The QR iteration method for quasiseparable matrices

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1 Introduction

2 Definitions

Let $\{a_k\}, k = 1, \ldots, N$ be a family of matrices of sizes $r_k \times r_{k-1}$. For positive integers $i, j,\ i > j$ define the operation $a^{\times}_{ij}$ as follows: $a^{\times}_{ij} = a_{i-1} \cdots a_{j+1}$ for $i > j+1$, $a^{\times}_{j+1,j} = I_{r_j}$.

Let $\{b_k\}, k = 1, \ldots, N$ be a family of matrices of sizes $r_{k-1} \times r_k$. For positive integers $i, j,\ j > i$ define the operation $b^{\times}_{ij}$ as follows: $b^{\times}_{ij} = b_{i+1} \cdots b_{j-1}$ for $j > i+1$, $b^{\times}_{i,i+1} = I_{r_i}$.

It is easy to see that

$$a^{\times}_{ik} = a^{\times}_{ij} a^{\times}_{j+1,k}, \quad i > j \geq k \tag{2.1}$$

and

$$b^{\times}_{kj} = b^{\times}_{k,i+1} b^{\times}_{i,j}, \quad k \leq i < j. \tag{2.2}$$

Let $R = \{R_{ij}\}_{i,j=1}^{N}$ be a matrix with block entries R_{ij} of sizes $m_i \times n_j$. Assume that the entries of this matrix are represented in the form

$$R_{ij} = \begin{cases} p_i a^{\times}_{ij} q_j, & 1 \leq j < i \leq N, \\ d_i, & 1 \leq i = j \leq N, \\ g_i b^{\times}_{ij} h_j, & 1 \leq i < j \leq N. \end{cases} \tag{2.3}$$

Here $p_i\ (i = 2, \ldots, N),\ q_j\ (j = 1, \ldots, N-1),\ a_k\ (k = 2, \ldots, N-1)$ are matrices of sizes $m_i \times r'_{i-1},\ r'_j \times n_j,\ r'_k \times r'_{k-1}$ respectively; these elements are said to be *lower generators of the matrix* R with *orders* r'_k $(k = 1, \ldots, N-1)$. The elements g_i $(i = 1, \ldots, N -$

1), h_j $(j=2,\dots,N)$, b_k $(k=2,\dots,N-1)$ are matrices of sizes $m_i \times r''_i$, $r''_{j-1} \times n_j$, $r''_{k-1} \times r''_k$ respectively; these elements are said to be *upper generators of the matrix* R with *orders* r''_k, $(k=1,\dots,N-1)$. The matrices d_k $(k=1,\dots,N)$ of sizes $m_k \times n_k$ are said to be *diagonal entries* of the matrix R. We define also orders of generators r'_k, r''_k for $k=0,\ N$ setting them to be zeros. For scalar matrices the generators p_i, g_i and q_j, h_j are rows and columns of the corresponding sizes. Set $n_L = \max_{1\le k\le N-1} r'_k$, $n_U = \max_{1\le k\le N-1} r''_k$, the matrix R is said to be *lower quasiseparable* of order n_L and *upper quasiseparable* of order n_U or *quasiseparable* of order (n_L, n_U).

Formally, we use some calculation rules with matrices that have blocks with dimension zero. Aside from obvious rules, the product of an "empty" matrix of dimension $m \times 0$ and an empty matrix of dimension $0 \times n$ is a matrix of dimension $m \times n$ with all elements equal to 0. All further rules of block matrix multiplication remain consistent. Such operations are used in MATLAB.

3 The QR factorization

Let $R = \{R_{ij}\}_{i,j=1}^N$ be a matrix with entries from $\mathbb{C}$ with given generators. We present here an algorithm for computing generators and diagonal entries of unitary matrix Q and upper triangular matrix S such that $R = QS$. The main part of the algorithm is based on the following result from [1].

