On characteristic polynomials, eigenvalues and eigenvectors of quasiseparable of order one matrices

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1 Introduction

In this paper we study spectral properties of quasiseparable of order one matrices. We derive various recursive relations for characteristic polynomials of the principal submatrices of such matrices and obtain a $O(N^2)$ algorithm for the computation of the coefficients of their characteristic polynomials. These recursions generalize some well-known results for the polynomials orthogonal on the real line and for the Szego polynomials. We obtain conditions in which an eigenvalue of the quasiseparable matrix is simple and derive the explicit formula for the corresponding eigenvector. Next we extend the method of computation of the eigenvectors on the case when these conditions are not valid. The concrete examples of Toeplitz, tridiagonal and unitary Hessenberg matrices are considered. Numerical methods based on the results of this paper will be suggested in another publication.

The paper contains four sections. Section 1 is the introduction. In Section 2 we define the class of quasiseparable of order one matrices and formulate the basic properties of this class obtained in [2]. In Section 3 we derive recursive relations for characteristic polynomials of principal submatrices of quasiseparable matrices. In Section 4 we obtain the conditions in which an eigenvalue of a quasiseparable matrix is simple and methods of computation of the eigenvectors when these conditions are valid and non-valid.

2 Definitions and basic properties of quasiseparable matrices

2.1 Definitions

Let $\{a_k\}, k = 1, \dots, N$ be a set of numbers. For positive integers i, j define the operation $a^{\times}_{ij}$ as follows: $a^{\times}_{ij} = a_{i-1} \cdots a_{j+1}$ for $N \geq i > j+1 \geq 2$, $a^{\times}_{ij} = a_{i+1} \cdots a_{j-1}$ for $N \geq j > i+1 \geq 2$, $a^{\times}_{k+1,k} = a^{\times}_{k,k+1} = 1$ for $1 \leq k \leq N-1$.

We consider a class of matrices R for which lower triangular and upper triangular parts have a special structure. Let R be a square matrix of size $N \times N$ with entries in the lower triangular part of the form

$$R_{ij} = p_i a^{\times}_{ij} q_j, \quad 1 \leq j < i \leq N,$$

where $p_i,\ q_j,\ a_k$ are given numbers. Then the matrix R is called *lower quasiseparable of order one* and the numbers $p_i\ (i = 2, \dots, N),\ q_j\ (j = 1, \dots, N-1),\ a_k\ (k = 2, \dots, N-1)$ are called *lower generators* of the matrix R.

Let R be a square matrix of size $N \times N$ with entries in the upper triangular part of the form

$$R_{ij} = g_i b^{\times}_{ij} h_j, \quad 1 \leq i < j \leq N,$$

where $g_i,\ h_j,\ b_k$ are given numbers. Then the matrix R is called *upper quasiseparable of order one* and the numbers $g_i\ (i = 1, \dots, N-1),\ h_j\ (j = 2, \dots, N),\ b_k\ (k = 2, \dots, N-1)$ are called *upper generators* of the matrix R.

If a matrix R of size $N \times N$ is lower quasiseparable of order one and upper quasiseparable of order one then it is called *quasiseparable of order one.* More precisely, quasiseparable of order one matrix is a matrix of the form

$$R_{ij} = \begin{cases} p_i a^{\times}_{ij} q_j, & 1 \leq j < i \leq N, \\ d_i, & 1 \leq i = j \leq N, \\ g_i b^{\times}_{ij} h_j, & 1 \leq i < j \leq N. \end{cases} \tag{2.1}$$

The elements $p_i\ (i = 2, \dots, N),\ q_j\ (j = 1, \dots, N-1),\ a_k\ (k = 2, \dots, N-1);\ g_i\ (i = 1, \dots, N-1),\ h_j\ (j = 2, \dots, N),\ b_k\ (k = 2, \dots, N-1);\ d_k\ (k = 1, \dots, N)$ are called *generators* of the matrix R.

The class of quasiseparable of order one matrices is a generalization of two well-known classes of structured matrices: tridiagonal matrices and diagonal plus semiseparable of order one matrices. If $a_k = b_k = 1,\ 2 \leq k \leq N-1$ then the matrix R is diagonal plus semiseparable of order one. If $a_k = b_k = 0,\ 2 \leq k \leq N-1$ then R is tridiagonal.

All statements below concern a quasiseparable of order one matrix R with generators denoted by $p_i\ (i = 2, \dots, N),\ q_j\ (j = 1, \dots, N-1),\ a_k\ (k = 2, \dots, N-1);\ g_i\ (i = 1, \dots, N-1),\ h_j\ (j = 2, \dots, N),\ b_k\ (k = 2, \dots, N-1);\ d_k\ (k = 1, \dots, N)$. This means that the entries of the matrix R have the form (2.1), where

$$a^{\times}_{ij} = a_{i-1} \cdots a_{j+1},\ N \geq i > j+1 \geq 2, \quad a^{\times}_{j+1,j} = 1,\ j = 1, \dots, N-1,$$

$$b_{ij}^{\times} = b_{i+1}\cdots b_{j-1},\ N \geq j > i+1 \geq 2, \quad b_{i,i+1}^{\times} = 1,\ i = 1,\ldots,N-1.$$

By the generators of the matrix R we define the vectors

$$P_k = \mathrm{col}(p_i a_{i,k-1}^{\times})_{i=k}^{N},\ H_k = \mathrm{row}(b_{k-1,i}^{\times} h_i)_{i=k}^{N}, \quad k = 2,\ldots,N; \tag{2.2}$$

$$Q_k = \mathrm{row}(a_{k+1,i}^{\times} q_i)_{i=1}^{k},\ G_k = \mathrm{col}(g_i b_{i,k+1}^{\times})_{i=1}^{k}, \quad k = 1,\ldots,N-1. \tag{2.3}$$

which are used frequently in the paper.

2.2 The basic properties of quasiseparable of order one matrices

We present here some results obtained in [2] for quasiseparable of order one matrices. These results are used essentially in this paper.

Theorem 2.1 *Let R be a quasiseparable of order one matrix R with generators p_i $(i = 2,\ldots,N)$, q_j $(j = 1,\ldots,N-1)$, a_k $(k = 2,\ldots,N-1)$; g_i $(i = 1,\ldots,N-1)$, h_j $(j = 2,\ldots,N)$, b_k $(k = 2,\ldots,N-1)$; d_k $(k = 1,\ldots,N)$. The matrix R admits the partitions*

$$R = \begin{pmatrix} A_k & G_k H_{k+1} \\ P_{k+1} Q_k & B_{k+1} \end{pmatrix}, \quad k = 1,\ldots,N-1, \tag{2.4}$$

where

$$A_k = R(1:k, 1:k),\ B_k = R(k:N, k:N), \quad k = 1,\ldots,N$$

and the elements P_k, Q_k, G_k, H_k are defined by (2.2), (2.3).

Next we consider some relations which follows directly from (2.4). Setting $k = N-1$ in (2.4) we obtain

$$R = \begin{pmatrix} A_{N-1} & G_{N-1} h_N \\ p_N Q_{N-1} & d_N \end{pmatrix}. \tag{2.5}$$

Applying this formula to the matrix A_k we get

$$A_k = \begin{pmatrix} A_{k-1} & G_{k-1} h_k \\ p_k Q_{k-1} & d_k \end{pmatrix}, \quad k = 1,\ldots,N-1. \tag{2.6}$$

Similarly setting in (2.4) $k = 1$ we obtain

$$R = \begin{pmatrix} d_1 & g_1 H_2 \\ P_2 q_1 & B_2 \end{pmatrix}. \tag{2.7}$$

Applying this formula to the matrix B_k we get

$$B_k = \begin{pmatrix} d_k & g_k H_{k+1} \\ P_{k+1} q_k & B_{k+1} \end{pmatrix}, \quad k = 1,\ldots,N-1. \tag{2.8}$$

Substituting (2.6) in (2.4) and using the relations

$$Q_k = \begin{pmatrix} a_kQ_{k-1} & q_k \end{pmatrix},\ G_k = \begin{pmatrix} G_{k-1}b_k \\ g_k \end{pmatrix}, \quad k = 2, \dots, N-1$$

we obtain the representations

$$R = \begin{pmatrix} A_{k-1} & G_{k-1}h_k & G_{k-1}b_kH_{k+1} \\ p_kQ_{k-1} & d_k & g_kH_{k+1} \\ P_{k+1}a_kQ_{k-1} & P_{k+1}q_k & B_{k+1} \end{pmatrix}, \quad k = 2, \dots, N-1. \tag{2.9}$$

Theorem 2.2 *Let $A_k = R(1:k, 1:k),\ B_k = R(k:N, k:N), \quad k = 1, \dots, N$ be principal submatrices of the matrix R and let*

$$\gamma_k = \det A_k, \quad \theta_k = \det B_k, \quad k = 1, \dots, N$$

and

$$f_k = Q_k(\mathrm{adj}A_k)G_k,\ k = 1, \dots, N-1; \quad z_k = H_k(\mathrm{adj}B_k)P_k,\ k = N, \dots, 2, \tag{2.10}$$

where the elements Q_k, G_k, P_k, H_k are defined by (2.2), (2.3).

Then the following recursive relations hold:

$$\begin{gathered} \gamma_1 = d_1, \quad f_1 = q_1g_1; \\ c_k = d_ka_kb_k - q_kp_kb_k - a_kh_kg_k, \\ f_k = c_kf_{k-1} + q_kg_k\gamma_{k-1}, \quad \gamma_k = d_k\gamma_{k-1} - p_kh_kf_{k-1}, \quad k = 2, \dots, N-1; \\ \gamma_N = d_N\gamma_{N-1} - p_Nh_Nf_{N-1} \end{gathered} \tag{2.11}$$

and

$$\begin{gathered} \theta_N = d_N, \quad z_N = h_Np_N, \\ z_k = c_kz_{k+1} + \theta_{k+1}h_kp_k, \quad \theta_k = d_k\theta_{k+1} - g_kq_kz_{k+1}, \quad k = N-1, \dots, 2; \\ \theta_1 = d_1\theta_2 - q_1g_1z_2. \end{gathered} \tag{2.12}$$

In the sequel we use also the vector $V_{N-1} = \mathrm{adj}A_{N-1}G_{N-1}$, where $A_{N-1} = R(1:N-1, 1:N-1)$ and G_{N-1} is defined in (2.3). This vector may be expressed explicitly via the numbers γ_k, f_k.

Theorem 2.3 *The vector $V_{N-1} = (\mathrm{adj}A_{N-1})G_{N-1}$ is given by the formula*

$$V_{N-1} = \mathrm{col}(v_i\delta^{\times}_{i,N})_{i=1}^{N-1}, \tag{2.13}$$

where

$$v_1 = g_1, \quad v_i = g_i\gamma_{i-1} - p_if_{i-1}b_i,\ \delta_i = d_ib_i - g_ih_i, \quad i = 2, \dots, N-1. \tag{2.14}$$

The proof may be found in the proof of Theorem 4.1 of [2].

3 The characteristic polynomial

In this section we derive recursive relations for for characteristic polynomials $\gamma_k(\lambda) = \det(R(1:k,1:k)-\lambda I)$ $\theta_k(\lambda) = \det(R(k:N,k:N)-\lambda I)$ $(k=1,\dots,N)$ of principal submatrices of a quasiseparable of order one matrix.

