Exercises in Economic Theory

Consumer and producer theory

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2024-1

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Contents

1 Preferences

We mainly follow for this section [1] and [4].

Definition 1. Given a set X, a preference $\succeq$ over X is a binary relation such that, for any $x, y \in X$

$$\underbrace{x \succeq y}_{x \text{ is at least as good a } y}.$$

From $\succeq$ we derive two other important relations on X:[1]

1. The strict preference relation $\succ$ defined by

$$\underbrace{x \succ y}_{x \text{ is preferred to } y} \Leftrightarrow x \succeq y \text{ but not } y \succeq x.$$

2. The indifference relation $\sim$ defined by

$$\underbrace{x \sim y}_{x \text{ is indifferent to } y} \Leftrightarrow x \succeq y \text{ and } y \succeq x.$$

Definition 2. We say that $\succeq$ is rational if it is

1. Complete: $\forall\ x, y \in X$, $x \succeq y$ or $y \succeq x$.
2. Transitive: if $x \succeq y$ and $y \succeq z$, then $x \succeq z$.

1. Prove that if $\succeq$ is rational, then

1. $\succ$ is both irreflexive ($x \succ x$ never holds) and transitive ($x \succ y$ and $y \succ z$ imply $x \succ z$).
2. $\sim$ is reflexive ($x \sim x$ for all x) and transitive ($x \sim y$ and $y \sim z$ imply $x \sim z$).
3. $x \succ y \succeq z$ then $x \succ z$.

Definition 3. A function $u : X \to \mathbb{R}$ is a utility function representing the

[1] Some sources, such as Federico Echenique's lecture notes, start defining a binary relation and call a preference relation a complete and transitive binary relation.

preference relation $\succeq$ if for all $x, y \in X$

$$x \succeq y \Leftrightarrow u(x) \geq u(y).$$

Note that a utility function that represent $\succeq$ is not unique. Given any strictly increasing function $f : \mathbb{R} \to \mathbb{R}$, $v(x) = f(u(x))$ is a new utility function representing the same preference. You may prove this.

2. Prove that a preference $\succeq$ can be represented by a utility function only if it is rational.

3. If u representes $\succeq$ and f is just increasing (not strictly), does $f \circ u$ is necessarily a utility function representing $\succeq$?

4. Consider a rational preference relation $\succeq$. Show that if $u(x) = u(y)$ implies $x \sim y$ and $u(x) > u(y)$ implies $x \succ y$, then u represents $\succeq$.

5. Proponga una función que represente las preferencias del siguiente enunciado: «Una persona nunca come pan solo. Siempre lo acompaña con mermelada, pero cuando no hay mermelada, usa mantequilla».

2 Consumer choice

We mainly follow for this section [1] and [4].

The number of commodities will be L and will be indexed by $\ell = 1, ..., L$. A commodity vector is $x = [x_1, \cdots, x_L]^T \in \mathbb{R}^L_+$.

Definition 4. Consumption set:

$$X = \mathbb{R}^L_+ = \{x \in \mathbb{R}^L : x_\ell \geq 0, \ \forall \ \ell = 1, ..., L\}.$$

1. Prove that X is convex.

2. Explain why the «classical» budget set is given by

$$B(p, w) = \left\{ x \in \mathbb{R}^L_+ : p \cdot x = \sum_{\ell=1}^{L} p_\ell x_\ell \leq w \right\}.$$

Here $p \in \mathbb{R}^L_{++}$ is the price of the commodities and I the income.

<u>Note:</u> $B(p, w)$ is also known as Walrasian set.

3. Draw the Walrasian set for

a) $w = 2$, $p_1 = 1$, $p_2 = 4$.

b) $w = 1$, $p_1 = p_2 = 2$ and $p_3 = 5$.

4. Prove (for the general case) that the Walrasian set is convex[2] and compact[3].

Definition 5. The consumer's Walrasian (or ordinary) demand correspondence[4] $x(p, w)$ assigns a set of chosen consumption bundles for each price-wealth pair (p, w).

Definition 6. A Walrasian demand correspondence $x(p, w)$ is homogeneous of degree one if $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and $\alpha > 0$.

Definition 7. A Walrasian demand correspondence $x(p, w)$ satisfies Walras Law if $p \cdot x = I$ for every $x \in x(p, w)$.

5. Suppose $L = 3$ and

$$
\begin{aligned}
x_1(p, I) &= \frac{p_2}{\sum_{i=1}^3 p_i} \frac{w}{p_1} \\
x_2(p, I) &= \frac{p_3}{\sum_{i=1}^3 p_i} \frac{w}{p_2} \\
x_3(p, I) &= \frac{\beta p_1}{\sum_{i=1}^3 p_i} \frac{w}{p_3}.
\end{aligned}
$$

Analyze for which values of $\beta \in [0, 1]$ the Walrasian demand satisfies Walras Law and degree one homogeneity.

