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A CONDITION FOR THE CLOSENESS OF THE SETS OF THE INVARIANT SUBSPACES OF CLOSE MATRICES IN TERMS OF THEIR JORDAN STRUCTURES

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1. In this paper we consider the problem of the behavior of the distance between the sets Inv A and Inv B of invariant subspaces of the operators A and B, acting in a finite-dimensional Hilbert space, in dependence on the Jordan structures of these operators and on the quantity $\|B - A\|$. By the distance between two subspaces we mean their gap, while the distance dist(Inv A, Inv B) is defined in the Hausdorff sense.

In [1] (see also [2]) one has obtained in certain cases estimates for dist(Inv A, Inv B) in terms of $\|B - A\|$ or in terms of a fractional power of this quantity. In particular, it has been proved there that for an arbitrary operator A we have

$$\sup \frac{\operatorname{dist}(\operatorname{Inv} A, \operatorname{Inv} B)}{\|B-A\|} < \infty, \tag{1}$$

where the supremum is taken over all operators B having the same Jordan structure as A. In [2] one has conjectured that the inequality (1) cannot hold if one considers in it operators B of a fixed Jordan structure, different from the Jordan structure of A.

The fundamental result of this paper is the determination of the conditions, satisfied by the Jordan structure of the operators B, in order that

$$\lim_{B \to A} \operatorname{dist}(\operatorname{Inv} A, \operatorname{Inv} B) = 0.$$

It turns out that a criterion for this is that the operators A and B should have the same Gohberg–Kaashoek numbers (see the definition in Sec. 3). It is also proved that

$$\inf \operatorname{dist}(\operatorname{Inv} A, \operatorname{Inv} B) > 0,$$

where the infinum is taken over all possible pairs of operators A and B, for which the Gohberg–Kaashoek numbers do not coincide. We give an example which refutes the above-mentioned conjecture from [2] (this example disproves also another conjecture made there).

2. Let F be the set of all nonincreasing, finite, nontrivial sequences of nonnegative integers. We denote by F_n the set of all finite collections

$$\Omega = \{(m_{ij})_{i=1}^{\infty}:\ j = 1, \ldots, q\} \quad ((m_{ij})_{i=1}^{\infty} \in F), \tag{2}$$

satisfying the condition $\sum^{\infty} \sum^{q}_{=1}$.* The collections occurring in F_n will be considered to be unordered, i.e., two collections are identified if they differ only by the order of the indexing of the sequences from F occurring in them. We define for $\Omega \in F_n$ the dual collection

$$D(\Omega) = \{(\max\{l:\ m_{lj} \geqslant i\})_{i=1}^{\infty}:\ j = 1, \ldots, q\},$$

and also the summarized sequence

$$\Sigma(\Omega) = \left(\sum_{j=}^{q} m_{ij}\right)_{i=1}^{\infty}.$$

*Something missing in Russian original – Editor.

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Obviously, $D(\Omega) \in F_n$, $\Sigma(\Omega) \in F$. It is easy to prove the following lemma.

LEMMA 1. Let $\Omega \in F_n$. The sequence $D(\Sigma(\Omega))$ is the union of the sequences from $D(\Omega)$, whose elements are in nonincreasing order.

Let $\Omega = (\alpha_i)_{i=1}^{\infty}$, $\Omega' = (\beta_i)_{i=1}^{\infty}$ be sequences from F. We shall write $\Omega \prec \Omega'$ if the following relations hold:

$$\sum_{i=1}^{r} \alpha_i \leqslant \sum_{i=1}^{r} \beta_i \quad (r = 1, 2, \ldots), \quad \sum_{i=1}^{\infty} \alpha_i = \sum_{i=1}^{\infty} \beta_i.$$

LEMMA 2. Let $\Omega, \Omega' \in F$. The following statements are equivalent:

1°) $\Omega \prec \Omega'$,

2°) $D(\Omega') \prec D(\Omega)$.

The proof of this lemma is also left to the reader.

