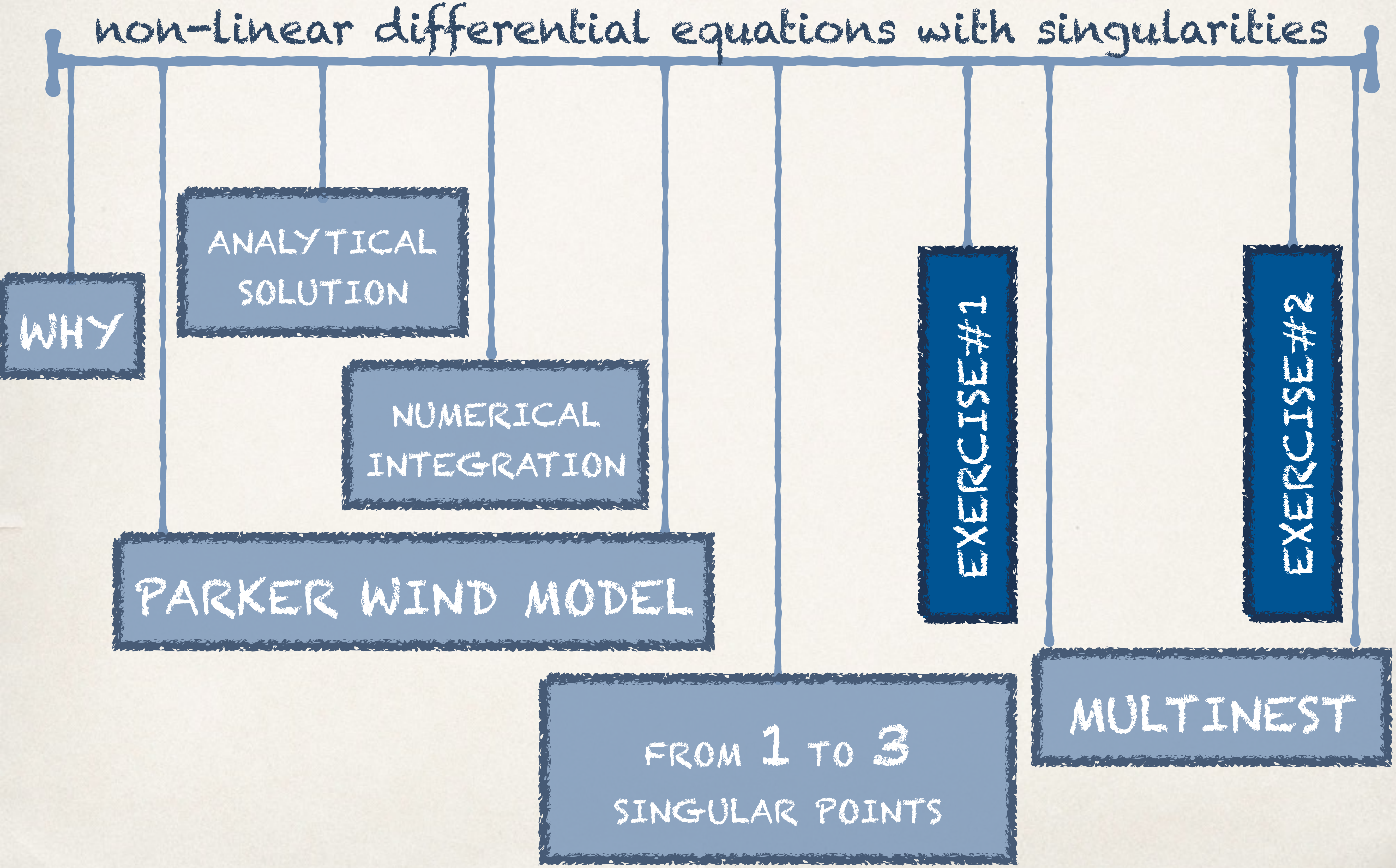
The Magere Brug - oil on canvas by Hubertine Heijermans

ADVANCING THEORETICAL ASTROPHYSICS
AMSTERDAM
JULY 15-26, 2019

Nonlinear differential equations with singular surfaces: the wind equation

Chiara Ceccobello - PD@Chalmers University of Technology

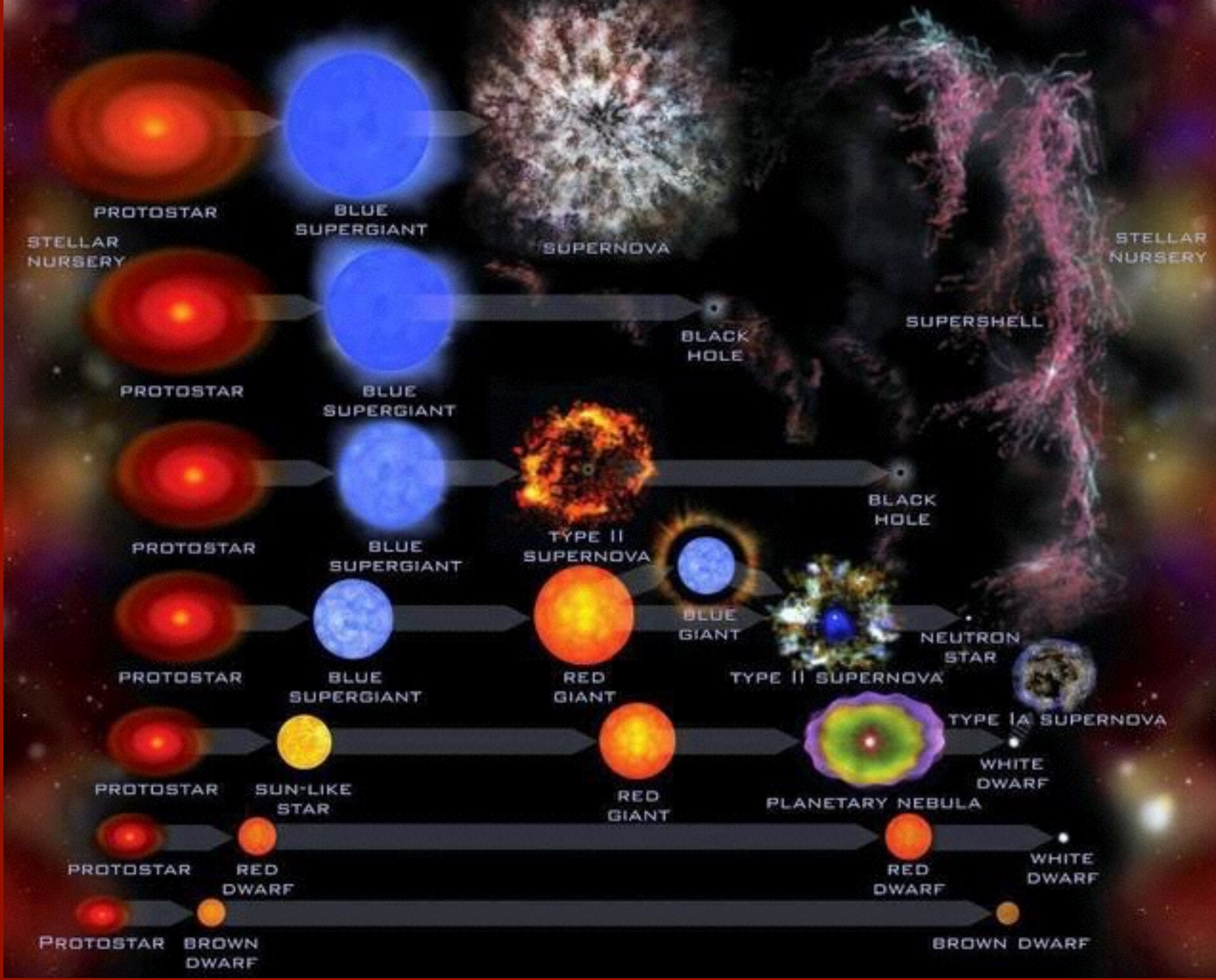
OUTLINE



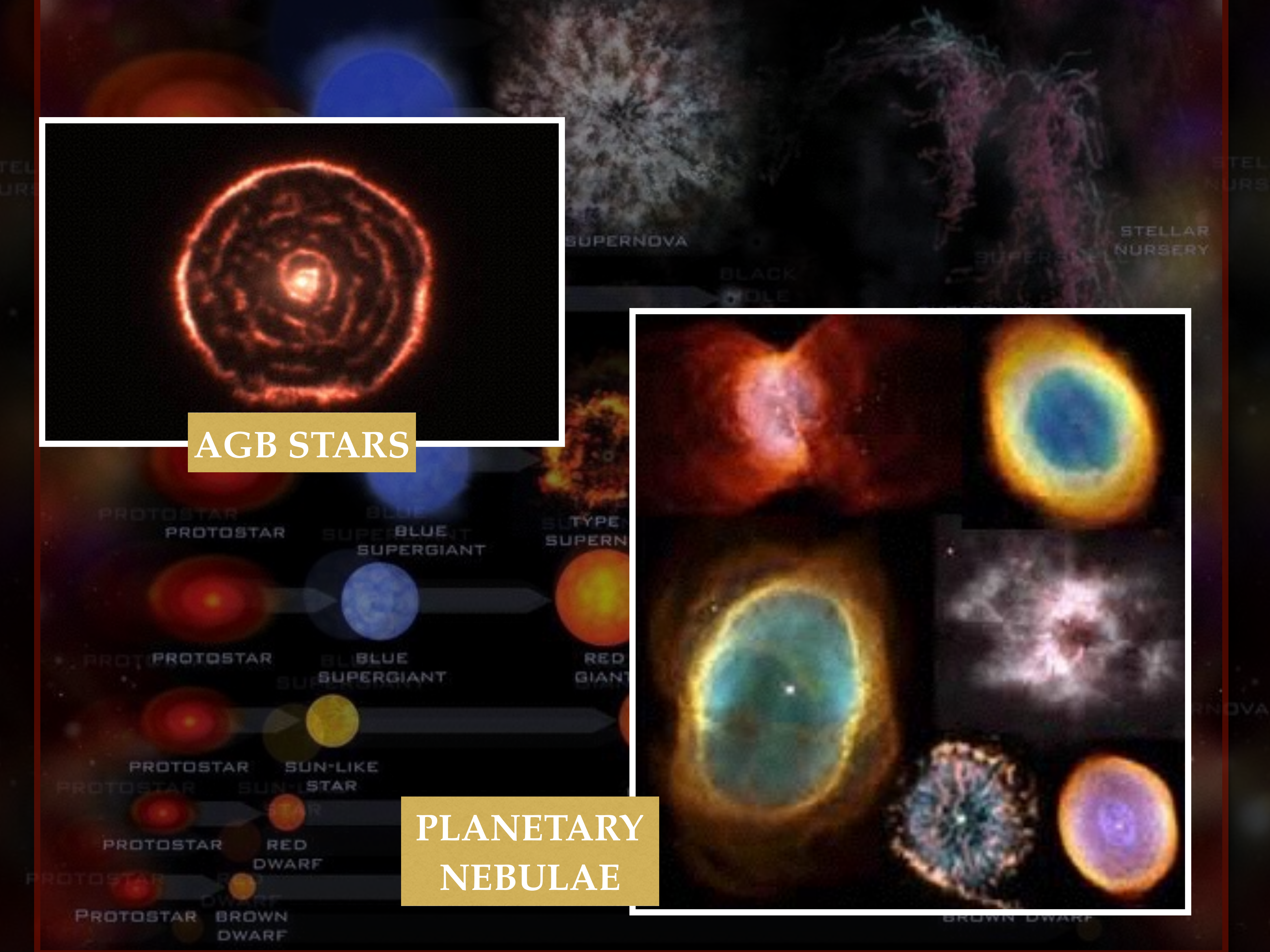
non-linear differential equations with singularities
WHY
ANALYTICAL SOLUTION
NUMERICAL INTEGRATION
PARKER WIND MODEL
FROM 1 TO 3 SINGULAR POINTS
EXERCISE#1
MULTINEST
EXERCISE#2

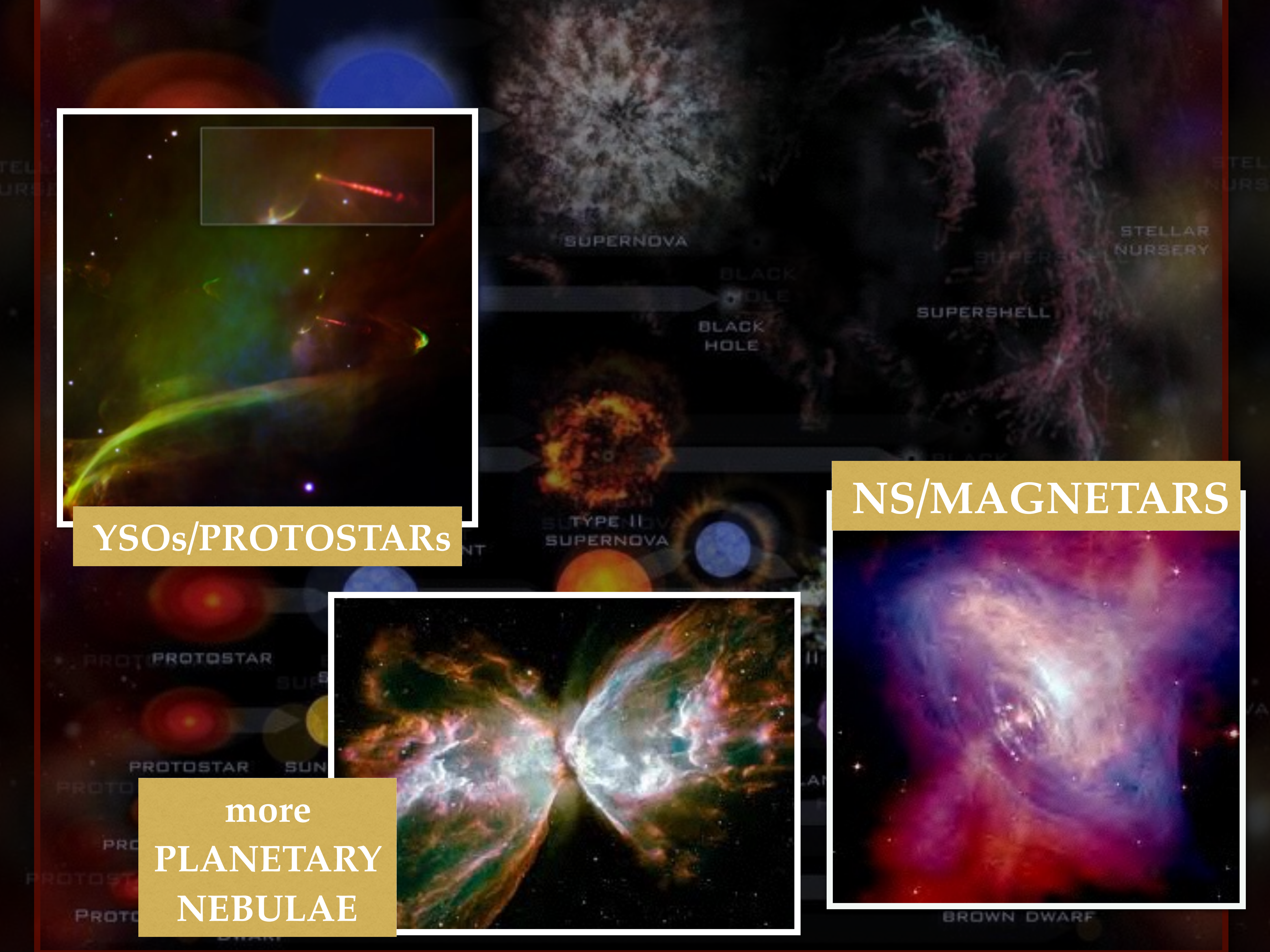
Why do we want/need to study **non-linear differential equations with singularities** in the first place???

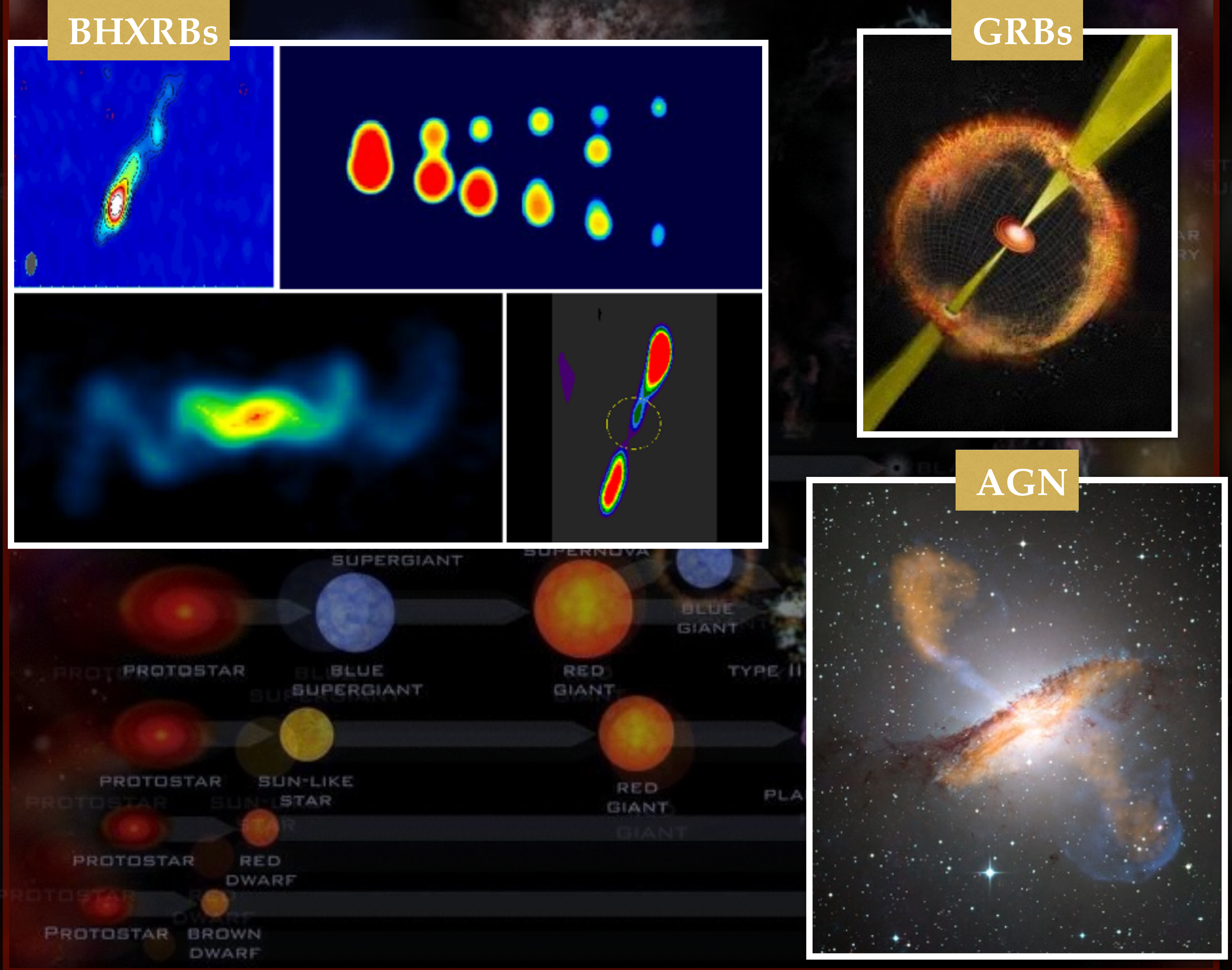
Among other things, they describe the accretion and ejection of matter into and away from an astrophysical object



STELLAR NURSERY
PROTOSTAR
BLUE SUPERGIANT
SUPERNOVA
SUPERSHELL
STELLAR NURSERY
PROTOSTAR
BLUE SUPERGIANT
BLACK HOLE
PROTOSTAR
BLUE SUPERGIANT
TYPE II SUPERNOVA
BLACK HOLE
PROTOSTAR
BLUE SUPERGIANT
RED GIANT
BLUE GIANT
TYPE II SUPERNOVA
NEUTRON STAR
TYPE IA SUPERNOVA
PROTOSTAR
SUN-LIKE STAR
RED GIANT
PLANETARY NEBULA
WHITE DWARF
PROTOSTAR
RED DWARF
RED DWARF
WHITE DWARF
PROTOSTAR
BROWN DWARF
BROWN DWARF


AGB STARS
PLANETARY NEBULAE
SUPERNOVA
STELLAR NURSERY
PROTOSTAR
BLUE SUPERGIANT
PROTOSTAR
BLUE SUPERGIANT
RED GIANT
PROTOSTAR
SUN-LIKE STAR
PROTOSTAR
RED DWARF
PROTOSTAR
BROWN DWARF


YSOs/PROTOSTARs
NS/MAGNETARS
more
PLANETARY
NEBULAE
SUPERNOVA
STELLAR
NURSERY
SUPERSHELL
BLACK
HOLE
TYPE II
SUPERNOVA
PROTOSTAR
PROTOSTAR
BROWN DWARF


BHXRBs
GRBs
AGN
SUPERGIANT
PROTOSTAR
BLUE SUPERGIANT
RED GIANT
BLUE GIANT
TYPE II
PROTOSTAR
SUN-LIKE STAR
RED GIANT
PROTOSTAR
RED DWARF
PROTOSTAR
BROWN DWARF

