Kharitonov's theorem and Hermite's criterion*

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Abstract

In this correspondence we answer a question posed in [WT99] and provide an elementary proof of the Kharitonov theorem deducing it from the classical Hermite criterion. The proof is based on the concept of a Bezoutian matrix. Generally, exploiting the special structure of such matrices (e.g., Bezoutians, Toeplitz, Hankel or Vandermonde matrices, etc.) can be interesting, e.g., leading to unified approaches in different cases, as well as to further generalizations. Here the concept of the Bezoutian matrix is used to provide a unified derivation of the Kharitonov-like theorems for the continuous-time and discrete-time settings. Finally, the (block) Anderson-Jury Bezoutians are used to propose a possible technique to attack an open difficult problem related to the robust stability in the MIMO case.

1 Introduction

I.1. Hermite's criterion. Perhaps the first solution to the polynomial stability problem was given by Hermite in 1856, see, e.g., [KN81]; we next provide a variant of his result .

Theorem 1 *[Hermite] The polynomial*

$$p(x) = p_0 + p_1 x + p_2 x^2 + \cdots + p_n x^n \tag{1}$$

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is stable (all roots are in the open left-half-plane) if and only if the matrix

$$B = \left[\ b_{kl}\ \right] \quad (b_{kl} \text{ are obtained from } \quad \sum_{k,l=0}^{n-1} b_{kl}x^k y^l = -\frac{j}{2}\cdot\frac{p(jx)\bar{p}(jy)-\bar{p}(jx)p(jy)}{x-y}) \tag{2}$$

is positive definite[1]*. Here*

$$\bar{p}(x) = p_0{}^* + p_1{}^*x + p_2{}^*x^2 + \cdots + p_n{}^*x^n. \tag{3}$$

Substituting

$$p(jx) = g(x) + jh(x) \tag{4}$$

into (2) one immediately sees that the above matrix B is the Bezoutian matrix

$$\mathrm{Bez}(p) = \mathrm{Bez}(h,g) = \left[\ b_{kl}\ \right] \quad \text{defined by} \quad \sum_{k,l=0}^{n-1} b_{kl}x^k y^l = \frac{h(x)g(y)-g(x)h(y)}{x-y}. \tag{5}$$

Hence theorem 1 can be immediately reformulated as follows.

Theorem 2 *The polynomial (1) is stable if and only if the Bezoutian matrix Bez(h,g) of the two "split" polynomials $g(x)$ and $h(x)$ in (4) is positive definite.*

I.2. The Kharitonov's theorem. We now turn to the stability of interval polynomials. In [K78] Kharitinov obtained the following fundamental result.

Theorem 3 *[Kharitonov] All polynomials of the form (1) satisfying $\underline{p_i} \le p_i \le \bar{p}_i$ are stable if and only if the following four polynomials are stable:*

$$k_1(x) = \widehat{e}(x) + \widehat{o}(x), \quad k_2(x) = \widehat{e}(x) + \check{o}(x), \quad k_3(x) = \check{e}(x) + \widehat{o}(x), \quad k_4(x) = \check{e}(x) + \check{o}(x), \tag{6}$$

where

$$\hat{e}(x) = \underline{p}_0 + \overline{p}_2x^2 + \underline{p}_4x^4 + \overline{p}_6x^6 + \ldots, \quad \check{e}(x) = \overline{p}_0 + \underline{p}_2x^2 + \overline{p}_4x^4 + \underline{p}_6x^6 + \ldots,$$
$$\hat{o}(x) = \underline{p}_1x + \overline{p}_3x^3 + \underline{p}_5x^5 + \overline{p}_7x^7 + \ldots, \quad \check{o}(x) = \overline{p}_1x + \underline{p}_3x^3 + \overline{p}_5x^5 + \underline{p}_7x^7 + \ldots,$$

Clearly, theorem 3 is equivalent to the following one.

Theorem 4 *[Kharitonov-Hermite] The matrix Bez(p) is positive definite if and only if four Bezoutians Bez(k_1),Bez(k_2),Bez(k_3),Bez(k_4) of the four polynomials in (6) are positive definite.*

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[1] It is worth noting that the expression on the left-hand side of (2) is a polynomial in x and y ($x-y$ cancels out) and the $n \times n$ matrix B is well-defined.