Hereafter a little break to study (smooth) comparative statics. Do not confuse with Monotone Comparative Statics. The wealth effect are represented as follows

$$
D_w x(p, w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial w_1} \\ \frac{\partial x_1(p,w)}{\partial w_2} \\ \vdots \\ \frac{\partial x_1(p,w)}{\partial w_n} \end{bmatrix} \in \mathbb{R}^L.
$$

---

[2] $\forall\ x_1, x_2 \in B(p, w)$ and $\theta \in [0, 1]$, $\theta x_1 + (1 - \theta)x_2 \in B(p, w)$.

[3] Closed and bounded under the usual topology of $\mathbb{R}^L$. See [4].

[4] A correspondence is a «point to set» map. This is, $\Gamma : X \to Y$ is a correspondence if for every $x \in X$, $\Gamma(x) \in 2^Y$.

On the other hand, the price effects, conveniently represented through a matrix, is

$$D_p x(p,w) = \begin{bmatrix} \frac{\partial x_1(p,w)}{\partial p_1} & \frac{\partial x_1(p,w)}{\partial p_2} & \cdots & \frac{\partial x_1(p,w)}{\partial p_L} \\ \frac{\partial x_2(p,w)}{\partial p_1} & \frac{\partial x_2(p,w)}{\partial p_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial x_L(p,w)}{\partial p_1} & \cdots & & \frac{\partial x_L(p,w)}{\partial p_L} \end{bmatrix}.$$

Proposition 8. If the Walrasian demand function $x(p,w)$ is homogeneous of degree zero, then for all p and w

$$\sum_{k=1}^{L} \frac{\partial x_\ell(p,w)}{\partial p_k} p_k + \frac{\partial x_\ell(p,w)}{\partial w} w = 0, \ \forall \ \ell = 1, ..., L. \tag{1}$$

Assume differentiability. In matrix notation,

$$D_p x(p,w)p + D_w x(p,w)w = 0.$$

Remark. Equation 1 means that increasing all prices (both good prices and wealth), summing and weighting with the prices, gives zero: no effect.

6. Obtain (1).

Hint: differentiate with respect to α: $x(p,w) = x(\alpha p, \alpha w)$.

Let

$$\varepsilon_{\ell k}(p,w) = \frac{\partial x_\ell(p,w)}{\partial p_k} \frac{p_k}{x_\ell(p,w)}$$
$$\varepsilon_{\ell w}(p,w) = \frac{\partial x_\ell(p,w)}{\partial w} \frac{w}{x_\ell(p,w)}.$$

These elasticities give the percetange change in demand for good ℓ per (marginal) percentage change in the price of good k or wealth,

$$\varepsilon_{\ell w} = \frac{\Delta x}{x} \frac{w}{\Delta w}.$$

7. Using elasticities, re-escribe (1).

Proposition 9. If the Walrasian demand function $x(p,w)$ satisfies Walras law, then for all p, w

$$\sum_{\ell=1}^{L} p_\ell \frac{x_\ell(p,w)}{\partial p_k} + x_k(p,w) = 0, \ \forall \ k = 1, ..., L. \tag{2}$$

8. Derive Equation 2, also known as Cournot aggregation. Interpret[5].

Hint: derive $p \cdot x = w$ with respect to p_k.

9. Prove Euler aggregation equation:

$$\sum_{\ell=1}^{L} p_\ell \frac{\partial x_\ell}{\partial w} = 1.$$

Hint: derive $p \cdot x = w$ with respect to w.

3 Weak Axiom of Revealed Preference (WARP)

Definition 10. The Walrasian demand function[6] $x(p, w)$ satisfies the weak axiom of revealed preference if the following property holds for any two prices-wealth situations (p, w) and (p', w')

$$p \cdot x(p', w') \leq w \text{ and } x(p', w') \neq x(p, w) \implies p' \cdot x(p, w) > w'. \tag{3}$$

1. Interprete (3) Note that $x(p', w')$ was available for the price-wealth configuration (p, w) and was not chosen. Hence, if we have $p'x(p, w) \leq w'$, $x(p, w)$ is available, and so is $x(p', w')$: it is logical to chose $x(p', w')$?.

WARP has significant implications for the effects of price changes on demand. Note that price changes affect the consume in two ways. First, they alter the relative cost of different commodities. But, second, they also change consumer's real wealth.

Proposition 11. Suppose thaty the Walrasian demand function $x(p, w)$ is homogeneous of degree zero and satisfies Walras law. Then, $x(p, w)$ satisfies the weak axiom if and only if the following property holds: for any compensated price change from an initial situation (p, w) to a new price-wealth pair $(p', w') = (p', p' \cdot x(p, w))$, we have

$$(p' - p)[x(p', w') - x(p, w)] \leq 0.$$

2. Prove Proposition 11.

[5] Total expenditure can change in response to a change in prices?

