case the conditions of Theorem 4.1 are valid for any eigenvalue of the matrix R.

Assume now that the number d is real and the number $b/\overline{a}$ is real and positive. In this case we have

$$\frac{p_{i+1}q_i}{\overline{g_i h_{i+1}}} = \frac{b^i b^{1-i}}{\overline{a^{1-i}a^i}} = \frac{b}{\overline{a}} > 0, \quad i = 1, \dots, N-1$$

and hence the matrix R satisfies the conditions (4.23). Applying the formulas (4.24) define the nonsingular diagonal matrix $D = \mathrm{diag}\,(\rho_i)_{i=1}^N$ with the entries

$$\rho_1 = 1, \quad \rho_k = \sqrt{\frac{a}{\overline{b}}}\rho_{k-1} = \left(\frac{a}{\overline{b}}\right)^{\frac{k-1}{2}}, \quad k = 2, \dots, N. \tag{4.42}$$

Moreover we have

$$\frac{p_k q_k}{a_k} = \frac{\overline{g_k h_k}}{\overline{b_k}} = 1, \quad k = 2, \dots, N-1$$

and hence in the same way as in Example 3) we obtain that the matrix $Q = D^{-1}RD$ is a Hermitian matrix. Furthermore using the formulas (4.27) and (4.42) we conclude that Q is a quasiseparable of order one matrix with generators $p_i^Q = (\overline{a}b)^{(i-1)/2}$ ($i = 2, \dots, N$), $q_j^Q = (\overline{a}b)^{(1-j)/2}$ ($j = 1, \dots, N-1$), $a_k^Q = 1$ ($k = 2, \dots, N-1$); $g_i^Q = (\overline{b}a)^{(1-i)/2}$ ($i = 1, \dots, N-1$), $h_j^Q = (\overline{b}a)^{(j-1)/2}$ ($j = 2, \dots, N$), $b_k^Q = 1$ ($k = 2, \dots, N-1$); $d_k^Q = d$ ($k = 1, \dots, N$). This implies that the matrix Q is a Toeplitz Hermitian matrix. Hence it follows that in the case under consideration the matrix R is diagonalizable and moreover from Theorem 4.1 we conclude that all eigenvalues of the matrix R are simple.

The arguments mentioned above imply, in particular, that for a real d and for $ab > 0$, $ab \neq 1$ the Toeplitz matrix R is diagonalizible and its eigenvalues are simple. Let us show that in the case $ab < 0$ the matrix R may have multiple eigenvalues. By virtue of Theorem 4.1 this implies that the matrix R is not diagonalizable. As an example we take the 3×3 matrix

$$R = \begin{bmatrix} d & a & a^2 \\ b & d & a \\ b^2 & b & d \end{bmatrix}$$

with $ab = -8$. Checking directly or applying the formula (3.21) we obtain the characteristic polynomial of this matrix

$$\gamma_3(\lambda) = (d-\lambda)^3 - (a^2b^2 + 2ab)(d-\lambda) + 2a^2b^2 = (d-\lambda)^3 - 48(d-\lambda) + 128 = -(\lambda - d + 4)^2(\lambda - d - 8).$$

It is clear that this polynomial has a multiple eigenvalue and then Theorem 4.1 implies that the matrix R is not diagonalizable.

Next we consider the case $ab = 1$. In this case the relations (3.20), (3.21) have the form

$$\gamma_1(\lambda) = d - \lambda, \quad \gamma_2(\lambda) = (d-\lambda)^2 - 1 = (d-\lambda+1)(d-\lambda-1);$$

$$\gamma_k(\lambda) = 2(d-\lambda-1)\gamma_{k-1}(\lambda) - (d-\lambda-1)^2\gamma_{k-2}(\lambda), \quad k = 3, \ldots, N.$$

¿From here one can derive easily by induction that

$$\gamma_k(\lambda) = (d-\lambda+k-1)(d-\lambda-1)^{k-1}, \quad k = 2, \ldots, N.$$

Hence it follows that in this case the matrix R has the simple eigenvalue $\lambda = d + N - 1$ and the multiple eigenvalue $\lambda = d - 1$. For $\lambda = d + N - 1$ the eigenvector is determined by the relation (4.6). For $\lambda = d - 1$ we obtain directly from (4.41) the eigenvectors

$$u_1 = \begin{pmatrix} 1 \\ -b \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; u_2 = \begin{pmatrix} 0 \\ 1 \\ -b \\ \vdots \\ 0 \end{pmatrix}, \; \ldots \; u_{N-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -b \end{pmatrix}.$$

Hence it follows that for $ab = 1$ the Toeplitz matrix R is diagonalizable.

4.3 Multiple eigenvalues

In this subsection we study the case when both the conditions (4.2) of Theorem 4.1 are non-valid.

Theorem 4.2 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \ldots, N)$, q_j $(j = 1, \ldots, N-1)$, a_k $(k = 2, \ldots, N-1)$; g_i $(i = 1, \ldots, N-1)$, h_j $(j = 2, \ldots, N)$, b_k $(k = 2, \ldots, N-1)$; d_k $(k = 1, \ldots, N)$ and let λ_0 be a number such that for some $m \in \{2, \ldots, N-1\}$,*

$$l_m(\lambda_0) = (d_m - \lambda_0)a_m - p_m q_m = 0, \quad \delta_m(\lambda_0) = (d_m - \lambda_0)b_m - g_m h_m = 0 \tag{4.43}$$

and

$$|p_m| + |d_m - \lambda_0| + |g_m| > 0, \quad |q_m| + |d_m - \lambda_0| + |h_m| > 0. \tag{4.44}$$