Assume that a collection $\Omega \in F_n$ has the form (2). By $P(\Omega)$ we denote the set of all collections

$$\Omega' = \{(m'_{ij})_{i=1}^{\infty}: j = 1, \ldots, q'\} \quad (\in F_n),$$

possessing the following conditions: the set $\{1, \ldots, q'\}$ can be partitioned into q subsets $\Delta_1, \ldots, \Delta_q$ such that

$$(m_{ij})_{i=1}^{\infty} \prec \left(\sum_{k \in \Delta_j} m'_{ik}\right)_{i=1}^{\infty} \quad (j = 1, \ldots, q). \tag{3}$$

Remark 1. Making use of Lemma 2, we can replace condition (3) in the definition of the set $P(\Omega)$ by its equivalent condition

$$D\left(\left(\sum_{k \in \Delta_j} m'_{ik}\right)_{i=1}^{\infty}\right) \prec D\left((m_{ij})_{i=1}^{\infty}\right) \quad (j = 1, \ldots, q).$$

3. Let $\mathfrak{H}$ be a Hilbert space of dimension n and let $\mathscr{L}(\mathfrak{H})$ be the algebra of all linear operator acting in $\mathfrak{H}$. We denote by $m_1(A, \lambda_0) \geqslant m_2(A, \lambda_0) \geqslant \ldots \geqslant m_r(A, \lambda_0)$ the partial multiplicities (the dimensions of the Jordan blocks) of the operator $A \in \mathscr{L}(\mathfrak{H})$, corresponding to its eigenvalue λ_0. For the sake of convenience we set $m_i(A, \lambda_0) = 0$ for $i > r$. If the spectrum $\sigma(A)$ of the operator A consists of q distinct eigenvalues $\{\lambda_i\}_1^q$, then the collection

$$\Omega(A) = \{(m_i(A, \lambda_j))_{i=1}^{\infty}: j = 1, \ldots, q\}$$

occurring in F_n, is called the Jordan structure of the operator A. The set of all operators from $\mathscr{L}(\mathfrak{H})$ of Jordan structure Ω will be denoted by $\mathscr{J}(\Omega)$. The elements $m_i(A)$ $(i = 1, 2, \ldots)$ of the sequence $\Sigma(\Omega(A))$ $(\in F)$ are called the Gohberg–Kaashoek numbers of the operator A. We denote by $k_i(A)$, $k_i(A, \lambda_j)$ the elements of the sequences $D\left((m_i(A))_{i=1}^{\infty}\right) = (k_i(A))_{i=1}^{\infty}$ and $D\left((m_i(A, \lambda_j))_{i=1}^{\infty}\right) = (k_i(A, \lambda_j))_{i=1}^{\infty}$ $(j = 1, \ldots, q)$.

LEMMA 3. Let $\Omega \in F_n$ be the Jordan structure of the operator $A \in \mathscr{L}(\mathfrak{H})$. Then

$$\sum_{i=1}^{r} k_i(A) = \max\left\{\dim \operatorname{Ker} \prod_{j=1}^{r} (A - \lambda_j I)\right\} \quad (r = 1, 2, \ldots), \tag{4}$$

where the maximum is taken over all possible (not necessarily distinct) $\lambda_1, \ldots, \lambda_r \in \sigma(A)$.

Proof. Obviously, if λ_0 is an eigenvalue of the operator A, then

$$\sum_{i=1}^{r} k_i(A, \lambda_0) = \dim \operatorname{Ker} (A - \lambda_0)^r \quad (r = 1, 2, \ldots),$$

and in order to obtain the equalities (4) one has to make use of Lemma 1.

In [3-5] one has obtained the following result. Let $\Omega \in F_n$ and assume that there is given an operator $A \in \mathscr{J}(\Omega)$. Then there exists $\varepsilon > 0$ such that the Jordan structure Ω' of an arbitrary operator B, satisfying the condition $\|B - A\| < \varepsilon$, in in $P(\Omega)$. We mention that this result can be easily proved by making use of Remark 1, Lemma 3, and the fact that, under a small perturbation of the operator, the dimension of the kernel cannot increase. In [3-5] it is also shown that for arbitrary $\Omega' \in P(\Omega)$ and $\varepsilon > 0$ one can construct an operator $B \in \mathscr{J}(\Omega')$, satisfying the condition $\|B - A\| < \varepsilon$.

