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DYNAMICAL SYSTEMS AND OPTIMAL CONTROL

FOR ECONOMISTS

DEDICATION

To my sister Ana Sofía

PREFACE

This book is the outcome of several years of teaching the courses Mathematics for Economists and Mathematics for Control Theory in both the Departments of Economics and Mathematics at the Pontificia Universidad Católica del Perú. It is intended for undergraduate and graduate students in economics, applied mathematics, and engineering with prior training in calculus and linear algebra. Additionally, it provides a valuable reference for instructors and researchers interested in dynamic economic modeling and optimal control theory.

Our book provides an accessible yet rigorous introduction to continuous-time dynamical systems and optimal control theory, emphasizing economic applications throughout. It is particularly notable for including more than 150 carefully structured exercises, distributed across the problem sets at the end of each section, often accompanied by hints to guide learning, reinforce understanding, and evaluate student progress. This extensive set of exercises also serves as an asset for instructors seeking comprehensive and practical teaching materials.

The book features a balanced mix of illustrative examples, both with and without economic context. These examples are diverse in content and carefully selected to clarify and reinforce theoretical concepts by showing their practical relevance. Each mathematical concept introduced is developed alongside one or more economic applications, ensuring that the theory is always grounded in concrete and meaningful contexts. Economic models are systematically integrated throughout the chapters, ranging from classical population dynamics and dynamic macroeconomic models to modern growth theory. This variety allows mathematics students interested in economics to engage deeply with applied topics, while economics students in search of formal and theoretical rigor will find the material both accessible and enriching. Many of the examples are based on seminal contributions, and even some inspired by recent publications from the early 21st century, particularly in the field of economic growth.

There is a strong emphasis on visual intuition and graphical analysis, supported by abundant, carefully crafted figures created using `TikZ`, MATLAB, GeoGebra, and Wolfram Mathematica. Phase diagrams are prominently included to facilitate a clear understanding of complex dynamic behaviors.

Our focus on continuous-time analysis allows for a coherent and self-contained exposition. To support this, we include a concise appendix that reviews essential topics from single-variable analysis and multivariable calculus. Virtually the entire book is accessible to students with a basic background in calculus and linear algebra. Exceptions include Sections 2.2, 3.5, 4.4, and 6.3, which are marked as optional or more appropriate for advanced or graduate-level readers. Moreover, the inclusion of less-standard topics—such as bifurcation theory, Goodwin's cyclical model, limit cycles, and economic growth models—sets this book apart from others in the current literature and broadens its relevance for students and researchers interested in modern applications of dynamical systems to economics.

The book is structured into six chapters:

- Chapter 1 introduces real dynamical systems via first-order differential equations, covering fundamental methods like separation of variables and Bernoulli equations, along with classic economic models including Domar's, Malthus, Verhulst, and Walrasian price adjustment.

- Chapter 2 extends the discussion to a general qualitative analysis of one-dimensional systems, addressing topics such as Solow's model, bifurcation theory, a labor market model, and numerical methods like Euler and Runge-Kutta.

- Chapter 3 delves into higher-dimensional linear systems, with a particular emphasis on two-dimensional cases, as they capture the core intuition of the general framework. We introduce matrix exponentials and the Jordan canonical form as practical tools for solving linear systems. The chapter also examines the structure of stable and unstable subspaces, laying the theoretical foundation for the nonlinear systems and optimal control theory addressed later in the book—especially in view of economic applications where these tools are crucial for formulating and analyzing policy dynamics.

- Chapter 4 addresses higher-dimensional nonlinear systems, focusing on the two-dimensional case through the application of the Hartman–Grobman Theorem. This framework allows for the analysis of models such as species competition and Goodwin's cyclical growth model. Economic systems exhibiting limit cycles are also explored, alongside the principal results of the theory and models that naturally fit this context, including Kaldor's model and the dual cross adjustment mechanism.

- Chapters 5 and 6 focus explicitly on dynamic optimization. Chapter 5 covers calculus of variations, providing foundational techniques, while Chapter 6 presents continuous-time optimal control theory, exploring economic applications such as the Ramsey–Cass–Koopmans model, AK growth model, and Tobin's Q.

- Two appendices provide concise reviews of single-variable and multivariable calculus. Familiarity with basic calculus, linear algebra, and finite-dimensional real analysis is recommended to fully appreciate the text.

This work was motivated by influential textbooks that have shaped the study of dynamical systems and economic modeling, including, for instance, Sydsaeter and Seierstad (1987), Léonard and Long (1981), Barro and i Martin (2003), Gandolfo (2009), Perko (2013), Acemoglu (2009), and Sethi (2019).

We believe this book fills a gap in the current literature by offering a dedicated and coherent treatment of continuous-time dynamical systems and optimal control theory specifically applied to economics. We also believe that is suitable for courses on dynamic macroeconomics, mathematical methods in economics, applied differential equations, and optimal control theory. It accommodates both one- and two-semester course formats, depending on instructional depth and pacing preferences. The accessibility of the material, combined with varied exercises and examples, ensures usefulness across different academic levels.

We hope this book will be highly useful for undergraduate and graduate students in economics, applied mathematics, engineering, physics or biology, as well as for instructors and researchers working in these areas.

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Symbols

The following is a brief list of the symbols which have been used in this book.

- $\mathbb{N}$: Set of natural numbers, $\mathbb{N} \triangleq \{1,2,\dots\}$.
- $\mathbb{Z}$: Set of integer numbers, $\mathbb{Z} \triangleq \{\dots,-3,-2,-1,0,1,2,3,\dots\}$.
- $\mathbb{Z}_0^+$: Set of positive integers including zero, $\mathbb{Z}_0^+ \triangleq \{0,1,2,3,\dots\}$.
- $\mathbb{R}$: Set of real numbers.
- A^c : If A is a set, A^c denotes its complement.
- $A \subset B$: Set A is included in set B.
- $\mathbb{R}^n$: Euclidean space of dimension $n \in \mathbb{N}$.
- $\|\mathbf{x}\|$: If $\mathbf{x}$ is a vector of a vector space V, $\|\mathbf{x}\|$ denotes its norm.
- $\mathcal{B}(\mathbf{x}_0, \epsilon)$: Open ball with center $\mathbf{x}_0$ and radius $\epsilon > 0$.
- $\mathcal{N}_\mathbf{x}$: Neighborhood of point $\mathbf{x}$, i.e., a set containing an open ball centered at $\mathbf{x}$.
- $\sup A$: Supremum of the set A.
- $\preceq$: Order relation.
- ∂A : Boundary of the set A.

- $x \cong y$: x almost equal to y.
- $\mathcal{M}_{n\times n}(\mathbb{R})$: Vector space of matrices of order $n \times n$, $n \in \mathbb{N}$, with real entries.
- $\mathrm{Ker}(A)$: Kernel of A.
- $|A| \equiv \det(A)$: Determinant of A.
- $\mathrm{tr}(A)$: Trace of A.
- $\mathrm{Span}(A)$: If A is a set in a vector space V, $\mathrm{Span}(A)$ denotes the space generated by A.
- $C^n(A)$: Function space of functions with n-th continuous derivative on A.
- $\sigma(A)$: Set of eigenvalues of matrix A, also known as *spectrum*.

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