The QR iteration method for Hermitian quasiseparable matrices of an arbitrary order

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Abstract

The QR iteration method for *tridiagonal* matrices is in the heart of one classical method to solve the general eigenvalue problem. In this paper we consider the more general class of *quasiseparable* matrices that includes not only tridiagonal but also companion, comrade, unitary Hessenberg and semiseparble matrices. A fast QR iteration method exploiting the Hermitian quasiseparable structure (and thus generalizing the classical tridiagonal scheme) is presented. The algorithm is based on an earlier work [9], and it applies to the general case of Hermitian quasiseparable matrices of an *arbitrary order*.

The algorithm operates on generators (i.e., a linear set of parameters defining the quasiseparable matrix), and the storage and the cost of one iteration are only linear. The results of some numerical experiments are presented.

An application of this method to solve the general eigenvalue problem via quasiseparable matrices will be analyzed separately elsewhere.

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1 Introduction

1.1. Tridiagonal, semiseparable and quasiseparable matrices.

Tridiagonal matrices are instrumental in a number of classical methods for solving the general eigenvalue problem. It is well-known [12] that the inverses of tridiagonal matrices are not sparse, but they have a $(1,1)$-semiseparable structure defined next.

A matrix A is called order (n_L, n_U)-*semiseparable* if

$$A = D + tril(A_L) + triu(A_U), \quad \text{where} \quad \text{rank}A_L = n_L, \quad \text{rank}A_U = n_U \tag{1.1}$$

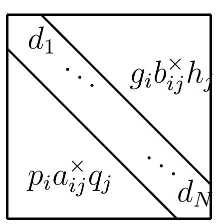
with some matrices A_L, A_U and a diagonal matrix D. Here $tril(A)$ and $triu(A)$ are the standard MATLAB notations standing for the strictly lower and strictly upper triangular parts of A, resp.

It is very easy to see that tridiagonal matrices are not semiseparable, we will return to this observation in a moment. In fact, both belong to the more general class of *quasiseparable* matrices defined next. Following [7] we refer to A as an *order-*(n_L, n_U)*-quasiseparable* matrix if

$$n_L = \max \text{rank}A_{21}, \qquad n_U = \max \text{rank}A_{12}, \tag{1.2}$$

where the maximum is taken over all *symmetric* partitions of the form $A = \left[\begin{array}{c|c} * & A_{12} \\ \hline A_{21} & * \end{array}\right]$. In case $n_U = n_L = r$ one refers to A as an order-r-quasiseparable [3] matrix.

1.2. Generators. A computationally important property of $N \times N$ quasiseparable matrices A is (see [5, p.55] and also [7]) that they can be represented by only $O((n_L + n_U)N)$ parameters via

$A =$ 

(1.3)

where $\quad a^{\times}_{ij} = a_{i-1} \cdot \ldots \cdot a_{j+1}, \quad b^{\times}_{ij} = b_{i+1} \cdot \ldots \cdot b_{j-1} \quad$ with $\quad a_{i+1,i} = b_{i,i+1} = 1.$

[3] Quasiseparable matrices are also known under different names such as matrices with a low Hankel rank [5], or weakly semiseparable matrices [16].

Here p_i $(i = 2, \dots, N)$ are vector rows and q_j $(j = 1, \dots, N-1)$ are vector columns of sizes r'_{i-1} and r'_j respectively, and a_k $(k = 2, \dots, N-1)$ are matrices of sizes $r'_k \times r'_{k-1}$;these elements are said to be *lower generators of the matrix* A with *orders* r'_k $(k = 1, \dots, N-1)$. The elements g_i $(i = 1, \dots, N-1)$ are vector rows and h_j $(j = 2, \dots, N)$ are vector columns of sizes r''_i and r''_{j-1} respectively, and b_k $(k = 2, \dots, N-1)$ are matrices of sizes $r''_{k-1} \times r''_k$; these elements are said to be *upper generators of the matrix* A with *orders* r''_k, $(k = 1, \dots, N-1)$. The numbers d_k $(k = 1, \dots, N)$ are the *diagonal entries* of the matrix A. Here $n_L = \max_{1 \le k \le N-1} r'_k$, $n_U = \max_{1 \le k \le N-1} r''_k$. Though we focus in this paper on algorithms for matrices with scalar entries, block quasiseparable matrices are also used in our proofs and computations.

Notice that for a given quasiseparable matrix generators are not unique. The appropriate choice of generators could be important in particular applications since it determines the numerical behaviour of the corresponding algorithms. Examples of making good numerical choices for generators can be found in [6,8] and in [18,17]. The analysis of all possible choices of generators for a given quasiseparable matrix will be performed in our subsequent publication.

1.3. The tridiagonal and quasiseparable QR iteration. It is well known that the QR iteration can be implemented very efficiently for tridiagonal A. The relevant facts are: (i) $A = QR$ can be computed cheaply; (ii) the next iterant $A_1 = RQ$ is also tridiagonal [15]. Both facts carry over to the more general Hermitian quasiseparable case: (i) we can also compute $A = QR$ fast (cf with [9]); (ii) the next iterant $A_1 = RQ$ has the same order of quasiseparability (cf with [7]). Moreover, as explained next the algorithms for computing Q, R, and $A_1 = RQ$ can be constructed from building blocks that have already been used in [7,9] for different purposes.