I.3. A connection between the Kharitonov and Hermite's results. There is a vast literature on the Kharitonov's theorem, focusing mainly on two issues. The first is constructing elementary proofs for the Kharitonov theorem(see, e.g., [B87],[V88],[WT99]) and the second is addressing its counterparts and generalizations. Typically the motivation for the first direction is to provide more insights into the connections of theorem 3 to classical results. Surprisingly enough, a connection between Kharitonov's theorem 3 and the classical Hermite theorem 1 was never fully elaborated. Moreover, Willems and Tempo [WT99] asked recently if it is possible to provide a direct proof for the theorem 4. One should mention that a brute-force approach does not work here since examples show that the matrix $B(p) - B(k_i)$ may not be positive definite for any of the four k_i's.

I.4. The structure of this correspondence. In the second section of our correspondence we answer the Willems-Tempo question and specify a proof of the theorem 4 based on the congruence of Bezoutians to particular diagonal matrices which can be analyzed further. Structured matrices (of which Bezoutians are a special case) have attracted some attention recently (see, e.g., a collection of papers [O01] and many references therein). Typically their use may offer the following two advantages. First, structured matrix formulations may lead to unified treatments of various special cases. Secondly, the matrix language can be used to formulate natural matrix generalizations. In the second section of this note we show that with little modifications the continuous-time results of the first section can be carried over to the discrete-time case, yielding the results of [V88]. In this way, the Bezoutian approach reveals a beautiful analogy between the continuous and discrete cases. All the above results were limited to the SISO case. In the fourth section we use the concept of the more general Anderson-Jury Bezoutian, and ask if it is possible to use its properties to obtain a solution to the Kharitonov-like problems in the MIMO case.

2 Continuous-time case

II.1. Properties of Bezoutians. To deduce theorem 4 from theorem 1 we need to establish the following elementary but crucial property of positive definite Bezoutian matrices.

Lemma 5 *If the Bezoutian $B = Bez(h, g)$ defined by (5) of two real polynomials is positive definite then the following five conditions hold.*

1. *The roots of $g(x)$ and $h(x)$ are all real.*
2. *The roots of each of the $h(x)$ and $g(x)$ are distinct.*
3. *Let us denote* $\begin{cases} a(x) \triangleq h(x), \quad b(x) \triangleq g(x) & \text{if } \deg h(x) \geq \deg g(x); \\ a(x) \triangleq g(x), \quad b(x) \triangleq -h(x) & \text{if } \deg h(x) < \deg g(x); \end{cases}$ *Then*

$$VBV^* = \mathrm{diag}(\begin{bmatrix} a'(x_1)b(x_1) & \cdots & a'(x_n)b(x_n) \end{bmatrix}) \tag{7}$$

where the nodes $\{x_i\}$ of the Vandermonde matrix $V_{ik} = \begin{bmatrix} x_i^{k-1} \end{bmatrix}$ are the roots of $a(x)$.

4. *The roots of $g(x)$ and $h(x)$ interlace.*
5. *The leading coefficients of $a(x)$ and $b(x)$ have the same sign.*

The conditions 1), 2), 4) and 5) are also sufficient for the positive definitness of B.

The proof is provided in the appendix below. We are now ready to prove theorem 4.

II.2. A Bezoutian Proof of the Hermite-Kharitonov theorem. Proof. Let us split polynomials $k_i(jx) = g_i(x) + jh_i(x)$ defined in (6) similarly to the splitting of $p(jx)$ in (4). Since all four matrices $B(h_i, g_k)$ are positive definite, all four pairs of polynomials $\{g_i(x), h_k(x)\}$ have the five properties stated in lemma 5 above. We now have to prove that the pair $\{g, h\}$ has the same properties so that lemma 5 could be applied to show that $B(h, g)$ is positive definite.