4. Let A be an operator from $\mathscr{L}(\mathfrak{H})$. A subspace $\mathfrak{M}\subset\mathfrak{H}$ is said to be invariant relative to the operator A (A-invariant) if $A(\mathfrak{M})\subset\mathfrak{M}$. In this case, by $A\mid\mathfrak{M}$ we shall denote the restriction of the operator A to the subspace $\mathfrak{M}$. We denote by Inv A the lattice (i.e., the set, partially ordered with respect to inclusion) of all A-invariant subspaces. By the gap between the subspaces $\mathfrak{M}$ and $\mathfrak{N}$ of the space $\mathfrak{H}$ we mean the quantity

$$\theta(\mathfrak{M},\mathfrak{N})=\max\left\{\max_{\substack{x\in\mathfrak{M}\\ \|x\|=1}}\rho(x,\mathfrak{N}),\ \max_{\substack{y\in\mathfrak{N}\\ \|y\|=1}}\rho(y,\mathfrak{M})\right\},$$

where $\rho(x,\mathfrak{N})=\min_{y\in\mathfrak{N}}\|x-y\|$. As it is known [2], we have $\theta(\mathfrak{M},\mathfrak{N})=\|P_{\mathfrak{M}}-P_{\mathfrak{N}}\|$, where $P_{\mathfrak{M}}$ is the orthogonal projection onto the subspace $\mathfrak{M}$. We recall that the set of all subspaces of the space $\mathfrak{H}$ is a compact metric space with the metric θ. Following [2], we introduce the Hausdorff distance between the lattices of the invariant subspaces of the operators A and B $(\in\mathscr{L}(\mathfrak{H}))$, i.e.,

$$\operatorname{dist}(\operatorname{Inv}A,\operatorname{Inv}B)=\max\left\{\max_{\mathfrak{M}\in\operatorname{Inv}A}\ \min_{\mathfrak{N}\in\operatorname{Inv}B}\theta(\mathfrak{M},\mathfrak{N}),\ \max_{\mathfrak{M}\in\operatorname{Inv}B}\ \min_{\mathfrak{N}\in\operatorname{Inv}A}\theta(\mathfrak{M},\mathfrak{N})\right\}.$$

LEMMA 4. Let $\Omega,\Omega'\in F_n$ such that $\Omega'\in P(\Omega)$ and $\Sigma(\Omega')=\Sigma(\Omega)$. For an operator $A_0\in\mathscr{J}(\Omega)$ and a subspace $\mathfrak{N}_0\in\operatorname{Inv}A_0$ there exists a number C > 0 such that for any $B\in\mathscr{J}(\Omega')$ we have

$$\min_{\mathfrak{M}\in\operatorname{Inv}B}\theta(\mathfrak{N}_0,\mathfrak{M})\leqslant C\|B-A_0\|^{1/\alpha}, \tag{5}$$

where $\alpha=\max\{m_1(A,\lambda):\lambda\in\sigma(A)\}$.

Proof. Since we always have $\theta(\mathfrak{M},\mathfrak{N})\leqslant 1$, it is sufficient to establish inequality (5) for an operator B satisfying the condition

$$\|B-A_0\|<\varepsilon, \tag{6}$$

where ε is some fixed positive number.

First we assume that the operator A_0 has only one eigenvalue λ_0; moreover, without loss of generality we can assume that $\lambda_0 = 0$. The absolute value of an eigenvalue λ of the operator B, satisfying condition (6), has the following estimate:

$$|\lambda|\leqslant C_1\|B-A_0\|^{1/\alpha}, \tag{7}$$

where C_1 depends only on A_0 and ε (see, for example, [2, Lemma 16.5.1]). It is well known [2, Theorem 13.5.1] that for any operator $S\in\mathscr{L}(\mathfrak{H})$ there exists a number $C_2 > 0$ such that from the condition dim Ker T = dim Ker S [$T\in\mathscr{L}(\mathfrak{H})$] there follows

$$\theta(\operatorname{Ker}T,\ \operatorname{Ker}S)\leqslant C_2\|T-S\|. \tag{8}$$

From $\Sigma(\Omega')=\Sigma(\Omega)$ there follows the equality

$$\sum_{i=1}^{r}k_i(B)=\sum_{i=1}^{r}k_i(A_0)\qquad(r=1,\ldots,\alpha),$$

while, by virtue of Lemma 3, there exist numbers $\lambda_1,\ldots,\lambda_\alpha\in\sigma(B)$ such that

$$\dim\operatorname{Ker}\prod_{j=1}^{r}(B-\lambda_jI)=\dim\operatorname{Ker}A_0^r\quad(r=1,\ldots,\alpha).$$

Therefore, by virtue of the inequalities (7), (8), there exists a number $C_3 > 0$, depending only on the operator A_0, such that

$$\theta\left(\operatorname{Ker}\prod_{j=1}^{r}(B-\lambda_jI),\ \operatorname{Ker}A_0^r\right)\leqslant C_3\|B-A_0\|^{1/\alpha}\quad(r=1,\ldots,\alpha). \tag{9}$$