Then $\lambda = \lambda_0$ is an eigenvalue of the matrix R if and only if at least one of the subspaces

$$\operatorname{Ker}\begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix}, \quad \operatorname{Ker}\begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix}, \tag{4.45}$$

where $A_m(\lambda_0) = R(1:m, 1:m) - \lambda_0 I$, $B_m(\lambda_0) = R(m+1:N, m+1:N) - \lambda_0 I$, $Q_m = \mathrm{row}(a^{\times}_{m+1,k} q_k)_{k=1}^{m}$, $H_m = \mathrm{row}(b^{\times}_{m-1,k} h_k)_{k=m}^{N}$, is nontrivial. Moreover if $\lambda = \lambda_0$ is an eigenvalue of the matrix R then the vector $x = \begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix}$, where x_1, x_2 are vectors of the sizes $m-1, N-m$ respectively, θ is a number, is an eigenvector of the matrix R corresponding to the eigenvalue λ_0 if and only if there exist numbers θ_1, θ_2 such that $\theta_1 + \theta_2 = \theta$ and

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} \in \operatorname{Ker}\begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix}, \quad \begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} \in \operatorname{Ker}\begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix}. \tag{4.46}$$

In the case of Hermitian matrix any eigenvalue λ_0 is real and hence we have $l_m(\lambda_0) = \overline{\delta(\lambda_0)}$. This implies that the conditions (4.2) and (4.43) cover all possibilities.

Proof. Assume that at least one of the subspaces (4.45) is nontrivial and the vectors $\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix}, \begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix}$ satisfy the condition (4.46). Applying the representation (2.4) with $k = m$ and with $k = m-1$ to the matrix $R_{\lambda_0} = R - \lambda_0 I$ we obtain

$$R_{\lambda_0} = \begin{pmatrix} A_m(\lambda_0) & G_m H_{m+1} \\ P_{m+1} Q_m & B_{m+1}(\lambda_0) \end{pmatrix} = \begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1} H_m \\ P_m Q_{m-1} & B_m(\lambda_0) \end{pmatrix}, \tag{4.47}$$

where $A_m(\lambda_0) = R(1:m, 1:m) - \lambda_0 I$, $B_m(\lambda_0) = R(m+1:N, m+1:N) - \lambda_0 I$ and the elements Q_k, G_k, P_k, H_k are defined by (2.2). From(4.47) we obtain the representations

$$R_{\lambda_0}(:, 1:m) = \begin{pmatrix} I_m & 0 \\ 0 & P_{m+1} \end{pmatrix} \begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix}, \tag{4.48}$$

$$R_{\lambda_0}(:, m:N) = \begin{pmatrix} G_{m-1} & 0 \\ 0 & I_{N-m+1} \end{pmatrix} \begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix}. \tag{4.49}$$

The condition (4.46) implies

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} \in \mathrm{Ker}(R_{\lambda_0}(:, 1:m)), \quad \begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} \in \mathrm{Ker}(R_{\lambda_0}(:, m:N)).$$

Hence it follows that the vectors $\begin{pmatrix} x_1 \\ \theta_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \theta_2 \\ x_2 \end{pmatrix}$ belong to the kernel of the matrix $R - \lambda_0 I$. This implies that $\lambda = \lambda_0$ is an eigenvalue of the matrix R and moreover the sum $\begin{pmatrix} x_1 \\ \theta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \theta_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix}$, where $\theta = \theta_1 + \theta_2$ is an eigenvector of R corresponding to the eigenvalue $\lambda = \lambda_0$.

Assume that λ_0 is an eigenvalue of the matrix R and let $\begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix}$ be the corresponding eigenvector. Using the representation (4.15) we obtain

$$\begin{cases} A_{m-1}(\lambda_0) x_1 + G_{m-1} h_m \theta + G_{m-1} b_m H_{m+1} x_2 = 0 \\ p_m Q_{m-1} x_1 + (d_m - \lambda_0)\theta + g_m H_{m+1} x_2 = 0 \\ P_{m+1} a_m Q_{m-1} x_1 + P_{m+1} q_m \theta + B_{m+1}(\lambda_0) x_2 = 0. \end{cases} \tag{4.50}$$

By the first inequality from (4.44) we obtain that at least one of the conditions $|p_m| + |d_m - \lambda_0| > 0$ or $|g_m| + |d_m - \lambda_0| > 0$ holds. Assume that $|p_m| + |d_m - \lambda_0| \neq 0$. The first equality from (4.43) implies

$$\det \begin{pmatrix} p_m & d_m - \lambda_0 \\ a_m & q_m \end{pmatrix} = 0$$

and hence there exists a number α such that

$$\begin{pmatrix} a_m & q_m \end{pmatrix} = \alpha \begin{pmatrix} p_m & d_m - \lambda_0 \end{pmatrix}. \tag{4.51}$$

Multiplying the second equation in (4.50) by $P_{m+1}\alpha$ and subtracting the result from the third equation we obtain the equivalent system

$$\begin{cases} A_{m-1}(\lambda_0) x_1 + G_{m-1} h_m \theta + G_{m-1} b_m H_{m+1} x_2 = 0 \\ p_m Q_{m-1} x_1 + (d_m - \lambda_0)\theta + g_m H_{m+1} x_2 = 0 \\ -P_{m+1} \alpha g_m H_{m+1} x_2 + B_{m+1}(\lambda_0) x_2 = 0. \end{cases} \tag{4.52}$$

Notice that $|p_m|+|d_m-\lambda_0|>0$ implies $|h_m|+|d_m-\lambda_0|>0$, otherwise we get $d_m-\lambda_0=0$, hence by virtue of (4.51) $q_m=0$ and therefore $|q_m|+|d_m-\lambda_0|+|h_m|=0$ contradicting the second inequality from (4.44). The second equality from (4.43) implies

$$\det\begin{pmatrix} h_m & b_m \\ d_m-\lambda_0 & g_m \end{pmatrix}=0$$

and hence there exists a number β such that

$$\begin{pmatrix} b_m \\ g_m \end{pmatrix}=\beta\begin{pmatrix} h_m \\ d_m-\lambda_0 \end{pmatrix}. \tag{4.53}$$