First note that (6) imply that for all $x \in R$ we have two properties: **(a)** Either $g_1(x) \leq g(x) \leq g_4(x)$ or $g_1(x) \geq g(x) \geq g_4(x)$. **(b)** Either $h_1(x) \leq h(x) \leq h_4(x)$ or $h_1(x) \geq h(x) \geq h_4(x)$. Hence, the roots of $g(x)$ and $h(x)$ are real (property 1)), distinct (property 2)). One can also see that the leading coefficients of $g(x)$ and $h(x)$ inherit their signs from $g_i(x)$ and $h_i(x)$ so that the property 5) also holds.

It remains to prove 4). To this end let us notice that the properties **(a) - (b)** imply

$$\text{Either} \quad z_{i,g_1} \leq z_{i,g} \leq z_{i,g_4}, \quad \text{or} \quad z_{i,g_1} \geq z_{i,g} \geq z_{i,g_4}. \tag{8}$$

$$\text{Either} \quad z_{i,h_1} \leq z_{i,h} \leq z_{i,h_4}, \quad \text{or} \quad z_{i,h_1} \geq z_{i,h} \geq z_{i,h_4}. \tag{9}$$

where z_{i,g_1} denotes the i'th root of $g_1(x)$ and the other quantities involved are defined similarly. Now, the roots of each $g_1(x)$ and $g_4(x)$ interlace with the roots of each $h_1(x)$

and $h_4(x)$. Hence (8) implies that the roots of g interlace with the roots of each $h_1(x)$ and $h_4(x)$. Finally, employing (9) one sees that the roots of $g(x)$ and $h(x)$ indeed interlace. $\Box$

One of the properties in lemma 5 is the interlacing property 4). One has to mention that it was used earlier in some direct proofs of the Kharitonov's theorem 3, see, e.g., [B87]. One advantage of its use in the context of the factorization of Bezoutians (7) is that it provides a beautiful unified approach to the continuous time and discrete time cases, the latter to be discussed in the next section. Moreover, it can provide a possible way to approach the problem of generalizing Kharitonov's theorem to the MIMO case, see sec. 4.

3 Discrete-time case

III.1. Discrete-time splitting, symmetric polynomials and Bezoutians. We start with recalling the dicrete-time analogues of the results described in the Introduction. The discrete-time counterpart of the Hermite's theorem 1 is the following theorem [S18].

Theorem 6 *[Schur] The polynomial (1) is discrete-time stable (has all its roots in the interior of the unit disk $\mathbb{T}$) if and only if the matrix*

$$B = \left[\ b_{kl}\ \right] \quad \text{defined by} \quad \sum_{k,l=0}^{n-1} b_{kl} x^k y^l = \frac{1}{2} \cdot \frac{p^{\#}(x)\bar{p}^{\#}(y) - p(x)\bar{p}(y)}{1-xy} \tag{10}$$

is positive definite. Here $\bar{p}(x)$ is defined as in (3) and $p^{\#}(x)$ is defined by

$$p^{\#}(x) = x^n \bar{p}(\frac{1}{x}) = {p_0}^* x^n + {p_1}^* x^{n-1} + \cdots + {p_n}^*. \tag{11}$$

$p(x)$ is called symmetric [KN81] if $p^{\#}(x) = p(x)$. A discrete-time analogue of (4) is

$$p(x) = g(x) + jh(x), \qquad \text{where } \{g(x), h(x)\} \text{ are both symmetric.} \tag{12}$$

Substituting (12) into (10) one immediately sees that the above matrix B is the Bezoutian

$$\mathrm{Bez}_{\mathbb{T}}(p) = \mathrm{Bez}_{\mathbb{T}}(h,g) = \left[\ b_{kl}\ \right] \quad \text{defined by} \quad \sum_{k,l=0}^{n-1} b_{kl} x^k y^l = j \frac{g(x)\bar{h}(y) - h(x)\bar{g}(y)}{1-xy}, \tag{13}$$

where $\bar{p}(x)$ is defined as in (3). Theorem 6 can now be reformulated as follows.

Theorem 7 *The polynomial (1) is stable in the discrete time sense if and only if the Bezoutian matrix $Bez_{\mathbb{T}}(h,g)$ of the $g(x)$ and $h(x)$ in (12) is positive definite.*

III.2. Properties of discrete-time Bezoutians. To formulate a discrete-time counterpart of lemma 5 we need to give two definitions.