Theorem 3.1 *Let $\gamma_k(\lambda) = \det(R(1:k,1:k)-\lambda I)$ $\theta_k(\lambda) = \det(R(k:N,k:N)-\lambda I)$ $(k=1,\dots,N)$ be characteristic polynomials of the principal submatrices of the matrix R.*

Then the following recursive relations hold:

$$\gamma_1(\lambda) = d_1 - \lambda, \quad f_1(\lambda) = q_1 g_1; \tag{3.1}$$

$$\gamma_k(\lambda) = (d_k-\lambda)\gamma_{k-1}(\lambda) - p_k h_k f_{k-1}(\lambda), \tag{3.2}$$

$$c_k = d_k a_k b_k - q_k p_k b_k - a_k h_k g_k, \tag{3.3}$$

$$f_k(\lambda) = (c_k - a_k b_k \lambda) f_{k-1}(\lambda) + q_k g_k \gamma_{k-1}(\lambda), \quad k=2,\dots,N-1; \tag{3.4}$$

$$\gamma_N(\lambda) = (d_N-\lambda)\gamma_{N-1}(\lambda) - p_N h_N f_{N-1}(\lambda) \tag{3.5}$$

and

$$\theta_N(\lambda) = d_N - \lambda, \quad z_N(\lambda) = h_N p_N, \tag{3.6}$$

$$\theta_k(\lambda) = (d_k-\lambda)\theta_{k+1}(\lambda) - g_k q_k z_{k+1}(\lambda), \tag{3.7}$$

$$z_k(\lambda) = (c_k - a_k b_k \lambda) z_{k+1}(\lambda) + \theta_{k+1}(\lambda) h_k p_k, \quad k=N-1,\dots,2; \tag{3.8}$$

$$\theta_1(\lambda) = (d_1-\lambda)\theta_2(\lambda) - q_1 g_1 z_2(\lambda). \tag{3.9}$$

Here $f_k(\lambda), z_k(\lambda)$ are auxiliary polynomials. Moreover the coefficients of the polynomials

$$\gamma_k(\lambda) = \sum_{i=0}^{k} A_i^{(k)} \lambda^i, \quad f_k(\lambda) = \sum_{i=0}^{k-1} F_i^{(k)} \lambda^i$$

are determined as follows:

$$\begin{gathered}
A_0^{(1)} = d_1, \quad A_1^{(1)} = -1, \quad F_0^{(1)} = q_1 g_1; \\
F_0^{(k)} = c_k F_0^{(k-1)} + q_k g_k A_0^{(k-1)}, \quad A_0^{(k)} = d_k A_0^{(k-1)} - p_k h_k F_0^{(k-1)}, \\
F_i^{(k)} = c_k F_i^{(k-1)} - a_k b_k F_{i-1}^{(k-1)} + q_k g_k A_i^{(k-1)}, \\
A_i^{(k)} = d_k A_i^{(k-1)} - A_{i-1}^{(k-1)} - p_k h_k F_i^{(k-1)}, \quad i=1,\dots,k-2, \\
F_{k-1}^{(k)} = -a_k b_k F_{k-2}^{(k-1)} + q_k g_k A_{k-1}^{(k-1)}, \\
A_{k-1}^{(k)} = d_k A_{k-1}^{(k-1)} - A_{k-2}^{(k-1)}, \ A_k^{(k)} = -A_{k-1}^{(k-1)}, \quad k=2,\dots,N-1; \\
A_0^{(N)} = d_N A_0^{(N-1)} - p_N h_N F_0^{(N-1)}, \\
A_i^{(N)} = d_N A_i^{(N-1)} - A_{i-1}^{(N-1)} - p_N h_N F_i^{(N-1)}, \quad i=1,\dots,N-2, \\
A_{N-1}^{(N)} = d_N A_{N-1}^{(N-1)} - A_{N-2}^{(N-1)}, \ A_N^{(N)} = -A_{N-1}^{(N-1)}.
\end{gathered} \tag{3.10}$$

The formulas (3.10) yield a $O(N^2)$ algorithm for computation of the coefficients of the characteristic polynomial of the matrix R.

Proof. The matrix $R-\lambda I$ is quasiseparable of order one with the same generators $p_i, q_j, a_k; g_i, h_j, b_k$ as the matrix R and diagonal entries $d_k-\lambda$. Substituting in (2.11) $d_k-\lambda$ instead of d_k we obtain (3.1)-(3.5). Similarly (3.6)-(3.9) follow from (2.12). The formulas (3.10) follow directly from (3.1)-(3.5). □

In assumption that some generators of the matrix R are zeros or nonzeros the recursive relations (3.1)-(3.5) may be given in a more convenient for our next purposes form.

In the sequel to simplify the exposition we use the auxiliary linear in λ expressions

$$d_k(\lambda) = d_k - \lambda, \quad k = 1, \ldots, N;$$
$$c_k(\lambda) = (d_k - \lambda)a_k b_k - q_k p_k b_k - h_k g_k a_k,$$
$$l_k(\lambda) = (d_k - \lambda)a_k - q_k p_k, \quad \delta_k(\lambda) = (d_k - \lambda)b_k - h_k g_k, \quad k = 2, \ldots, N-1.$$

The expressions $l_k(\lambda), \delta_k(\lambda)$ play essential role in the main theorem below. One can check easily that

$$d_k(\lambda)c_k(\lambda) = l_k(\lambda)\delta_k(\lambda) - q_k p_k h_k g_k. \tag{3.11}$$

Theorem 3.2 *Let for some $k \in \{3, \ldots, N-1\}$ the inequality $p_{k-1}h_{k-1} \neq 0$ holds. Then the polynomials $\gamma_j(\lambda) = \det(R(1:j, 1:j) - \lambda I)$ $j = k-2, k-1, k$ satisfy the three-term recurrence relation:*

$$\gamma_k(\lambda) = \psi_k(\lambda)\gamma_{k-1}(\lambda) - \varphi_k(\lambda)\gamma_{k-2}(\lambda), \tag{3.12}$$

where

$$\psi_k(\lambda) = d_k - \lambda + \frac{p_k h_k}{p_{k-1}h_{k-1}} c_{k-1}(\lambda),$$

$$\varphi_k(\lambda) = \frac{p_k h_k}{p_{k-1}h_{k-1}} l_{k-1}(\lambda)\delta_{k-1}(\lambda). \tag{3.13}$$

Proof. Substituting $k-1$ instead of k in (3.4) and (3.2) we obtain

$$f_{k-1}(\lambda) = c_{k-1}(\lambda)f_{k-2}(\lambda) + q_{k-1}g_{k-1}\gamma_{k-2}(\lambda), \tag{3.14}$$

$$\gamma_{k-1}(\lambda) = d_{k-1}(\lambda)\gamma_{k-2}(\lambda) - p_{k-1}h_{k-1}f_{k-2}(\lambda). \tag{3.15}$$

The relation (3.15) implies

$$f_{k-2}(\lambda) = \frac{1}{p_{k-1}h_{k-1}}[d_{k-1}(\lambda)\gamma_{k-2}(\lambda) - \gamma_{k-1}(\lambda)]. \tag{3.16}$$

Substituting (3.16) in (3.14) and taking into account (3.11) we obtain

$$
\begin{aligned}
f_{k-1}(\lambda) &= \frac{c_{k-1}(\lambda)}{p_{k-1}h_{k-1}}[d_{k-1}(\lambda)\gamma_{k-2}(\lambda) - \gamma_{k-1}(\lambda)]q_{k-1}g_{k-1}\gamma_{k-2}(\lambda) \\
&= -\frac{c_{k-1}(\lambda)}{p_{k-1}h_{k-1}}\gamma_{k-1}(\lambda) + \left[\frac{c_{k-1}(\lambda)}{p_{k-1}h_{k-1}}d_{k-1}(\lambda) + g_{k-1}h_{k-1}\right]\gamma_{k-2}(\lambda) \\
&= -\frac{c_{k-1}(\lambda)}{p_{k-1}h_{k-1}}\gamma_{k-1}(\lambda) + \frac{1}{p_{k-1}h_{k-1}}l_{k-1}(\lambda)\delta_{k-1}(\lambda)\gamma_{k-2}(\lambda).
\end{aligned}
$$

Substitution of the last expression in (3.2) yields (3.12). □

Corollary 3.3 *Let* $p_i h_i \neq 0$ $(i = 2, \ldots, N-1)$*. Then the polynomials* $\gamma_k(\lambda) = \det(R(1 : k, 1 : k) - \lambda I)$ $(k = 1, \ldots, N)$ *satisfy the three-term recurrence relations:*

$$\gamma_1(\lambda) = d_1 - \lambda, \quad \gamma_2(\lambda) = (d_2 - \lambda)(d_1 - \lambda) - p_2 q_1 g_1 h_2; \tag{3.17}$$

$$\gamma_k(\lambda) = \psi_k(\lambda)\gamma_{k-1}(\lambda) - \varphi_k(\lambda)\gamma_{k-2}(\lambda), \quad k = 3, \ldots, N, \tag{3.18}$$

where

$$\psi_k(\lambda) = d_k - \lambda + \frac{p_k h_k}{p_{k-1}h_{k-1}}c_{k-1}(\lambda), \quad \varphi_k(\lambda) = \frac{p_k h_k}{p_{k-1}h_{k-1}}l_{k-1}(\lambda)\delta_{k-1}(\lambda).$$

Proof. The relations (3.17) directly follow from (3.1) and from (3.2) with $k = 2$. For $k = 3, \ldots, N$ the relations (3.18) follow from Theorem 3.2. □

This corollary generalizes the well-known results for tridiagonal matrices and the corresponding result of the paper [3] obtained for symmetric semiseparable matrices represented as a sum of a diagonal matrix and a matrix with a tridiagonal inverse.

Next we combine the cases $p_k h_k = 0$ and $p_k h_k \neq 0$. We use here the following auxiliary relations.

Lemma 3.4 *For any* $2 \leq s \leq k \leq N-1$ *the polynomials* $f_k(\lambda)$ *defined in (3.1)-(3.4) satisfy the relations*

$$f_k(\lambda) = (c_{k+1,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s}^{k}(c_{k+1,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda). \tag{3.19}$$

Proof. The proof is by induction on k. For $k = s$ by virtue of (3.4) we have

$$f_k(\lambda) = f_s(\lambda) = c_s(\lambda)f_{s-1}(\lambda) + q_s g_s \gamma_{s-1}(\lambda) = (c_{s+1,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + (c_{s+1,j}(\lambda))^{\times} q_s g_s \gamma_{s-1}(\lambda)$$

which implies (3.19). Assume that (3.19) is valid for some $k \geq s$. By virtue of (3.4) and the equality

$$c_{k+1}(\lambda)(c_{k+1,s-1}(\lambda))^{\times} = (c_{k+2,s-1}(\lambda))^{\times}$$

we have

$$
\begin{aligned}
f_{k+1}(\lambda) &= c_{k+1}(\lambda) f_k(\lambda) + q_{k+1} g_{k+1}(\lambda)\gamma_k(\lambda) \\
&= c_{k+1}(\lambda)[(c_{k+1,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s}^{k}(c_{k+1,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda)] + q_{k+1} g_{k+1}\gamma_k(\lambda) \\
&= (c_{k+2,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s}^{k}(c_{k+2,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda) + (c_{k+2,k+1}(\lambda))^{\times} q_{k+1} g_{k+1}\gamma_k(\lambda) \\
&= (c_{k+2,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s}^{k+1}(c_{k+2,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda)
\end{aligned}
$$

which completes the proof of the lemma. □

Theorem 3.5 *Assume that for some* $2 \leq s < k-1 \leq N-1$, $p_s h_s \neq 0$, $p_{s+1}h_{s+1} = \cdots = p_{k-1}h_{k-1} = 0$. *Then*

$$\gamma_k(\lambda) = \psi_{k,s}(\lambda)\gamma_s(\lambda) - \varphi_{k,s}(\lambda)\gamma_{s-1}(\lambda), \tag{3.20}$$

where

$$\psi_{k,s}(\lambda) = (d_{k+1,s}(\lambda))^{\times} + \frac{p_k h_k}{p_s h_s}(c_{k,s-1}(\lambda))^{\times} - p_k h_k \sum_{j=s+1}^{k-1}(c_{k,j}(\lambda))^{\times} q_j g_j (d_{j,s}(\lambda))^{\times},$$

$$\varphi_{k,s}(\lambda) = \frac{p_k h_k}{p_s h_s}(c_{k,s}(\lambda))^{\times} l_s(\lambda)\delta_s(\lambda).$$