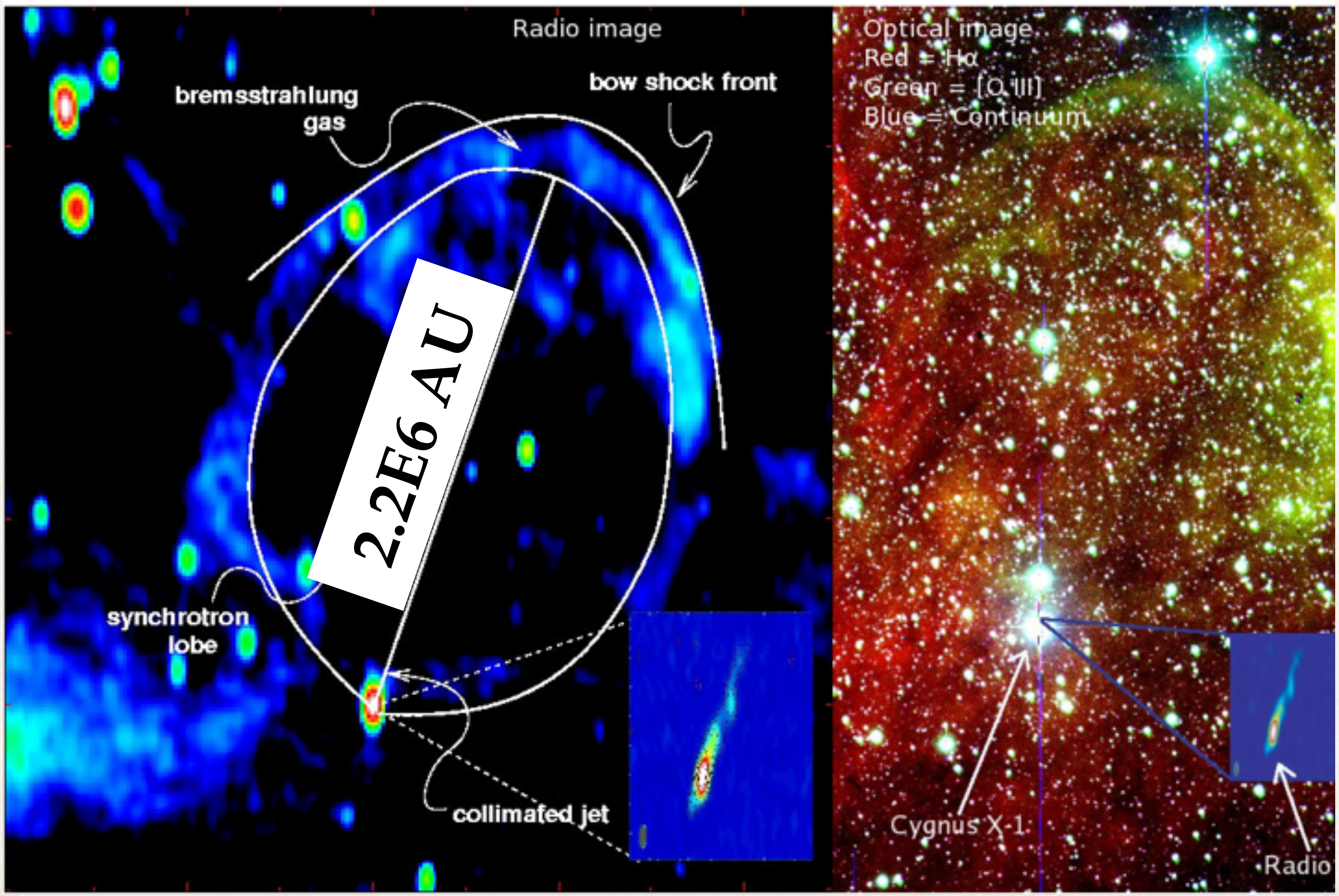
Impact of outflows

Interaction and feedback

8.2E5 AU

Impact of outflows

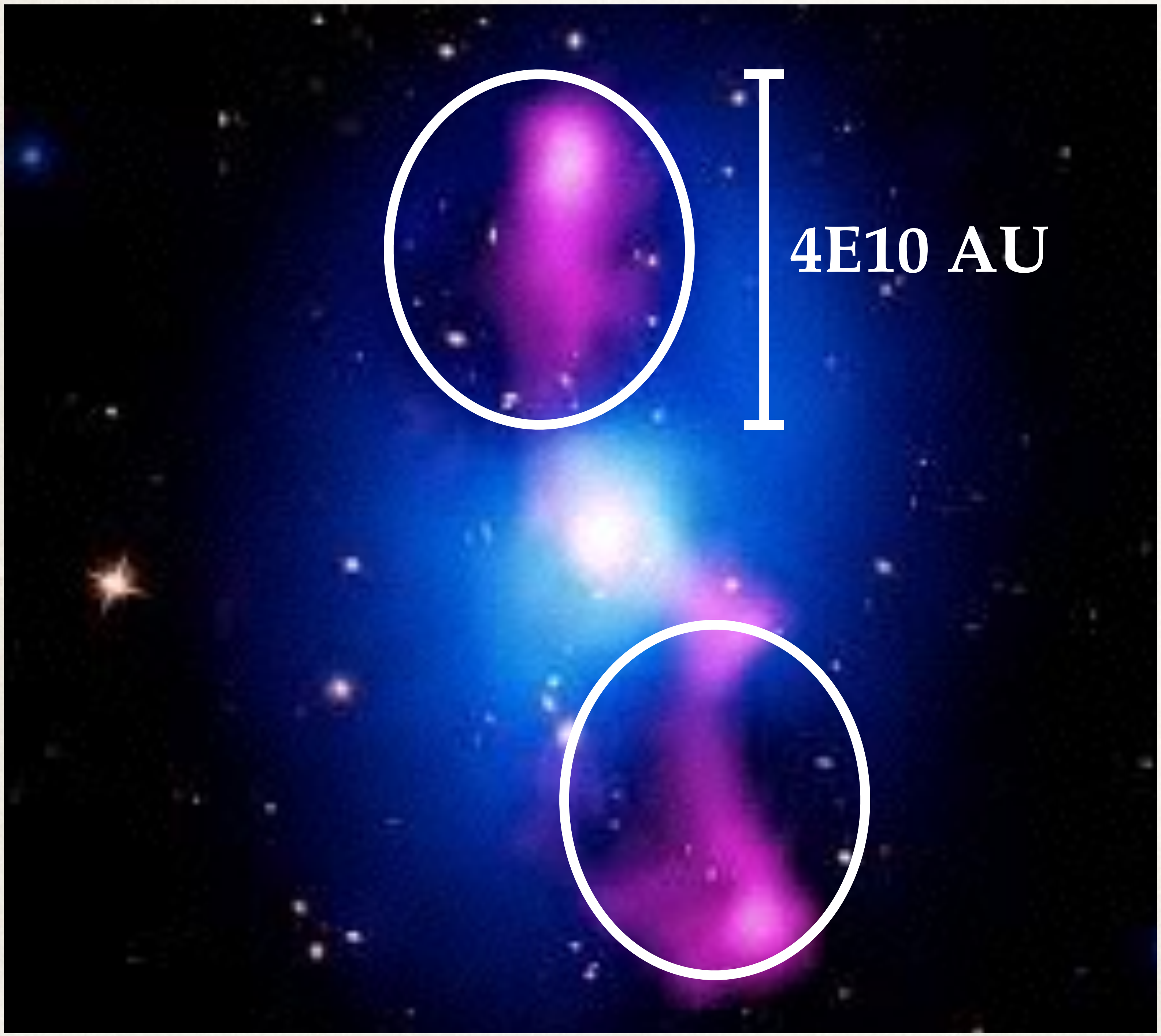
Interaction and feedback



Black Hole X-Ray Binary - Cygnus X-1: Gallo et al. 2005, Russell et al. 2007

Impact of outflows

Interaction and feedback



BLUE: X-ray
PINK: radio
BG: optical

Credit:
X-ray: NASA/CXC/Univ. of Waterloo/
A.Vantyghem et al;
Optical: NASA/STScI;
Radio: NRAO/VLA

WINDS
OUTFLOWS
JETS

HOW DO THEY FORM?

HOW MUCH ENERGY DO THEY CARRY?

WHAT ARE THEY MADE OF?

HOW DO THEY RADIATE?

HOW DO THEY FORM?

HOW MUCH ENERGY DO THEY CARRY?

WINDS
OUTFLOWS
JETS

WHAT ARE THEY MADE OF?

HOW DO THEY RADIATE?

How do outflows(/inflows) form?

The MHD system of equations

We saw that

- stars can loose mass through stellar wind
- gravitating objects can attract matter from the interstellar medium or a companion
- accreting objects can launch outflows of different types (collimated/uncollimated winds, jets,…)

To study both accretion/ejection of gas into/away from a source, we need to solve ***a set of equations describing a decelerating/accelerating fluid in the gravitational field of the source and possibly in the presence of large-scale magnetic fields***

Hydro-/MagnetoHydro Dynamics (HD or MHD)

Parker HD wind model (1958)

The simplest case of study is a radial flow, either towards or away from an object of mass M.

We assume the following:

- **Spherically symmetric** ⇨ radial flow, i.e. $v = v(r)$
- **Perfect, non self-gravitating gas** ⇨ $\boxed{\Phi = -\frac{GM}{r}}$
- **Isothermal** ⇨ constant sound speed, i.e. $\boxed{c_s^2 = \frac{p}{\rho}}$

Parker HD wind model (1958)

The governing equations are:

- **Mass conservation**: $\boxed{4\pi r^2 \rho v = -\dot{M}}$
  - if $v > 0$: **STELLAR WIND**
  - if $v < 0$: **ACCRETION FLOW**
  - the mass-loss/-accretion rate, $\dot{M}$, is constant
- **Equation of state**: $\boxed{p = K\rho^{\Gamma}}$
  - which reduces to $c_s^2 = p/\rho$ in the isothermal case
- **Equation of motion**: $\boxed{\rho v \frac{dv}{dr} = -\rho \frac{d\Phi}{dr} - \frac{dp}{dr}}$

Analytical integration

$$\rho v \frac{dv}{dr} = -\rho \frac{d\Phi}{dr} - \frac{dp}{dr}$$

- To perform the integration of the Parker wind equation, we first need to get rid of as many unknown as possible by using the constraints we know from the other equations

- Namely, we want to find a version of the above equation which is only function of *v*,*r* + constants

$$\frac{dv}{dr} = \mathcal{F}(r, v; c_s, G, M)$$

Analytical integration

$$\rho v \frac{dv}{dr} = -\rho \frac{d\Phi}{dr} - \frac{dp}{dr}$$

- Let's rearrange it and use $c_s^2 = p/\rho$ to eliminate p

$$v \frac{dv}{dr} + \frac{c_s^2}{\rho} \frac{d\rho}{dr} + \frac{GM}{r^2} = 0$$

Analytical integration

$$\boxed{\rho v \frac{dv}{dr} = -\rho \frac{d\Phi}{dr} - \frac{dp}{dr}}$$

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$$v\frac{dv}{dr} + \frac{c_s^2}{\rho}\frac{d\rho}{dr} + \frac{GM}{r^2} = 0$$

- Then we use the mass conservation law to get rid of ρ

$$\frac{d}{dr}(4\pi r^2 \rho v) = 0, \qquad \frac{d}{dr}(r^2\rho v) = \rho\frac{d}{dr}(r^2 v) + r^2 v\frac{d\rho}{dr} = 0$$

$$\Rightarrow \frac{1}{r^2 v}\frac{d}{dr}(r^2 v) = -\frac{1}{\rho}\frac{d\rho}{dr}$$

Analytical integration

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$$\Rightarrow \quad \frac{1}{r^2 v} \frac{d}{dr}(r^2 v) = -\frac{1}{\rho} \frac{d\rho}{dr}$$

- so the wind equation becomes

$$v \frac{dv}{dr} - \frac{c_s^2}{r^2 v} \frac{d}{dr}(r^2 v) + \frac{GM}{r^2} = 0 \Rightarrow v \frac{dv}{dr} - \frac{c_s^2}{r^2 v} \left(2rv + r^2 \frac{dv}{dr} \right) + \frac{GM}{r^2} = 0$$

Analytical integration

$$\boxed{\rho v \frac{dv}{dr} = -\rho \frac{d\Phi}{dr} - \frac{dp}{dr}}$$

We obtain this form

$$\left(v - \frac{c_s^2}{v}\right)\frac{dv}{dr} = \frac{2c_s^2}{r} - \frac{GM}{r^2}$$

Rearranging the equation in the form of ***dv*/*dr* = N/D**

WIND EQUATION

$$\boxed{\frac{dv}{dr} = \frac{v}{r}\frac{(2c_s^2 - GM/r)}{(v^2 - c_s^2)}}$$

we show that it has a singular point (0/0), called the **SONIC POINT** at $v = c_s$ and $r = r_c$ with $r_c = GM/(2c_s^2)$

Analytical integration

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We can rewrite it in terms of the **Mach number** $\mathcal{M} = \frac{v}{c_s}$

WIND EQUATION

$$\boxed{\frac{d\mathcal{M}^2}{dr} = \frac{4\mathcal{M}^2}{r^2}\frac{(r - r_c)}{(\mathcal{M}^2 - 1)}}$$

Analytical integration

$$\boxed{\rho v \frac{dv}{dr} = -\rho \frac{d\Phi}{dr} - \frac{dp}{dr}}$$

We can now separate the variables

$$\int \left(v - \frac{c_s^2}{v} \right) dv = 2 \int \frac{c_s^2}{r^2} \left(r - r_c \right) dr$$

Analytical integration

$$\rho v \frac{dv}{dr} = -\rho \frac{d\Phi}{dr} - \frac{dp}{dr}$$

We can now separate the variables

$$\int \left(v - \frac{c_s^2}{v} \right) dv = 2 \int \frac{c_s^2}{r^2} \left(r - r_c \right) dr$$

Changing the variable on the lhs to $(v/c_s)^2$ and to r/r_c, we get

$$\int \frac{1}{2v} \left(v - \frac{c_s^2}{v} \right) 2v \frac{dv}{c_s^2} = 2 \int \left(\frac{r_c}{r} - \frac{r_c^2}{r^2} \right) d \left(\frac{r}{r_c} \right)$$

$$\frac{1}{2} \int d \left(\frac{v^2}{c_s^2} \right) - \frac{1}{2} \int \frac{c_s^2}{v^2} d \left(\frac{v^2}{c_s^2} \right) = 2 \int \frac{r_c}{r} d \left(\frac{r}{r_c} \right) - 2 \int \frac{r_c^2}{r^2} d \left(\frac{r}{r_c} \right)$$

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$$\frac{1}{2} \int d \left(\frac{v^2}{c_s^2} \right) - \frac{1}{2} \int \frac{c_s^2}{v^2} d \left(\frac{v^2}{c_s^2} \right) = 2 \int \frac{r_c}{r} d \left(\frac{r}{r_c} \right) - 2 \int \frac{r_c^2}{r^2} d \left(\frac{r}{r_c} \right)$$

$$\boxed{\left(\frac{v}{c_s} \right)^2 - \ln \left(\frac{v^2}{c_s^2} \right) = 4 \ln \left(\frac{r}{r_c} \right) + 4 \frac{r_c}{r} + C}$$

when $v = c_s$ and $r = r_c$
SONIC POINT
C = -3

Analytical integration

$$\rho v \frac{dv}{dr} = -\rho \frac{d\Phi}{dr} - \frac{dp}{dr}$$

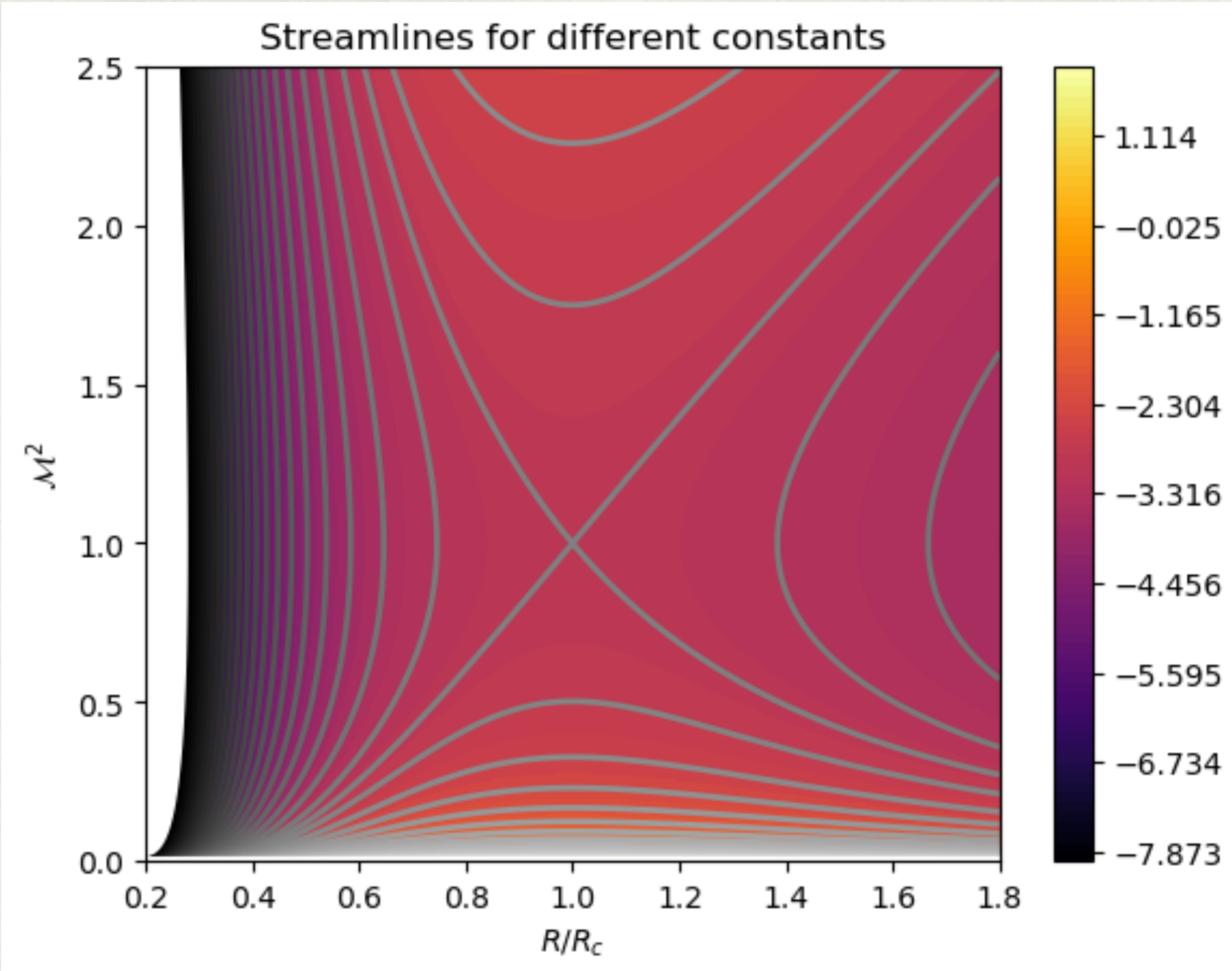
$$\left(\frac{v}{c_s}\right)^2 - \ln\left(\frac{v^2}{c_s^2}\right) = 4\ln\left(\frac{r}{r_c}\right) + 4\frac{r_c}{r} + C$$

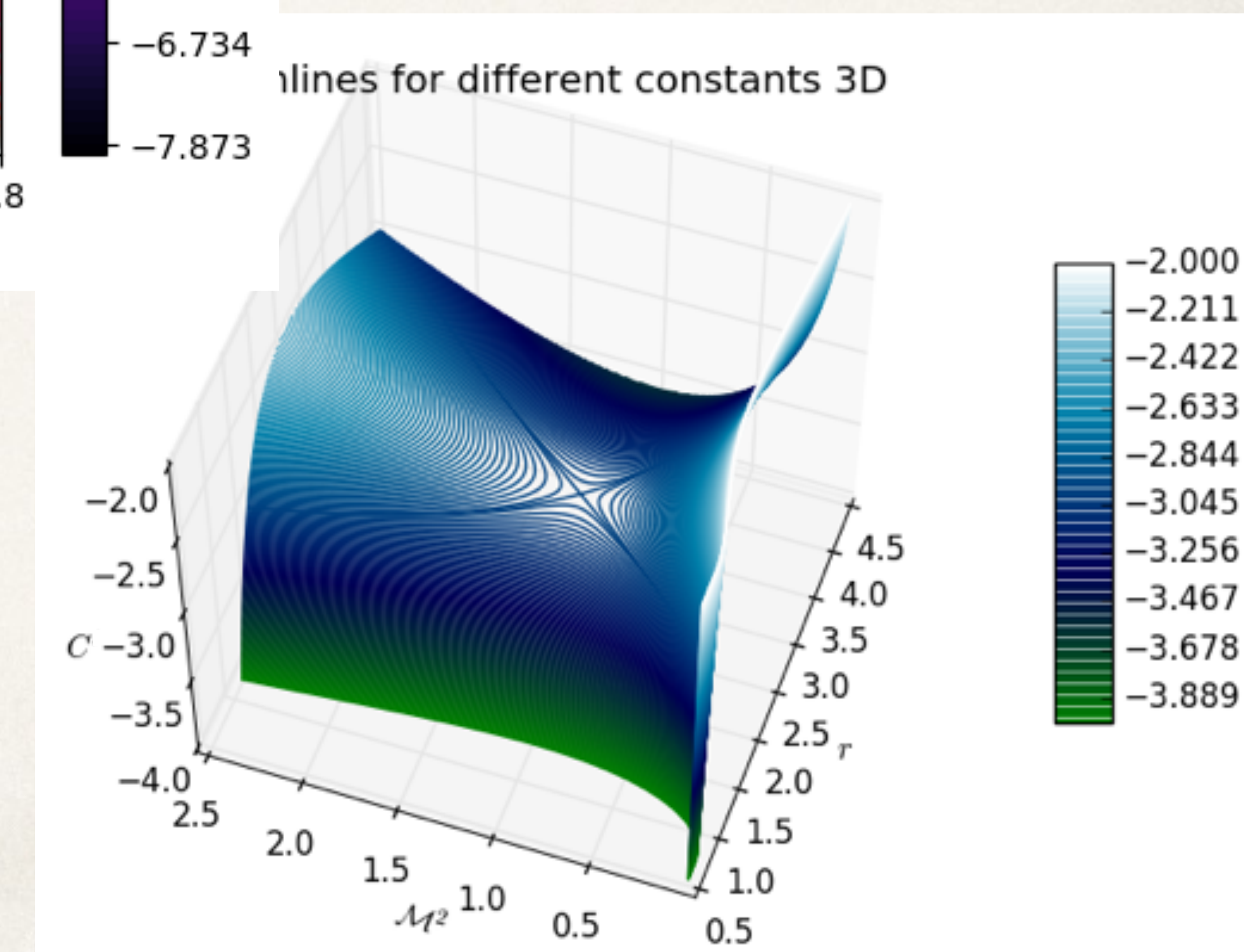
when $v = c_s$ and $r = r_c$
SONIC POINT
C = -3

or in the Mach number variable $\mathcal{M} = \dfrac{v}{c_s}$

$$\mathcal{M}^2 - \ln\mathcal{M}^2 - 4\ln\left(\frac{r}{r_c}\right) - 4\frac{r_c}{r} + C' = 0$$

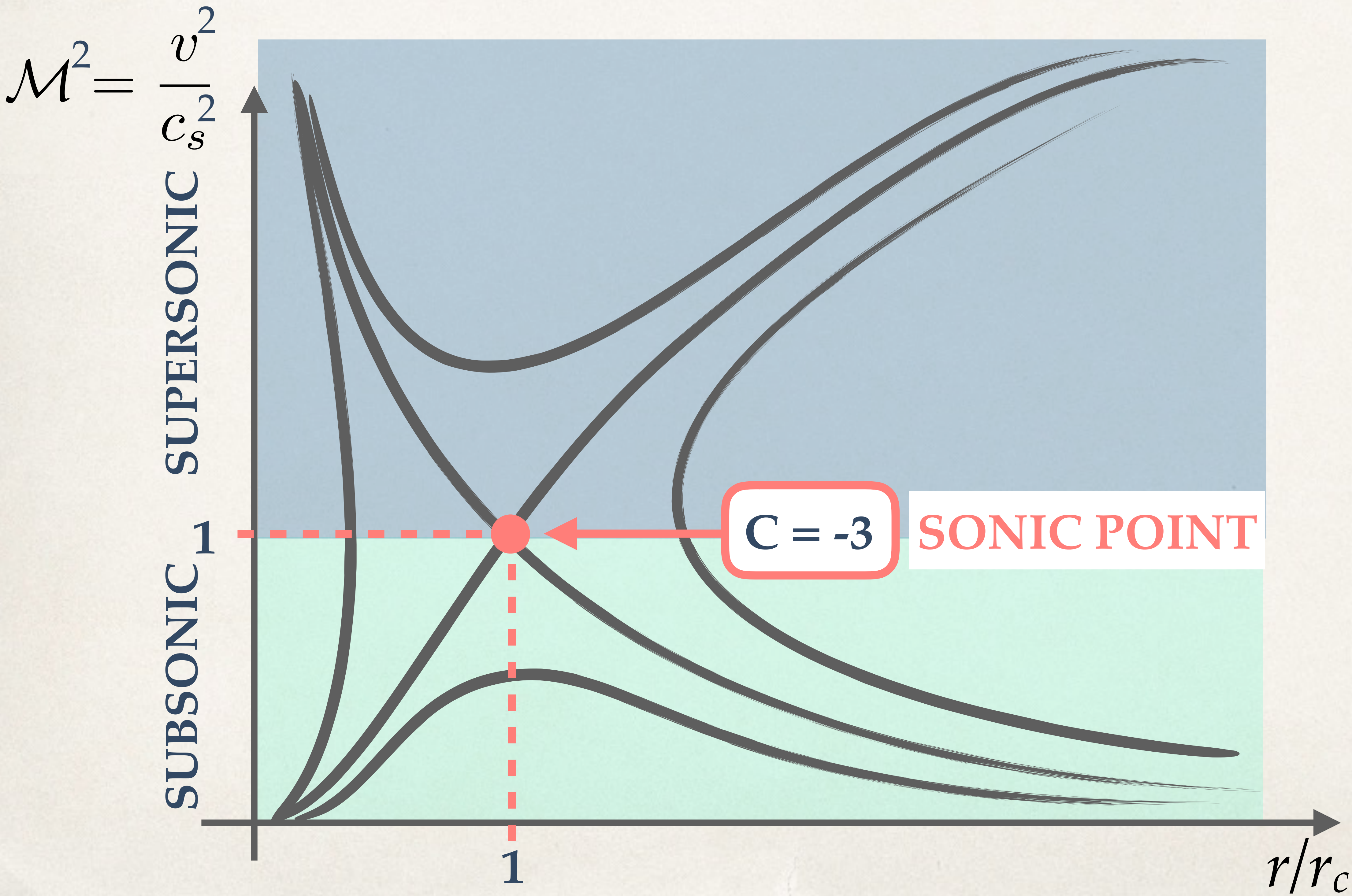
Let's now make some contour plots of the solution space.