1.4. The structure of the Q and R factors. To compute the generators of the Q and R factors we adapt the algorithm of [9] (derived there via applying the more general Dewilde-Van der Veen method [5] to *finite* quasiseparable matrices). Specifically, given generators of an order (n_L, n_U) quasiseparable matrix A, the algorithm of [9] computes

$$\underbrace{A}_{(n_L,n_U)} = \underbrace{V}_{(n_L,*)} \cdot \underbrace{U}_{(*,n_L)} \cdot \underbrace{R}_{(0,n_L+n_U)}$$

where the order of quasiseparability is shown in the bottom line. Here V, U are unitary matrices and R is an upper triangular matrix.

Matrices V and U are quasiseparable, so it is not surprising that $Q = VU$ has some structure as well. It can be shown that the factor $Q = VU$ is order (n_L, n_L) quasiseparable. Hence in order to convert an algorithm of [9] into a form computing the generators of Q and R one just needs to provide a structure-exploiting method for multiplying V and U. A similar method for multiplication of quasiseparable matrices has already been used in [7], and here we adapt the approach to produce an $O(N)$ algorithm for computing generators of the factors in

$$\underbrace{A}_{(n_L,n_U)} = \underbrace{Q}_{(n_L,n_L)} \cdot \underbrace{R.}_{(0,n_L+n_U)} \tag{1.4}$$

1.4. Quasiseparable QR iteration. If A is Hermitian, then (1.4) becomes

$$\underbrace{A}_{(n,n)} = \underbrace{Q}_{(n,n)} \cdot \underbrace{R.}_{(0,2n)} \tag{1.5}$$

Now, observe that the Hermitian quasiseparable structure is inherited under QR iteration, i.e., that the order of quasiseparability of A_1 is also n:

$$\underbrace{A_1}_{(n,n)} = \underbrace{R}_{(0,2n)} \cdot \underbrace{Q.}_{(n,n)} \tag{1.6}$$

Though this fact follows from the multiplication algorithms of [7] (applied to the particular patterns of Q and R shown in (1.6)), we provide here a different and a more transparent explanation. Let us partition A and A_1 as in

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix} \cdot \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix}$$

and

$$A_1 = \begin{bmatrix} A_{11}^{(1)} & A_{12}^{(1)} \\ A_{21}^{(1)} & A_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} R_{11} & R_{12} \\ 0 & R_{22} \end{bmatrix} \cdot \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix},$$

so that $A_{21} = Q_{21}R_{11}$ and $A_{21}^{(1)} = R_{22}Q_{21}$. Assume, for simplicity, that R is invertible (this assumption is not used in the paper). Then it is immediate to see that $rank A_{21} = rank A_{21}^{(1)}$. In the Hermitian case it means that A_1 inherits the order of quasiseparability of A. Again, the techniques of [7] for multiplication of quasiseparable matrices can be adapted to compute $A_1 = RQ$, or A_1 for the shifted QR iteration method

$$\begin{cases} A - \sigma I = QR, \\ A_1 = \sigma I + RQ, \end{cases}$$

Notice that the total complexity of one step of QR iteration is $O(n^3N)$ as it is shown in Section 6.

1.5. The context and the structure of the paper.

Various numerical methods for quasiseparable matrices are being suggested in parallel papers [1–4,11,13,14,17]. Their QR-iteration methods were studied for several special cases of quasi-separable matrices, mostly for the order-one matrices and their derivatives. For example, a special perturbation of a symmetric order-one matrix is considered in [2]. A symmetric matrix represented as a sum of a diagonal matrix and a semiseparable of order one Green (Hessenberg like) matrix is studied in [17,18], in these works the implicit shift technique was applied. Some types of quasiseparable matrices invariant under QR iterations were considered also in [1,10]. Remark that tridiagonal matrices are not semiseparable and hence the semiseparable algorithms are not generalizations of the classical methods rather their alternatives.

The algorithms presented here differ from their in three aspects. First, they are not limited to the low *order of semiseparability*. Secondly, they apply to the more general Hermitian *quasiseparable* structure and hence generalize both classical tridiagonal and semi-separable methods. Finally, they are based on an the earlier work and *explicitly* employ the ingredients available in the literature. An application of our method to solve the general eigenvalue problem via quasiseparable matrices will appear separately elsewhere.

The paper consists of seven sections. Section 1 is the introduction. Section 2 contains the definitions. In Section 3 we present without proof an algorithm which is a modification of the Dewilde-Van der Veen method and based on this algorithm derive an algorithm for QR factorization of a quasiseparable matrix. In Section 4 we consider one step of QR iteration for quasiseparable matrices and derive an algorithm to compute the lower generators and the diagonal entries of the next iterant via the generators of the previous one. In Section 5 we consider the case of a Hermitian quasiseparable matrix. Section 6 contains expressions for complexities of the obtained algorithms. In Section 7 we presents results of numerical experiments.

2 Definitions

Let $\{a_k\}, k = 1, \dots, N$ be a family of matrices of sizes $r_k \times r_{k-1}$. For positive integers $i, j, \ i > j$ define the operation $a^{\times}_{ij}$ as follows: $a^{\times}_{ij} = a_{i-1} \cdots a_{j+1}$ for $i > j+1$, $a^{\times}_{j+1,j} = I_{r_j}$.

Let $\{b_k\}, k = 1, \dots, N$ be a family of matrices of sizes $r_{k-1} \times r_k$. For positive integers $i, j, \ j > i$ define the operation $b^{\times}_{ij}$ as follows: $b^{\times}_{ij} = b_{i+1} \cdots b_{j-1}$ for $j > i+1$, $b^{\times}_{i,i+1} = I_{r_i}$.