Proof. By virtue of Lemma 3.4 we have

$$f_k(\lambda) = (c_{k,s-1}(\lambda))^{\times} f_{s-1}(\lambda) + \sum_{j=s+1}^{k-1}(c_{k,j}(\lambda))^{\times} q_j g_j \gamma_{j-1}(\lambda) + (c_{k,s}(\lambda))^{\times} q_s g_s \gamma_{s-1}(\lambda). \tag{3.21}$$

As a direct consequence of (3.2) we get that if for some $1 \leq i < j \leq N$, $p_{i+1}h_{i+1} = \cdots = p_j h_j = 0$ then

$$\gamma_j(\lambda) = (d_j - \lambda)\cdots(d_{i+1} - \lambda)\gamma_i(\lambda). \tag{3.22}$$

Hence it follows that

$$\gamma_{j-1}(\lambda) = \gamma_s(\lambda)(d_{j,s}(\lambda))^{\times}, \quad j = s+1, \ldots, k. \tag{3.23}$$

Next by virtue of (3.2) we have

$$f_{s-1}(\lambda) = \frac{1}{p_s h_s}[(d_s - \lambda)\gamma_{s-1}(\lambda) - \gamma_s(\lambda)]. \tag{3.24}$$

Substituting (3.24) and (3.23) in (3.21) we obtain

$$f_{k-1}(\lambda) = \frac{(c_{k,s-1}(\lambda))^{\times}}{p_s h_s}(d_s - \lambda)\gamma_{s-1}(\lambda) - \frac{(c_{k,s-1}(\lambda))^{\times}}{p_s h_s}\gamma_s(\lambda)$$
$$+ \sum_{j=s+1}^{k-1}(c_{k,j}(\lambda))^{\times} q_j g_j (d_{j,s}(\lambda))^{\times}\gamma_s(\lambda) + (c_{k,s}(\lambda))^{\times} q_s g_s \gamma_{s-1}(\lambda).$$

From here using the equalities

$$(c_{k,s-1}(\lambda))^{\times} = (c_{k,s}(\lambda))^{\times} c_s(\lambda), \quad \frac{c_s(\lambda)}{p_s h_s}(d_s - \lambda) + q_s g_s = \frac{l_s(\lambda)\delta_s(\lambda)}{p_s h_s}$$

we obtain

$$f_{k-1}(\lambda) = \Big[-\frac{(c_{k,s-1}(\lambda))^{\times}}{p_s h_s} + \sum_{j=s+1}^{k-1}(c_{k,j}(\lambda))^{\times} q_j g_j (d_{j,s}(\lambda))^{\times}\Big]\gamma_s(\lambda) + \frac{(c_{k,s}(\lambda))^{\times}}{p_s h_s} l_s(\lambda)\delta_s(\lambda)\gamma_{s-1}(\lambda). \tag{3.25}$$

Next using (3.23) we get

$$(d_k - \lambda)\gamma_{k-1}(\lambda) = (d_k - \lambda)(d_{k,s}(\lambda))^{\times}\gamma_s(\lambda) = (d_{k+1,s}(\lambda))^{\times}\gamma_s(\lambda). \tag{3.26}$$

Substituting (3.25) and (3.26) in (3.2) we obtain (3.20). □

4 Eigenvectors and multiplicities

In this section we study eigenvalues and eigenvectors of quasiseparable of order one matrices.

4.1 The simple eigenvalues

The next theorem is one of the main results of this paper.

Theorem 4.1 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$. Assume that $\lambda = \lambda_0$ is an eigenvalue of the matrix R and*

$$|d_1 - \lambda_0| + |q_1| > 0, \quad |d_1 - \lambda_0| + |g_1| > 0; \tag{4.1}$$

$$l_k(\lambda_0) = (d_k - \lambda_0)a_k - p_k q_k \neq 0, \quad \delta_k(\lambda_0) = (d_k - \lambda_0)b_k - g_k h_k \neq 0, \quad k = 2, \dots, N-1; \tag{4.2}$$

$$|d_N - \lambda_0| + |p_N| > 0, \quad |d_N - \lambda_0| + |h_N| > 0. \tag{4.3}$$

Then the eigenvalue λ_0 has the geometric multiplicity one. Moreover the corresponding to λ_0 eigenvector v may be determined as follows:

1)if $p_2h_2 = \cdots = p_Nh_N = 0$ then

$$v = \begin{pmatrix} 1 \\ B_2(\lambda_0)^{-1}y_2 \end{pmatrix}, \tag{4.4}$$

where $B_2(\lambda_0) = R(2:N, 2:N) - \lambda_0 I,\ y_2 = -R(1, 2:N)$ and $\theta_2(\lambda_0) = \det B_2(\lambda_0) \neq 0$;

2)if for some $m,\ 2 \leq m \leq N-1$ we have $p_mh_m \neq 0$ and $p_{m+1}h_{m+1} = \cdots = p_Nh_N = 0$ then

$$v = \begin{pmatrix} A_{m-1}(\lambda_0)^{-1}x_m \\ 1 \\ B_{m+1}(\lambda_0)^{-1}y_m \end{pmatrix}, \tag{4.5}$$

where $A_{m-1}(\lambda_0) = R(1:m-1, 1:m-1) - \lambda_0 I,\ x_m = -R(1:m-1, m),\ B_{m+1}(\lambda_0) = R(m+1:N, m+1:N) - \lambda_0 I,\ y_m = P_{m+1}(a_m \frac{f_{m-1}(\lambda_0)}{\gamma_{m-1}(\lambda_0)} h_m - q_m),\ P_{m+1} = \mathrm{col}(p_k a^{\times}_{km})_{k=m+1}^{N}$ and

$$\gamma_{m-1}(\lambda_0) = \det A_{m-1}(\lambda_0) \neq 0, \quad \theta_{m+1} = \det B_{m+1}(\lambda_0) \neq 0;$$

3)if $p_Nh_N \neq 0$ then

$$v = \begin{pmatrix} A_{N-1}(\lambda_0)^{-1}x_N \\ 1 \end{pmatrix}, \tag{4.6}$$

where $A_{N-1}(\lambda_0) = R(1:N-1, 1:N-1) - \lambda_0 I,\ x_N = -R(1:N-1, N)$ and $\gamma_{N-1}(\lambda_0) = \det A_{N-1}(\lambda_0) \neq 0$. Furthermore in this case the eigenvector may be also determined by the values of the polynomials $\gamma_k(\lambda),\ f_k(\lambda)$ at the point λ_0 as follows:

$$v = \begin{pmatrix} -V_{N-1}(\lambda_0)h_N \\ \gamma_{N-1}(\lambda_0) \end{pmatrix}, \tag{4.7}$$

where $V_{N-1}(\lambda_0) = \mathrm{col}(v_i(\lambda_0)\delta^{\times}_{iN}(\lambda_0))_{i=1}^{N-1}$,

$$v_1(\lambda_0) = g_1, \quad v_i(\lambda_0) = g_i\gamma_{i-1}(\lambda_0) - p_i f_{i-1}(\lambda_0) b_i,\ i = 1, \ldots, N-1.$$

Proof. We start proving the part 1). At first let us show that $\theta_2(\lambda_0) \neq 0$. By virtue of (3.6) we have $z_N(\lambda_0) = 0$ and next by virtue of (3.8) we obtain $z_{N-1}(\lambda_0) = \cdots = z_2(\lambda_0) = 0$ and using (3.6)and (3.7) we obtain

$$\theta_2(\lambda_0) = (d_N - \lambda_0)\ldots(d_2 - \lambda_0).$$

By virtue of conditions (4.2), (4.3) and equalities $p_2h_2 = \cdots = p_Nh_N = 0$ we conclude that $d_j - \lambda_0 \neq 0,\ j = 2, \ldots, N$ and hence $\theta_2(\lambda_0) \neq 0$.

Consider the partition of the matrix $R - \lambda_0 I$ in the form

$$R - \lambda_0 I = \begin{pmatrix} d_1 - \lambda_0 I & r_2' \\ r_2 & B_2(\lambda_0) \end{pmatrix}, \tag{4.8}$$

where $B_2(\lambda_0) = R(2:N, 2:N) - \lambda_0 I$, $r_2 = R(2:N, 1)$, $r_2' = R(1, 2:N)$. Let v be a corresponding to λ_0 eigenvector. We represent this vector in the form $v = \begin{pmatrix} \alpha \\ v' \end{pmatrix}$ where v' is a $N-1$-dimensional vector and α is a scalar. From the equality $(R - \lambda_0)v = 0$ using the representation (4.8) we obtain $B_2(\lambda_0)v' + \alpha r_2 = 0$. This means $v' = -B_2^{-1}(\lambda_0)\alpha r_2$, i.e. $v = \alpha v_0$, where $v_0 = \alpha \begin{pmatrix} 1 \\ B_2^{-1}(\lambda_0)y_2 \end{pmatrix}$. This implies that the eigenvalue λ_0 has the geometric multiplicity one. Taking $\alpha = 1$ we obtain (4.4).

2)In the same way as in the proof of the part 1) we obtain

$$z_N(\lambda_0) = z_{N-1}(\lambda_0) = \cdots = z_m(\lambda_0) = 0 \tag{4.9}$$

and $\theta_m(\lambda_0) \neq 0$.

By virtue of (3.22) and the fact that λ_0 is an eigenvalue of the matrix R we have

$$(d_N - \lambda_0)\cdots(d_{m+1} - \lambda_0)\gamma_m(\lambda_0) = \gamma_N(\lambda_0) = 0. \tag{4.10}$$

The equalities $p_{m+1}h_{m+1} = \cdots = p_N h_N$ and the conditions (4.2), (4.3) imply $d_j - \lambda_0 \neq 0$, $j = m+1, \ldots, N$ and therefore from (4.10) we conclude that $\gamma_m(\lambda_0) = 0$. Let us prove that $\gamma_{m-1}(\lambda_0) \neq 0$. Assume that it is not true. We will show that in this case the equalities

$$\gamma_1(\lambda_0) = \gamma_2(\lambda_0) = \cdots = \gamma_{m-1}(\lambda_0) = \gamma_m(\lambda_0) = 0 \tag{4.11}$$

hold. We have the following cases:

(a)$m = 2$. By the assumption we have $\gamma_1(\lambda_0) = \gamma_2(\lambda_0) = 0$ and hence (4.11) follows.

(b)$p_2h_2 = \cdots = p_{m-1}h_{m-1} = 0$. By virtue of (3.22) we have

$$\gamma_{j+1}(\lambda) = (d_{j+1} - \lambda)\gamma_j(\lambda), \quad j = m-2, \ldots, 1$$

and since by virtue of the conditions (4.2) the inequalities $d_{j+1} - \lambda_0 \neq 0$, $j = m-2, \ldots, 1$ hold, we obtain (4.11).

(c)$p_{m-1}h_{m-1} \neq 0$. Applying Theorem 3.2 we obtain

$$\gamma_m(\lambda) = \psi_m(\lambda)\gamma_{m-1}(\lambda) - \varphi_m(\lambda)\gamma_{m-2}(\lambda), \tag{4.12}$$

where

$$\varphi_m(\lambda) = \frac{p_m h_m}{p_{m-1}h_{m-1}} l_{m-1}(\lambda)\delta_{m-1}(\lambda).$$

Here $p_m h_m = 0$ by the condition, $l_{m-1}(\lambda_0)\delta_{m-1}(\lambda_0) \neq 0$ by virtue of (4.11) and hence $\varphi_m(\lambda_0) \neq 0$. Since $\gamma_m(\lambda_0) = \gamma_{m-1}(\lambda_0) = 0$ the equality (4.12) implies $\gamma_{m-2}(\lambda_0) = 0$.

