FAST STATE SPACE ALGORITHMS FOR MATRIX NEHARI AND NEHARI-TAKAGI INTERPOLATION PROBLEMS [1]

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Numerical algorithms with complexity $O(n^2)$ operations are proposed for solving matrix Nehari and Nehari-Takagi problems with n interpolation points. The algorithms are based on explicit formulas for the solutions and on theorems about cascade decomposition of rational matrix function given in a state space form. The method suggests also fast algorithms for LDU factorizations of structured matrices. The numerical behavior of the designed algorithms is studied for a wide set of examples.

1. MATRIX NEHARI PROBLEM

Let there be given rational $M \times N$ matrix function $K(z)$ such that $\sup_{z \in i\mathbf{R}} \|K(z)\| < \infty$ and we seek $R(z)$ in the set $\mathcal{R}_{M\times N}(\Pi^- \cup i\mathbf{R})$ of all rational $M \times N$ matrix functions with no poles in open left half plane Π^- and on imaginary line $i\mathbf{R}$, such that

$$\sup_{z \in i\mathbf{R}} \|K(z) - R(z)\| < 1. \tag{1.1}$$

We consider here the generic case of Nehari problem, when function $K(z)$ has only simple poles $z_1, ... z_n \in \Pi^-$. Namely, the local Laurent series of $K(z)$ in a neighborhood of every pole z_i has the form

$$K(z) = (z - z_i)^{-1} \cdot \gamma_i \cdot w_i + [\text{analytic at } z_i] \qquad (1 \leq i \leq n),$$

where $\gamma_1, ..., \gamma_n \in \mathbf{C}^{M\times 1}$ be nonzero column vectors, and $w_1, ..., w_n \in \mathbf{C}^{1\times N}$ be nonzero row vectors. Set

$$P = \left[-\frac{w_i \cdot w_j^*}{z_i + z_j^*} \right]_{1\leq i,j\leq n}, \qquad Q = \left[-\frac{\gamma_i^* \cdot \gamma_j}{z_i^* + z_j} \right]_{1\leq i,j\leq n}, \tag{1.2}$$

[1] In : Integral Equations and Operator Theory, **20**, (1994), 44–83.

and suppose that the matrix $I_n - PQ$ is nonsingular. Then [BGR, Theorem 20.3.2] (see also [BGR1]) there exists $R(z)$, which satisfies (1.1) if and only if the maximal eigenvalue $\lambda_1(PQ)$ of PQ is less than 1. (Both P and Q are positive definite and therefore all eigenvalues of PQ are positive). If $\lambda_1(PQ) < 1$, then in accordance with [BGR, Theorem 20.3.2] all the solutions $F(z) = K(z) - R(z)$ of matrix Nehari problem are parameterized by the formula

$$F(z) = (\Theta_{11}(z) \cdot G(z) + \Theta_{12}(z)) \cdot (\Theta_{21}(z) \cdot G(z) + \Theta_{22}(z))^{-1} \tag{1.3}$$

with arbitrary $G(z) \in \mathcal{R}_{M\times N}(\Pi^- \cup i\mathbf{R})$, $\sup_{z\in i\mathbf{R}} \|G(z)\| \le 1$, and

$$\Theta(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix},$$

which is given by

$$\Theta(z) = I_{M+N} + \begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix} \cdot \begin{bmatrix} (zI - A)^{-1} & 0 \\ 0 & (zI + A^*)^{-1} \end{bmatrix} \cdot \Lambda^{-1} \cdot \begin{bmatrix} -C^* & 0 \\ 0 & B \end{bmatrix}, \tag{1.4}$$

where

$$\Lambda = \begin{bmatrix} Q & I \\ I & P \end{bmatrix}, \quad A = \begin{bmatrix} z_1 & & & \\ & z_2 & & 0 \\ & & \ddots & \\ 0 & & & z_n \end{bmatrix}, \quad B = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad C = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix}. \tag{1.5}$$

Formula (1.4) allows to obtain the solution of matrix Nehari problem via solving $M + N$ linear systems of equations

$$\Lambda \cdot X = \begin{bmatrix} -C^* & 0 \\ 0 & B \end{bmatrix}, \qquad (X \in \mathbf{C}^{2n\times(M+N)}) \tag{1.6}$$

with matrix $\Lambda \in \mathbf{C}^{2n\times 2n}$. Solving these systems by Gaussian elimination requires $O(n^3)$ arithmetic operations.

In this paper we propose several $O(n^2)$ algorithms for solution of the matrix Nehari problem. A few of them are based on representing $\Theta(z)$ from (1.4) as a cascade decomposition

$$\Theta(z) = \Theta_1(z) \cdot \ldots \cdot \Theta_r(z), \tag{1.7}$$

where $r \le 2n$ and each cascade factor $\Theta_i(z)$ has the form

$$\Theta_i(z) = I_{M+N} + \sum_{k=1}^{s} \frac{1}{d_k} \cdot \frac{1}{z - t_k} \cdot \psi_k \cdot \varphi_k, \tag{1.8}$$

with $d_k \in \mathbf{C}$, $\quad t_k \in \mathbf{C}$, $\quad \psi_k \in \mathbf{C}^{(M+N)\times 1}$, $\quad \varphi_k \in \mathbf{C}^{1\times(M+N)}$ and the number s in the sum in (1.8) does not exceed 2. Obviously, (1.7) allows to compute in $O(n)$ operations the value of the solution $F(z)$ of matrix Nehari problem at a given point.

Numerical features of computing (1.7) are sensitive to the ordering of interpolation points. We suggest a few heuristics for the permutation of interpolation data, which correspond to partial and symmetric pivoting in LDU factorization of matrix Λ in (1.4).

The other $O(n^2)$ algorithms proposed in this paper are based on new schemes for solving structured linear systems of equations (1.6) and then making use of the formulas (1.4) and (1.3). The numerical behaviour of designed algorithms is studied in a wide set of computer experiments.

In the end of this paper the same algorithms are used for solving the more general matrix Nehari-Takagi problem. The numerical features of the designed algorithms are analyzed also for this case.

2. FACTORIZATION OF RATIONAL MATRIX FUNCTIONS

Let us start with a review of some properties of rational matrix functions and set the notations, we refer to a monograph [BGR] for more information.

Let $W(z)$ be a rational $p \times p$ matrix function with identity value I_p at infinity. Then $W(z)$ admits a *realization* of the form

$$W(z) = I_p + C \cdot (zI_N - A)^{-1} \cdot B, \tag{2.1}$$

where A, B, and C are matrices of adequate sizes. The inverse rational matrix function $W(z)$ is then given by

$$W(z)^{-1} = I_p - C \cdot (zI_N - A^{\times})^{-1} \cdot B, \qquad \text{where} \qquad A^{\times} = A - B \cdot C. \tag{2.2}$$

The smallest possible N in (2.1) is called the *McMillan degree* of $W(z)$ and if N is equal to McMillan degree, then realization (2 .1) is called *minimal*.

A factorization $W(z) = W_1(z) \cdot W_2(z)$ is called *minimal* if McMillan degree of the product $W(z)$ is equal to the sum of McMillan degrees of the factors $W_1(z)$ and $W_2(z)$. The criteria for a minimal factorization is given in the following well known theorem, which is stated here in a block matrix form. The irrelevant block entries are designated by asterisk *.

THEOREM 2.1 ([BGK, Theorem 4.8]) *Let*

$$W(z) = I_p + C \cdot (zI_N - A)^{-1} \cdot B$$

be a minimal realization. If matrices $A \in \mathbf{C}^{N\times N}, B \in \mathbf{C}^{N\times p}, C \in \mathbf{C}^{p\times N}$ can be partitioned such that matrix A is upper triangular :

$$C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \qquad A = \begin{bmatrix} A_1 & * \\ 0 & A_2 \end{bmatrix}, \qquad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

and matrix $A^{\times} = A - B \cdot C$ is lower triangular :

$$A^{\times} = \begin{bmatrix} A_1^{\times} & 0 \\ * & A_2^{\times} \end{bmatrix},$$

then rational matrix function $W(z)$ admits a minimal factorization $W(z) = W_1(z) \cdot W_2(z)$, where

$$W_1(z) = I_p + C_1 \cdot (zI_{N_1} - A_1)^{-1} \cdot B_1, \qquad W_2(z) = I_p + C_2 \cdot (zI_{N_2} - A_2)^{-1} \cdot B_2.$$

Moreover, each minimal factorization of $W(z)$ can be obtained in this way.

We will specialize the assertions of the above theorem for our purposes in this paper. To this end let us proceed with the definitions. A pair of matrices (C_π, A_π) is called a *global right pole pair* for $W(z)$ if it admits a minimal realization

$$W(z) = I_p + C_\pi \cdot (zI_N - A_\pi)^{-1} \cdot B_\pi$$

with some matrix B_π. Analogously, a pair of matrices (A_ζ, B_ζ) is called a *global left null pair* for $W(z)$ if there exists matrix C_ζ such that $W(z)^{-1}$ admits a minimal realization

$$W(z)^{-1} = I_p - C_\zeta \cdot (zI_N - A_\zeta)^{-1} \cdot B_\zeta.$$

Given global right pole pair (C_π, A_π) and global left null pair (A_ζ, B_ζ) for rational matrix function $W(z)$, then there exists a unique invertible solution of Sylvester equation

$$S \cdot A_\pi - A_\zeta \cdot S = B_\zeta \cdot C_\pi, \tag{2.3}$$

which satisfies the additional constraint :

$$W(z) = I_p + C_\pi \cdot (zI_N - A_\pi)^{-1} \cdot S^{-1} \cdot B_\zeta.$$

This invertible matrix S is referred to as *null-pole coupling matrix* for $W(z)$ (associated with global right pole pair (C_π, A_π) and global left null pair (A_ζ, B_ζ)). The above collection of matrices

$$\tau = (C_\pi, A_\pi, A_\zeta, B_\zeta, S) \tag{2.4}$$

is called a *global left null-pole triple* for $W(z)$. Obviously, from (2.3) follows that $A_\pi^\times = S^{-1} \cdot A_\zeta \cdot S$ and hence

$$W(z)^{-1} = I_p - C_\pi \cdot S^{-1} \cdot (zI_N - A_\zeta)^{-1} \cdot B_\zeta.$$

It is easy see that for any choice of invertible square matrices T_1 and T_2 of compatible sizes, a new collection of matrices

$$\tilde{\tau} = (\tilde{C}_\pi, \tilde{A}_\pi, \tilde{A}_\zeta, \tilde{B}_\zeta, \tilde{S}) \tag{2.5}$$

with

$$\begin{gathered} \tilde{C}_\pi = C_\pi \cdot T_1, \qquad \tilde{A}_\pi = T_1^{-1} \cdot A_\pi \cdot T_1, \\ \tilde{A}_\zeta = T_2^{-1} \cdot A_\zeta \cdot T_2 \qquad \tilde{B}_\zeta = T_2^{-1} \cdot B_\zeta, \\ \tilde{S} = T_2^{-1} \cdot S \cdot T_1 \end{gathered} \tag{2.6}$$

again forms a left global null-pole triple. Triples (2.4) and (2.5), related as in (2.6) are called *similar*. For a fixed rational matrix function the amount of nonuniqueness in a global left null-pole triple is restricted by the relation of similarity. More precisely, given two global left null-pole triples (2.4) and (2.5) for the same rational matrix function $W(z)$, then equalities (2.6) hold with some invertible matrices T_1 and T_2.

Theorem 2.1 gives the criteria for factorization in terms of realization. The following theorem describes the factorization of rational matrix function in terms of global left null-pole triples.

THEOREM 2.2 ([S]). *Let*

$$W(z) = I_p + C_\pi \cdot (zI_N - A_\pi)^{-1} \cdot B_\pi \tag{2.7}$$

and

$$W(z)^{-1} = I_p - C_\zeta \cdot (zI_N - A_\zeta)^{-1} \cdot B_\zeta \tag{2.8}$$

be two minimal realizations and R be invertible solution of matrix equation

$$A_\pi \cdot R - R \cdot A_\zeta = B_\pi \cdot C_\zeta, \tag{2.9}$$

such that

$$R \cdot B_\zeta = B_\pi, \qquad C_\pi \cdot R = C_\zeta.$$

Suppose also that matrices in (2.7), (2.8) *and* (2.9) *be partitioned as*

$$C_\pi = \begin{bmatrix} C_{\pi,1} & C_{\pi,2} \end{bmatrix}, \qquad A_\pi = \begin{bmatrix} A_{\pi,1} & * \\ 0 & A_{\pi,2} \end{bmatrix}, \qquad B_\pi = \begin{bmatrix} B_{\pi,1} \\ B_{\pi,2} \end{bmatrix},$$

$$C_\zeta = \begin{bmatrix} C_{\zeta,1} & C_{\zeta,2} \end{bmatrix}, \qquad A_\zeta = \begin{bmatrix} A_{\zeta,1} & 0 \\ * & A_{\zeta,2} \end{bmatrix}, \qquad B_\zeta = \begin{bmatrix} B_{\zeta,1} \\ B_{\zeta,2} \end{bmatrix},$$

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{bmatrix} \qquad \text{with} \qquad R^{-1} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

If the matrix R_{22} is invertible then $W(z)$ admits a minimal factorization $W(z) = W_1(z) \cdot W_2(z)$, where

$$W_1(z) = I_p + C_{\pi,1} \cdot (zI_{N_1} - A_{\pi,1})^{-1} \cdot S_{11}^{-1} \cdot B_{\zeta,1}, \tag{2.10}$$

$$W_2(z) = I_p + C_{\zeta,2} \cdot R_{22}^{-1} \cdot (zI_{N_2} - A_{\pi,2})^{-1} \cdot B_{\pi,2}. \tag{2.11}$$

Let us remark, that matrix R in (2.9) is in fact the inverse of the null-pole coupling matrix S of $W(z)$, corresponding to global right pole pair (C_π, A_π) in (2.7) and global left null pair (A_ζ, B_ζ) in (2.8). Moreover, matrix R_{22} is a Schur complement of (1,1) block entry in the matrix S, i.e. $R_{22} = S_{22} - S_{21} \cdot S_{11}^{-1} \cdot S_{12}$. Thus, the latter theorem makes a connection between the factorization of a rational matrix function and the Schur complementation in a null-pole coupling matrix.

Theorem 2.2 can be used for recursive computing the cascade decomposition for rational matrix function $\Theta(z)$ in (1.4) and it is more appropriate to our purposes in this paper. However, to write down both factors $W_1(z)$ and $W_2(z)$ by (2.10) and (2.11), one has to know not only global left null-pole triple $(C_\pi, A_\pi, A_\zeta, B_\zeta, S)$ for $W(z)$, but also global left null-pole triple $(C_\zeta, A_\zeta, A_\pi, B_\pi, R)$ for the inverse $W(z)^{-1}$. For this reason the straightforward application of the Theorem 2.2 for computing the cascade decomposition of $\Theta(z)$ in (1.4) is too expensive. Indeed,the input data

$$z_1, z_2, ..., z_n \in \Pi^-, \qquad w_1, w_2, ..., w_n \in \mathbf{C}^{1\times N}, \qquad \gamma_1, \gamma_2, ..., \gamma_n \in \mathbf{C}^{M\times 1}$$

for matrix Nehari problem define the entries of the first four matrices

$$C_\pi = \begin{bmatrix} C & 0 \\ 0 & B^* \end{bmatrix}, \quad A_\pi = \begin{bmatrix} A & 0 \\ 0 & -A^* \end{bmatrix}, \quad A_\zeta = \begin{bmatrix} -A^* & 0 \\ 0 & A \end{bmatrix}, \quad B_\zeta = \begin{bmatrix} -C^* & 0 \\ 0 & B \end{bmatrix},$$

in the global left null pole triple $(C_\pi, A_\pi, A_\zeta, B_\zeta, \Lambda)$ of $\Theta(z)$.

On the basis of these data one can write the first cascade factor using (2.10). At the same time, formula (2.11) for the quotient involves the entries of two more matrices

$$C_\zeta = C_\pi \cdot \Lambda^{-1} \qquad \text{and} \qquad B_\pi = \Lambda^{-1} \cdot B_\zeta,$$

i.e. requires solving linear systems (1.6) with matrix Λ already at the first step of the recursion. Since the same operation is repeated at each step of the recursion, hence computing the cascade decomposition for $\Theta(z)$ on the basis of Theorem 2.2 requires $O(n^4)$ operations. As it was mentioned in the section 1, solving linear systems (1.6) and using the above solution and the formula (1.4), already leads to $O(n^3)$ algorithm for matrix Nehari problem.