It is easy to see that

$$a^{\times}_{ik} = a^{\times}_{ij} a^{\times}_{j+1,k}, \quad i > j \geq k \tag{2.1}$$

and

$$b^{\times}_{kj} = b^{\times}_{k,i+1} b^{\times}_{i,j}, \quad k \le i < j. \tag{2.2}$$

Let $R = \{R_{ij}\}_{i,j=1}^{N}$ be a matrix with block entries R_{ij} of sizes $m_i \times n_j$. Assume that the entries of this matrix are represented in the form

$$R_{ij} = \begin{cases} p_i a^{\times}_{ij} q_j, & 1 \le j < i \le N, \\ d_i, & 1 \le i = j \le N, \\ g_i b^{\times}_{ij} h_j, & 1 \le i < j \le N. \end{cases} \tag{2.3}$$

Here p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$ are matrices of sizes $m_i \times r'_{i-1}$, $r'_j \times n_j$, $r'_k \times r'_{k-1}$ respectively; these elements are said to be *lower generators of the matrix* R with *orders* r'_k $(k = 1, \dots, N-1)$. The elements g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$ are matrices of sizes $m_i \times r''_i$, $r''_{j-1} \times n_j$, $r''_{k-1} \times r''_k$ respectively; these elements are said to be *upper generators of the matrix* R with *orders* r''_k, $(k = 1, \dots, N-1)$. The matrices d_k $(k = 1, \dots, N)$ of sizes $m_k \times n_k$ are said to be *diagonal entries* of the matrix R. We define also orders of generators r'_k, r''_k for $k = 0, N$ setting them to be zeros. For scalar matrices the generators p_i, g_i and q_j, h_j are rows and columns of the corresponding sizes. Set $n_L = \max_{1 \le k \le N-1} r'_k$, $n_U = \max_{1 \le k \le N-1} r''_k$, the matrix R is said to be *order* n_L *lower quasiseparable* and *order* n_U *upper quasiseparable* or (n_L, n_U) *quasiseparable*.

Formally, we use some calculation rules with matrices that have blocks with dimension zero. Aside from obvious rules, the product of an "empty" matrix of dimension $m \times 0$ and an empty matrix of dimension $0 \times n$ is a matrix of dimension $m \times n$ with all elements equal to 0. All further rules of block matrix multiplication remain consistent. Such operations are used in MATLAB.

3 The QR factorization

Let $R = \{R_{ij}\}_{i,j=1}^{N}$ be a matrix with entries from $\mathbb{C}$ with given generators. We present here an algorithm for computing generators and diagonal entries of unitary matrix Q and upper triangular matrix S such that $R = QS$. The main part of the algorithm is based on the following result from [9]. This algorithm from [9] yields the factorization of the matrix R in the form $R = VUS$, where V, U are block unitary matrices, V is block lower triangular, U is block upper triangular, S is an upper triangular matrix. The matrices V, U, S are given by their generators which are computed using QR factorizations for the matrices of small sizes obtained via generators of the original matrix.

Theorem 3.1 *Let* $R = \{R_{ij}\}_{i,j=1}^{N}$ *be a scalar matrix with lower generators*

p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$ of orders r'_k $(k = 1, \dots, N-1)$, upper generators g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$ of orders r''_k $(k = 1, \dots, N-1)$ and diagonal entries d_k $(k = 1, \dots, N)$. Let us define the numbers ρ_k via recursive relations $\rho_N = 0$, $\rho_{k-1} = \min\{1 + \rho_k,\ r'_{k-1}\}$, $k = N, \dots, 2$, $\rho_0 = 0$ and the numbers $m_k = 1, n_k = 1, \nu_k = 1 + \rho_k - \rho_{k-1}$, $\rho'_k = r''_k + \rho_k$, $k = 1, \dots, N$.

The matrix R admits the factorization

$$R = VUS,$$

where V is a unitary matrix represented in the block lower triangular form with blocks of sizes $m_i \times \nu_j$ $(i, j = 1, \dots, N)$, lower generators $(p_V)_i$ $(i = 2, \dots, N)$, $(q_V)_j$ $(j = 1, \dots, N-1)$, $(a_V)_k$ $(k = 2, \dots, N-1)$ of orders ρ_k $(k = 1, \dots, N-1)$ and diagonal entries $(d_V)_k$ $(k = 1, \dots, N)$, U is a unitary matrix represented in the block upper triangular form with blocks of sizes $\nu_i \times n_j$ $(i, j = 1, \dots, N)$, upper generators $(g_U)_i$ $(i = 1, \dots, N-1)$, $(h_U)_j$ $(j = 2, \dots, N)$, $(b_U)_k$ $(k = 2, \dots, N-1)$ of orders ρ_k $(k = 1, \dots, N-1)$ and diagonal entries $(d_U)_k$ $(k = 1, \dots, N)$ and S is an upper triangular matrix with upper generators $(g_S)_i$ $(i = 1, \dots, N-1)$, $(h_S)_j$ $(j = 2, \dots, N)$, $(b_S)_k$ $(k = 2, \dots, N-1)$ of orders ρ'_k $(k = 1, \dots, N-1)$ and diagonal entries $(d_S)_k$ $(k = 1, \dots, N)$.

The generators and the diagonal entries of the matrices V, U, S are determined using the following algorithm.