Streamlines for different constants
2.5
2.0
1.5
1.0
0.5
0.0
$\mathcal{M}^2$
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
R/R_c
1.114
−0.025
−1.165
−2.304
−3.316
−4.456
−5.595
−6.734
−7.873


lines for different constants 3D
−2.0
−2.5
C −3.0
−3.5
−4.0
2.5
2.0
1.5
1.0
0.5
$\mathcal{M}^2$
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
r
−2.000
−2.211
−2.422
−2.633
−2.844
−3.045
−3.256
−3.467
−3.678
−3.889

Families of solutions

$$\mathcal{M}^2 - \ln \mathcal{M}^2 - 4 \ln\left(\frac{r}{r_c}\right) - 4\frac{r_c}{r} + C' = 0$$



Families of solutions

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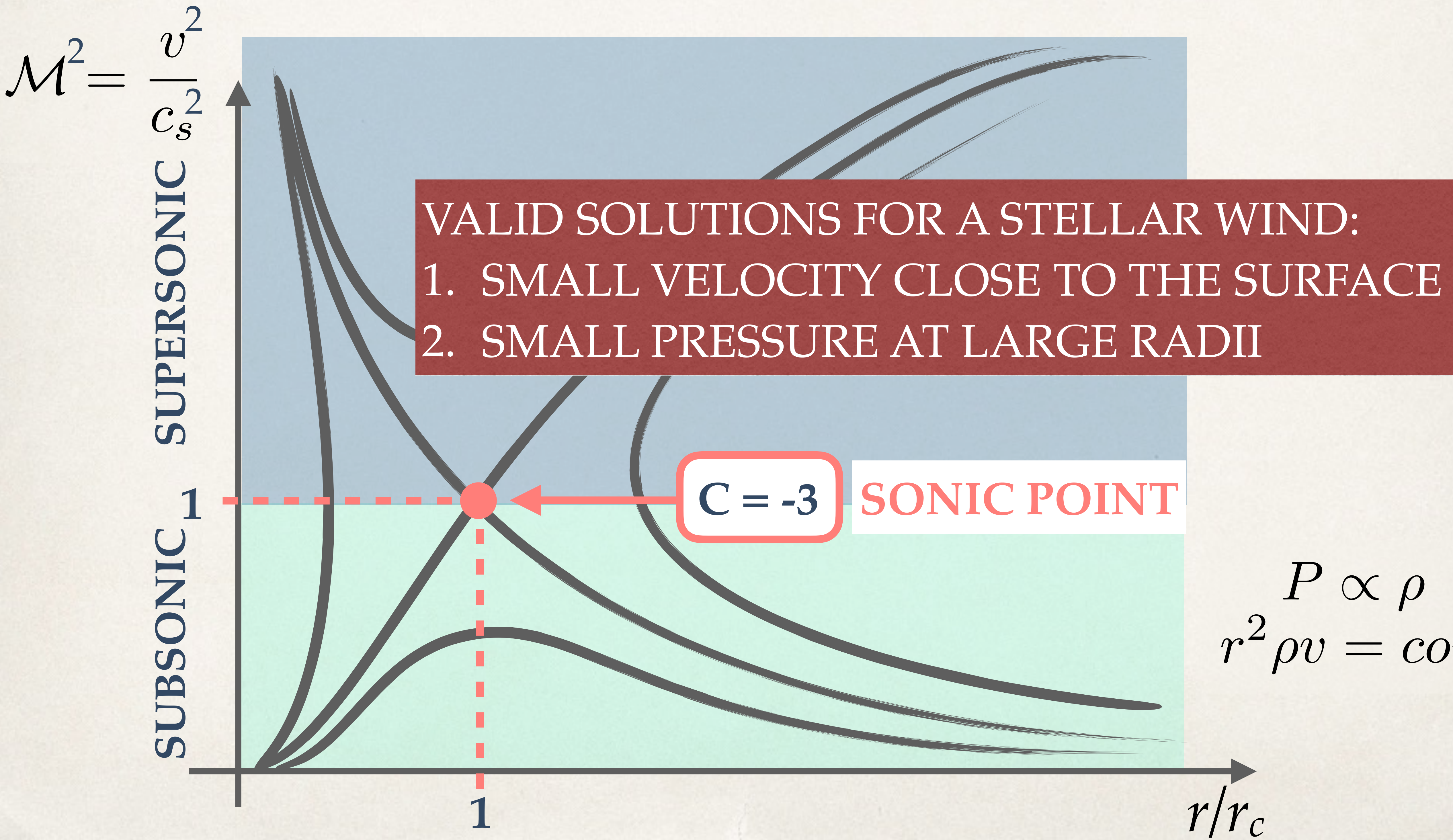
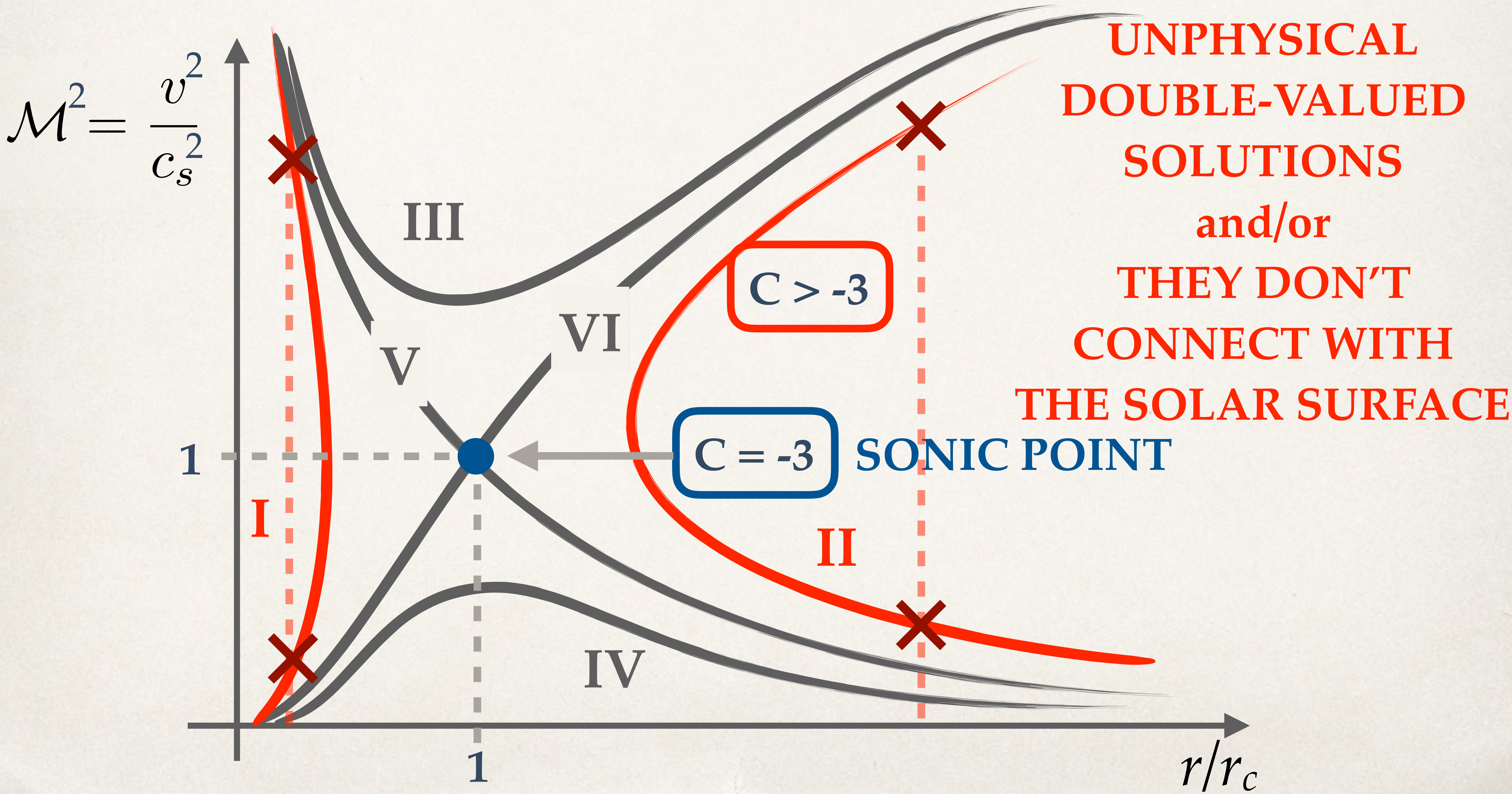


$P \propto \rho$

$r^2 \rho v = const$

Families of solutions

$$\mathcal{M}^2 - \ln \mathcal{M}^2 - 4\ln\left(\frac{r}{r_c}\right) - 4\frac{r_c}{r} + C' = 0$$



Families of solutions

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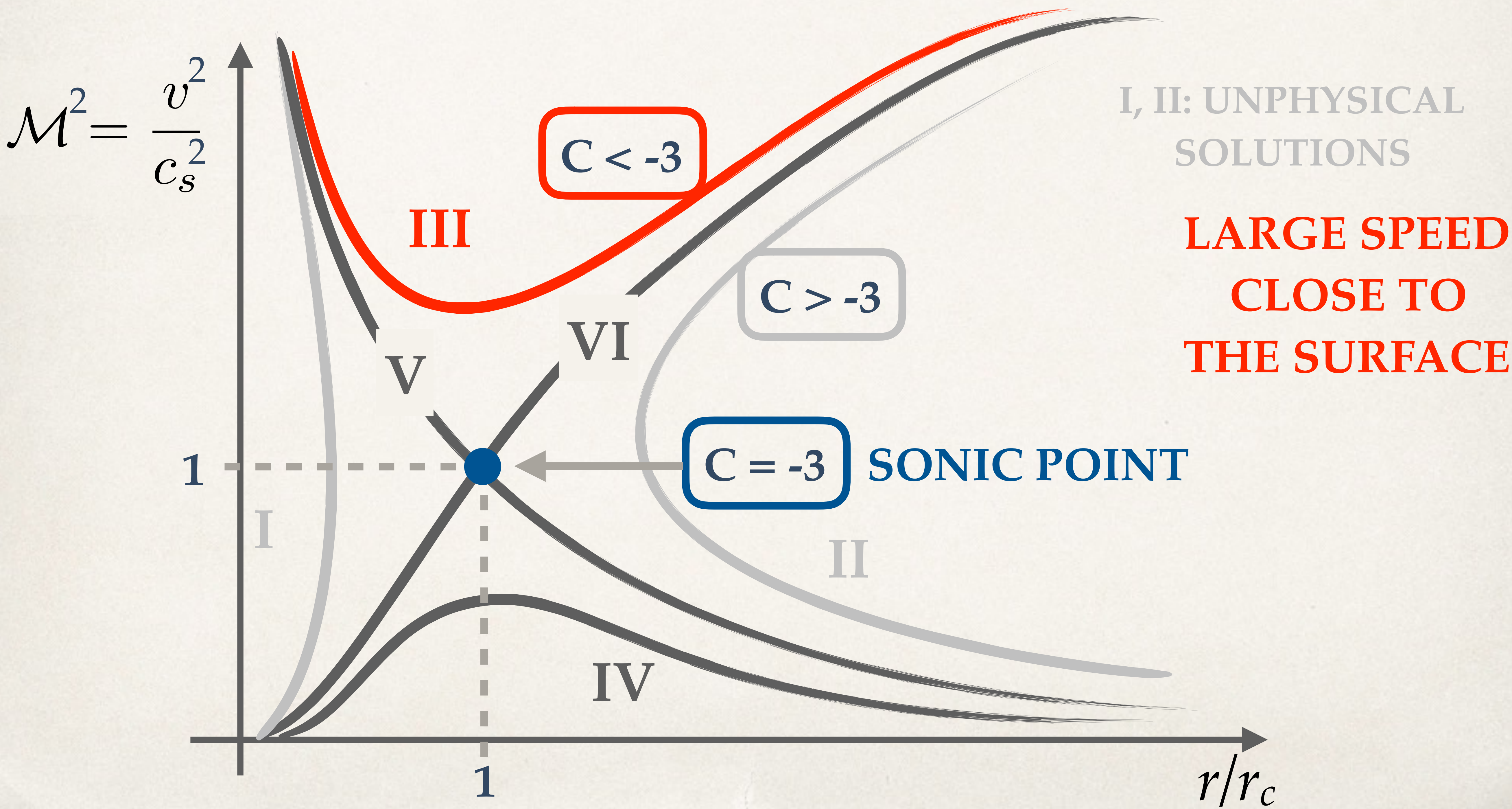
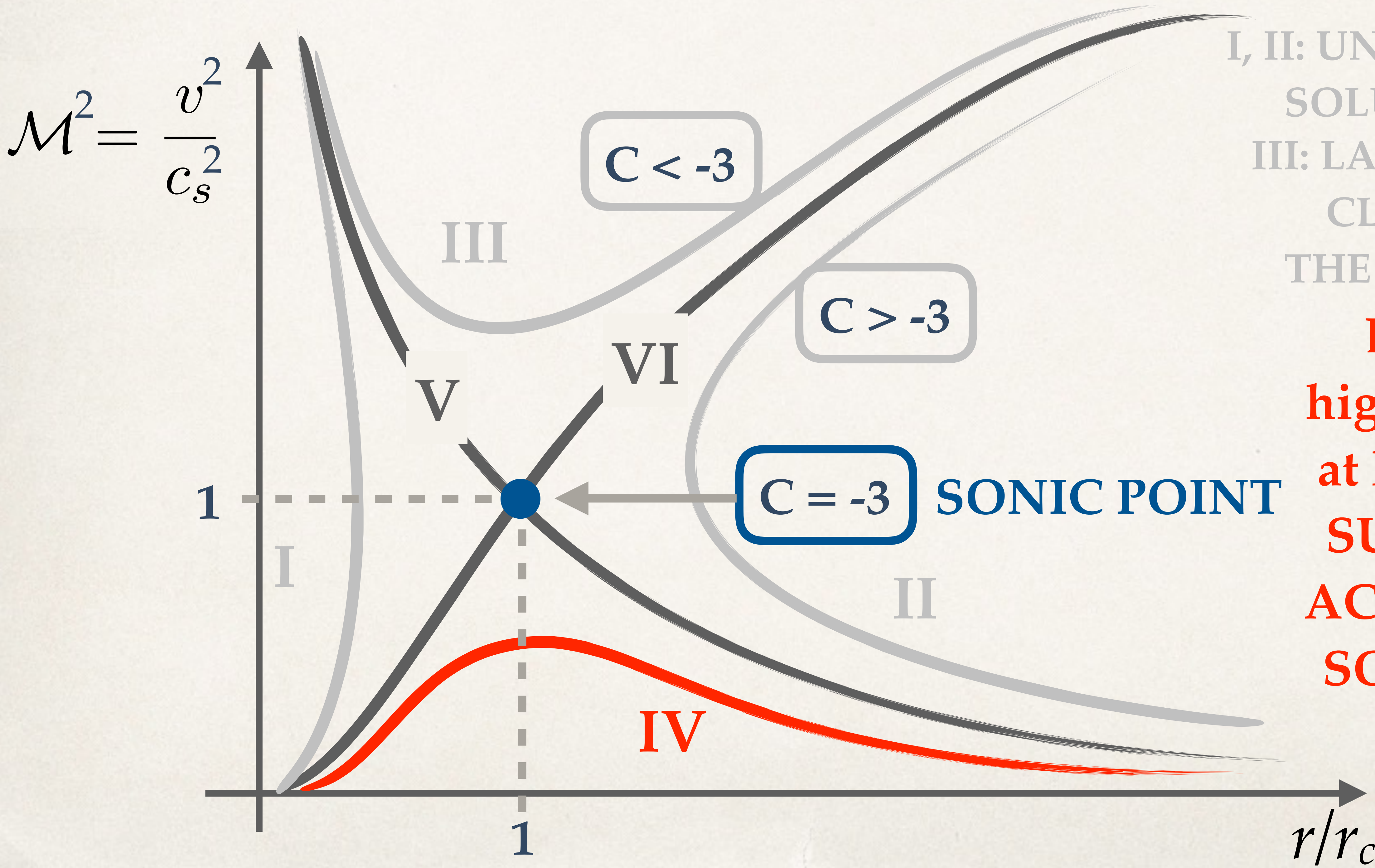


Families of solutions

$$\mathcal{M}^2 - \ln \mathcal{M}^2 - 4\ln\left(\frac{r}{r_c}\right) - 4\frac{r_c}{r} + C' = 0$$



I, II: UNPHYSICAL SOLUTIONS

III: LARGE SPEED CLOSE TO THE SURFACE

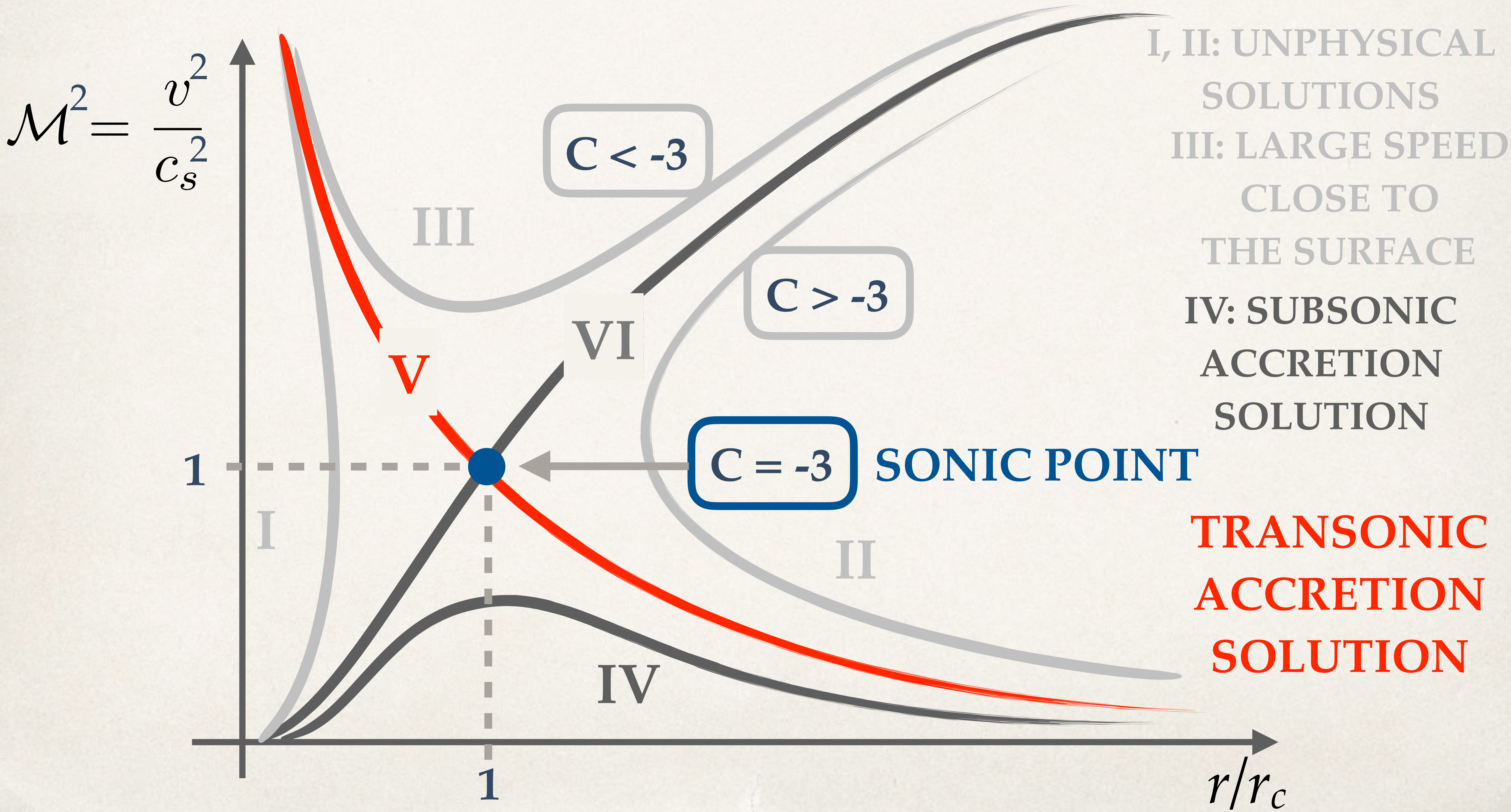
BREEZE

high pressure at large radii

SUBSONIC ACCRETION SOLUTION

Families of solutions

$$\mathcal{M}^2 - \ln \mathcal{M}^2 - 4\ln\left(\frac{r}{r_c}\right) - 4\frac{r_c}{r} + C' = 0$$