(d)$p_s h_s \neq 0$, $p_{s+1}h_{s+1} = \cdots = p_{m-1}h_{m-1}$ for some s such that $2 \leq s < m-1$. By virtue of (3.22) we have

$$\gamma_{j+1}(\lambda) = (d_{j+1} - \lambda)\gamma_j(\lambda), \quad j = s, \ldots, m-2$$

and since by virtue of the conditions (4.2), the inequalities $d_{j+1} - \lambda_0 \neq 0$, $j = m-2, \ldots, 1$ hold, we obtain

$$\gamma_s(\lambda_0) = \gamma_{s+1}(\lambda_0) = \cdots = \gamma_{m-1}(\lambda_0) = 0.$$

Next apply Theorem 3.5. We have

$$\gamma_m(\lambda) = \psi_{m,s}(\lambda)\gamma_s(\lambda) - \varphi_{m,s}(\lambda)\gamma_{s-1}(\lambda), \tag{4.13}$$

where

$$\varphi_{m,s}(\lambda) = \frac{p_m h_m}{p_s h_s} c_{m-1}(\lambda)\cdots c_{s+1}(\lambda) l_s(\lambda)\delta_s(\lambda).$$

Here $p_m h_m \neq 0$ by the condition, $l_s(\lambda_0)\delta_s(\lambda_0) \neq 0$ by virtue of (4.2) and $c_j(\lambda_0) \neq 0$ by virtue of (4.2) and the equality (3.11). Thus $\varphi_{m,s}(\lambda_0) \neq 0$ and since $\gamma_m(\lambda_0) = 0,\ \gamma_s(\lambda_0) = 0$ from (4.13) we conclude that $\gamma_{s-1}(\lambda_0) = 0$.

In the cases (a) and (b) the proof of (4.11) is finished. In the case (c) we apply the method described above to the polynomials $\gamma_{m-1}(\lambda), \gamma_{m-2}(\lambda)$ and in the case (d) we apply this method to the polynomials $\gamma_s(\lambda), \gamma_{s-1}(\lambda)$. We go on in this way and finally obtain (4.11).

Thus we have the equalities (4.11). Hence in particular follows that $\gamma_1(\lambda_0) = d_1 - \lambda_0 = 0$ and by virtue of the condition (4.1) and the second equality in (3.1) we have $f_1(\lambda_0) = q_1 g_1 \neq 0$. Let t be the minimal index such that $p_t h_t \neq 0$. It is clear that $t \leq k$. Using (3.2) and (4.11) we obtain

$$\gamma_t(\lambda_0) = (d_t - \lambda_0)\gamma_{t-1}(\lambda_0) - p_t h_t f_{t-1}(\lambda_0) = -p_t h_t f_{t-1}(\lambda_0). \tag{4.14}$$

By virtue of Lemma 3.4 and (4.11) we have

$$f_{t-1}(\lambda_0) = (c_{t,1}(\lambda_0))^{\times} f_1(\lambda_0) + \sum_{j=2}^{t-1} (c_{t,1}(\lambda_0))^{\times} q_j g_j \gamma_{j-1}(\lambda_0) = c_{t-1}(\lambda_0)\cdots c_2(\lambda_0) f_1(\lambda_0).$$

From the conditions (4.2) and the equalities (3.11) we obtain $c_j(\lambda_0) \neq 0,\ j = 2, \ldots, t-1$ and hence $f_{t-1}(\lambda_0) \neq 0$. But from (4.14) we conclude that $\gamma_t(\lambda_0) \neq 0$ which is a contradiction. Thus we have proved that $\gamma_{m-1}(\lambda_0) \neq 0$.

Applying the partition (2.9) with $k = m$ to the matrix $R - \lambda_0 I$ we get

$$R - \lambda_0 I = \begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1}h_m & G_{m-1}b_m H_{m+1} \\ p_m Q_{m-1} & d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1}a_m Q_{m-1} & P_{m+1}q_m & B_{m+1}(\lambda_0) \end{pmatrix}, \tag{4.15}$$

where $A_{m-1}(\lambda_0) = A_{m-1} - \lambda_0 I,\ B_{m+1}(\lambda_0) = B_{m+1} - \lambda_0 I$ and the elements Q_m, G_m, P_m, H_m are defined by (2.2), (2.3). Moreover from (2.10) we obtain

$$Q_{m-1}A_{m-1}^{-1}(\lambda_0)G_{m-1} = \frac{f_{m-1}(\lambda_0)}{\gamma_{m-1}(\lambda_0)}, \quad H_{m+1}B_{m+1}^{-1}(\lambda_0)P_{m+1} = \frac{z_{m+1}(\lambda_0)}{\theta_{m+1}(\lambda_0)}. \tag{4.16}$$

Let v be a corresponding to λ_0 eigenvector. We represent this vector in the form $v = \begin{pmatrix} v' \\ \alpha \\ v'' \end{pmatrix}$, where v', v'' are $m-1$ and $N-m$-dimensional vectors and α is a scalar. From the equality

$(R-\lambda_0 I)v=0$ using the representation (4.15) we obtain

$$\begin{aligned} A_{m-1}(\lambda_0)v'+G_{m-1}(h_m\alpha+b_mH_{m+1}v'')=0,\\ P_{m+1}(a_mQ_{m-1}v'+q_m\alpha)+B_{m+1}(\lambda_0)v''=0.\end{aligned} \tag{4.17}$$

By virtue of invertibility of the matrices $A_{m-1}(\lambda_0), B_{m+1}(\lambda_0)$ we may rewrite (4.17) in the form

$$v'=-A^{-1}_{m-1}(\lambda_0)G_{m-1}(h_m\alpha+b_mH_{m+1}v''), \tag{4.18}$$

$$v''=-B^{-1}_{m+1}(\lambda_0)P_{m+1}(a_mQ_{m-1}v'+q_m\alpha). \tag{4.19}$$

Multiplying (4.19) by H_{m+1} and using the second equality from (4.16) and the equality $z_m=0$ from (4.9) we obtain $H_{m+1}v''=0$. Substituting this in (4.18) we obtain

$$v'=-A^{-1}_{m-1}(\lambda_0)G_{m-1}h_m\alpha=A^{-1}_{m-1}(\lambda_0)x_m\alpha. \tag{4.20}$$

Substituting (4.20) in (4.19) we obtain

$$v''=B^{-1}_{m+1}(\lambda_0)P_{m+1}(a_mQ_{m-1}A^{-1}_{m-1}(\lambda_0)G_{m-1}h_m-q_m)\alpha$$

and using the first equality from (4.16) we conclude that

$$v''=B^{-1}_{m+1}(\lambda_0)P_{m+1}(a_m\frac{f_{m-1}(\lambda_0)}{\gamma_{m-1}(\lambda_0)}h_m-q_m)\alpha=B^{-1}_{m+1}(\lambda_0)y_m\alpha. \tag{4.21}$$

Thus from (4.20), (4.21) we conclude that $v=\alpha v_0$, where $v_0=\alpha\begin{pmatrix} A^{-1}_{m-1}(\lambda_0)x_m\\ 1\\ B^{-1}_{m+1}(\lambda_0)y_m\end{pmatrix}$. This implies that the eigenvalue λ_0 has the geometric multiplicity one. Taking $\alpha=1$ we obtain (4.5).

3)In the same way as in the proof of part 2) we show that $\gamma_{N-1}(\lambda_0)=0$. Next for the matrix $R-\lambda_0 I$ consider the partition

$$R-\lambda_0 I=\begin{pmatrix} A_{N-1}(\lambda_0) & r_{N-1}\\ r'_{N-1} & d_N-\lambda_0\end{pmatrix}, \tag{4.22}$$

where $A_{N-1}(\lambda_0)=R(1:N-1,1:N-1)-\lambda_0 I$, $r_{N-1}=R(1:N-1,N)$, $r'_{N-1}=R(N,1:N-1)$. Let v be an eigenvector corresponding to the eigenvalue λ_0. We represent this vector in the form $v=\begin{pmatrix} v'\\ \alpha\end{pmatrix}$, where v' is a $N-1$-dimensional vector α is a scalar. Using (4.22) we obtain $v'=-\alpha A^{-1}_{N-1}(\lambda_0)r_{N-1}$ which implies $v=\alpha v_0$, where $v_0=\alpha\begin{pmatrix} -A^{-1}_{N-1}(\lambda_0)r_{N-1}\\ 1\end{pmatrix}$. This implies that the eigenvalue λ_0 has the geometric multiplicity one. Taking $\alpha=1$ we obtain (4.6). Next by virtue of (2.5) we have $x_N = -R(1:N-1, 1:N-1) = -G_{N-1}h_N$ and moreover using the formula

$$A_{N-1}(\lambda_0)^{-1} = \frac{1}{\gamma_{N-1}(\lambda_0)}(\mathrm{adj}A_{N-1}(\lambda_0))$$

we obtain

$$v = \frac{1}{\gamma_{N-1}(\lambda_0)}\begin{pmatrix} -V_{N-1}(\lambda_0)h_N \\ \gamma_{N-1}(\lambda_0) \end{pmatrix},$$

where $V_{N-1}(\lambda_0) = (\mathrm{adj}A_{N-1}(\lambda_0))G_{N-1}$. Applying Theorem 2.3 to the matrix $R - \lambda_0 I$ and reducing the factor $\frac{1}{\gamma_{N-1}(\lambda_0)}$ we obtain the representation (4.7) for the eigenvector. $\square$

Remark. Using a fast algorithm suggested in [2] one can compute the eigenvector given by the formulas (4.4)- (4.6) in $O(N)$ arithmetic operations.

Examples. We consider some cases when the validity of the conditions of Theorem 4.1 is guaranteed for any eigenvalue.

1)The Hermitian quasiseparable of order one matrix R with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; $\overline{q_i}$ $(i = 1, \dots, N-1)$, $\overline{p_j}$ $(j = 2, \dots, N)$, a_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$ satisfying the conditions

$$\overline{p_k q_k} \neq p_k q_k, \ \overline{a_k} = a_k, \quad k = 2, \dots, N-1$$

. In this case we have

$$l_k(\lambda) = (d_k - \lambda)a_k - p_k q_k, \ \delta_k(\lambda) = (d_k - \lambda)a_k - \overline{p_k q_k}, \quad k = 2, \dots, N-1.$$

If λ is an eigenvalue of R then the number $(\lambda - d_k)a_k$ is real and the numbers $p_k q_k, \overline{p_k q_k}$ are pure imaginary. Hence it follows that the condition (4.2) holds for any eigenvalue of R.

Assume that $q_1 \neq 0$, $p_N \neq 0$. Then all the conditions of Theorem 4.1 hold and since for a Hermitian matrix the geometric multiplicity of any eigenvalue is equal to its algebraic multiplicity we conclude that all eigenvalues of the matrix R are simple.

2)The irreducible tridiagonal matrix, i.e. the tridiagonal matrix with non-zero entries on the lower and upper subdiagonals. In this case we have $a_k = b_k = 0$, $R_{k+1,k} = p_{k+1}q_k \neq 0$, $R_{k,k+1} = g_k h_{k+1} \neq 0$ $(k = 1, \dots, N-1)$ and therefore

$$l_k(\lambda) = -p_k q_k \neq 0, \ \delta_k(\lambda) = -g_k h_k \neq 0, \quad k = 2, \dots, N-1.$$

and hence the conditions of Theorem 4.1 hold for any λ.