In order to design $O(n^2)$ algorithm for matrix Nehari problem, we actually need the following modification of the above two theorems, where the initial rational matrix function as well as both factors are described in terms of global left null-pole triples.

THEOREM 2.3 *Let $\tau = (C_\pi, A_\pi, A_\zeta, B_\zeta, S)$ be a global left null-pole triple for rational matrix function $W(z)$ with identity value at infinity. Let matrices in τ be partitioned as*

$$C_\pi = \begin{bmatrix} C_{\pi,1}, & C_{\pi,2} \end{bmatrix}, \qquad A_\pi = \begin{bmatrix} A_{\pi,1} & * \\ 0 & A_{\pi,2} \end{bmatrix},$$
$$A_\zeta = \begin{bmatrix} A_{\zeta,1} & 0 \\ * & A_{\zeta,2} \end{bmatrix}, \qquad B_\zeta = \begin{bmatrix} B_{\zeta,1} \\ B_{\zeta,2} \end{bmatrix}, \tag{2.12}$$
$$S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}.$$

If matrix S_{11} is invertible then

$$W(z) = I_p + C_\pi \cdot (zI_N - A_\pi)^{-1} \cdot S^{-1} \cdot B_\zeta$$

admits a minimal factorization $W(z) = W_1(z) \cdot W_2(z)$, where

$$\tau_1 = (C_{\pi,1}, A_{\pi,1}, A_{\zeta,1}, B_{\zeta,1}, S_{11}) \tag{2.13}$$

is a global left null-pole triple for $W_1(z)$, and

$$\tau_2 = (C_{\pi,2} - C_{\pi,1} \cdot S_{11}^{-1} \cdot S_{12}, \quad A_{\pi,2}, \quad A_{\zeta,2}, \quad B_{\zeta,2} - S_{21} \cdot S_{11}^{-1} \cdot B_{\zeta,1}, \quad S_{22} - S_{21} \cdot S_{11}^{-1} \cdot S_{12}) \tag{2.14}$$

is a global left null-pole triple for $W_2(z)$. In this case

$$W_1(z) = I_p + C_{\pi,1} \cdot (zI_{N_1} - A_{\pi,1})^{-1} \cdot S_{11}^{-1} \cdot B_{\zeta,1}, \tag{2.15}$$

$$W_2(z) = I_p + (C_{\pi,2} - C_{\pi,1} \cdot S_{11}^{-1} \cdot S_{12}) \times$$
$$(zI_{N_2} - A_{\pi,2})^{-1} \cdot (S_{22} - S_{21} \cdot S_{11}^{-1} \cdot S_{12})^{-1} \cdot (B_{\zeta,2} - S_{21} \cdot S_{11}^{-1} \cdot B_{\zeta,1}) \tag{2.16}$$

and

$$W_1(z)^{-1} = I_p - C_{\pi,1} \cdot S_{11}^{-1} \cdot (zI_{N_1} - A_{\zeta,1})^{-1} \cdot B_{\zeta,1}, \tag{2.17}$$

$$W_2(z)^{-1} = I_p - (C_{\pi,2} - C_{\pi,1} \cdot S_{11}^{-1} \cdot S_{12}) \times$$
$$(S_{22} - S_{21} \cdot S_{11}^{-1} \cdot S_{12})^{-1} \cdot (zI_{N_2} - A_{\zeta,2})^{-1} \cdot (B_{\zeta,2} - S_{21} \cdot S_{11}^{-1} \cdot B_{\zeta,1}) \tag{2.18}$$

PROOF. Let

$$T_1 = \begin{bmatrix} I & -S_{11}^{-1} \cdot S_{12} \\ 0 & I \end{bmatrix}, \qquad T_2 = \begin{bmatrix} I & 0 \\ S_{21} \cdot S_{11}^{-1} & I \end{bmatrix},$$

then

$$T_1^{-1} = \begin{bmatrix} I & S_{11}^{-1} \cdot S_{12} \\ 0 & I \end{bmatrix}. \qquad T_2^{-1} = \begin{bmatrix} I & 0 \\ -S_{21} \cdot S_{11}^{-1} & I \end{bmatrix},$$

Let us pass from $\tau = (C_\pi, A_\pi, A_\zeta, B_\zeta, S)$ to the similar global left null-pole triple $\tilde{\tau} = (\tilde{C}_\pi, \tilde{A}_\pi, \tilde{A}_\zeta, \tilde{B}_\zeta, \tilde{S})$, where

$$\tilde{C}_\pi = C_\pi \cdot T_1 = \begin{bmatrix} C_{\pi,1}, & C_{\pi,2} - C_{\pi,1} \cdot S_{11}^{-1} \cdot S_{12} \end{bmatrix}, \qquad \tilde{A}_\pi = T_1^{-1} \cdot A_\pi \cdot T_1 = \begin{bmatrix} A_{\pi,1} & * \\ 0 & A_{\pi,2} \end{bmatrix},$$

$$\tilde{A}_\zeta = T_2^{-1} \cdot A_\zeta \cdot T_2 = \begin{bmatrix} A_{\zeta,1} & 0 \\ * & A_{\zeta,2} \end{bmatrix}, \qquad \tilde{B}_\zeta = T_2^{-1} \cdot B_\zeta = \begin{bmatrix} B_{\zeta,1} \\ B_{\zeta,2} - S_{21} \cdot S_{11}^{-1} \cdot B_{\zeta,1} \end{bmatrix},$$

$$\tilde{S} = T_2^{-1} \cdot S \cdot T_1 = \begin{bmatrix} S_{11} & 0 \\ 0 & S_{22} - S_{21} \cdot S_{11}^{-1} \cdot S_{12} \end{bmatrix}. \tag{2.19}$$

By definition, the matrices in the new global left null-pole triple $\tilde{\tau}$ satisfy matrix equation

$$\tilde{S} \cdot \tilde{A}_\pi - \tilde{A}_\zeta \cdot \tilde{S} = \tilde{B}_\zeta \cdot \tilde{C}_\pi \tag{2.20}$$

and define the minimal factorization

$$W(z) = I_p + \begin{bmatrix} C_{\pi,1}, & C_{\pi,2} - C_{\pi,1} \cdot S_{11}^{-1} \cdot S_{12} \end{bmatrix} \cdot (zI - \begin{bmatrix} A_{\pi,1} & * \\ 0 & A_{\pi,2} \end{bmatrix})^{-1} \times$$

$$\begin{bmatrix} S_{11}^{-1} & 0 \\ 0 & (S_{22} - S_{21} \cdot S_{11}^{-1} \cdot S_{12})^{-1} \end{bmatrix} \cdot \begin{bmatrix} B_{\zeta,1} \\ B_{\zeta,2} - S_{21} \cdot S_{11}^{-1} \cdot B_{\zeta,1} \end{bmatrix}$$

with upper triangular matrix $\tilde{A}_\pi = \begin{bmatrix} A_{\pi,1} & * \\ 0 & A_{\pi,2} \end{bmatrix}$. Equation (2.20) implies that

$$\tilde{A}_\pi^\times = \tilde{A}_\pi - \tilde{S}^{-1} \cdot \tilde{B}_\zeta \cdot \tilde{C}_\pi = \tilde{S}^{-1} \cdot (\tilde{S} \cdot \tilde{A}_\pi - \tilde{B}_\zeta \cdot \tilde{C}_\pi) = \tilde{S}^{-1} \cdot \tilde{A}_\zeta \cdot \tilde{S}.$$

From the latter equality and partition in (2.19) it follows that the matrix $\tilde{A}^\times$ has a lower triangular form. Therefore from Theorem 2.1 follows that rational matrix function $W(z)$ admits a minimal factorization $W(z) = W_1(z) \cdot W_2(z)$ with the factors (2.15) and (2.16). Furthermore, using partition in (2.19) and equating the upper left block entries in both sides of (2.20) one can obtain that

$$S_{11} \cdot A_{\pi,1} - A_{\zeta,1} \cdot S_{11} = B_{\zeta,1} \cdot C_{\pi,1} \tag{2.21}$$

The latter equality and formula (2.15) which is already proved, imply the equality $A_{\pi,1}^\times = S_{11}^{-1} \cdot A_{\zeta,1} \cdot S_{11}$ and formula (2.17). Finally, from (2.15), (2.17) and (2.21) follows, that collection of matrices τ_1 in (2.13) forms a global left null-pole triple for the first factor $W_1(z)$.

Analogously, using partition in (2.19) and equating the lower right block entries in both sides of (2.20) we deduce that

$$(S_{22}-S_{21}\cdot S_{11}^{-1}\cdot S_{12})\cdot A_{\pi,2}-A_{\zeta,2}\cdot(S_{22}-S_{21}\cdot S_{11}^{-1}\cdot S_{12})=$$
$$(B_{\zeta,2}-S_{21}\cdot S_{11}^{-1}\cdot B_{\zeta,1})\cdot(C_{\pi,2}-C_{\pi,1}\cdot S_{11}^{-1}\cdot S_{12}). \tag{2.22}$$

The latter equation and formula (2.16) which is already proved, imply the equality

$$A_{\pi,2}^{\times}=(S_{22}-S_{21}\cdot S_{11}^{-1}\cdot S_{12})^{-1}\cdot A_{\zeta,2}\cdot(S_{22}-S_{21}\cdot S_{11}^{-1}\cdot S_{12})$$

and formula (2.18). Finally, from the formulas (2.16), (2.18) and (2.22) follows that collection of matrices τ_2 in (2.14) forms a global left null-pole triple for the second factor $W_2(z)$. Theorem 2.3 is now completely proved.

□

The following corollary specifies the above theorem for rational matrix function with simple poles and zeros.

COROLLARY 2.4 *Let* $t_1,t_2,...,t_m,\ r_1,r_2,...,r_m$ *be* $2m$ *complex numbers (not necessarily distinct),* $\psi_1^{(1)},\psi_2^{(1)},\ ...,\psi_m^{(1)}\in\mathbf{C}^{p\times 1}$ *be* m *nonzero column vectors,* $\varphi_1^{(1)},\varphi_2^{(1)}$, $...,\varphi_m^{(1)}\in\mathbf{C}^{1\times p}$ *be* m *nonzero row vectors, and* S_1 *be an invertible solution of matrix Sylvester equation*

$$S_1\cdot\begin{bmatrix} t_1 & & 0\\ & \ddots & \\ 0 & & t_m\end{bmatrix}-\begin{bmatrix} r_1 & & 0\\ & \ddots & \\ 0 & & r_m\end{bmatrix}\cdot S_1=\begin{bmatrix}\varphi_1^{(1)}\\ \varphi_2^{(1)}\\ \vdots\\ \varphi_m^{(1)}\end{bmatrix}\cdot\begin{bmatrix}\psi_1^{(1)} & \psi_2^{(1)} & \cdots & \psi_m^{(1)}\end{bmatrix}. \tag{2.23}$$

If (1,1) *entry* d_1 *of* $S_1=\begin{bmatrix} d_1 & u_1\\ l_1 & S_{22}\end{bmatrix}$ *is nonzero, then the rational matrix function*

$$W_1(z)=I_p+\begin{bmatrix}\psi_1^{(1)} & \psi_2^{(1)} & \cdots & \psi_m^{(1)}\end{bmatrix}\cdot\begin{bmatrix}(z-t_1)^{-1} & & 0\\ & \ddots & \\ 0 & & (z-t_m)^{-1}\end{bmatrix}\cdot S_1^{-1}\cdot\begin{bmatrix}\varphi_1^{(1)}\\ \varphi_2^{(1)}\\ \vdots\\ \varphi_m^{(1)}\end{bmatrix}, \tag{2.24}$$

admits a minimal factorization $W_1(z)=\Theta_1(z)\cdot W_2(z)$ *with*

$$\Theta_1(z)=I_p+\frac{1}{d_1}\cdot\frac{1}{(z-t_1)}\cdot\psi_1^{(1)}\cdot\varphi_1^{(1)} \tag{2.25}$$

and

$$W_2(z)=I_p+\begin{bmatrix}\psi_2^{(2)} & \psi_3^{(2)} & \cdots & \psi_m^{(2)}\end{bmatrix}\cdot\begin{bmatrix}(z-t_2)^{-1} & & 0\\ & \ddots & \\ 0 & & (z-t_m)^{-1}\end{bmatrix}\cdot S_2^{-1}\cdot\begin{bmatrix}\varphi_2^{(2)}\\ \varphi_3^{(2)}\\ \vdots\\ \varphi_m^{(2)}\end{bmatrix}, \tag{2.26}$$

where

$$\begin{bmatrix} \psi_2^{(2)} & \psi_3^{(2)} & \cdots & \psi_m^{(2)} \end{bmatrix} = \begin{bmatrix} \psi_2^{(1)} & \psi_3^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix} - \psi_1^{(1)} \cdot u_1 \cdot \frac{1}{d_1}, \tag{2.27}$$

$$\begin{bmatrix} \varphi_2^{(2)} \\ \varphi_3^{(2)} \\ \vdots \\ \varphi_m^{(2)} \end{bmatrix} = \begin{bmatrix} \varphi_2^{(1)} \\ \varphi_3^{(1)} \\ \vdots \\ \varphi_m^{(1)} \end{bmatrix} - \frac{1}{d_1} \cdot l_1 \cdot \varphi_1^{(1)} \tag{2.28}$$

and invertible matrix

$$S_2 = S_{22} - \frac{1}{d_1} \cdot l_1 \cdot u_1 \tag{2.29}$$

satisfies the Sylvester matrix equation

$$S_2 \cdot \begin{bmatrix} t_2 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix} - \begin{bmatrix} r_2 & & 0 \\ & \ddots & \\ 0 & & r_m \end{bmatrix} \cdot S_2 = \begin{bmatrix} \varphi_2^{(2)} \\ \varphi_3^{(2)} \\ \vdots \\ \varphi_m^{(2)} \end{bmatrix} \cdot \begin{bmatrix} \psi_2^{(2)} & \psi_3^{(2)} & \cdots & \psi_m^{(2)} \end{bmatrix}. \tag{2.30}$$

PROOF. To show that Corollary 2.4 is a specification of the Theorem 2.3 one need only check that the collection of matrices

$$(C_\pi, A_\pi, A_\zeta, B_\zeta, S_1) =$$

$$(\begin{bmatrix} \psi_1^{(1)} & \psi_2^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix}, \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix}, \begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_m \end{bmatrix}, \begin{bmatrix} \varphi_1^{(1)} \\ \varphi_2^{(1)} \\ \vdots \\ \varphi_m^{(1)} \end{bmatrix}, S_1) \tag{2.31}$$

forms a global left null-pole triple of rational matrix function $W_1(z)$ in (2.24). According to [BGR, Theorem 4.3.1] it is so in the case where both matrices

$$\begin{bmatrix} C_\pi \\ C_\pi \cdot A_\pi \\ \vdots \\ C_\pi \cdot A_\pi^{m-1} \end{bmatrix}, \qquad \begin{bmatrix} B_\zeta, & A_\zeta \cdot B_\zeta, & \cdots, & A_\zeta^{m-1} \cdot B_\zeta \end{bmatrix} \tag{2.32}$$

are of the full rank. Consider the first matrix

$$\begin{bmatrix} \psi_1^{(1)} & \psi_2^{(1)} & \cdots & \psi_m^{(1)} \\ \psi_1^{(1)} \cdot t_1 & \psi_2^{(1)} \cdot t_2 & \cdots & \psi_m^{(1)} \cdot t_m \\ \vdots & \vdots & & \vdots \\ \psi_1^{(1)} \cdot t_1^{m-1} & \psi_2^{(1)} \cdot t_2^{m-1} & \cdots & \psi_m^{(1)} \cdot t_m^{m-1} \end{bmatrix} \tag{2.33}$$

in (2.32) and show that all its columns are linear independent. Let k-th entry of nonzero vector $\psi_j^{(1)}$ differ from zero. Extracting from the matrix in (2.33) the rows $k, k+m, \ldots,$

$k+(m-1)m$ one obtains the Vandermonde matrix with the nodes $t_1, t_2, ..., t_m$, multiplied from the right by a diagonal matrix with nonzero (j,j) entry. Hence the j-th column in (2.33) is linear independent from the others. Therefore, the first matrix in (2.32) is of the full rank. Similarly, one can deduce the same statement regarding the second matrix in (2.32).

□

Fast $O(n^2)$ algorithm for computing the cascade decomposition of rational matrix function with simple poles and zeros, based on the above corollary will be given in section 7. Then this algorithm will be used for solving matrix Nehari problem via decomposition of $\Theta(z)$ in (1.4). The latter rational matrix function shares zeros and poles at the same points. Note that in the simplest case, when zeros $r_1, ..., r_m$ and poles $t_1, ..., t_m$ of rational matrix function are two disjoint sets, formulas (2.27) and (2.28) can also be deduced from recursions (18) and (19) in [KS].