Families of solutions

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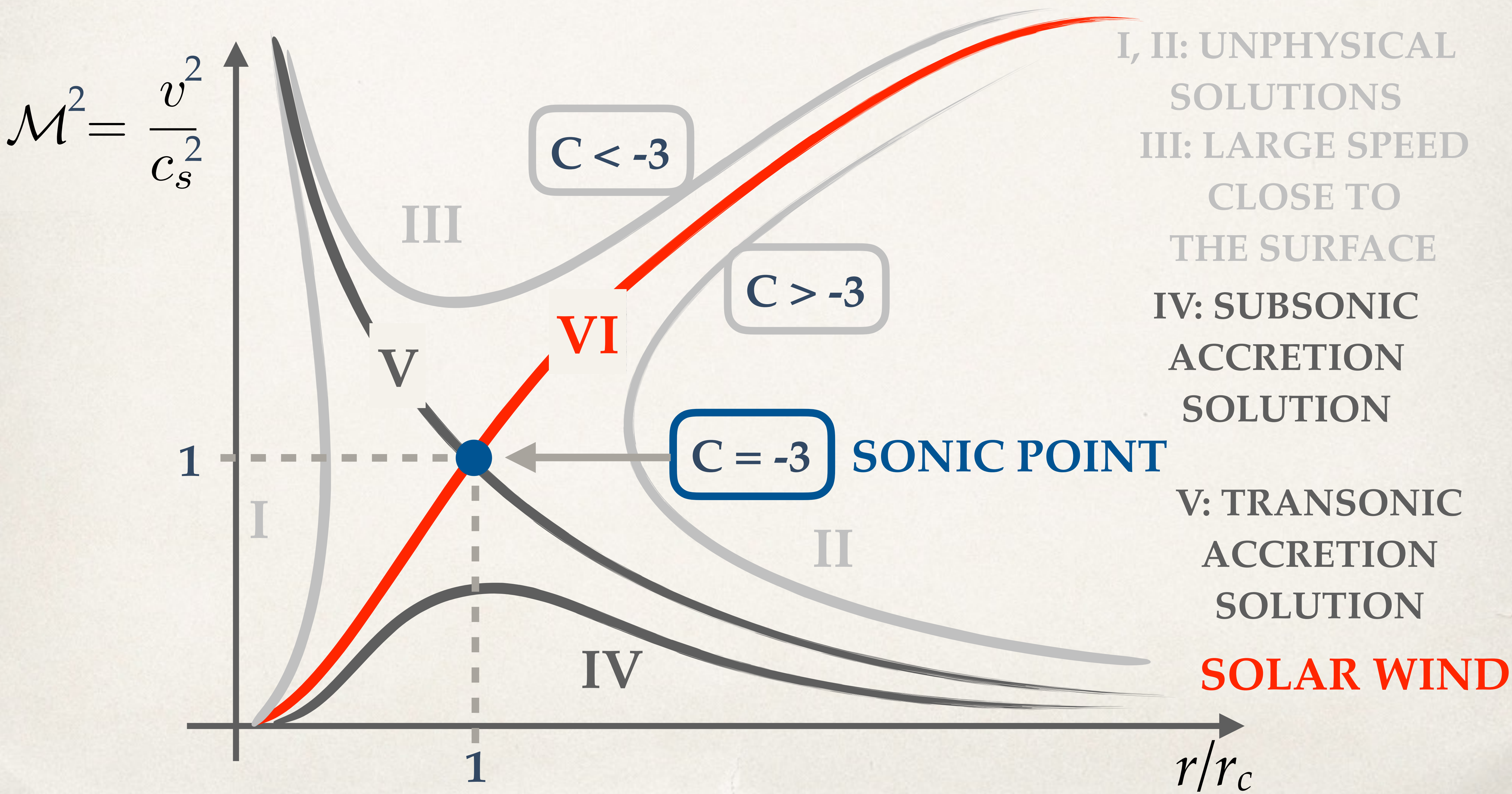


What if…

Let's assume that:

- There is **no analytical solution** to the Parker wind equation
- **We don't know there is an intermediate critical point**

We want to integrate the equation

$$\frac{d\mathcal{M}^2}{dr} = \frac{4\mathcal{M}^2}{r^2}\frac{(r - r_c)}{(\mathcal{M}^2 - 1)}$$

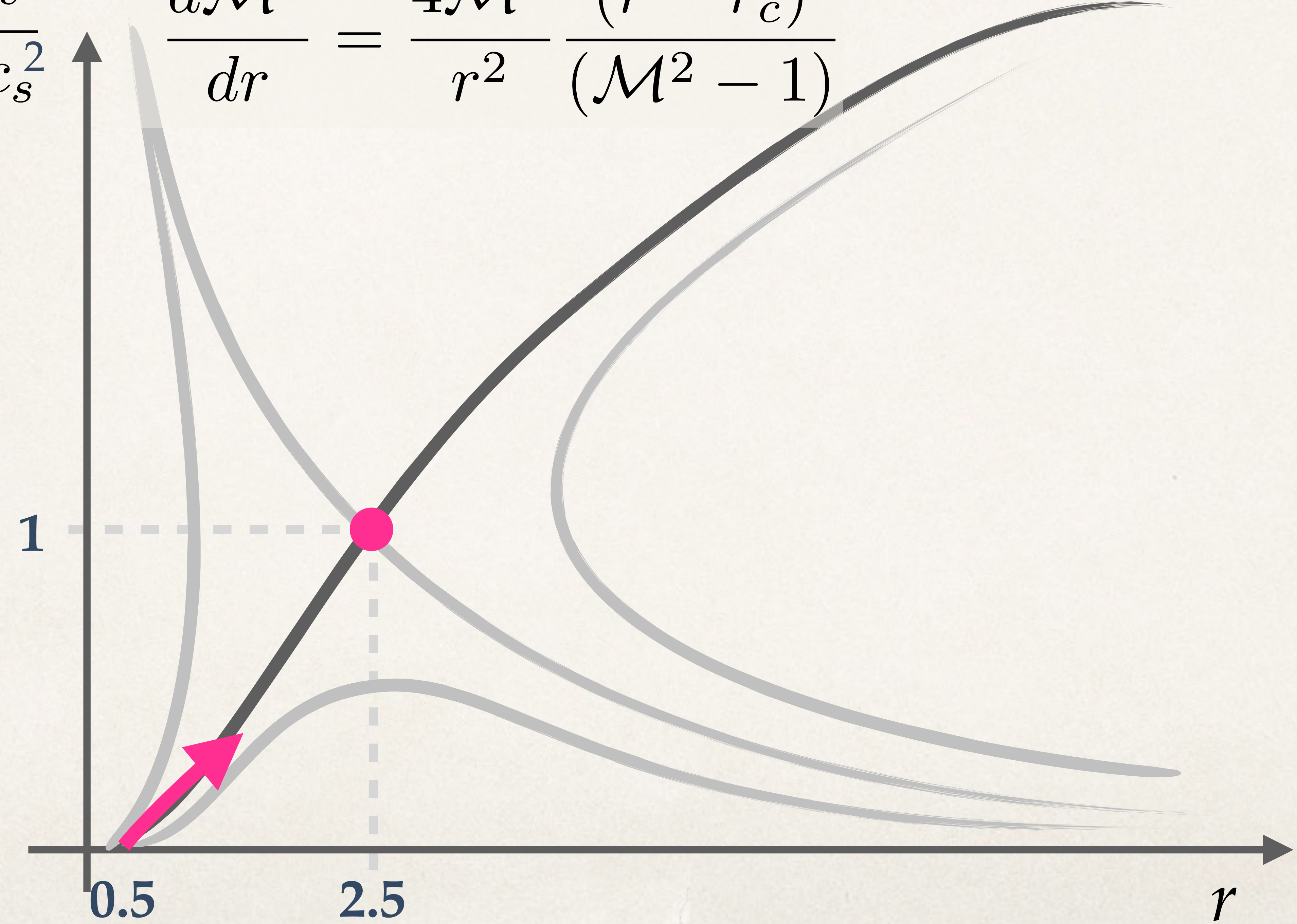
from $r = 0.5$ (to avoid to take the log 0) with some arbitrary value of the critical radius, let's say $r_c = 2.5$.

Excluding troublesome behaviour at the boundary is easier.

Integrating towards singularities

The case of unknown intermediate singularities

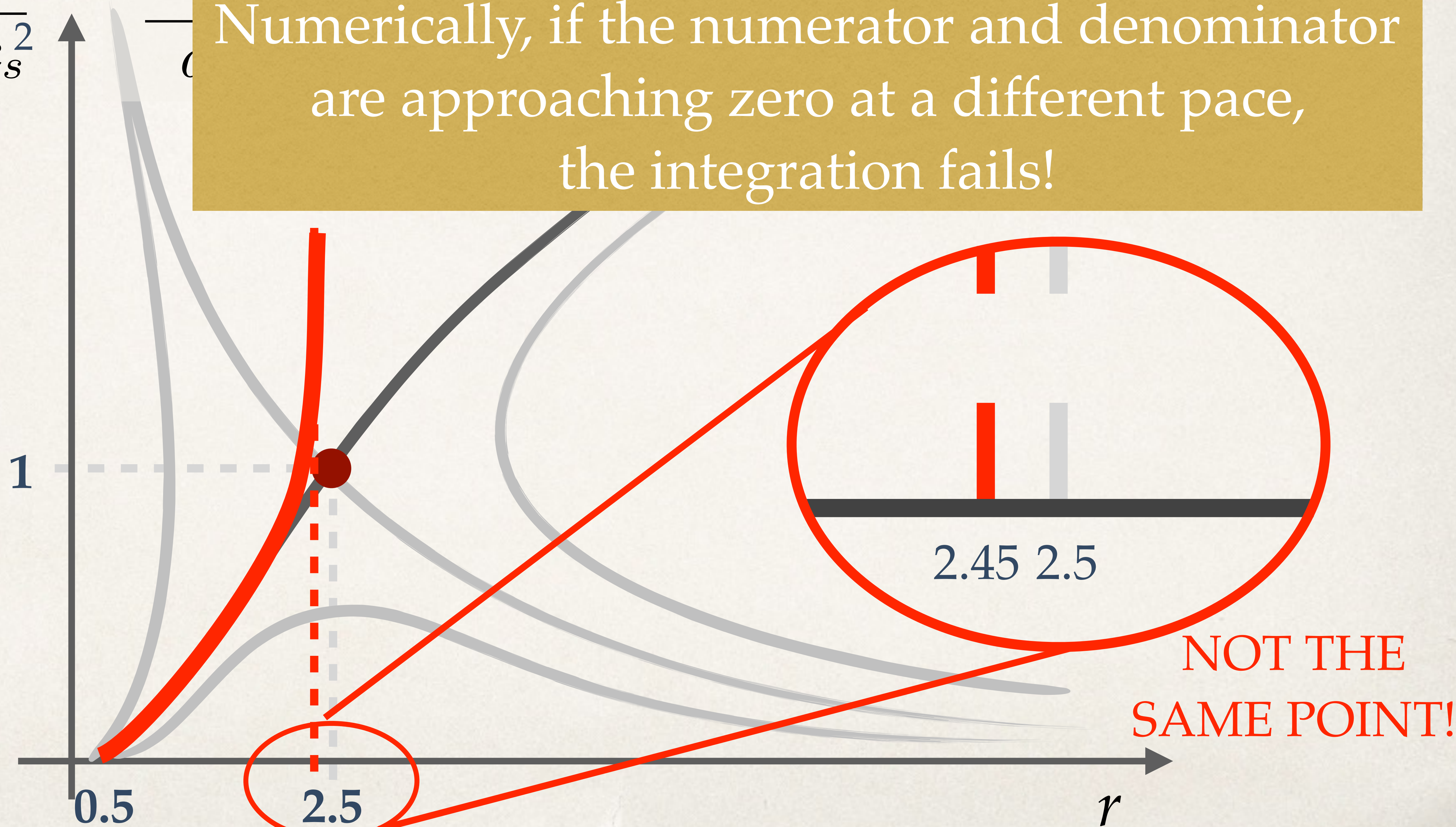
$$\mathcal{M}^2 = \frac{v^2}{c_s^2} \qquad \frac{d\mathcal{M}^2}{dr} = \frac{4\mathcal{M}^2}{r^2}\frac{(r-r_c)}{(\mathcal{M}^2-1)}$$



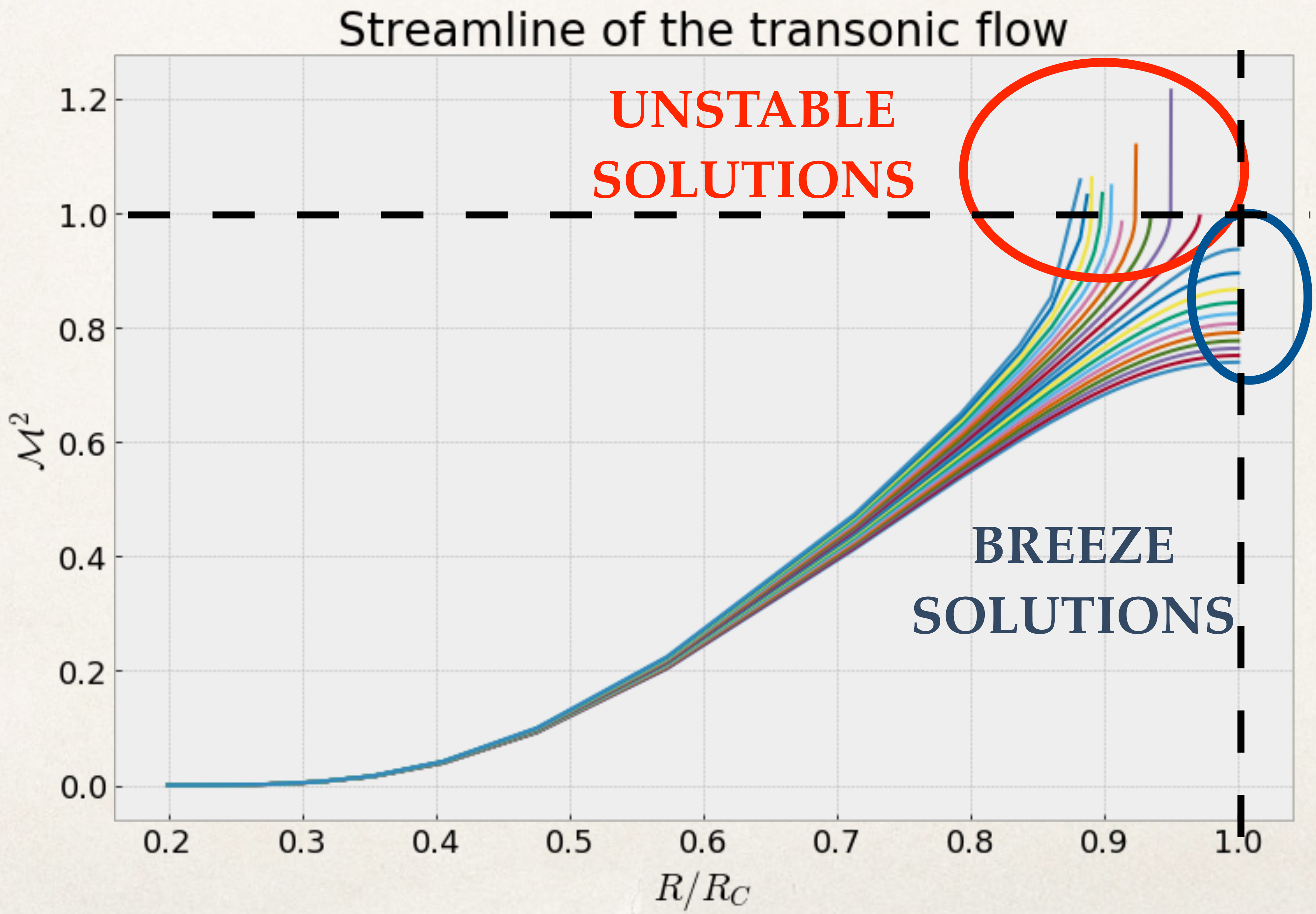
Integrating towards singularities

$$\mathcal{M}^2 = \frac{v^2}{c_s^2}$$

Numerically, if the numerator and denominator are approaching zero at a different pace, the integration fails!



Integrating towards singularities



Alternative approach needed!

- In physical problems with a finite solution, if there is a singularity in the integration interval ($0/0$, $\pm\infty/\pm\infty$), usually there is a **regularity condition** which gives the finite value of the solution at the point.
- Integrating numerically an equation with intermediate unknown singularities is equivalent to **solving a problem with intermediate boundary conditions** where special values of the variables need to satisfy the regularity condition there and we cannot formulate this condition analytically.
- We need to get creative!

de l'Hôpital's rule

L'Hôpital's rule states that for functions f and g which are differentiable on an open interval $\boldsymbol{I}$ except possibly at a point $a \in \boldsymbol{I}$, if $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$ or $\pm\infty$

for all $x \in \boldsymbol{I}$ with $x \neq a$ and the limit $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

de l'Hôpital's rule

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Given the wind equation

$$\frac{d\mathcal{M}^2}{dr} = \frac{4\mathcal{M}^2}{r^2} \frac{(r - r_c)}{(\mathcal{M}^2 - 1)}$$

we can define f(r) and g(r) as follows and calculate their derivatives wrt r

$$f(r) = 4\mathcal{M}^2(r - r_c)$$
$$g(r) = r^2(\mathcal{M}^2 - 1)$$

$$f'(r) = 4\frac{d\mathcal{M}^2}{dr}(r - r_c) + 4\mathcal{M}^2$$
$$g'(r) = r^2\frac{d\mathcal{M}^2}{dr} + \frac{dr^2}{dr}(\mathcal{M}^2 - 1)$$

de l'Hôpital's rule

$$\boxed{\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)}}$$

Given the wind equation

$$\frac{d\mathcal{M}^2}{dr} = \frac{4\mathcal{M}^2}{r^2}\frac{(r-r_c)}{(\mathcal{M}^2-1)}$$

we can define f(r) and g(r) as follows and calculate their derivatives wrt r

$$f(r) = 4\mathcal{M}^2(r-r_c)$$
$$g(r) = r^2(\mathcal{M}^2-1)$$

$$f'(r) = 4\frac{d\mathcal{M}^2}{dr}(r-r_c) + 4\mathcal{M}^2$$
$$g'(r) = r^2\frac{d\mathcal{M}^2}{dr} + \frac{dr^2}{dr}(\mathcal{M}^2-1)$$

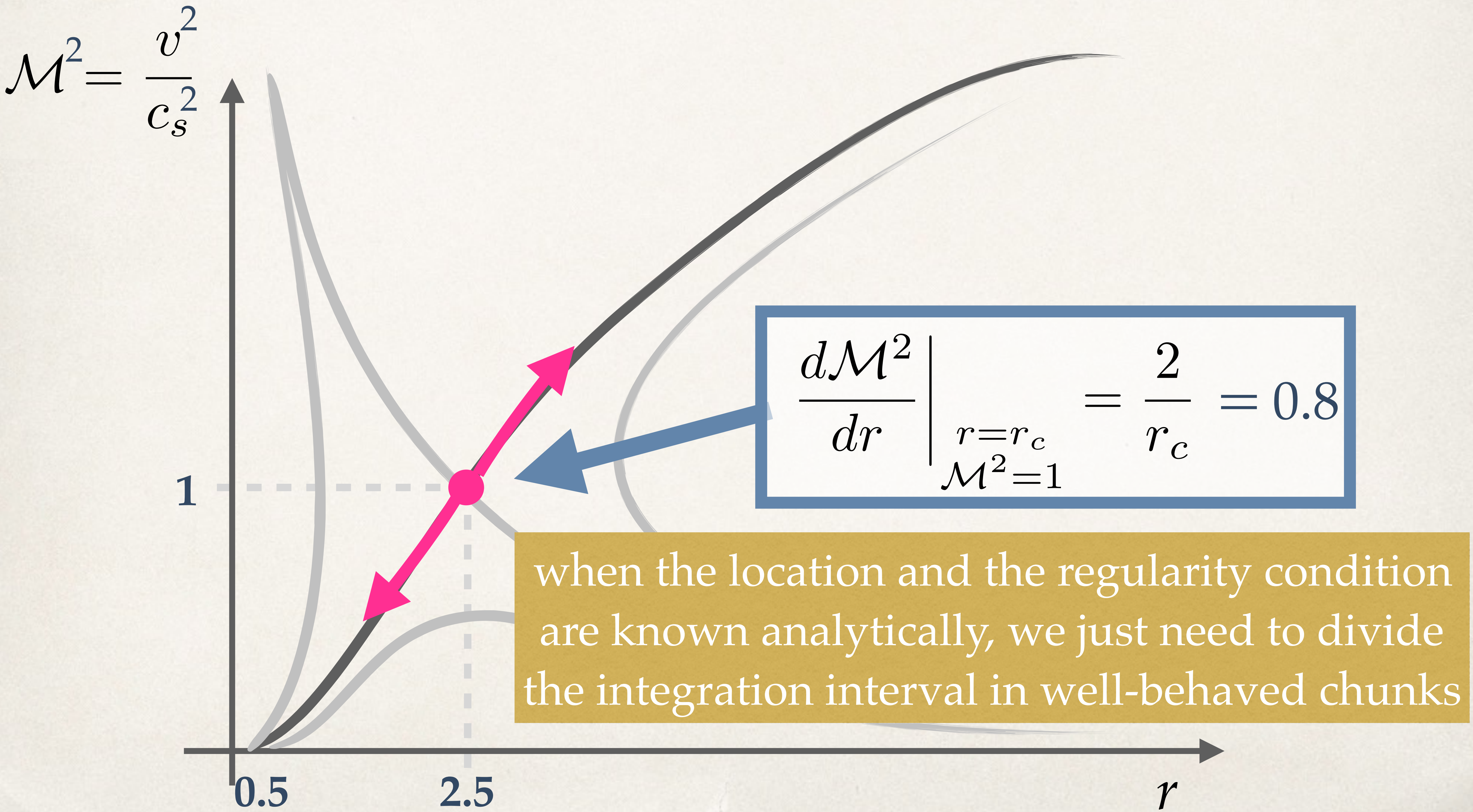
The resulting limit of the ratio of the derivatives for *r* approaching r_c is finite

$$\lim_{\substack{r\to r_c\\ \mathcal{M}^2\to 1}}\frac{f'(x)}{g'(x)} = \frac{4}{r_c^2}\frac{1}{\left(\frac{d\mathcal{M}^2}{dr}\right)}$$

