Universality of three fermions

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The aim of this paper is to study the cutoff dependence of the One-Pion-Exchange. Before one can execute this task one has to verify the grid points and have to calculate the right parameters for different cutoff's of the One-Pion-Exchange-Potential.

I. MOTIVATION

The theory of Quantum chromodynamics(QCD) is widely accepted but is hard to handle because we can not solve it with pertubation theory. One way to solve it is the lattice simulation but until now it is at the beginning for systems with more than two nucleons [1][2] and therefore only for limited use.
It is possible to use chiral perturbation theory(ChPT) where we make use of the spontaneously-broken chiral symmetry of QCD[3][4]. With this theory we are able to formulate an effective field theory(EFT) involving nucleons and Goldstone bosons related to chiral-symmetry breaking called the pion(π). The advantage of the ChPT is that one can processes with different numbers of pions. In the case of the paper, we only look at One-Pion-Exchange(OPE). If we follow the theory one will find the non-perturbative nature of the bound states in a few body problem. Weinberg [5][6] reconized it that this is caused by an infrared enhancement and he suggested a two step calculation. In the first step, he defines the nuclear potential as a sum of sub-diagrams that do not contain purely-nucleonic intermediate states. In the second step one wants to expand it to all orders by using the Lippmann-Schwinger(LS) equation. The Potential has to include pion exchange(s) and contact interactions. These contact interactions represent the contribution of the massiv degree of freedom. We can assume that the contact interaction has only a finite number of pion exchanges because it is finite dimensional. Thus our requirements that we only use the OPE is justified. Because of this power counting one can see that two-nucleon interactions are more important than three-nucleon interactions. For simplicity we neglect the three-nucleon interaction term[7].
The Weinberg theory has been criticized. In any EFT we need a regularization procedure to separate high- and low-energy physics. The problem is that we need sufficient counterterms at each order to absorb any cutoff dependence. A cutoff is a term in the potential such that for high momenta the potential vanishes. This allows us to integrate the potential till infinity. To study the cutoff dependence one has to use the LS equation which is numerical in character. Thus the cutoff dependence is challenging. There are a few attempts to avoid these cutoff dependence. For example Kaplan and his coworkers tried to to treat the OPE in finite order pertubation theory[8][9][10]. However this theory fails in some partial waves at momenta comparable to the pion mass. One other is for small momenta to integrate the pion mass out and create a "pionless" EFT[4]. Again the problem is that a lot of interesting interactions take place of the order of the pion mass.
The problem with the cutoff dependence still holds. Generally the cutoff dependence is observed in higher partial waves[11]. For small cutoff variations, the phase shift decreases with increasing order. In this work we want to show these cutoff dependence of the OPE and compare it to the One-Boson-Exchange(OBE). Before we can act out the calculation we have to verify the number of grid points, we have to take into account. This we have to do for the OBE as well as the OPE.

In this paper, we can describe theoretical concept and method on section II. After we present the two different potentials in section III. Moreover, we can present and analysed our finding in section IV regarding the cutoff dependence. Finally, we conclude our results in section V.

II. THREE BODY BOUND STATE

In this section we discuss the theory behind the three body bound state. Let's start with the Schrödinger equation

$$\hat{H}\,|\psi\rangle = (\hat{H}_0 + \hat{V}_{12} + \hat{V}_{23} + \hat{V}_{31} + \hat{V}_{123}) \tag{1}$$

with $|\psi\rangle$ a physical state, $\hat{V}_{ij}$ the potential between two particles, $\hat{H}_0$ the internal kinetic energy and $\hat{V}_{123}$ the potential for a three body interaction. In this paper we neglect the three body force. We only focus on two body interactions. The three particles will be identical fermions. Thus we have to take the Pauli principle into account. It is easier to express the equation with only one pair of fermions. This is possible because the particles are identical. Thus we can write the potential $V_{23} = P_{12}P_{23}\ V_{12}\ P_{23}P_{12}$ and $V_{23} = P_{12}P_{23}\ V_{12}\ P_{23}P_{12}$ where P_{ij} are permutation operators. With this help the Schrödinger equation simplifies to

$$(E - H_0)\,|\psi\rangle = (\mathbb{1} + P)V_{12}\,|\psi\rangle \ \text{ ,with } P = P_{12}P_{23} + P_{13}P_{23} \tag{2}$$