We select in the subspace $\mathfrak{N}_0$ some Jordan bases $\{f_{ij}: i=1,\ldots,l;\ j=1,\ldots,m_i\}$ of the operator A_0, i.e., $A_0f_{i1}=0$, $A_0f_{ij}=f_{ij-1}$ $(i=1,\ldots,l;\ j=2,\ldots,m_i)$. Obviously, each of the vectors f_{im_i} is in $\operatorname{Ker}A_0^{m_i}$ and it is not in $\operatorname{Ker}A_0^{m_i-1}$ $(i=1,\ldots,l)$. From (9) there follows that one can select a vector g_{im_i}, that is in $\operatorname{Ker}\prod_{j=1}^{m_i}(B-\lambda_jI)$ but not in $\operatorname{Ker}\prod_{j=1}^{m_i-1}(B-\lambda_jI)$, such that

$$\|g_{im_i}-f_{im_i}\|\leqslant C_3\|B-A_0\|^{1/\alpha}\quad(i=1,\ldots,l). \tag{10}$$

We set

$$g_{ij}=B^{m_i-j}g_{im_i}\quad(i=1,\ldots,l;\ j=1,\ldots,m_i),$$

and by $\mathfrak{M}$ we denote the linear span of the vectors $\{g_{ij}: i=1, \ldots, l;\ j=1, \ldots, m_i\}$. Obviously, $\mathfrak{M} \in \operatorname{Inv} B$. By virtue of the inequalities (10), there exists a number $C > 0$, depending only on A_0 and $\mathfrak{N}_0$, for which

$$\theta(\mathfrak{N}_0, \mathfrak{M}) \leqslant C\, \|B - A_0\|^{1/\alpha}.$$

Thus, the lemma is proved for the case when $\sigma(A_0)$ consists of one point.

Now we proceed to the general case. Let $\lambda_1, \ldots, \lambda_q$ be all the distinct eigenvalues of the operator A_0. We consider the circles G_j with centers at the points λ_j $(j = 1, \ldots, q)$ and such small radii that $G_k \cap G_j = \varnothing$ for $k \neq j$. As it is known, there exists a number $\varepsilon > 0$ such that the sums of the multiplicities of the eigenvalues, lying in G_j, of an operator B, satisfying condition (6), and of the operator A_0 coincide $(j = 1, \ldots, q)$. As already mentioned, it is sufficient to carry out the proof only for such B. We consider the operator

$$S = I - \sum_{j=1}^{q} (P_j(A_0) - P_j(B))\, P_j(A_0),$$

where

$$P_j(B) = -\frac{1}{2\pi i} \int_{\partial G_j} (B - \lambda I)^{-1} d\lambda$$

is the Riesz projection onto the direct sum $\mathscr{R}_j(B)$ of the root subspaces of B, corresponding to the eigenvalues lying inside G_j $(j = 1, \ldots, q)$. In a similar manner one defines $P_j(A_0)$ and $\mathscr{R}_j(A_0)$. It is easy to see that the operator S maps $\mathscr{R}_j(A_0)$ into $\mathscr{R}_j(B)$ and, in addition, there exists a number $C_1 > 0$ such that

$$\|I - S\| \leqslant C_1 \|B - A_0\|, \tag{11}$$

and C_1 depends only on the operator A_0 and the circles G_j $(j = 1, \ldots, q)$. By virtue of the inequality (11), we can assume that the operator S is invertible.

Let $\mathfrak{N}_0$ be an initial A_0-invariant subspace. Then $\mathfrak{N}_0 = \mathfrak{N}_1 \dotplus \ldots \dotplus \mathfrak{N}_q$, where $\mathfrak{N}_j = \mathfrak{N}_0 \cap \mathscr{R}_j(A_0)$ $(j = 1, \ldots, q)$. By virtue of what has been already proved and of the fact that $\mathscr{R}_j(A_0) = \mathscr{R}_j(S^{-1}BS)$, for each $\mathfrak{N}_j \in \operatorname{Inv}(A_0 | \mathscr{R}_j(A_0))$ there exists $\mathfrak{M}_j \in \operatorname{Inv}(S^{-1}BS | \mathscr{R}_j(A_0))$ such that