3. J-UNITARY FACTORIZATION

Let J be a $n \times n$ signature matrix, i.e. $J = J^*$ and $J^2 = I$, and $\triangle$ stand for open right half plane Π^+, or for open left half plane Π^-, or for open unit disk $\mathcal{D}$.

Rational matrix function $U(z)$ is called J-unitary on $\partial\triangle$ if it satisfies the equality

$$(U(z))^* \cdot J \cdot U(z) = J \qquad z \in \partial\triangle. \tag{3.1}$$

Below we observe briefly basic properties of a J-unitary on $i\mathbf{R}$ rational matrix function, and refer to [BGR] and [AG] for more information. In fact, global right pole pair and null-pole coupling matrix uniquely define a J-unitary rational on $i\mathbf{R}$ matrix function. More precisely, given global right pole pair (C_π, A_π) of J-unitary on $i\mathbf{R}$ rational matrix function $U(z)$. Then $(-A_\pi^*, -C_\pi^* \cdot J)$ is global left null pair for $U(z)$. In the global left null-pole triple $(C_\pi, A_\pi, A_\zeta, B_\zeta, S)$ the null-pole coupling matrix S appears as a (possibly nonunique) solution of Lyapunov equation

$$A_\pi^* \cdot S + S \cdot A_\pi = -C_\pi^* \cdot J \cdot C_\pi. \tag{3.2}$$

Moreover S turns out to be a Hermitian matrix. This matrix is called the *associated Hermitian matrix* (associated with global right pole pair (C_π, A_π)). Thus, J-unitary on $i\mathbf{R}$ rational matrix function with global right pole pair (C_π, A_π) and associated Hermitian matrix S has the form

$$U(z) = I - C_\pi \cdot (zI - A_\pi)^{-1} \cdot S^{-1} \cdot C_\pi^* \cdot J. \tag{3.3}$$

Moreover, any function of the form (3.3), where the matrix

$$\begin{bmatrix} C_\pi \\ C_\pi \cdot A_\pi \\ \vdots \\ C_\pi \cdot A_\pi^l \end{bmatrix}$$

is of the full rank for some $l \in \mathbf{N}$, and matrix S is invertible Hermitian solution of Lyapunov equation (3.2), is a J-unitary on $i\mathbf{R}$ rational matrix function [BGR, Theorem 6.1.4].

If in addition to (3.1), function $U(z)$ also satisfies the condition

$$(U(z))^* \cdot J \cdot U(z) \leq J \qquad z \in \triangle, \tag{3.4}$$

then $U(z)$ is referred to as a J-inner over $\triangle$ rational matrix function. According to [AG, Theorem 2.16], $U(z)$ is J-inner over Π^+ rational matrix function if and only if its associated Hermitian matrix is positive definite. Note that all the poles of J-inner over Π^+ rational matrix function are located in Π^-. Similarly, rational matrix function $U(z)$ is J-inner over Π^- if and only if its associated Hermitian matrix is negative definite. Remark that all the poles of J-inner over Π^- rational matrix function are located in Π^+.

Using the above arguments Theorem 2.3 and Corollary 2.4 can be specified for J-unitary case as follows.

THEOREM 3.1 *Let*

$$(C = \begin{bmatrix} C_{\pi,1} & C_{\pi,2} \end{bmatrix} \qquad A_\pi = \begin{bmatrix} A_{\pi,1} & * \\ 0 & A_{\pi,2} \end{bmatrix}) \qquad \text{and} \qquad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^* & S_{22} \end{bmatrix}$$

be a global right pole pair and an associated Hermitian matrix for J-unitary on $i\mathbf{R}$ rational matrix function

$$U(z) = I_p - C_\pi \cdot (zI_N - A_\pi)^{-1} \cdot S^{-1} \cdot C_\pi^* \cdot J.$$

If matrix S_{11} is invertible,then $U(z)$ admits a minimal factorization $U(z) = U_1(z) \cdot U_2(z)$ and

$$(C_{\pi,1}, A_{\pi,1}) \qquad \text{and} \qquad S_{11}$$

are the global right pole pair and the associated Hermitian matrix for J-unitary on $i\mathbf{R}$ rational matrix function

$$U_1(z) = I_p - C_{\pi,1} \cdot (zI_{N_1} - A_{\pi,1})^{-1} \cdot S_1^{-1} \cdot C_{\pi,1}^* \cdot J, \tag{3.5}$$

and

$$(C_{\pi,2} - C_{\pi,1} \cdot S_{11}^{-1} \cdot S_{12}, \;\; A_{\pi,1}) \qquad \text{and} \qquad S_{22} - S_{12}^* \cdot S_{11}^{-1} \cdot S_{12} \tag{3.6}$$

are the global right pole pair and the associated Hermitian matrix for J-unitary on $i\mathbf{R}$ rational matrix function

$$\begin{aligned} U_2(z) = I_p - (C_{\pi,2} - C_{\pi,1} \cdot S_{11}^{-1} \cdot S_{12}) \cdot (zI_{N_2} - A_{\pi,2})^{-1} \times \\ (S_{22} - S_{12}^* \cdot S_{11}^{-1} \cdot S_{12})^{-1} \cdot (C_{\pi,2} - C_{\pi,1} \cdot S_{11}^{-1} \cdot S_{12})^* \cdot J. \end{aligned} \tag{3.7}$$

If, in addition, rational matrix function $U(z)$ is a J-inner over Π^+ (or over Π^-) rational matrix function, then both factors $U_1(z)$ and $U_2(z)$ are J-inner over the same domain.

COROLLARY 3.2 *Let* $J = \begin{bmatrix} I_M & 0 \\ 0 & -I_N \end{bmatrix}$ *be a signature matrix, $t_1, ..., t_m$ be m complex numbers, $\psi_i^{(1)} \in \mathbf{C}^{(M+N)\times 1}$ be m nonzero column vectors, and*

$$U_1(z) = I_{M+N} - \begin{bmatrix} \psi_1^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix} \cdot \begin{bmatrix} (z-t_1)^{-1} & & 0 \\ & \ddots & \\ 0 & & (z-t_m)^{-1} \end{bmatrix} \cdot S_1^{-1} \cdot \begin{bmatrix} (\psi_1^{(1)})^* \\ \vdots \\ (\psi_m^{(1)})^* \end{bmatrix} \cdot J \tag{3.8}$$

be a J-unitary rational matrix function, where $S_1 \in \mathbf{C}^{m\times m}$ is invertible Hermitian matrix satisfying

$$\begin{bmatrix} t_1^* & & 0 \\ & \ddots & \\ 0 & & t_m^* \end{bmatrix} \cdot S_1 + S_1 \cdot \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix} = - \begin{bmatrix} (\psi_1^{(1)})^* \\ \vdots \\ (\psi_m^{(1)})^* \end{bmatrix} \cdot J \cdot \begin{bmatrix} \psi_1^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix}. \tag{3.9}$$

Then the following statements hold:

(i) *Let matrix S_1 be of the partitioned as $S_1 = \begin{bmatrix} d_1 & l_1^* \\ l_1 & S_{22} \end{bmatrix}$ and suppose that its (1,1) scalar entry d_1 is nonzero. Then $U_1(z)$ admits a minimal factorization $U_1(z) = \Theta_1(z) \cdot U_2(z)$ with J-unitary factors*

$$\Theta_1(z) = I_{M+N} - \frac{1}{d_1} \cdot \frac{1}{z-t_1} \cdot \psi_1^{(1)} \cdot (\psi_1^{(1)})^* \cdot J \tag{3.10}$$

and

$$U_2(z) = I_{M+N} - \begin{bmatrix} \psi_2^{(2)} & \cdots & \psi_m^{(2)} \end{bmatrix} \cdot \begin{bmatrix} (z-t_2)^{-1} & & 0 \\ & \ddots & \\ 0 & & (z-t_m)^{-1} \end{bmatrix} \cdot S_2^{-1} \cdot \begin{bmatrix} (\psi_2^{(2)})^* \\ \vdots \\ (\psi_m^{(2)})^* \end{bmatrix} \cdot J, \tag{3.11}$$

where

$$\begin{bmatrix} \psi_2^{(2)} & \cdots & \psi_m^{(2)} \end{bmatrix} = \begin{bmatrix} \psi_2^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix} - \frac{1}{d_1} \cdot \psi_1^{(1)} \cdot l_1^* \tag{3.12}$$

and invertible Hermitian matrix

$$S_2 = S_{22} - \frac{1}{d_1} \cdot l_1 \cdot l_1^* \tag{3.13}$$

satisfies

$$\begin{bmatrix} t_2^* & & 0 \\ & \ddots & \\ 0 & & t_m^* \end{bmatrix} \cdot S_2 + S_2 \cdot \begin{bmatrix} t_2 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix} = - \begin{bmatrix} (\psi_2^{(2)})^* \\ \vdots \\ (\psi_m^{(2)})^* \end{bmatrix} \cdot J \cdot \begin{bmatrix} \psi_2^{(2)} & \cdots & \psi_m^{(2)} \end{bmatrix}. \tag{3.14}$$

(ii) *Let S_1 be of the form $S_1 = \begin{bmatrix} 0 & d_1^* & l_1^* \\ d_1 & 0 & l_2^* \\ l_1 & l_2 & S_{22} \end{bmatrix}$ and suppose that its (2,1) scalar entry d_1 is nonzero. Then $U_1(z)$ admits a minimal factorization $U_1(z) = \Theta_1(z) \cdot U_2(z)$ with J-unitary factors*

$$\Theta_1(z) = I_{M+N} - \frac{1}{d_1} \cdot \frac{1}{z-t_1} \cdot \psi_1^{(1)} \cdot (\psi_2^{(1)})^* \cdot J - \frac{1}{d_1^*} \cdot \frac{1}{z-t_2} \cdot \psi_2^{(1)} \cdot (\psi_1^{(1)})^* \cdot J \tag{3.15}$$

and

$$U_2(z) = I_{M+N} - \left[\begin{array}{ccc} \psi_3^{(2)} & \cdots & \psi_m^{(2)} \end{array}\right] \cdot \left[\begin{array}{ccc} (z-t_3)^{-1} & & 0 \\ & \ddots & \\ 0 & & (z-t_m)^{-1} \end{array}\right] \cdot S_2^{-1} \cdot \left[\begin{array}{c} (\psi_3^{(2)})^* \\ \vdots \\ (\psi_m^{(2)})^* \end{array}\right] \cdot J, \tag{3.16}$$

where

$$\left[\begin{array}{ccc} \psi_3^{(2)} & \cdots & \psi_m^{(2)} \end{array}\right] = \left[\begin{array}{ccc} \psi_3^{(1)} & \cdots & \psi_m^{(1)} \end{array}\right] - \frac{1}{d_1} \cdot \psi_1^{(1)} \cdot l_2^* - \frac{1}{d_1^*} \cdot \psi_2^{(1)} \cdot l_1^* \tag{3.17}$$

and invertible Hermitian matrix

$$S_2 = S_{22} - \frac{1}{d_1} \cdot l_1 \cdot l_2^* - \frac{1}{d_1^*} \cdot l_2 \cdot l_1^* \tag{3.18}$$

satisfies

$$\left[\begin{array}{ccc} t_3^* & & 0 \\ & \ddots & \\ 0 & & t_m^* \end{array}\right] \cdot S_2 + S_2 \cdot \left[\begin{array}{ccc} t_3 & & 0 \\ & \ddots & \\ 0 & & t_m \end{array}\right] = - \left[\begin{array}{c} (\psi_3^{(2)})^* \\ \vdots \\ (\psi_m^{(2)})^* \end{array}\right] \cdot J \cdot \left[\begin{array}{ccc} \psi_3^{(2)} & \cdots & \psi_m^{(2)} \end{array}\right]. \tag{3.19}$$

Corollary 3.2 will serve the basis for the design of the fast $O(n^2)$ algorithms for computing the cascade decomposition of J-inner (Algorithm 5.1) and J-unitary (Algorithm 6.1) rational matrix function with simple zeros and poles.

4. J-INNER FACTORIZATION FOR $\Theta(z)$

Let

$$C_\pi = \left[\begin{array}{cc} C & 0 \\ 0 & B^* \end{array}\right], \quad A_\pi = \left[\begin{array}{cc} A & 0 \\ 0 & -A^* \end{array}\right], \quad J = \left[\begin{array}{cc} I_M & 0 \\ 0 & -I_N \end{array}\right],$$

where corresponding block entries are defined in (1.5). With the same arguments as in the proof of Corollary 2.4 one can deduce that the matrix

$$\left[\begin{array}{c} C_\pi \\ C_\pi \cdot A_\pi \\ \vdots \\ C_\pi \cdot A_\pi^{2n} \end{array}\right]$$

is of the full rank. Furthermore, the matrix Λ in (1.5) appears as a (nonunique) Hermitian solution of Lyapunov equation (3.2). Therefore, $\Theta(z)$ in (1.4) has the form (3.3), and hence it is a J-unitary on $i\mathbf{R}$ rational matrix function with global right pole pair (C_π, A_π) and associated Hermitian matrix Λ. Since the latter matrix is not a definite one, hence $\Theta(z)$ is not a J-inner rational matrix function. However, the following statement holds.

THEOREM 4.1 *Rational matrix function $\Theta(z)$ in (1.4) admits a minimal factorization*

$$\Theta(z) = \Phi(z) \cdot \Psi(z), \tag{4.1}$$

with J-inner over Π^+ rational matrix function

$$\Phi(z) = I_{M+N} + \begin{bmatrix} C \\ 0 \end{bmatrix} \cdot (zI_n - A)^{-1} \cdot Q^{-1} \cdot \begin{bmatrix} -C^* & 0 \end{bmatrix}, \tag{4.2}$$

and J-inner over Π^- rational matrix function

$$\Psi(z) = I_{M+N} + \begin{bmatrix} -C \cdot Q^{-1} \\ B^* \end{bmatrix} \cdot (zI_n + A^*)^{-1} \cdot (P - Q^{-1})^{-1} \cdot \begin{bmatrix} Q^{-1} \cdot C^*, & B \end{bmatrix}. \tag{4.3}$$

PROOF. As it was shown in [BGR] (see also [BGR1]), the upper left block Q in the associated Hermitian matrix Λ is a positive definite matrix and hence invertible. Then by Theorem 2.3, rational matrix function $\Theta(z)$ admits a minimal factorization (4.1) with J-unitary factors $\Phi(z)$ and $\Psi(z)$. Since the associated Hermitian matrix Q of the first factor $\Phi(z)$ is a positive definite matrix, hence $\Phi(z)$ is J-inner over Π^+. Furthermore, maximal eigenvalue $\lambda_1(PQ)$ of PQ is less than 1 and hence all the eigenvalues of the matrix $PQ-I$ are strictly negative. The same statement holds for similar Hermitian matrix $Q^{\frac{1}{2}}(PQ-I)Q^{-\frac{1}{2}} = Q^{\frac{1}{2}}PQ^{\frac{1}{2}} - I$ (here $Q^{\frac{1}{2}}$ is square root of positive definite matrix Q). From here follows that the congruent matrix $Q^{-\frac{1}{2}}(Q^{\frac{1}{2}}PQ^{\frac{1}{2}} - I)Q^{-\frac{1}{2}} = P - Q^{-1}$ is strictly negative. The latter matrix appears as an associated Hermitian matrix for the second factor $\Psi(z)$. Therefore $\Psi(z)$ is J-inner over Π^- rational matrix function.

□

Observe that function $\Phi(z)$ in (4.1) has the form

$$\Phi(z) = \begin{bmatrix} \Phi_{11}(z) & 0 \\ 0 & I_N \end{bmatrix}, \qquad \text{where} \qquad \Phi_{11}(z) = I_M - C \cdot (zI_n - A) \cdot Q^{-1} \cdot C^*,$$

and $\Phi_{11}(z)$ is inner over Π^+ rational matrix function.

As it is well known [P] (see also [AG], [GVKDM] and [LAK]), each J-inner rational matrix function can be decomposed into the cascade of first degree factors. This fact and Theorem 4.1 suggest an algorithm for matrix Nehari problem via computing the cascade decomposition for J-inner factors $\Phi(z)$ and $\Psi(z)$.