Assume additionally that the conditions

$$\overline{d_i} = d_i, \quad i = 1, \dots, N; \quad \frac{p_{i+1}q_i}{g_i h_{i+1}} > 0, \quad i = 1, \dots, N-1 \tag{4.23}$$

are valid. Let $D = \mathrm{diag}\,(\rho_i)_{i=1}^N$ be a nonsingular diagonal matrix with the entries

$$\rho_1 = 1, \quad \rho_k = \sqrt{\frac{p_k q_{k-1}}{g_{k-1}h_k}}\rho_{k-1}, \ k = 2, \dots, N \tag{4.24}$$

and let $Q = D^{-1}RD$. The matrix Q is a tridiagonal Hermitian matrix. Indeed the entries of the matrix Q have the form

$$Q_{ij} = \begin{cases} \rho_{i+1}^{-1} p_{i+1} q_i \rho_i, & i = j+1, j = 1, \dots, N-1 \\ d_i, & 1 \le i = j \le N, \\ \rho_i^{-1} g_i h_{i+1} \rho_{i+1}, & j = i+1, i = 1, \dots, N-1 \\ 0, & |i-j| > 1. \end{cases}$$

The relations (4.24) imply

$$p_{i+1} q_i \rho_i^2 = \overline{g_i h_{i+1}} \rho_{i+1}^2, \quad i = 1, \dots, N-1 \tag{4.25}$$

which means

$$\rho_{i+1}^{-1} p_{i+1} q_i \rho_i = \overline{\rho_i^{-1} g_i h_{i+1} \rho_{i+1}}, \quad i = 1, \dots, N-1.$$

Thus from here and from Theorem 4.1 we obtain the known result that in conditions (4.23) the tridiagonal matrix R is diagonalizable and its eigenvalues are real and simple.

Now we apply Corollary 3.3 to the case of a tridiagonal matrix. We have

$$c_k(\lambda) = 0, \quad l_k(\lambda) = -p_k q_k, \; \delta_k(\lambda) = -g_k h_k,$$

and hence

$$\psi_k(\lambda) = d_k - \lambda, \quad \varphi_k(\lambda) = \frac{p_k h_k}{p_{k-1} h_{k-1}} p_{k-1} q_{k-1} g_{k-1} h_{k-1} = p_k q_{k-1} g_{k-1} h_k.$$

Hence the recursions (3.17), (3.18) one can write down in the form

$$\gamma_0(\lambda) = 1, \quad \gamma_1(\lambda) = d_1 - \lambda;$$

$$\gamma_k(\lambda) = (d_k - \lambda)\gamma_{k-1}(\lambda) - (p_k q_{k-1})(g_{k-1} h_k)\gamma_{k-2}(\lambda), \quad k = 2, \dots, N.$$

Note that it is well known (see for instance [1, p. 121]) that in the case

$$d_k = \overline{d_k}, \; k = 1, \dots, N; \quad (p_k q_{k-1})(g_{k-1} h_k) > 0, \quad k = 2, \dots, N-1$$

these recursions defines polynomials orthogonal with some weight on the real line.

3)The invertible matrix with generators satisfying the conditions

$$a_k \ne 0, b_k \ne 0, \; d_k a_k - p_k q_k = 0, \; d_k b_k - g_k h_k = 0, \quad k = 2, \dots, N-1. \tag{4.26}$$

In this case we have

$$l_k(\lambda) = -\lambda a_k, \; \delta_k(\lambda) = -\lambda b_k, \quad k = 2, \dots, N-1$$

and since by the assumption the zero is not the eigenvalue of R the condition (4.2) holds for any eigenvalue of the matrix. Suppose also that $p_{k+1} q_k \ne 0, \; g_k h_{k+1} \ne 0$ $(k = 1, \dots, N-1)$. Then all the conditions of Theorem 4.1 hold for any eigenvalue of the matrix R.

Now as in the previous example assume that the conditions (4.23) are valid and define the nonsingular diagonal matrix $D = \mathrm{diag}\,(\rho_i)_{i=1}^N$ via the relations (4.24). Let us show that in this case as well as in the previous example the matrix $Q = D^{-1}RD$ is a Hermitian matrix. Indeed the entries of the matrix Q have the form

$$Q_{ij} = \begin{cases} \rho_i^{-1} p_i a_{ij}^{\times} q_j \rho_j, & 1 \le j < i \le N, \\ d_i, & 1 \le i = j \le N, \\ \rho_i^{-1} g_i b_{ij}^{\times} h_j \rho_j, & 1 \le i < j \le N. \end{cases} \tag{4.27}$$

One should check that

$$p_i a_{ij}^{\times} q_j \rho_j^2 = \overline{g_j} \overline{b_{ji}^{\times}} \overline{h_j} \rho_i^2, \quad 1 \le j < i \le N. \tag{4.28}$$

From the relations (4.25) it follows that

$$\frac{\rho_i^2}{\rho_j^2} = \frac{\rho_i^2}{\rho_{i-1}^2} \cdots \frac{\rho_{j+1}^2}{\rho_j^2} = \frac{p_i q_{i-1}}{\overline{g_{i-1} h_i}} \frac{p_{i-1} q_{i-2}}{\overline{g_{i-2} h_{i-1}}} \cdots \frac{p_{j+1} q_j}{\overline{g_j h_{j+1}}}$$

which means

$$\frac{\rho_i^2}{\rho_j^2} = \frac{p_i}{\overline{h_i}} \frac{q_j}{\overline{g_j}} \frac{p_{ij}^{\times} q_{ij}^{\times}}{\overline{g_{ij}^{\times} h_{ij}^{\times}}}, \quad 1 \le j < i \le N. \tag{4.29}$$

By virtue of (4.26) and the assumption that the numbers d_k are real we have

$$d_k = \frac{p_k q_k}{a_k} = \frac{g_k h_k}{b_k} = \frac{\overline{g_k h_k}}{\overline{b_k}}, \quad k = 2, \ldots, N-1. \tag{4.30}$$

Hence it follows that

$$\frac{p_{ij}^{\times} q_{ij}^{\times}}{\overline{g_{ij}^{\times} h_{ij}^{\times}}} = \frac{a_{ij}^{\times}}{\overline{b_{ij}^{\times}}}, \quad 1 \le j < i \le N. \tag{4.31}$$

Comparing (4.29) and (4.31) we obtain

$$\frac{\rho_i^2}{\rho_j^2} = \frac{p_i q_j}{\overline{g_j} \overline{h_i}} \frac{a_{ij}^{\times}}{\overline{b_{ij}^{\times}}}, \quad 1 \le j < i \le N$$

which is equivalent to (4.28). Thus we conclude that in the case under consideration the matrix R is similar to a Hermitian matrix and hence R is diagonalizable and its eigenvalues are real. Moreover by Theorem 4.1 the eigenvalues of R are simple.

4)The unitary Hessenberg matrix

$$R_Q = \begin{bmatrix} -\rho_1 \rho_0^* & -\rho_2 \mu_1 \rho_0^* & -\rho_3 \mu_2 \mu_1 \rho_0^* & \cdots & -\rho_{N-1} \mu_{N-2} \cdots \mu_1 \rho_0^* & -\rho_N \mu_{N-1} \cdots \mu_1 \rho_0^* \\ \mu_1 & -\rho_2 \rho_1^* & -\rho_3 \mu_2 \rho_1^* & \cdots & -\rho_{N-1} \mu_{N-2} \cdots \mu_2 \rho_1^* & -\rho_N \mu_{N-1} \cdots \mu_2 \rho_1^* \\ 0 & \mu_2 & -\rho_3 \rho_2^* & \cdots & -\rho_{N-1} \mu_{N-2} \cdots \mu_3 \rho_2^* & -\rho_N \mu_{N-1} \cdots \mu_3 \rho_2^* \\ \vdots & \ddots & \mu_3 & & \vdots & \vdots \\ \vdots & & \ddots & \ddots & -\rho_{N-1} \rho_{N-2}^* & -\rho_N \mu_{N-1} \rho_{N-2}^* \\ 0 & \cdots & \cdots & 0 & \mu_{N-1} & -\rho_N \rho_{N-1}^* \end{bmatrix}, \tag{4.32}$$

where $\mu_k > 0,\ |\rho_k|^2 + \mu_k^2 = 1\ (k = 1, \dots, N-1),\ \rho_0 = -1,\ |\rho_N| = 1$. Such matrices were studied by various authors, see for instance [4] where it was proved that the matrix of the form (4.32) is unitary.

The matrix R_Q may be treated as a quasiseparable of order one matrix with generators $d_k = -\rho_k\rho_{k-1}^*,\ a_k = 0,\ b_k = \mu_{k-1},\ p_i = 1,\ q_j = \mu_j,\ g_i = -\rho_{i-1}^*,\ h_j = \mu_{j-1}\rho_j$. In this case we have $p_k q_k = \mu_k,\ g_k h_k = -\rho_{k-1}^*\mu_{k-1}\rho_k$ and next

$$l_k(\lambda) = -\mu_k, \quad \delta_k(\lambda) = -(\lambda + \rho_k\rho_{k-1}^*)\mu_{k-1} + \rho_{k-1}^*\mu_{k-1}\rho_k = -\lambda\mu_{k-1}, \quad k = 2, \dots, N-1.$$

From here since the matrix R_Q is invertible we conclude that the condition (4.2) holds for any eigenvalue of this matrix. The other conditions of Theorem 4.1 are also valid for any eigenvalue of the matrix R_Q. Since for a unitary matrix the geometric multiplicity of any eigenvalue is equal to its algebraic multiplicity Theorem 4.1 implies that all eigenvalues of the matrix R_Q are simple.

Next consider the recursions (3.1)-(3.4). We have

$$\gamma_1(\lambda) = \rho_1 - \lambda, \quad f_1(\lambda) = \mu_1; \tag{4.33}$$

$$\gamma_k(\lambda) = (-\rho_k\rho_{k-1}^* - \lambda)\gamma_{k-1}(\lambda) - \mu_{k-1}\rho_k f_{k-1}(\lambda), \tag{4.34}$$

$$f_k(\lambda) = -\mu_{k-1}\mu_k f_{k-1}(\lambda) - \mu_k\rho_{k-1}^*\gamma_{k-1}(\lambda), \quad k = 2, \dots, N-1. \tag{4.35}$$

To compare with some known results we consider the polynomials $\tilde{\gamma}_k(\lambda) = \det(\lambda I - R(1 : k, 1 : k)$ instead of $\gamma_k(\lambda) = \det(R(1 : k, 1 : k) - \lambda I)$, i.e. $\tilde{\gamma}_k(\lambda) = (-1)^k\gamma_k(\lambda)$ and set $\tilde{f}_k(\lambda) = (-1)^{k+1} f_k(\lambda)$. From the recursions (4.33)-(4.35) we obtain

$$\tilde{\gamma}_0(\lambda) = 1, \quad \tilde{f}_0(\lambda) = 0; \tag{4.36}$$

$$\tilde{\gamma}_k(\lambda) = (\lambda + \rho_k\rho_{k-1}^*)\tilde{\gamma}_{k-1}(\lambda) - \mu_{k-1}\rho_k\tilde{f}_{k-1}(\lambda), \tag{4.37}$$

$$\tilde{f}_k(\lambda) = \mu_{k-1}\mu_k\tilde{f}_{k-1}(\lambda) - \mu_k\rho_{k-1}^*\gamma_{k-1}(\lambda), \quad k = 1, \dots, N-1 \tag{4.38}$$

(we set here $\mu_0 = 0$). Set

$$G_k(\lambda) = -\rho_k^*\tilde{\gamma}_k(\lambda) + \mu_k\tilde{f}_k(\lambda), \quad k = 0, \dots, N-1. \tag{4.39}$$