Theorem 3.1 *Let $R = \{R_{ij}\}_{i,j=1}^N$ be a scalar matrix with lower generators p_i $(i=2,\dots,N)$, q_j $(j=1,\dots,N-1)$, a_k $(k=2,\dots,N-1)$ of orders r'_k $(k=1,\dots,N-1)$, upper generators g_i $(i=1,\dots,N-1)$, h_j $(j=2,\dots,N)$, b_k $(k=2,\dots,N-1)$ of orders r''_k $(k=1,\dots,N-1)$ and diagonal entries d_k $(k=1,\dots,N)$. Let us define the numbers ρ_k via recursive relations $\rho_N = 0$, $\rho_{k-1} = \min\{1+\rho_k,\ r'_{k-1}\}$, $k=N,\dots,2$, $\rho_0 = 0$ and the numbers $m_k = 1, n_k = 1, \nu_k = 1+\rho_k-\rho_{k-1}$, $\rho'_k = r''_k + \rho_k$, $k=1,\dots,N$.*

The matrix R admits the factorization

$$R = VUS,$$

where V is a unitary matrix represented in the block lower triangular form with blocks of sizes $m_i \times \nu_j$ $(i,j=1,\dots,N)$, lower generators $(p_V)_i$ $(i=2,\dots,N)$, $(q_V)_j$ $(j=1,\dots,N-1)$, $(a_V)_k$ $(k=2,\dots,N-1)$ of orders ρ_k $(k=1,\dots,N-1)$ and diagonal entries $(d_V)_k$ $(k=1,\dots,N)$, U is a unitary matrix represented in the block upper triangular form with blocks of sizes $\nu_i \times n_j$ $(i,j=1,\dots,N)$, upper generators $(g_U)_i$ $(i=1,\dots,N-1)$, $(h_U)_j$ $(j=2,\dots,N)$, $(b_U)_k$ $(k=2,\dots,N-1)$ of orders ρ_k $(k=1,\dots,N-1)$ and diagonal entries $(d_U)_k$ $(k=1,\dots,N)$ and S is an upper triangular matrix with upper generators $(g_S)_i$ $(i=1,\dots,N-1)$, $(h_S)_j$ $(j=2,\dots,N)$, $(b_S)_k$ $(k=2,\dots,N-1)$ of orders ρ'_k $(k=1,\dots,N-1)$ and diagonal entries $(d_S)_k$ $(k=1,\dots,N)$.

The generators and the diagonal entries of the matrices V, U, S are determined using the following algorithm.

1.1. If $r'_{N-1} > 0$ *set*

$$X_N = p_N, \quad (p_V)_N = 1, \quad (h_S)_N = \begin{bmatrix} h_N \\ d_N \end{bmatrix},$$

$(d_v)_N$ *to be* 1×0 *empty matrix,* Δ_N *to be* 0×1 *empty matrix;*

if $r'_{N-1} = 0$ *set* X_N *to be* 0×0 *empty matrix,* $(p_V)_N$ *to be* 1×0 *empty matrix,*

$$(d_V)_N = 1, \quad (h_S)_N = h_N, \quad \Delta_N = d_N.$$

1.2. For $k = N-1, \dots, 2$ *perform the following. Compute the QR factorization*

$$\begin{bmatrix} p_k \\ X_{k+1}a_k \end{bmatrix} = V_k \begin{pmatrix} X_k \\ 0 \end{pmatrix},$$

where V_k *is a unitary matrix of sizes* $(1+\rho_k) \times (1+\rho_k)$*,* X_k *is a matrix of sizes* $\rho_{k-1} \times r'_{k-1}$*. Determine matrices* $(p_V)_k,\ (a_V)_k,\ (d_V)_k,\ (q_V)_k$ *of sizes* $1 \times \rho_{k-1},\ \rho_k \times \rho_{k-1},\ 1 \times \nu_k,\ \rho_k \times \nu_k$ *from the partition*

$$V_k = \begin{bmatrix} (p_V)_k & (d_V)_k \\ (a_V)_k & (q_V)_k \end{bmatrix}.$$

Compute

$$h'_k = (p_V)^*_k d_k + (a_V)^*_k X_{k+1} q_k, \quad (h_S)_k = \begin{bmatrix} h_k \\ h'_k \end{bmatrix}, \quad (b_S)_k = \begin{pmatrix} b_k & 0 \\ (p^*_V)_k g_k & (a_V)^*_k \end{pmatrix},$$
$$\Theta_k = \begin{bmatrix} (d_V)^*_k g_k & (q_V)^*_k \end{bmatrix}, \quad \Delta_k = (d_V)^*_k d_k + (q_V)^*_k X_{k+1} q_k.$$