5. ALGORITHM FOR MATRIX NEHARI PROBLEM VIA J-INNER CASCADE DECOMPOSITION

Let J be a signature matrix and $U_1(z)$ be a J-inner over either Π^+ or Π^- rational matrix function with global right pole pair

$$(C_\pi = \begin{bmatrix} \psi_1^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix}, A_\pi = \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix}) \tag{5.1}$$

and definite associated Hermitian matrix S_1. Since in this case all the poles $t_1, ..., t_m$ of $U_1(z)$ are located from the one side of imaginary axis $i\mathbf{R}$, hence the associated Hermitian matrix S_1 is uniquely recovered from the equation

$$A_\pi^* \cdot S_1 + S_1 \cdot A_\pi = -C_\pi^* \cdot J \cdot C_\pi \tag{5.2}$$

as

$$S_1 = \left[\; -\frac{(\psi_i^{(1)})^* \cdot J \cdot \psi_j^{(1)}}{t_i^* + t_j} \; \right]_{1 \le i,j \le n}. \tag{5.3}$$

Thus, J-inner function $U_1(z)$ is completely defined by its global right pole pair in (5.1).

A fast algorithm for cascade decomposition of J-inner function $U_1(z)$ can be designed on the basis of the part (i) of Corollary 3.2. This algorithms starts with the data contained in the global right pole pair (5.1) of $U_1(z)$ and it consists of m recursive steps. At first step one has to compute, using (5.3), the entries of the first column of the matrix $S_1 = \begin{bmatrix} d_1 & l_1^* \\ l_1 & S_{22} \end{bmatrix}$. Since S_1 is a definite matrix, hence its (1,1) entry d_1 is nonzero. Therefore one can write down the first cascade factor $\Theta(z)$ by (3.10) and to compute by (3.12) the entries of the first matrix in the global right pole pair

$$(\begin{bmatrix} \psi_2^{(2)} & \cdots & \psi_m^{(2)} \end{bmatrix}, \begin{bmatrix} t_2 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix})$$

of the J-inner quotient $U_2(z)$ in (3.11). Afterwards, one has to proceed with $U_2(z)$ exactly in the same way and to compute the cascade decomposition for $U_1(z)$ in m recursive steps.

The described algorithm starts with the associated matrix S_1 of J-inner function $U_1(z)$, and at its first step it computes by (5.3) the entries of the first column of the matrix S_1. From the Schur complementation formula

$$S_1 = \begin{bmatrix} d_1 & l_1^* \\ l_1 & S_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{d_1} \cdot l_1 & I \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 \\ 0 & S_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{d_1} \cdot l_1^* \\ 0 & I \end{bmatrix}, \tag{5.4}$$

where $S_2 = S_{22} - \frac{1}{d_1} \cdot l_1 \cdot l_1^*$, it is clear, that at its first step this algorithm computes in fact the (1,1) entry d_1 of diagonal matrix D and the first column $\begin{bmatrix} 1 \\ \frac{1}{d_1} \cdot l_1 \end{bmatrix}$ of lower triangular matrix L and in LDL^* factorization for associated Hermitian matrix S_1. Moreover, since this algorithm proceeds similarly with the associated matrix S_2 of J-inner quotient $U_2(z)$, hence it finally computes above LDL^* factorization for S_1.

Let $U_1(z)$ be a J-inner rational matrix function of the form (3.8) with global right pole pair (C_π, A_π) as in (5.1) and with definite associated matrix S_1 as in (5.3). It is easy to see that interchanging in (5.1) and in (5.3) the roles of t_1 and t_q and the roles of $\psi_1^{(1)}$ and $\psi_q^{(1)}$ has no influence on rational matrix function $U_1(z)$ in (3.8). Moreover, such a reordering is equivalent to passing from the global left null-pole triple $(C_\pi, A_\pi, -A_\pi^*, -C_\pi^* \cdot J, S_1)$ of $U_1(z)$ to the similar global left null-pole triple $(C_\pi \cdot P_1^T, \;\; P_1 \cdot A_\pi \cdot P_1^T, \;\; -P_1 \cdot A_\pi^* \cdot P_1^T, \;\; -P_1 \cdot C_\pi^* \cdot J, \;\; P_1 \cdot S_1 \cdot P_1^T)$. Here P_1 is a permutation of the first and the q-th entries. Such reordering allows different versions $P_1 \cdot S_1 \cdot P_1^T$ of associated Hermitian matrix, and therefore different selections of its (1,1) entry d_1. This freedom means that one can choose the pole for the first cascade factor $\Theta_1(z)$ in a nonunique way. Since from a numerical point of view it is desirable to avoid large vectors $\psi_i^{(2)}$ computed by (3.12), hence it is reasonable to choose the pivot d_1 with possibly large magnitude. Therefore, at first step of the algorithm one has to find, say in the (q,q) position, the entry with maximal magnitude on the main diagonal of matrix S_1 and then to swap vectors $\psi_1^{(1)}$ and $\psi_q^{(1)}$ and points t_1 and t_q. With this modification, the algorithm ends up with a cascade decomposition of J-inner rational matrix function $U_1(z)$ and triangular factorization

$$P \cdot S_1 \cdot P^T = L \cdot D \cdot L^* \tag{5.5}$$

of permuted version of matrix S_1 in (5.3). Here $P = P_m \cdot ... \cdot P_1$, where P_k is a permutation of the k-th step of the recursion. Moreover, this choice of permutation matrices P_k completely corresponds to *symmetric pivoting* [GL] in LDL^* factorization of positive definite matrix S_1.

Here is the record of the algorithm.

ALGORITHM 5.1 Cascade decomposition of J-inner rational matrix function

Input: *The* $J = \begin{bmatrix} I_M & 0 \\ 0 & -I_N \end{bmatrix}$*-inner over either* Π^+ *or* Π^- *rational matrix function* $U_1(z)$ *in* (3.8)*, given by global right pole pair*
$$\left(\begin{bmatrix} \psi_1^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix}, \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix}\right).$$

Output: **(i)** *The decomposition* $U_1(z) = \Theta_1(z) \cdot \Theta_2(z) \cdot ... \cdot \Theta_m(z)$*, where*
$\Theta_i(z) = I_{M+N} - \frac{1}{d_i} \cdot \frac{1}{z-t_i} \cdot \psi_i^{(i)} \cdot (\psi_i^{(i)})^* \cdot J.$
(ii) *Triangular factorization* $S_1 = P^T \cdot L \cdot D \cdot L^* \cdot P$ *of permuted version of associated Hermitian matrix* S_1*, where* P *is permutation, matrix* $L = \begin{bmatrix} l_{ij} \end{bmatrix}_{i,j=1}^m$ *is lower triangular and* $D = \mathrm{diag}(d_1, d_2, ..., d_m)$.

Complexity: $3(M+N)m^2$ *operations.*

Initialization: *Set* $L = \begin{bmatrix} l_{ij} \end{bmatrix}_{i,j=1}^m$ *to be zero matrix and* P *to be identity matrix.*

FOR $i = 1 : m$
  FOR $k = i : m$
    $s_{kk}^{(i)} = -\frac{(\psi_k^{(i)})^* \cdot J \cdot \psi_k^{(i)}}{t_k^* + t_k}$
  ENDFOR
  FIND $i \le q \le m$ so that $|s_{qq}^{(i)}| = \max_{i \le k \le m} |s_{kk}^{(i)}|$
  $d_i = s_{qq}^{(i)}$
  SWAP t_i and t_q
  SWAP $\psi_i^{(i)}$ and $\psi_q^{(i)}$
  SWAP i-th and q-th rows in L
  SWAP i-th and q-th rows in P
  $l_{ii} = 1$
  FOR $k = i+1 : m$
    $s_{ki}^{(i)} = -\frac{(\psi_k^{(i)})^* \cdot J \cdot \psi_i^{(i)}}{t_k^* + t_i}$

$$l_{ki} = s_{ki}^{(i)} \cdot \frac{1}{d_i}$$
$$\psi_k^{(i+1)} = \psi_k^{(i)} - \psi_i^{(i)} \cdot l_{ki}^*$$

ENDFOR

ENDFOR

The above algorithm and Theorem 4.1 suggest the following algorithm for matrix Nehari problem. At its first stage, the cascade decomposition

$$\Theta(z) = \Theta_1(z) \cdot \Theta_2(z) \cdot \ldots \cdot \Theta_{2n}(z) \tag{5.6}$$

is computed for rational matrix function $\Theta(z)$ in (1.4), using the factorization (4.1) into the product of two J-inner factors $\Phi(z)$ and $\Psi(z)$. The entries of the global right pole pair $(\begin{bmatrix} C \\ 0 \end{bmatrix}, A)$ of the first J-inner factor $\Phi(z)$ are defined as in (1.5) by the data of matrix Nehari problem. Applying Algorithm 5.1 for $\Phi(z)$, one will compute the cascade decomposition $\Phi(z) = \Theta_1(z) \cdot \Theta_2(z) \cdot \ldots \cdot \Theta_n(z)$ into the product of J-inner over Π^+ first degree factors and simultaneously the triangular factorization

$$Q = P^T \cdot L \cdot D \cdot L^* \cdot P \tag{5.7}$$

for permuted version of the associated Hermitian matrix Q. To apply Algorithm 5.1 for computing the cascade decomposition for the second J-inner factor $\Psi(z)$ one has to compute the entries of its global right pole pair $(\begin{bmatrix} -C \cdot Q^{-1} \\ B^* \end{bmatrix}, -A^*)$. Using triangular decomposition (5.7) of Q it can be done by the cost $O(n^2)$ operations via forward and back substitution [GL]. Applying then Algorithm 5.1 with these data, one will compute the cascade decomposition $\Psi(z) = \Theta_{n+1}(z) \cdot \Theta_{n+2}(z) \cdot \ldots \cdot \Theta_{2n}(z)$ into the product of J-inner over Π^- first degree factors.

At the second stage of the proposed algorithm one has to compute using (5.6)

$$\begin{bmatrix} \Theta_{12}(z_0) \\ \Theta_{22}(z_0) \end{bmatrix} = \Theta_1(z_0) \cdot \Theta_2(z_0) \cdot \ldots \cdot \Theta_{2n}(z_0) \cdot \begin{bmatrix} 0 \\ I \end{bmatrix}.$$

Then using (1.3) with $G(z) = 0$ compute the value of $F(z_0) = \Theta_{12}(z_0) \cdot \Theta_{22}(z_0)^{-1}$ at given point z_0. It can be done in $O(n)$ arithmetic operations. Here is the record of the algorithm.

ALGORITHM 5.2 Matrix Nehari problem via J-inner cascade decomposition

Input: *Data :* $z_1, z_2, \ldots, z_n \in \Pi^-$, $\gamma_1, \gamma_2, \ldots, \gamma_n \in \mathbf{C}^{M\times 1}$, $w_1, w_2, \ldots, w_n \in \mathbf{C}^{1\times N}$. *Point* z_0.

Output: *Value of* $F(z)$ *in* (1.3) *with* $G(z) = 0$ *at the point* z_0.

Complexity: $3(2M+N)n^2$ *operations.*

1. *Compute using Algorithm* 5.1 *:*

1.1. *Cascade decomposition* $\Phi_{11}(z) = \Phi_1(z) \cdot ... \cdot \Phi_n(z)$ *with*

$$\Phi_k(z) = I_M - \frac{1}{d_k} \cdot \frac{1}{z - t_k} \cdot \chi_k \cdot (\chi_k)^*$$

for for the inner over Π^+ *rational matrix function* $\Phi_{11}(z)$, *given by global right pole pair*

$$(\begin{bmatrix} \gamma_1 & \cdots & \gamma_m \end{bmatrix}, \begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}).$$

1.2. *Triangular factorization*

$$Q = P^T \cdot L \cdot D \cdot L^* \cdot P \tag{5.8}$$

for associated Hermitian matrix Q.

2. *Set* $\psi_k = \begin{bmatrix} \chi_k \\ 0 \end{bmatrix} \in \mathbf{C}^{(M+N)\times 1}$.

3. *Using* (5.8) *solve* M *linear systems*

$$Q \cdot X = -C^* \qquad (X \in \mathbf{C}^{n\times M})$$

with matrix Q *by forward and back substitution* [GL].

4. *Compute using Algorithm* 5.1 *the cascade decomposition* $\Psi(z) = \Theta_{n+1}(z) \cdot ... \cdot \Theta_{2n}(z)$ *with*

$$\Theta_k(z) = I_{M+N} - \frac{1}{d_k} \cdot \frac{1}{z - t_k} \cdot \psi_k \cdot (\psi_k)^* \cdot J$$

for J*-inner over* Π^- *rational matrix function* $\Psi(z)$ *given by global right pole pair*

$$(\begin{bmatrix} & X^* & \\ w_1^* & \cdots & w_n^* \end{bmatrix}, \begin{bmatrix} -z_1^* & & 0 \\ & \ddots & \\ 0 & & -z_n^* \end{bmatrix}).$$

5. *Set* $\begin{bmatrix} \Theta_{12}^{(0)} \\ \Theta_{22}^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ I_N \end{bmatrix} \in \mathbf{C}^{(M+N)\times N}$.

6. *For* $i = 1, 2, ..., 2n$ *compute* $\begin{bmatrix} \Theta_{12}^{(i)} \\ \Theta_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix} - \frac{1}{d_i} \cdot \frac{1}{z_0 - t_i} \cdot \psi_i \cdot (\psi_i)^* \cdot J \cdot \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix}$.

7. *Compute* $F(z_0) = \Theta_{12}^{(2n)} \cdot (\Theta_{22}^{(2n)})^{-1}$ *using one of standard inversion algorithms.*

6. ALGORITHM FOR MATRIX NEHARI PROBLEM VIA J-UNITARY CASCADE DECOMPOSITION

In this section we propose another algorithm for matrix Nehari problem, also based on computing the cascade decomposition for $\Theta(z)$ in (1.4). This algorithm ignores the J-inner factorization in (4.1) and it is based on direct computing the cascade decomposition of J-unitary on $i\mathbf{R}$ rational matrix function with simple zeros and poles. The difference between the J-inner case and the situation here is that not every J-unitary rational matrix function admits a decomposition into a cascade of the first degree factors. However, according to [AD] (see also [AG, Theorem 2.14]), any J-unitary on $i\mathbf{R}$ rational matrix function $U_1(z)$ with simple poles can be decomposed into the cascade

$$U_1(z) = \Theta_1(z) \cdot \Theta_1(z) \cdot \ldots \cdot \Theta_r(z), \tag{6.1}$$

where each factor $\Theta_i(z)$ is of McMillan degree δ_i less than or equal to 2.

Let $U_1(z)$ be a J-unitary on $i\mathbf{R}$ rational matrix function in (3.8). As it was explained in the section 3, function $U_1(z)$ is comletely determined by its global right pole pair

$$(C_\pi = \begin{bmatrix} \psi_1^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix}, A_\pi = \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix}) \tag{6.2}$$

and by associated Hermitian matrix $S_1 = \begin{bmatrix} s_{ij}^{(1)} \end{bmatrix}_{1 \leq i,j \leq m}$. Observe that the above data contain duplicate information. Indeed, if $t_i \neq -t_j^*$ then (i,j) entry of matrix S_1 can be recovered from the equation (3.9) as

$$s_{ij}^{(1)} = -\frac{(\psi_i^{(1)})^* \cdot J \cdot \psi_j^{(1)}}{t_i^* + t_j} \qquad (t_i \neq -t_j^*). \tag{6.3}$$

In case $t_i = -t_j^*$ we will refer to the (i,j) entry of S_1 as a (i,j) *coupling number* ρ_{ij}. Thus, J-inner rational matrix function in (3.8) is completely described by its global right pole pair and by coupling numbers.

The algorithm proposed in this section starts with $U_1(z)$ as in (3.8), given by the points $t_1, t_2, \ldots, t_m$, vectors $\psi_1^{(1)}, \psi_1^{(1)}, \ldots, \psi_m^{(1)}$ and by coupling array $\begin{bmatrix} \sigma_1 & \cdots & \sigma_m \\ \rho_{1,\sigma_1}^{(1)} & \cdots & \rho_{m,\sigma_m}^{(1)} \end{bmatrix}$. Here $\sigma_i = 0$, $\rho_{i,\sigma_i}^{(1)} = 0$ if $t_i \neq -t_j^*$ for $1 \leq j \leq m$; and in case $t_i = -t_j^*$ for some $1 \leq j \leq m$, then $\sigma_i = j$ and $\rho_{i,\sigma_i}^{(1)}$ is the (i,j) coupling number.

The algorithm is based on both statements in Corollary 3.2 and consists of at most m recursive steps. The computations of the first step of the recursion are organized in a different ways for each of the following two cases.