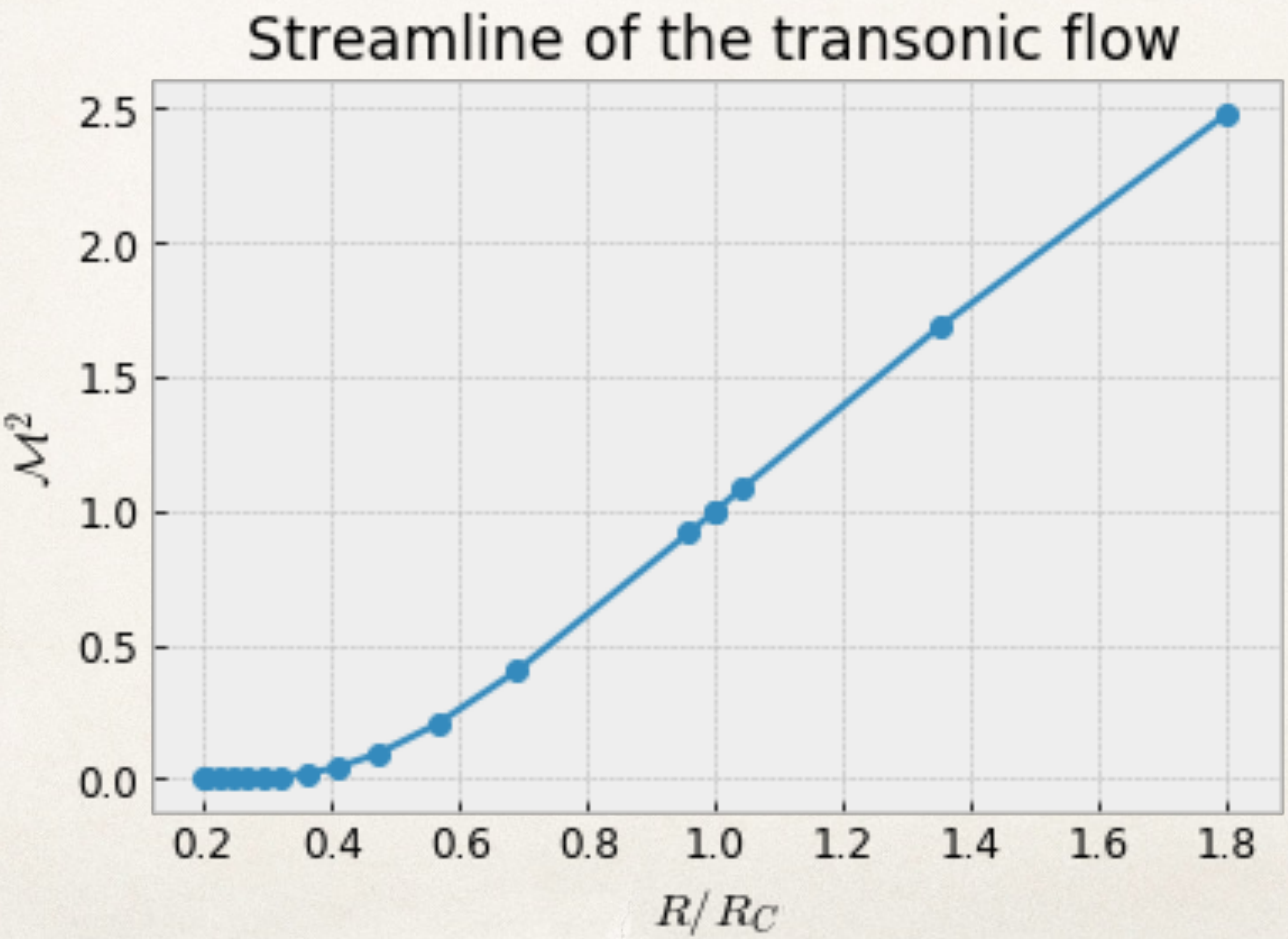
$$\boxed{\left.\frac{d\mathcal{M}^2}{dr}\right|_{\substack{r=r_c\\ \mathcal{M}^2=1}} = \frac{2}{r_c}}$$

REGULARITY CONDITION AT SP

Integrating away from singularities



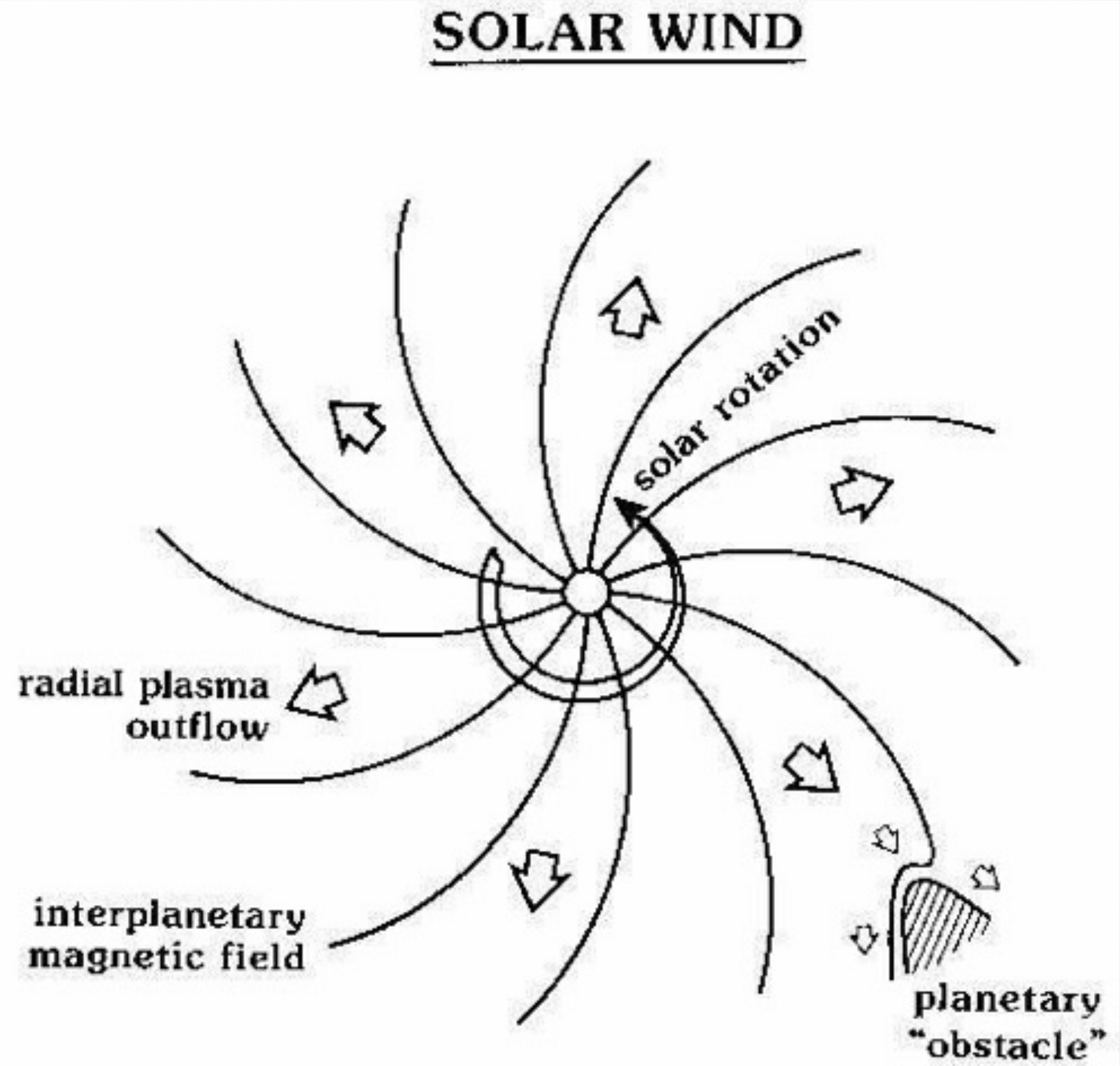
Integrating away from singularities



Parker vs Reality

Introducing rotation

- The Parker wind speed increases with increasing temperature
- This trend is not observed, so we miss some important physics!
- First of all, **rotation**, but even more importantly…



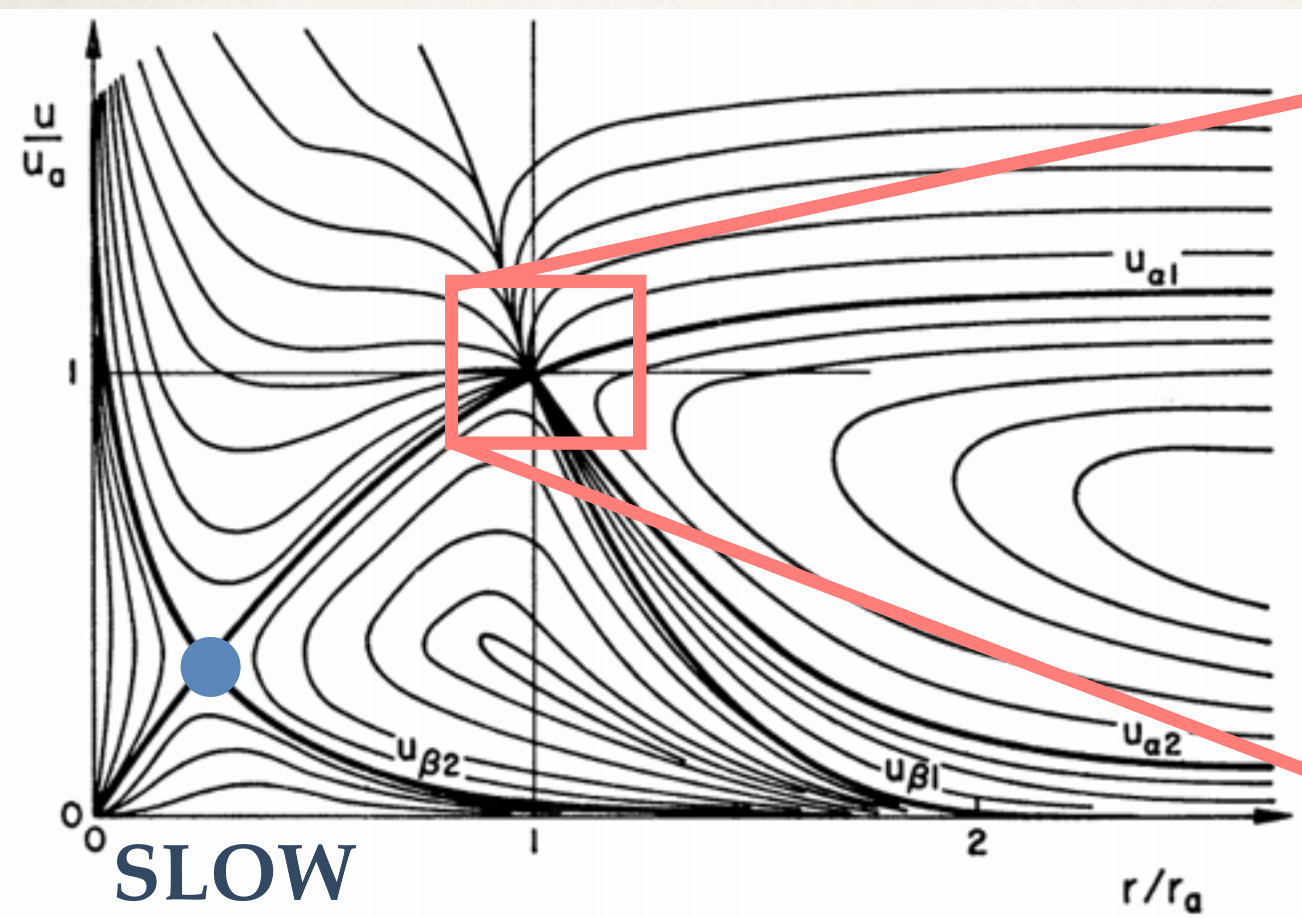
Parker's spiral

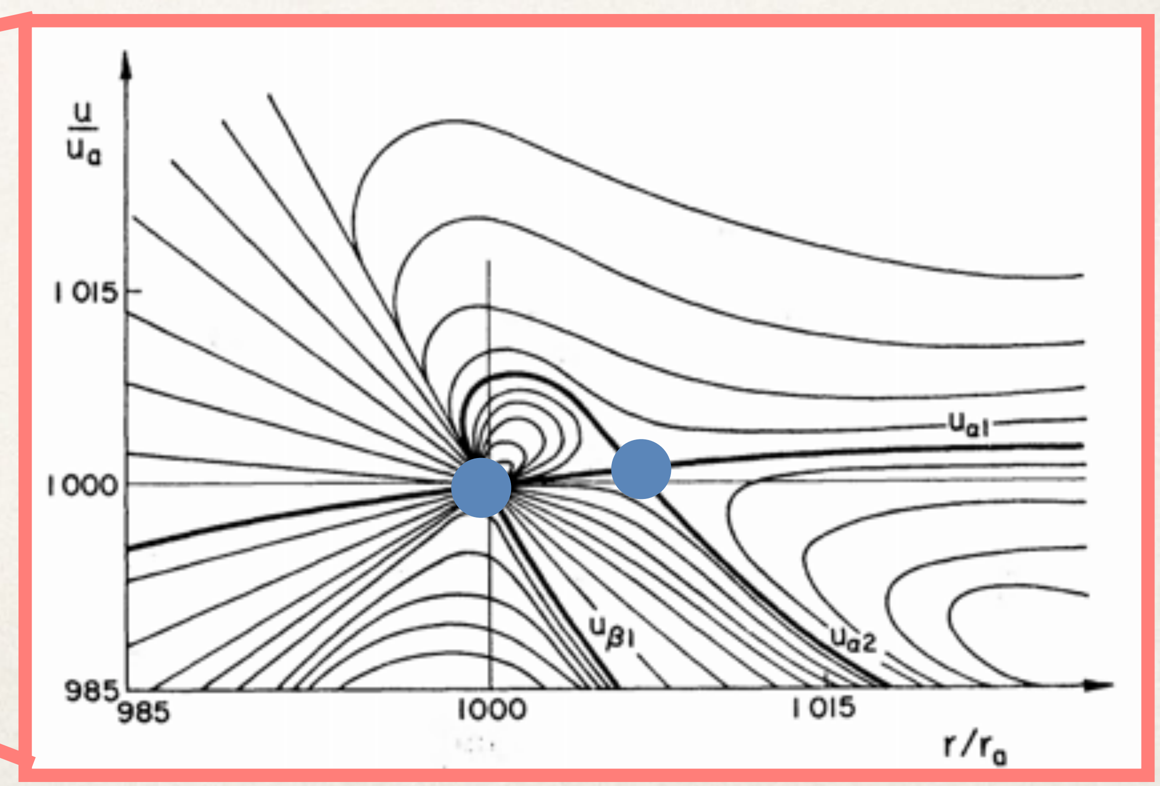
Introducing magnetic fields

Weber & Davis 1967: 1D wind; Sakurai 1985: 2D collimated wind

- The wind equation of a MHD wind has three singular points/surfaces: SLOW, ALFVEN and FAST
- The SLOW point reduces to the sonic point when B=0.
- The rotating object has to be a **fast magnetic rotator (E_0<<M)**: regular stars are too slow (E_0>>M)!

$E = K+T+G+M = E_0 + M$





ALFVEN FAST

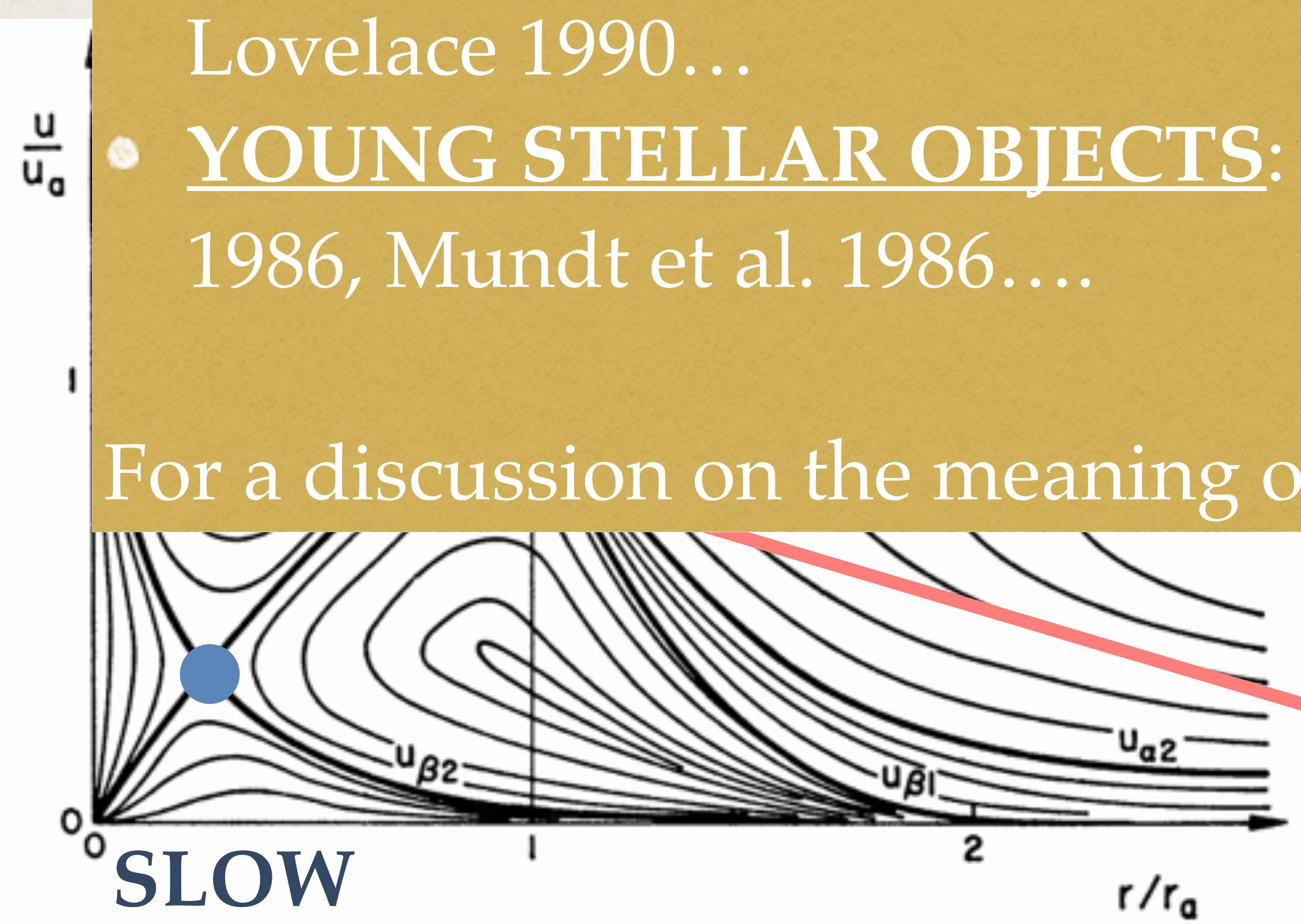
Introducing magnetic fields

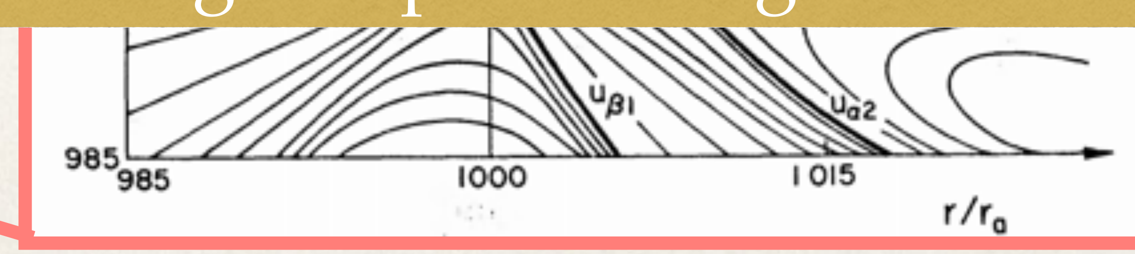
Weber & Davis 1967: 1D wind; Sakurai 1985: 2D collimated wind

APPLICATIONS

- **AGN:** Blandford & Payne 1982, Camenzind 1986a, 1986b, Ferrari et al. 1985, 1986, Konigl 1989, Mobarry & Lovelace 1986, Takahashi et.al. 1990 and Beskin & Par'ev 1993…
- **PULSARS**: Michel 1969, Camenzind 1989, Beskin 1993, Sulkanen & Lovelace 1990…
- **YOUNG STELLAR OBJECTS**: Pelletier 1992, Pudritz & Norman 1986, Mundt et al. 1986….