Set $\theta_2=-\beta H_{m+1}x_2,\ \theta_1=\theta-\theta_2$. We should prove that $\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix}\in \operatorname{Ker}\begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix}$. Using the definition of θ_2, the formula (4.53) and the representation $H_m=\begin{pmatrix} h_m & b_mH_{m+1} \end{pmatrix}$ we get

$$H_m\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix}=h_m\theta_2+b_mH_{m+1}x_2=-h_m\beta H_{m+1}x_2+h_m\beta H_{m+1}x_2=0. \tag{4.54}$$

and

$$(d_m-\lambda_0)\theta_2+g_mH_{m+1}x_2=-\beta(d_m-\lambda_0)H_{m+1}x_2+\beta(d_m-\lambda_0)H_{m+1}x_2=0. \tag{4.55}$$

Next by virtue of the third equation from (4.50) and the relations (4.51), (4.53) we obtain

$$\begin{aligned} P_{m+1}q_m\theta_2+B_{m+1}(\lambda_0)x_2&=-P_{m+1}\alpha(d_m-\lambda_0)\beta H_{m+1}x_2+B_{m+1}(\lambda_0)x_2 \\ &=-P_{m+1}\alpha g_mH_{m+1}x_2+B_{m+1}(\lambda_0)x_2=0 \end{aligned}$$

which completes the proof of the second inclusion from (4.46).

Let us prove the first inclusion from (4.46). Using (4.54) and the first equation from (4.50) we have

$$A_{m-1}(\lambda_0)x_1+G_{m-1}h_m\theta_1+G_{m-1}(h_m\theta_2+b_mH_{m+1}x_2)=A_{m-1}(\lambda_0)x_1+G_{m-1}h_m\theta_1=0$$

and similarly from (4.55) and the second equation from (em.4) we conclude that

$$p_mQ_{m-1}x_1+(d_m-\lambda_0)\theta_1=0.$$

Moreover from here using the representation $Q_m=\begin{pmatrix} a_mQ_{m-1} & q_m \end{pmatrix}$ and the relation (4.51) we get

$$Q_m\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix}=\begin{pmatrix} a_m & q_m \end{pmatrix}\begin{pmatrix} Q_{m-1}x_1 \\ \theta_1 \end{pmatrix}=\alpha(p_mQ_{m-1}x_1+(d_m-\lambda_0)\theta_1)=0$$

which completes the proof. $\square$

Theorem 4.3 *Let R be quasiseparable of order one matrix with generators p_i $(i=2,\dots,N)$, q_j $(j=1,\dots,N-1)$, a_k $(k=2,\dots,N-1)$; g_i $(i=1,\dots,N-1)$, h_j $(j=2,\dots,N)$, b_k $(k=2,\dots,N-1)$; d_k $(k=1,\dots,N)$ and let $\lambda=\lambda_0$ be an eigenvalue of the matrix R. Assume that the following conditions are valid:*

$$|d_1-\lambda_0|+|q_1|>0,\quad |d_1-\lambda_0|+|g_1|>0, \tag{4.56}$$

$$|d_N-\lambda_0|+|p_N|>0,\quad |d_N-\lambda_0|+|h_N|>0; \tag{4.57}$$

for $m=j_1,j_2,\dots,j_k,\ j_1,\dots,j_k\in\{2,\dots,N-1\}$:

$$l_m(\lambda_0)=(d_m-\lambda_0)a_m-p_mq_m=0,\quad \delta_m(\lambda_0)=(d_m-\lambda_0)b_m-g_mh_m=0, \tag{4.58}$$

$$|p_m|+|d_m-\lambda_0|+|g_m|>0,\quad |q_m|+|d_m-\lambda_0|+|h_m|>0 \tag{4.59}$$

and for the other values of $m \in \{2, \dots, N-1\}$*:*

$$l_m(\lambda_0) \neq 0, \quad \delta_m(\lambda_0) \neq 0. \tag{4.60}$$

Then the corresponding to λ_0 *linear independent eigenvectors* u_i *of the matrix* R *are given via the following algorithm:*

1)Set $s = 0,\ \theta = 1,\ j_0 = 1$*; Set* $R^{(1)} = R$*, i.e. define the quasiseparable matrix* $R^{(1)}$ *via generators* $p_i^{(1)}$ $(i = 2, \dots, N)$*,* $q_j^{(1)}$ $(j = 1, \dots, N-1)$*,* $a_k^{(1)}$ $(k = 2, \dots, N-1)$*;* $g_i^{(1)}$ $(i = 1, \dots, N-1)$*,* $h_j^{(1)}$ $(j = 2, \dots, N)$*,* $b_k^{(1)}$ $(k = 2, \dots, N-1)$*;* $d_k^{(1)}$ $(k = 1, \dots, N)$ *which are equal to the corresponding generators of the matrix* R*.*

2)If $\theta = 0$ *stop.*

3)Set $m = j_{s+1} - j_s$*. Using the generators of the matrix* $R^{(s)}$ *compute by the formulas (3.1)-(3.4) the values* $\gamma_{m-1}^{(s)}(\lambda_0), f_{m-1}^{(s)}(\lambda_0)$*.*

