Practical Programming and Numerical Methods

Exam problem

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Among the list of exam problems, the one chosen was number 16: "Classical 4th order Runge-Kutta ODE stepper with step-doubling error estimate". We were told to implement the 4th order classical Runge-Kutta method using the step-doubling error estimate, which is also known as the *Runge's principle*. The 4th order Runge-Kutta method, hereafter RK4, has the following **k** values:

$$\begin{aligned}
\mathbf{k}_0 &= \mathbf{f}\left(x_0, \mathbf{y}_0\right) \\
\mathbf{k}_1 &= \mathbf{f}\left(x_0 + \tfrac{h}{2}, \mathbf{y}_0 + \tfrac{1}{2}h\mathbf{k}_0\right) \\
\mathbf{k}_2 &= \mathbf{f}\left(x_0 + \tfrac{h}{2}, \mathbf{y}_0 + \tfrac{1}{2}h\mathbf{k}_1\right) \\
\mathbf{k}_1 &= \mathbf{f}\left(x_0 + h, \mathbf{y}_0 + h\mathbf{k}_2\right) \\
\mathbf{k} &= \frac{\mathbf{k}_0 + 2\mathbf{k}_1 + 2\mathbf{k}_2 + \mathbf{k}_3}{6}
\end{aligned} \tag{1}$$

Then, the step towards the solution is given as:

$$\mathbf{y}_{i+1} = \mathbf{y}_i + h\mathbf{k} \tag{2}$$

where h is the step size. The step size chosen was

$$h = \frac{b-a}{20} \tag{3}$$

where a and b are the initial and final values of the interval of interest, respectively.

The step-doubling error estimate consists in comparing the solution obtained with one full step integration to the one obtained with two consecutive half-step integrations. These half-step integrations were calculated as follows

$$\begin{aligned}
\mathbf{k}_0 &= \mathbf{f}\left(x_0, \mathbf{y}_0\right) \\
\mathbf{k}_1 &= \mathbf{f}\left(x_0 + \tfrac{h}{4}, \mathbf{y}_0 + \tfrac{1}{4}h\mathbf{k}_0\right) \\
\mathbf{k}_2 &= \mathbf{f}\left(x_0 + \tfrac{h}{4}, \mathbf{y}_0 + \tfrac{1}{4}h\mathbf{k}_1\right) \\
\mathbf{k}_1 &= \mathbf{f}\left(x_0 + \tfrac{h}{2}, \mathbf{y}_0 + \tfrac{1}{2}h\mathbf{k}_2\right) \\
\mathbf{k} &= \frac{\mathbf{k}_0 + 2\mathbf{k}_1 + 2\mathbf{k}_2 + \mathbf{k}_3}{12}
\end{aligned} \tag{4}$$

And the step towards the solution is

$$\mathbf{y}_{i+1} = \mathbf{y}_i + 2h\mathbf{k} \tag{5}$$

In the Runge's principle, we assume that if h is small, we can write the errors of the full step and the two half steps as such:

$$\delta \mathbf{y}_{\text{full\_step}} = Ch^{p+1} \tag{6}$$

$$\delta \mathbf{y}_{\text{two\_half\_steps}} = \frac{Ch^{p+1}}{2^p} \tag{7}$$

where p is the order of the method and C is an unknown constant. Omitting some intermediate steps, we find that the error of our integration can be calculated as

$$\delta \mathbf{y} = \frac{\mathbf{y}_{\text{full\_step}} - \mathbf{y}_{\text{two\_half\_steps}}}{2^p - 1} \tag{8}$$

We are not going to go into details on how the method works. To check if our implementation worked, we solved the ordinary differential equation

$$y'' = -y \tag{9}$$

with the initial conditions $y(a) = 0$ and $y'(a) = 1$. To show the result we made a plot:

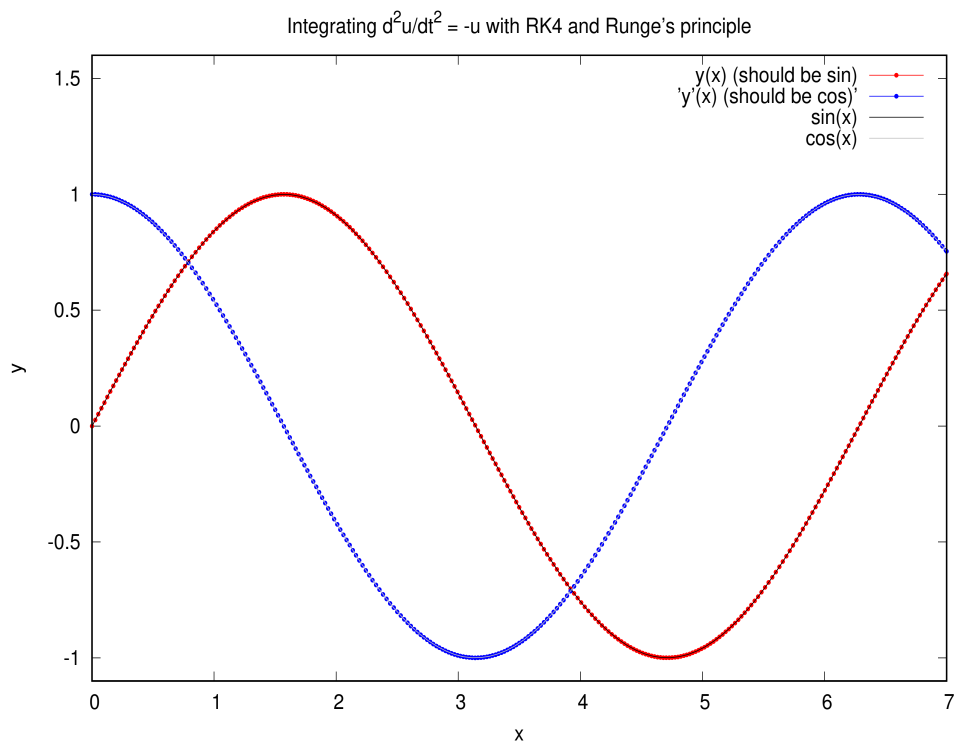

Figure 1: Plot with the solution of the ODE given by equation (9).

We also tested our implementation as an integral calculator, and found that it was in agreement with the exact known results. These results are shown in a .txt file in attachment with this pdf and also in a repository on github, with the link below.

Link for github repository with all the assignments:

`https://github.com/MarceloAron/PPNM`