- Let $a(x) = a^{\#}(x) = \sum_{k=0}^{n} a_k x^k$ is symmetric, $a_0 \neq 0$ and $a_n \neq 0$, and n is even. Then

$$\frac{a(x)}{x^{\frac{n}{2}}} = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \widetilde{a}_k z^k \text{ with } \widetilde{a}_k = \begin{cases} a_{k+\frac{n}{2}} & if \quad k \geq 0; \\ (\widetilde{a}_{-k})^* & if \quad k < 0. \end{cases}$$

From $a_k = a^*_{n-k}$ ($a(x)$ is symmetric) it follows that the following function is real

$$\tilde{a}(\theta) = \frac{a(e^{j\theta})}{e^{jn\theta/2}} = \sum_{k=-\frac{n}{2}}^{\frac{n}{2}} \widetilde{a}_k e^{jk\theta} \tag{14}$$

- If n is odd then we define the corresponding real function by

$$\tilde{a}(\theta) = \frac{a(e^{j\theta})}{e^{jn\theta/2}} = \sum_{k=1}^{\frac{n+1}{2}} (\widetilde{a}_{-k} e^{-j\frac{(2k-1)}{2}\theta} + \widetilde{a}_k e^{j\frac{(2k-1)}{2}\theta}) \text{ with } \widetilde{a}_k = \begin{cases} a_{k+\frac{n-1}{2}} & if \quad k > 0; \\ (\widetilde{a}_{-k})^* & if \quad k < 0. \end{cases} \tag{15}$$

Lemma 8 *If the Bezoutian $Bez_{\mathbb{T}}(h,g)$ (5) of two symmetric polynomials is positive definite then the following five conditions hold.*

1. The roots of $g(x)$ and $h(x)$ all lie on the unit circle.

2. The roots of each of the $h(x)$ and $g(x)$ are distinct, and ged $h(x) = \deg g(x)$.

3. Let $\{x_k = e^{j\theta_k}\}$ be the roots of $h(x)$, and $V = \left[\begin{array}{c} x_i^{j-1} \end{array}\right]$. Then

$$VBV^* = j\text{diag}(\left[\begin{array}{ccc} \frac{h'(x_1)g(x_1)}{x_1^{n-1}} & \cdots & \frac{h'(x_n)g(x_n)}{x_1^{n-1}} \end{array}\right] = \text{diag}\left[\begin{array}{ccc} \tilde{h}'(\theta_1)\tilde{g}(\theta_1) & \cdots & \tilde{h}'(\theta_n)\tilde{g}(\theta_n) \end{array}\right]. \tag{16}$$

4. The roots of $g(x)$ and $h(x)$ interlace on the unit circle. The same is true for $\tilde{g}(\theta)$ and $\tilde{h}(\theta)$ on the interval $\theta \in [0, 2\pi]$.

5. The quantities $\tilde{h}'(0)$ and $\tilde{g}(0)$ should have the same sign. The conditions 1), 2), 4) and 5) are also sufficient for the positive definitness of B.

The proof is provided in the appendix below.

III.3. A Bezoutian proof of Vaidyanathan's Theorem. It is known that there is no direct counterpart of Kharitonov's theorem for the discrete case. The Bezoutian approach shows one reason for this. Formula (16) indicates that the analogue would be more naturally formulated in terms of the coefficients of $\widetilde{g}(\theta)$ and $\widetilde{h}(\theta)$. To formulate the corresponding result by Vaidyanathan we need to express $\widetilde{g}(\theta)$ and $\widetilde{h}(\theta)$ in a more appropriate form needed later to define the discrete-time "interval polynomials."

Lemma 9 *Let $p(x) = g(x) + jh(x)$, with symmetric $g(x)$ and $h(x)$. The functions $\widetilde{g}(\theta)$ and $\widetilde{h}(\theta)$ can be represented as follows:*

$$\widetilde{g}(\theta) = \Sigma_k c_{2k}^{(p)} cos^{2k}(\frac{\theta}{2}), \qquad \widetilde{h}(\theta) = sin(\theta)\Sigma_k d_{2k}^{(p)} cos^{2k}(\frac{\theta}{2}) \tag{17}$$

Due to the symmetry of $\widetilde{g}(\theta)$ it is possible to pair off the symmetric terms from its expansion so that a cosine is produced from each pair. Then, trigonometric transformations can be used to represent the result as a sum of powers of cosine. Similarly, the terms of $\widetilde{h}(\theta)$ can be paired off and then transformed using trigonometric identities to give the needed result. For details of this construction, we refer the reader to [V88].

