Magnetohydrodynamics
and
Turbulence

Blakesley Burkhart

These lecture notes are largely based on *Plasma Physics for Astrophysics* by Russell Kulsrud, *Lectures in Magnetohydrodynamics* by the late Dalton Schnack, *Ideal Magnetohydrodynamics* by Jerey Freidberg, *Magnetic Reconnection* by Eric Priest and Terry Forbes, course notes from similar classes taught by Ellen Zweibel and Chris Hegna, and lecture notes from Nick Murphy

Astrophysical proverbs

If we don't understand it, invoke magnetic fields.

If we still don't understand it, invoke turbulence.

Astrophysical Magnetized Turbulence: A Difficult Problem to Study

Fact 1: It is difficult to measure magnetic fields in the ISM/IGM
Fact 2: The ISM/solar wind are complicated MHD flows with a range of temperatures, scales; instabilities…

This makes the study of turbulence & magnetic fields unpopular with some astronomers…

Quotes from anonymous IAU participants:

Indifference... "Magnetic fields are too complicated for our models…they are too numerically expensive.."

Disgust... "You study turbulence and magnetic fields…ugh!"

Aversion... "I hate magnetic fields!!"

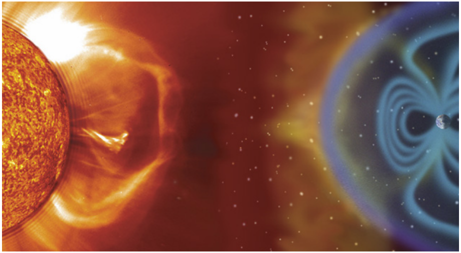
Outline

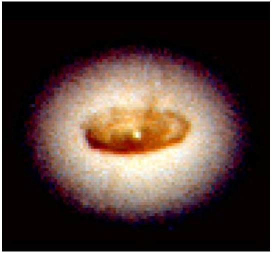
- Overview of MHD
  - Approximation
  - Usefulness
  - Applications
- The equations of MHD and their physical meaning
  - Continuity equation
  - Momentum equation
  - Energy equation
  - Faraday's law
  - Ohm's law

Why study plasma astrophysics?

- Most of the baryonic matter in the universe is plasma
- Magnetic fields play vital roles in astrophysical processes
  - Star formation, thermal conduction, accretion, turbulence, particle acceleration, dynamos, etc.
- Astrophysical magnetic fields directly impact our increasingly technological civilization
  - Space weather
- Plasma astrophysics allows the study of phenomena at extreme regions of parameter space that are inaccessible in the laboratory

Applications of plasma astrophysics





- Planetary and exoplanetary magnetospheres
- Stellar, planetary, and galactic dynamos
- Solar and stellar flares
- Interplanetary and near-Earth space plasmas
- Accretion disks and jets
- Neutron star magnetospheres
- Interstellar medium
- Supernovae, supernova remnants, and cosmic rays

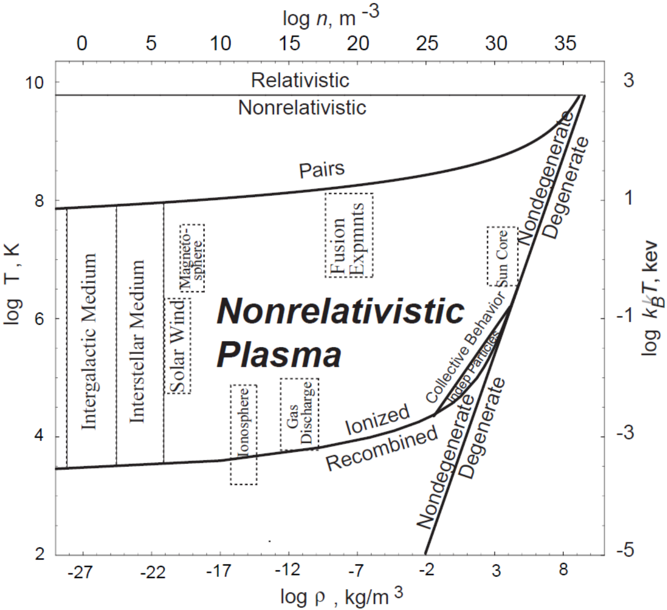
Plasma physics is interdisciplinary

- Laboratory plasma physics
  - Study of basic plasma processes
  - Fusion
  - Laser produced plasmas
- Heliophysics and space weather
  - Solar eruptions (flares, coronal mass ejections)
  - Interaction between solar wind and Earth's magnetosphere
- Astrophysics
  - ISM, accretion disks/jets, galaxy clusters, neutron star magnetospheres
  - Space weather around exoplanets
- Lightning
- Material science
  - Plasmas are used to etch circuits on microchips
- Plasma screen TVs
  - I actually have no idea how these work

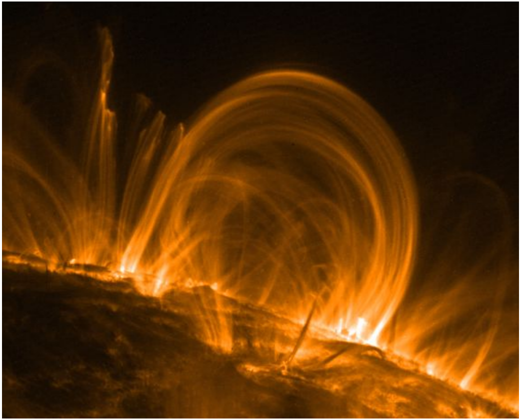
Fundamental processes in plasma astrophysics

- Waves
- Shocks
- Instabilities
- Turbulence
- Particle acceleration
- Dynamo
  - Converts kinetic energy to magnetic energy
- Reconnection
  - Converts magnetic energy to kinetic/thermal energy and particle energization
  - Alters magnetic field connectivity

The density-temperature regime of plasmas



What is MHD?



MHD couples Maxwell's equations with hydrodynamics to describe the macroscopic behavior of highly conducting fluids such as plasmas

Ideal MHD at a glance (cgs units)

Continuity Equation	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Momentum Equation	$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p$
Ampere's law	$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}$
Faraday's law	$\frac{\partial \mathbf{B}}{\partial t} = -c \nabla \times \mathbf{E}$
Ideal Ohm's law	$\mathbf{E} + \frac{\mathbf{V} \times \mathbf{B}}{c} = 0$
Divergence constraint	$\nabla \cdot \mathbf{B} = 0$
Adiabatic Energy Equation	$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{p}{\rho^\gamma} \right) = 0$

Definitions: **B**, magnetic field; **V**, plasma velocity; **J**, current density; **E**, electric field; ρ, mass density; p, plasma pressure; γ, ratio of specific heats (usually 5/3); t, time.

MHD is a low-frequency, long-wavelength approximation

- MHD is valid on time scales longer than the inverses of the plasma frequencies and cyclotron frequencies for both ions and electrons:

$$\tau \gg \omega_{pe}^{-1}, \omega_{pi}^{-1}, \Omega_{ce}^{-1}, \Omega_{ci}^{-1} \tag{1}$$

- MHD is valid on length scales longer than the Debye length and electon/ion gyroradii:

$$L \gg \lambda_D, r_{Le}, r_{Li} \tag{2}$$

- MHD assumes quasineutrality (since $L \gg \lambda_D$)

MHD is a low-frequency, long-wavelength approximation

- MHD assumes that collisions are frequent enough for the particle distribution function to be Maxwellian with $T_i = T_e$
- Ideal MHD assumes an adiabatic equation of state
  - No additional heating, cooling, or dissipation
- MHD assumes that the plasma is fully ionized
- MHD ignores the most important advances in physics since $\sim$1860
  - Ignore relativity (assume $V^2 \ll c^2$)
  - Ignore quantum mechanics
  - Ignore displacement current in Ampere's law (assume $V^2 \ll c^2$)

When is MHD useful?

- MHD traditionally describes macroscopic force balance, equilibria, and dynamics
- MHD is a good predictor of stability
  - The most catastrophic instabilities are unstable in ideal MHD
  - Important in laboratory plasmas, solar atmosphere, etc.
- Systems that are often described using MHD include:
  - Solar wind, heliosphere, and Earth's magnetosphere[1]
  - Inertial range of plasma turbulence
  - Neutron star magnetospheres
- MHD is a reasonably good approximation in many astrophysical plasmas
  - However, extensions are often needed

[1] On large scales!

