TOO OLD TO DEMONSTRATE, TO YOUNG TO DIE

MARCEL TAUDE

ABSTRACT. Playing with integer equations is really nice. This paper is a ... hmmm ... demonstration of an old adage *if you don't practice mathematics for a long time, it's hard to come back to it.* Even when listening cool jazz from *Lee Konitz.* At least, I played with LaTeX and *gcc.*

At first, I wanted to find an algorithm to find integers which verify $a^3+b^3+c^3 = 1000^2.a + 1000.b + c$ with $a, b, c \in [0, 1000[$. There is an obvious lazy algorithm which tries every values of a, b and c in the range $[0, 1000[$. I wanted a clever one, not sure I found it, but read.

Contents

1. THE PROBLEM

Find all integers which verify :

$$a^3 + b^3 + c^3 = 1000^2.a + 1000.b + c, \text{ with } a, b, c \in [0, 1000[\tag{1}$$

We can *easily* find all the integers (a, b, c) verfiying (1) by computing the two members of (1) for every values (a, b, c) from 0 to 999. Can we find a more clever way to get these values?

We can transform (1) in :

$$\begin{aligned} c^3 - c &= 1000^2.a + 1000.b - a^3 - b^3 \\ c(c-1)(c+1) &= a(1000-a)(1000+a) + b(1000-b^2) \end{aligned} \tag{2}$$

It becomes more obvious that the triplets $(0, 0, 0)$ and $(0, 0, 1)$ are two solutions of (1).

2. ... AND THIS IS THE PLACE WHERE I BECAME MAD

So much time without playing with mathematics makes things harder than I thought. Music helps me to keep cool, some good old jazz played by *Lee Konitz* in the late fifties, and that's all. At least, I am playing with LaTeXand *gcc*...

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3. REBOOT

We will do it in a lighter way. We started with power 3, bad idea.

$$a^2 + b^2 = 100.a + b \text{ with } a, b \in [0, 100[\tag{3}$$

We can make this transformation :

$$b(b-1) = a(100-a) \tag{4}$$

We can see that the left side of (5) must be even, so a is even too.

3.1. the even side.

We begin with the case of b even :

$$\begin{aligned} a &= 2\alpha \\ b &= 2\beta \\ 2\beta(2\beta-1) &= 2\alpha(100-2\alpha) \\ \beta(2\beta-1) &= 2\alpha(50-\alpha) \end{aligned} \tag{5}$$

If so, β must be even, *i.e.* :

$$\begin{aligned} \beta &= 2\gamma \\ 2\gamma(4\gamma-1) &= 2\alpha(50-\alpha) \\ \gamma(4\gamma-1) &= \alpha(50-\alpha) \end{aligned} \tag{6}$$

The good news is :

$$\begin{aligned} &\gamma \in [0, 25[\\ &\alpha \in [0, 50[\\ &a = 2\alpha \\ &b = 4\gamma \end{aligned} \tag{7}$$

We have less couples (α, γ) to test.

3.2. the odd side.

Now, we are looking at odd values of b:

$$\begin{aligned} a &= 2\alpha \\ b &= 2\beta+1 \\ 2\beta(2\beta+1) &= 2\alpha(100-2\alpha) \\ \beta(2\beta+1) &= 2\alpha(50-\alpha) \end{aligned} \tag{8}$$

We comme back to the previous case with few changes :

$$\begin{aligned} &\gamma \in [0, 25] \\ &\alpha \in [0, 50[\\ &a = 2\alpha \\ &b = 4\gamma+1 \end{aligned} \tag{9}$$

3.3. ... and now.

It's time to write some code to test all these beautiful formulas.

4. NOTES

I wrote this document with LaTeX, using the AMS packages (*cf.* [AMSInstr2004] and [AMSUsersG2002]). The *href* package (*cf.* [HREFPackage]) gives me the links.

References

[AMSInstr2004] *Instructions for Preparation of Papers and Monographs*, AMS ftp://ftp.ams.org/pub/author-info/documentation/amslatex/instr-l.pdf (2004)

[AMSUsersG2002] *User's Guide for the amsmath Package*, AMS ftp://ftp.ams.org/pub/tex/doc/amsmath/amsldoc.pdf (1999–2002)

[HREFPackage] *LaTeX/Hyperlinks*, Wikipedia https://en.wikibooks.org/wiki/LaTeX/Hyperlinks#Commands