Case 1. Nonzero main diagonal. As a first step one has to compute, using (6.3), the entries on the main diagonal of $S_1$and to determine the position (q,q) of the one with maximal magnitude. In the case under consideration this (q,q) entry is nonzero. Then one has to replace it in (1,1) position, i.e. to swap points t_1 and t_q, vectors $\psi_1^{(1)}$ and $\psi_q^{(1)}$ and also to attribute the coupling numbers corresponding to t_1 and t_q with new indexes. Then one has to compute by (6.3) the entries of the first column of S_1, which are not coupling numbers (the latter are given as a part of input data), and to write down by (3.10) the first cascade factor $\Theta_1(z)$. Then one has to compute for the quotient $U_2(z)$ the same set of data as for initial rational matrix function $U_1(z)$. It should be organized as follows. Vectors $\psi_2^{(2)}, \psi_3^{(3)}, ..., \psi_m^{(2)}$ are computed by (3.12) and new coupling numbers $\rho_{i,\sigma_i}^{(2)}$ are computed via (3.13).

It is easy to see that this first step of the recursion corresponds to the following factorization of permuted version of associated Hermitian matrix

$$P_1 \cdot S_1 \cdot P_1^T = \begin{bmatrix} d_1 & l_1^* \\ l_1 & S_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{d_1} \cdot l_1 & I \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 \\ 0 & S_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{d_1} \cdot l_1^* \\ 0 & I \end{bmatrix}, \tag{6.4}$$

where P_1 is the permutation of the first and the q-th entries, and $S_2 = S_{22} - \frac{1}{d_1} \cdot l_1 \cdot l_1^*$ is a Schur complement of upper left entry d_1 in a matrix S_1.

Case 2. Zero main diagonal. Now let us turn to the case, when the attempt to find a nonzero entry on the main diagonal of S_1 failed. In the latter situation the computations are based on the part (ii) of Corollary 3.2 and are organized as follows. If $t_1 = -t_j^*$ for some j, then $(j,1)$ entry of S_1 is given as $(j,1)$ coupling number ρ_{j1}. The other entries of the first column of S_1 are computed by (6.3). Since S_1 is an invertible matrix, hence there is a nonzero entry in its first column. Suppose that this entry occupies the $(q,1)$ position. Then one has to replace this entry in $(2,1)$ position by swapping the points t_2 and t_q, vectors $\psi_2^{(1)}$ and $\psi_q^{(1)}$ and attributing respectively the coupling numbers (if any), corresponding to t_2 and t_q. After such manipulation a new version of matrix S_1 has the form as in the statement (ii) of Corollary 3.2. Then one has to write down the first cascade factor $\Theta_1(z)$ by (3.15) and to compute new vectors $\psi_3^{(2)}, \psi_4^{(2)}, ..., \psi_m^{(2)}$ using (3.17). New coupling numbers $\rho_{i,\sigma_i}^{(2)}$ are computed via (3.18). Obviously, this step of the recursion corresponds to the following factorization of permuted version of associated Hermitian matrix

$$P_1 \cdot S_1 \cdot P_1^T = \begin{bmatrix} 0 & d_1^* & l_1^* \\ d_1 & 0 & l_2^* \\ l_1 & l_2 & S_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \frac{1}{d_1^*} \cdot l_2 & \frac{1}{d_1} \cdot l_1 & I \end{bmatrix} \cdot \begin{bmatrix} 0 & d_1^* & 0 \\ d_1 & 0 & 0 \\ 0 & 0 & S_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & \frac{1}{d_1} \cdot l_2^* \\ 0 & 1 & \frac{1}{d_1^*} \cdot l_1^* \\ 0 & 0 & I \end{bmatrix}, \tag{6.5}$$

where $S_2 = S_{22} - \frac{1}{d_1} \cdot l_1 \cdot l_2^* - \frac{1}{d_1^*} \cdot l_2 \cdot l_1^*$ is a Schur complement of upper left block entry $\begin{bmatrix} 0 & d_1^* \\ d_1 & 0 \end{bmatrix}$ in a matrix S_1.

Furthermore, after the first step, described above, the quotient $U_2(z)$ in (3.16) is represented by the same set of data as input rational matrix function. One has to proceed with the recursion for $U_2(z)$ similarly, and to compute finally the cascade decomposition (6.1) in at most m recursive steps. From the factorizations (6.4) and (6.5) follows that the described algorithm computes simultaneously the factorization

$$P \cdot S_1 \cdot P^T = L \cdot D \cdot L^*$$

where L is a lower triangular matrix and $D = \text{diag}(D_1, ..., D_r)$, where each D_i is 1×1 or 2×2 block.

Here is the record of the algorithm.

ALGORITHM 6.1 Cascade decomposition of J-unitary rational matrix function.

<u>Input</u> *The* $J = \begin{bmatrix} I_M & 0 \\ 0 & -I_N \end{bmatrix}$*-unitary on* $i\mathbf{R}$ *rational matrix function* $U_1(z)$ *in* (3.8)*, given by global right pole pair* $\left(\begin{bmatrix} \psi_1^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix}, \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix}\right)$*,* *and coupling array* $\begin{bmatrix} \sigma_1 & \cdots & \sigma_m \\ \rho_1^{(1)} & \cdots & \rho_m^{(1)} \end{bmatrix}$.

<u>Output</u> **(i)** *The numbers* $\delta_1, ..., \delta_r$ $(r \le m)$*, where each* δ_i *equals to* 1 *or to* 2*. The cascade decomposition* $U_1(z) = \Theta_1(z) \cdot \Theta_2(z) \cdot ... \cdot \Theta_r(z)$*. If* $\delta_i = 1$ *then* $\Theta_i(z) = I_{M+N} - \frac{1}{d_i} \cdot \frac{1}{z-t_i} \cdot \psi_i^{(i)} \cdot (\psi_i^{(i)})^* \cdot J$*. If* $\delta_i = 2$ *then* $\Theta_i(z) = I_{M+N} - \frac{1}{d_i} \cdot \frac{1}{z-t_i} \cdot \psi_i^{(i)} \cdot (\psi_{i+1}^{(i)})^* \cdot J - \frac{1}{d_i^*} \cdot \frac{1}{z-t_{i+1}} \cdot \psi_{i+1}^{(i)} \cdot (\psi_i^{(i)})^* \cdot J$.
(ii) *Triangular factorization* $S_1 = P^T \cdot L \cdot D \cdot L^* \cdot P$ *of permuted version of associated Hermitian matrix* S_1*, where* P *is permutation, matrix* $L = \begin{bmatrix} l_{ij} \end{bmatrix}_{i,j=1}^m$ *is lower triangular, and* $D = \operatorname{diag}(D_1, D_2, ..., D_r)$ *is block diagonal matrix. If* $\delta_i = 1$ *then* $D_i = d_i$*, and in case* $\delta_i = 2$ *then* $D_i = \begin{bmatrix} 0 & d_i^* \\ d_i & 0 \end{bmatrix}$.

<u>Complexity</u> $3(M+N)m^2$ *operations.*

<u>Initialization</u> *Set* $L = \begin{bmatrix} l_{ij} \end{bmatrix}_{i,j=1}^m$ *to be zero matrix and* P *to be identity matrix.*

i = 1
START :
 IF $i > m$ THEN
  STOP
 ENDIF
 FOR $k = i : m$
  IF $\sigma_k = k$ THEN
   $s_{kk}^{(i)} = \rho_k^{(i)}$
  ELSE
   $s_{kk}^{(i)} = -\frac{(\psi_k^{(i)})^* \cdot J \cdot \psi_k^{(i)}}{t_k^* + t_k}$
  ENDIF
 ENDFOR
 FIND $i \le q \le m$ so that $|s_{qq}^{(i)}| = max_{i \le k \le m} |s_{kk}^{(i)}|$
 CASE $s_{qq}^{(i)} \ne 0$
  $\delta_i = 1$
  $d_i = s_{qq}^{(i)}$
  SWAP t_i and t_q
  SWAP $\psi_i^{(i)}$ and $\psi_q^{(i)}$
  $\sigma_{\sigma_i} = q$
  $\sigma_{\sigma_q} = i$
  SWAP σ_i and σ_q
  SWAP $\rho_i^{(i)}$ and $\rho_q^{(i)}$
  SWAP i-th and q-th rows in L
  SWAP i-th and q-th rows in P

$l_{ii} = 1$
FOR $k = i+1 : m$
IF $\sigma_k = i$ THEN
$s_{ki}^{(i)} = \rho_k^{(i)}$
ELSE
$s_{ki}^{(i)} = -\frac{(\psi_k^{(i)})^* \cdot J \cdot \psi_i^{(i)}}{t_k^* + t_i}$
ENDIF
$l_{ki} = s_{ki}^{(i)} \cdot \frac{1}{d_i}$
$\psi_k^{(i+1)} = \psi_k^{(i)} - \psi_i^{(i)} \cdot l_{ki}^*$
IF $k \leq \sigma_k$ THEN
$\rho_k^{(i+1)} = \rho_k^{(i)} - \frac{1}{d_i} \cdot s_{ki} \cdot (s_{\sigma_k,i})^*$
ELSE
$\rho_k^{(i+1)} = (\rho_{\sigma_k}^{(i+1)})^*$
ENDIF
ENDFOR
$i = i+1$
GO TO START
ENDCASE
CASE $s_{qq}^{(i)} = 0$
$\delta_i = 2$
FOR $k = i+1 : m$
IF $\sigma_k = i$ THEN
$s_{ki}^{(i)} = \rho_k^{(i)}$
ELSE
$s_{ki}^{(i)} = -\frac{(\psi_k^{(i)})^* \cdot J \cdot \psi_i^{(i)}}{t_k^* + t_i}$
ENDIF
ENDFOR
FIND $i+1 \leq q \leq m$ so that $|s_{qi}^{(i)}| = max_{i+1 \leq k \leq m} |s_{ki}^{(i)}|$
$d_i = s_{qi}^{(i)}$
SWAP t_{i+1} and t_q
SWAP $\psi_{i+1}^{(i)}$ and $\psi_q^{(i)}$
$\sigma_{\sigma_{i+1}} = q$
$\sigma_{\sigma_q} = i+1$
SWAP σ_{i+1} and σ_q
SWAP ρ_{i+1} and ρ_q
SWAP $(i+1)$-th and q-th rows in L
SWAP $(i+1)$-th and q-th rows in P
FOR $k = i+2 : m$
IF $\sigma_k = i+1$ THEN
$s_{k,i+1}^{(i)} = \rho_k^{(i)}$
ELSE
$s_{k,i+1}^{(i)} = -\frac{(\psi_k^{(i)})^* \cdot J \cdot \psi_{i+1}^{(i)}}{t_k^* + t_{i+1}}$
ENDIF
ENDFOR
$l_{ii} = 1$
$l_{i+1,i} = 0$
$l_{i+1,i+1} = 1$
FOR $k = i+2 : m$
$l_{k,i} = s_{k,i+1}^{(i)} \cdot \frac{1}{d_i^*}$

$l_{k,i+1} = s_{k,i}^{(i)} \cdot \frac{1}{d_i}$
$\psi_k^{(i+1)} = \psi_k^{(i)} - \psi_i^{(i)} \cdot l_{ki}^* - \psi_{i+1}^{(i)} \cdot l_{k,i+1}^*$
IF $k \leq \sigma_k$ THEN
$\rho_k^{(i+1)} = \rho_k^{(i)} - \frac{1}{d_i} \cdot s_{ki}^{(i)} \cdot (s_{\sigma_k,i+1})^* - \frac{1}{d_i^*} \cdot s_{k,i+1} \cdot (s_{\sigma_k,i})^*$
ELSE
$\rho_k^{(i+1)} = (\rho_{\sigma_k}^{(i+1)})^*$
ENDIF
ENDFOR
$i = i+2$
GO TO START
ENDCASE

Algorithm 6.1 suggests computing the cascade decomposition for J-unitary rational matrix function $\Theta(z)$ in (1.4) and then solving the matrix Nehari problem by making use of the formula (1.3) with $G(z) = 0$. Here is the record of the algorithm.

ALGORITHM 6.2 Matrix Nehari problem via J-unitary cascade decomposition

Input	*Data :* $z_1, z_2, ..., z_n \in \Pi^-$, $\gamma_1, \gamma_2, ..., \gamma_n \in \mathbf{C}^{M\times 1}$, $w_1, w_2, ..., w_n \in \mathbf{C}^{1\times N}$. *Point* z_0.
Output	*Value of* $F(z)$ *in* (1.3) *with* $G(z) = 0$ *at the point* z_0.
Complexity	$12(M+N)n^2$ *operations.*

1. *For J-unitary on $i\mathbf{R}$ rational matrix function $\Theta(z)$ in (1.4), given by global right pole pair*

$$(\begin{bmatrix} \gamma_1 & \cdots & \gamma_n & 0 & \cdots & 0 \\ 0 & \cdots & 0 & w_1^* & \cdots & w_n^* \end{bmatrix}, \mathrm{diag}(z_1, ..., z_n, -z_1^*, ..., -z_n^*))$$

and coupling array $\begin{bmatrix} n+1 & \cdots & 2n & 1 & \cdots & n \\ 1 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}$, *compute by Algorithm 6.1 numbers $\delta_1, , ..., \delta_r$ and cascade decomposition $\Theta(z) = \Theta_1(z) \cdot ... \cdot \Theta_r(z)$, where for $\delta_i = 1$*

$$\Theta_i(z) = I_{M+N} - \frac{1}{d_i} \cdot \frac{1}{z - t_i} \cdot \psi_i \cdot (\psi_i)^* \cdot J,$$

and for $\delta_i = 2$

$$\Theta_i(z) = I_{M+N} - \frac{1}{d_i} \cdot \frac{1}{z - t_i} \cdot \psi_i \cdot (\varphi_i)^* \cdot J - \frac{1}{d_i^*} \cdot \frac{1}{z - t_{i+1}} \cdot \varphi_i \cdot (\psi_i)^* \cdot J.$$

2.

$\begin{bmatrix} \Theta_{12}^{(0)} \\ \Theta_{22}^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ I_M \end{bmatrix} \in \mathbf{C}^{(M+N)\times N}$
FOR $i = 1 : r$
IF $\delta_i = 1$ THEN
$\begin{bmatrix} \Theta_{12}^{(i)} \\ \Theta_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix} - \frac{1}{d_i} \cdot \frac{1}{z_0 - t_i} \cdot \psi_i \cdot (\psi_i)^* \cdot J \cdot \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix}$

ELSE

$$\begin{bmatrix} \Theta_{12}^{(i)} \\ \Theta_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix} - \frac{1}{d_i} \cdot \frac{1}{z_0 - t_i} \cdot \psi_i \cdot (\varphi_i)^* \cdot J \cdot \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix} -$$
$$\frac{1}{d_i^*} \cdot \frac{1}{z_0 - t_{i+1}} \cdot \varphi_i \cdot (\psi_i)^* \cdot J \cdot \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix}$$

ENDIF

ENDFOR

3. *Compute* $F(z_0) = \Theta_{12}^{(p)} \cdot (\Theta_{22}^{(p)})^{-1}$ *using one of standard inversion algorithms.*

7. ALGORITHM FOR MATRIX NEHARI PROBLEM VIA NONSYMMETRIC CASCADE DECOMPOSITION

In this section we propose one more algorithm for matrix Nehari problem. This algorithm does not pay attention to the fact that $\Theta(z)$ in (1.4) is a J-unitary rational matrix function. According to [BGK, Theorem 1.6], any rational matrix function with simple poles admits a decomposition into a cascade of the first degree factors. The algorithm proposed in this section is based on computing the decomposition of $\Theta(z)$ into such a cascade of non-J-unitary factors.

Let $W_1(z)$ be rational matrix function in (2.24), given by its global left null-pole triple

$$(C_\pi, A_\pi, A_\zeta, B_\zeta, S_1) = (\begin{bmatrix} \psi_1^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix}, \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix}, \begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_m \end{bmatrix}, \begin{bmatrix} \varphi_1^{(1)} \\ \vdots \\ \varphi_m^{(1)} \end{bmatrix}, S_1). \tag{7.1}$$

Observe, that the above data contain the duplicate information. Indeed, if $t_j \neq r_i$ then (i,j) entry of a matrix S_1 can be recovered from the equation (2.23) as

$$s_{ij}^{(1)} = \frac{\varphi_i^{(1)} \cdot \psi_j^{(1)}}{t_j - r_i}. \tag{7.2}$$

In case $t_j = r_i$ the (i,j) entry of a matrix S_1 is referred to as (i,j) coupling number ρ_{ij}. Thus, rational matrix function $W_1(z)$ in (2.24) is completely described by its global right pole pair, global left null pair and by coupling numbers.