$$\theta(\mathfrak{N}_j, \mathfrak{M}_j) \leqslant C_2 \,\|\, (S^{-1}BS - A_0) \,|\, \mathscr{R}_j(A_0) \,\|^{1/\alpha},$$

where C_2 depends only on the operator A_0 and the subspace $\mathfrak{N}_0$. We set $\mathfrak{M}' = \mathfrak{M}_1 \dotplus \ldots \dotplus \mathfrak{M}_q$. Obviously, $\mathfrak{M}' \in \operatorname{Inv} S^{-1}BS$ and $\theta(\mathfrak{N}_0, \mathfrak{M}') \leqslant C_3 \|S^{-1}BS - A_0\|^{1/\alpha}$, where the number C_3 depends only on A_0 and $\mathfrak{N}_0$. As one can easily see, $\mathfrak{M} = S\mathfrak{M}'$ is a B-invariant subspace. From (11) there follows that $\|S^{-1}BS - A_0\| \leqslant C_4 \|B - A_0\|$. Thus,

$$\theta(\mathfrak{N}_0, \mathfrak{M}) \leqslant \theta(\mathfrak{N}_0, \mathfrak{M}') + \theta(\mathfrak{M}', \mathfrak{M}) \leqslant C_3 \|S^{-1}BS - A_0\|^{1/\alpha} + C_5 \|I - S\| \leqslant C \|B - A_0\|^{1/\alpha}.$$

<u>THEOREM 1.</u> Assume that there are given $\Omega, \Omega' \in F_n$ such that $\Omega' \in P(\Omega)$ and let $A_0 \in \mathscr{J}(\Omega)$. The following statements are equivalent:

1°) $\lim\limits_{\substack{B \to A_0 \\ B \in \mathscr{J}(\Omega')}} \operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B) = 0,$

2°) $\Sigma(\Omega') = \Sigma(\Omega)$.

<u>Proof.</u> The implication 1° ⇒ 2° is a consequence of the implication 1° ⇒ 2° of Theorem 2, proved below. We show that 2° ⇒ 1°. If this is not true, then there exists a sequence $B_n \in \mathscr{J}(\Omega')$ such that $B_n \to A_0$ and

$$\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B_n) \geqslant \delta > 0 \quad (n = 1, 2, \ldots).$$

According to the definition of the quantity dist, this can mean the following. Either a) there exists a sequence of subspaces $\mathfrak{N}_n \in \operatorname{Inv} A_0$ $(n = 1, 2, \ldots)$ such that

$$\min_{\mathfrak{M} \in \operatorname{Inv} B_n} \theta(\mathfrak{N}_n, \mathfrak{M}) \geqslant \delta \quad (n = 1, 2, \ldots), \tag{12}$$

or b) there exists a sequence $\mathfrak{M}_n \in \operatorname{Inv} B_n$ $(n = 1, 2, \ldots)$ such that

$$\min_{\mathfrak{N} \in \operatorname{Inv} A_0} \theta(\mathfrak{M}_n, \mathfrak{N}) \geqslant \delta \quad (n = 1, 2, \ldots). \tag{13}$$

By virtue of the compactness of the set of all subspaces of the space $\mathfrak{H}$, we can select subsequences $\mathfrak{N}_{n_k} \in \operatorname{Inv} A_0$ and $\mathfrak{M}_{n_k} \in \operatorname{Inv} B_{n_k}$, converging to some subspaces $\mathfrak{N}_0$ and $\mathfrak{M}_0$, respectively:

$$\lim_{k\to\infty}\theta(\mathfrak{N}_{n_k},\mathfrak{N}_0)=\lim_{k\to\infty}\theta(\mathfrak{M}_{n_k},\mathfrak{M}_0)=0.$$

It is easy to see that $\mathfrak{N}_0, \mathfrak{M}_0 \in \operatorname{Inv} A_0$. By virtue of (12) and (13), the subspaces $\mathfrak{N}_0$ and $\mathfrak{M}_0$ must satisfy the inequalities

$$\min_{\mathfrak{M}\in\operatorname{Inv} B_{n_k}}\theta(\mathfrak{N}_0,\mathfrak{M})\geqslant\delta;\quad \min_{\mathfrak{N}\in\operatorname{Inv} A_0}\theta(\mathfrak{M}_0,\mathfrak{N})\geqslant\delta.$$

For large k, the first of these inequalities contradicts Lemma 4, while the second one is obviously false.