If

$$(d_m^{(s)} - \lambda_0)\gamma_{m-1}^{(s)}(\lambda_0) - p_m^{(s)} f_{m-1}^{(s)}(\lambda_0) h_m^{(s)} = 0, \quad q_m^{(s)}\gamma_{m-1}^{(s)}(\lambda_0) - a_m^{(s)} f_{m-1}^{(s)}(\lambda_0) h_m^{(s)} = 0$$

compute the eigenvector u_s *of the matrix* R *as follows. Define the matrix* $\widetilde{R}^{(s)}$ *of sizes* $m \times m$ *via generators: set*

$$\widetilde{q}_j^{(s)} = q_j^{(s)},\ \widetilde{g}_j^{(s)} = g_j^{(s)},\ \widetilde{d}_j^{(s)} = d_j^{(s)}, \quad j = 1, \dots, m-1,$$

$$\widetilde{p}_i^{(s)} = p_i^{(s)},\ \widetilde{h}_i^{(s)} = h_i^{(s)},\ \widetilde{a}_j^{(s)} = a_i^{(s)},\ \widetilde{b}_j^{(s)} = b_i^{(s)}, \quad i = 2, \dots, m-1, \widetilde{h}_m^{(s)} = h_m^{(s)},$$

and if $|p_m^{(s)}| + |d_m^{(s)} - \lambda_0| > 0$ *then set* $\widetilde{p}_m^{(s)} = p_m^{(s)},\ \widetilde{d}_m^{(s)} = d_m^{(s)}$*, else set* $\widetilde{p}_m^{(s)} = a_m^{(s)},\ \widetilde{d}_m^{(s)} = q_m^{(s)} + \lambda_0$*. Compute the eigenvector* $\widetilde{u}_s$ *of the matrix* $\widetilde{R}^{(s)}$ *using the corresponding formula from Theorem 4.1. Set*

$$u_s = \begin{pmatrix} 0_{j_s - 1} \\ \widetilde{u}_s \\ 0_{N - j_{s+1}} \end{pmatrix}.$$

4) Using the generators of the matrix $R^{(s)}$ *compute by the formulas (3.6)-(3.8) the values* $\theta_{m+1}^{(s)}(\lambda_0), z_{m+1}^{(s)}(\lambda_0)$*.*

If

$$(d_m^{(s)} - \lambda_0)\theta_{m-1}^{(s)}(\lambda_0) - g_m^{(s)} z_{m+1}^{(s)}(\lambda_0) q_m^{(s)} = 0, \quad h_m^{(s)}\theta_{m+1}^{(s)}(\lambda_0) - q_m^{(s)} z_{m+1}^{(s)}(\lambda_0) b_m^{(s)} = 0$$

then define the matrix $R^{(s+1)}$ *of sizes* $N - j_{s+1} + 1 \times N - j_{s+1} + 1$ *via generators: set*

$$q_1^{(s+1)} = q_m^{(s)},\ q_j^{(s+1)} = q_{j+m-1}^{(s)},\ g_j^{(s+1)} = g_{j+m-1}^{(s)},\ d_j^{(s+1)} = d_{j+m-1}^{(s)}, \quad j = 1, \dots, N - j_{s+1},$$

$$p_i^{(s+1)} = p_{i+m-1}^{(s)},\ h_i^{(s+1)} = h_{i+m-1}^{(s)},\ a_i^{(s+1)} = a_{i+m-1}^{(s)},\ b_i^{(s)} = b_{i+m-1}^{(s)}, \quad i = 2, \dots, N - j_{s+1} + 1$$

and if $|g_m^{(s)}| + |d_m^{(s)} - \lambda_0| > 0$ *then set* $g_1^{(s+1)} = g_m^{(s)},\ d_1^{(s+1)} = d_m^{(s)}$*, else set* $g_m^{(s+1)} = b_m^{(s)},\ d_m^{(s+1)} = h_m^{(s)} + \lambda_0$*;*

else set $\theta = 0$*.*

5)$s := s + 1$*;*

6)If $s = k + 1$ *set* $\theta = 0$*;*

7)Go to 2).

Proof. Take $m = j_1$. By Theorem 4.2 each eigenvector of the matrix R corresponding to the eigenvalue λ_0 has the form

$$u = \begin{pmatrix} \widetilde{u}_1 \\ 0_{N-m} \end{pmatrix} + \begin{pmatrix} 0_{m-1} \\ \hat{u}_2 \end{pmatrix}$$

with

$$\widetilde{u}_1 \in \operatorname{Ker} \begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix}, \quad \hat{u}_2 \in \operatorname{Ker} \begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix},$$

where $A_m(\lambda_0)=R(1:m,1:m)-\lambda_0 I,\ B_m(\lambda_0)=R(m+1:N,m+1:N)-\lambda_0 I,\ Q_m=\mathrm{row}(a^{\times}_{m+1,k}q_k)_{k=1}^{m},\ H_m=\mathrm{row}(b^{\times}_{m-1,k}h_k)_{k=m}^{N}$. Moreover by virtue of (4.48), (4.49) the vectors $\begin{pmatrix} \widetilde{u}_1 \\ 0_{N-m} \end{pmatrix}$, $\begin{pmatrix} 0_{m-1} \\ \hat{u}_2 \end{pmatrix}$ belong to $\mathrm{Ker}(R-\lambda_0)I$. Notice that such a nonzero vector $\widetilde{u}_1$ exists if and only if

$$\mathrm{Ker}\begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix}=\mathrm{Ker}\begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1}h_m \\ p_mQ_{m-1} & d_m-\lambda_0 \\ a_mQ_{m-1} & q_m \end{pmatrix} \tag{4.61}$$

is a nontrivial subspace. Here we used the representation (2.6) for the matrix $A_m(\lambda_0)$ and the equality $Q_m=\begin{pmatrix} a_mQ_{m-1} & q_m \end{pmatrix}$. Since by virtue of the first equality from (4.58) we have

$$\det\begin{pmatrix} p_m & d_m-\lambda_0 \\ a_m & q_m \end{pmatrix}=0$$

this subspace is non-trivial if and only if

$$\det\begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1}h_m \\ p_mQ_{m-1} & d_m-\lambda_0 \end{pmatrix}=\det\begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1}h_m \\ a_mQ_{m-1} & q_m \end{pmatrix}=0.$$