[6] Let us assume for simplicity that we deal with functions and no with correspondences.

3. Consider the consumption of a consumer in two different periods, period 0 and period 1. Period t prices, wealth and consumption are $p^t, w^t, x^t(p^t, w^t)$. The Laspeyres quantity index computes the change in quantity using period 0 prices as weights:

$$L_Q = \frac{p_0 \cdot x^1}{p_0 \cdot x^0}$$

while Paasche quantity index wieghts using prices in period 1:

$$P_Q = \frac{p_1 \cdot x^1}{p_1 \cdot x^0}.$$

Finally, the consumer's expenditure change is just $E_Q = \frac{p^1 \cdot x^1}{p^0 \cdot x^0}$.

1. Prove that, if $L_Q < 1$, then the consumers reveals preference for x^0 over x^1.

2. Prove that, if $P_Q > 1$, then the consumers reveals preference for x^1 over x^0.

3. We cannot conclude if $E_Q < 1$ or $E_Q > 1$ (not enough information).

4. Consider the following Walrasian demand:

$$x_k(p, w) = \frac{w}{\sum_{\ell=1}^{L} p_\ell}, \ \ell = k, ..., L.$$

Awnser the following item:

1. Is the demand homogeneous of degree 0 in (p, w)?

2. Does it satisfies Walras Law?

3. Does it satisfies WARP?

4 Classical Demand Theory

We mainly follow for this section [1], [2] and [4].

Definition 12. The preference relation $\succeq$ is rational on $X \subset \mathbb{R}^L$ if it possesses the following two properties:

1. Completeness: $\forall\, x, y \in X$, $x \succeq y$ or $y \succeq x$.

2. Transitivity: $\forall\, x, y, z \in X$, $x \succeq y$ and $y \succeq z$ implies $x \succeq z$.

Definition 13. he preference relation $\succeq$ is monotone on $X \subset \mathbb{R}^L$ if $x \in X$ and $y > x$ (strict inequality in each entry) implies $y \succ x$. It is strongly monotone if $y \geq x$, $y \neq x$ implies $y \succ x$.

1. Prove that, if $u : \mathbb{R}^L_+ \to \mathbb{R}$, C^1, represents $\succ$ and $\succ$ is strongly monotone, then $\frac{\partial u}{\partial x_i} > 0$.

Definition 14. The preference relation on X, $\succeq$ is locally nonsatiated if for every $x \in X$ and every $\varepsilon > 0$, there is $y \in X$ such that $||y - x|| \leq \varepsilon$ and $y \succ x$.

2. Prove that if $\succeq$ is monotone, then it is locally nonsatiated.

Definition 15. Given the preference relation $\succeq$ and a consumption bundle x, we can define three related sets of consumptions bundles. The indifference set containing point x is the set $\{y \in X : x \sim y\}$. The upper contour is $\{y \in X : y \succeq x\}$ and the lower contour is $\{y \in X : x \succeq y\}$.

Definition 16. The preference relation $\succeq$ on X is convex if for every $x \in X$ the upper contour set $\{y \in X : y \succeq x\}$ is convex: that is, $y \succeq x, z \succeq x$, then

$$\theta y + (1 - \theta) z \succeq x, \ \forall\, \theta \in [0, 1].$$

3. Prove that if $\succeq$ is convex and u represents $\succeq$, then u is quasi-concave.

Definition 17. The preference relation $\succeq$ on X is strictly convex if for every $x, y, z \in X$ such that $y \succeq x, z \succeq x$, then

$$\theta y + (1 - \theta) z \succ x, \ \forall\, \theta \in [0, 1].$$

Definition 18. A monotone preference relation $\succeq$ on $X = \mathbb{R}^L_+$ is homothetic if all indifference sets are related by proportional expansion along rays: that is, if $x \sim y$ then $\alpha x \sim \alpha y$ for all $\alpha \geq 0$.

Definition 19. The preference relation $\succeq$ on

$$X = (-\infty, \infty) \times \mathbb{R}^{L-1}_+$$

is quasilinear with respect to commodity 1 if

1. All the indifference sets are parallel displacement of each other along the axis of commodity 1. That is, $x \sim y$ then $x + \alpha e_1 \sim y + \alpha e_1$, $e_1 = (1, 0, \cdots, 0)$, and any $\alpha \in \mathbb{R}$.

2. Good 1 is desirable: $x + \alpha e_1 \succ x$ for all x and $\alpha > 0$.

Remark. From now, we assume that all preferences are rational.

Definition 20. The preference relation $\succeq$ on X is continuous if it is preserved under limits. That is, for any sequence of pairs $\{(x^n, y^n)\}_{n\in\mathbb{N}}$ with $x^n \succeq y^n$, for all $n \in \mathbb{N}$, $x = \lim_n x^n$, $y = \lim_n y^n$, we have $x \succeq y$.