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with E the binding Energy and the Faddeev component is given as

$$|\psi_{12}\rangle = \frac{1}{E - H_0} V_{12}(\mathbb{1} + P)\,|\psi_{12}\rangle\,. \tag{3}$$

With the help of the Lippman-Schwinger-equation it is possible to determine the Faddeev equation:

$$|\psi_{12}\rangle = \frac{1}{E - H_0} t_{12} P\,|\psi_{12}\rangle \tag{4}$$

where t_{12} is the non trivial part of the scattering matrix. Now we have to consider the partial waves and the coupling between different states. Because we examine a fermionic system we have to have anti-symmetric wave function. We will describe the fermion system as

$$|\alpha\rangle = |\vec{p}_{12}\ \vec{p}_3((l_{12}S_{12})\ J_{12}(l_3S_3)I_3)\rangle \tag{5}$$

with the Jacobi momenta $\vec{p}_{12} = \frac{1}{2}(\vec{k}_1 - \vec{k}_2)$, $\vec{p}_3 = \frac{2}{3}\vec{k}_3 - \frac{1}{3}(\vec{k}_1 + \vec{k}_2)$ where $\vec{k}_i$ is the single particle momenta, l is the angular momentum, S the spin and J the total angular momentum. The indices represent the subsystem for these parameters. We can project the Lippmann-Schwinger equation for the transfer matrix t_{12} on the partial wave basis and get

$$\begin{aligned} t_{12}(\vec{p}_1'2\alpha_{12}';\vec{p}_1 2\alpha_{12};\vec{p}_3) = V_{12}(\vec{p}_1'2\alpha_{12}';\vec{p}_1 2\alpha_{12}) + \\ \sum_{\alpha''}\int \mathrm{d}^3\vec{p}''\, V_{12}(\vec{p}_1'2\alpha_{12}';\vec{p}_1''2\alpha_{12}'') \\ \frac{t_{12}(\vec{p}_1''2\alpha_{12}'';\vec{p}_1 2\alpha_{12};\vec{p}_3)}{\widetilde{E} - \frac{\vec{p}_{12}''^2}{2m}}, \end{aligned} \tag{6}$$

where $\widetilde{E} = E - \frac{3p_3^2}{4m}$ and $|\alpha_{12}\rangle = |(l_{12}S_{12})J_{12}\rangle$. After a long derivation which can be read in *"The Quantum Mechanical Few Body Problem"* written by Prof. Dr. Walter Glöckle [12] we arrive at

$$\begin{aligned} \psi(p_{12}'p_3'\alpha') = \sum_{\alpha}\int \mathrm{d}p_{12}\mathrm{d}p_3 p_{12}^2 p_3^2 \\ \int_{-1}^{+1}\mathrm{d}x \frac{\delta(p_{12} - \pi_{12}(p_3p_3'x))}{p_{12}^2}\frac{\delta(p_{12} - \pi_{12}'(p_3p_3'x))}{p_3^2} \\ \cdot(1 + 2G_{\alpha'\alpha}(p_3'p_3x))\psi(p_{12}p_3\alpha) \end{aligned} \tag{7}$$

with $\pi_{12} = |-p_3' - \frac{1}{2}p_3|$ and $\pi_{12}' = |p_3 + \frac{1}{2}p_3'|$ and the $G_{\alpha'\alpha}(p_3'p_3x))$ function is given in the appendix A1. In the following step one has to discretize the equation 7 to work with it on a computational basis. Therefore we get:

$$\begin{aligned} \psi(p_i'q_j'\alpha') = \sum_{p_iq_j\alpha}\sum_k w_k(1 + 2G_{\alpha'\alpha}(q_j'q_jx)) \\ \delta(p_i - \pi_{12}(q_jq_j'x)\delta(q_j - \pi_{12}'(q_jq_j'x_k))\psi(p_iq_j\alpha), \end{aligned} \tag{8}$$

where w_k are the weights from the gaussian integration, $p_i \widehat{=} p_{12}$ and $q_j \widehat{=} p_3$.