Let us show that the polynomials $\tilde{\gamma}_k(\lambda), G_k(\lambda)$ satisfy the two-terms recurrence relations

$$\begin{pmatrix} G_0(\lambda) \\ \tilde{\gamma}_0(\lambda) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}; \quad \begin{pmatrix} G_k(\lambda) \\ \tilde{\gamma}_k(\lambda) \end{pmatrix} = \begin{pmatrix} 1 & -\rho_k^*\lambda \\ -\rho_k & \lambda \end{pmatrix} \begin{pmatrix} G_{k-1}(\lambda) \\ \tilde{\gamma}_{k-1}(\lambda) \end{pmatrix}, \quad k = 1, \dots, N-1. \tag{4.40}$$

These recursions define Szego polynomials orthogonal with the corresponding weight on the unit circle (see for instance [1, p. 176]) . From (4.37) and (4.39) we obtain

$$\tilde{\gamma}_k(\lambda) = \lambda\gamma_{k-1}(\lambda) - \rho_k G_{k-1}(\lambda), \quad k = 1, 2, \ldots, N.$$

Next using (4.39), (4.37), (4.38) and the equalities $\mu_k^2 + |\rho_k|^2 = 1$ we obtain

$$G_k(\lambda) = -\rho_k^*\tilde{\gamma}_k(\lambda) + \mu_k\tilde{f}_k(\lambda)$$
$$= -\rho_k^*(\lambda + \rho_k\rho_{k-1}^*)\tilde{\gamma}_{k-1}(\lambda) + \mu_{k-1}\rho_k^*\rho_k\tilde{f}_{k-1}(\lambda) + \mu_k(\mu_{k-1}\mu_k\tilde{f}_{k-1}(\lambda) - \mu_k\rho_{k-1}^*\gamma_{k-1}(\lambda))$$
$$= -\rho_k^*\lambda\tilde{\gamma}_{k-1}(\lambda) - |\rho_k|^2\rho_{k-1}^*\gamma_{k-1}(\lambda) + \mu_{k-1}|\rho_k|^2\tilde{f}_{k-1}(\lambda) + \mu_k^2\mu_{k-1}\tilde{f}_{k-1}(\lambda) - \mu_k^2\rho_{k-1}^*\tilde{f}_{k-1}(\lambda)$$
$$= -\rho_k^*\lambda\tilde{\gamma}_{k-1}(\lambda) - \rho_{k-1}^*\tilde{\gamma}_{k-1}(\lambda) + \mu_{k-1}\tilde{f}_{k-1}(\lambda) = -\rho_k^*\lambda\tilde{\gamma}_{k-1}(\lambda) + G_{k-1}(\lambda), \quad k = 1, \ldots, N-1$$

which completes the proof of (4.40). This result is also contained in [4].

5)The Toeplitz matrix

$$R = \begin{bmatrix} d & a & a^2 & \ldots & a^{n-1} \\ b & d & a & \ldots & a^{n-2} \\ b^2 & b & d & \ldots & a^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \ldots & d \end{bmatrix}$$

with $a \neq 0,; b \neq 0$. This is a quasiseparable of order one matrix with generators $p_i = b^{i-1}$ $(i = 2, \ldots, N)$, $q_j = b^{1-j}$ $(j = 1, \ldots, N-1)$, $a_k = 1$ $(k = 2, \ldots, N-1)$; $g_i = a^{1-i}$ $(i = 1, \ldots, N-1)$, $h_j = a^{j-1}$ $(j = 2, \ldots, N)$, $b_k = 1$ $(k = 2, \ldots, N-1)$; $d_k = d$ $(k = 1, \ldots, N)$. We have

$$l_k(\lambda) = -\delta_k(\lambda) = d - \lambda - 1, \quad k = 2, \ldots, N-1.$$

Assume that $ab \neq 1$. Then (see [5]) the matrix

$$R - (d-1)I = \begin{bmatrix} 1 & a & a^2 & \ldots & a^{n-1} \\ b & 1 & a & \ldots & a^{n-2} \\ b^2 & b & 1 & \ldots & a^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b^{n-1} & b^{n-2} & b^{n-3} & \ldots & 1 \end{bmatrix} \tag{4.41}$$

is invertible, i.e. the number $d-1$ is not an eigenvalue of the matrix R. Thus in this case the conditions of Theorem 4.1 are valid for any eigenvalue of the matrix R.

Assume now that the number d is real and the number $b/\overline{a}$ is real and positive. In this case we have

$$\frac{p_{i+1}q_i}{\overline{g_i h_{i+1}}} = \frac{b^i b^{1-i}}{\overline{a^{1-i}a^i}} = \frac{b}{\overline{a}} > 0, \quad i = 1, \ldots, N-1$$

and hence the matrix R satisfies the conditions (4.23). Applying the formulas (4.24) define the nonsingular diagonal matrix $D = \text{diag}\,(\rho_i)_{i=1}^N$ with the entries

$$\rho_1 = 1, \quad \rho_k = \sqrt{\frac{a}{\overline{b}}}\rho_{k-1} = \left(\frac{a}{\overline{b}}\right)^{\frac{k-1}{2}}, \quad k = 2, \ldots, N. \tag{4.42}$$

Moreover we have

$$\frac{p_k q_k}{a_k} = \frac{\overline{g_k h_k}}{\overline{b_k}} = 1, \quad k = 2, \dots, N-1$$

and hence in the same way as in Example 3) we obtain that the matrix $Q = D^{-1}RD$ is a Hermitian matrix. Furthermore using the formulas (4.27) and (4.42) we conclude that Q is a quasiseparable of order one matrix with generators $p_i^Q = (\overline{a}b)^{(i-1)/2}$ $(i = 2, \dots, N)$, $q_j^Q = (\overline{a}b)^{(1-j)/2}$ $(j = 1, \dots, N-1)$, $a_k^Q = 1$ $(k = 2, \dots, N-1)$; $g_i^Q = (\overline{b}a)^{(1-i)/2}$ $(i = 1, \dots, N-1)$, $h_j^Q = (\overline{b}a)^{(j-1)/2}$ $(j = 2, \dots, N)$, $b_k^Q = 1$ $(k = 2, \dots, N-1)$; $d_k^Q = d$ $(k = 1, \dots, N)$. This implies that the matrix Q is a Toeplitz Hermitian matrix. Hence it follows that in the case under consideration the matrix R is diagonalizable and moreover from Theorem 4.1 we conclude that all eigenvalues of the matrix R are simple.

The arguments mentioned above imply, in particular, that for a real d and for $ab > 0$, $ab \neq 1$ the Toeplitz matrix R is diagonalizible and its eigenvalues are simple. Let us show that in the case $ab < 0$ the matrix R may have multiple eigenvalues. By virtue of Theorem 4.1 this implies that the matrix R is not diagonalizable. As an example we take the 3×3 matrix

$$R = \begin{bmatrix} d & a & a^2 \\ b & d & a \\ b^2 & b & d \end{bmatrix}$$

with $ab = 8$. Checking directly or applying the formula (3.18) we obtain the characteristic polynomial of this matrix

$$\gamma_3(\lambda) = (d-\lambda)^3 - (a^2b^2 - 2ab)(d-\lambda) + 2a^2b^2 = (d-\lambda)^3 - 48(d-\lambda) + 128 = -(\lambda - d + 4)^2(\lambda - d - 8).$$

It is clear that this polynomial has a multiple eigenvalue and then Theorem 4.1 implies that the matrix R is not diagonalizable.

Next we consider the case $ab = 1$. In this case the relations (3.17), (3.18) have the form

$$\gamma_1(\lambda) = d - \lambda, \quad \gamma_2(\lambda) = (d-\lambda)^2 - 1 = (d - \lambda + 1)(d - \lambda - 1);$$

$$\gamma_k(\lambda) = 2(d - \lambda - 1)\gamma_{k-1}(\lambda) - (d - \lambda - 1)^2 \gamma_{k-2}(\lambda), \quad k = 3, \dots, N.$$

From here one can derive easily by induction that

$$\gamma_k(\lambda) = (d - \lambda + k - 1)(d - \lambda - 1)^{k-1}, \quad k = 2, \dots, N.$$

Hence it follows that in this case the matrix R has the simple eigenvalue $\lambda = d + N - 1$ and the multiple eigenvalue $\lambda = d - 1$. For $\lambda = d + N - 1$ the eigenvector is determined by the relation (4.6). For $\lambda = d - 1$ we obtain directly from (4.41) the eigenvectors

$$u_1 = \begin{pmatrix} 1 \\ -b \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ u_2 = \begin{pmatrix} 0 \\ 1 \\ -b \\ \vdots \\ 0 \end{pmatrix}, \ \dots \ u_{N-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ -b \end{pmatrix}.$$

Hence it follows that for $ab = 1$ the Toeplitz matrix R is diagonalizable.

4.2 The multiple eigenvalues

In this subsection we study the case when the conditions of Theorem 4.1 are non-valid.

Theorem 4.2 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$. Assume that $\lambda = \lambda_0$ is an eigenvalue of the matrix R and for some $m \in \{2, \dots, N-1\}$,*

$$l_m(\lambda_0) = (d_m - \lambda_0)a_m - p_m q_m = 0, \quad \delta_m(\lambda_0) = (d_m - \lambda_0)b_m - g_m h_m = 0 \tag{4.43}$$

and

$$|p_m| + |d_m - \lambda_0| + |g_m| > 0, \quad |q_m| + |d_m - \lambda_0| + |h_m| > 0. \tag{4.44}$$

The vector $x = \begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix}$, where x_1, x_2 are vectors of the sizes $m-1, N-m$ respectively, θ is a number, is an eigenvector of the matrix R corresponding to the eigenvalue λ_0 if and only if there exist numbers θ_1, θ_2 such that $\theta_1 + \theta_2 = \theta$ and

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix}, \quad \begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} \in \operatorname{Ker} \begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix}, \tag{4.45}$$

where $A_m(\lambda_0) = R(1:m, 1:m) - \lambda_0 I$, $B_m(\lambda_0) = R(m+1:N, m+1:N) - \lambda_0 I$, $Q_m = \operatorname{row}(a^{\times}_{m+1,k} q_k)_{k=1}^{m}$, $H_m = \operatorname{row}(b^{\times}_{m-1,k} h_k)_{k=m}^{N}$.

Proof. Assume that the vectors $\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix}$, $\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix}$ satisfy the condition (4.45). Applying the representation (2.4) with $k = m$ and with $k = m-1$ to the matrix $R - \lambda_0 I$ we obtain

$$R = \begin{pmatrix} A_m(\lambda_0) & G_m H_{m+1} \\ P_{m+1} Q_m & B_{m+1}(\lambda_0) \end{pmatrix} = \begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1} H_m \\ P_m Q_{m-1} & B_m(\lambda_0) \end{pmatrix}, \tag{4.46}$$

where $A_m(\lambda_0) = R(1:m, 1:m) - \lambda_0 I$, $B_m(\lambda_0) = R(m+1:N, m+1:N) - \lambda_0 I$ and the elements Q_k, G_k, P_k, H_k are defined by (2.2). From(4.46) we obtain the representations

$$R(:, 1:m) = \begin{pmatrix} I_m & 0 \\ 0 & P_{m+1} \end{pmatrix} \begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix}, \tag{4.47}$$

$$R(:, m:N) = \begin{pmatrix} G_{m-1} & 0 \\ 0 & I_{N-m+1} \end{pmatrix} \begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix}. \tag{4.48}$$

The condition (4.45) implies

$$\begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} \in \operatorname{Ker}(R(:, 1:m)), \quad \begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} \in \operatorname{Ker}(R(:, m:N)).$$

Hence it follows that the vectors $\begin{pmatrix} x_1 \\ \theta_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ \theta_2 \\ x_2 \end{pmatrix}$ belong to the kernel of the matrix $R-\lambda_0 I$ and therefore the sum $\begin{pmatrix} x_1 \\ \theta_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \theta_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix}$, where $\theta = \theta_1 + \theta_2$, also belongs to $\operatorname{Ker}(R - \lambda_0 I)$.