Applying the formula (3.2) to these determinants we conclude that the subspace (4.61) is non-trivial if and only if

$$(d_m-\lambda_0)\gamma_{m-1}(\lambda_0)-p_mf_{m-1}(\lambda_0)h_m=q_m\gamma_{m-1}(\lambda_0)-a_mf_{m-1}(\lambda_0)h_m=0.$$

If the last holds we should consider two cases. The first one is $|p_m|+|d_m-\lambda_0|>0$. In this case we have also $|h_m|+|d_m-\lambda_0|>0$, otherwise we have $d_m-\lambda_0=0,\ p_m\neq 0$, from the second inequality from (4.59) we get $q_m\neq 0$ contradicting the first equality from (4.58). Thus the matrix

$$A_m=R(1:m,1:m)=\begin{pmatrix} A_{m-1} & G_{m-1}h_m \\ p_mQ_{m-1} & d_m \end{pmatrix}$$

and the number λ_0 satisfy the conditions of Theorem 4.1. Hence it follows that the number λ_0 is an eigenvalue of the matrix A_m of the geometric multiplicity one and the corresponding eigenvector $\widetilde{u}_1$ is obtained by the corresponding formula from Theorem 4.1. If $|p_m|+|d_m-\lambda_0|=0$ then we have $d_m-\lambda_0=0$, moreover by virtue of the first inequality from (4.59) we get $g_m\neq 0$ and from the second equality from (4.58) we obtain $h_m=0$ and thus from the second inequality from (4.59) we get $q_m\neq 0$. Hence it follows that the matrix

$$\begin{pmatrix} A_{m-1} & G_{m-1}h_m \\ p_mQ_{m-1} & d_m+\lambda_0 \end{pmatrix}$$

and the number λ_0 satisfy the conditions of Theorem 4.1. Therefore the number λ_0 is an eigenvalue of this matrix of the geometric multiplicity one and the corresponding eigenvector $\widetilde{u}_1$ is obtained by the corresponding formula from Theorem 4.1.

Next a nonzero vector $\hat{u}_2$ exists if and only if

$$\mathrm{Ker}\begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix}=\mathrm{Ker}\begin{pmatrix} h_m & b_mH_{m+1} \\ d_m-\lambda_0 & g_mH_{m+1} \\ P_{m+1}q_m & B_{m+1}(\lambda_0) \end{pmatrix} \tag{4.62}$$

is a nontrivial subspace. Here we used the representation (2.8) for the matrix $B_m(\lambda_0)$ and the equality $H_m=\begin{pmatrix} h_m & b_mH_{m+1} \end{pmatrix}$.

Since by virtue of the second equality from (4.58) we have

$$\det\begin{pmatrix} h_m & b_m \\ d_m-\lambda_0 & g_m \end{pmatrix}=0$$

this subspace is non-trivial if and only if

$$\det \begin{pmatrix} d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1} q_m & B_{m+1}(\lambda_0) \end{pmatrix} = \det \begin{pmatrix} h_m & b_m H_{m+1} \\ P_{m+1} q_m & B_{m+1}(\lambda_0) \end{pmatrix} = 0$$

Applying the formula (3.7) to these determinants we conclude that the subspace (4.61) is non-trivial if and only if

$$(d_m - \lambda_0)\theta_{m+1}(\lambda_0) - g_m z_{m+1}(\lambda_0) q_m = h_m \theta_{m+1}(\lambda_0) - b_m z_{m+1}(\lambda_0) q_m = 0. \tag{4.63}$$

If the last is valid we should consider two cases. The first one is $|g_m| + |d_m - \lambda_0| > 0$. In this case we have also $|q_m| + |d_m - \lambda_0| > 0$, otherwise we have $d_m - \lambda_0 = 0,\ q_m \neq 0$, from the second inequality from (4.59) we get $h_m \neq 0$ contradicting the second equality from (4.58). Thus in this case we set

$$R^{(2)} = B_m = R(m:N, m:N) = \begin{pmatrix} d_m & g_m H_{m+1} \\ P_{m+1} q_m & B_{m+1} \end{pmatrix}.$$

If $|g_m| + |d_m - \lambda_0| = 0$ then we have $d_m - \lambda_0 = 0$, moreover by virtue of the second inequality from (4.59) we get $g_m \neq 0$ and from the second equality from (4.58) we obtain $h_m = 0$ and thus from the first inequality from (4.59) we get $h_m \neq 0$. In this case we set

$$R^{(2)} = \begin{pmatrix} h_m + \lambda_0 & b_m H_{m+1} \\ P_{m+1} q_m & B_{m+1} \end{pmatrix}$$

Next we apply the same procedure to the matrix $R^{(2)}$ and so on until we get that the condition (4.63) is not valid or take $m = k$.

Since each of the eigenvectors u_s contains one in the position in which the previous vectors contain zero, it is clear that these vectors are linear independent. □

Remark. In the conditions of Theorem 4.3 the geometric multiplicity of the eigenvalue λ_0 is less than or equal to $k+1$.

The following theorem completes the results of Theorem 4.1 and Theorem 4.3 in some additional assumptions.