LEMMA 5. Assume that there are given a subspace $\mathfrak{A}\subset\mathfrak{H}$ and a number $m\in\mathbf{N}$. There exists a number $\delta > 0$ such that for any m subspaces $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$ $(\subset\mathfrak{H})$, each of them having dimension less than the dimension of $\mathfrak{A}$, there exists a vector $g\in\mathfrak{A}$ $(\|g\|=1)$ having the property $\rho(g, \mathfrak{A}_i)\geqslant\delta$ $(i = 1,\ldots,m)$.

For the case when $\dim\mathfrak{A}=2$, this assertion is established in [2, p. 508]. In the general case the proof is similar.

THEOREM 2. For any collections $\Omega, \Omega'\in F_n$ the following statements are equivalent:

1°) $\inf\limits_{\substack{A\in\mathcal{J}(\Omega)\\ B\in\mathcal{J}(\Omega')}}\operatorname{dist}(\operatorname{Inv} A, \operatorname{Inv} B)=0,$

2°) $\Delta(\Omega')=\Sigma(\Omega)$.

Proof. First we show that $2^\circ\Rightarrow 1^\circ$. If $\Omega'\in P(\Omega)$ or $\Omega\in P(\Omega')$, then the assertion follows at once from the implication $2^\circ\Rightarrow 1^\circ$ of Theorem 1. If, however, none of these conditions holds, then we set $\Omega''=\Sigma(\Omega)$. Obviously, $\Omega, \Omega'\in P(\Omega'')$ and for an arbitrary operator $C\in\mathcal{J}(\Omega'')$ we can select $A_n\in\mathcal{J}(\Omega)$, $B_n\in\mathcal{J}(\Omega')$ such that $A_n\to C$, $B_n\to C$ for $n\to\infty$. By the triangle inequality we have

$$\operatorname{dist}(\operatorname{Inv} A_n, \operatorname{Inv} B_n)\leqslant\operatorname{dist}(\operatorname{Inv} A_n, \operatorname{Inv} C)+\operatorname{dist}(\operatorname{Inv} C, \operatorname{Inv} B_n).$$

By virtue of the implication $2^\circ\Rightarrow 1^\circ$ of Theorem 1, the right-hand side of this inequality tends to zero when $n\to\infty$; from here there follows statement 1°.

We show that $1^\circ\Rightarrow 2^\circ$. Let $l\leqslant m_1(A)$. We denote by $\mathcal{E}_l(A)$ the set of all subspaces of the form $\operatorname{Ker}\prod_{j=1}^{l}(A-\lambda_j I)$, where $\lambda_1, \ldots, \lambda_l$ is a collection of eigenvalues of the operator A (not necessarily distinct).

We assume that assertion 1° holds and 2° does not. Let $r=\min\{i:\ k_i(A)\neq k_i(B)\}$. For the sake of definiteness we assume that $k_r(A) > k_r(B)$. By Lemma 3, there exist $\lambda_1,\ldots,\lambda_r\in\sigma(A)$ such that $\dim\mathfrak{A}=\sum_{i=1}^{r}k_i(A)$, where

$$\mathfrak{A}=\operatorname{Ker}\prod_{j=1}^{r}(A-\lambda_j I)\quad(\in\mathcal{E}_r(A)).$$

From Lemma 3 and the inequality $k_r(A) > k_r(B)$ there follows also that for an arbitrary $\mathfrak{B}\in\mathcal{E}_r(B)$ we have the inequality $\dim\mathfrak{B}<\dim\mathfrak{A}$. Obviously, $\mathcal{E}_r(B)$ contains at most C_n^r subspaces. It is easy to see that for each r-dimensional subspace $\mathfrak{M}\in\operatorname{Inv} B$, there exists $\mathfrak{B}\in\mathcal{E}_r(B)$ such that $\mathfrak{M}\subset\mathfrak{B}$. In addition, each vector of the subspace $\mathfrak{A}$ belongs at least to one-dimensional A-invariant subspace. From these considerations and from Lemma 5 we obtain at once the assertion of the theorem.

COROLLARY 1. We have the inequality

$$\inf\operatorname{dist}(\operatorname{Inv} A, \operatorname{Inv} B)>0, \tag{14}$$

where the infimum is taken over all possible pairs of operators $A, B\in\mathcal{L}(\mathfrak{H})$, having different Gohberg–Kaashoek numbers.

For the case when the infimum is taken over all $A, B\in\mathcal{L}(\mathfrak{H})$ such that $m_2(A)\neq 0$ and $m_2(B)=0$, inequality (14) has been proved in [2, Theorem 16.6.1].