Assume that $\begin{pmatrix} x_1 \\ \theta \\ x_2 \end{pmatrix} \in \operatorname{Ker}(R - \lambda_0 I)$. Using the representation (4.15) we obtain

$$\begin{cases} A_{m-1}(\lambda_0)x_1 + G_{m-1}h_m\theta + G_{m-1}b_mH_{m+1}x_2 = 0 \\ p_mQ_{m-1}x_1 + (d_m - \lambda_0)\theta + g_mH_{m+1}x_2 = 0 \\ P_{m+1}a_mQ_{m-1}x_1 + P_{m+1}q_m\theta + B_{m+1}(\lambda_0)x_2 = 0. \end{cases} \tag{4.49}$$

By the first inequality from (4.44) we obtain that at least one of the conditions $|p_m| + |d_m - \lambda_0| > 0$ or $|g_m| + |d_m - \lambda_0| > 0$ holds. Assume that $|p_m| + |d_m - \lambda_0| \neq 0$. The first equality from (4.43) implies

$$\det \begin{pmatrix} p_m & d_m - \lambda_0 \\ a_m & q_m \end{pmatrix} = 0$$

and hence there exists a number α such that

$$\begin{pmatrix} a_m & q_m \end{pmatrix} = \alpha \begin{pmatrix} p_m & d_m - \lambda_0 \end{pmatrix}. \tag{4.50}$$

Multiplying the second equation in (4.49) by $P_{m+1}\alpha$ and subtracting the result from the third equation we obtain the equivalent system

$$\begin{cases} A_{m-1}(\lambda_0)x_1 + G_{m-1}h_m\theta + G_{m-1}b_mH_{m+1}x_2 = 0 \\ p_mQ_{m-1}x_1 + (d_m - \lambda_0)\theta + g_mH_{m+1}x_2 = 0 \\ -P_{m+1}\alpha g_mH_{m+1}x_2 + B_{m+1}(\lambda_0)x_2 = 0. \end{cases} \tag{4.51}$$

Notice that $|p_m| + |d_m - \lambda_0| > 0$ implies $|h_m| + |d_m - \lambda_0| > 0$, otherwise we get $d_m - \lambda_0 = 0$, hence by virtue of (4.50) $q_m = 0$ and therefore $|q_m| + |d_m - \lambda_0| + |h_m| = 0$ contradicting the second inequality from (4.44). The second equality from (4.43) implies

$$\det \begin{pmatrix} h_m & b_m \\ d_m - \lambda_0 & g_m \end{pmatrix} = 0$$

and hence there exists a number β such that

$$\begin{pmatrix} b_m \\ g_m \end{pmatrix} = \beta \begin{pmatrix} h_m \\ d_m - \lambda_0 \end{pmatrix}. \tag{4.52}$$

Set $\theta_2 = -\beta H_{m+1}x_2$, $\theta_1 = \theta - \theta_2$. We should prove that $\begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} \in \operatorname{Ker}(R(:, m : N))$. Using the definition of θ_2, the formula (4.52) and the representation $H_m = \begin{pmatrix} h_m & b_mH_{m+1} \end{pmatrix}$ we get

$$H_m \begin{pmatrix} \theta_2 \\ x_2 \end{pmatrix} = h_m\theta_2 + b_m H_{m+1}x_2 = -h_m\beta H_{m+1}x_2 + h_m\beta H_{m+1}x_2 = 0. \tag{4.53}$$

and

$$(d_m - \lambda_0)\theta_2 + g_m H_{m+1}x_2 = -\beta(d_m - \lambda_0)H_{m+1}x_2 + \beta(d_m - \lambda_0)H_{m+1}x_2 = 0. \tag{4.54}$$

Next by virtue of the third equation from (4.49) and the relations (4.50), (4.52) we obtain

$$\begin{aligned} P_{m+1}q_m\theta_2 + B_{m+1}(\lambda_0)x_2 &= -P_{m+1}\alpha(d_m - \lambda_0)\beta H_{m+1}x_2 + B_{m+1}(\lambda_0)x_2 \\ &= -P_{m+1}\alpha g_m H_{m+1}x_2 + B_{m+1}(\lambda_0)x_2 = 0 \end{aligned}$$

which completes the proof of the second inclusion from (4.45).

Let us prove the second inclusion from (4.45). Using (4.53) and the first equation from (em.4) we have

$$A_{m-1}(\lambda_0)x_1 + G_{m-1}h_m\theta_1 + G_{m-1}(h_m\theta_2 + b_m H_{m+1}x_2) = A_{m-1}(\lambda_0)x_1 + G_{m-1}h_m\theta_1 = 0$$

and similarly from (4.54) and the second equation from (em.4) we conclude that

$$p_m Q_{m-1}x_1 + (d_m - \lambda_0)\theta_1 = 0.$$

Moreover from here using the representation $Q_m = \begin{pmatrix} a_m Q_{m-1} & q_m \end{pmatrix}$ and the relation (4.50) we get

$$Q_m \begin{pmatrix} x_1 \\ \theta_1 \end{pmatrix} = \begin{pmatrix} a_m & q_m \end{pmatrix} \begin{pmatrix} Q_{m-1}x_1 \\ \theta_1 \end{pmatrix} = \alpha(p_m Q_{m-1}x_1 + (d_m - \lambda_0)\theta_1) = 0$$

which completes the proof. $\square$

Theorem 4.3 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$; g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$; d_k $(k = 1, \dots, N)$ and let $\lambda = \lambda_0$ be an eigenvalue of the matrix R. Assume that the following conditions are valid:*

$$|d_1 - \lambda_0| + |q_1| > 0, \quad |d_1 - \lambda_0| + |g_1| > 0, \tag{4.55}$$

$$|d_N - \lambda_0| + |p_N| > 0, \quad |d_N - \lambda_0| + |h_N| > 0; \tag{4.56}$$

for $m = j_1, j_2, \dots, j_k,\ j_1, \dots, j_k \in \{2, \dots, N-1\}$:

$$l_m(\lambda_0) = (d_m - \lambda_0)a_m - p_m q_m = 0, \quad \delta_m(\lambda_0) = (d_m - \lambda_0)b_m - g_m h_m = 0, \tag{4.57}$$

$$|p_m| + |d_m - \lambda_0| + |g_m| > 0, \quad |q_m| + |d_m - \lambda_0| + |h_m| > 0 \tag{4.58}$$

and for the other values of $m \in \{2, \dots, N-1\}$*:*

$$l_m(\lambda_0) \neq 0, \quad \delta_m(\lambda_0) \neq 0. \tag{4.59}$$

Then the corresponding to λ_0 *linear independent vectors* u_i *of the matrix* R *are given via the following algorithm:*

1)Set $s = 0,\ \theta = 1,\ j_0 = 1$*; Set* $R^{(1)} = R$*, i.e. define the quasiseparable matrix* $R^{(1)}$ *via generators* $p_i^{(1)}$ $(i = 2, \dots, N)$, $q_j^{(1)}$ $(j = 1, \dots, N-1)$, $a_k^{(1)}$ $(k = 2, \dots, N-1)$; $g_i^{(1)}$ $(i = 1, \dots, N-1)$, $h_j^{(1)}$ $(j = 2, \dots, N)$, $b_k^{(1)}$ $(k = 2, \dots, N-1)$; $d_k^{(1)}$ $(k = 1, \dots, N)$ *which are equal to the corresponding generators of the matrix* R*.*

2)If $\theta = 0$ *stop.*

3)Set $m = j_{s+1} - j_s$*. Using the generators of the matrix* $R^{(s)}$ *compute by the formulas (3.1)-(3.4) the values* $\gamma_{m-1}^{(s)}(\lambda_0), f_{m-1}^{(s)}(\lambda_0)$*.*

If

$$(d_m^{(s)} - \lambda_0)\gamma_{m-1}^{(s)}(\lambda_0) - p_m^{(s)} f_{m-1}^{(s)}(\lambda_0) h_m^{(s)} = 0, \quad q_m^{(s)}\gamma_{m-1}^{(s)}(\lambda_0) - a_m^{(s)} f_{m-1}^{(s)}(\lambda_0) h_m^{(s)} = 0$$

compute the eigenvector u_s *of the matrix* R *as follows. Define the matrix* $\tilde{R}^{(s)}$ *of sizes* $m \times m$ *via generators: set*

$$\tilde{q}_j^{(s)} = q_j^{(s)},\ \tilde{g}_j^{(s)} = g_j^{(s)},\ \tilde{d}_j^{(s)} = d_j^{(s)}, \quad j = 1, \dots, m-1,$$

$$\tilde{p}_i^{(s)} = p_i^{(s)},\ \tilde{h}_i^{(s)} = h_i^{(s)},\ \tilde{a}_j^{(s)} = a_i^{(s)},\ \tilde{b}_j^{(s)} = b_i^{(s)}, \quad i = 2, \dots, m-1, \tilde{h}_m^{(s)} = h_m^{(s)},$$

and if $|p_m^{(s)}| + |d_m^{(s)} - \lambda_0| > 0$ *then set* $\tilde{p}_m^{(s)} = p_m^{(s)},\ \tilde{d}_m^{(s)} = d_m^{(s)}$*, else set* $\tilde{p}_m^{(s)} = a_m^{(s)},\ \tilde{d}_m^{(s)} = q_m^{(s)} + \lambda_0$*. Compute the eigenvector* $\tilde{u}_s$ *of the matrix* $\tilde{R}^{(s)}$ *using the corresponding formula from Theorem 4.1. Set*

$$u_s = \begin{pmatrix} 0_{j_s - 1} \\ \tilde{u}_s \\ 0_{N - j_{s+1}} \end{pmatrix}.$$

4) Using the generators of the matrix $R^{(s)}$ *compute by the formulas (3.6)-(3.8) the values* $\theta_{m+1}^{(s)}(\lambda_0), z_{m+1}^{(s)}(\lambda_0)$*.*

If

$$(d_m^{(s)} - \lambda_0)\theta_{m-1}^{(s)}(\lambda_0) - g_m^{(s)} z_{m+1}^{(s)}(\lambda_0) q_m^{(s)} = 0, \quad h_m^{(s)}\theta_{m+1}^{(s)}(\lambda_0) - q_m^{(s)} z_{m+1}^{(s)}(\lambda_0) b_m^{(s)} = 0$$

then define the matrix $R^{(s+1)}$ *of sizes* $N - j_{s+1} + 1 \times N - j_{s+1} + 1$ *via generators: set*

$$q_1^{(s+1)} = q_m^{(s)},\ q_j^{(s+1)} = q_{j+m-1}^{(s)},\ g_j^{(s+1)} = g_{j+m-1}^{(s)},\ d_j^{(s+1)} = d_{j+m-1}^{(s)}, \quad j = 1, \dots, N - j_{s+1},$$

$$p_i^{(s+1)} = p_{i+m-1}^{(s)},\ h_i^{(s+1)} = h_{i+m-1}^{(s)},\ a_i^{(s+1)} = a_{i+m-1}^{(s)},\ b_i^{(s)} = b_{i+m-1}^{(s)}, \quad i = 2, \dots, N - j_{s+1} + 1$$

and if $|g_m^{(s)}| + |d_m^{(s)} - \lambda_0| > 0$ *then set* $g_1^{(s+1)} = g_m^{(s)},\ d_1^{(s+1)} = d_m^{(s)}$*, else set* $g_m^{(s+1)} = b_m^{(s)},\ d_m^{(s+1)} = h_m^{(s)} + \lambda_0$*;*

else set $\theta = 0$*.*

5)$s := s + 1$*;*

6)If $s = k + 1$ *set* $\theta = 0$*;*

7)Go to 2).