1.3. Set $V_1 = I_{\nu_1}$ *and define matrices* $(d_V)_1,\ (q_V)_1$ *of sizes* $1 \times \rho_1,\ \rho_1 \times \nu_1$ *from the partition*

$$V_1 = \begin{bmatrix} (d_V)_1 \\ (q_V)_1 \end{bmatrix};$$

compute

$$\Delta_1 = \begin{pmatrix} d_1 \\ X_2 q_1 \end{pmatrix}, \ \Theta_1 = \begin{pmatrix} g_1 & 0 \\ 0 & I_{\rho_1} \end{pmatrix}.$$

Thus we have computed generators and diagonal entries of the natrix V *and generators* $(b_S)_k,\ (h_S)_k$ *of the matrix* S*.*

2.1. Compute the QR factorization

$$\begin{bmatrix} \Delta_1 & \Theta_1 \end{bmatrix} = U_1 \begin{bmatrix} (d_S)_1 & (g_S)_1 \\ 0 & Y_1 \end{bmatrix},$$

where U_1 *is a unitary matrix of sizes* $\nu_1 \times \nu_1$*,* $(d_S)_1$ *is a number,* $(g_S)_1$ *is a row of size* ρ'_1*,* Y_1 *is a matrix of sizes* $\rho_1 \times \rho'_1$*. Determine matrices* $(d_U)_1,\ (g_U)_1$ *of sizes* $\nu_1 \times 1,\ \nu_1 \times \rho'_1$ *from the partition*

$$U_1 = \begin{bmatrix} (d_U)_1 & (g_U)_1 \end{bmatrix}.$$

2.2. For $k = 2, \dots, N-1$ perform the following. Compute the QR factorization

$$\begin{bmatrix} Y_{k-1}(h_S)_k & Y_{k-1}(b_S)_k \\ \Delta_k & \Theta_k \end{bmatrix} = U_k \begin{bmatrix} (d_S)_k & (g_S)_k \\ 0 & Y_k \end{bmatrix},$$

where U_k is a unitary matrix of sizes $(1+\rho_k) \times (1+\rho_k)$, $(d_S)_k$ is a number, $(g_S)_k$ is a row of size ρ'_k, Y_k is a matrix of sizes $\rho_k \times \rho'_k$.

2.3. If $r'_{N-1} > 0$ set $(d_U)_N = 1$ and $(h_U)_N$ to be 0×1 empty matrix;
if $r'_{N-1} = 0$ set $(h_U)_N = 1$ and $(d_U)_N$ to be 0×1 empty matrix;
compute

$$(d_S)_N = \begin{bmatrix} Y_{N-1}(h_S)_N \\ \Delta_N \end{bmatrix}.$$

Thus we have computed generators and diagonal entries of the matrix U and generators $(g_S)_k$ and diagonal entries $(d_S)_k$ of the matrix S.

Theorem 3.1 yields the QR-factorization of the matrix R, i.e. representation of R in the form $R = QS$ with the unitary matrix $Q = UV$ and the upper triangular matrix S. For the next considerations we should obtain generators of the matrix Q explicitly.

Theorem 3.2 *Let $R = \{R_{ij}\}_{i,j=1}^N$ be a scalar matrix with lower generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$ of orders r'_k $(k = 1, \dots, N-1)$, upper generators g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$ of orders r''_k $(k = 1, \dots, N-1)$ and diagonal entries d_k $(k = 1, \dots, N)$. Let us define the numbers ρ_k via recursive relations $\rho_N = 0$, $\rho_{k-1} = \min\{1 + \rho_k, \ r'_{k-1}\}$, $k = N, \dots, 2$, $\rho_0 = 0$ and the numbers $\rho'_k = r''_k + \rho_k$, $k = 1, \dots, N$.*

The matrix R admits the factorization

$$R = QS,$$

where Q is a unitary matrix with lower generators $(p_Q)_i$ $(i = 2, \dots, N)$, $(q_Q)_j$ $(j = 1, \dots, N-1)$, $(a_Q)_k$ $(k = 2, \dots, N-1)$ of orders ρ_k $(k = 1, \dots, N-1)$, upper generators $(g_Q)_i$ $(i = 1, \dots, N-1)$, $(h_Q)_j$ $(j = 2, \dots, N)$, $(b_Q)_k$ $(k = 2, \dots, N-1)$ of orders ρ_k $(k = 1, \dots, N-1)$ also and diagonal entries $(d_Q)_k$ $(k = 1, \dots, N)$ and S is an upper triangular matrix with upper generators $(g_S)_i$ $(i = 1, \dots, N-1)$, $(h_S)_j$ $(j = 2, \dots, N)$, $(b_S)_k$ $(k = 2, \dots, N-1)$ of orders ρ'_k $(k = 1, \dots, N-1)$ and diagonal entries $(d_S)_k$ $(k = 1, \dots, N)$.