1.1. If $r'_{N-1} > 0$ set

$$X_N = p_N, \quad (p_V)_N = 1, \quad (h_S)_N = \begin{bmatrix} h_N \\ d_N \end{bmatrix},$$

$(d_V)_N$ to be 1×0 empty matrix, Δ_N to be 0×1 empty matrix;

if $r'_{N-1} = 0$ set X_N to be 0×0 empty matrix, $(p_V)_N$ to be 1×0 empty matrix,

$$(d_V)_N = 1, \quad (h_S)_N = h_N, \quad \Delta_N = d_N.$$

1.2. For $k = N-1, \dots, 2$ perform the following. Compute the QR factorization

$$\begin{bmatrix} p_k \\ X_{k+1} a_k \end{bmatrix} = V_k \begin{pmatrix} X_k \\ 0 \end{pmatrix},$$

where V_k is a unitary matrix of sizes $(1 + \rho_k) \times (1 + \rho_k)$, X_k is a matrix of sizes $\rho_{k-1} \times r'_{k-1}$. Determine matrices $(p_V)_k$, $(a_V)_k$, $(d_V)_k$, $(q_V)_k$ of sizes

$1 \times \rho_{k-1},\ \rho_k \times \rho_{k-1},\ 1 \times \nu_k,\ \rho_k \times \nu_k$ *from the partition*

$$V_k = \begin{bmatrix} (p_V)_k & (d_V)_k \\ (a_V)_k & (q_V)_k \end{bmatrix}.$$

Compute

$$h'_k = (p_V)^*_k d_k + (a_V)^*_k X_{k+1} q_k, \quad (h_S)_k = \begin{bmatrix} h_k \\ h'_k \end{bmatrix}, \quad (b_S)_k = \begin{pmatrix} b_k & 0 \\ (p^*_V)_k g_k & (a_V)^*_k \end{pmatrix},$$
$$\Theta_k = \begin{bmatrix} (d_V)^*_k g_k & (q_V)^*_k \end{bmatrix}, \quad \Delta_k = (d_V)^*_k d_k + (q_V)^*_k X_{k+1} q_k.$$

1.3. Set $V_1 = I_{\nu_1}$ *and define matrices* $(d_V)_1,\ (q_V)_1$ *of sizes* $1 \times \nu_1,\ \rho_1 \times \nu_1$ *from the partition*

$$V_1 = \begin{bmatrix} (d_V)_1 \\ (q_V)_1 \end{bmatrix};$$

compute

$$\Delta_1 = \begin{pmatrix} d_1 \\ X_2 q_1 \end{pmatrix}, \ \Theta_1 = \begin{pmatrix} g_1 & 0 \\ 0 & I_{\rho_1} \end{pmatrix}.$$

Thus we have computed generators and diagonal entries of the matrix V *and generators* $(b_S)_k,\ (h_S)_k$ *of the matrix* S.

2.1. Compute the QR factorization

$$\begin{bmatrix} \Delta_1 & \Theta_1 \end{bmatrix} = U_1 \begin{bmatrix} (d_S)_1 & (g_S)_1 \\ 0 & Y_1 \end{bmatrix},$$

where U_1 *is a unitary matrix of sizes* $\nu_1 \times \nu_1$, $(d_S)_1$ *is a number,* $(g_S)_1$ *is a row of size* ρ'_1, Y_1 *is a matrix of sizes* $\rho_1 \times \rho'_1$. *Determine matrices* $(d_U)_1,\ (g_U)_1$ *of sizes* $\nu_1 \times 1,\ \nu_1 \times \rho'_1$ *from the partition*

$$U_1 = \begin{bmatrix} (d_U)_1 & (g_U)_1 \end{bmatrix}.$$

2.2. For $k = 2, \dots, N-1$ *perform the following. Compute the QR factorization*

$$\begin{bmatrix} Y_{k-1}(h_S)_k & Y_{k-1}(b_S)_k \\ \Delta_k & \Theta_k \end{bmatrix} = U_k \begin{bmatrix} (d_S)_k & (g_S)_k \\ 0 & Y_k \end{bmatrix},$$

where U_k is a unitary matrix of sizes $(1+\rho_k)\times(1+\rho_k)$, $(d_S)_k$ is a number, $(g_S)_k$ is a row of size ρ'_k, Y_k is a matrix of sizes $\rho_k \times \rho'_k$. Determine matrices $(d_U)_k,\ (g_U)_k,\ (h_U)_k,\ (b_U)_k$ of sizes $(1+\rho_k-\rho_{k-1})\times 1,\ (1+\rho_k-\rho_{k-1})\times \rho_k,\ \rho_{k-1}\times 1,\ \rho_{k-1}\times\rho_k$ from the partition

$$U_k = \begin{bmatrix} (h_U)_k & (b_U)_k \\ (d_U)_k & (g_U)_k \end{bmatrix}.$$

2.3. If $r'_{N-1} > 0$ set $(h_U)_N = 1$ and $(d_U)_N$ to be 0×1 empty matrix;

if $r'_{N-1} = 0$ set $(d_U)_N = 1$ and $(h_U)_N$ to be 0×1 empty matrix;

compute

$$(d_S)_N = \begin{bmatrix} Y_{N-1}(h_S)_N \\ \Delta_N \end{bmatrix}.$$

Thus we have computed generators and diagonal entries of the matrix U and generators $(g_S)_k$ and diagonal entries $(d_S)_k$ of the matrix S.