4. Not easy: prove that $\succeq$ is continuous if and only if $\{y \in X : x \succeq y\}$ is closed.

5. Define the Lexicographic preference. Then, prove that it is rational but it is not continuous.

Hint: consider $(1/n, 0, ..., 0)$ and $(0, 1 + 1/n, 0, ..., 0)$.

Theorem 21. Suppose a rational preference relation on X is continuous. Then, there is a continuous utility function $u(x)$ that represents $\succeq$.

Proof. See [4] or [6] □

Remark. Differentiability if a much more complicated matter. See a discussion in [1] and think about Leontief preferences:

$$(x_1, x_2) \succeq (y_1, y_2) \Leftrightarrow \min\{x_1, x_2\} \geq \min\{y_1, y_2\}.$$

6. Consider a continuous preference relation $\succeq$ over $X = \mathbb{R}^L_+$ ($\mathbb{R} \times \mathbb{R}^{L-1}_+$ respectively). Prove that

1. $\succeq$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one: $u(\alpha x) = \alpha u(x)$.

2. $\succeq$ is quasi-linear with respect to the first commodity if and only if it admits a utility function $u(x)$ of the form

$$u(x_1, \cdots, x_n) = x_1 + \phi(x_2, \cdots, x_L).$$

Now we study in detail the **utility maximization problem**:

$$\mathcal{P}_u : \begin{cases} \max & u(x) \\ \text{s.t.} & p \cdot x \leq w \\ & x \geq 0. \end{cases}$$

The problem $\mathcal{P}_u$ will be, some times, named UMP.

7. Prove that, if u is continuous, $\mathcal{P}_u$ posses always a solution.

8. Explain carefully $\mathcal{P}_u$.

9. For $L = 2$, $p_1 = p_2 = 1$ and $I = 10$, solve the problem if $u(x_1, x_2) = x_1 + 2x_2$.

Definition 22. The Walrasian Demand Correspondence Function. The rule that assigns the set of optimal consumption vectors in the UMP to each price-wealth situation $(p, w) > 0$ is denoted by $x(p, w) \in \mathbb{R}^L_+$ and is known as the Walrasian demand correspondence.

Proposition 23. Suppose that $u(\cdot)$ is continuos utility function representing a loclaly nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. Then the Walrasian demand correspondence $x(p, w)$ possesses the following properties:

1. Homogeneity of degree zero in (p, w): $x(\alpha p, \alpha w) = x(p, w)$ for any p, w and scalar $\alpha > 0$.
2. Walras law: $p \cdot x = w$, for all $x \in x(p, w)$.
3. Convexity/uniqueness: if $\succeq$ is convex, so that $u(\cdot)$ is quasi-concave, then $x(p, w)$ is convex. Moreover, if u is strictly quasi-concave, $x(p, w)$ has a single element (unique solution).

10. Prove Proposition 23.

If $u(\cdot)$ is continuously differentiable, an optimal consumption bundle $x^* \in x(p, w)$ can be characterized in a very usesul manner by means of first order conditions. The Kuhn-Tucker (necessary) conditions:[7]

$$\frac{\partial u(x^*)}{\partial x_\ell} - \lambda p_\ell \leq 0, \ \forall \ \ell = 1, ..., L, \quad \text{with equality if } x^*_\ell > 0$$

[7]Prove that these are indeed the FOC.

$$x_\ell^* \left[\frac{\partial u(x^*)}{\partial x_\ell} - \lambda^* p_\ell \right] = 0$$
$$\lambda^*(w - px^*) = 0.$$

Together with $x_\ell^* \geq 0$ for all $\ell = 1, ..., L$.

11. Prove that if $u(\cdot)$ satisfies Inada conditions, $x_\ell^* > 0$.

12. Solve the UMP for

$$u(x_1, x_2) = \alpha \ln x_1 + (1-\alpha) \ln x_2, \;\; \alpha \in (0,1).$$

13. Prove that if $x_\ell^* > 0$ for all $\ell = 1, ..., L$, the optimality condition is

$$\frac{\frac{\partial u(x^*)}{\partial x_\ell}}{\frac{\partial u(x^*)}{\partial x_k}} = \frac{p_\ell}{p_k}.$$

14. Define

$$v(p, w) = \max_{x \geq 0, \; p \cdot x \leq w} u(x).$$

Prove that the indirect utility function $v : \mathbb{R}^L_{++} \times \mathbb{R}_+$ satisfies the following properties

a) Homogeneous of degree zero.

b) Strictly increasing in w and non increasing in p_ℓ.

c) Quasi-convex: $\{(p, w) : v(p, w) \leq \overline{v}\}$ is convex for all $\overline{v}$.

d) Continuous in p, w.