The generators and the diagonal entries of the matrices Q and S are determined using the following algorithm.

1. Using the algorithm from Theorem 3.1 compute generators and diagonal entries of the upper triangular matrix S and of the unitary block triangular matrices V and U such that $R = VUS$.

2. Compute generators and diagonal entries of the matrix $Q = VU$ using generators and diagonal entries of the matrices V, U as follows.

2.1. Compute

$$z_1 = (q_V)_1 (g_U)_1,$$

$$(q_Q)_1 = (q_V)_1(d_U)_1, \quad \alpha_1 = (a_V)_2 z_1, \tag{3.1}$$

$$(d_Q)_1 = (d_V)_1(d_U)_1, \quad \beta_1 = z_1, \tag{3.2}$$

$$(g_Q)_1 = (d_V)_1(g_U)_1, \quad \gamma_1 = z_1(b_U)_2. \tag{3.3}$$

Set $(a_V)_N = 0_{0\times\rho_{N-1}}$, $(b_V)_N = 0_{\rho_{N-1}\times 0}$.

2.2. For $i = 2, \dots, N-1$ *perform the following. Set*

$$(p_Q)_i = (p_V)_i, \quad (a_Q)_i = (a_V)_i, \quad (b_Q)_i = (b_U)_i, \quad (h_Q)_i = (h_U)_i.$$

Compute

$$z_i = (q_V)_i(g_U)_i,$$

$$(q_Q)_i = (q_V)_i(d_U)_i + \alpha_{i-1}(h_U)_i, \quad \alpha_i = (a_V)_{i+1}[z_i + \alpha_{i-1}(b_U)_i], \tag{3.4}$$

$$(d_Q)_i = (d_V)_i(d_U)_i + (p_V)_i\beta_{i-1}(h_U)_i, \quad \beta_i = z_i + (a_V)_i\beta_{i-1}(b_U)_i, \tag{3.5}$$

$$(g_Q)_i = (d_V)_i(g_U)_i + (q_V)_i\gamma_{i-1}, \quad \gamma_i = [z_i + (a_V)_i\gamma_{i-1}](b_U)_{i+1}. \tag{3.6}$$

2.3. Set $(p_Q)_N = (p_V)_N$, $(h_Q)_N = (h_U)_N$. *Compute*

$$(d_Q)_N = (d_V)_N(d_U)_N + (p_V)_N\beta_{N-1}(h_U)_N. \tag{3.7}$$

Proof. We should justify the second stage of the algorithm. Let $Q = \{Q_{ij}\}_{i,j=1}^N$, $V = \{V_{ij}\}_{i,j=1}^N$, $U = \{U_{ij}\}_{i,j=1}^N$. For $N \geq i > j \geq 1$ since U is an upper triangular matrix and $(p_V)_i$ $(i = 2, \dots, N)$, $(q_V)_j$ $(j = 1, \dots, N-1)$, $(a_V)_k$ $(k = 2, \dots, N-1)$ are lower generators of the matrix V we have

$$Q_{ij} = \sum_{k=1}^{j} V_{ik}U_{kj} = \sum_{k=1}^{j}(p_V)_i(a_V)_{ik}^{\times}(q_V)_k U_{kj}.$$

Using the equality (2.1) we obtain

$$Q_{ij} = (p_V)_i(a_V)_{ij}^{\times}(q_Q)_j, \quad 1 \leq j < i \leq N$$

where

$$(q_Q)_j = \sum_{k=1}^{j}(a_V)_{j+1,k}^{\times}(q_V)_k U_{kj}, \quad j = 1, \dots, N-1. \tag{3.8}$$