Theorem 3.1 yields the QR-factorization of the matrix R, i.e. representation of R in the form $R = QS$ with the unitary matrix $Q = VU$ and the upper triangular matrix S. For the next considerations we should obtain generators of the matrix Q explicitly. To do it we should only compute generators of the product $Q = VU$ of two block triangular quasiseparable matrices.

Theorem 3.2 *Let $R = \{R_{ij}\}_{i,j=1}^N$ be a scalar matrix with lower generators $p_i\ (i = 2,\dots,N),\ q_j\ (j = 1,\dots,N-1),\ a_k\ (k = 2,\dots,N-1)$ of orders $r'_k\ (k = 1,\dots,N-1)$, upper generators $g_i\ (i = 1,\dots,N-1),\ h_j\ (j = 2,\dots,N),\ b_k\ (k = 2,\dots,N-1)$ of orders $r''_k\ (k = 1,\dots,N-1)$ and diagonal entries $d_k\ (k = 1,\dots,N)$. Let us define the numbers ρ_k via recursive relations $\rho_N = 0,\ \rho_{k-1} = \min\{1+\rho_k,\ r'_{k-1}\},\ k = N,\dots,2,\ \rho_0 = 0$ and the numbers $\rho'_k = r''_k + \rho_k,\ k = 1,\dots,N$.*

The matrix R admits the factorization

$$R = QS,$$

where Q is a unitary matrix with lower generators $(p_Q)_i\ (i = 2,\dots,N),\ (q_Q)_j\ (j = 1,\dots,N-1),\ (a_Q)_k\ (k = 2,\dots,N-1)$ of orders $\rho_k\ (k = 1,\dots,N-1)$, upper generators $(g_Q)_i\ (i = 1,\dots,N-1),\ (h_Q)_j\ (j = 2,\dots,N),\ (b_Q)_k\ (k = 2,\dots,N-1)$ of orders $\rho_k\ (k = 1,\dots,N-1)$ also and diagonal entries $(d_Q)_k\ (k = 1,\dots,N)$ and S is an upper triangular matrix with upper generators $(g_S)_i\ (i = 1,\dots,N-1),\ (h_S)_j\ (j = 2,\dots,N),\ (b_S)_k\ (k = 2,\dots,N-1)$ of orders $\rho'_k\ (k = 1,\dots,N-1)$ and diagonal entries $(d_S)_k\ (k = 1,\dots,N)$.

The generators and the diagonal entries of the matrices Q and S are determined using the following algorithm.

1. Using the algorithm from Theorem 3.1 compute generators and diagonal entries of the upper triangular matrix S and of the unitary block triangular matrices V and U such that $R = VUS$.

2. Compute generators and diagonal entries of the matrix $Q = VU$ using generators and diagonal entries of the matrices V, U as follows.

2.1. Compute

$$z_1 = (q_V)_1 (g_U)_1,$$

$$(q_Q)_1 = (q_V)_1 (d_U)_1, \quad \alpha_1 = (a_V)_2 z_1, \tag{3.1}$$

$$(d_Q)_1 = (d_V)_1 (d_U)_1, \quad \beta_1 = z_1, \tag{3.2}$$

$$(g_Q)_1 = (d_V)_1 (g_U)_1, \quad \gamma_1 = z_1 (b_U)_2. \tag{3.3}$$

Set $(a_V)_N = 0_{0\times\rho_{N-1}}$, $(b_V)_N = 0_{\rho_{N-1}\times 0}$.

2.2. For $i = 2, \dots, N-1$ perform the following. Set

$$(p_Q)_i = (p_V)_i, \quad (a_Q)_i = (a_V)_i, \quad (b_Q)_i = (b_U)_i, \quad (h_Q)_i = (h_U)_i.$$

Compute

$$z_i = (q_V)_i (g_U)_i, \tag{3.4}$$

$$(q_Q)_i = (q_V)_i (d_U)_i + \alpha_{i-1} (h_U)_i, \quad \alpha_i = (a_V)_{i+1} [z_i + \alpha_{i-1} (b_U)_i], \tag{3.5}$$

$$(d_Q)_i = (d_V)_i (d_U)_i + (p_V)_i \beta_{i-1} (h_U)_i, \quad \beta_i = z_i + (a_V)_i \beta_{i-1} (b_U)_i, \tag{3.6}$$

$$(g_Q)_i = (d_V)_i (g_U)_i + (p_V)_i \gamma_{i-1}, \quad \gamma_i = [z_i + (a_V)_i \gamma_{i-1}] (b_U)_{i+1}. \tag{3.7}$$

2.3. Set $(p_Q)_N = (p_V)_N$, $(h_Q)_N = (h_U)_N$. Compute

$$(d_Q)_N = (d_V)_N (d_U)_N + (p_V)_N \beta_{N-1} (h_U)_N. \tag{3.8}$$

Proof. We should justify the second stage of the algorithm. Let $Q = \{Q_{ij}\}_{i,j=1}^N$, $V = \{V_{ij}\}_{i,j=1}^N$, $U = \{U_{ij}\}_{i,j=1}^N$. For $N \geq i > j \geq 1$ since U is an upper triangular matrix and $(p_V)_i$ $(i = 2, \dots, N)$, $(q_V)_j$ $(j = 1, \dots, N-1)$, $(a_V)_k$ $(k = 2, \dots, N-1)$ are lower generators of the matrix V we have

$$Q_{ij} = \sum_{k=1}^{j} V_{ik} U_{kj} = \sum_{k=1}^{j} (p_V)_i (a_V)^{\times}_{ik} (q_V)_k U_{kj}.$$