Theorem 4.4 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$ and assume that the generators of R satisfy the following conditions:*

$$\overline{d_i} = d_i, \quad i = 1, \dots, N; \quad \frac{p_{i+1} q_i}{\overline{g_i h_{i+1}}} > 0, \quad i = 1, \dots, N-1 \tag{4.64}$$

$$a_k \neq 0,\ b_k \neq 0, \quad \frac{p_k q_k}{a_k} = \frac{g_k h_k}{b_k} = \frac{\overline{g_k h_k}}{\overline{b_k}}, \quad k = 2, \dots, N-1. \tag{4.65}$$

Let also λ_0 be an eigenvalue of the matrix R.

If

$$l_k(\lambda_0) = (\lambda_0 - d_k) a_k + p_k q_k \neq 0, \quad k = 2, \dots, N-1 \tag{4.66}$$

then the eigenvalue λ_0 is simple and the corresponding eigenvector is given by the formula (4.6). If for $m = j_1, j_2, \dots, j_k,\ j_1, \dots, j_k \in \{2, \dots, N-1\}$:

$$l_m(\lambda_0) = (d_m - \lambda_0) a_m - p_m q_m = 0, \tag{4.67}$$

and for the other values of $m \in \{2, \dots, N-1\}$

$$l_m(\lambda_0) \neq 0 \tag{4.68}$$

then the multiplicity of λ_0 is less than or equal to $k+1$ and the corresponding eigenvectors may be computed by the algorithm from Theorem 4.3.

Proof. In the same way as in Example 3) from Subsection 4.1 we show that the matrix R is similar to a Hermitian matrix. Hence it follows that the matrix R is diagonalizable. Moreover the condition (4.65) implies that for any λ,

$$\delta_k(\lambda) = (\lambda - d_k)b_k + g_k h_k = b_k(\lambda - d_k + \frac{g_k h_k}{b_k}) = b_k(\lambda - d_k + \frac{p_k q_k}{a_k}) = \frac{b_k}{a_k} l_k(\lambda),\ k = 2, \dots, N-1.$$

Hence it follows that for any λ and for any $k \in \{2, \dots, N-1\}$, $l_k(\lambda) = 0$ if and only if $\delta_k(\lambda) = 0$. Therefore if the condition (4.66) holds then taking also into account that by virtue of the condition (4.64) we have $q_1 \neq 0,\ p_N \neq 0,\ h_N \neq 0$ we conclude all conditions of Theorem 4.1 are valid. Hence by Theorem 4.1 and by virtue of the fact that the matrix R is diagonalizable we conclude that the eigenvalue λ_0 is simple. Moreover since we have $p_N h_N \neq 0$ the corresponding eigenvector is given by the formula (4.6). In the case when the conditions (4.67), (4.68) are valid we have

$$l_m(\lambda_0) = \delta_m(\lambda_0) = 0$$

for $m = j_1, j_2, \dots, j_k,\ j_1, \dots, j_k \in \{2, \dots, N-1\}$ and

$$l_m(\lambda_0) \neq 0, \quad \delta_m(\lambda_0) \neq 0.$$

for the other values of $m \in \{2, \dots, N-1\}$. Taking into account that by virtue of the condition (4.64) we have $q_k \neq 0,\ g_k \neq 0\ (k = 1, \dots, N-1),\ p_k \neq 0,\ h_k \neq 0\ (k = 2, \dots, N)$ we conclude that all conditions of Theorem 4.3 are valid. Hence by Theorem 4.3 and by virtue of the fact that the matrix R is diagonalizable we conclude that the multiplicity of λ_0 is less than or equal to $k+1$ and the corresponding eigenvectors may be computed by the algorithm from Theorem 4.3. $\square$

Example

As an example of quasiseparable of order one matrix with a multiple eigenvalue we consider the matrix

$$R = \begin{bmatrix} \alpha & 1 & 1 & a & a & a \\ 1 & 2 & 1 & a & a & a \\ 1 & 1 & 1 & a & a & a \\ a & a & a & 1 & 1 & 1 \\ a & a & a & 1 & 2 & 1 \\ a & a & a & 1 & 1 & \beta \end{bmatrix}.$$

Here $\alpha, \beta, a \neq 0$ are the parameters. This matrix is quasiseparable of order one with generators $p_2 = p_3 = 1,\ p_4 = p_5 = p_6 = a;\ q_1 = q_2 = q_3 = 1,\ q_4 = q_5 = a^{-1};\ a_k = 1,\ k = 2, \dots, 5;\ g_1 = g_2 = g_3 = 1,\ g_4 = g_5 = a^{-1};\ h_2 = h_3 = 1,\ h_4 = h_5 = h_6 = a;\ a_k = 1,\ k = 2, \dots, 5;\ b_k = 1,\ k = 2, \dots, 5;\ d_1 = \alpha,\ d_2 = 2,\ d_3 = d_4 = 1,\ d_5 = 2,\ d_6 = \beta$. Take $\lambda_0 = 0$. We have $l_3(\lambda_0) = \delta_3(\lambda_0) = l_4(\lambda_0) = \delta_4(\lambda_0) = 0$. By Theorem 4.2 we conclude that $\lambda_0 = 0$ is an eigenvalue of the matrix R of the multiplicity three if

$$\det \begin{bmatrix} \alpha & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & \beta \end{bmatrix} = 0,$$

i.e. if $\alpha = 1,\ \beta = 1,\ a = 1$. In the case $\alpha = 1,\ \beta = 1,\ a \neq 1$ we obtain that the multiplicity of $\lambda_0 = 0$ equals two. In the same way one can obtain other cases when the multiplicity of $\lambda_0 = 0$ equals two and also the cases when this eigenvalue is simple.

4.4 An intermediate case

Next we consider an intermediate case when at least one of the conditions (4.2) of Theorem 4.1 does not hold but another one may be valid or non-valid, i.e. for some $m \in \{2, \dots, N-1\}$, $l_m(\lambda_0)\delta_m(\lambda_0) = 0$. We assume for simplicity that $d_m - \lambda_0 \neq 0$ which implies the conditions (4.44). We obtain here an analog of

Theorem 4.2. However we do not suggest here an algorithm for computation of the eigenvectors as it was done in Theorem 4.3.