When is MHD not useful?

- MHD has limited applicability when:
  - Non-fluid or kinetic effects are important
    - Dissipation in the turbulent solar wind
    - Magnetic reconnection
    - Small-scale dynamics in Earth's magnetosphere
  - Particle distribution functions are non-Maxwellian
    - Cosmic rays
  - The plasma is weakly ionized
    - Solar photosphere/chromosphere, molecular clouds, protoplanetary disks, Earth's ionosphere, some laboratory plasmas
- MHD is mediocre at describing the dynamics of laboratory plasmas but remains a good predictor of stability

Ideal MHD at a glance (cgs units)

Continuity Equation	$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$
Momentum Equation	$\rho \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \frac{\mathbf{J} \times \mathbf{B}}{c} - \nabla p$
Ampere's law	$\mathbf{J} = \frac{c}{4\pi} \nabla \times \mathbf{B}$
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Definitions: **B**, magnetic field; **V**, plasma velocity; **J**, current density; **E**, electric field; ρ, mass density; p, plasma pressure; γ, ratio of specific heats (usually 5/3); t, time.

The second golden rule of astrophysics

The *density* of wombats

times the *velocity* of wombats

gives the *flux* of wombats.

The continuity equation in conservative form

- The continuity equation in conservative form is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0 \tag{7}$$

- Conservative form is usually given by

$$\frac{\partial}{\partial t}(\text{stuff}) + \nabla \cdot (\text{flux of stuff}) = 0 \tag{8}$$

- Source and sink terms go on the RHS
  - *Example:* In a partially ionized plasma, there are continuity equations for both the ions and neutrals. Ionization acts as a source term in the ion continuity equation and a sink term in the neutral continuity equation.
- The mass flux is given by $\rho \mathbf{V}$

The momentum equation is derived from Newton's 2nd law

- Newton's second law of motion for a fluid element is

$$\rho \frac{\mathrm{d}\mathbf{V}}{\mathrm{d}t} = \mathbf{F} \tag{15}$$

  where $\mathbf{F}$ is the force per unit volume
- Example forces include
  - Lorentz force: $\mathbf{F}_L = \frac{\mathbf{J}\times\mathbf{B}}{c}$
  - Pressure gradient force: $\mathbf{F}_p = -\nabla p$
  - Gravity: $\mathbf{F}_g = -\rho\mathbf{g}$ or $\mathbf{F}_g = -\nabla\phi$ for gravitational potental ϕ
  - Viscosity: $\mathbf{F}_V = \nabla\cdot\mathbf{\Pi}$, where $\mathbf{\Pi}$ is the viscous stress tensor
- The ideal MHD momentum equation in Eulerian form is

$$\rho\left(\frac{\partial}{\partial t} + \mathbf{V}\cdot\nabla\right)\mathbf{V} = \frac{\mathbf{J}\times\mathbf{B}}{c} - \nabla p \tag{16}$$

  where we neglect gravity and ignore viscous forces

Where does the Lorentz force come from?

- The Lorentz force acting on a single particle is

$$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{V}\times\mathbf{B}}{c}\right) \tag{17}$$

- The current density is given by

$$\mathbf{J} = \sum_{\alpha} n_\alpha q_\alpha \mathbf{V}_\alpha \tag{18}$$

where α includes all species of ions and electrons. For a quasineutral plasma with electrons and singly charged ions, this becomes

$$\mathbf{J} = en\left(\mathbf{V}_i - \mathbf{V}_e\right) \tag{19}$$

where $n = n_e = n_i$, $\mathbf{V}_i$ is the ion velocity, and $\mathbf{V}_e$ is the electron velocity.

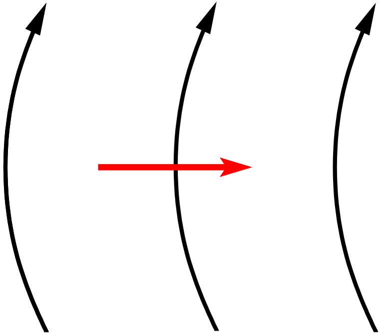
The Lorentz force includes a magnetic tension force and a magnetic pressure force

- Use Ampere's law and vector identities to decompose the Lorentz force term into two components

$$\begin{aligned}\underbrace{\frac{\mathbf{J} \times \mathbf{B}}{c}}_{\text{Lorentz force}} &= \frac{(\nabla \times \mathbf{B}) \times \mathbf{B}}{4\pi} \\ &= \underbrace{\frac{\mathbf{B} \cdot \nabla \mathbf{B}}{4\pi}}_{\sim\text{magnetic tension}} - \underbrace{\nabla \left(\frac{B^2}{8\pi} \right)}_{\sim\text{magnetic pressure}} \end{aligned} \quad (20)$$

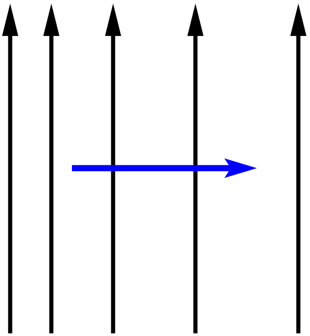
- While the Lorentz force must be orthogonal to be **B**, both of these terms may have components along **B**. The parallel component of the above tension term cancels out the parallel part of the magnetic pressure term (Kulsrud §4.2).

The magnetic tension force wants to straighten magnetic field lines



- The magnetic tension force is directed radially inward with respect to magnetic field line curvature

Regions of high magnetic pressure exert a force towards regions of low magnetic pressure



- The magnetic pressure is given by $p_B \equiv \frac{B^2}{8\pi}$

The ratio of the plasma pressure to the magnetic pressure is an important dimensionless number

- Define plasma β as

$$\beta \equiv \frac{\text{plasma pressure}}{\text{magnetic pressure}} \equiv \frac{p}{B^2/8\pi}$$

- If $\beta \ll 1$ then the magnetic field dominates
  - Solar corona
  - Poynting flux driven jets
  - Tokamaks ($\beta \lesssim 0.1$)
- If $\beta \gg 1$ then plasma pressure forces dominate
  - Stellar interiors
- If $\beta \sim 1$ then pressure/magnetic forces are both important
  - Solar chromosphere
  - Parts of the solar wind and interstellar medium
  - Some laboratory plasma experiments

Key Properties of Ideal MHD

- *Frozen-in condition:* if two parcels of plasma are attached by a field line at one time, they will continue to be attached by a field line at future times
  - Magnetic topology is preserved
- Mass, momentum, and energy are conserved
- Helicity and cross-helicity are conserved
- Adiabatic equation of state
- No dissipation!
  - No resistivity, viscosity, or thermal conduction
- Ideal MHD is scale-free

How do we approach a problem when MHD is insufficient?

- Extended MHD
  - Keep the fluid approximation
  - Add terms including effects beyond MHD
  - Resistivity, viscosity, anisotropic thermal conduction, separate ion and electron temperatures, neutrals, etc.
- Kinetic theory/particle-in-cell simulations
  - Abandon the fluid approximation
  - Track particle distribution functions
- A hybrid approach
  - Keep some parts of the fluid approximation
  - Express other parts kinetically