Using the equality (2.1) we obtain

$$Q_{ij} = (p_V)_i (a_V)^{\times}_{ij} (q_Q)_j, \quad 1 \leq j < i \leq N$$

where

$$(q_Q)_j = \sum_{k=1}^{j} (a_V)^{\times}_{j+1,k}(q_V)_k U_{kj}, \quad j = 1, \dots, N-1. \tag{3.9}$$

This implies that the matrix Q has the lower generators $(p_Q)_i = (p_V)_i$ ($i = 2, \dots, N$), $(a_Q)_k = (a_V)_k$ ($k = 2, \dots, N-1$) and $(q_Q)_j$ ($j = 1, \dots, N-1$) defined in (3.9). This in particular means that the orders ρ_k ($k = 1, \dots, N-1$) of these generators are the same as for the matrix V. Now we must check that the generators $(q_Q)_j$ satisfy the relations (3.1), (3.5). Indeed for $j = 1$ we have

$$(q_Q)_1 = (a_V)^{\times}_{2,1}(q_V)_1 U_{11} = (q_V)_1(d_U)_1$$

and for $j = 2, \dots, N-1$ using $U_{jj} = (d_U)_j$ and the fact that $(g_U)_i$ ($i = 1, \dots, N-1$), $(h_U)_j$ ($j = 2, \dots, N$), $(b_U)_k$ ($k = 2, \dots, N-1$) are the upper generators of the matrix U we get

$$(q_Q)_j = \sum_{k=1}^{j-1} (a_V)^{\times}_{j+1,k}(q_V)_k(g_U)_k(b_U)^{\times}_{kj}(h_U)_j + (a_V)^{\times}_{j+1,j}(q_V)_j(d_U)_j = \alpha_{j-1}(h_U)_j + (q_V)_j(d_U)_j,$$

where

$$\alpha_{j-1} = \sum_{k=1}^{j-1} (a_V)^{\times}_{j+1,k}(q_V)_k(g_U)_k(b_U)^{\times}_{kj}.$$

We have

$$\alpha_1 = (a_V)^{\times}_{3,1}(q_V)_1(g_U)_1(b_U)^{\times}_{2,1} = (a_V)_2(q_V)_1(g_U)_1$$

and using the relations (2.1), (2.2) we obtain

$$\alpha_j = \sum_{k=1}^{j} (a_V)^{\times}_{j+2,k}(q_V)_k(g_U)_k(b_U)^{\times}_{k,j+1}$$

$$= (a_V)^{\times}_{j+2,j}(q_V)_j(g_U)_j(b_U)^{\times}_{j,j+1} + (a_V)_{j+1}\left(\sum_{k=1}^{j-1}(a_V)^{\times}_{j+1,k}(q_V)_k(g_U)_k(b_U)^{\times}_{kj}\right)(b_U)_j$$

$$= (a_V)_{j+1}(q_V)_j(g_U)_j + (a_V)_{j+1}\alpha_{j-1})(b_U)_j$$

which completes the proof of (3.1), (3.5).

For diagonal entries of the matrix Q we have

$$(d_Q)_1 = Q_{11} = V_{11}U_{11} = (d_V)_1(d_U)_1$$

and for $i = 2, \dots, N$

$$Q_{ii} = \sum_{k=1}^{i} V_{ik}U_{ki} = V_{ii}U_{ii} + \sum_{k=1}^{i-1} V_{ik}U_{ki} = (d_V)_i(d_U)_i + (p_V)_i\beta_{i-1}(h_U)_i,$$

where

$$\beta_{i-1} = \sum_{k=1}^{i-1} (a_V)^{\times}_{ik}(q_V)_k(g_U)_k(b_U)^{\times}_{ki}$$

We have $\beta_1 = (q_V)_1(g_U)_1$ and using the relations (2.1), (2.2) we obtain

$$\beta_i = \sum_{k=1}^{i}(a_V)^{\times}_{i+1,k}(q_V)_k(g_U)_k(b_U)^{\times}_{k,i+1}$$
$$= (a_V)^{\times}_{i+1,i}(q_V)_i(g_U)_i(b_U)^{\times}_{i,i+1} + (a_V)_i(\sum_{k=1}^{j-1}(a_V)^{\times}_{ik}(q_V)_k(g_U)_k(b_U)^{\times}_{ki})(b_U)_i$$
$$= (q_V)_i(g_U)_i + (a_V)_i\beta_{i-1})(b_U)_i$$

which completes the proof of (3.2), (3.6), (3.8).

The proof of the relations (3.3), (3.7) is performed in the same way as the proof of (3.1), (3.5). □

Corollary 1 *Let R be a (n_L, n_U) quasiseparable matrix with scalar entries and let $R = QS$ be the factorization obtained in Theorem 3.2. Then the unitary matrix Q is (n_L, n_L) quasiseparable and the upper triangular matrix S is $n_L + n_U$ upper quasiseparable.*

Proof. By Theorem 3.2 the matrix Q has lower and upper generators of the orders ρ_k $(k = 1, \ldots, N-1)$ defined by the relations

$$\rho_N = 0, \ \rho_{k-1} = \min\{1 + \rho_k, \ r'_{k-1}\}, \ k = N, \ldots, 2 \tag{3.10}$$

and by Theorem 3.1 the matrix S has upper generators of orders

$$\rho'_k = r''_k + \rho_k, \ k = 1, \ldots, N-1. \tag{3.11}$$

From the inequalities $r'_k \leq n_L$ $(k = 1, \ldots, N-1)$ and the relations (3.10) it follows that

$$\rho_k \leq r'_k \leq n_L, \quad k = 1, \ldots, N-1 \tag{3.12}$$

and hence the maximal order of generators of the matrix Q is not greater than n_L. Next from (3.11) and (3.12) we conclude that the maximal order of upper generators of the matrix S is not greater than $n_L + n_U$. □