The next analogue of Kharitonov's theorem [V88] uses these new coefficients c_k and d_k.

Theorem 10 *[Vaidyanathan] Let $\{g_1(x), g_2(x), h_1(x), h_2(x)\}$ be four symmetric polynomials such that four related polynomials polynomials $k_{mp}(x) = g_m(x) + jh_p(x)$, $(m, p = 1, 2)$ are stable in the discrete-time sense, and satisfy $c_m^{(k_{11})} \leq c_i^{(k_{22})}, d_i^{(k_{11})} \leq d_i^{(k_{22})}$ (where real $c_i^{(k_{ss})}$ and $d_i^{(k_{tt})}$ are defined similarly to (17)). Then all polynomials $p(x)$ satisfying*

$$c_i^{(k_{11})} \leq c_i^{(p)} \leq c_i^{(k_{22})}, \qquad d_i^{(k_{11})} \leq d_i^{(p)} \leq d_i^{(k_{22})} \tag{18}$$

are stable in the discrete-time sense.

We shall prove the following immediate reformulation of the latter theorem.

Theorem 11 *[Schur-Vaidyanathan] If $\mathrm{Bez}_{\mathbb{T}}(h_k, g_m) > 0$ for $k, m = 1, 2$ then $\mathrm{Bez}_{\mathbb{T}}(h, g) > 0$ where h and g are constrained by (18).*

Proof. As in the continuous time case, the fact that the four Bezoutians $Bez(h_m, g_p)$ are positive definite implies stability of the four polynomials k_{mp}. Hence (18) implies that for all $\theta \in [0, 2\pi]$ we have

(a) Either $\widetilde{g}_1(\theta) \leq \widetilde{g}(\theta) \leq \widetilde{g}_2(\theta)$ or $\widetilde{g}_1(\theta) \geq \widetilde{g}(\theta) \geq \widetilde{g}_2(\theta)$.

(b) Either $\widetilde{h}_1(\theta) \leq \widetilde{h}(\theta) \leq \widetilde{h}_2(\theta)$ or $\widetilde{h}_1(\theta) \geq \widetilde{h}(\theta) \geq \widetilde{h}_2(\theta)$.

Hence the roots of $g(x), h(x)$ lie on the unit circle and distinct (properties 1,2 of lemma 8). We need to show property 4). To this end notice that the properties **(a)-(b)** above imply

$$\text{Either} \quad \theta_{i,g_1} \leq \theta_{m,g} \leq \theta_{i,g_2}, \ \text{ or } \ \theta_{i,g_1} \geq \theta_{i,g} \geq \theta_{i,g_2}. \tag{19}$$

$$\text{Either} \quad \theta_{i,h_1} \leq \theta_{i,h} \leq \theta_{i,h_2}, \ \text{ or } \ \theta_{i,h_1} \geq \theta_{i,h} \geq \theta_{i,h_2}. \tag{20}$$

where θ_{i,g_1} denotes the i'th root of $\widetilde{g}_1(\theta)$ and the other quantities are defined similarly.

Now, the roots of each $\widetilde{g}_1(\theta)$ and $\widetilde{g}_2(\theta)$ interlace with the roots of each $\widetilde{h}_1(\theta)$ and $\widetilde{h}_2(\theta)$. It follows (19) that the roots of $\widetilde{g}(\theta)$ interlace with the roots of each $\widetilde{h}_1(\theta)$ and $\widetilde{h}_2(\theta)$. Hence (20) implies that the roots of $\widetilde{g}(\theta)$ and $\widetilde{h}(\theta)$ interlace (property 4)).