III. OBE- AND OPE-POTENTIAL

Before we can start calculating anything we have to check for all the possible wave functions depending on the angular momentum, spin, isospin and total angular momentum. Because we have a fermionic system we have to have anti-symmetric wave functions. If this is done we can start with the potential.

A. One-Boson-Exchange-Potential(OBE)

We will examine the three body system with two different potentials. The first one is the One-Boson-Potential. A realistic case for the OBE would be the following:

$$\begin{aligned} V(\vec{p},\vec{p}') = \frac{A}{2\pi}\frac{1}{(\vec{p}-\vec{p}')^2 + m_B^2}\exp\left(-\frac{(\vec{p}-\vec{p}')^2 + m_B^2}{\Lambda^2}\right) \\ + \frac{C_0}{4\pi}\exp\left(-\frac{p^2 + p'^2}{\Lambda^2}\right) \end{aligned} \tag{9}$$

where Λ is the the cutoff and $\vec{p}$ and $\vec{p}'$ are the incoming and outgoing momenta. We need Λ because we can not integrate till infinity on a computer. Thus we have to stop at one value. To compensate the early stop we adjust the potential with the so called regulator $f(\vec{p},\vec{p}') = \exp\left(-\frac{p^2+p'^2}{\Lambda^2}\right)$. The parameters A and C_0 we have to determine with a known binding Energy for example the Deuteron. We know that the ${}^3S_1$ state has a binding Energie of $-2.225\,\mathrm{MeV}$ and the ${}^1S_0$ state is not a bound state. In the OBE potential we assume that an exchange in form of a boson is taken place. It is a nucleon-nucleon interaction for the short to intermediate region compared to the long range of the One-Pion-Exchange-Potential(OPE). In this potential only partial waves with the same angular momentum can couple. Thus Spin, total angular momentum and isospin can change. The potential only depends on the incoming and outgoing momenta and the angular momentum.

1. *Stability of the grid points for the OBE-Potential.*

We have to verify that the grid points we take are enough. For the momenta p and position q we use three intervals with the following arrangement for the grid points:

$$0\underbrace{\ldots}_{np_1/2}p_a\underbrace{\ldots}_{np_1/2}p_b\underbrace{\ldots}_{np_2}p_c. \tag{10}$$

We set the internals to $p_a = 1\mathrm{fm}^{-1}$, $p_b = 5\mathrm{fm}^{-1}$ and $p_c = 20\mathrm{fm}^{-1}$. Analog we do the same for the position q. So we take the parameters $q_a = 1\mathrm{fm}$, $q_b = 5\mathrm{fm}$ and $q_c = 20\mathrm{fm}$.We also use nx grid points for the integration over the angle. In tabular I one can see the stability.

parameter set	nx	np1	np2	nq1	nq2	energy in MeV
1	4	6	6	6	6	-20.5841
2	6	10	6	10	6	-19.8241
3	10	15	10	15	10	-19.7137
4	10	20	12	20	12	-19.6959

We will take the third parameter set. The error is under one percent and the calculation does not take so long.

TABLE I. Stability of grid points with the Parameters $\Lambda = 500$,$C_0 = 0.020167$,$A = -1.0/6.4749$, $l_{12,max} = 2$ and $l_{3,max} = 1$.

t	$S=1$	$l=J-1$	$l=J$	$l=J+1$
$t=1$	$l'=J-1$	X	O	X
	$l'=J$	O	X	O
	$l'=J+1$	X	O	X
$t=0$	$l'=J-1$	X	O	X
	$l'=J$	O	X	O
	$l'=J+1$	X	O	X

TABLE II. Possible partial waves that couple to each other. l and l' are the incoming and outgoing angular momentum, S is the total spin, J the total angular momentum and t the isospin. X marks a possible coupling and O marks an impossible transition.