4 The QR iteration

We consider the QR iteration algorithm for matrices defined via generators. In each iteration step for a given matrix R and for a given real number σ the new iterant R_1 is obtained by the rule

$$\begin{cases} R - \sigma I = QS, \\ R_1 = \sigma I + SQ, \end{cases}$$

where Q is a unitary matrix and S is an upper triangular matrix. We show that the matrix R_1 has lower generators with the same orders as for the lower generators of the matrix Q and hence these orders are not greater than the orders of the corresponding generators of the matrix R and obtain an algorithm for computation of these generators and the diagonal entries of the matrix R_1.

Theorem 4.1 *Let $R = \{R_{ij}\}_{i,j=1}^{N}$ be a scalar matrix with lower generators p_i $(i = 2, \dots, N)$, q_j $(j = 1, \dots, N-1)$, a_k $(k = 2, \dots, N-1)$ of orders r'_k $(k = 1, \dots, N-1)$, upper generators g_i $(i = 1, \dots, N-1)$, h_j $(j = 2, \dots, N)$, b_k $(k = 2, \dots, N-1)$ of orders r''_k $(k = 1, \dots, N-1)$ and diagonal entries d_k $(k = 1, \dots, N)$ and σ be a real number. Let us define the numbers ρ_k via recursive relations $\rho_N = 0,\ \rho_{k-1} = \min\{1 + \rho_k,\ r'_{k-1}\},\ k = N, \dots, 2,\ \rho_0 = 0$. Define the matrix R_1 by the rule*

$$\begin{cases} R - \sigma I = QS, \\ R_1 = \sigma I + SQ, \end{cases}$$

where Q is a unitary matrix and S is an upper triangular matrix.

The matrix R_1 has lower generators of orders ρ_k $(k = 1, \dots, N-1)$. These lower generators $p_i^{(1)}$ $(i = 2, \dots, N)$, $q_j^{(1)}$ $(j = 1, \dots, N-1)$, $a_k^{(1)}$ $(k = 2, \dots, N-1)$ and the diagonal entries $d_k^{(1)}$ $(k = 1, \dots, N)$ of the matrix R are determined using the following algorithm.

1. Apply to the matrix $R - \sigma I$, which has the same lower and upper generators as the matrix R and the diagonal entries $d_k - \sigma$ $(k = 1, \dots, N)$, the algorithm from Theorem 3.2, to compute the lower generators $(p_Q)_i$ $(i = 2, \dots, N)$, $(q_Q)_j$ $(j = 1, \dots, N-1)$, $(a_Q)_k$ $(k = 2, \dots, N-1)$ and the diagonal entries $(d_Q)_k$ $(k = 1, \dots, N)$ of the matrix Q and the upper generators $(g_S)_i$ $(i = 1, \dots, N-1)$, $(h_S)_j$ $(j = 2, \dots, N)$, $(b_S)_k$ $(k = 2, \dots, N-1)$ and the diagonal entries $(d_S)_k$ $(k = 1, \dots, N)$ of the matrix S.

2. Compute the lower generators and the diagonal entries of the matrix R_1 as follows.

2.1. Compute

$$z_N = (h_S)_N (p_Q)_N,$$

$$p_N^{(1)} = (d_S)_N (p_Q)_N, \quad \alpha_N = z_N (a_Q)_{N-1}, \tag{4.1}$$

$$d_N^{(1)} = (d_S)_N (d_Q)_N + \sigma, \quad \beta_N = z_N. \tag{4.2}$$

Set $(a_Q)_1 = 0_{\rho_1 \times 0}$.

2.2. For $i = N-1, \dots, 2$ perform the following. Set

$$q_i^{(1)} = (q_Q)_i, \quad a_i^{(1)} = (a_Q)_i.$$

Compute

$$z_i = (h_S)_i (p_Q)_i, \tag{4.3}$$

$$p_i^{(1)} = (d_S)_i (p_Q)_i + (g_S)_i \alpha_{i+1}, \quad \alpha_i = [z_i + (b_S)_i \alpha_{i+1}](a_Q)_{i-1}, \tag{4.4}$$

$$d_i^{(1)} = (d_S)_i (d_Q)_i + (g_S)_i \beta_{i+1} (q_Q)_i + \sigma, \quad \beta_i = z_i + (b_S)_i \beta_{i+1} (a_Q)_i. \tag{4.5}$$

2.3. Set $q_1^{(1)} = (q_Q)_1$. *Compute*

$$d_1^{(1)} = (d_S)_1 (d_Q)_1 + (g_S)_1 \beta_2 (q_Q)_1 + \sigma. \tag{4.6}$$

*Proof.*We should justify the second stage of the algorithm. Let $Q = \{Q_{ij}\}_{i,j=1}^N$, $S = \{S_{ij}\}_{i,j=1}^N$ and $R_1 = \{R_{ij}^{(1)}\}_{i,j=1}^N$. For $N \geq i > j \geq 1$ using the fact S is an upper triangular matrix and $(p_Q)_i$ $(i = 2, \ldots, N)$, $(q_Q)_j$ $(j = 1, \ldots, N-1)$, $(a_Q)_k$ $(k = 2, \ldots, N-1)$ are lower generators of the matrix Q we have

$$R_{ij}^{(1)} = \sum_{k=i}^N S_{ik} Q_{kj} = \sum_{k=i}^N S_{ik} (p_Q)_k (a_Q)_{kj}^{\times} (q_Q)_j.$$