For (d) you may require a strong result known as Maximum Theorem. See [7]

Analogous to the UMP, we have the Expenditure Minimization Problem (EMP)

$$\mathcal{P}_e : \begin{cases} \min & p \cdot x \\ \text{s.t.} & u(x) \geq \overline{u} \\ & x \geq 0. \end{cases}$$

Proposition 24. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$ and that the price vector is $p > 0$. Thus

1. If x^* is the optimal in the UMP when $w > 0$, then x^* is optimal in the EMP when $\overline{u} = u(x^*)$. Moreover, $e(p, \overline{u}) = w$.

2. If x^* is optimal in the EMP when the required utility level is $\overline{u} > u(0)$, then x^* is optimal in the UMP when $w = p \cdot x^*$. Moreover, $v(p, w) = \overline{u}$.

15. Prove Proposition 24.

Proposition 25. Suppose $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. The expenditure function is

1. Homogeneous of degree one un p.

2. Strictly increasing in $\overline{u}$ and nondecreasing in p_ℓ for any $\ell = 1, ..., L$.

3. Concave in p.

4. Continuous in p and $\overline{u}$.

16. Prove Proposition 25.

Remark. It follows from our previous discussion that

$$e(p, v(p, w)) = w \text{ and } \ v(p, e(p, \overline{u})) = \overline{u}.$$

Definition 26. The Hicksian Compensated Demand is the set of optimal commodity in the EMP.

Proposition 27. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. Then, for any $p > 0$, the Hicksian demand correspondence $h(p, \overline{u})$ possesses the following properties:

1. Homogeneity of degree zero in p: $h(\alpha p, \overline{u}) = h(p, \overline{u})$ for all $\alpha > 0$ and for any $p, \overline{u}$.

2. No excess utility: for any $x \in h(p, \overline{u})$, $u(x) = \overline{u}$.

3. Convexity/uniqueness: if $\succeq$ is convex, then $h(p, \overline{u})$ is a convex set. If $\succeq$ is strictly convex, then there is a unique element in $h(p, \overline{u})$.

17. Prove Proposition 27.

18. Show that the FOC for the EMP are

$$p - \lambda \nabla u(x^*) \geq 0 \;\wedge\; x^*[p - \lambda u(x^*)] = 0,$$

for some $\lambda \geq 0$.

Remark. From our previous discussion, it follows that

$$h(p, \overline{u}) = x(p, e(p, \overline{u}))$$

and

$$h(p, v(p, w)) = x(p, w).$$

19. Solve the EMP for $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$, $\alpha \in (0, 1)$. Obtain the expenditure function.

Proposition 28. Compensated law of demand. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated preference relation $\succeq$ and $h(p, \overline{u})$ consists of a single element for all $p > 0$. Then, the Hicksian demand function $h(p, \overline{u})$ satisfies the compensated law of demand:

$$\forall\, p', p'' : \; (p' - p'') \cdot (h(p'', \overline{u}) - h(p', \overline{u})) \leq 0.$$

The following results are classical in consumer theory and have analogous results in producer theory. Their proof uses the classical Envelope Theorem [4].

Lemma 29. Shepard's Lema. Suppose that $u(\cdot)$ is a continuous utility function representing a preference locally non satiated and strictly convex preference relation $\succeq$ defined on $X = \mathbb{R}^L_+$. For all p and $\overline{u}$, the Hicksian demand $h(p, \overline{u})$ and the expenditure function satisfies the following relation

$$h(p, \overline{u}) = \nabla_p e(p, \overline{u}).$$

20. Prove Shepard's Lemma.

Hint: Shepard's Lema consists on proving

$$\frac{\partial e(p,\overline{u})}{\partial p_\ell} = h_\ell(p,\overline{u}),\ \forall\ \ell = 1, ..., L.$$

Proposition 30. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that $h(\cdot,\overline{u})$ is continuously differentiable at $(p,\overline{u})$ and denote its $L \times L$ Jacobian matriz by $D_p h(p,\overline{u})$. Then,

1. $D_p h(p,\overline{u}) = D_p^2 e(p,\overline{u})$.
2. $D_p h(p,\overline{u})$ is negative semidefinite matrix.
3. $D_p h(p,\overline{u})$ is a symmetric matrix.
4. $D_p h(p,\overline{u})p = 0$.

21. Prove Proposition 30.

Hint: use Shepard's Lema.

Proposition 31. Slutsky Equation. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. Then, for all (p,w) and $\overline{u} = v(p,w)$ we have

$$\frac{\partial h_\ell(p,\overline{u})}{\partial p_k} = \frac{\partial x_\ell(p,w)}{\partial p_k} + \frac{\partial x_\ell(p,w)}{\partial w} x_k(p,w),\ \forall\ \ell, k.$$

21. Obtain Slutsky Equation.

Hint: set $h(p,\overline{u}) = x(p, e(p,\overline{u}))$.