The algorithm proposed in this section starts with $W_1(z)$ in (2.24), given by the points $t_1, ..., t_m, r_1, ..., r_m$, vectors $\psi_1^{(1)}, ..., \psi_m^{(1)}, \varphi_1^{(1)}, ..., \varphi_m^{(1)}$ and a coupling array $\begin{bmatrix} \sigma_1 & \cdots & \sigma_m \\ \rho_{1,\sigma_1}^{(1)} & \cdots & \rho_{m,\sigma_m}^{(1)} \end{bmatrix}$. Here $\sigma_i = 0$, $\rho_{i,\sigma_i}^{(1)} = 0$ if $t_j \neq r_i$ for $1 \leq j \leq m$; and in case $t_j = r_i$ for some $1 \leq j \leq m$, then $\sigma_i = j$ and $\rho_{i,\sigma_i}^{(1)}$ is the (i,j) coupling number.

The algorithm is based on Corollary 2.4 and it consists of the m recursive steps. As a first step one has to compute by (7.2) the entries in the first column and first row of the matrix S_1, which are not coupling numbers (the latter are part of input data). And then to determine the position $(q,1)$ of the entry $s_{q1}^{(1)}$ with maximal magnitude in the first column of S_1. Since matrix S_1 is invertible, hence $s_{q1}^{(1)}$ is nonzero. Then one has to replace it in the (1,1) position by swapping the points r_1 and r_q, the vectors $\varphi_1^{(1)}$ and $\varphi_q^{(1)}$ and the columns $\begin{bmatrix} \sigma_1 \\ \rho_{1,\sigma_1} \end{bmatrix}$ and $\begin{bmatrix} \sigma_q \\ \rho_{q,\sigma_q} \end{bmatrix}$ in the coupling array. In fact the above permutation does not change the rational matrix function $\Theta(z)$ in (1.4). Moreover, it is equivalent to passing from the global left null-pole triple in (7.1) to the similar global null-pole triple $(C_\pi,\ A_\pi,\ P_1 \cdot A_\zeta \cdot P_1^T,\ P_1^T \cdot B_\zeta,\ P_1 \cdot S_1)$, where P_1 is the permutation of the 1-st and q-th entries.

Then one has to write down by (2.25) the first cascade factor $\Theta_1(z)$ and to compute by (2.27) and (2.28) the new vectors $\psi_i^{(2)}$ and $\varphi_i^{(2)}$ $(i = 2, 3, ..., m)$ for the quotient. The new coupling numbers $\rho_{i,\sigma_i}^{(2)}$ are computed via (2.29).

Furthermore, after the above first step, the quotient $W_2(z)$ is represented by the same set of data as input rational matrix function $W_1(z)$. One has to proceed with the recursion for $W_2(z)$ and to compute finally the cascade decomposition of $W_1(z)$ in m recursive steps.

It is easy to see that this first step of the recursion corresponds to the factorization

$$P_1 \cdot S_1 = \begin{bmatrix} d_1 & u_1 \\ l_1 & S_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{1}{d_1} \cdot l_1 & I \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 \\ 0 & S_2 \end{bmatrix} \cdot \begin{bmatrix} 1 & \frac{1}{d_1} \cdot u_1 \\ 0 & I \end{bmatrix}, \tag{7.3}$$

where $S_2 = S_{22} - \frac{1}{d_1} \cdot l_1 \cdot u_1$ is a Schur complement of upper left entry d_1 in a matrix S_1. From (7.3) it can be easily seen, that the described algorithm computes simultaneously the LDU factorization

$$P \cdot S_1 = L \cdot D \cdot U$$

for permuted version of null-pole coupling matrix S_1. Here $P = P_n \cdot ... \cdot P_1$, where P_k is the permutation of the k-th step of the recursion. Moreover this choice of permutation matrices P_k completely corresponds to *partial pivoting* [GL].

Here is the record of the algorithm.

ALGORITHM 7.1 Cascade decomposition of rational matrix function.

Input *The rational matrix function* $W_1(z)$ *in* (2.24), *given by global right pole pair* $\left(\begin{bmatrix} \psi_1^{(1)} & \cdots & \psi_m^{(1)} \end{bmatrix}, \begin{bmatrix} t_1 & & 0 \\ & \ddots & \\ 0 & & t_m \end{bmatrix} \right)$, *global left null pair* $\left(\begin{bmatrix} r_1 & & 0 \\ & \ddots & \\ 0 & & r_m \end{bmatrix}, \begin{bmatrix} \varphi_1^{(1)} \\ \vdots \\ \varphi_m^{(1)} \end{bmatrix} \right)$ *and coupling array* $\begin{bmatrix} \sigma_1 & \cdots & \sigma_m \\ \rho_1^{(1)} & \cdots & \rho_m^{(1)} \end{bmatrix}$.

Output **(i)** *The cascade decomposition* $W_1(z) = \Theta_1(z) \cdot \Theta_2(z) \cdot \ldots \cdot \Theta_m(z)$, *where* $\Theta_i(z) = I_p + \frac{1}{d_i} \cdot \frac{1}{z-t_1} \cdot \psi_i^{(i)} \cdot \varphi_i^{(i)}$.
(ii) *Triangular factorization* $S_1 = P^T \cdot L \cdot D \cdot U$ *of permuted version of null-pole coupling matrix* S_1, *where* P *is permutation, matrix* $L = \left[\; l_{ij} \; \right]_{i,j=1}^{m}$ *is lower triangular, matrix* $U = \left[\; u_{ij} \; \right]_{i,j=1}^{m}$ *is upper triangular and* $D = \mathrm{diag}(d_1, d_2, ..., d_m)$.

Complexity $4pm^2$ *operations.*

Initialization *Set* $L = \left[\; l_{ij} \; \right]_{i,j=1}^{m}$, $U = \left[\; u_{ij} \; \right]_{i,j=1}^{m}$ *to be zero matrices and* P *to be identity matrix*

```
FOR i = 1 : m
        FOR k = i : m
                IF σ_k = i    THEN
                        s_ki^(i) = ρ_k^(i)
                ELSE
                        s_ki^(i) = φ_k^(i)·ψ_i^(i) / (t_i − r_k)
                ENDIF
        ENDFOR
        FIND i ≤ q ≤ m     so that     |s_qi^(i)| = max_{i≤k≤m} |s_ki^(i)|
        d_i = s_qi^(i)
        SWAP r_i and r_q
        SWAP φ_i^(i) and φ_q^(i)
        SWAP s_ii^(i) and s_qi^(i)
        SWAP σ_i and σ_q
        SWAP ρ_i^(i) and ρ_q^(i)
        SWAP i-th and q-th rows in L
        SWAP i-th and q-th rows in P
        FOR k = i + 1 : m
                IF σ_i = k THEN
                        s_ik^(i) = ρ_i^(i)
                ELSE
                        s_ik^(i) = φ_i^(i)·ψ_k^(i) / (t_k − r_i)
                ENDIF
        ENDFOR
        l_ii = 1
        u_ii = 1
        FOR k = i + 1 : m
                l_{k,i} = s_{k,i}^(i) · 1/d_i
                u_{i,k} = s_{i,k}^(i) · 1/d_i
                ψ_k^(i+1) = ψ_k^(i) − ψ_i^(i) · u_ik
                φ_k^(i+1) = φ_k^(i) − φ_i^(i) · l_ki
                IF σ_k ≠ 0 THEN
                        ρ_k^(i+1) = ρ_k^(i) − 1/d_i · s_ki^(i) · s_{i,σ_k}^(i)
                ENDIF
        ENDFOR
ENDFOR
```

Algorithm 7.1 suggests computing the cascade decomposition for J-unitary rational matrix function $\Theta(z)$ in (1.4) and then solving the matrix Nehari problem by making use of the formula (1.3) with $G(z) = 0$. Here is the record of the algorithm.

ALGORITHM 7.2 Matrix Nehari problem via nonsymmetric cascade decomposition

Input	*Data :* $z_1, z_2, ..., z_n \in \Pi^-$, $\gamma_1, \gamma_2, ..., \gamma_n \in \mathbf{C}^{M\times 1}$, $\quad w_1, w_2, ..., w_n \in \mathbf{C}^{1\times N}$. *Point* z_0.
Output	*Value of* $F(z)$ *in* (1.3) *with* $G(z) = 0$ *at the point* z_0.
Complexity	$16(M+N)n^2$ *operations.*

1. *For rational matrix function* $\Theta(z)$ *in* (1.4), *given by the global right pole pair*

$$(\begin{bmatrix} \gamma_1 & \cdots & \gamma_n & 0 & \cdots & 0 \\ 0 & \cdots & 0 & w_1^* & \cdots & w_n^* \end{bmatrix}, \mathrm{diag}(z_1, ..., z_n, -z_1^*, ..., -z_n^*));$$

global left null pair

$$(\mathrm{diag}(-z_1^*, ..., -z_n^*, z_1, ..., z_n), \begin{bmatrix} -\gamma_1^* & 0 \\ \vdots & \vdots \\ -\gamma_n^* & 0 \\ 0 & w_1 \\ \vdots & \vdots \\ 0 & w_n \end{bmatrix}),$$

and coupling array $\begin{bmatrix} n+1 & \cdots & 2n & 1 & \cdots & n \\ 1 & \cdots & 1 & 1 & \cdots & 1 \end{bmatrix}$, *compute by Algorithm* 7.1 *the cascade decomposition* $\Theta(z) = \Theta_1(z) \cdot ... \cdot \Theta_{2n}(z)$, *with* $\Theta_i(z) = I_{M+N} + \frac{1}{d_i} \cdot \frac{1}{z-t_i} \cdot \psi_i \cdot \varphi_i$.

2.

$$\begin{bmatrix} \Theta_{12}^{(0)} \\ \Theta_{22}^{(0)} \end{bmatrix} = \begin{bmatrix} 0 \\ I_M \end{bmatrix} \in \mathbf{C}^{(M+N)\times N}$$

FOR $i = 1 : 2n$

$$\begin{bmatrix} \Theta_{12}^{(i)} \\ \Theta_{22}^{(i)} \end{bmatrix} = \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix} + \frac{1}{d_i} \cdot \frac{1}{z_0 - t_i} \cdot \psi_i \cdot \varphi_i \cdot \begin{bmatrix} \Theta_{12}^{(i-1)} \\ \Theta_{22}^{(i-1)} \end{bmatrix}$$

ENDFOR

3. *Compute* $F(z_0) = \Theta_{12}^{(p)} \cdot (\Theta_{22}^{(p)})^{-1}$ *using one of standard inversion algorithms.*

8. MATRIX NEHARI PROBLEM FOR THE UNIT DISK : REDUCTION TO HALF PLANE CASE

In this section we show how matrix Nehari problem over the unit disk $\mathcal{D}$ can be solved using the algorithms, designed in the first part of this paper for the half plane case.

In this case we are given rational $M \times N$ matrix function $K(z)$, which has only simple poles in $\mathcal{D}$ and no poles on $\partial\mathcal{D}$. The Nehari interpolation problem for the unit disk is : Describe all $R(z)$ in the set $\mathcal{R}_{M\times N}(\mathcal{D})$ of all rational matrix functions with no poles in $\mathcal{D} \cup \partial\mathcal{D}$, such that

$$\sup_{z\in\partial\mathcal{D}} \|K(z) - R(z)\| \leq 1. \tag{8.1}$$

As with the half plane case, we consider the generic case of matrix Nehari problem. Let $z_1, ..., z_n$ be all the poles of $K(z)$ in $\mathcal{D}$ ($z_i \neq 0$),and $\gamma_1, ..., \gamma_n \in \mathbf{C}^{M\times 1}$ and $w_1, .., w_n \in \mathbf{C}^{1\times N}$ be nonzero vectors, such that

$$K(z) = (z - z_i)^{-1} \cdot \gamma_i \cdot w_i + [\text{analytic} \quad \text{at} \quad z_i] \qquad (1 \leq i \leq n).$$

Let

$$P = \left[\frac{w_i \cdot w_j^*}{1 - z_i \cdot z_j^*} \right]_{1\leq i,j\leq n}, \qquad Q = \left[\frac{\gamma_i^* \cdot \gamma_j}{1 - z_i^* \cdot z_j} \right]_{1\leq i,j\leq n}$$

be $n \times n$ matrices (both P and Q turn out to be positive definite). Then [BGR, Theorem 20.3.2] there exists $R(z)$, which satisfies (8.1) if and only if $\lambda_1(PQ)$, the maximal eigenvalue of PQ does not exceed 1. If $\lambda_1(PQ) < 1$, then the rational matrix functions $F(z) = K(z) - R(z)$ with the property (8.1) are characterized as matrix functions of the form

$$F(z) = (\Theta_{11}(z) \cdot G(z) + \Theta_{12}(z)) \cdot (\Theta_{21}(z) \cdot G(z) + \Theta_{22}(z))^{-1} \tag{8.2}$$

with arbitrary $G(z) \in \mathcal{R}_{M\times N} \in \mathcal{D} \cup \partial\mathcal{D}$, $\sup_{z\in\partial\mathcal{D}} \|G(z)\| \leq 1$, and

$$\Theta(z) = \begin{bmatrix} \Theta_{11}(z) & \Theta_{12}(z) \\ \Theta_{21}(z) & \Theta_{22}(z) \end{bmatrix}, \tag{8.3}$$

which is given by

$$\Theta(z) = D + \begin{bmatrix} C & 0 \\ 0 & -B^* \cdot (A^*)^{-1} \end{bmatrix} \cdot \begin{bmatrix} (zI - A)^{-1} & 0 \\ 0 & (zI - (A^*)^{-1})^{-1} \end{bmatrix} \times$$

$$\Lambda^{-1} \cdot \begin{bmatrix} (A^*)^{-1} \cdot C^* & 0 \\ 0 & B \end{bmatrix} \cdot D,$$

where

$$\Lambda = \begin{bmatrix} -Q & I \\ I & -P \end{bmatrix}, \quad A = \begin{bmatrix} z_1 & & & 0 \\ & z_2 & & \\ & & \ddots & \\ 0 & & & z_n \end{bmatrix}, \quad B = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}, \quad C = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{bmatrix},$$

and

$$D = I_{M+N} - \begin{bmatrix} C & 0 \\ 0 & -B^* \cdot (A^*)^{-1} \end{bmatrix} \cdot \Lambda^{-1} \times$$

$$\begin{bmatrix} (\alpha I - (A^*)^{-1})^{-1} & 0 \\ 0 & (\alpha I - A)^{-1} \end{bmatrix} \cdot \begin{bmatrix} (A^*)^{-1} \cdot C^* & 0 \\ 0 & B \end{bmatrix}.$$

Here $\alpha \in \partial\mathcal{D}$ is a chosen point. In fact, $\Theta(z)$ was built in [BGR] as $J = \begin{bmatrix} I_M & 0 \\ 0 & -I_N \end{bmatrix}$-unitary on $\partial\mathcal{D}$ rational matrix function. Set

$$K_\alpha = \begin{bmatrix} A + \alpha I & 0 \\ 0 & (A^*)^{-1} + \alpha I \end{bmatrix} \cdot \begin{bmatrix} \alpha I - A & 0 \\ 0 & \alpha I - (A^*)^{-1} \end{bmatrix}^{-1},$$

$$M_\alpha = \sqrt{2}\alpha \cdot \begin{bmatrix} C & 0 \\ 0 & -B^* \cdot (A^*)^{-1} \end{bmatrix} \cdot \begin{bmatrix} \alpha I - A & 0 \\ 0 & \alpha I - (A^*)^{-1} \end{bmatrix}^{-1},$$

$$N_\alpha = \Theta(\alpha).$$

Then, in accordance with [AG, Theorem 3.6], the function

$$\Psi(z) = \Theta(\alpha\frac{z-1}{z+1}) = (I - M_\alpha \cdot (zI - K_\alpha)^{-1} \cdot \Lambda^{-1} \cdot M_\alpha^* \cdot J) \cdot N_\alpha \tag{8.4}$$

is a J-unitary on $i\mathbf{R}$ rational matrix function. Furthermore, since

$$P - A \cdot P \cdot A^* = B \cdot B^*, \qquad Q - A^* \cdot Q \cdot A = C^* \cdot C,$$

hence

$$\Lambda - \begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix} \cdot \Lambda \cdot \begin{bmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{bmatrix} = \begin{bmatrix} -C^* \cdot C & 0 \\ 0 & A^{-1} \cdot B \cdot B^* \cdot (A^*)^{-1} \end{bmatrix}.$$

Rewriting the latter equality as

$$\begin{bmatrix} A & 0 \\ 0 & (A^*)^{-1} \end{bmatrix} - \Lambda^{-1} \cdot \begin{bmatrix} (A^*)^{-1} \cdot C^* \cdot C & 0 \\ 0 & -B \cdot B^* \cdot (A^*)^{-1} \end{bmatrix} = \Lambda^{-1} \cdot \begin{bmatrix} (A^*)^{-1} & 0 \\ 0 & A \end{bmatrix} \cdot \Lambda,$$

and using formula (2.2), written for the matrix function $\Theta(z) \cdot D^{-1}$ one can be easily deduce, that

$$N_\alpha = \Theta(\alpha) = I_{M+N}.$$

From here follows, that $\Psi(z)$ in (8.4) is a J-inner on $i\mathbf{R}$ rational matrix function, which is given by the global right pole pair (M_α, K_α) and by its coupling numbers ρ_{ij}, where

$$M_\alpha = \begin{bmatrix} \frac{\sqrt{2}\alpha\gamma_1}{\alpha - z_1} & \cdots & \frac{\sqrt{2}\alpha\gamma_n}{\alpha - z_n} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{-\sqrt{2}\alpha w_1^*}{\alpha z_1^* - 1} & \cdots & \frac{-\sqrt{2}\alpha w_n^*}{\alpha z_n^* - 1} \end{bmatrix},$$

$$K_\alpha = \mathrm{diag}(\frac{\alpha + z_1}{\alpha - z_1}, ..., \frac{\alpha + z_n}{\alpha - z_n}, \frac{\alpha z_1^* + 1}{\alpha z_1^* - 1}, ..., \frac{\alpha z_n^* + 1}{\alpha z_n^* - 1}),$$

and $\rho_{i,i+n} = \rho_{i+n,i} = 1$ $(i = 1, 2, ..., n)$. Therefore, Algorithms 6.1 and 7.1 can be applied for computing the cascade decomposition of the function $\Psi(z)$. Then this cascade decomposition can be used for computing the value of

$$\Theta(z_0) = \Psi(\frac{\alpha + z_0}{\alpha - z_0}),$$

and using (8.3) and (8.2) with $G(z) = 0$, one can compute the value of $F(z_0)$ at a given point z_0. We omit here the records for the analogs of Algorithms 6.2 and 7.2 for the unit disk case.