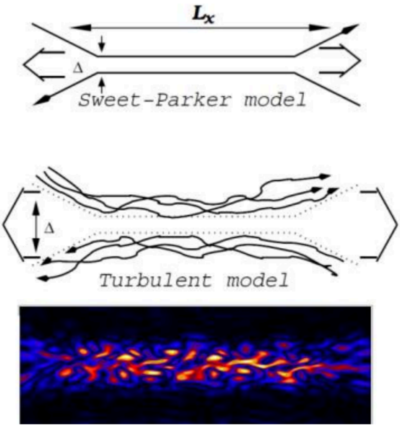
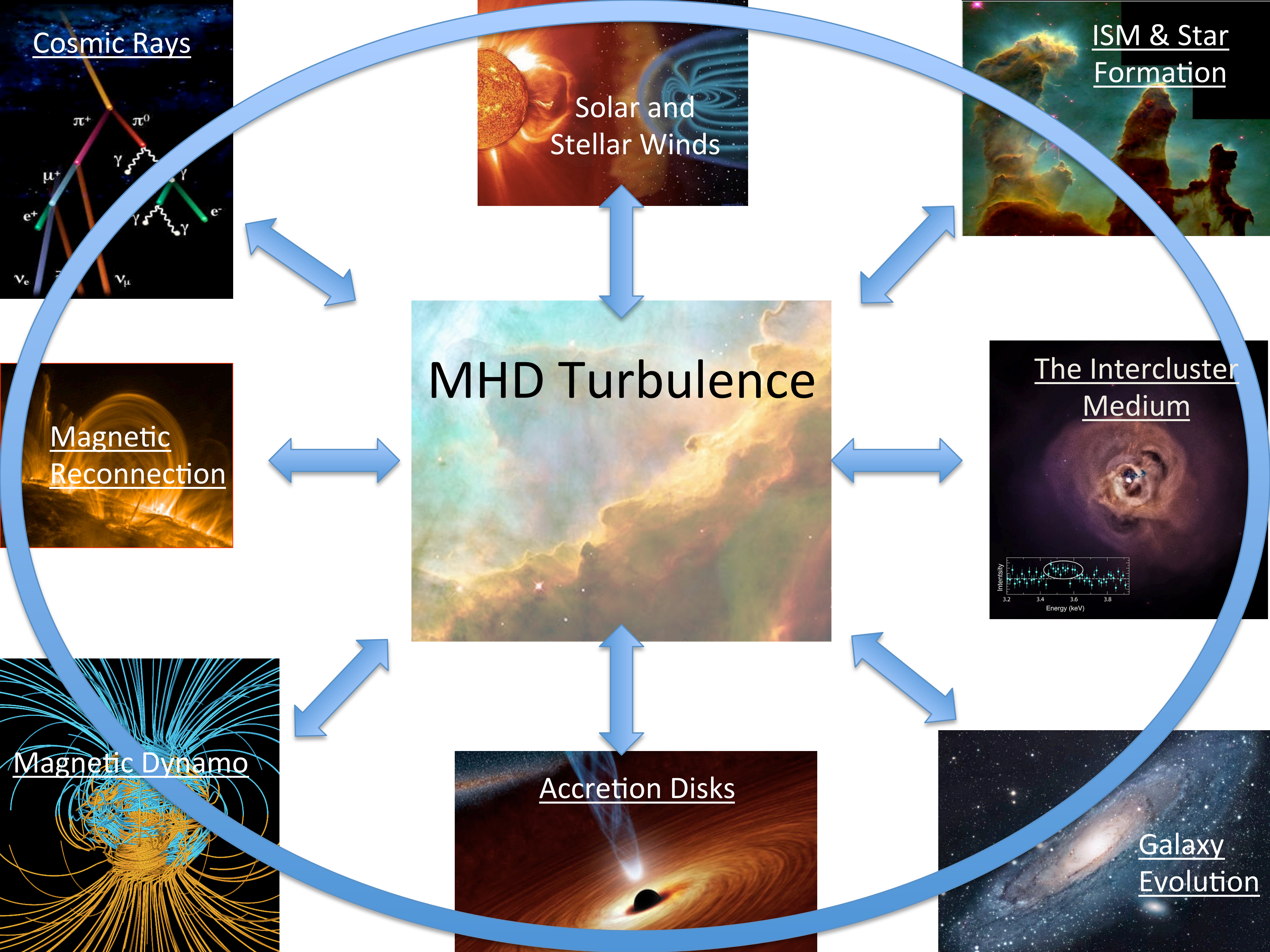


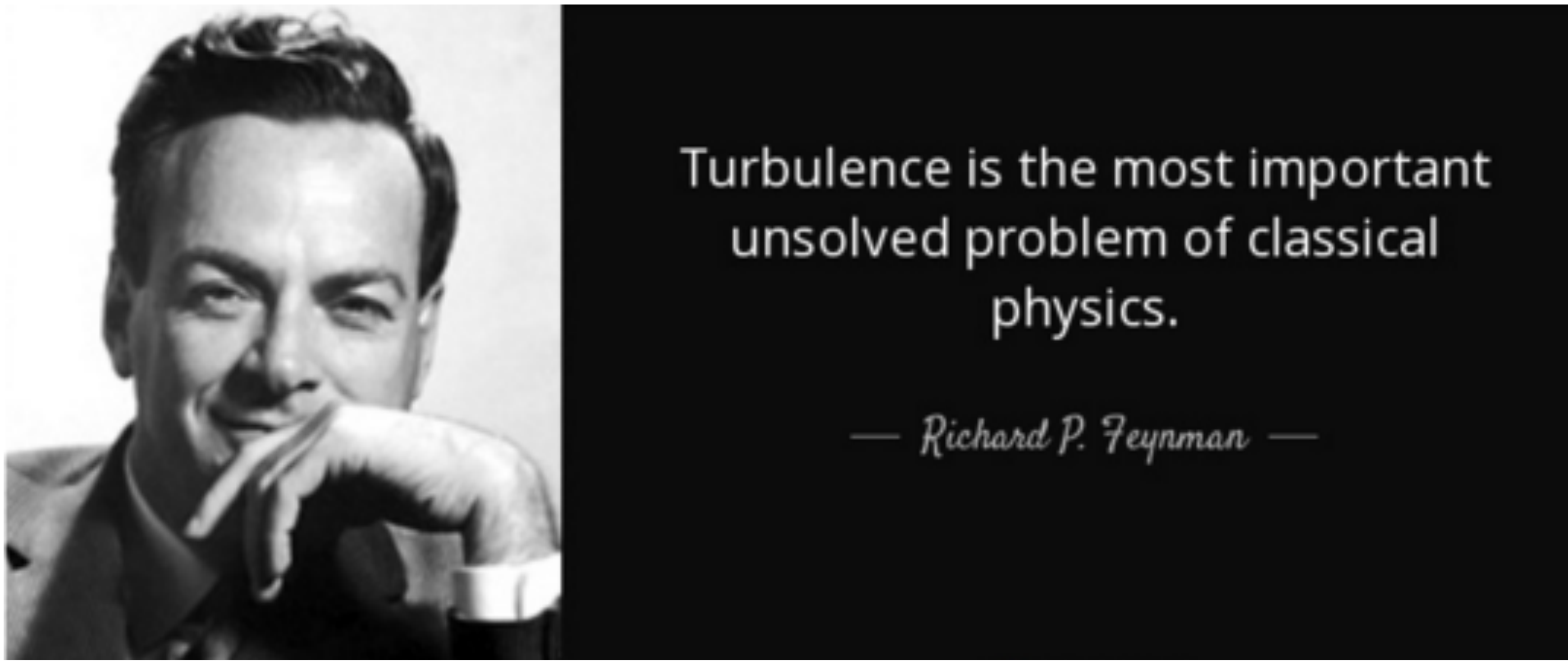
Figure 1.— Top panel: Sweet–Parker model of reconnection. The outflow is limited by a thin slot Δ determined by Ohmic diffusivity. The other scale is an astrophysical scale $L_x >> \Delta$. Middle panel: fast reconnection model in presence of turbulence according to Lazarian & Vishiniac (1999) (extracted from Lazarian et al. 2004). Bottom panel: 3D MHD numerical simulation of fast turbulent reconnection (from Kowal et al. 2009).