Finally, (18) yields that the signs at 0 of $\widetilde{h}'(\theta)$ and $\widetilde{g}(\theta)$ are inherited from the signs at 0 of $\widetilde{h}_i'(\theta)$ and $\widetilde{g}_i(\theta)$, $i = 1, 2$, implying property 5. □

4 The MIMO Case: the Anderson-Jury Bezoutian and the Lerer-Tismenetsky results

We conclude this note with an open question about robust stability in the MIMO case. Let $L(\lambda) = A_0 + A_1\lambda + \cdots A_n\lambda^n$ be a matrix polynomial with A_k be $m \times m$ matrices. We refer, e.g., to [GLR82] to the theory of matrix polynomials, and here only briefly recall some definitions. A number μ is called an eigenvalue of $L(\lambda)$ if $\det L(\mu) = 0$. Clearly, if A_n is nonsingular $L(\lambda)$ has nm eigenvalues. A matrix polynomial is called stable if all of its eigenvalues lie in the open left half plane. A matrix polynomial is called stable in the discrete-time case if all its eigenvalues lie in the interior of the unit disk. The natural counterparts of theorems 3 and 10 can be immediately formulated. For example, one can ask if stability of the entire set of matrix polynomials satisfying

$$\underline{A}_k \leq A_k \leq \overline{A}_k$$

could be reduced to checking a finite number of matrix polynomials.

In this section we describe one approach that might be useful to attack this problem. Let $M(\lambda)$ be any polynomial satisfying

$$L^*(-\lambda)L(\lambda) = M^*(-\lambda)M(\lambda). \tag{21}$$

Define

$$B(\lambda,\mu) = \frac{L(-\lambda)L(\mu) - M(-\lambda)M(\mu)}{\lambda - \mu} = \Sigma_{i,k=0}^{m-1,n-1}\Gamma_{ik}\lambda^i\mu^k.$$

The latter matrix B is the Anderson-Jury Bezoutian [AJ76]. Lerer and Tismenetsky showed that it plays a crucial role in studying stability of matrix polynomials. We next describe a special case of their results [LT82]. For a given $L(\lambda)$ there many polynomials $M(\lambda)$ satisfying (21). Lerer and Tismenetsky showed that

$$\gamma_+(L) = \pi(\hat{B}) + \gamma_+(N),$$

Where $N(\lambda)$ is the greatest common divisor of $L(\lambda)$ and $M(\lambda)$. In case $L(\lambda)$ is stable $M(\lambda)$ can be chosen to be antistable[2] (so that $L(\lambda)$ and $M(\lambda)$ are relatively prime). Thus positive definite Bezoutaians correspond to the stable matrix polynomials. The paper [LT82] contains a counterpart for the discrete-time case. We therefore ask if these facts can be used to generalize the results of sec. 2 and 3 to matrix polynomials.

5 Appendix: Proofs of Lemmas

5.1 Proof of Lemma 5

1. The condition $B > 0$ implies $uBu^* > 0$ for any row vector u. Assuming z is not real, let us evaluate $B(z, z^*) =$

$$\begin{bmatrix} 1 & z & \cdots & z^{n-2} & z^{n-1} \end{bmatrix} B \begin{bmatrix} 1 & z & \vdots & z^{n-2} & z^{n-1} \end{bmatrix}^* = \frac{a(z)b(z^*) - a(z^*)b(z)}{z - z^*}. \tag{22}$$

If z is a non-real zero of one of the real polynomials $a(x)$ or $b(x)$ then the expression in (22) must be zero, which would mean that B is not positive definite. Hence all $\{x_i\}$ must be real.

[2] All eigenvalues are in the right-half plane.

2. Let us now evaluate the expression in (22) for z being a real root of $a(x)$. Clearly, it is equal to

$$B(z, z^*) = lim_{y \to z} \frac{a(z)b(y) - a(y)b(z)}{z - y} = a'(z)b(z) \tag{23}$$

Thus, if z would be a multiple root of $a(x)$ the expression in (23) would be zero. Then an argument similar to the one above would imply that the matrix B is not positive definite. Hence all roots of $a(x)$ are simple.

3. Since V stacks the rows of the form $\left[\begin{array}{cccc} 1 & x_i & \cdots & x_i^n \end{array} \right]$ shown in (22) we have $VBV^* = \left[\begin{array}{c} B(x_i, x_k^*) \end{array} \right]$. Evaluating the entries of the latter matrix and using (23) one obtains (7).

