Simulation of 2-Dimensional Ising Model

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1 Introduction

In this exercise we want to examine the Ising model in two Dimensions. We already did the Analysis in one Dimension. So we just have to expand our Analysis. After we got the Analysis part right we want to look for the magnitization per spin in dependency of the coupling J. Also we look for the avarage energie per site.

2 Theory

We can still start with the Hamiltonian:

$$H(s) = -J \sum_{<x,y>} s_x s_y - h \sum_x s_x$$

$< x, y >$ denotes the nearest-neighbor pair x and y on the chain, J is the coupling constant and we assume periodic boundary conditions in x- and y- axis. That means that spins couple only to the nearest neighbor and s_{N-1} couples with s_0 with N the number of total spin-particles. The partition function stays the same in two dimensions:

$$Z = \sum_{s'} \exp\left(-\frac{H(s')}{k_B T}\right)$$

It is possible to examine the the magnitization for $h = 0$ analytically

$$|m| = \begin{cases} (1 - \frac{1}{\sinh(2J)^4})^{\frac{1}{8}}, & J > J_c \\ 0, & J \leq J_c \end{cases} .$$

We can also examine the energy per site analytically

$$\epsilon = -J \coth(2J) \left[1 + \frac{2}{\pi}(2\tanh 2J^2 - 1) K(4\frac{\tanh(2J)^2}{\cosh 2J^2})\right]$$

where K(m) is the complete elliptic integral of the first kind.
For $h \neq 0$ the analytic solution is not known yet.

3 Analysis

We first want to answer some impotant questions to write the code in a smart way. The numerical cost to calculate the energy is propotional to $\Lambda = N_x \cdot N_y$, Where $N_{x/y}$ is the number of spin particles in the axis x or y. Because we look at a qudratic field of spin particles the numerical cost is N^2. On the other hand if we flip one Spin and examine the difference in energy ΔS the numerical cost is independently of N because we have to only examine the difference to the nearest neighbours of the flipped spin. We get the formula

$$\Delta S = 2 \cdot s_{xy}(h + s_{(x+1),y} + s_{(x-1),y} + s_{x,(y+1)} + s_{x,(y-1)})$$

where x and y are the coordinates for the flipped spin. In this case we also take boundary conditions.
We saw that the magnitization depends on $J_c = \frac{1}{2}\log\big(1+\sqrt{2}\big) \approx 0.44069$. If the coupling is higher than J_c then the matrial will start building a magnetic field without an outside source. Is the coupling smaller than J_c then the coupling is to small to start an magnetic field on its own.
In this exercise we want to use the Metropolis-Hasting algorithm. That means we do the following steps:

1 Choose some random spin s_i

2 Flip the spin $s_i = (-1) \cdot s_i$

3 Determine the change in energy ΔS

4 Metropolis-Hasting step:
   - if $\Delta S < 0$ ACCPEPT the spin flip
   - else sample $Y \propto U(0,1)$
     * if $y \leq \exp\{(-\Delta S)\}$ ACCEPT the spin flip
     * otherwise REJECT the spin flip

5 repeat steps 1-4 for Λ times- This is calles a "sweep" through the lattice

6 Make a measurment and store result $O_1, ..., O_{Nmeas}$

7 Go back to 1

After we done these steps a few time we can estimate the observale. $\bar{O} = \frac{1}{N_{meas}} \sum_{i=1}^{N_{meas}} O_i$.
In the following we put $J = 0.3$, $N_{x/y} = 20$ and vary h. The result of the Metropolis-Hasting algorithm is in graph 1. We see that the result looks familiar. It looks like the result from sheet 1. We also can see that with higher N we get a greater magnitization.

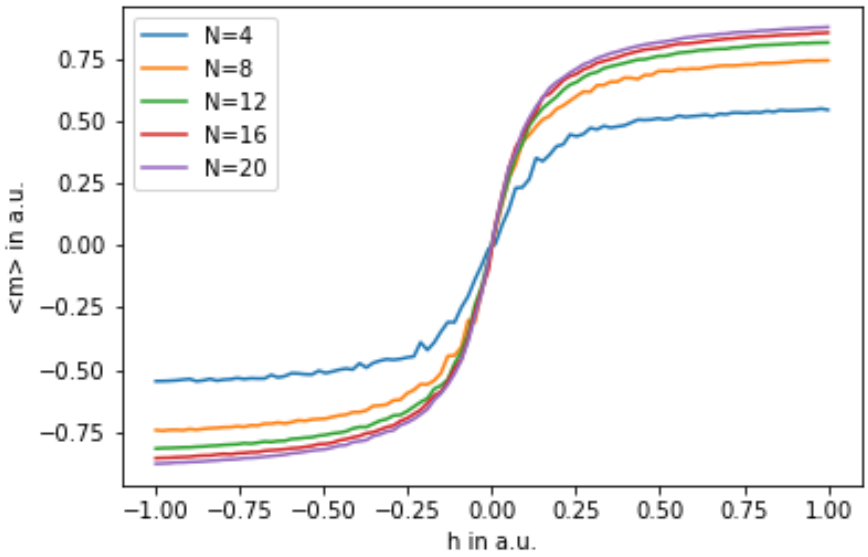

Figure 1: Magnitization dependancy of h.