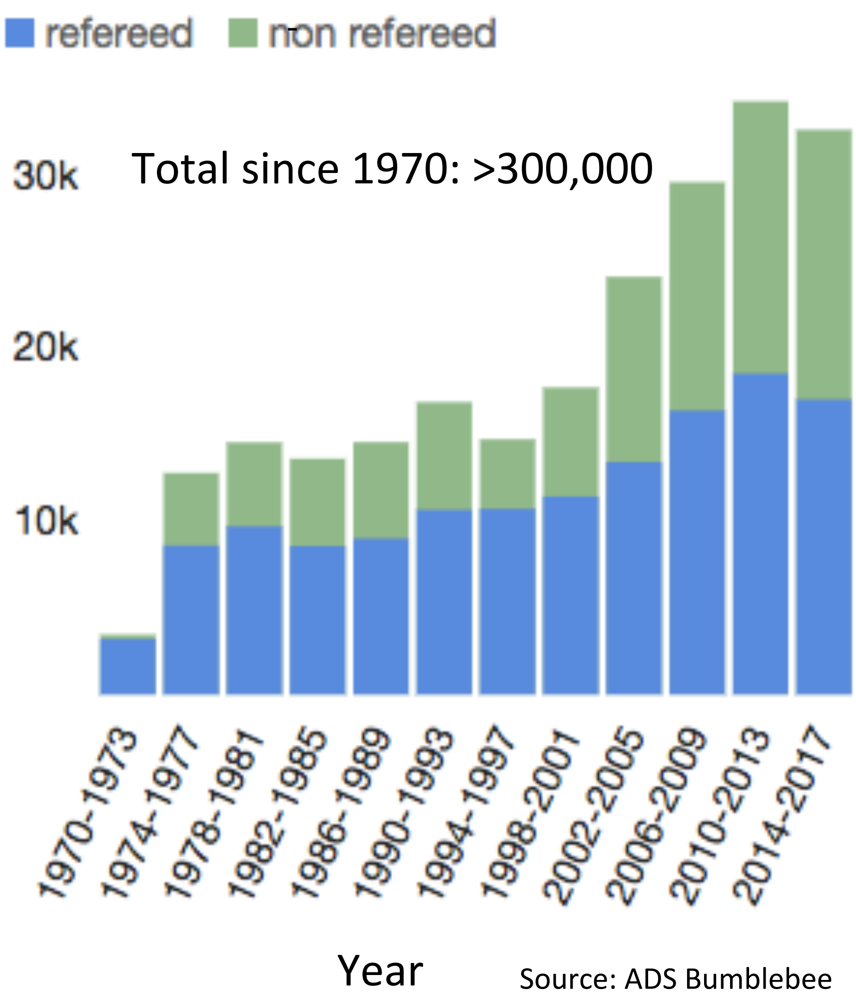
Cosmic Rays
Solar and Stellar Winds
ISM & Star Formation
MHD Turbulence
Magnetic Reconnection
The Intercluster Medium
Intensity
Energy (keV)
Magnetic Dynamo
Accretion Disks
Galaxy Evolution

When I meet God, I am going to ask him two questions: Why relativity ? And why turbulence ? I really believe he will have an answer for the first.

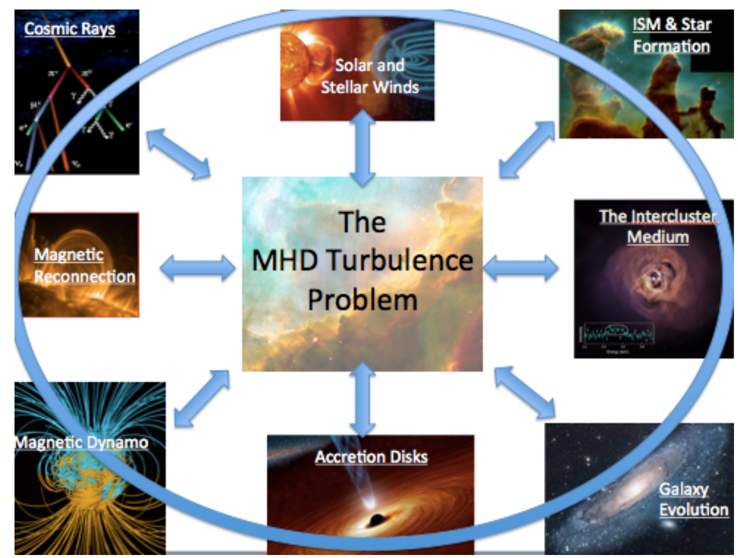
Werner Heisenberg


Turbulence is the most important unsolved problem of classical physics.
— Richard P. Feynman —

(Astro)physics Papers w/ "Turbulence" in Abstract



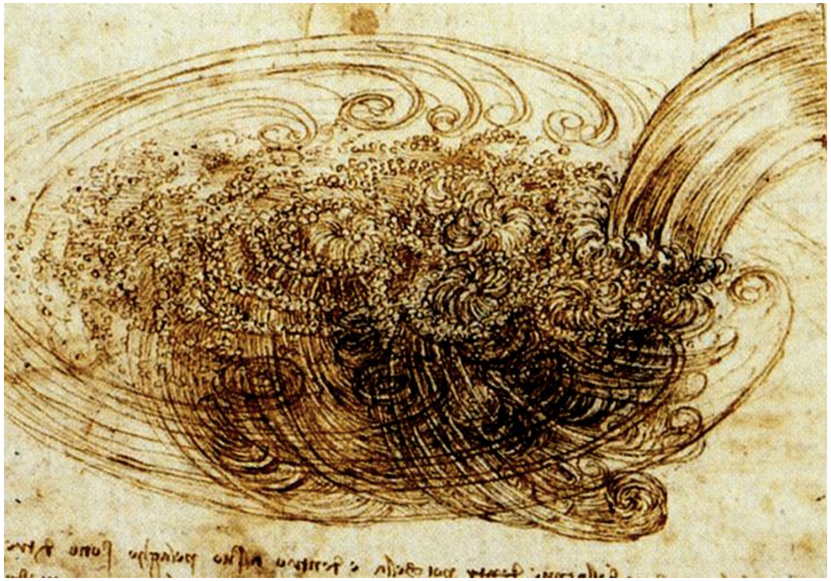
Source: ADS Bumblebee



Burkhart 2019, ASP Reviews

- Question: What is turbulence?

The word "turbulence" (Latin: turbulentia) originally refers to the disorderly motion of a crowd (turba).



What is Turbulence? Reynolds Number

- Reynolds number: $Re = VL/\nu$ ← $\boxed{(V^2/L) / (\nu V/L^2)}$

$$\frac{\partial \mathbf{v}}{\partial t} = -(\mathbf{v} \cdot \nabla)\mathbf{v} + \nu \nabla^2 \mathbf{v}$$

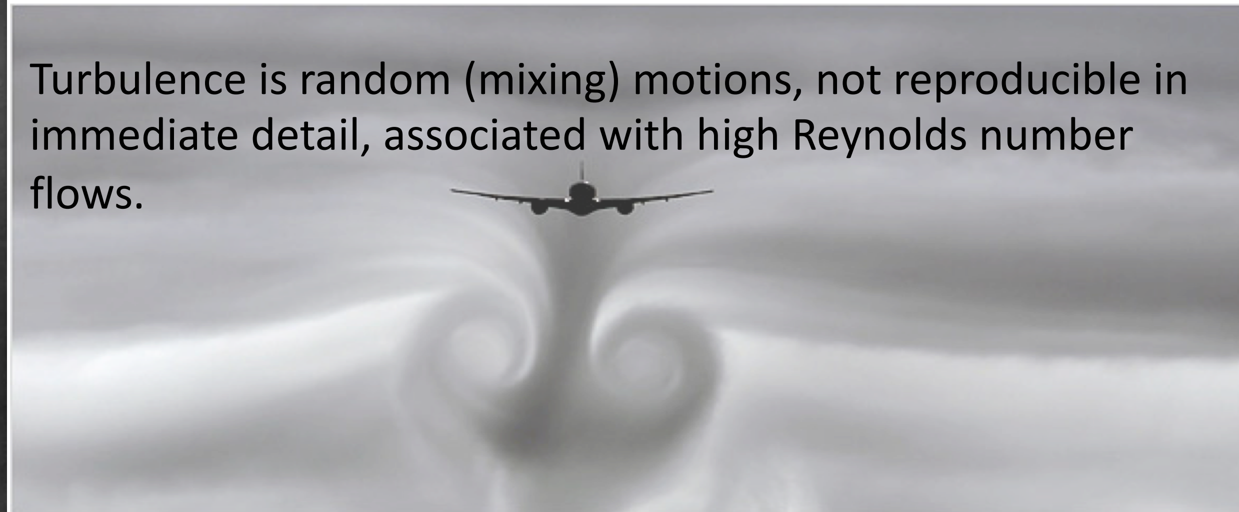
V^2/L $\qquad$ $\nu V/L^2$

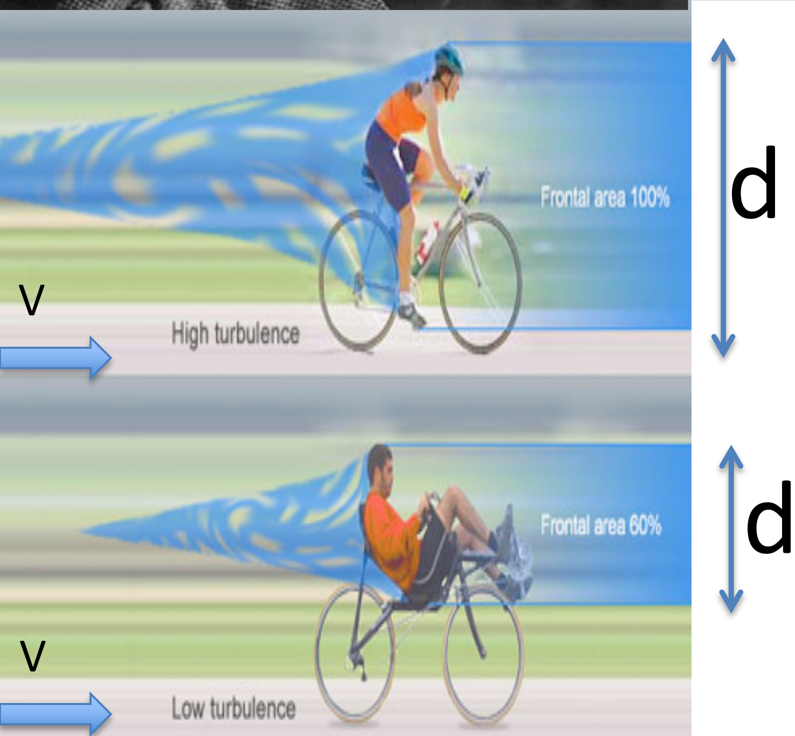
- When $Re \ll Re_{critical}$, flow = laminar
  When $Re \gg Re_{critical}$, flow = turbulent