9. NUMERICAL EXPERIMENTS FOR MATRIX NEHARI PROBLEM

We have carried out a wide set of computer experiments with the algorithms presented in this paper applied for solving matrix Nehari problem. All the tests were performed on SUN Sparc-1 workstation using MATLAB and C language. SUN Sparc-1 supports IEEE standard single and double precision arithmetic with about 7 and 15 significant digits, respectively.

The data were generated in the double precision using MATLAB ver. 4.0a in the following way. We tried various choices of the nodes $z_1, ..., z_n \in \Pi^-$ such as random complex points, various variants of real negative points, etc. We also tested different sizes $M \times N$ for the given rational function $K(z)$, and various choices of maximal eigenvalue $\lambda_1(PQ)$ of the matrix PQ. The vectors $\gamma_1, ..., \gamma_n \in \mathbf{C}^{M\times 1}$ and $w_1, ..., w_n \in \mathbf{C}^{1\times n}$ were chosen with random real and imaginary parts, and then scaled to make the maximal eigenvalue of the matrix PQ to be equal to the assigned in advance value in the interval (0,1).

For each set of the data we computed the value of the solution at point $z_0 = -2+3i$ for solution $F(z)$ of matrix Nehari problem via the following algorithms :

(1) **(J-I)** Algorithm 5.2 (J-inner).
(2) **(J-U)** Algorithm 6.2 (J-unitary).
(3) **(WP)** Version of Algorithm 6.2 without pivoting.
(4) **(S-J-U** Solving structured linear systems (1.6) by forward and backsubstitution, using LDL* factorization of matrix Λ, computed by Algorithm 6.1 and then using the above solution with the formulas (1.3) and (1.4)
(5) **(NS)** Algorithm 7.2 (nonsymmetric).
(6) **(S-NS)** Solving structured linear systems (1.6) by forward and backsubstitution, using LDU factorization of matrix Λ, computed via Algorithm 7.1 and then using the above solution with the formulas (1.4) and (1.3).
(7) **(S-G)** Solving linear systems (1.6) via Gaussian elimination with complete pivoting (C routine from [PFTV]) and then using the above solution with the formulas (1.4) and (1.3). This algorithm requires $O(n^3)$ operations.

All the algorithms listed above were implemented in the C language in the single precision. The Algorithms **(NS)** and **(J-U)** were also implemented in the double precision.

Denote by

$$F^{d\mathbf{NS}} = \left[\ f_{ij}^{d\mathbf{NS}}\ \right]_{\substack{1\le i\le M\\ 1\le j\le N}} \qquad \text{and} \qquad F^{d\mathbf{J-U}} = \left[\ f_{ij}^{d\mathbf{J-U}}\ \right]_{\substack{1\le i\le M\\ 1\le j\le N}}$$

the $M \times N$ matrices computed in double precision by Algorithms **(NS)** and **(J-U)**, respectively. In each computed example we measured the relative componentwise error

$$ERR^{(\mathbf{DBL})} = \max_{\substack{1\le i\le M\\ 1\le j\le N}} \frac{|f_{ij}^{d\mathbf{NS}} - f_{ij}^{d\mathbf{J-U}}|}{|f_{ij}^{d\mathbf{NS}}|},$$

which turned out to be of the order of 10^{-11} at most. It means, that in the entries of the solutions $F^{d\mathbf{NS}}$ and $F^{d\mathbf{J-U}}$ computed in double precision by these two algorithms, the first 11 out of 16 significant digits were the same. On this basis we assume the first 7 digits in the entries of the matrix $F^{d\mathbf{NS}}$ to be exact and relied on them, when measuring the relative componentwise error for the single precision versions of the algorithms listed above. For example, for the solution $F^{\mathbf{J-I}} = \left[\ f_{ij}^{\mathbf{J-I}}\ \right]_{\substack{1\le i\le M\\ 1\le j\le N}}$, computed in single precision via Algorithm **(J-I)** the relative componentwise error was computed by

$$ERR^{(\mathbf{J-I})} = \max_{\substack{1\le i\le M\\ 1\le j\le N}} \frac{|f_{ij}^{d\mathbf{NS}} - f_{ij}^{\mathbf{J-I}}|}{|f_{ij}^{d\mathbf{NS}}|}.$$

The relative componentwise errors for the single precision implementations of the other algorithms listed above were measured in the same way. Here are some of the computed examples with the complex points with random real and imaginary parts in the rectangle $\{z| -20 \le \mathrm{Re}z \le -1; \quad -10 \le \mathrm{Im}z \le 10\}$.

	ERR							
n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
10	2.6e-15	2.8e-03	3.2e-06	5.2e-05	2.4e-07	3.2e-06	4.5e-07	2.6e- 07
20	4.5e-15	2.5e-01	2.5e-06	1.5e+00	4.4e-07	4.7e-06	3.2e-07	8.5e- 08
30	1.3e-15	1.1e+00	1.1e-06	1.3e-05	3.2e-07	7.3e-07	3.2e-07	4.3e- 07
40	3.8e-15	Inf	3.2e-06	Inf	3.8e-07	2.3e-06	3.8e-07	4.2e-07
50	3.3e-15	Inf	1.2e-06	Inf	1.2e-06	1.7e-06	1.2e-06	1.7e-07
60	7.0e-14	Inf	1.5e-04	Inf	1.5e-07	1.6e-04	1.2e-07	2.1e-07
70	3.3e-15	Inf	1.7e-06	Inf	7.5e-08	2.0e-06	2.8e-07	1.8e-07

Table 1. $M = 1$, $N = 1$, $\lambda_1(PQ) = 0.9$

	-ERR							
n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
10	8.3e-14	2.0e-05	2.1e-05	5.9e-05	1.0e-05	2.1e-05	1.0e-05	1.5e- 06
20	3.8e-15	7.1e-04	1.8e-06	2.8e-06	8.4e-07	1.7e-06	3.9e-07	6.9e- 07
30	3.8e-15	1.4e-01	3.3e-06	9.0e-04	1.2e-06	4.9e-06	3.2e-06	3.8e- 06
40	1.2e-14	Inf	4.8e-06	Inf	3.1e-06	5.5e-06	1.3e-06	4.9e-06
50	9.3e-15	2.6e+00	1.8e-05	7.7e-01	5.9e-06	2.2e-05	1.2e-06	1.6e- 06
60	7.4e-15	1.6e+00	4.1e-06	5.8e-01	5.2e-07	3.1e-06	7.7e-07	7.2e- 07
70	6.0e-15	Inf	4.6e-06	Inf	1.9e-06	2.8e-06	1.7e-06	6.7e-07

Table 2. $M = 2,\ N = 2,\ \lambda_1(PQ) = 0.9$

	-ERR							
n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
10	7.8e-15	1.6e-05	3.2e-06	8.7e-06	1.5e-06	3.1e-06	1.5e-06	9.2e- 07
20	2.4e-14	Inf	1.6e-05	Inf	2.1e-06	6.3e-06	2.6e-06	1.5e-06
30	1.2e-14	6.5e-03	2.6e-06	5.4e-05	2.0e-06	4.7e-06	2.7e-06	3.0e- 06
40	1.6e-14	Inf	9.6e-06	Inf	1.6e-06	6.1e-06	1.2e-06	1.8e-06
50	2.4e-14	8.5e-01	1.3e-05	1.3e-03	2.4e-06	6.0e-06	1.3e-06	4.7e- 06
60	3.2e-14	6.9e+00	5.2e-06	7.1e-04	3.1e-06	5.3e-06	2.0e-06	2.9e- 06
70	7.4e-14	2.5e+00	7.1e-05	1.9e-01	1.7e-06	8.0e-05	1.5e-06	2.3e- 06

Table 3. $M = 4,\ N = 3,\ \lambda_1(PQ) = 0.9$

	-ERR							
n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
10	6.8e-15	2.0e-03	1.6e-06	3.4e-05	5.5e-07	1.1e-06	3.3e-07	2.7e -07
20	2.0e-14	2.6e-01	2.4e-05	1.7e+00	1.3e-06	2.3e-05	1.2e-06	1.5e -07
30	1.9e-15	9.9e-01	1.2e-06	7.3e-06	4.0e-07	4.3e-07	5.3e-08	4.0e -07
40	1.1e-15	Inf	4.6e-07	Inf	1.1e-07	4.8e-07	4.4e-07	7.1e-07
50	2.3e-15	Inf	1.3e-06	Inf	7.3e-07	1.1e-06	1.2e-06	2.0e-07
60	6.5e-14	Inf	9.1e-06	Inf	1.5e-06	8.6e-06	1.1e-06	5.2e-07
70	7.6e-15	Inf	9.7e-06	Inf	3.4e-07	1.2e-05	2.9e-07	8.5e-07

Table 4. $M = 1,\ N = 1,\ \lambda_1(PQ) = 0.99$

	-ERR							
n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
10	2.2e-14	7.0e-05	4.1e-05	3.0e-05	4.7e-06	1.5e-05	1.2e-05	1.1e -05
20	2.2e-13	1.7e-03	6.3e-05	6.6e-05	1.1e-05	4.7e-05	2.9e-05	3.0e -05
30	7.2e-14	1.2e-01	3.9e-05	7.7e-04	2.3e-05	2.7e-05	3.3e-05	1.7e -05
40	6.7e-14	2.6e+00	1.4e-04	2.0e-01	2.2e-05	3.7e-05	4.5e-06	2.6e -05
50	6.7e-13	1.7e+00	2.4e-04	1.3e+00	2.0e-05	1.2e-04	7.5e-06	7.3e -06
60	1.9e-13	8.9e-01	9.2e-05	4.4e+00	3.2e-05	6.1e-05	1.3e-05	2.4e -05
70	5.0e-12	Inf	5.6e-03	Inf	2.9e-05	2.4e-04	9.5e-06	1.7e-05

Table 5. $M = 2,\ N = 2,\ \lambda_1(PQ) = 0.99$

n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
	-ERR							
10	1.7e-13	5.7e-05	5.9e-05	4.4e-05	3.2e-05	2.0e-05	4.6e-05	4.7e -05
20	1.3e-13	5.5e-05	8.5e-05	4.2e-05	1.1e-05	2.1e-05	1.7e-05	2.0e -05
30	5.0e-12	2.1e-02	4.0e-03	1.6e-04	1.5e-04	2.4e-04	1.3e-04	8.5e -05
40	2.0e-13	1.2e-02	1.0e-04	6.4e-05	2.3e-05	4.3e-05	1.5e-05	2.0e -05
50	1.7e-13	Inf	8.5e-05	Inf	3.4e-05	4.2e-05	9.6e-06	1.5e-05
60	3.9e-12	8.0e-01	1.9e-03	4.3e-02	7.3e-05	1.3e-03	5.5e-05	1.6e -04
70	1.4e-12	8.2e+00	3.4e-04	2.8e-02	5.0e-05	6.1e-05	4.3e-05	6.2e -05

Table 6. $M = 4$, $N = 3$, $\lambda_1(PQ) = 0.99$

n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
	-ERR							
10	9.5e-15	1.4e-03	2.0e-06	1.5e-06	2.0e-06	3.1e-06	1.1e-06	1.0 e-06
20	1.1e-14	3.1e-01	2.7e-06	1.1e-03	2.3e-06	4.7e-06	2.7e-06	2.7 e-07
30	1.4e-15	2.3e+00	5.2e-07	3.6e-01	1.8e-07	2.9e-07	3.7e-07	3.5 e-07
40	1.9e-15	Inf	3.0e-06	Inf	1.7e-06	1.8e-06	9.0e-07	1.6e-07
50	5.0e-14	Inf	2.0e-05	Inf	3.0e-06	2.0e-05	3.9e-06	6.4e-07
60	1.8e-14	Inf	2.5e-05	Inf	1.7e-06	5.8e-06	3.8e-07	5.1e-07
70	1.4e-14	Inf	6.4e-06	Inf	7.3e-07	8.6e-06	4.4e-07	6.6e-07

Table 7. $M = 1$, $N = 1$, $\lambda_1(PQ) = 0.999$

n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
	-ERR							
10	1.3e-13	1.2e-04	4.6e-05	4.9e-05	3.8e-05	1.3e-04	2.5e-05	1.8 e-05
20	1.2e-11	1.3e-02	4.7e-03	1.0e-03	3.1e-04	2.4e-04	1.3e-04	1.2 e-04
30	5.9e-13	1.2e-01	6.2e-04	6.9e-04	7.2e-05	2.1e-04	3.7e-05	1.6 e-04
40	4.0e-12	4.0e+00	2.4e-03	7.4e-02	2.2e-04	5.2e-04	2.2e-04	4.6 e-04
50	1.3e-10	7.4e+00	5.3e-02	2.8e+01	9.7e-04	4.3e-03	3.2e-04	6.4 e-04
60	6.8e-11	2.2e+01	1.5e-02	4.7e+00	2.7e-04	1.5e-03	1.0e-03	1.0 e-03
70	5.0e-13	Inf	2.0e-04	Inf	1.4e-04	2.1e-04	1.2e-04	1.8e-04

Table 8. $M = 2$, $N = 2$, $\lambda_1(PQ) = 0.999$

n	DBL	J-I	J-U	WP	NS	S-J-U	S-NS	S-G
	-ERR							
10	5.4e-13	2.0e-04	3.5e-04	3.2e-04	1.6e-04	2.2e-04	8.9e-05	1.6 e-04
20	1.6e-12	1.0e-03	1.0e-03	4.5e-04	3.8e-04	2.6e-04	2.7e-04	2.5 e-04
30	2.5e-13	2.4e-02	1.4e-04	1.9e-04	6.4e-05	1.4e-04	4.8e-05	6.6 e-05
40	3.0e-13	1.8e-02	1.7e-04	4.7e-04	1.1e-04	4.4e-04	1.2e-04	9.6 e-05
50	1.0e-11	4.4e-01	5.0e-03	2.5e-03	1.6e-04	8.9e-04	3.3e-04	5.1 e-04
60	7.8e-13	7.8e-01	7.1e-04	5.0e-02	1.8e-04	3.1e-04	2.2e-04	1.5 e-04
70	2.9e-12	Inf	6.1e-03	Inf	4.4e-04	7.2e-04	3.7e-04	3.0e-04

Table 9. $M = 4$, $N = 3$, $\lambda_1(PQ) = 0.999$

10. CONCLUSIONS

From the data in the above tables, as well as from the results of many other computer experiments, follows that in computed examples $O(n^2)$ Algorithm (**NS**) and $O(n^3)$ Algorithm (**S-G**) showed the best numerical behavior. These two algorithms allows the accurate solution of matrix Nehari problem with interpolation data given in a large number of the points.