For the next Graph 2 we plotted the absolut value of the magnitization per particle against J^{-1}. We can see that for higher N the curve goes to the analytical one and we see for lower N the error seems to be high. In this graph the magnitization is zero until the J_c is reached. Then we see a fast rise in the magnitization. This is because with h=0 an external field will only build if the coupling constant is high enough. For the graph on the right side we did not plot the absolut value. We can see that it is for higher J it is mirrowed on the x-axis.

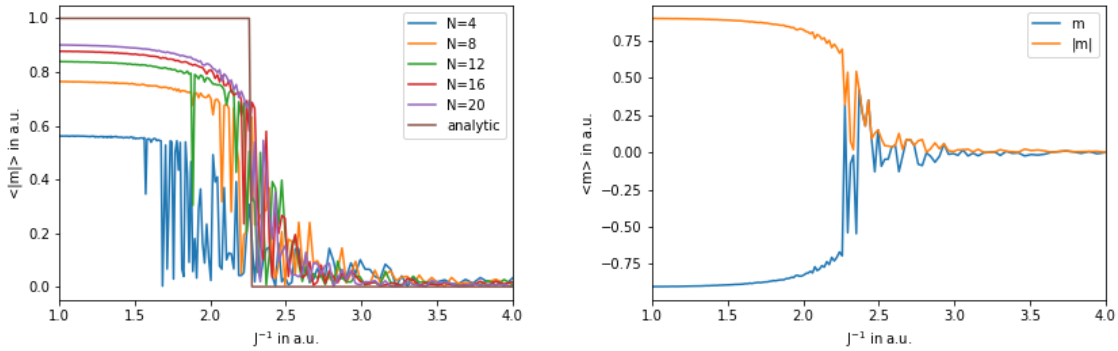

Figure 2: Magnitization dependancy of J^{-1}.

For the next part we want to look at the energy per site. We plottet for different N the energy in graph 3. Again we can see with lower N the error gets higher. We can also see that until the critical coupling is reached all graphs are nearly the same. After the critical coupling we can see that the thermodynamic limit is way lower than the rest. This makes sense because the thermodynamic limit is for $N \to \infty$ and with higher N the energy per site can get higher.

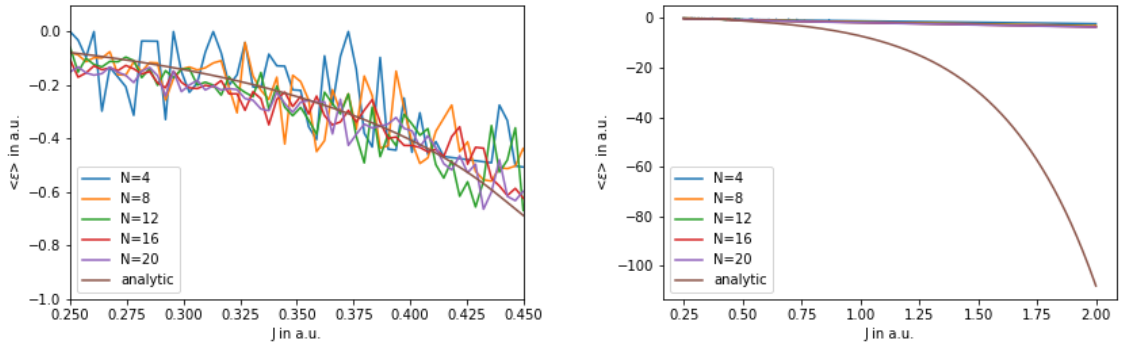

Figure 3: Energy per site compared to J.

For the extra homework we should look at the specific heat $C = \Lambda \cdot (< \epsilon^2 > - < \epsilon >^2)$. In the Graph 4 we plottet it out. We again see that we get a change in curve by $\frac{1}{J_c} \approx 2.27$. If N is higher we get a greater value of C.

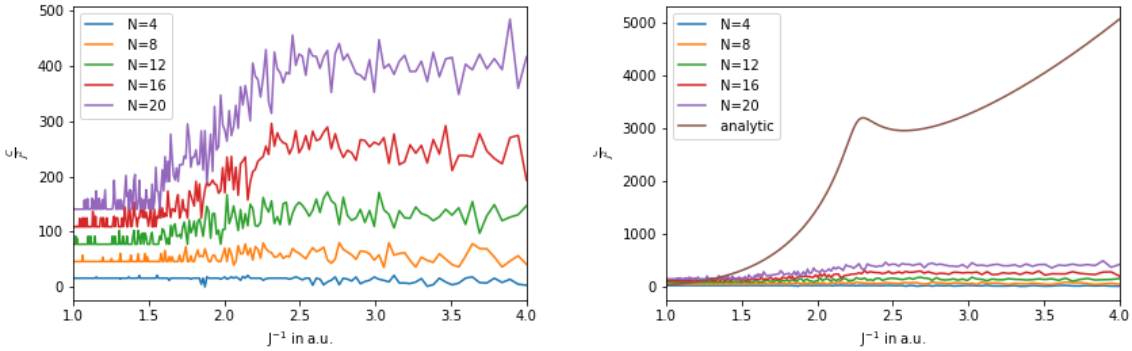

Figure 4: Specific heat

4 conclusion

In this exercise we saw that we get an internal magnitization if the coupling constant is high enough and we could replicate the graph from week one but now for two dimensions. Futhermore we saw that with lower N the error ist a lot higher. Lastly we can see that the energy per site scales with Λ, so it scales with how many sites are there.

References

[1] Thomas Luu, Andreas Nogga, Marcus Petschlies and Andreas Wirzba, Exercise-sheet, 2020.