5. In Lemma 4 we have established that, under the condition $\Sigma(\Omega')=\Sigma(\Omega)$, for each A_0-invariant subspace $\mathfrak{N}_0$ $[A_0\in\mathcal{J}(\Omega)]$ the quantity

$$\min_{\mathfrak{M}\in\operatorname{Inv} B}\theta(\mathfrak{N}_0,\mathfrak{M})\ (B\in\mathcal{J}(\Omega'))$$

tends to zero when $B \to A_0$, not slower than some power of $\|B - A_0\|$. In [2] it has been proved that similar estimates are valid for the quantity $\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B)$, but under more rigid restrictions on Ω'. In particular, according to [2, Theorem 16.3.1], for an operator $A_0 \in \mathcal{J}(\Omega)$ we have

$$\sup \frac{\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B)}{\|B - A_0\|} < \infty, \tag{15}$$

where the supremum is taken over all operators B from $\mathcal{J}(\Omega)$.

We denote by $\mathcal{D}(A)$ the direct sum of the root subspaces of the operator A, containing at least two linearly independent eigenvectors. Let $\Omega, \Omega' \in F_n$. We consider $A \in \mathcal{J}(\Omega)$ and $B \in \mathcal{J}(\Omega')$. If $\Omega(A|\mathcal{D}(A)) = \Omega(B|\mathcal{D}(B))$, then we say [2] that the operators A and B have the same derogatory Jordan structure (or, equivalently, Ω and Ω' have the same derogatory parts). By the height $h(A)$ of the operator $A \in \mathcal{L}(\mathfrak{H})$ we mean the maximal multiplicity of the eigenvalues which correspond to a single independent eigenvector. If the operator A does not have such eigenvalues, then we define $h(A) = 1$.

In [2, Theorem 16.4.1] it has been proved that for a given operator $A_0 \in \mathcal{L}(\mathfrak{H})$ we have

$$\sup \frac{\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B)}{\|B - A_0\|^{1/h(A_0)}} < \infty, \tag{16}$$

where the supremum is taken over all operators $B \in \mathcal{L}(\mathfrak{H})$ having the same derogatory Jordan structure as A_0.

Remark 2. Let $\Omega \in F_n$ and let Q be some closed set of operators of Jordan structure Ω. It is easy to show that the supremum in (15) is bounded from above by the same quantity for all operators A_0 from Q. The same is true regarding the supremum in (16).

Remark 3. Let $\Omega, \Omega' \in F_n$ be such that $\Omega \in P(\Omega')$, $\Sigma(\Omega') \neq \Sigma(\Omega)$. For an arbitrary operator $C \in \mathcal{J}(\Omega')$ one can indicate operators $A_n, B_n \in \mathcal{J}(\Omega)$ $(n = 1, 2, \ldots)$, such that $A_n \to C$, $B_n \to C$ when $n \to \infty$, but $\inf_n \operatorname{dist}(\operatorname{Inv} A_n, \operatorname{Inv} B_n) > 0$. From here we obtain, in particular, the following. Assume that there is given some set of operators $Q \subset \mathcal{J}(\Omega)$ such that $I \notin \mathcal{J}(\Omega)$. We also assume that Q contains some operator $C \in \mathcal{J}(\Omega')$ such that $\Sigma(\Omega') \neq \Sigma(\Omega)$. For example, this will be so if $Q = \mathcal{J}(\Omega)$. Then none of the suprema in (15) and (16) can be bounded from above by the same number for all operators $A_0 \in Q$.

6. The following conjectures have been formulated in [2, pp. 512, 513].

Conjecture A. Let $\Omega \in F_n$ and $A_0 \in \mathcal{J}(\Omega)$. Then for any $\Omega' \in P(\Omega)$, different from Ω, there exists a sequence of operators $B_n \in \mathcal{J}(\Omega')$ $(n = 1, 2, \ldots)$, converging to A_0, such that

$$\lim_{n\to\infty} \frac{\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B_n)}{\|B_n - A_0\|} = \infty.$$

Conjecture B. Let $\Omega \in F_n$ and $A_0 \in \mathcal{J}(\Omega)$. Then for any $\Omega' \in P(\Omega)$ such that Ω' and Ω have different derogatory Jordan structures, there exists a sequence of operators $B_n \in \mathcal{J}(\Omega')$ $(n = 1, 2, \ldots)$, converging to A_0, such that

$$\lim_{n\to\infty} \frac{\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B_n)}{\|B_n - A_0\|^{1/h(A_0)}} = \infty.$$

Theorem 1 shows that in the case when $\Sigma(\Omega') \neq \Sigma(\Omega)$, both conjectures hold. Nevertheless, in the general case this is not so, as shown by the following example.