4. Since all the roots $\{x_i\}$ of $a(x)$ are simple, hence the sequence $\{a'(x_i)\}$ is sign-interchanging. Positive definiteness of B and (7) imply that $\{b(x_i)\}$ must be sign-interchanging as well. Hence the zeros of $a(x)$ and $b(x)$ interlace. In particular, this means that the roots of $b(x)$ are simple as well.

5. Finally, all the elements $a'(x_i)b(x_i)$ of the matrix on the right-hand side of (7) must be positive. Hence the leading coefficients of $a(x)$ and $b(x)$have the same sign.

After the above discussion the sufficiency of 1), 2), 4) and 5) is obvious.

5.2 Proof of Lemma 8

1. Assuming $z \neq \frac{1}{z^*}$ (i.e., it is not on the unit circle), let us evaluate

$$\left[\begin{array}{cccc} 1 & z & \cdots & z^{n-1} \end{array} \right] B \left[\begin{array}{cccc} 1 & z & \cdots & z^{n-1} \end{array} \right]^* = B(z, z^*) = j \frac{g(z)\bar{h}(z^*) - h(z)\bar{g}(z^*)}{1 - zz^*}. \tag{24}$$

the expression in (24) must be zero, but this would mean that B is not positive definite. Hence all $\{x_i\}$ must lie on the unit circle.

2. Let us now evaluate the expression in (24) for the z being a unimodular root of $h(x)$.

$$B(z, z^*) = lim_{y \to z^*} j \frac{g(z)\bar{h}(y) - h(z)\bar{g}(y)}{1 - zy}. \tag{25}$$

Making the substitution $y = \frac{1}{u}$ and using the fact that both $h(x)$ and $g(x)$ are symmetric, one can show that (25) reduces to

$$(z^*)^{n-1} lim_{u \to z} j \frac{g(z)h(u) - h(z)g(u)}{u - z} = j \frac{h'(z)g(z)}{(z)^{n-1}} \tag{26}$$

Thus, if z would be a multiple root of $h(x)$ the expression in (26) would be zero. Then the matrix B would not be positive definite. Hence all roots of $h(x)$ are simple.

According to Schur's theorem, the positive definiteness of Bez(h, g) implies that $p(x) = h(x) + jg(x)$ is stable. The stability of $p(x)$ implies that $|p_0| \neq |p_n|$. From this it follows that $\deg g(x) = \deg h(x) = \deg p(x)$.

3. Formula (24) implies $VBV^* = \left[\ B(x_i, x_k^*) \ \right]$. Evaluating the entries of the latter matrix and using (26) one obtains the first equality in (16).

   Using $\frac{\partial h}{\partial x} = \frac{\partial h}{\partial \theta} \frac{\partial \theta}{\partial x} = \frac{\partial h}{\partial \theta} \frac{-i}{x}$ we see that the expression on the right-hand-side of (26) is equal to $\frac{h'(\theta)}{e^{j\theta n/2}} \cdot \frac{g(\theta)}{e^{j\theta n/2}}$. Using the definitions (14) and (15) and the fact that x is a root of $h(x)$ (so that θ is the root of $\widetilde{h}$ we obtain the second equality in (16).

4. Since all the roots $\{x_i\}$ of $h(x)$ are simple, hence the sequence $\{\widetilde{h}'(\theta_i)\}$ is sign-interchanging. Positive definiteness of B and (16) imply that $\{\widetilde{g}(\theta_i)\}$ must be sign-interchanging as well. Hence the zeros of $\widetilde{h}(\theta)$ and $\widetilde{g}(\theta)$ interlace. In particular, this means that the roots of $\widetilde{g}(\theta)$ are simple as well.

5. Finally, all the elements $\widetilde{h}'(\theta_i)\widetilde{g}(\theta_i)$ of the matrix on the right-hand side of (16) must be positive. Because 0 is always a root of $\widetilde{h}(\theta)$ according to (17), it follows that $\widetilde{h}'(0)$ and $\widetilde{g}(0)$ must have the same sign.

After the above discussion the sufficiency of 1), 2), 4) and 5) is obvious.

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