Definition 32. Substitution Effect: this captures how the quantity demanded of a good changes as consumers switch away from goods that have become relatively more expensive towards those that are relatively cheaper, holding utility constant (i.e., the change in consumption that would occur if the consumer were compensated to keep their original level of utility).

Income Effect: this reflects how the quantity demanded changes in response to a change in purchasing power caused by the price change, holding prices constant.

21. Identify in Slutsky Equation both substitution and income effect.

Another result derived from the Envelope Theorem is the following.

Proposition 33. Roy's identity. Suppose that $u(\cdot)$ is a continuous utility function representing a locally nonsatiated and strictly convex preference relation $\succeq$ defined on the consumption set $X = \mathbb{R}^L_+$. Suppose also that the indirect utility function is differentiable at $(\overline{p}, \overline{w}) > 0$. Then

$$x(\overline{p}, \overline{w}) = -\frac{1}{\nabla_w v(\overline{p}, \overline{w})} \nabla_p v(\overline{p}, \overline{w}).$$

This is,

$$x_\ell(\overline{p}, \overline{w}) = -\frac{1}{\frac{\partial v(\overline{p}, \overline{w})}{\partial w}} \frac{\partial v(\overline{p}, \overline{w})}{\partial p}.$$

22. Prove Roy's identity. Use the Envelope Theorem to do so.

The purpose now, is to recover a preference relation from (by means of) $e(p, \overline{u})$. More recent work address this issue from a more advanced framework.

For each utility level $\overline{u}$ let $V_{\overline{u}} \subset \mathbb{R}^L$ be an at-least-as-good set such that $e(p, \overline{u})$ is the minimal expenditure required for the consumer to purchase a bundle in V_u at prices $p > 0$. This is

$$e(p, \overline{u}) = \min_{x \geq 0} p \cdot x$$
$$x \in V_u.$$

Proposition 34. Suppose that $e(p, \overline{u})$ is strictly increasing in $\overline{u}$ and is continuous increasing, homogeneous of degree one, concave and differentiable in p. Then, for every utility level $\overline{u}$,

$$V_{\overline{u}} = \{x \in \mathbb{R}^L_+ : p \cdot x \geq e(p, \overline{u}),\ \forall\ p > 0\}.$$

Remark. The following system of partial differential equations is derived using Shepard's Lemma:

$$\frac{\partial e(p)}{\partial p_1} = x_1(p, e(p))$$
$$\vdots$$
$$\frac{\partial e(p)}{\partial p_L} = x_L(p, e(p)),$$

for initial conditions p^0 and $e(p^0) = w^0$.

23. Explain why in order to ensure a solution to the PDE system presented right before it is required to $S(p, e(p))$ to be symmetric.

To conclude with this section, we present the basics of welfare evaluation of economic changes (the normative sude of consumer theory). We shall consider a consumer with a rational preference relation $\succeq$. Whenever it is convenient, it will be assumed that both the indirect utility and expenditure function are differentiable.

In a first stage, we assume that a consumer has a fixed wealth $w > 0$ and faces prices p^0. Then, prices change to p^1. The invidivual is worse when

$$v(p^1, w) - v(p^0, w) < 0.$$

Now, $e(p, v(p, w))$ is the wealth required to achieves a utility level $e(p, v(p, w))$ when prices are p. Hence,

$$e(p, v(p^1, w)) - e(p, v(p^0, w))$$

provides a measure of welfare change expressed in monetary units.

A money metric indirect utility function can be constructed in this manner for any price vector $p > 0$. Let $u^0 = v(p^0, w)$ and $u^1 = v(p^1, w)$, and note that $e(p^0, u^0) = e(p^1, w^1)$. We define the equivalent variation and the compensated variation.

$$\begin{aligned} EV(p^0, p^1, w) &= e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w \\ CV(p^0, p^1, w) &= e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0). \end{aligned}$$

In the equivalent variation, we work with initial prices, and in the compensated variation, we work with final prices. The equivalence variation is the u.m. amount that the consumer would be indifferent about accepting in lieu of the price change: that is, it is the change in the wealth that would be equivalent to the price change in terms of its welfare impact. Therefore, it is negative if the price change would make the consumer worse off). Thus,

$$v(p^0, w + EV) = u^1 = v(p^1, w).$$

Compensated variation on the other hand measures the net revenue of a planner who must compensate the consumer for the price change after it occurs, bringing the consumers utility level to the original u^0. Hence, the compensating variation is negative if the planner would have to pay the consumer a positive level of compensation because the price change makes the individual worse off. Hence,

$$v(p^1, w - CV) = u^0.$$

24. Prove that if only price of the good 1 changes,

$$EV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^1) dp_1.$$

Hint: $e(p^0, u^1) - e(p^0, u^0) = e(p^0, u^1) - w = e(p^0, u^1) - e(p^1, u^1)$.