This implies that the matrix Q has the lower generators $(p_Q)_i = (p_V)_i$ $(i = 2, \dots, N)$, $(a_Q)_k = (a_V)_k$ $(k = 2, \dots, N-1)$ and $(q_Q)_j$ $(j = 1, \dots, N-1)$ defined in (3.8). This in particular means that the orders ρ_k $(k = 1, \dots, N-1)$ of these generators are the same as for the matrix V. Now we must check that the generators $(q_Q)_j$ satisfy the relations (3.1), (3.4). Indeed for $j = 1$ we have

$$(q_Q)_1 = (a_V)_{2,1}^{\times}(q_V)_1 U_{11} = (q_V)_1(d_U)_1$$

and for $j=2,\dots,N-1$ using $U_{jj}=(d_U)_j$ and the fact that $(g_U)_i$ $(i=1,\dots,N-1)$, $(h_U)_j$ $(j=2,\dots,N)$, $(b_U)_k$ $(k=2,\dots,N-1)$ are the upper generators of the matrix U we get

$$(q_Q)_j=\sum_{k=1}^{j-1}(a_V)^{\times}_{j+1,k}(q_V)_k(g_U)_k(b_U)^{\times}_{kj}(h_U)_j+(a_V)^{\times}_{j+1,j}(q_V)_j(d_U)_j=\alpha_{j-1}(h_U)_j+(q_V)_j(d_U)_j,$$

where

$$\alpha_{j-1}=\sum_{k=1}^{j-1}(a_V)^{\times}_{j+1,k}(q_V)_k(g_U)_k(b_U)^{\times}_{kj}.$$

We have

$$\alpha_1=(a_V)^{\times}_{3,1}(q_V)_1(g_U)_1(b_U)^{\times}_{2,1}=(a_V)_2(q_V)_1(g_U)_1$$

and using the relations (2.1), (2.2) we obtain

$$\begin{aligned}\alpha_j&=\sum_{k=1}^{j}(a_V)^{\times}_{j+2,k}(q_V)_k(g_U)_k(b_U)^{\times}_{k,j+1}\\
&=(a_V)^{\times}_{j+2,j}(q_V)_j(g_U)_j(b_U)^{\times}_{j,j+1}+(a_V)_{j+1}(\sum_{k=1}^{j-1}(a_V)^{\times}_{j+1,k}(q_V)_k(g_U)_k(b_U)^{\times}_{kj})(b_U)_j\\
&=(a_V)_{j+1}(q_V)_j(g_U)_j+(a_V)_{j+1}\alpha_{j-1})(b_U)_j\end{aligned}$$

which completes the proof of (3.1), (3.4).

For diagonal entries of the matrix Q we have

$$(d_Q)_1=Q_{11}=V_{11}U_{11}=(d_V)_1(d_U)_1$$

and for $i=2,\dots,N$

$$Q_{ii}=\sum_{k=1}^{i}V_{ik}U_{ki}=V_{ii}U_{ii}+\sum_{k=1}^{i-1}V_{ik}U_{ki}=(d_V)_i(d_U)_i+(p_V)_i\beta_{i-1}(h_U)_i,$$

where

$$\beta_{i-1}=\sum_{k=1}^{i-1}(a_V)^{\times}_{ik}(q_V)_k(g_U)_k(b_U)^{\times}_{ki}$$

We have $\beta_1=(q_V)_1(g_U)_1$ and using the relations (2.1), (2.2) we obtain

$$\begin{aligned}\beta_i&=\sum_{k=1}^{i}(a_V)^{\times}_{i+1,k}(q_V)_k(g_U)_k(b_U)^{\times}_{k,i+1}\\
&=(a_V)^{\times}_{i+1,i}(q_V)_i(g_U)_i(b_U)^{\times}_{i,i+1}+(a_V)_i(\sum_{k=1}^{j-1}(a_V)^{\times}_{ik}(q_V)_k(g_U)_k(b_U)^{\times}_{ki})(b_U)_i\\
&=(q_V)_i(g_U)_i+(a_V)_i\beta_{i-1})(b_U)_i\end{aligned}$$

which completes the proof of (3.2), (3.5), (3.7).