At first we show that in the case under consideration the value of the characteristic polynomial of a matrix at the point λ_0 may be expressed via the product of the values at this point of the characteristic polynomial of the corresponding principal submatrices.

Lemma 4.5 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$ and let λ_0 be a number such that for some $m \in \{2, \dots, N-1\}$, $l_m(\lambda_0)\delta_m(\lambda_0) = 0$. Then the relation*

$$\det(R - \lambda_0 I) = \frac{1}{d_m - \lambda_0}\gamma_m(\lambda_0)\theta_m(\lambda_0), \tag{4.69}$$

where $\gamma_m(\lambda_0) = \det(R(1:m, 1:m) - \lambda_0 I)$, $\theta_m(\lambda_0) = \det(R(m+1:N, m+1:N) - \lambda_0 I)$, holds.

Proof. Assume that $l_m(\lambda_0) = 0$, i.e.

$$\det\begin{pmatrix} p_m & d_m - \lambda_0 \\ a_m & q_m \end{pmatrix} = 0.$$

Since $d_m - \lambda_0 \neq 0$ there exists a number α such that

$$\begin{pmatrix} a_m & q_m \end{pmatrix} = \alpha \begin{pmatrix} p_m & d_m - \lambda_0 \end{pmatrix}. \tag{4.70}$$

By virtue of the partition (4.15) we have

$$\det(R - \lambda_0 I) = \det\begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1}h_m & G_{m-1}b_m H_{m+1} \\ p_m Q_{m-1} & d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1}a_m Q_{m-1} & P_{m+1}q_m & B_{m+1}(\lambda_0) \end{pmatrix},$$

where $A_{m-1}(\lambda_0) = R(1:m-1, 1:m-1) - \lambda_0 I$, $B_{m+1}(\lambda_0) = R(1:m+1, 1:m+1) - \lambda_0 I$. Multiplying the second row of this determinant by $P_{m+1}\alpha$ and subtracting the result from the third row we obtain

$$\det(R - \lambda_0 I) = \det\begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1}h_m & G_{m-1}b_m H_{m+1} \\ p_m Q_{m-1} & d_m - \lambda_0 & g_m H_{m+1} \\ 0 & 0 & \widetilde{B}_{m+1}(\lambda_0) \end{pmatrix},$$

where $\widetilde{B}_{m+1}(\lambda_0) = B_{m+1}(\lambda_0) - P_{m+1}\alpha g_m H_{m+1}$ which implies

$$\det(R - \lambda_0 I) = \det A_m(\lambda_0) \cdot \det\widetilde{B}_{m+1}(\lambda_0) = \gamma_m(\lambda_0) \cdot \det\widetilde{B}_{m+1}(\lambda_0). \tag{4.71}$$

Next consider the determinant

$$\theta_m(\lambda_0) = \det\begin{pmatrix} d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1}q_m & B_{m+1}(\lambda_0) \end{pmatrix}.$$

Multiplying the first row of this determinant by $P_{m+1}\alpha$ and subtracting the result from the second row we obtain

$$\theta_m(\lambda_0) = \det\begin{pmatrix} d_m - \lambda_0 & g_m H_{m+1} \\ 0 & \widetilde{B}_{m+1}(\lambda_0) \end{pmatrix} = (d_m - \lambda_0) \cdot \det\widetilde{B}_{m+1}(\lambda_0). \tag{4.72}$$

Comparing (4.71) and (4.72) we obtain (4.69).

In the same way one can proceed in the case $\delta_m(\lambda_0) = 0$. □

Next we provide an analog of Theorem 4.2 for the eigenvectors of a matrix satisfying for some $m \in \{2, \dots, N-1\}$ the condition $l_m(\lambda_0)\delta_m(\lambda_0) = 0$.

Theorem 4.6 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$ and let λ_0 be a number such that for some $m \in \{2, \dots, N-1\}$,*

$$l_m(\lambda_0)\delta_m(\lambda_0) = 0. \tag{4.73}$$

Then $\lambda = \lambda_0$ is an eigenvalue of the matrix R if and only if

$$\gamma_m(\lambda_0)\theta_m(\lambda_0) = 0,, \tag{4.74}$$

where $\gamma_m(\lambda_0) = \det A_m(\lambda_0)$, $\theta_m(\lambda_0) = \det B_m(\lambda_0)$ and $A_m(\lambda_0) = R(1:m, 1:m) - \lambda_0 I$, $B_m(\lambda_0) = R(m+1:N, m+1:N) - \lambda_0 I$. Moreover if $\lambda = \lambda_0$ is an eigenvalue of the matrix R and $l_m(\lambda_0) = 0$ then the vector $x = \begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix}$, where x_1, x_2 are vectors of the sizes $m-1, N-m$ respectively, θ is a number, is an eigenvector of the matrix R corresponding to the eigenvalue λ_0 if and only if there exist numbers θ_1, θ_2 such that $\theta_1 + \theta_2 = \theta$ and

$$B_m(\lambda_0)\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} = 0, \quad A_m(\lambda_0)\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = -\begin{pmatrix} G_{m-1} \\ 0 \end{pmatrix}\frac{\delta_m(\lambda_0)}{d_m - \lambda_0}(H_{m+1}x_2). \tag{4.75}$$

Furthermore if $\lambda = \lambda_0$ is an eigenvalue of R and $\delta_m(\lambda_0) = 0$ then the vector x is an eigenvector of the matrix R corresponding to the eigenvalue λ_0 if and only if there exist numbers θ_1, θ_2 such that $\theta_1 + \theta_2 = \theta$ and

$$A_m(\lambda_0)\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = 0, \quad B_m(\lambda_0)\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} = -\begin{pmatrix} 0 \\ P_{m+1} \end{pmatrix}\frac{l_m(\lambda_0)}{d_m - \lambda_0}(Q_{m-1}x_1). \tag{4.76}$$

Here the vectors P_k, Q_k, G_k, H_k are defined via (2.2), (2.3).