24. Prove that if only price of the good 1 changes,

$$CV(p^0, p^1, w) = \int_{p_1^1}^{p_1^0} h_1(p_1, p_{-1}, u^0) dp_1.$$

Hint: $e(p^1, u^1) - e(p^1, u^0) = w - e(p^1, u^0) = e(p^0, u^0) - e(p^1, u^0)$.

25. Following Consumer's Surplus, Price Instability, and Consumer Welfare, prove, using the indirect utility function (and Roy's identity), that it is only convenient to stabilize a price if:

$$\varepsilon_{xp_x} + s_x(\varepsilon_{xw} - r_r) < 0.$$

- High risk aversion (volatility or low income).
- Low wealth elasticity (necessary good).
- Share not low.
- Price elasticity is low.

26. Prove that the sum of elasticities is zero for the following demand functions:

$$x(p_1, p_2, w) = \frac{\alpha w}{p_1}$$
$$x(p_1, p_2, w) = \frac{\alpha w}{(ap_1 + bp_2)}.$$

27. Consider the following utility function

$$u(x_1, x_2) = x_1^{0.5} + x_2^{0.5}.$$

a) Find the ordinary demands, indirect utility function and the expenditure function.

b) If initial prices are ($p_1^0 = p_2^0 = 2$) but then $p_1^1 = 3$ (keeping $p_2^1 = 2$ and considering $w = 100$) find the compensated variation and equivalent variation.

28. Prove that if $u : \mathbb{R}^2 \to \mathbb{R}$, is C^2 and quasi-concave, the MRS $\frac{u_{x_1}}{u_{x_2}}$ is decreasing.

29. Establish the following two results:

1. A continuous $\succeq$ is homothetic if and only if it admits a utility function $u(x)$ that is homogeneous of degree one, i.e., $u(\alpha x) = \alpha u(x)$, for all $\alpha > 0$.

2. A continuous $\succeq$ on $\mathbb{R} \times \mathbb{R}_+^{L-1}$ is quasi-linear with respect to the first commodity if and only if it admits a utility function $u(x)$ of the form

$$x_1 + \phi(x_2, ..., x_{L-1}).$$

30. Suppose that in a two commodity world, the consumer's utility function takes the form

$$u(x) = [\alpha_1 x_1^\rho + \alpha_2 x_2^\rho]^{1/\rho},\ \rho \neq 0,\ \alpha_i > 0. \tag{4}$$

This is, a constant elasticity substitution utility function (CES). Prove the following:

a) When $\rho = 1$, the utility becomes linear.

b) When $\rho \to 0$, the utility comes to present the same preferences as the Cobb-Douglas utility function $u(x) = x_1^{\alpha_1} x_2^{\alpha_2}$.

c) When $\rho \to -\infty$, the utility comes to present the same preferences as the Leontief utility function $\min\{x_1, x_2\}$.

Try to generalize this result for the L commodity world.

31. Consider the CES utility function (4) with $\alpha_1 = \alpha_2 = 1$.

a) Compute the Walrasian demand an indirect utility function.

b) Compute the Hicksian demand and expenditure function.

c) Check if Shepard's Lema and Roy's identity are satisfied.

d) Prove that the elasticity of substitution[8] between goods 1 and 2, defined as

$$\xi_{1,2}(p,w) = -\frac{\partial[x_1(p,w)/x_2(p,w)]}{\partial[p_1/p_2]}\frac{p_1/p_2}{x_1(p,w)/x_2(p,w)}$$

if for the CES utility function $\frac{1}{1-\rho}$.

e) Compute ξ_{12} for the linear, Leontief and Cobb-Douglas utility function.

32. Consider the Stone-Geary utility function

$$u(x) = \prod_{i=1}^{n}(x_i - a_i)^{\alpha_i},\ a_i > 0,\ \alpha_i > 0.$$

Obtain the Walrasian demand, indirect utility and verify Roy's identity.

Hint: you may want to use $\ln(u(x))$.

33. A utility function $u(x)$ is additively separable if it has the form

$$u(x) = \sum_{\ell=1}^{L} u_\ell(x_\ell)$$

a) Show that additive separability is a cardinal property that is preserved only by linear transformations of the utility function.

b) Show that the induced ordering on any group of commodities is independent of whatever fixed values we attach to the remaining ones.

c) Show that the Walrasian and Hicksian demand function generated by an additively separable utility function admit no inferior good[9] if the functions $u_\ell(\cdot)$ are strictly concave. Assume differentiability and interior solutions.

[8] Given an original allocation/combination and a specific substitution on allocation/combination for the original one, the larger the magnitude of the elasticity of substitution (the marginal rate of substitution elasticity of the relative allocation) means the more likely to substitute. It measures the curvature of an indifference curve. Since $MRS = p_1/p_2$, $\xi_{12} = \frac{d\ln(x_2/x_1)}{d\ln(p_1/p_2)}$.