For a discussion on the meaning of the singular points: Bogovalov 1994



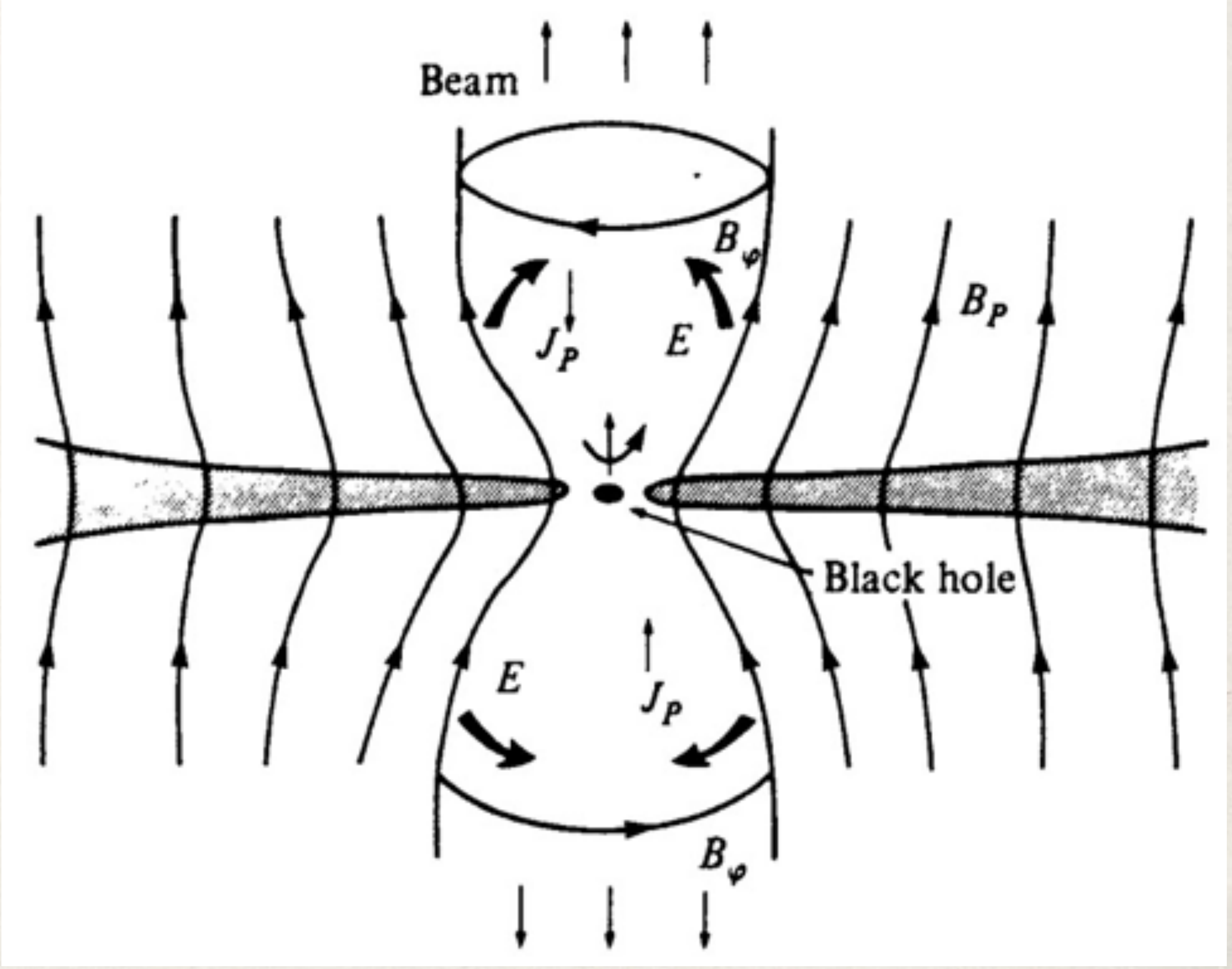


Disk-driven jets

Blandford & Payne 1982

BP82 model describe a radial *self-similar* disk-driven wind/jet which is launched via magneto-centrifugal forces

- The thermal energy of the plasma is NOT taken into account, so these are **cold outflows** and they **don't cross** the **magnetosonic slow surface**
- **They don't cross the magnetosonic fast surface either**



Disk-driven jets

- **Heyvaerts & Norman's 1989, 2003**: asymptotic analyses of rotating/magnetized outflows
- **Li et al. 1992**: relativistic self-similar cold MHD jets, no gravity
- **Contopoulos & Lovelace 1994**: different field line geometries
- **Ostriker 1997**: self-similar disk winds with cylindrical asymptotics
- **Sauty & Tsinganos 1994**: meridional self-similar MHD jets and criterion for collimation
- **Vlahakis & Tsinganos 1998**:construction of self-similar models of MHD outflows
- **Vlahakis et al. 2000**: crossing of ALL the critical surfaces, non-relativistic + gravity
- **Vlahakis & Königl 2003, 2004**: relativistic self-similar warm MHD jets, no gravity
- **Sauty, Trussoni, Tsinganos (1994 - 2005)**: on analytical modelling self-similar MHD outflows
- **Ferreira, Casse, Keppens, Romanova (1997, 2002, 2004, 2006)**: magnetized accretion-ejection structures
- **Polko et al. (2010 - 2014)**: radial self-similar relativistic warm MHD jets with gravity crossing all the critical surfaces
- **Ceccobello et al. 2018**: extended parameter study of the Polko et al. eqns

The self-similarity assumption

All the streamlines filling the space are the same: they never cross each other

If we know one streamline, we can build all the others. One streamline needs to cross smoothly all the three critical surfaces.

- In a spherical- or axi- symmetric system, the natural variables are the poloidal Mach number and the magnetic flux function (M, A), which are functions of the preferred coordinate system (x_1, x_2). i.e. M(x_1, x_2) and A(x_1, x_2).
- **ANSATZ**: We want to separate the variables in such a way that, i.e. M = M(χ), where what χ is is defined by the specific self-similar assumption we make. In spherical coordinates $x_1=r$, $x_2=\theta$, so we have:
- ***$\chi=r$:* meridional self-similar models with spherical critical surfaces**
- ***$\chi=\theta$:* radial self-similar models with conical critical surfaces**

The self-similarity assumption

All the streamlines filling the space are the same: they never cross each other

If we know one streamline, we can build all the others. One streamline needs to cross smoothly all the three critical surfaces.

- In a spherical- or axi- symmetric system, the natural variables are the

When the problem is reduced to 1D, the critical surfaces are **POINTS** on a streamline and*, **because of the self-similar assumption,*** the locations of the slow and fast magneto sonic points are not coincidental with the classical ones, so they are called ***"modified" slow/fast points.***

- make. In spherical coordinates $x_1=r$, $x_2=\theta$, so we have:

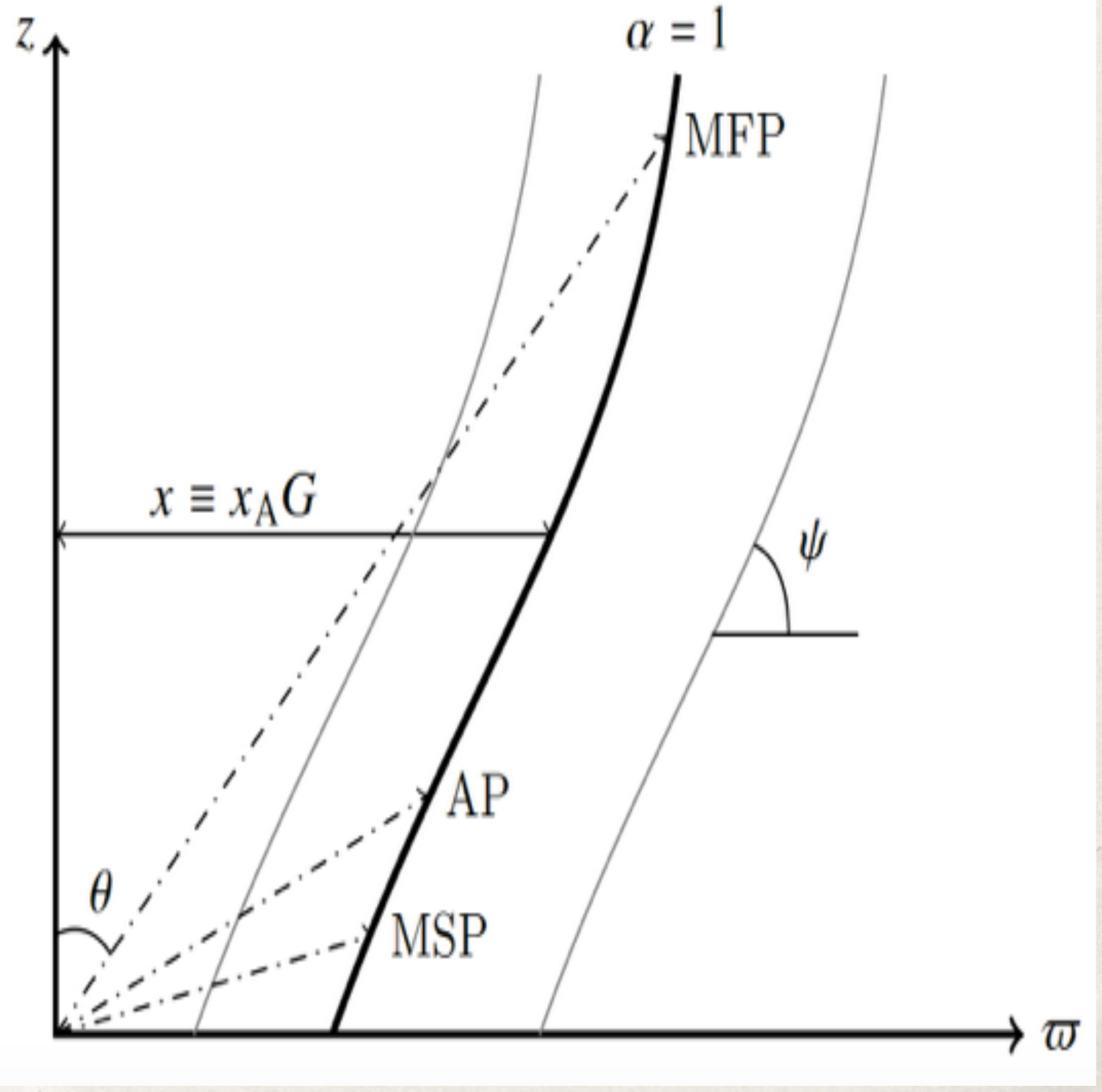
- ***$\chi=r$:* meridional self-similar models with spherical critical surfaces**

- ***$\chi=\theta$:* radial self-similar models with conical critical surfaces**

NRMHD Vlahakis et al. 2000, Ceccobello et al. in prep.
RMHD Vlahakis & Königl, 2003; Polko et al. 2010,2013,2014; Ceccobello et al. 2018

- The non-relativistic and relativistic set of equations described in the papers listed above include **gravity** and **thermal energy** of the plasma, therefore they describe **warm/hot disk-driven jets with different velocities**

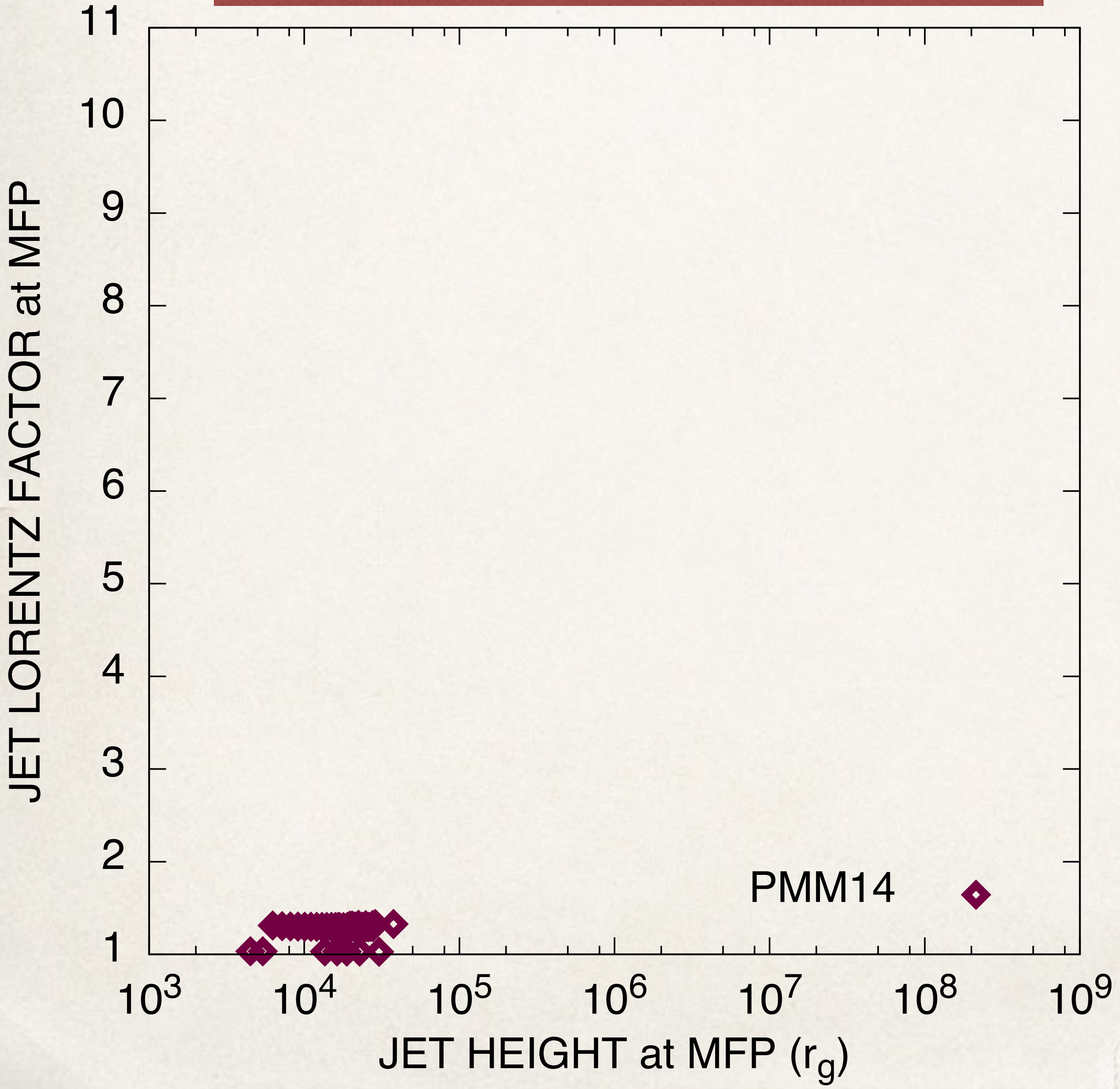
- The most recently found **solutions** of both problems a**re all crossing smoothly the three singular points** and cover large volumes of the parameter space



NRMHD Vlahakis et al. 2000, Ceccobello et al. in prep.

RMHD Vlahakis & Königl, 2003; Polko et al. 2010,2013,2014; Ceccobello et al. 2018

INTEGRATING **TOWARDS** UNKNOWN SINGULARITIES

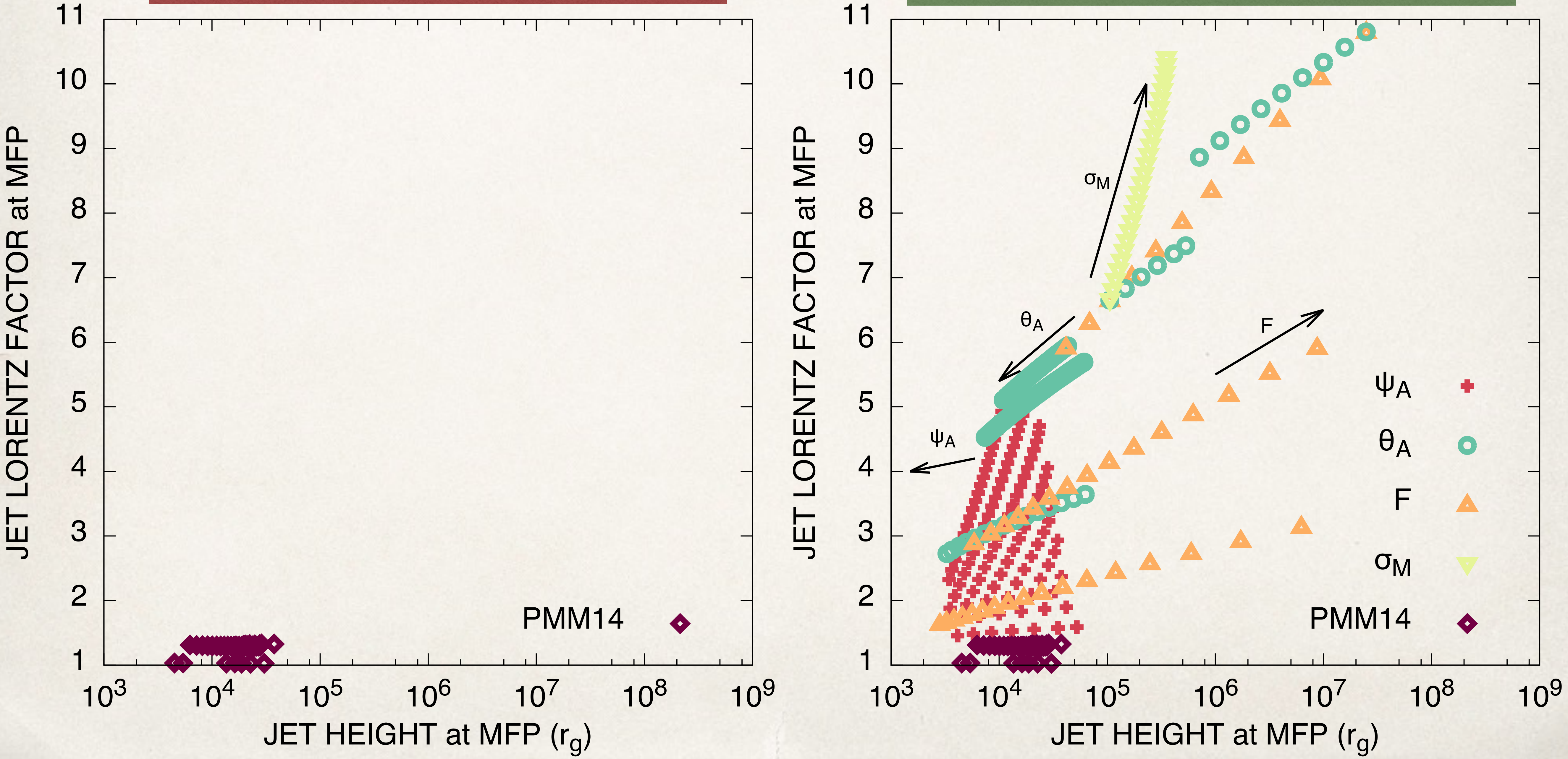


NRMHD Vlahakis et al. 2000, Ceccobello et al. in prep.

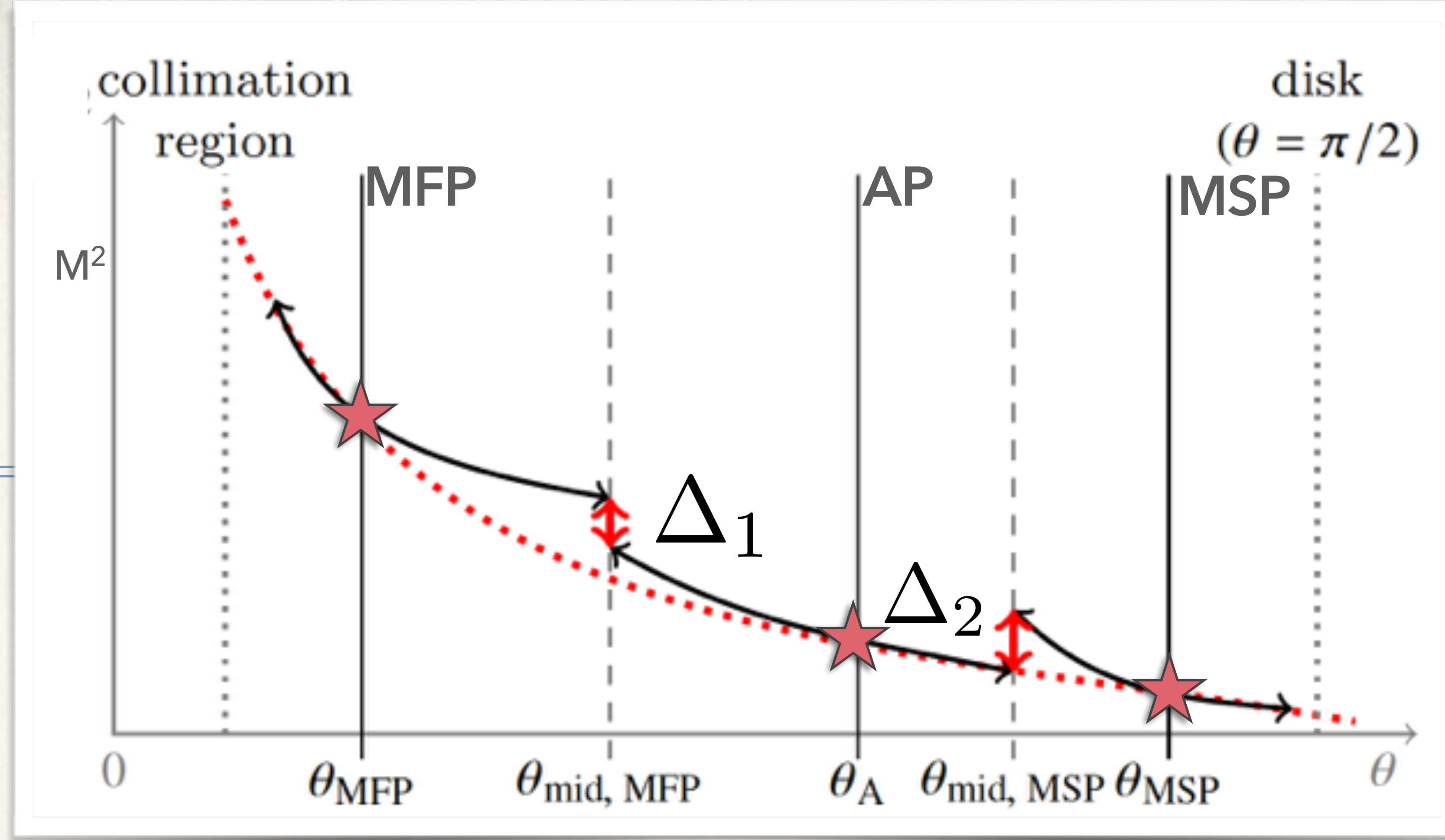
RMHD Vlahakis & Königl, 2003; Polko et al. 2010,2013,2014; Ceccobello et al. 2018

INTEGRATING **TOWARDS** UNKNOWN SINGULARITIES

INTEGRATING **AWAY** FROM UNKNOWN SINGULARITIES



The method



- We guess $\theta_{\rm MSP}$ and $\theta_{\rm MFP}$ and derive values for M^2,G^2 and their derivatives which satisfy the condition that $\left.\frac{d\mathcal{M}^2}{dr}\right|_* = \frac{0}{0}$
- We integrate away from AP, MSP and MFP towards the midpoints and upstream of MSP and downstream of MFP
- We use **MultiNest** to minimise the offsets at the midpoints to finally find the best set of parameters to describe a given solution.