Proof. Take $m = j_1$. By Theorem 4.2 each eigenvector of the matrix R corresponding to the eigenvalue λ_0 has the form

$$u = \begin{pmatrix} \tilde{u}_1 \\ 0_{N-m} \end{pmatrix} + \begin{pmatrix} 0_{m-1} \\ \hat{u}_2 \end{pmatrix}$$

with

$$\tilde{u}_1 \in \operatorname{Ker} \begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix}, \quad \hat{u}_2 \in \operatorname{Ker} \begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix},$$

where $A_m(\lambda_0) = R(1 : m, 1 : m) - \lambda_0 I,\ B_m(\lambda_0) = R(m+1 : N, m+1 : N) - \lambda_0 I,\ Q_m = \operatorname{row}(a^{\times}_{m+1,k} q_k)_{k=1}^{m},\ H_m = \operatorname{row}(b^{\times}_{m-1,k} h_k)_{k=m}^{N}$. Moreover by virtue of (4.47), (rprz.3) the vectors $\begin{pmatrix} \tilde{u}_1 \\ 0_{N-m} \end{pmatrix}$, $\begin{pmatrix} 0_{m-1} \\ \hat{u}_2 \end{pmatrix}$ belong to $\operatorname{Ker}(R - \lambda_0)I$. Notice that such a nonzero vector $\tilde{u}_1$ exists if and only if

$$\operatorname{Ker} \begin{pmatrix} A_m(\lambda_0) \\ Q_m \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1} h_m \\ p_m Q_{m-1} & d_m - \lambda_0 \\ a_m Q_{m-1} & q_m \end{pmatrix} \tag{4.60}$$

is a nontrivial subspace. Here we used the representation (2.6) for the matrix $A_m(\lambda_0)$ and the equality $Q_m = \begin{pmatrix} a_m Q_{m-1} & q_m \end{pmatrix}$. Since by virtue of the first equality from (4.57) we have

$$\det \begin{pmatrix} p_m & d_m - \lambda_0 \\ a_m & q_m \end{pmatrix} = 0$$

this subspace is non-trivial if and only if

$$\det \begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1} h_m \\ p_m Q_{m-1} & d_m - \lambda_0 \end{pmatrix} = \det \begin{pmatrix} A_{m-1}(\lambda_0) & G_{m-1} h_m \\ a_m Q_{m-1} & q_m \end{pmatrix} = 0.$$

Applying the formula (3.2) to these determinants we conclude that the subspace (4.60) is non-trivial if and only if

$$(d_m - \lambda_0)\gamma_{m-1}(\lambda_0) - p_m f_{m-1}(\lambda_0) h_m = q_m \gamma_{m-1}(\lambda_0) - a_m f_{m-1}(\lambda_0) h_m = 0.$$

If the last holds we should consider two cases. The first one is $|p_m| + |d_m - \lambda_0| > 0$. In this case we have also $|h_m| + |d_m - \lambda_0| > 0$, otherwise we have $d_m - \lambda_0 = 0,\ p_m \neq 0$, from the second inequality from (4.58) we get $q_m \neq 0$ contradicting the first equality from (4.57). Thus the matrix

$$A_m = R(1 : m, 1 : m) = \begin{pmatrix} A_{m-1} & G_{m-1} h_m \\ p_m Q_{m-1} & d_m \end{pmatrix}$$

and the number λ_0 satisfy the conditions of Theorem 4.1. Hence it follows that the number λ_0 is an eigenvalue of the matrix A_m of the geometric multiplicity one and the corresponding eigenvector $\tilde{u}_1$ is obtained by the corresponding formula from Theorem 4.1. If $|p_m| + |d_m - \lambda_0| = 0$ then we have $d_m - \lambda_0 = 0$, moreover by virtue of the first inequality from (4.58) we get $g_m \neq 0$ and from the second equality from (4.57) we obtain $h_m = 0$ and thus from the second inequality from (4.58) we get $q_m \neq 0$. Hence it follows that the matrix

$$\begin{pmatrix} A_{m-1} & G_{m-1}h_m \\ p_m Q_{m-1} & d_m + \lambda_0 \end{pmatrix}$$

and the number λ_0 satisfy the conditions of Theorem 4.1. Therefore the number λ_0 is an eigenvalue of this matrix of the geometric multiplicity one and the corresponding eigenvector $\tilde{u}_1$ is obtained by the corresponding formula from Theorem 4.1.

Next a nonzero vector $\hat{u}_2$ exists if and only if

$$\operatorname{Ker}\begin{pmatrix} H_m \\ B_m(\lambda_0) \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} h_m & b_m H_{m+1} \\ d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1} q_m & B_{m+1}(\lambda_0) \end{pmatrix} \tag{4.61}$$

is a nontrivial subspace. Here we used the representation (2.8) for the matrix $B_m(\lambda_0)$ and the equality $H_m = \begin{pmatrix} h_m & b_m H_{m+1} \end{pmatrix}$.

Since by virtue of the second equality from (4.57) we have

$$\det\begin{pmatrix} h_m & b_m \\ d_m - \lambda_0 & g_m \end{pmatrix} = 0$$

this subspace is non-trivial if and only if

$$\det\begin{pmatrix} d_m - \lambda_0 & g_m H_{m+1} \\ P_{m+1} q_m & B_{m+1}(\lambda_0) \end{pmatrix} = \det\begin{pmatrix} h_m & b_m H_{m+1} \\ P_{m+1} q_m & B_{m+1}(\lambda_0) \end{pmatrix} = 0$$

Applying the formula (3.7) to these determinants we conclude that the subspace (4.60) is non-trivial if and only if

$$(d_m - \lambda_0)\theta_{m+1}(\lambda_0) - g_m z_{m+1}(\lambda_0) q_m = h_m \theta_{m+1}(\lambda_0) - b_m z_{m+1}(\lambda_0) q_m = 0. \tag{4.62}$$

If the last is valid we should consider two cases. The first one is $|g_m| + |d_m - \lambda_0| > 0$. In this case we have also $|q_m| + |d_m - \lambda_0| > 0$, otherwise we have $d_m - \lambda_0 = 0,\ q_m \neq 0$, from the second inequality from (4.58) we get $h_m \neq 0$ contradicting the second equality from (4.57). Thus in this case we set

$$R^{(2)} = B_m = R(m:N, m:N) = \begin{pmatrix} d_m & g_m H_{m+1} \\ P_{m+1} q_m & B_{m+1} \end{pmatrix}.$$

If $|g_m| + |d_m - \lambda_0| = 0$ then we have $d_m - \lambda_0 = 0$, moreover by virtue of the second inequality from (4.58) we get $g_m \neq 0$ and from the second equality from (4.57) we obtain $h_m = 0$ and thus from the first inequality from (4.58) we get $h_m \neq 0$. In this case we set

$$R^{(2)} = \begin{pmatrix} h_m + \lambda_0 & b_m H_{m+1} \\ P_{m+1} q_m & B_{m+1} \end{pmatrix}$$

Next we apply the same procedure to the matrix $R^{(2)}$ and so on until we get that the condition (4.62) is not valid or take $m = k$.

Since each of the eigenvectors u_s contains one in the position in which the previous vectors contain zero, it is clear that these vectors are linear independent. $\square$

Remark. In the conditions of Theorem 4.3 the geometric multiplicity of the eigenvalue λ_0 is less than or equal to $k+1$.

The following theorem completes the results of Theorem 4.1 and Theorem 4.3 in some additional assumptions.

Theorem 4.4 *Let R be quasiseparable of order one matrix with generators p_i $(i = 2, \ldots, N)$, q_j $(j = 1, \ldots, N-1)$, a_k $(k = 2, \ldots, N-1)$; g_i $(i = 1, \ldots, N-1)$, h_j $(j = 2, \ldots, N)$, b_k $(k = 2, \ldots, N-1)$; d_k $(k = 1, \ldots, N)$ and assume that the generators of R satisfy the following conditions:*

$$\overline{d_i} = d_i, \quad i = 1, \ldots, N; \quad \frac{p_{i+1}q_i}{\overline{g_i h_{i+1}}} > 0, \quad i = 1, \ldots, N-1 \tag{4.63}$$

$$a_k \neq 0,\ b_k \neq 0, \quad \frac{p_k q_k}{a_k} = \frac{g_k h_k}{b_k} = \frac{\overline{g_k h_k}}{\overline{b_k}}, \quad k = 2, \ldots, N-1. \tag{4.64}$$

Let also λ_0 be an eigenvalue of the matrix R.

If

$$l_k(\lambda_0) = (\lambda_0 - d_k)a_k + p_k q_k \neq 0, \quad k = 2, \ldots, N-1 \tag{4.65}$$

then the eigenvalue λ_0 is simple and the corresponding eigenvector is given by the formula (4.6). If for $m = j_1, j_2, \ldots, j_k$, $j_1, \ldots, j_k \in \{2, \ldots, N-1\}$:

$$l_m(\lambda_0) = (d_m - \lambda_0)a_m - p_m q_m = 0, \tag{4.66}$$

and for the other values of $m \in \{2, \ldots, N-1\}$

$$l_m(\lambda_0) \neq 0 \tag{4.67}$$

then the multiplicity of λ_0 is less than or equal to $k+1$ and the corresponding eigenvectors may be computed by the algorithm from Theorem 4.3.

Proof. In the same way as in Example 3) from Subsection 4.1 we show that the matrix R is similar to a Hermitian matrix. Hence it follows that the matrix R is diagonalizable. Moreover the condition (4.64) implies that for any λ,

$$\delta_k(\lambda) = (\lambda - d_k)b_k + g_k h_k = b_k(\lambda - d_k + \frac{g_k h_k}{b_k}) = b_k(\lambda - d_k + \frac{p_k q_k}{a_k}) = \frac{b_k}{a_k} l_k(\lambda),\ k = 2, \ldots, N-1.$$

Hence it follows that for any λ and for any $k \in \{2, \ldots, N-1\}$, $l_k(\lambda) = 0$ if and only if $\delta_k(\lambda) = 0$. Therefore if the condition (4.65) holds then taking also into account that by virtue of the condition (4.63) we have $q_1 \neq 0,\ p_N \neq 0,\ h_N \neq 0$ we conclude all conditions of Theorem 4.1 are valid. Hence by Theorem 4.1 and by virtue of the fact that the matrix R is diagonalizable we conclude that the eigenvalue λ_0 is simple. Moreover since we have $p_N h_N \neq 0$ the corresponding eigenvector is given by the formula (4.6). In the case when the conditions (4.66), (4.67) are valid we have

$$l_m(\lambda_0) = \delta_m(\lambda_0) = 0$$

for $m = j_1, j_2, \ldots, j_k,\ j_1, \ldots, j_k \in \{2, \ldots, N-1\}$ and

$$l_m(\lambda_0) \neq 0, \quad \delta_m(\lambda_0) \neq 0.$$

for the other values of $m \in \{2, \ldots, N-1\}$. Taking into account that by virtue of the condition (4.63) we have $q_k \neq 0,\ g_k \neq 0\ (k = 1, \ldots, N-1),\ p_k \neq 0,\ h_k \neq 0\ (k = 2, \ldots, N)$ we conclude that all conditions of Theorem 4.3 are valid. Hence by Theorem 4.3 and by virtue of the fact that the matrix R is diagonalizable we conclude that the multiplicity of λ_0 is less than or equal to $k+1$ and the corresponding eigenvectors may be computed by the algorithm from Theorem 4.3. □

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