Using the equality (2.1) we obtain

$$R_{ij}^{(1)} = p_i^{(1)} (a_Q)_{ij}^{\times} (q_Q)_j, \quad 1 \leq j < i \leq N$$

where

$$p_i^{(1)} = \sum_{k=i}^N S_{ik} (p_Q)_k (a_Q)_{k,i-1}^{\times}, \quad i = 2, \ldots, N. \tag{4.7}$$

This implies that the matrix $R^{(1)}$ has the lower generators $a_k^{(1)} = (a_Q)_k$ $(k = 2, \ldots, N-1)$, $q_j^{(1)} = (q_Q)_j$ $(j = 1, \ldots, N-1)$ and $p^{(1)}$ $(i = 2, \ldots, N)$ defined in (4.7). This in particular means that the orders ρ_k $(k = 1, \ldots, N-1)$ of these generators are the same as for the matrix Q. Now we must check that the generators $p_i^{(1)}$ satisfy the relations (4.1), (4.4). Indeed for $i = N$ we have

$$p_N^{(1)} = S_{NN} (p_Q)_N (a_Q)_{N,N-1}^{\times} = (d_S)_N (p_Q)_N$$

and for $i = N-1, \ldots, 2$ using $S_{jj} = (d_S)_j$ and the fact that $(g_S)_i$ $(i = 1, \ldots, N-1)$, $(h_S)_j$ $(j = 2, \ldots, N)$, $(b_S)_k$ $(k = 2, \ldots, N-1)$ are the upper generators of the matrix S we get

$$p_i^{(1)} = (g_S)_i \sum_{k=i+1}^N (b_S)_{ik}^{\times} (h_S)_k (p_Q)_k (a_Q)_{k,i-1}^{\times} + (d_S)_i (p_Q)_i (a_Q)_{i,i-1}^{\times} = (d_S)_i (p_Q)_i + (g_S)_i \alpha_{i+1},$$

where

$$\alpha_{i+1} = \sum_{k=i+1}^N (b_S)_{ik}^{\times} (h_S)_k (p_Q)_k (a_Q)_{k,i-1}^{\times}.$$

We have

$$\alpha_N = (b_S)^{\times}_{N-1,N}(h_S)_N(p_Q)_N(a_Q)^{\times}_{N,N-2} = (h_S)_N(p_Q)_N(a_Q)_{N-1}$$

and using the relations (2.1), (2.2) we obtain

$$\alpha_i = \sum_{k=i}^{N}(b_S)^{\times}_{i-1,k}(h_S)_k(p_Q)_k(a_Q)^{\times}_{k,i-2}$$

$$= (b_S)^{\times}_{i-1,i}(h_S)_i(p_Q)_i(a_Q)^{\times}_{i,i-2} + (b_S)_i(\sum_{k=i+1}^{N}(b_S)^{\times}_{ik}(h_S)_k(p_Q)_k(a_Q)^{\times}_{k,i-1})(a_Q)_{i-1}$$

$$= [(h_S)_i(p_Q)_i + (b_S)_i\alpha_{i+1}](a_Q)_{i-1}$$

which completes the proof of (4.1), (4.4).

For diagonal entries of the matrix S we have

$$d^{(1)}_N = R^{(1)}_{NN} = S_{NN}Q_{NN} = (d_S)_N(d_Q)_N$$

and for $i = N-1, \dots, 1$

$$R^{(1)}_{ii} = \sum_{k=i}^{N} S_{ik}Q_{ki} = S_{ii}Q_{ii} + \sum_{k=i+1}^{N} S_{ik}Q_{ki} = (d_S)_i(d_Q)_i + (g_S)_i\beta_{i+1}(h_S)_i,$$

where

$$\beta_{i+1} = \sum_{k=i+1}^{N}(b_S)^{\times}_{ik}(h_S)_k(p_Q)_k(a_Q)^{\times}_{ki}$$

We have $\beta_1 = (q_V)_1(g_U)_1$ and using the relations (2.1), (2.2) we obtain

$$\beta_i = \sum_{k=i}^{N}(b_S)^{\times}_{i-1,k}(h_S)_k(p_Q)_k(a_Q)^{\times}_{k,i-1}$$

$$= (b_S)^{\times}_{i-1,i}(h_S)_i(p_Q)_i(a_Q)^{\times}_{i,i-1} + (b_S)_i(\sum_{k=i+1}^{N}(b_S)^{\times}_{ik}(h_S)_k(p_Q)_k(a_Q)^{\times}_{ki})(a_Q)_i =$$

$$(h_S)_i(p_Q)_i + (b_S)_i\beta_{i+1})(a_Q)_i$$

which completes the proof of (4.2), (4.5), (4.6). $\square$

Corollary 2 *Let R be a lower n_L quasiseparable matrix with scalar entries and let R_1 be the matrix obtained in Theorem 3.2. Then the unitary matrix Q is (n_L, n_L) quasiseparable matrix and the upper triangular matrix S is $n_L + n_U$ upper quasiseparable.*

Proof follows directly from Theorem 4.1 and Corollary 1.