* = MSP or MFP

$$\frac{dM^2}{d\theta} = -2\frac{\sin(\theta+\psi)}{\cos(\theta+\psi)}\left\{-\frac{k^2_{\mathrm{VTST}}\sin\theta}{G} - \mu_{\mathrm{VTST}}(F-2)M^{4-2\Gamma}\right.$$
$$+\frac{M^4}{G^4}(1-M^2)\frac{\cos\psi\sin\theta}{\sin(\theta+\psi)} - \frac{M^4}{G^4}(F-2)\frac{\sin^2\theta}{\cos^2(\theta+\psi)}$$
$$-\lambda^2_{\mathrm{VTST}}\frac{M^4}{G^2}(F-2)\left(\frac{1-G^2}{1-M^2}\right) + \lambda^2_{\mathrm{VTST}}\frac{M^2}{G^2}\frac{G^4-M^2}{1-M^2}$$
$$\left.-\lambda^2_{\mathrm{VTST}}\frac{\cos\psi}{\sin\theta\sin(\theta+\psi)}\frac{(2M^2-1)G^4-M^4}{G^2(1-M^2)}\right\}$$
$$\times\left\{\Gamma\mu_{\mathrm{VTST}}(1-M^2)M^{-2\Gamma} - 2\lambda^2_{\mathrm{VTST}}\frac{M^2}{G2}\left(\frac{1-G^2}{1-M^2}\right)^2\right.$$
$$\left.+2\frac{M^4\sin^2\theta}{G^4}\left(1-\frac{1}{M^2\cos^2(\theta+\psi)}\right)\right\}^{-1}$$

radial self-similar NRMHD WIND EQUATION for warm jets with gravity

$$\frac{dG^2}{d\theta} = \frac{2\,G^2\cos\psi}{\sin\theta\cos(\theta+\psi)}$$

equation for the cylindrical radius of the streamline

Alfven Regularity Condition

$$(F-2)(4\lambda^2_{\mathrm{VTST}} + p_\star^2\sin^2\theta_\star)\tan^2(\theta_\star+\psi_\star)$$
$$+(F-2)\left(\mu_{\mathrm{VTST}}p_\star^2 + p_\star^2\sin^2\theta_\star + \frac{4\lambda^2_{\mathrm{VTST}}}{\tan^2\theta_\star}\right)$$
$$+\left(p_\star^3\sin^2\theta_\star + 4\lambda^2_{\mathrm{VTST}}p_\star + 8\lambda^2_{\mathrm{VTST}}\frac{F-2}{\tan\theta_\star}\right)\tan(\theta_\star+\psi_\star)$$
$$+k^2_{\mathrm{VTST}}p_\star^2\sin\theta_\star - \lambda^2_{\mathrm{VTST}}p_\star\left(p_\star - \frac{4}{\tan\theta_\star}\right) = 0$$

Exercise #1

Integration of the Parker wind equation

Check out the python notebook: parker.ipynb

USE:

$ python -m jupyter notebook --no-browser --ip=000.000.00.00 --port=8890

The number in ip has to be changed into the IP of your VM

MultiNest

Bayesian Inference

$$\begin{aligned} \text{Likelihood} \times \text{Prior} &= \text{Evidence} \times \text{Posterior} \\ L(\theta) \times \pi(\theta) d\theta &= Z \times p(\theta) d\theta \end{aligned}$$

- The **prior distribution** is the distribution of the parameter(s) before any data is observed
- The sampling distribution or **likelihood** is the distribution of the observed data conditional on its parameters
- The **posterior distribution** is the distribution of the parameter(s) after taking into account the observed data
- The **marginal likelihood (or evidence)** is the distribution of the observed data *marginalized* over the parameter(s)

Bayesian Inference

- **Parameter estimation:** often we use bayesian inference to obtain the posterior distribution of one or more parameters that better explains the observed data. In this case, **the marginal likelihood is only a normalisation and can be neglected**

posterior ∝ likelihood x prior

- **Model selection:** we have to choose between two models on the basis of observed data D, the plausibility of the two different models H_0 and H_1, parametrised by model parameter vectors, Θ_0 and Θ_1, is assessed by the **Bayes factor K** given by

$$\frac{\text{posterior of } H_1}{\text{posterior of } H_0} = \frac{\boxed{\text{marginal likelihood of } H_1} \times \text{prior of } H_1}{\boxed{\text{marginal likelihood of } H_0} \times \text{prior of } H_0}$$

Bayesian Inference

- **Parameter estimation:** often we use bayesian i posterior distribution of one or more parameters tha observed data. In this case, **the marginal likelihood is** **isation and can be neglected**

Markov Chain Monte Carlo

$$\textbf{posterior} \propto \textbf{likelihood} \times \textbf{prior}$$

- **Model selection:** we have to choose between two m observed data D, the plausibility of the two different mo parametrised by model parameter vectors, Θ_0 and Θ_1, is as Bayes factor K given by

Nested Sampling

$$K = \frac{\text{posterior of } H_1}{\text{posterior of } H_0} = \frac{\text{marginal likelihood of } H_1 \times \text{prior of } H_1}{\text{marginal likelihood of } H_0 \times \text{prior of } H_0}$$

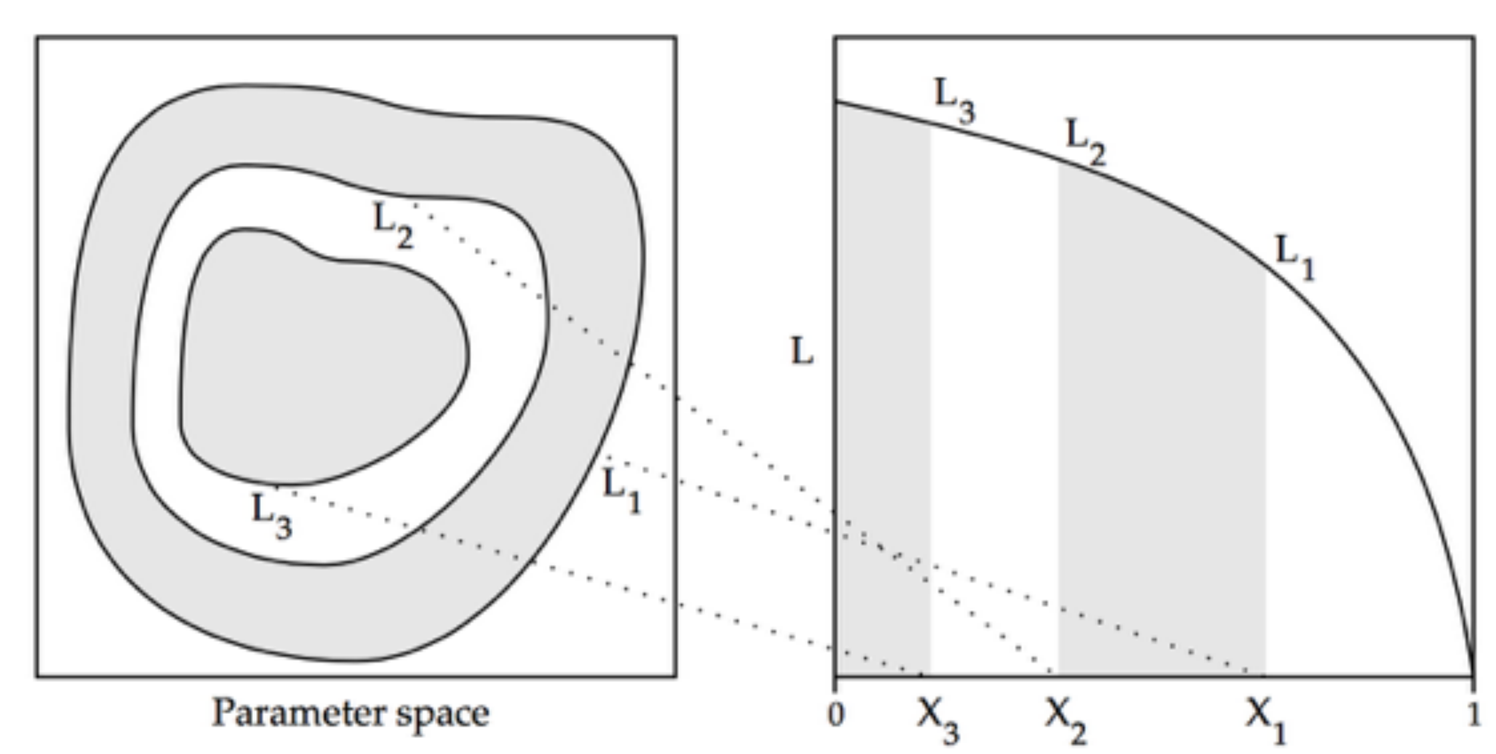
Nested Sampling (Skilling, 2004)

Still a Monte Carlo technique, but….

$$\begin{array}{ccccc} \text{Likelihood} \times & \text{Prior} & = & \text{Evidence} \times \text{Posterior} \\ L(\theta) \times & \pi(\theta)d\theta & = & Z \times p(\theta)d\theta \end{array}$$

- Problem: when the dimension of θ is high, the calculation of the evidence is challenging, but needed especially for model comparison problems.
- Solution: transform a multi-D integral into a 1D integral!

2D 1D



Skilling, 2006

Nested Sampling (Skilling, 2004)

Still a Monte Carlo technique, but….

$$Z = \int L(\theta)\pi(\theta)d\theta$$

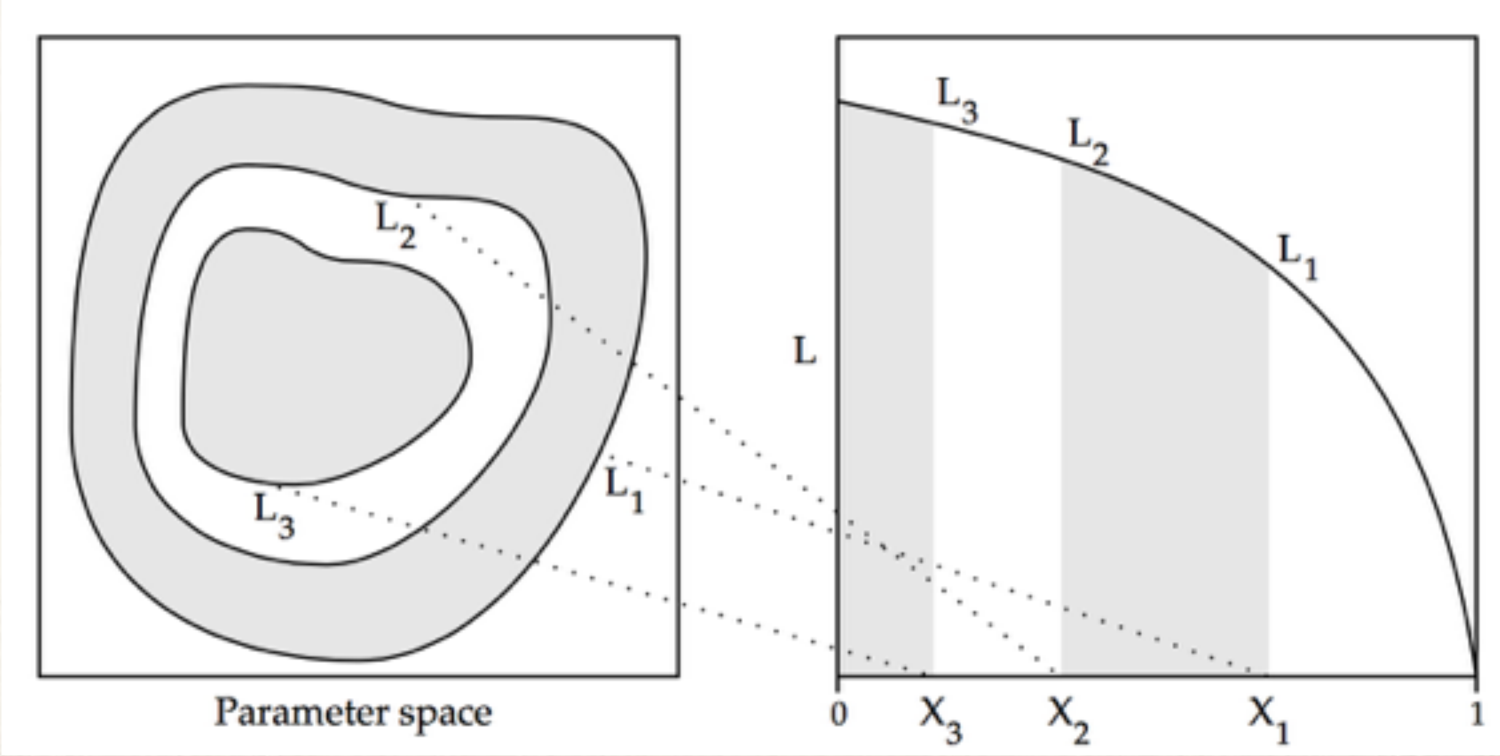
X is the amount of prior probability with likelihood greater than **λ**.

Roughly, X is the volume with likelihood above **λ**.

$$dX = \pi(\theta)d\theta$$

$$X(\lambda) = \int_{L(\theta)>\lambda} \pi(\theta)d\theta$$

$$Z = \int_0^1 L(X)dX$$



Skilling, 2006

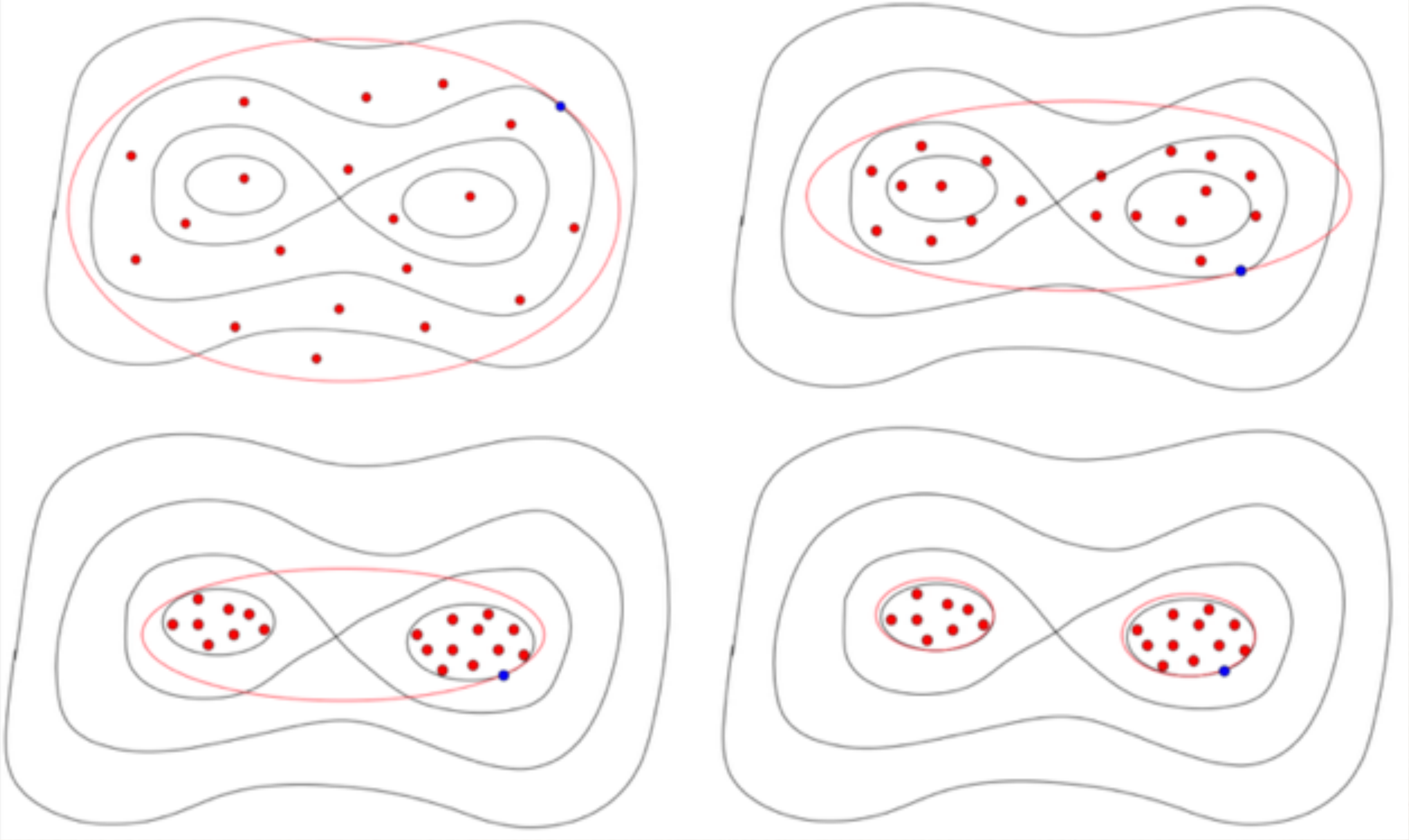
Nested sampling: pros and cons

- Nested sampling is good for irregular likelihood landscapes and more efficient than MCMC, but it's still not enough…
- There is still the risk of not finding the global maximum.
- If sampling the space uniformly in a high dimensional problem, it can be slow.

MultiNest

Feroz & Hobson 2008; Feroz et al. 2009, 2013

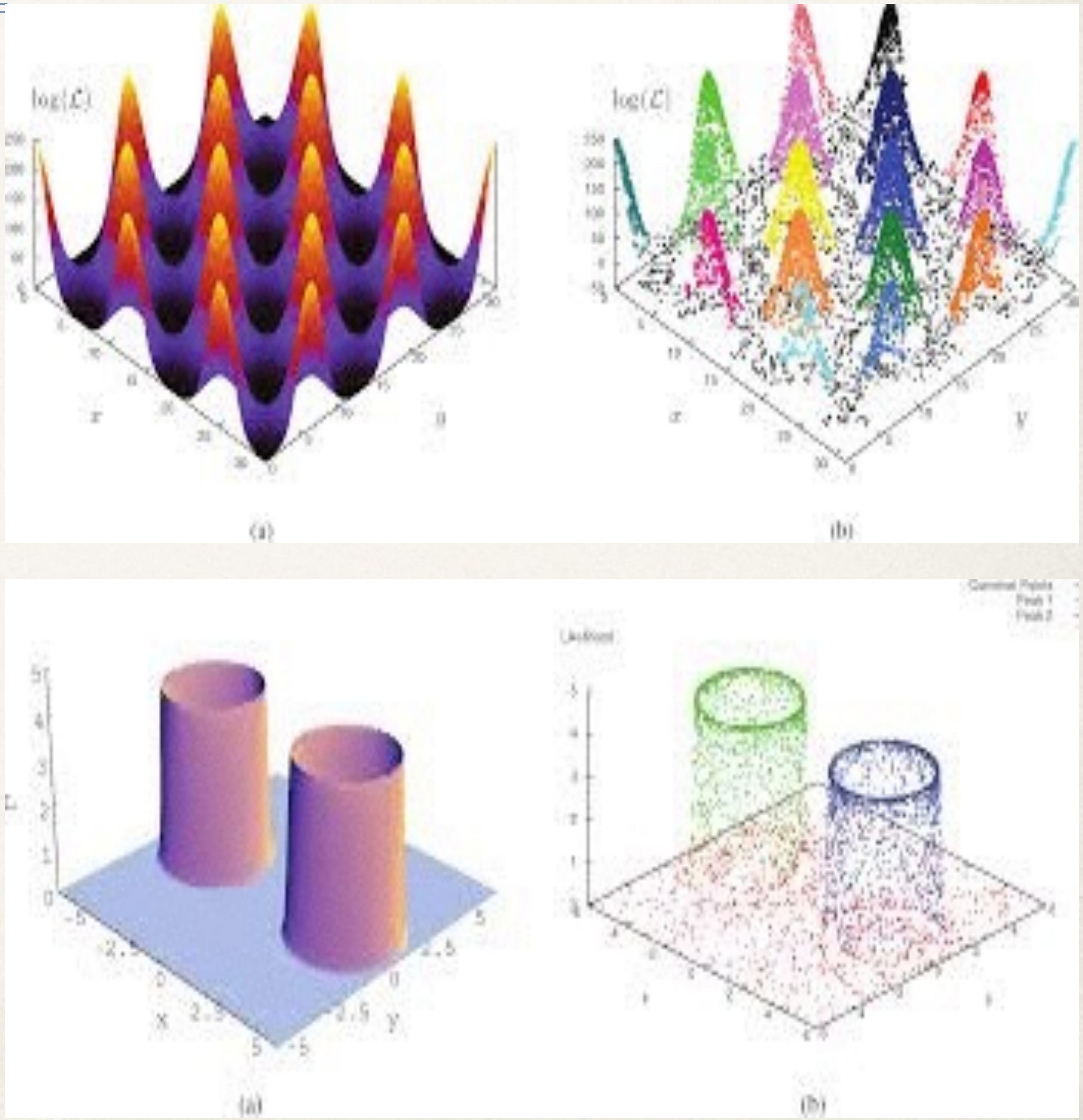
- Solid algorithm for multi-modal likelihood landscapes
- Instead of using a multi-dimensional uniform prior for each replacement point, use an n-dimensional ellipsoid for resampling



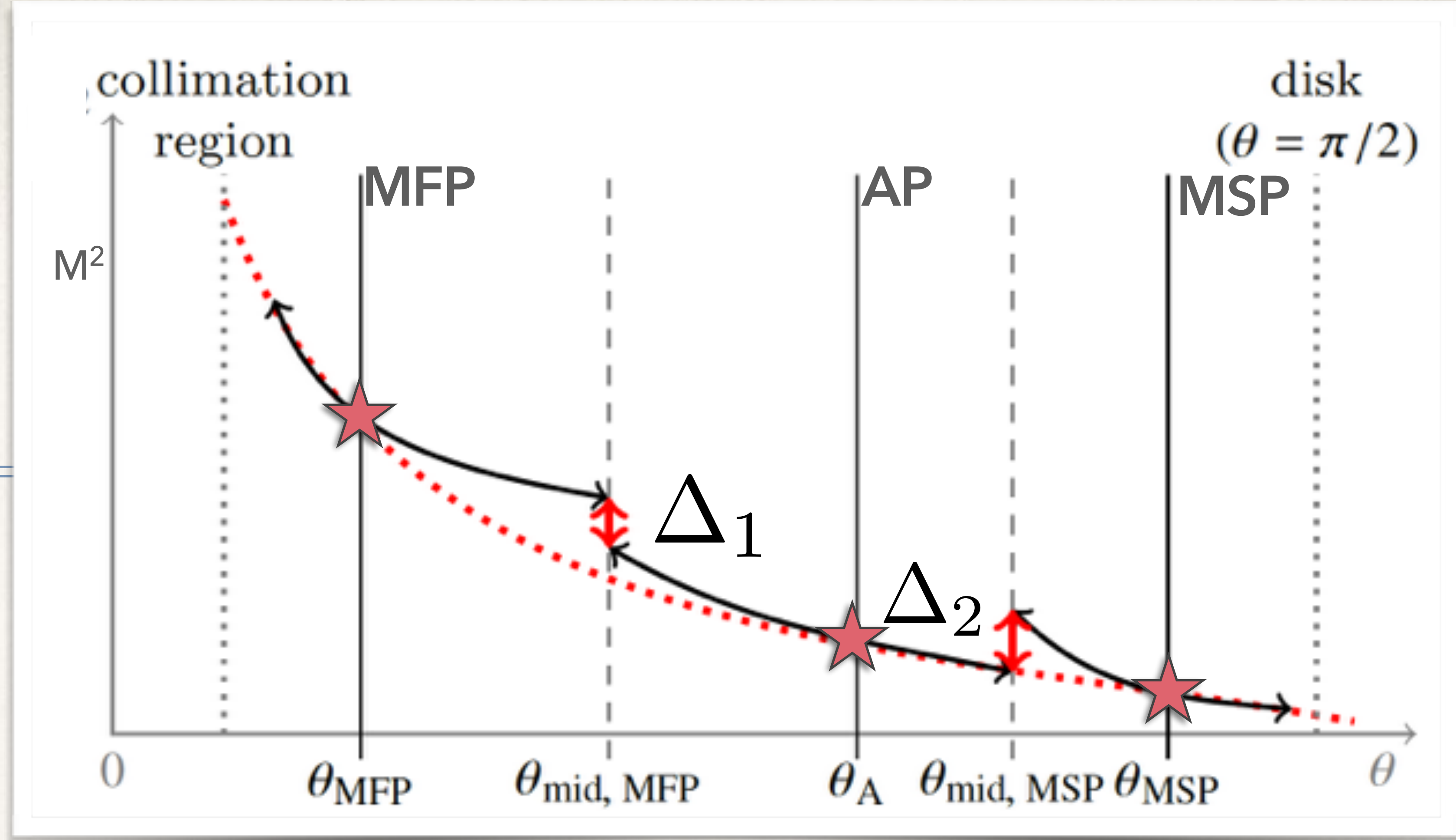
MultiNest

Feroz & Hobson 2008; Feroz et al. 2009, 2013

Disjoint regions as well as multi-dimensional multi-modal regions can be found efficiently without continual resampling of the whole space.