Astrosphere regions/ISM can have flows $Re>10^{10}$.

What is Turbulence?

Turbulence is random (mixing) motions, not reproducible in immediate detail, associated with high Reynolds number flows.





The Reynolds number is the ratio of inertial forces vs. kinematic viscosity:

$$\mathrm{Re} = \frac{V \cdot d}{\upsilon}$$

Turbulence and Mixing

Without turbulence:

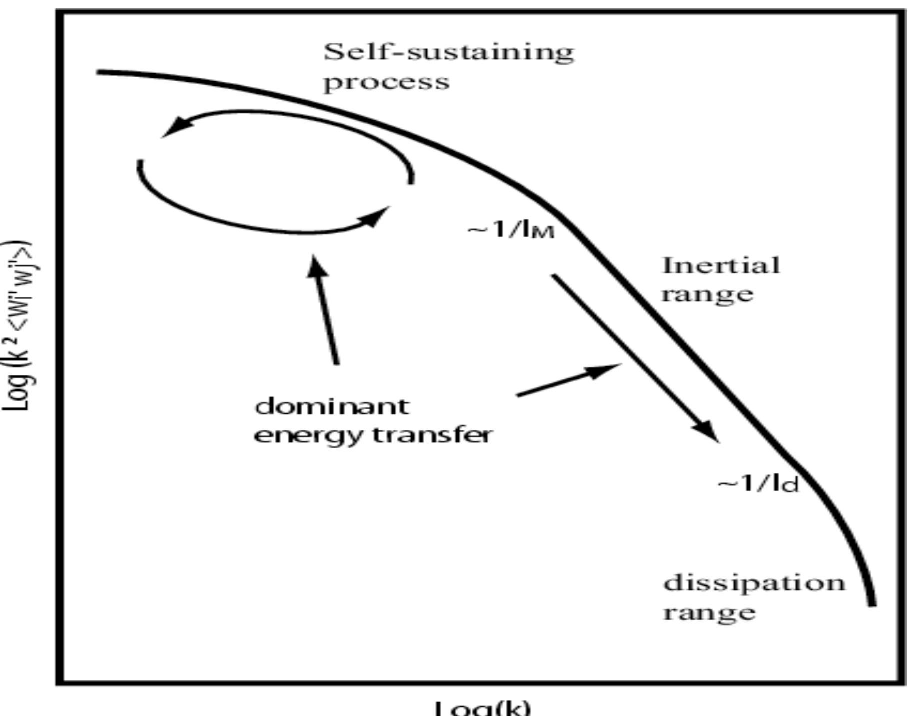
molecular diffusion coefficient D ~10^{-5} cm^2/sec

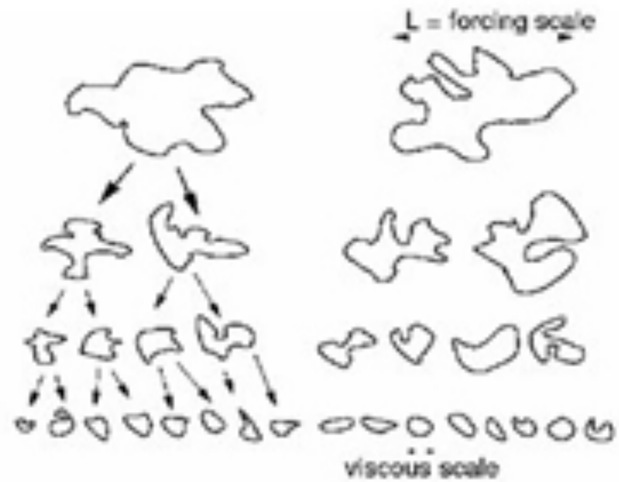
(← It's for small molecules in water.)

➔ Mixing time ~ (size of the cup)^2/D ~ 10^7 sec ~ 0.3 year !

What is Turbulence?.....Most importantly:

Turbulence is not just ‘chaos’. It has specific statistical properties which can be seen when averaged over time and space.





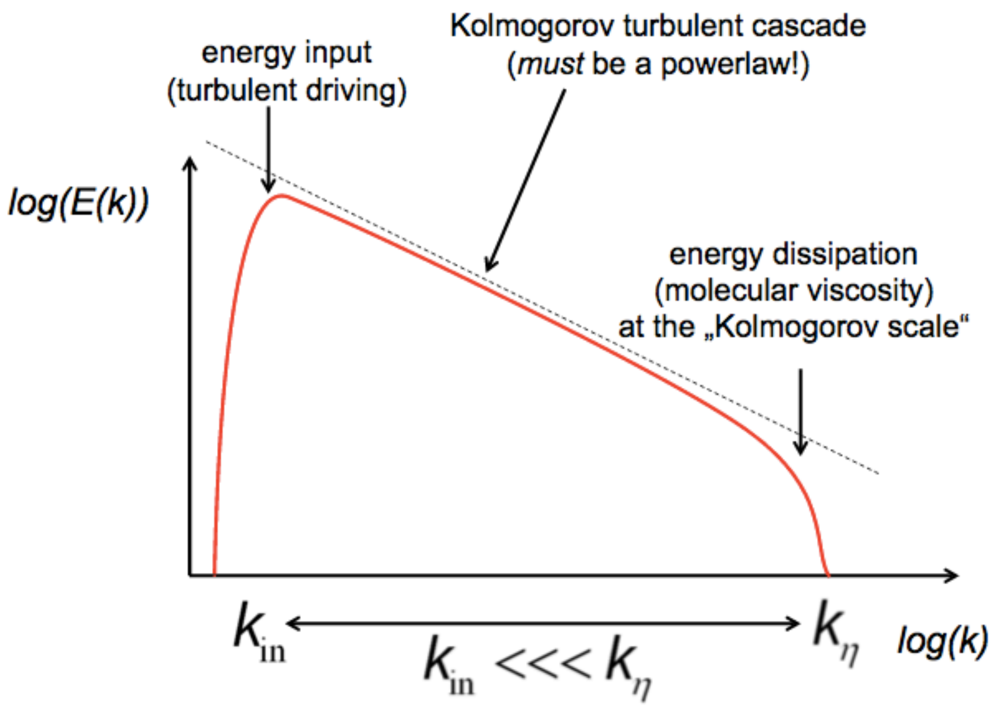
" Big whorls have little whorls
That feed on their velocity,
And little whorls have lesser whorls
And so on to viscosity"

Lewis Fry Richardson (1920)