B. One-Pion-Exchange-Potential(OPE)

The second potential is the One-Pion-Potential. In this potential an exchange in form of one Pion is taken place. This potential is true for larger distance than the OBE and is given as

$$V(\vec{q}=\vec{p}-\vec{p}') = -\frac{1}{(2\pi)^3}(\frac{g_A}{2f_\pi})^2\tau_1\tau_2\frac{(\vec{\sigma}_1\cdot\vec{q})(\vec{\sigma}_2\cdot\vec{q})}{\vec{q}^2+m_\pi^2} \quad (11)$$

with m_π the pion mass, the axial-coupling constant $g_A = 1.26$, pion-decay constant $f_\pi = 92.4$MeV and $\vec{\sigma}$ and τ are the Pauli matrices in spin and isospin space. In addition to the pion exchange we also get a short range interaction. The simplest are two contact interactions

$$V_c = \frac{1}{4\pi}\frac{1}{(2\pi)^3}(C_0P_s + C_1P_t), \quad (12)$$

where P_t and P_s are the projecters onto spin-triplet and spin-singlet states. The two parameters C_0 and C_1 can again be determine of scattering data. In our case we take again the Deuteron ${}^3S_1$ and ${}^1S_0$ state. For the numerical solution, again we need to introduce an regulator $f(\vec{p},\vec{p}') = \exp\left(-\frac{\vec{p}^2+\vec{p}'^2}{\Lambda^2}\right)$ to compensate the early stop of the integral. The regulator only depends on the incoming and outgoing momenta. Because of that the regulator has no influence on the partial-wave decomposition. For the partial waves that couple to each other one gets the possibilities in table II. The total spin, isospin and total angular momentum must be obtain. Only the angular momentum can change by transition to another state.

Λ	C_0	C_1	bound state of ${}^3S_1$ in MeV
500	-0.196	-0.187035	-2.2251
600	-0.181	-0.13535	-2.2262
700	-0.169	-0.086018	-2.2251
800	-0.161	-0.02955	-2.2245
900	-0.155	0.05115	-2.2231
1000	-0.149	0.2091	-2.2250

TABLE III. Parameter sets for the OPE-Potential

1. *Parameters for the OPE-Potential*

In this section we try to determine the parameters for the OPE-Potential. We start with the known fact that the ${}^3S_1$ state has an binding energy of -2.225 MeV but has no bound state in ${}^1S_0$. With this knowledge we get the parameters in table III.

2. *Stability of the grid points for the OPE-Potential.*

This section is analogue to section III A 1. Thus we take the same Intervalls and get the following result in table IV

parameter set	nx	np1	np2	nq1	nq2	energy in MeV
1	4	6	6	6	6	-13.4611
2	6	10	6	10	6	-13.1791
3	10	15	10	15	10	-13.1239
4	10	20	12	20	12	-13.1151

TABLE IV. Stability of grid points with the Parameters $\Lambda = 500$,$C_0 = -0.196$,$C_1 = -0.18691$, $l_{12,max} = 2$ and $l_{3,max} = 1$.

Since the calculation is taking quit long. We try to take as little grid points as possible. Thus for our Problem we will take parameter set 2. We are less than one percent off, so it is a good approximation.

IV. CUTOFF DEPENDENCE

We want to examine if the binding energy changes with a contribution of higher partial waves and with different regulators. Since the OPE is for larger distances we expect to see a bigger difference with different cutoffs.

A. OBE-Potential

In table V we see the binding energy for different partial waves and different regulators for the OBE.
One can only see a small difference between different regulators and a small difference if we take higher partial waves into account. This difference can not be explained with not enough grid points because the difference is

possible partial waves	$\Lambda = 700$	$\Lambda = 1200$
$l_{12,max} = 2, l_{3,max} = 0$	-19.7138	-19.2964
$l_{12,max} = 2, l_{3,max} = 2$	-19.7664	-19.3589
$l_{12,max} = 3, l_{3,max} = 2$	-19.8699	-19.4698
$l_{12,max} = 4, l_{3,max} = 2$	-19.8757	-19.4798

TABLE V. Energie[MeV] for the OBE-Potential with different Λ and different partial waves.

much larger. But one has to say that the difference is only a few percents.