The proof of the relations (3.3), (3.6) is performed in the same way as the proof of (3.1), (3.4). $\square$

Corollary 3.3 *Let R be a quasiseparable of order (n_L, n_U) matrix with scalar entries and let $R = QS$ be the factorisation obtained in Theorem 3.2. Then the unitary matrix Q is quasiseparable of order (n_L, n_L) at most and the upper triangular matrix S is upper quasiseparable of order $n_L + n_U$ at most.*

Proof. By Theorem 3.2 the matrix Q has lower and upper generators of the orders ρ_k ($k = 1, \dots, N-1$) defined by the relations

$$\rho_N = 0, \ \rho_{k-1} = \min\{1 + \rho_k, \ r'_{k-1}\}, \ k = N, \dots, 2 \tag{3.9}$$

and by Theorem 3.1 the matrix S has upper generators of orders

$$\rho'_k = r''_k + \rho_k, \ k = 1, \dots, N-1. \tag{3.10}$$

From the inequalities $r'_k \leq n_L$ ($k = 1, \dots, N-1$) and the relations 3.9 it follows that

$$\rho_k \leq r'_k \leq n_L, \quad k = 1, \dots, N-1 \tag{3.11}$$

and hence the maximal order of generators of the matrix Q is not greater than n_L. Next from (3.10) and (3.11) we conclude that the maximal order of upper generators of the matrix S is not greater than $n_L + n_U$. $\square$

4 The QR iteration

We consider the QR iteration algorithm for matrices defined via generators. In each iteration step for a given matrix R and for a given real number σ the new iterant R_1 is obtained by the rule

$$\begin{cases} R - \sigma I = QS, \\ R_1 = \sigma I + QR, \end{cases}$$

where Q is a unitary matrix and S is an upper triangular matrix. We show that the matrix R_1 has lower generators with the same order as the lower generators of the matrix Q and hence these orders are not greater that the corresponding generators of the matrix R and obtain an algorithm for computation of these generators and the diagonal entries of the matrix R_1.

Theorem 4.1 *Let $R = \{R_{ij}\}_{i,j=1}^N$ be a scalar matrix with lower generators p_i ($i = 2, \dots, N$), q_j ($j = 1, \dots, N-1$), a_k ($k = 2, \dots, N-1$) of orders r'_k ($k = 1, \dots, N-1$), upper generators g_i ($i = 1, \dots, N-1$), h_j ($j = 2, \dots, N$), b_k ($k = 2, \dots, N-1$) of orders r''_k ($k = 1, \dots, N-1$) and diagonal entries d_k ($k = 1, \dots, N$) and σ be a real number. Let us define the numbers*

ρ_k via recursive relations $\rho_N = 0,\ \rho_{k-1} = \min\{1+\rho_k,\ r'_{k-1}\},\ k = N,\dots,2,\ \rho_0 = 0$. Define the matrix R_1 by the rule

$$\begin{cases} R - \sigma I = QS, \\ R_1 = \sigma I + QR, \end{cases}$$

where Q is a unitary matrix and S is an upper triangular matrix.

The matrix R_1 has lower generators of orders ρ_k $(k = 1,\dots,N-1)$. These lower generators $p_i^{(1)}$ $(i = 2,\dots,N)$, $q_j^{(1)}$ $(j = 1,\dots,N-1)$, $a_k^{(1)}$ $(k = 2,\dots,N-1)$ and the diagonal entries $d_k^{(1)}$ $(k = 1,\dots,N)$ of the matrix R are determined using the following algorithm.

1. Apply to the matrix $R - \sigma I$, which has the same lower and upper generators as the matrix R and the diagonal entries $d_k - \sigma$ $(k = 1,\dots,N)$, the algorithm from Theorem 3.2, to compute the lower generators $(p_Q)_i$ $(i = 2,\dots,N)$, $(q_Q)_j$ $(j = 1,\dots,N-1)$, $(a_Q)_k$ $(k = 2,\dots,N-1)$ and the diagonal entries $(d_Q)_k$ $(k = 1,\dots,N)$ of the matrix Q and the upper generators $(g_S)_i$ $(i = 1,\dots,N-1)$, $(h_S)_j$ $(j = 2,\dots,N)$, $(b_S)_k$ $(k = 2,\dots,N-1)$ and the diagonal entries $(d_S)_k$ $(k = 1,\dots,N)$ of the matrix S.

2. Compute the lower generators and the diagonal entries of the matrix Q as follows.

2.1. Compute

$$z_N = (h_S)_N (p_Q)_N,$$

$$p_N^{(1)} = (d_S)_N (p_Q)_N, \quad \alpha_N = z_N (a_Q)_{N-1}, \tag{4.1}$$

$$d_N^{(1)} = (d_S)_N (d_Q)_N, \quad \beta_N = z_N, \tag{4.2}$$

Set $(a_Q)_1 = 0_{\rho_1 \times 0}$.