Unfortunately we have not in our disposal a simple criteria and an algorithm for solution of the equations (4.75), (4.76) with singular matrices and non-zero right hand parts. In the additional assumption that $|l_m(\lambda_0)| + |\delta_m(\lambda_0)| > 0$ implies $|\gamma_m(\lambda_0)| + |\delta_m(\lambda_0| > 0$ we obtain in the equations (4.75), (4.76) either invertible matrices or zero right-hand parts. In this case one can obtain an algorithm similar to one from Theorem 4.3 but essentially more laborious.

Proof. From Lemma 4.5 we conclude that $\lambda = \lambda_0$ is an eigenvalue of the matrix R if and only if (4.74) holds.

Assume that λ_0 is an eigenvalue of R. Let $l_m(\lambda_0) = 0$ and let the vectors $\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix}$, $\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix}$ satisfy the condition (4.75). Using the representations

$$A_m(\lambda_0) = \begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1}h_m \\ p_m Q_{m-1} & d_m - \lambda_0 \end{pmatrix}, \quad B_m(\lambda_0) = \begin{pmatrix} d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1}q_m & B_{m+1}(\lambda_0) \end{pmatrix}$$

we obtain

$$A_{m-1}(\lambda_0)x_1 + G_{m-1}h_m\theta_1 + G_{m-1}\frac{\delta_m(\lambda_0)}{d_m - \lambda_0}H_{m+1}x_2 = 0, \tag{4.77}$$

$$p_m Q_{m-1}x_1 + (d_m - \lambda_0)\theta_1 = 0, \tag{4.78}$$

$$(d_m - \lambda_0)\theta_2 + g_m H_{m+1}x_2 = 0, \tag{4.79}$$

$$P_{m+1}q_m\theta_2 + B_{m+1}(\lambda_0)x_2 = 0. \tag{4.80}$$

From (4.79) we get $\theta_2 = -\frac{1}{d_m - \lambda_0}g_m H_{m+1}x_2$ and the substitution to (4.77) yields

$$A_{m-1}(\lambda_0)x_1 + G_{m-1}h_m(\theta_1 + \theta_2) + G_{m-1}\left(-\frac{g_m h_m}{d_m - \lambda_0} + \frac{\delta_m(\lambda_0)}{d_m - \lambda_0}\right)H_{m+1}x_2 = 0.$$

¿From here using the equality $\delta_m(\lambda_0) = b_m(d_m - \lambda_0) + g_m h_m$ and setting $\theta = \theta_1 + \theta_2$ we obtain

$$A_{m-1}(\lambda_0)x_1 + G_{m-1}h_m\theta + G_{m-1}b_m H_{m+1}x_2 = 0. \tag{4.81}$$

Moreover from (4.78), (4.79) it follows that

$$p_m Q_{m-1}x_1 + (d_m - \lambda_0)\theta + g_m H_{m+1}x_2 = 0. \tag{4.82}$$

Next using the relations (4.78) and (4.70) we obtain

$$a_m Q_{m-1}x_1 + q_m\theta_1 = 0$$

which implies

$$P_m a_m Q_{m-1}x_1 + P_m q_m\theta_1 = 0.$$

The last equality together with (4.80) yields

$$P_m a_m Q_{m-1}x_1 + (d_m - \lambda_0)\theta + g_m H_{m+1}x_2 = 0. \tag{4.83}$$

¿From (4.81)-(4.83) we conclude that the vector $x = \begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix}$ belongs to $\mathrm{Ker}(R - \lambda_0 I)$.

Now assume that $l_m(\lambda_0) = 0$ and the vector $x = \begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix}$, where x_1, x_2 are vectors of the sizes $m-1, N-m$ respectively, θ is a number, is an eigenvector of the matrix R corresponding to the eigenvalue λ_0. In the same way as in the proof of Theorem 4.2 we obtain that the components of the vector x satisfy the system (4.52). Moreover the number α in the third equation from (4.52) is defined via the relation (4.70) which implies $\alpha = q_m/(d_m - \lambda_0)$. Set

$$\theta_2 = -(g_m H_{m+1}x_2)/(d_m - \lambda_0). \tag{4.84}$$

We obtain

$$B_m(\lambda_0)\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1}Q_m & B_{m+1}(\lambda_0) \end{pmatrix}\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} = 0.$$

Furthermore setting $\theta_1 = \theta - \theta_2$ and using the first and the second equation from (4.52) and the equality (4.84) we get

$$\begin{cases} A_{m-1}(\lambda_0)x_1 + G_{m-1}h_m\theta_1 + G_{m-1}\left(-\frac{h_m g_m}{d_m - \lambda_0} + b_m\right)H_{m+1}x_2 = 0 \\ p_m Q_{m-1}x_1 + (d_m - \lambda_0)\theta_1 = 0. \end{cases}$$

Hence it follows that

$$\begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1}h_m \\ p_m Q_{m-1} & d_m - \lambda_0 \end{pmatrix}\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = -\begin{pmatrix} G_{m-1}\frac{\delta_m(\lambda_0)}{d_m - \lambda_0}(H_{m+1}x_2) \\ 0 \end{pmatrix}$$

which completes the proof of (4.75).

In the case $\delta_m(\lambda_0) = 0$ one can proceed in the same way. □

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