The Power Spectrum

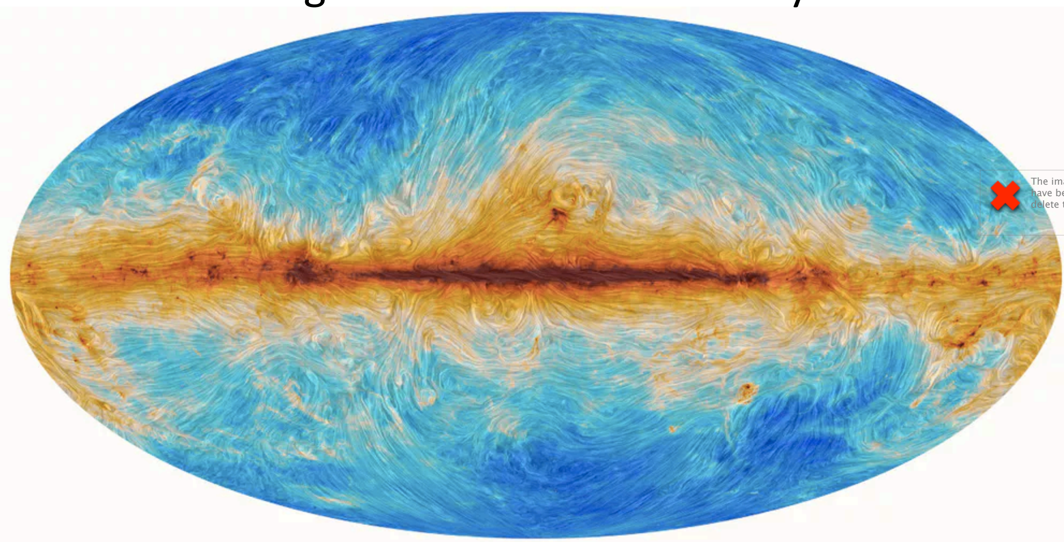
Fourier amplitude (no phase information) as a function of spatial scale.

$$P(\vec{k}) = \sum_{\vec{k}=\text{const.}} \tilde{A}(\vec{k}) \cdot \tilde{A}^*(\vec{k}). \qquad k = \frac{2\pi}{l}$$

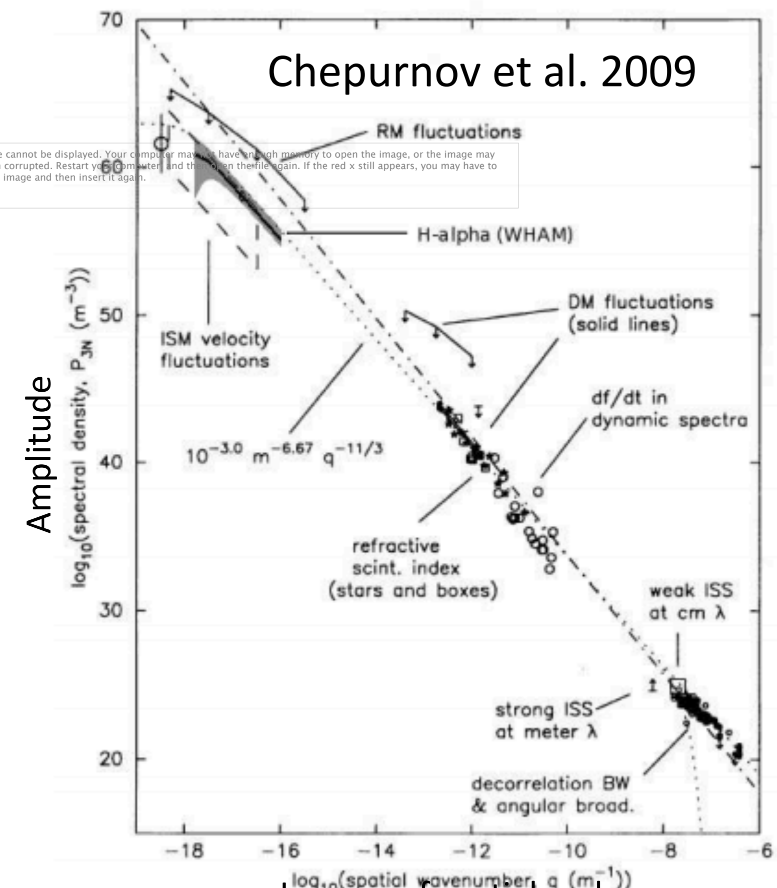


Galaxy is Magnetized: “Magneto”hydrodynamic Turbulence

The Galactic magnetic field as revealed by *Planck*



ESA and the Planck Collaboration



Power law in density fluctuations over ~10 orders of magnitude in scale

Kolmogorov turbulence

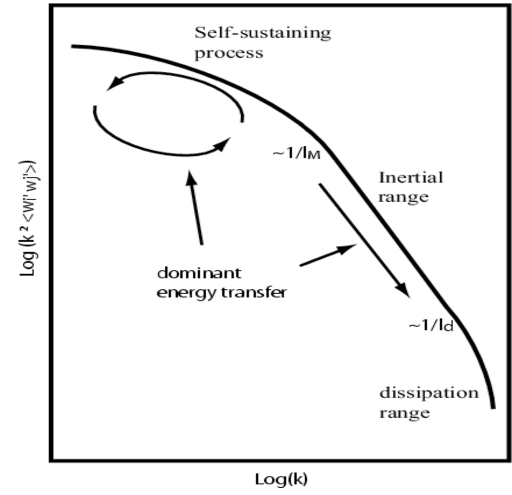
Driving turbulence with energy input ε [erg/gram.s]

For scales $k_{\rm in} < k < k_\eta$ i.e. $l_{\rm in} > l > l_\eta$ we can use dimensional analysis to get the powerlaw slope. The question is: What combination of *k* and ε gives *E*? Dimensions (using erg=gram cm^2 s^{-2}):

$$k = \frac{1}{\text{length}} \qquad \varepsilon = \frac{\text{length}^2}{\text{time}^3} \qquad E(k) = \frac{\text{length}^3}{\text{time}^2}$$

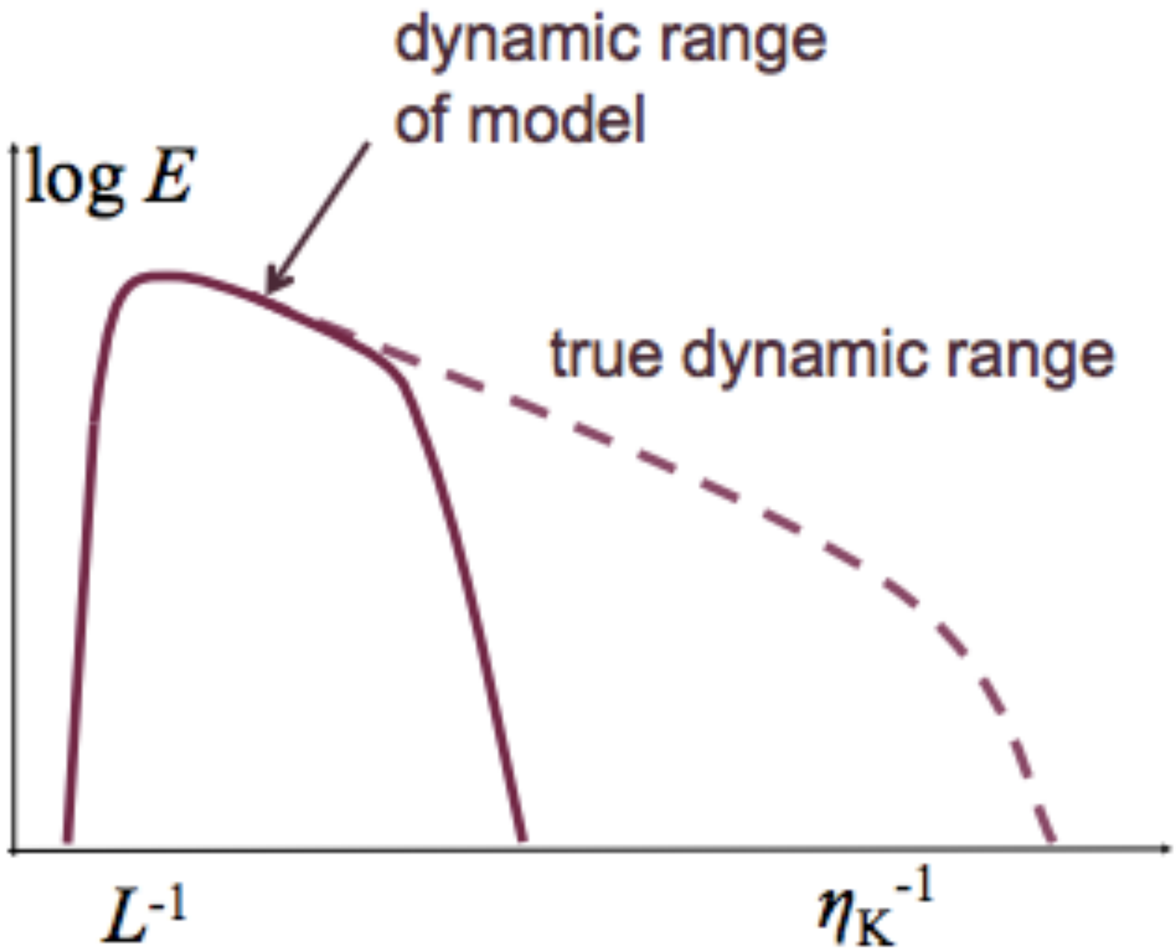
Only possible combination with the right dimensions:

$$\boxed{E(k) \propto \varepsilon^{2/3} k^{-5/3}}$$



caveat emptor: this theory only applies to incompressible unmagnetized fluids

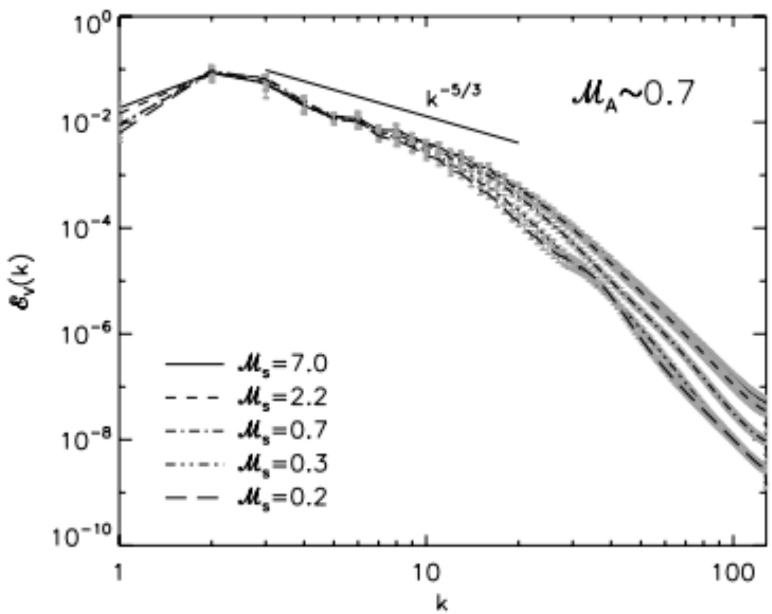
Limitations of simulations



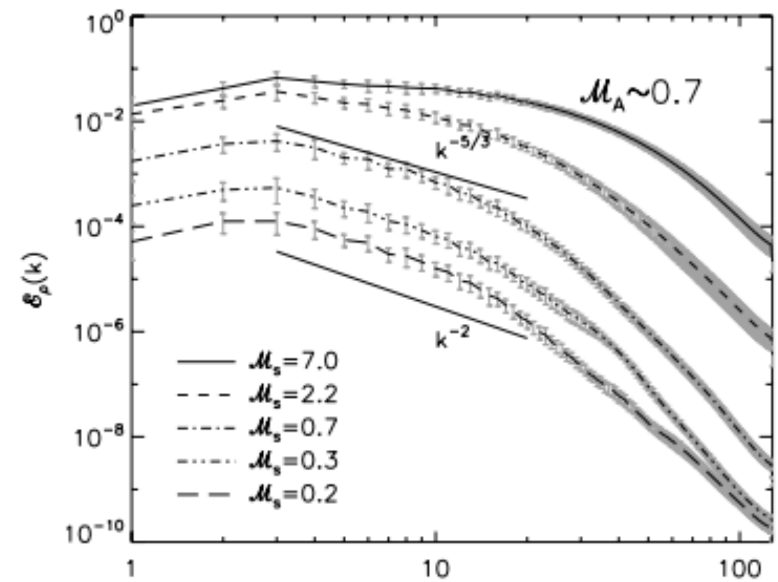
Supersonic Power Spectra

$$M_s \equiv \frac{V_L}{V_s}$$

Velocity Power Spectrum



Density Power Spectrum

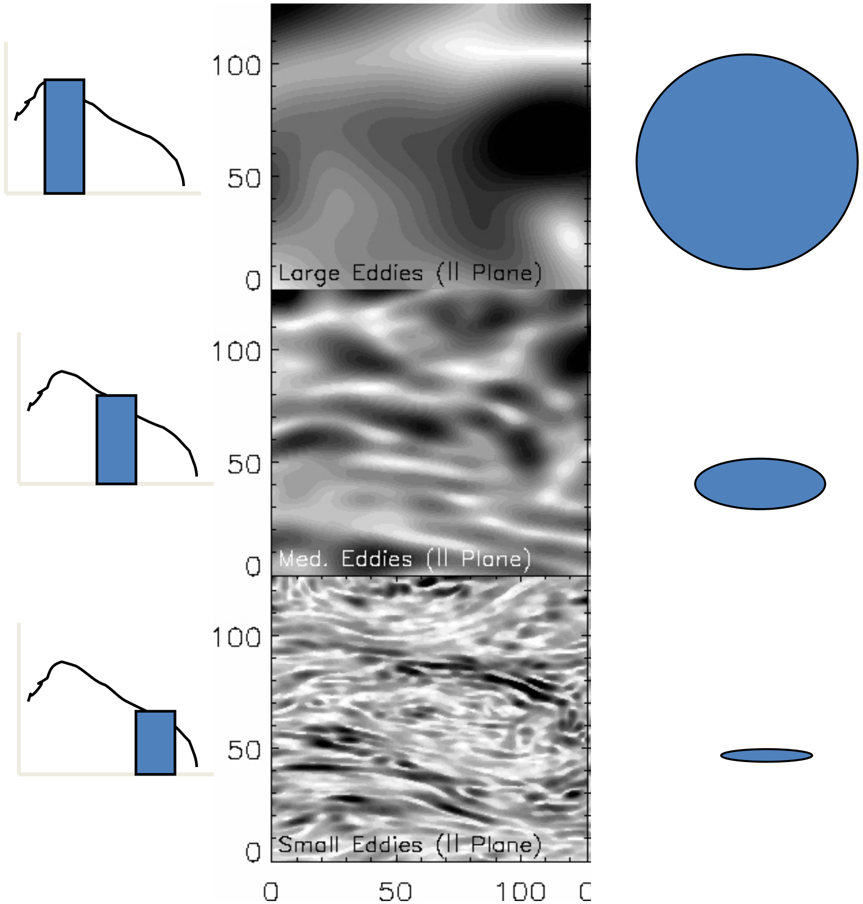


Kowal & Lazarian 2010
Burkhart et al. 2010

Fleck 1996 model of compressible turbulence: velocity steepens and density shallows relative to the incompressible kolmogorov slope