2.2. For $i = N-1,\dots,2$ perform the following. Set

$$q_i^{(1)} = (q_Q)_i, \quad a_i^{(1)} = (a_Q)_i.$$

Compute

$$z_i = (h_S)_i (p_Q)_i,$$

$$p_i^{(1)} = (d_S)_i (p_Q)_i + (g_S)_i \alpha_{i+1}, \quad \alpha_i = [(h_S)_i (p_Q)_i + (b_S)_i \alpha_{i+1}](a_Q)_{i-1}, \tag{4.3}$$

$$d_i^{(1)} = (d_S)_i (d_Q)_i + (g_S)_i \beta_{i+1} (q_Q)_i, \quad \beta_i = z_i + (b_S)_i \beta_{i+1} (a_Q)_i. \tag{4.4}$$

2.3. Set $q_1^{(1)} = (q_Q)_1$. Compute

$$d_1^{(1)} = (d_S)_1 (d_Q)_1 + (g_S)_1 \beta_2 (q_Q)_1. \tag{4.5}$$

*Proof.*We should justify the second stage of the algorithm. Let $Q = \{Q_{ij}\}_{i,j=1}^N$, $S = \{S_{ij}\}_{i,j=1}^N$ and $R_1 = \{R_{ij}^{(1)}\}_{i,j=1}^N$. For $N \geq i > j \geq 1$ using the fact S is an upper triangular matrix and $(p_Q)_i$ $(i = 2,\dots,N)$, $(q_Q)_j$ $(j = 1,\dots,N-1)$, $(a_Q)_k$ $(k = 2,\dots,N-1)$ are lower generators of the matrix Q we have

$$R_{ij}^{(1)} = \sum_{k=i}^{N} S_{ik} Q_{kj} = \sum_{k=i}^{N} S_{ik} (p_Q)_k (a_Q)_{kj}^{\times} (q_Q)_j.$$

Using the equality (2.1) we obtain

$$R^{(1)}_{ij} = p^{(1)}_i (a_Q)^{\times}_{ij} (q_Q)_j, \quad 1 \le j < i \le N$$

where

$$p^{(1)}_i = \sum_{k=i}^{N} S_{ik} (p_Q)_k (a_Q)^{\times}_{k,i-1}, \quad i = 2, \dots, N. \tag{4.6}$$

This implies that the matrix $R^{(1)}$ has the lower generators $a^{(1)}_k = (a_Q)_k$ $(k = 2, \dots, N-1)$, $q^{(1)}_j = (q_Q)_j$ $(j = 1, \dots, N-1)$ and $p^{(1)}$ $(i = 2, \dots, N)$ defined in (4.6). This in particular means that the orders ρ_k $(k = 1, \dots, N-1)$ of these generators are the same as for the matrix Q. Now we must check that the generators $p^{(1)}_i$ satisfy the relations (4.1), (4.3). Indeed for $i = N$ we have

$$p^{(1)}_N = S_{NN} (p_Q)_N (a_Q)^{\times}_{N,N-1} = (d_S)_N (p_Q)_N$$

and for $i = N-1, \dots, 2$ using $S_{jj} = (d_S)_j$ and the fact that $(g_S)_i$ $(i = 1, \dots, N-1)$, $(h_S)_j$ $(j = 2, \dots, N)$, $(b_S)_k$ $(k = 2, \dots, N-1)$ are the upper generators of the matrix S we get

$$p^{(1)}_i = (g_S)_i \sum_{k=i+1}^{N} (b_S)^{\times}_{ik} (h_S)_k (p_Q)_k (a_Q)^{\times}_{k,i-1} + (d_S)_i (p_Q)_i (a_Q)^{\times}_{i,i-1} = (d_S)_i (p_Q)_i + (g_S)_i \alpha_{i+1},$$

where

$$\alpha_{i+1} = \sum_{k=i+1}^{N} (b_S)^{\times}_{ik} (h_S)_k (p_Q)_k (a_Q)^{\times}_{k,i-1}.$$