The method



- We guess $\boldsymbol{\theta}_{\mathrm{MSP}}$ and $\boldsymbol{\theta}_{\mathrm{MFP}}$ and derive values for M^2,G^2 and their derivatives which satisfy the condition that $\left.\frac{d\mathcal{M}^2}{dr}\right|_* = \frac{0}{0}$
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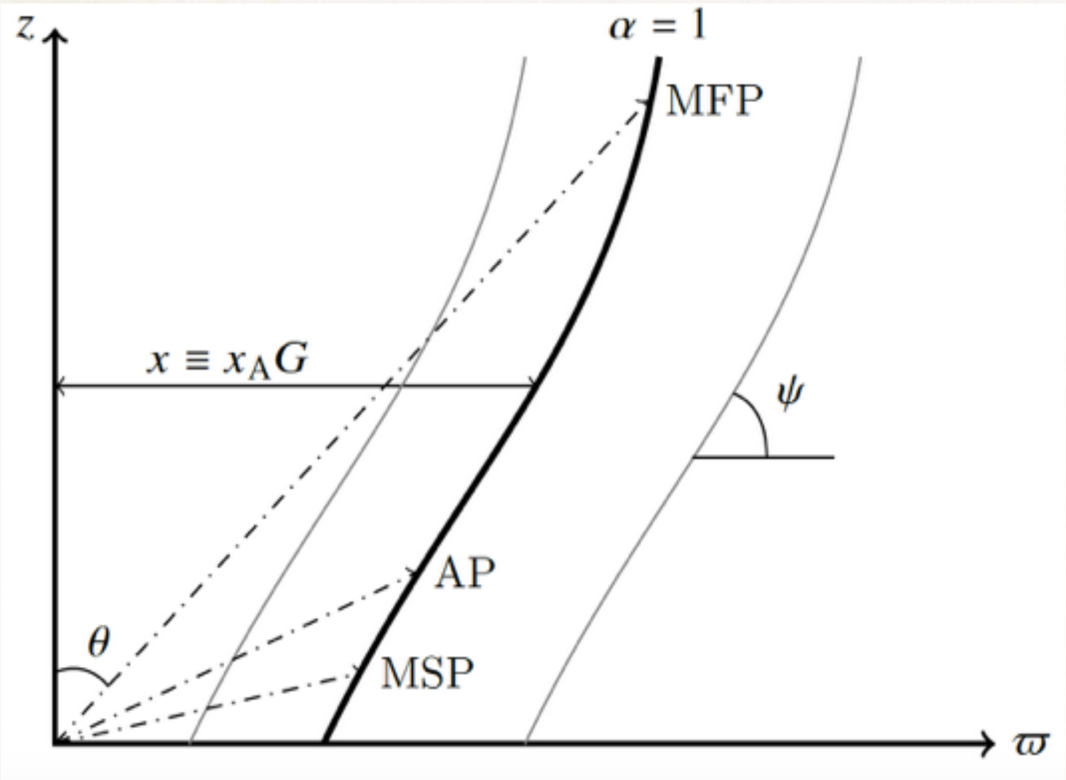
* = MSP or MFP

Fitted parameters and fitness function in the NRMHD problem

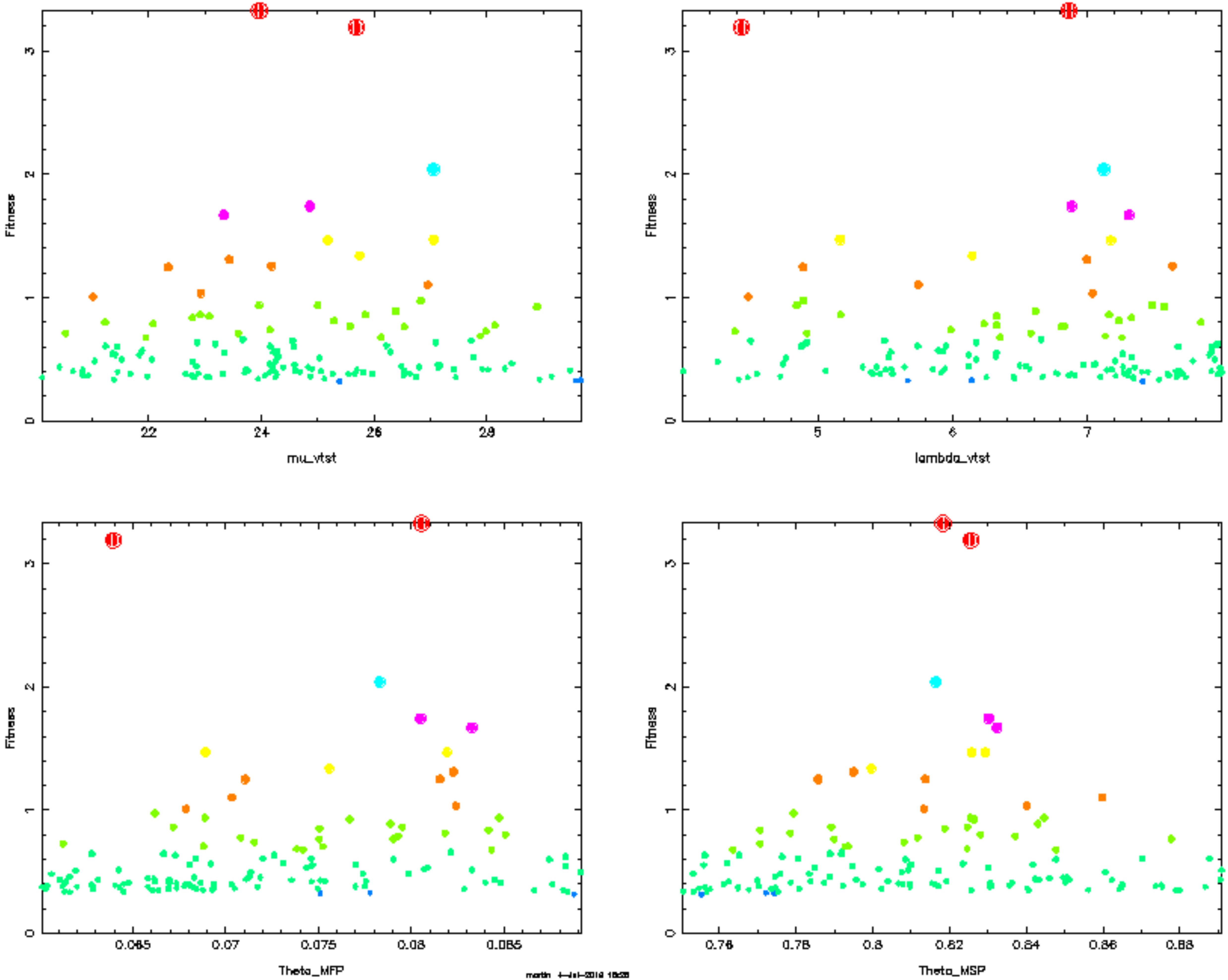


$$\mathcal{F} = \left[f_{\mathrm{G,F}}^2 + f_{\mathrm{M,F}}^2 + f_{\mathrm{G,S}}^2 + f_{\mathrm{M,S}}^2\right]^{-1/2}$$

$$f_{\mathrm{G},*} = \left.\frac{2(G_L^2 - G_R^2)}{(G_L^2 + G_R^2)}\right|_{\theta_{\mathrm{mid},*}}$$

$$f_{\mathrm{M},*} = \left.\frac{2(M_L^2 - M_R^2)}{(M_L^2 + M_R^2)}\right|_{\theta_{\mathrm{mid},*}}$$

Input parameters	
F	determines the shape of the magnetic field at the base
Γ	polytropic index of the gas
$\theta_\star$	angular distance of the AP from the jet axis
$\psi_\star$	inclination of the stream line with respect to the horizontal axis at the AP
k_{VTST}	mass loss parameter at AP
Fitted parameters	
θ_{MFP}	angular distance of MFP from the jet axis
θ_{MSP}	angular distance of MSP from the jet axis
μ_{VTST}	scaling of the plasma-β at AP
λ_{VTST}	specific angular momentum in units of $V_\star \varpi_\star$


Fitness
mu_vtst
22
24
26
28
lambda_vtst
5
6
7
Theta_MFP
0.065
0.07
0.075
0.08
0.085
Theta_MSP
0.76
0.78
0.8
0.82
0.84
0.86
0.88
0
1
2
3
martin

Exercise #2

MultiNest: optimization of the parameters

How to run the NRMHD code: I

Setting the parameters and boxes and compile

- $ cd Multinest/Code/

- Open the file **params.f90** with your favourite editor (VIM, EMACS, etc.) and set **next_maxIter = 100**

- Open the file **main.f90** and set the intervals for the input parameters as in the table

- $ ./Compile-mhd %this will create a mhd.exe file one folder above. Every time you change one of the files in Code, you **need to run** this command **again** and create a new exe file

	main.f90
μ_{VTST}	20.0 - 30.0
λ_{VTST}	1 - 8
θ_{MFP}	0,06 - 0,08
θ_{MSP}	0,75 - 0,90

How to run the NRMHD code: II

Running the code and check the best fit (so far)

- $ cd ..
- $ ./mhd.exe >> mhd.log %while running the code, you are writing a log file and every time you run it again, it will append the new results after the old ones.
- $ fgrep Fit_ness mhd.log | sort -g -k 3 > fitness_XXX.log %takes all the fitness values from the log file, sort them in increasing order and place them in the file fitness_XXX.log . Remember to change the XXX in fitness_XXX.log each run!
- $ tail fitness_XXX.log %only shows the end of the file where the highest fitness solution is.
- **Repeat all the steps so far a few times, e.g. 5, to get solutions with increasing fitness.**

How to run the NRMHD code: III

Prepare a run with more iterations to find a **true solution** for comparison

- $ cd Multinest/Code/
- Open the file **params.f90** with a text editor (VIM, EMACS, etc.) and set **next_maxIter = 5000**
- $ ./Compile-mhd %this will create a mhd.exe file one folder above. Every time you change one of the files in Code, you **need to run** this command **again** and create a new exe file

How to run the NRMHD code: IV

Running the code and check the best fit

- $ cd ..
- $ rm Chains/* %to clean the results of the last run in case something went wrong before restarting a new run
- $ ./mhd.exe >> mhd_true.log %while running the code, you are writing a log file and every time you run it again, it will append the new results after the old ones. You can run this in the background by adding the character & to the command as ./mhd.exe >> mhd_true.log &
- $ fgrep Fit_ness mhd_true.log | sort -g -k 3 > fitness_true.log %takes all the fitness values from the log file, sort them in increasing order and place them in the file fitness_true.log
- $ tail fitness_true.log %only shows the end of the file where the highest fitness solution is.
- Copy the last solution in the fitness_true.log into Solutions/collective-data.dat

How to run the NRMHD code

Generate solution files

- Choose a few solutions with different low fitness, e.g. 3, 7,12, 25, from your fitness_*.log files and add those to the file Solutions/collective-data.dat
- $ cd Solutions
- $ ./mhd-generate.exe %it creates the solution files, e.g. solution_XXXX.dat, and the collective-results.dat.
- If the above step is done **before** the last long run is over, you need to remove the collective-results.dat and solution_*.dat files and run the mhd-generate.exe script again.
- Now we can run the notebook multinest.ipynb to plot the results
- $ python -m jupyter notebook --no-browser -- ip=000.000.00.00 -- port=8890

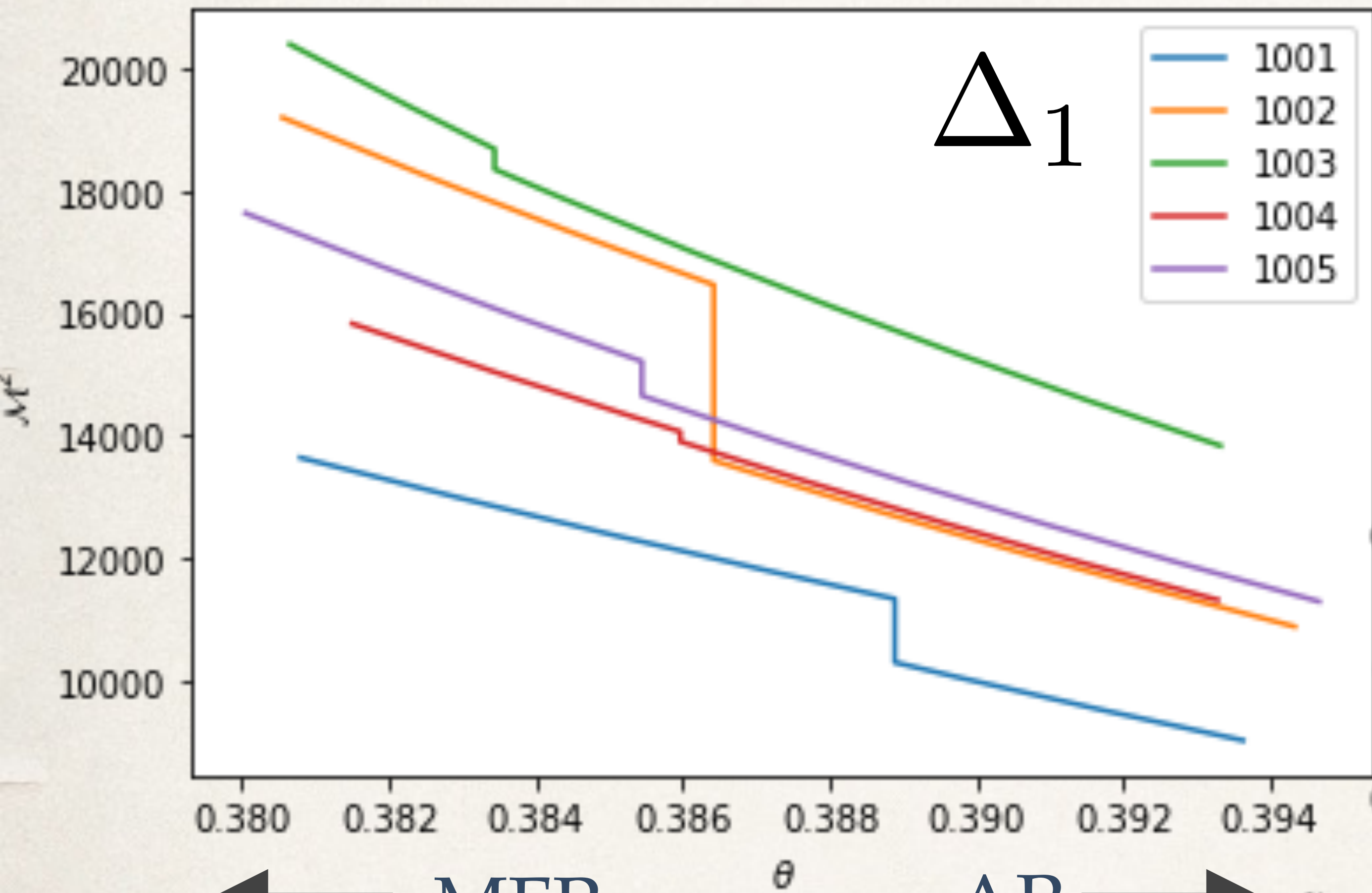
Priors, iter and best fit results

	main.f90
μ_{VTST}	20.0 - 30.0
λ_{VTST}	1.0 - 8.0
θ_{MFP}	0,06 - 0,08
θ_{MSP}	0,75 - 0,90

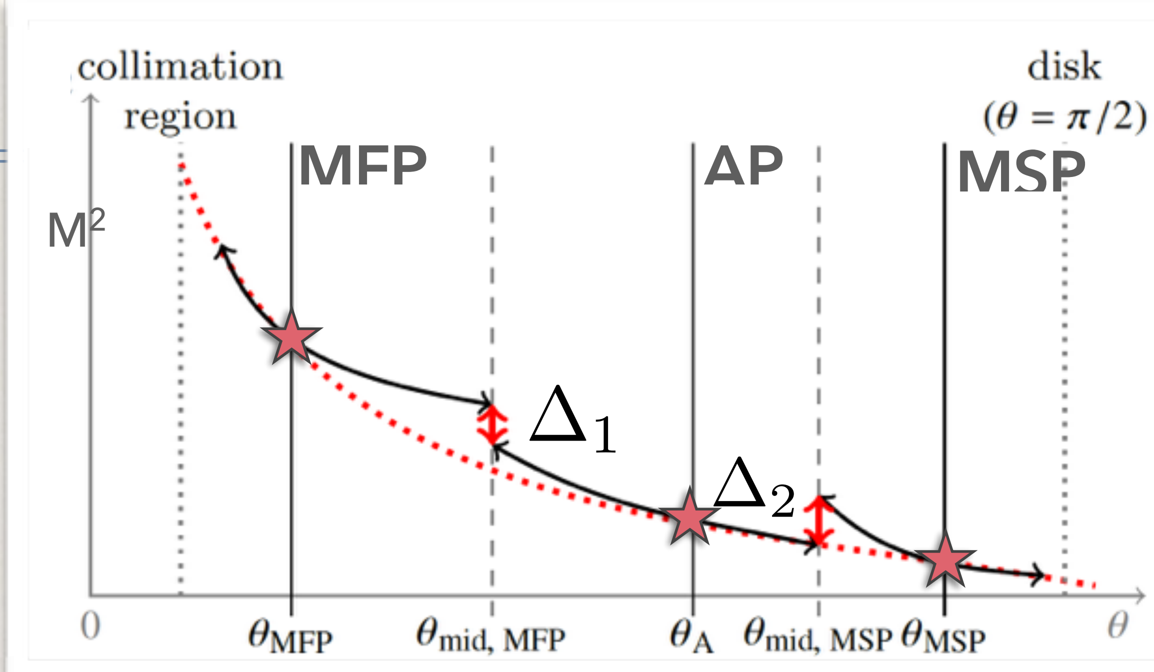
	params.f90	params.f90
next_maxIter	100(x5)	5000(x1)

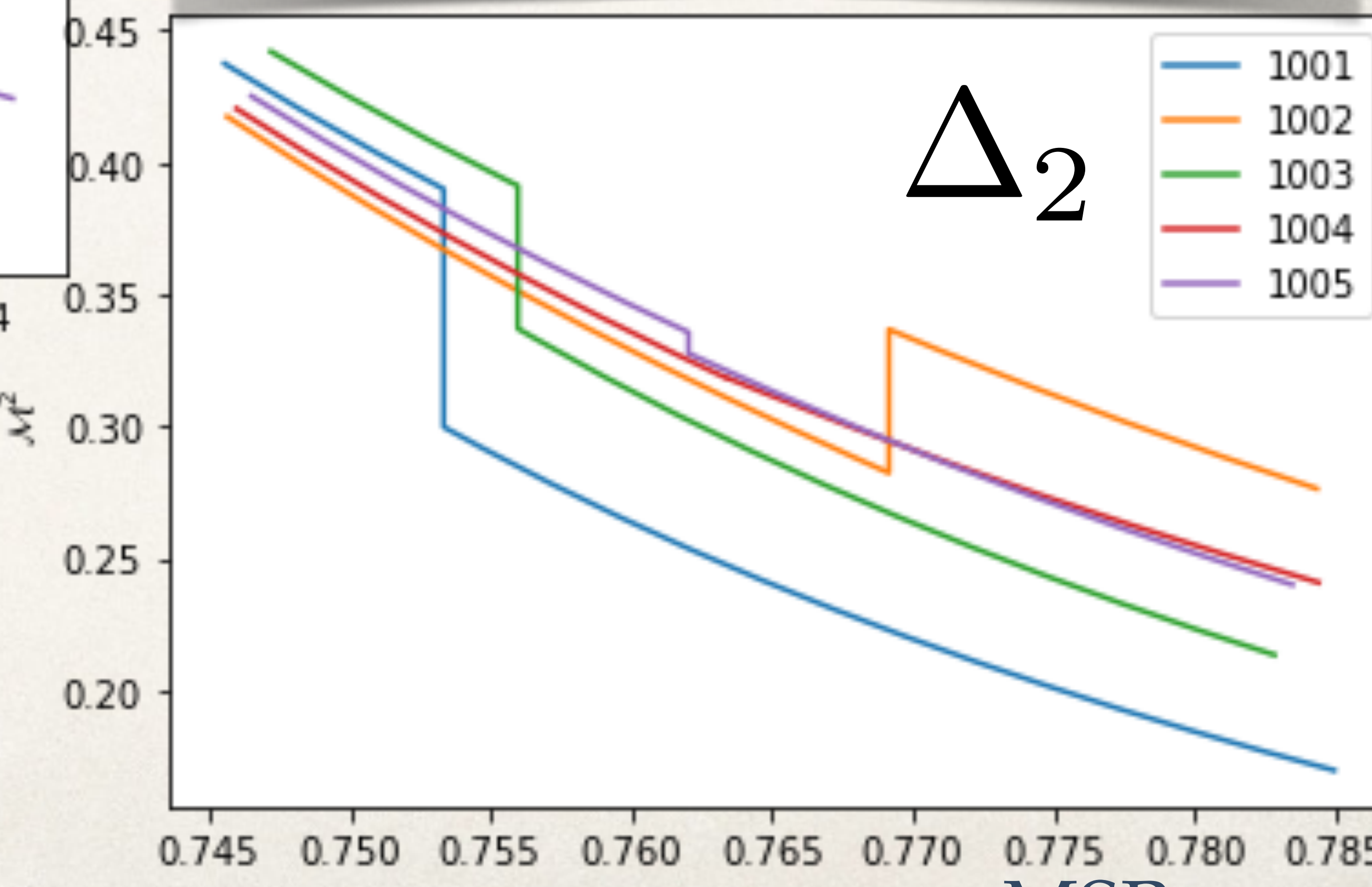
	results
μ_{VTST}	26,54
λ_{VTST}	5,783
θ_{MFP}	0,072
θ_{MSP}	0,828
θ_{A}	40° (0,698)
ψ_{A}	55°
Γ	4/3
k_{VTST}	5,0

Examples of low-fitness offsets



MFP AP





AP MSP

THE END

and don't forget to enjoy Amsterdam!