Example. Let $n = 3$, $\Omega = \{(2, 1, 0, \ldots)\}$; $\Omega' = \{(1, 1, 0, \ldots), (1, 0, \ldots)\}$. Clearly, $\Omega' \in P(\Omega)$. We consider the operator $A_0 \in \mathcal{L}(\mathfrak{H})$, defined by the matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

in the orthonormal basis $\{e_i\}_1^3$. Obviously, $A_0 \in \mathcal{J}(\Omega)$. In [2, Example 16.6.1] it has been shown that there exists a number $C_1 > 0$ for which

$$\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B) \leqslant C_1 \|B - A_0\|, \tag{17}$$

if the operator $B \in \mathcal{J}(\Omega')$ is defined by the matrix

$$\begin{pmatrix} \mu & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad (\mu \neq 0). \tag{18}$$

We show that inequality (17) holds not only for these B but also for all B from $\mathcal{J}(\Omega')$. By virtue of the boundedness of the quantity dist, it is sufficient to prove the inequality (17) only for operators $B \in \mathcal{J}(\Omega')$, satisfying the condition $\|B - A_0\| < \varepsilon$, where $\varepsilon > 0$ is some fixed number. We select ε so small that we have the inequality

$$\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B) < 1. \tag{19}$$

This is possible by virtue of the inequality $\Sigma(\Omega') = \Sigma(\Omega)$ and Theorem 1. From condition (19) there follows that the one-dimensional subspace, spanned by e_3, has a B-invariant complement. This means that the right lower element of the matrix of the operator B in the basis $\{e_i\}_1^3$ is an eigenvalue λ_1 of the operator B. Therefore,

$$|\lambda_1| = |(Be_3, e_3)| = |((B - A_0)e_3, e_3)| \leqslant \|B - A_0\|.$$

Further, the absolute value of the second eigenvalue λ_2 of the operator B has a similar estimate:

$$|\lambda_2| \leqslant |2\lambda_1 + \lambda_2| + 2|\lambda_1| = |\operatorname{tr} B| + 2|\lambda_1| = |\operatorname{tr}(B - A_0)| + 2|\lambda_1| \leqslant 5\|B - A_0\|.$$

Here $\operatorname{tr} B$ is the trace of the operator B. With the aid of arguments, similar to those given in the proof of Lemma 4, we can see that

$$\theta(\operatorname{Ker} A_0, \operatorname{Ker}(B - \lambda_1 I)) \leqslant C_2\|B - A_0\|.$$

Therefore, for the vector $e_3 \in \operatorname{Ker} A_0$ one can select a vector $g_3 \in \operatorname{Ker}(B - \lambda_1 I)$ such that $\|g_3 - e_3\| \leqslant C_2\|B - A_0\|$. We set $g_2 = e_2$, $g_1 = (B - \lambda_1 I)g_2$. It is easy to see that $\|g_i - e_i\| \leqslant C_3\|B - A_0\|$ (i = 1, 2, 3). We define an operator S on the vectors of the basis $\{e_i\}_1^3$: $Se_i = g_i$ (i = 1, 2, 3). Then $\|I - S\| \leqslant C_4\|B - A_0\|$. In addition, the matrix of the operator $S^{-1}BS - \lambda_1 I$ has the form (18) with $\mu = \lambda_2 - \lambda_1$. Further, making use of (17), we obtain

$$\operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv} B) \leqslant \operatorname{dist}(\operatorname{Inv} A_0, \operatorname{Inv}(S^{-1}BS - \lambda_1 I)) +$$
$$+ \operatorname{dist}(\operatorname{Inv} S^{-1}BS, \operatorname{Inv} B) \leqslant C_1\|S^{-1}BS - \lambda_1 I - A_0\| + C_5\|I - S\| \leqslant C\|B - A_0\|.$$

Thus, inequality (17) is proved for all operators B from $\mathcal{J}(\Omega')$. Since $\Omega' \neq \Omega$ and, moreover, Ω' and Ω have different derogatory parts, this example shows that Conjectures A and B are not true.

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