We have

$$\alpha_N = (b_S)^{\times}_{N-1,N} (h_S)_N (p_Q)_N (a_Q)^{\times}_{N,N-2} = (h_S)_N (p_Q)_N (a_Q)_{N-1}$$

and using the relations (2.1), (2.2) we obtain

$$\begin{aligned}\alpha_i &= \sum_{k=i}^{N} (b_S)^{\times}_{i-1,k} (h_S)_k (p_Q)_k (a_Q)^{\times}_{k,i-2} \\ &= (b_S)^{\times}_{i-1,i} (h_S)_i (p_Q)_i (a_Q)^{\times}_{i,i-2} + (b_S)_i \Big(\sum_{k=i+1}^{N} (b_S)^{\times}_{ik} (h_S)_k (p_Q)_k (a_Q)^{\times}_{k,i-1} \Big) (a_Q)_{i-1} \\ &= [(h_S)_i (p_Q)_i + (b_S)_i \alpha_{i+1}] (a_Q)_{i-1}\end{aligned}$$

which completes the proof of (4.1), (4.3).

For diagonal entries of the matrix S we have

$$d^{(1)}_N = R^{(1)}_{NN} = S_{NN} Q_{NN} = (d_S)_N (d_Q)_N$$

and for $i = N-1, \dots, 1$

$$R_{ii}^{(1)} = \sum_{k=i}^{N} S_{ik}Q_{ki} = S_{ii}Q_{ii} + \sum_{k=i+1}^{N} S_{ik}Q_{ki} = (d_S)_i(d_Q)_i + (g_S)_i\beta_{i+1}(h_S)_i,$$

where

$$\beta_{i+1} = \sum_{k=i+1}^{N} (b_S)_{ik}^{\times}(h_S)_k(p_Q)_k(a_Q)_{ki}^{\times}$$

We have $\beta_1 = (q_V)_1(g_U)_1$ and using the relations (2.1), (2.2) we obtain

$$\beta_i = \sum_{k=i}^{N}(b_S)_{i-1,k}^{\times}(h_S)_k(p_Q)_k(a_Q)_{k,i-1}^{\times}$$

$$= (b_S)_{i-1,i}^{\times}(h_S)_i(p_Q)_i(a_Q)_{i,i-1}^{\times} + (b_S)_i(\sum_{k=i+1}^{N}(b_S)_{ik}^{\times}(h_S)_k(p_Q)_k(a_Q)_{ki}^{\times})(a_Q)_i =$$

$$(h_S)_i(p_Q)_i + (b_S)_i\beta_{i+1})(a_Q)_i$$

which completes the proof of (4.2), (4.4), (4.5). $\square$

Corollary 4.2 *Let R be a lower quasiseparable of order n_L matrix with scalar entries and let R_1 be the matrix obtained in Theorem 3.2. Then the unitary matrix Q is quasiseparable of order (n_L, n_L) at most and the upper triangular matrix S is upper quasiseparable of order $n_L + n_U$ at most.*

Proof folows directly from Theorem 4.1 and Corollary 3.3.

Now assume that the matrix R is Hermitian. Then the new iterant R_1 is a Hermitian matrix which is quasiseparable of the same order as the matrix R. This means that for a quasiseparable of a given order Hermitian matrix, the result of QR iteration has the same structure as the original matrix. Moreover an algorithm for computation of this structure is given.

Theorem 4.3 *Let $R = \{R_{ij}\}_{i,j=1}^{N}$ be a scalar Hermitian quasiseparable of order (n,n) matrix with lower generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$ of orders r'_k $(k = 1, \dots, N-1)$, upper generators q_i^* $(i = 1, \dots, N-1)$, p_j^* $(j = 2, \dots, N)$, a_k^* $(k = 2, \dots, N-1)$ and diagonal entries d_k $(k = 1, \dots, N)$ and σ be a real number. Define the matrix R_1 by the rule*

$$\begin{cases} R - \sigma I = QS, \\ R_1 = \sigma I + QR, \end{cases}$$

where Q is a unitary matrix and S is an upper triangular matrix.

Then R_1 is a Hermitian quasiseparable of order (n,n) at most matrix and generators and diagonal entries of this matrix are obtained using the algorithm from Theorem 4.1.

References

[1] Y. Eidelman and I. Gohberg, A modification of the Dewilde-van der Veen method for inversion of finite structured matrices. *Linear Algebra and Application* 343-344: 419-450 (2002).