Algorithm (**J-U**) was accompanied by a slightly larger error accumulation, but it also can be ascribed to the group of reliable algorithms.

Algorithms (**NS**) and (**S-NS**), both based on the same Algorithm 7.1, have very similar numerical behavior. Similarly, Algorithms (**J-U**) and (**S-J-U**), both based on Algorithm 6.1 also have the same numerical features in the computed examples.

In all computed examples Algorithm (**J-I**) was accompanied by essential error accumulation, despite pivoting in computing the cascade decomposition of J-inner factors $\Phi(z)$ and $\Psi(z)$ in (4.1). The pivots d_i in Algorithm 5.1 decreased rapidly with the growth of the number n of the points, which was reflected in the underflows and overflows (denoted by *Inf* in the above tables). This strongly suggests the conclusion about the unstable nature of Algorithm (**J-I**).

Pivoting plays a crucial role in achieving high accuracy in the computed solution of matrix Nehari problem, and partial pivoting (Algorithm (**NS**)) is preferable to symmetric pivoting (Algorithm (**J-U**)).

The propagation of the roundoff errors grows slightly with the number of input points. Furthermore, the accumulated error is larger for the matrix case (cases $M = N = 2$; and $M = 4$, $N = 3$ in the tables), in comparison with the scalar case ($M = N = 1$). And, finally, the error accumulation is increasing when the maximal eigenvalue $\lambda_1(PQ)$ of the matrix PQ tends to unity.

11. MATRIX NEHARI-TAKAGI PROBLEM

In this section we use the algorithms designed in the first part of this paper for the matrix Nehari problem in order to solve a more general Nehari-Takagi problem.

Let rational $M \times N$ matrix function $K(z)$ satisfying $\sup_{z \in i\mathbf{R}} \|K(z)\| \leq \infty$ be given. In the matrix Nehari-Takagi problem we seek a rational $M \times N$ matrix function $R(z)$ with at most k (counting multiplicities) poles in Π^- such that

$$\sup_{z \in i\mathbf{R}} \|K(z) - R(z)\| < 1. \tag{11.1}$$

If $k = 0$, we obtain the matrix Nehari problem, studied above.

As in section 1 we will assume that $K(z)$ has the form

$$K(z) = (z - z_i)^{-1} \cdot \gamma_i \cdot w_i + [\text{analytic} \quad \text{at} \quad z_i] \qquad (1 \leq i \leq n),$$

where $z_1, ..., z_n$ be all the poles of $K(z)$ in Π^-, and $\gamma_1, ..., \gamma_n \in \mathbf{C}^{M \times 1}$ be nonzero column vectors, and $w_1, ..., w_n \in \mathbf{C}^{1 \times N}$ be nonzero row vectors. Let P and Q be positive definite matrices from (1.2) and assume that 1 is not an eigenvalue of PQ. Let k_0 be the number of eigenvalues (counting multiplicities) of the matrix PQ, which are bigger than 1. then in accordance with [BGR, Theorem 20.5.1], all rational matrix functions

$$F(z) = K(z) - R(z) \tag{11.2}$$

satifying (1.11) and such that $R(z)$ has precisely k_0 poles in Π^-, are parameterized by the formula (1.3) with arbitrary rational matrix function $G(z)$ satisfying $\sup_{z\in\Pi^-} \|G(z)\| < 1$, and $\Theta(z)$ given by (1.4). In other words, the form of the solution of matrix Nehari problem is inherited in a more general matrix Nehari-Takagi problem. The difference is that the latter problem is solvable for a more wide set of input data

$$z_1, ..., z_n \in \mathbf{C}; \qquad \gamma_1, ..., \gamma_n \in \mathbf{C}^{M\times 1}; \qquad w_1, ..., w_n \in \mathbf{C}^{1\times N}.$$

Moreover, it is straightforward to see that Algorithms 6.2 and 7.2 are applicable in this new situation, when matrix PQ has eigenvalues bigger then 1.

Let ν_+ and ν_- be the numbers of the positive and negative eigenvalues (counted with multiplicities) of associated Hermitian matrix Λ of $\Theta(z)$. Using the standard Schur complementation formula

$$\Lambda = \begin{bmatrix} Q & I \\ I & P \end{bmatrix} = \begin{bmatrix} I & O \\ Q^{-1} & I \end{bmatrix} \cdot \begin{bmatrix} Q & 0 \\ 0 & P - Q^{-1} \end{bmatrix} \cdot \begin{bmatrix} I & Q^{-1} \\ 0 & I \end{bmatrix}$$

and the arguments similar to those in the end of the proof of theorem 4.1, one can easily deduce that the number k_0 of the poles in Π^- of $R(z)$ from (11.2) satisfies the equality

$$k_0 = \nu_+ - n.$$

Algorithm 6.2 involves the Algorithm 6.1, which computes for the associated Hermitian matrix Λ of $\Theta(z)$ the factorization

$$\Lambda = L \cdot D \cdot L^*, \tag{11.3}$$

where L is a lower triangular matrix and $D = \mathrm{diag}(D_1, ...D_r)$, where each D_i is 1×1 or 2×2 block. Recall (see Algorithm 6.1), that in the above factorization 2×2 blocks are of the form $D_i = \begin{bmatrix} 0 & d_i^* \\ d_i & 0 \end{bmatrix}$, and hence ν_+ and ν_- can be counted using (11.3) by taking into account only 1×1 blocks. Therefore Algorithm (**J-U**) ends up with the solution $F(z_0) = K(z_0) - R(z_0)$ of matrix Nehari-Takagi problem and it allows to obtain for free the above number k_0 of the poles of $R(z)$ in Π^-.

Finally, let us remark, that the matrix Nehari-Takagi problem for the unit disk is reduced to the half plane problem with exactly the same formulas, as in section 8.

12. NUMERICAL EXPERIMENTS WITH NEHARI-TAKAGI PROBLEM

For matrix Nehari-Takagi problem we performed a large number of computer experiments similar to those described in sections 9 – 10. We tried various choices of the points in Π^-, different sizes $M \times N$ for the given function $K(z)$ and various variants for the number k_0 of the eigenvalues of PQ, which are bigger than 1. The vectors $\gamma_1, ..., \gamma_n \in \mathbf{C}^{M\times 1}$ and $w_1, ..., w_n \in \mathbf{C}^{1\times N}$ were generated with random real and imaginary parts, and then scaled to reach the desired number k_0 of the eigenvalues of PQ, which are bigger than 1.

For each set of data we computed the solution of matrix Nehari-Takagi problem at the point $z_0 = -2 + 3i$. In the following tables we keep all the notations introduced in the section 9. The points are with random real and imaginary parts in the rectangle $\{z| -20 \leq \mathrm{Re}z \leq -1; \quad -10 \leq \mathrm{Im}z \leq 10\}$.

	-ERR-						
n	DBL	J-U	WP	NS	S-J-U	S-NS	S-G
10	8.9e-13	1.6e-06	3.1e-05	1.8e-04	6.4e-06	3.4e-04	1.7e-04
20	5.4e-13	4.7e-06	7.2e-04	6.4e-04	1.1e-05	6.0e-04	7.9e-05
30	3.6e-15	2.4e-06	8.2e-01	4.3e-06	1.1e-06	6.6e-06	1.4e-05
40	3.1e-14	2.9e-06	Inf	6.1e-06	3.8e-07	4.6e-06	6.5e-05
50	1.1e-14	1.0e-05	Inf	3.6e-06	8.3e-06	4.0e-06	6.2e-06
60	1.3e-12	9.0e-05	Inf	5.0e-06	1.6e-04	1.9e-05	1.7e-05
70	3.5e-14	6.3e-05	Inf	1.9e-05	3.7e-05	5.1e-05	1.5e-04

Table 10. $M = 1,\ N = 1,\ k_0 = 5$

	-ERR-						
n	DBL	J-U	WP	NS	S-J-U	S-NS	S-G
15	3.2e-09	1.5e-06	6.4e-06	1.1e+00	1.9e-02	1 .9e+00	1.0e+00
20	6.3e-08	1.2e-06	5.9e-03	1.3e+00	7.5e-03	6 .3e+00	1.0e+00
30	6.3e-10	6.5e-05	4.1e-03	1.8e-01	2.1e-03	2 .3e-01	2.8e-01
40	3.9e-13	2.7e-06	1.8e-01	8.1e-04	2.2e-05	1.0e- 03	9.7e-04
50	2.2e-12	3.5e-05	Inf	2.2e-03	1.5e-04	3.4e-03	6.6e-03
60	6.2e-11	6.9e-05	Inf	4.8e-02	4.4e-03	5.8e-02	4.7e-01
70	8.2e-11	5.2e-06	Inf	8.9e-02	4.7e-03	6.7e-02	4.8e-01

Table 11. $M = 1,\ N = 1,\ k_0 = 10$

n	DBL	J-U	WP	NS	S-J-U	S-NS	S-G
20	1.2e-07	3.7e-06	4.4e-03	6.5e+00	7.4e-02	2 .5e+01	1.0e+00
30	3.2e-08	6.3e-06	2.2e-01	6.7e+00	8.6e-03	2 .1e+00	1.0e+00
40	2.3e-08	2.0e-06	6.5e-01	1.6e+00	4.2e-02	1.3e+ 01	1.0e+00
50	1.6e-07	3.9e-06	Inf	1.0e+01	1.3e-01	2.6e+01	1.0e+00
60	2.5e-10	8.1e-06	Inf	5.3e-01	3.5e-02	5.2e-01	9.8e-01
70	2.1e-09	1.6e-04	Inf	1.9e-01	6.4e-03	1.8e-01	1.2e+00

Table 12. $M = 1,\ N = 1,\ k_0 = 15$

n	DBL	J-U	WP	NS	S-J-U	S-NS	S-G
10	4.6e-14	6.3e-06	1.6e-04	8.8e-06	3.6e-06	9.3e-06	2.2e-06
20	3.3e-13	1.8e-04	9.7e-04	3.8e-05	1.9e-04	4.2e-05	1.9e-05
30	1.0e-14	2.2e-06	2.2e-04	2.4e-06	2.5e-06	2.5e-06	1.6e-06
40	9.6e-14	1.4e-05	Inf	4.0e-05	1.6e-05	4.3e-05	2.8e-05
50	4.9e-13	1.7e-04	5.2e+00	1.5e-04	1.5e-04	1.0e-04	3.7e-05
60	1.3e-13	2.0e-05	1.9e+00	3.9e-05	2.2e-05	5.0e-05	1.8e-05
70	2.7e-13	4.1e-05	Inf	8.6e-05	4.2e-05	8.8e-05	4.1e-05

Table 13. $M = 2,\ N = 2,\ k_0 = 5$

n	DBL	J-U	WP	NS	S-J-U	S-NS	S-G
10	4.6e-14	6.3e-06	1.6e-04	8.8e-06	3.6e-06	9.3e-06	2.2e-06
20	1.7e-12	3.2e-05	1.5e-03	2.2e-03	1.3e-04	2.0e-03	2.2e-03
30	7.0e-13	2.6e-06	2.4e-01	9.2e-04	3.1e-05	8.3e-04	3.0e-04
40	1.7e-12	2.3e-05	1.1e+00	1.3e-03	5.8e-05	1.1e-03	2.9e-04
50	1.5e-11	2.8e-05	2.6e+00	1.7e-03	9.0e-05	1.6e-03	2.8e-04
60	4.8e-13	4.4e-05	2.0e+00	2.1e-04	7.3e-05	3.6e-04	1.1e-04
70	5.8e-12	2.4e-04	3.3e+00	4.4e-03	2.7 e-04	4.5e-03	3.9e-04

Table 14. $M = 2,\ N = 2,\ k_0 = 10$

n	DBL	J-U	WP	NS	S-J-U	S-NS	S-G
20	7.9e-11	4.9e-06	2.4e-03	1.5e-02	6.2e-04	2.4e-02	1.2e-01
30	8.4e-09	5.7e-04	7.3e-04	3.3e+00	8.8e-03	3.5e+00	1.0e+00
40	1.2e-10	3.0e-06	6.5e-04	1.3e-02	1.9e-03	1.0e-02	1.7e-01
50	2.8e-11	2.3e-05	2.8e-01	2.1e-02	2.2e-04	2.1e-02	1.2e-02
60	2.9e-11	2.5e-05	1.5e+00	2.2e-02	2.6e-03	2.1e-02	7.5e-02
70	5.9e-11	8.3e-05	3.9e+00	3.4e-02	8.4e-04	3.6e-02	2.2e-02

Table 15. $M = 2,\ N = 2,\ k_0 = 15$

n	-ERR- DBL	J-U	WP	NS	S-J-U	S-NS	S-G
15	2.8e-13	1.2e-05	Inf	1.4e-04	6.5e-05	1.6e-04	1.5e-04
20	4.5e-13	3.4e-05	1.1e+00	2.5e-04	5.5e-05	2 .0e-04	2.9e-04
30	2.7e-12	1.9e-04	8.2e-01	1.3e-03	4.5e-04	1 .5e-03	2.8e-03
40	1.6e-13	2.5e-05	1.3e-03	1.6e-05	2.2e-05	5 .3e-05	4.5e-05
50	7.7e-13	1.2e-04	4.9e-01	3.7e-04	1.2e-04	3 .0e-04	1.5e-04
60	4.3e-13	9.7e-05	2.9e-01	5.1e-05	1.1e-04	7 .5e-05	1.7e-04
70	4.1e-13	3.8e-04	Inf	8.4e-05	3.1e-04	1.0e-04	4.5e-05

Table 16. $M = 4,\ N = 3,\ k_0 = 10$

n	-ERR- DBL	J-U	WP	NS	S-J-U	S-NS	S-G
20	2.6e-12	5.7e-06	2.8e-04	4.1e-03	2.6e-04	4 .4e-03	1.9e-02
30	1.9e-12	2.0e-05	5.0e-01	1.0e-03	2.3e-04	1 .0e-03	2.4e-03
40	6.8e-12	1.1e-04	2.1e-02	8.0e-03	4.6e-04	1 .6e-02	2.8e-02
50	1.1e-11	7.6e-04	4.4e-01	1.5e-02	6.3e-04	1 .5e-02	1.3e-02
60	9.7e-12	2.2e-04	9.6e-02	6.4e-03	3.3e-04	4 .3e-03	4.3e-03
70	8.8e-13	2.0e-04	2.9e-01	6.1e-04	2.9e-04	3 .6e-04	1.5e-03

Table 17. $M = 4,\ N = 3,\ k_0 = 15$

n	-ERR- DBL	J-U	WP	NS	S-J-U	S-NS	S-G
25	1.0e-09	1.2e-05	3.2e-05	7.6e-01	8.3e-03	9 .2e-01	3.4e+00
30	1.0e-11	2.5e-05	1.0e-02	1.2e-02	6.4e-04	1 .2e-02	6.0e-02
40	2.8e-11	8.3e-05	8.8e-02	3.5e-02	1.6e-03	3 .1e-02	1.3e-01
50	1.0e-10	6.4e-05	Inf	4.1e-02	8.7e-04	2.8e-02	2.2e-01
60	2.3e-11	1.0e-03	1.6e-02	3.5e-02	3.9e-03	3 .3e-02	6.7e-02
70	1.5e-11	1.6e-04	2.3e+00	3.0e-03	2.6e-04	4 .7e-03	1.0e-02

Table 18. $M = 4,\ N = 3,\ k_0 = 20$

12. CONCLUSIONS

Studing tables 10 - 18 one may conclude that the numerical behavior of the compared algorithms applied for solving matrix Nehari-Takagi differs from the one in examples for matrix Nehari problem. Below we list some specific peculiarities revealed in the above examples, when the number k_0 is not small in comparison with the number n of interpolation points.

Algorithms (**NS**) and (**S-G**) lose their stable behavior, showed in the tables 1 - 9.

Algorithm (**J-U**) became preferable and it allows the accurate solution for matrix Nehari-Takagi problem, with interpolation data given in a large number of the points.

Algorithms (**J-U**) and (**S-J-U**) both involve Algorithm 6.1. Nevertheless, Algorithm (**J-U**) (based on the cascade decomposition) is accompanied by an error accumulation, which is lower than error accumulation of Algorithm (**S-J-U**) (based on LDL* factorization).

And, finally, we remark that for the scalar case $M = N = 1$ the mentioned peculiarities are more transparent.

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Submitted : November, 28, 1993.