$$E(k) \sim k^{-5/3-2\alpha}.$$

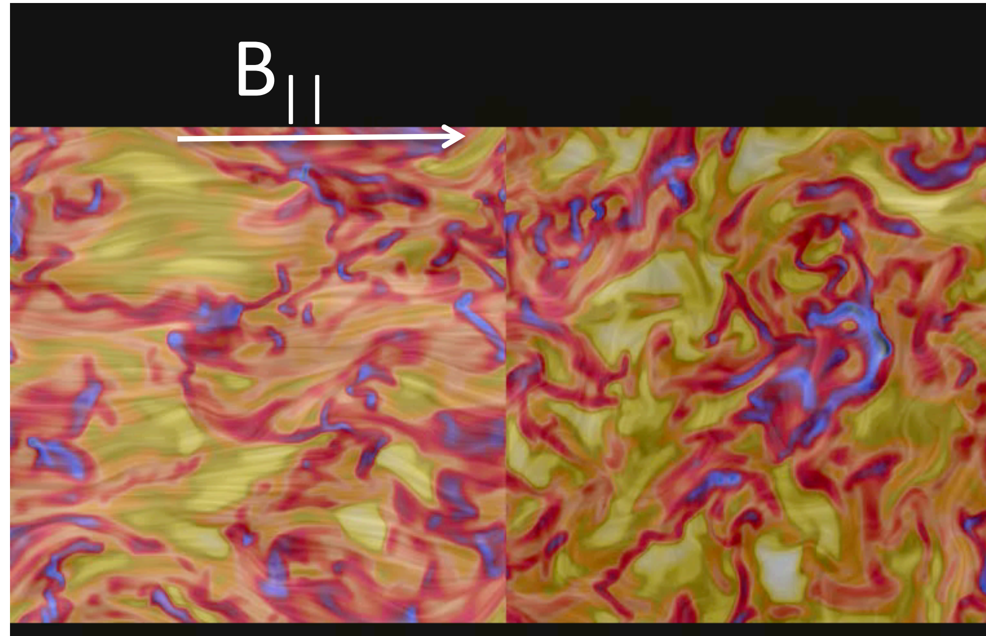
Magnetohydrodynamic Turbulence



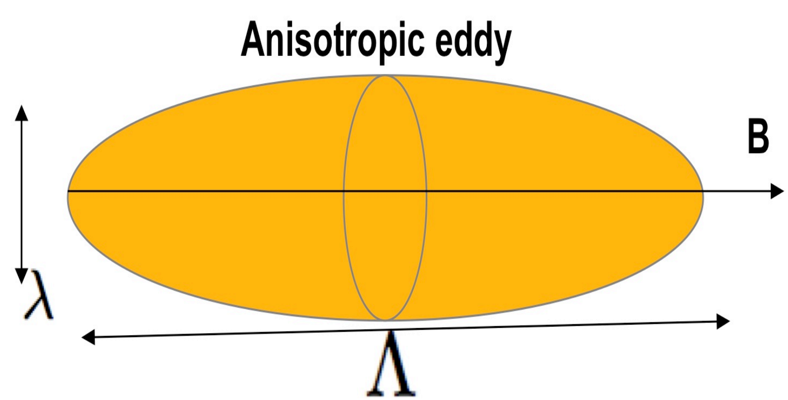
Magnetic field

B_0

Goldreich & Sridhar 1995, Cho et al. 2002

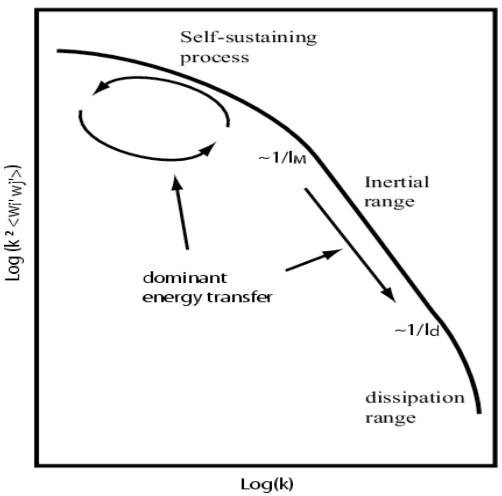


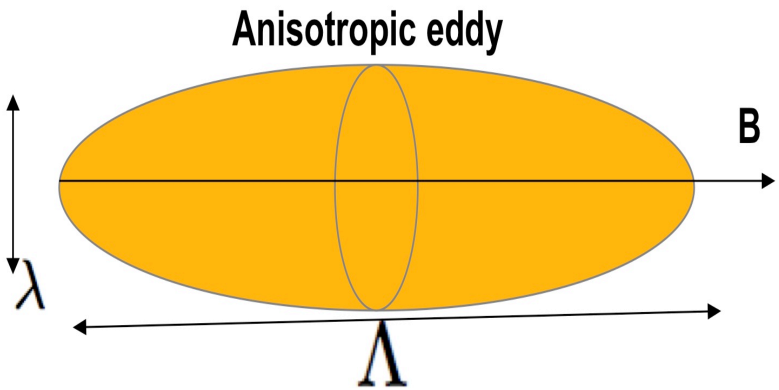
Density Field MHD Simulation, Burkhart et al. 2009

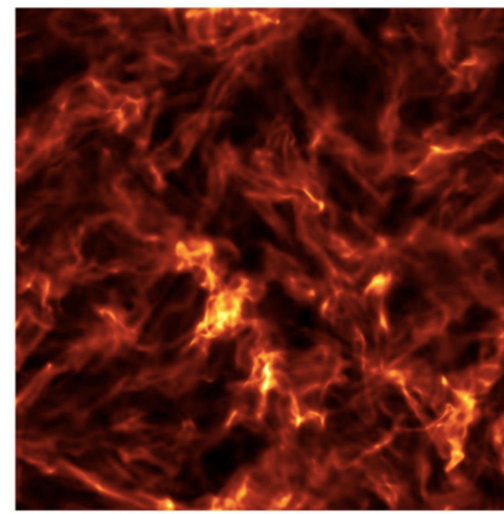


What should we measure?

$E(k)$ $\qquad$ $M_A = V/V_A$ $\qquad$ $M_s = V/c_s$







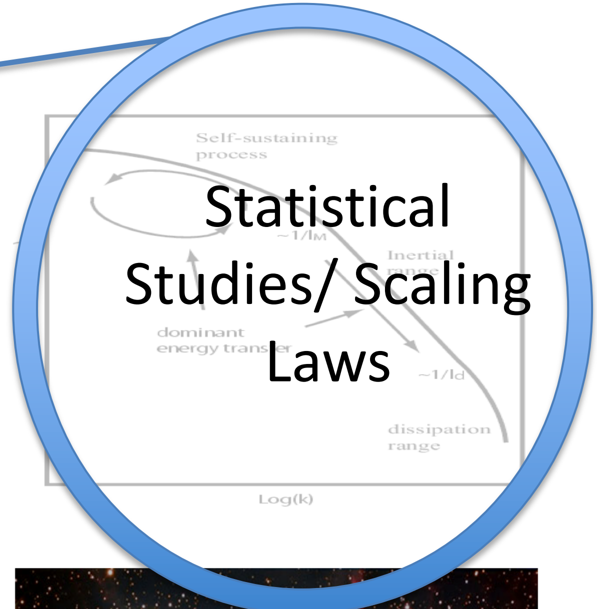
Summary for incompressible MHD

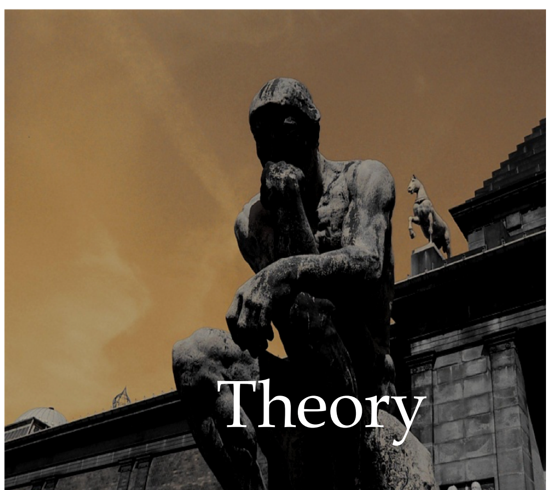
- Spectrum: $E(k) \sim k^{-5/3}$
- Anisotropy: $l_{||} \sim l_{\perp}^{2/3}$
- Theory: Goldreich-Sridhar (1995)

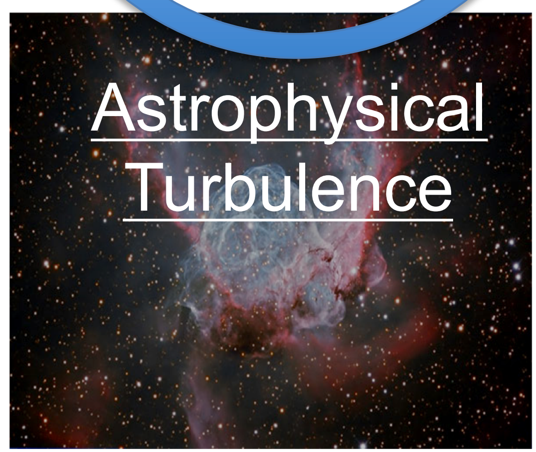
  Numerical test: Cho-Vishniac (2000)

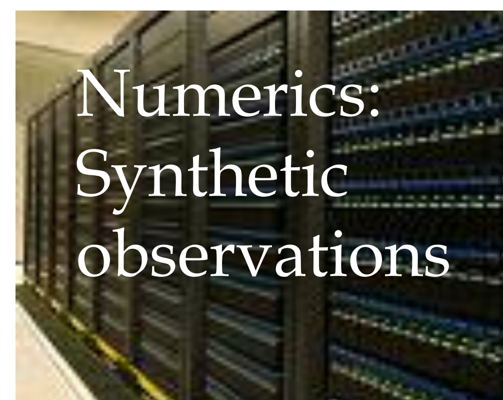
How should we measure properties of turbulence?

Tool box for studying MHD turbulence:









Turbulence Statistics and their Dependencies