B. OPE-Potential

In this part we examine the OPE-Potential and how the bound energies changes with different partial waves and with different cutoff's. The results are in table VI. We can see that both regulators converge to a value but

possible partial waves	$\Lambda = 500$	$\Lambda = 1000$
$l_{12,max} = 2, l_{3,max} = 0$	-13.1239	-7.9445
$l_{12,max} = 2, l_{3,max} = 1$	-13.4127	-15.7252
$l_{12,max} = 2, l_{3,max} = 2$	-14.1299	-13.2799
$l_{12,max} = 3, l_{3,max} = 2$	-14.1267	-12.9590
$l_{12,max} = 4, l_{3,max} = 3$	-14.1387	-12.6420
$l_{12,max} = 5, l_{3,max} = 3$	-14.1386	-12.6321
$l_{12,max} = 5, l_{3,max} = 4$	-14.1386	-12.6819

TABLE VI. Energie[MeV] for the OPE-Potential with different Λ and different partial waves.

they are quite far away. We can also see that the convergence of the OPE is slower with higher partial waves than the OBE.
Because of the different values we can verify that the potential depends on the regulator.

V. CONCLUSION

First of all, we checked for the grid points. We saw that not many grid points are enough so that the value converge. This is essential because the calculation takes a lot of time, especially the Wigner symbols. After that, we got enough grid points, and we could successfully determine the parameters for the different cutoffs. We, finally, examined the dependence on the cutoff. We found that the OBE-Potential is far less dependent on the cutoff compared to the heavy dependence of the OPE-Potential. The cutoff dependence is expected in the OPE-Potential since we have not enough counterterms. We have only one counterterm in the S-wave. Also, with higher cutoffs the dependence should vanish or if one formulates counterterms for each partial wave. Moreover, we expected that the dependence should vanish. However, It is quite complicated but one can read more about it in A. Nogga, R.G.E. Timmermans, and U. van Kolck in *"Renormalization of One-Pion Exchange and Power Counting"*[13].
We have found that the OPE-Potential without preparation for the cutoffs is only from limited use, and we have found an idea of the final bound state energy but the error is too large to make use of it.

Appendix A

$$G_{\alpha\alpha\prime}(\vec{p}_3'\vec{p}_3 x) = \sum_{L=l_{12}+l_3}\sum_{S=s_{12}+s_3}\sum_{t}\hat{S}\sqrt{\hat{J}_{12}\hat{J}'_{12}\hat{I}_3\hat{I}'_3}$$
$$\begin{Bmatrix} l'_{12} & s'_{12} & J'_{12} \\ l'_3 & \frac{1}{2} & I'_3 \\ L & S & J \end{Bmatrix}\begin{Bmatrix} l_{12} & s_{12} & J_{12} \\ l_3 & \frac{1}{2} & I_3 \\ L & S & J \end{Bmatrix}(-1)^{S_{12}}\sqrt{\hat{s}_{12}\hat{s}'_{12}}$$
$$\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & s'_{12} \\ \frac{1}{2} & S & s_{12} \end{Bmatrix}(-1)^{t_{12}}\sqrt{\hat{t}_{12}\hat{t}'_{12}}\begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & t'_{12} \\ \frac{1}{2} & T & t_{12} \end{Bmatrix}\left(\frac{8\pi^2}{2L+1}\right)$$
$$\sum_{M,L} Y^{\star l\ M_l}_{l'_{12}l'_3}\left(\widehat{\vec{p}_3 + \frac{1}{2}\vec{p}_3\hat{p}_3}\right)Y^{\star l\ M_l}_{l'_{12}l'_3}\left(\widehat{-\vec{p}_3 - \frac{1}{2}\vec{p}_3\hat{p}_3}\right) \tag{A1}$$

The Symbol $\left\{\right\}$ Is the 9j-symbol or 6j-symbol.

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[13] A. Nogga et al. *Re-normalization of One-Pion Exchange and Power Counting. 2 Jun 2005*
[14] The program for these results can be found here: *https://github.com/MarcelSchindler/Universality-of-three-fermions.git*