[9] The demand drops when income rises.

34. Consider the following (intertemporal) utility function

$$u(x) = \sum_{t=1}^{T} \beta^t \sqrt{x_t}.$$

1. For $\beta = 1$, obtain the Walrasian demand and the indirect utility function.

2. For $\beta \in (0,1)$, prove that

$$x_t^* = \frac{\delta^t (1-\delta^2)}{1-\delta^{2(T+1)}}.$$

35. By means of comparative statics, with respect to the utility maximization problem

$$\begin{aligned} &\max\ u(x_1, x_2) \\ &\text{s.t. } p_1 x_1 + p_2 x_2 \leq w \\ &x_1, x_2 \geq 0 \end{aligned}$$

obtain

$$\frac{\partial x_1}{\partial w}.$$

You may assume that preferences are monotone and $u \in C^2$.

Hint: recall Cramer's rule and see [4].

36. With respect to the expenditure minimization problem for two goods, find by means of comparative statics

$$\frac{\partial x_1}{\partial p_1}.$$

Is it true that the substitution effect is always negative? You may assume that preferences are monotone and $u \in C^2$.

37. Consider the following expenditure function

$$e(p, \overline{u}) = \exp\left\{ \sum_{\ell=1}^{L} \alpha_\ell \ln(p_\ell) + \left(\prod_{\ell=1}^{L} p_\ell^{\beta_\ell} \right) \overline{u} \right\}.$$

a) What restrictions on $\alpha_1, ..., \alpha_L, \beta_1, ..., \beta_L$ are necessary for this to be derivable from the expenditure minimization problem?

b) Find the indirect utility function that corresponds to it. Hint: use duality theorems.

c) Verify Roy's Identity.

5 Producer Theory

We follow again [1].

A input-output production plan is a vector $y = (y_1, ..., y_L) \in \mathbb{R}^L$ that describes the (net) outputs of the L commodities from a production function.

Example 35. Suppose that $L = 5$, then

$$y = (-5, 2, -6, 3, 0)$$

means that 2 and 3 units of goods 2 and 4, respectively are produced, while 5 and 6 units of goods 1 and 3, respectively, are used. Good 5 is neither used as an input or produced in this production vector.

We need to identify which production vectors are technologically possible, i.e., plans that belong to the production set $Y \subset \mathbb{R}^L$, known as technology. Any $y \in Y$ is possible and $y \notin Y$ is not.

Sometimes, it is convenient to write Y by means of a production function $F(\cdot)$, called the transformation function. This function has the property that

$$Y = \{y \in \mathbb{R}^L : F(y) \leq 0\}$$

and $F(y) = 0$ if and only if $y \in \partial Y$. The set of boundary points of Y

$$\{y \in Y : F(y) = 0\}.$$

If $F(\cdot)$ is differentiable, and if the production vector $\overline{y}$ is such that $F(\overline{y}) = 0$, Then, for all ℓ and k

$$MRT_{\ell k}(\overline{y}) = \frac{\frac{\partial F(\overline{y})}{\partial y_\ell}}{\frac{\partial F(\overline{y})}{\partial y_k}}.$$

This is, the MRT of good ℓ for good k at $\overline{y}$.

One of the most frequently encountered production models is that in which there is a single output. A single output technology is commonly described by means of a production function $f(z)$ that gives the maximum amount q of output that can be produced using inputs amount $(z_1, ..., z_{L-1}) \geq 0$. Hence

$$Y = \{(-z_1, ..., -z_{L-1}, q) : q \leq f(z_1, ..., z_{L-1}),\ (z_1, ..., z_{L-1}) \geq 0\}.$$

Hereafter some important definitions regarding production sets:

1. Y is nonempty.

2. Y is closed: the set includes its boundary. Thus, the limit of a sequence belonging to Y, let us say y^n, is $y \in Y$.

3. No free lunch: if $y \in Y$ and $y \geq 0$, then $y = 0$.

4. Possibility of inaction: $0 \in Y$.

5. Free disposal: if $y \in Y$ and $y' \leq y$, then $y' \in Y$. This means that it is possible to produce with the same amount of inputs less output.

6. Irreversibility: suppose $y \in Y$ and $y \neq 0$. Then the irreversibility says that $-y \in Y$.

7. Nonincreasing returns to scale: $\forall\, y \in Y$, $\alpha \in Y$ for all scalars $\alpha \in [0, 1]$.

8. Nondecreasing returns to scale: $\forall\, y \in Y$, $\alpha \in Y$ for all scalars $\alpha \geq 1$.

9. Constant returns to scale: Y is a cone, i.e., $\forall\, y \in Y$, and $\alpha \geq 0$, $\alpha y \in Y$.

10. Additive (or free entry): if $y \in Y$ and $y' \in Y$, then $y + y' \in Y$. More succintly, $Y + Y \subset Y$. This means for instance that for any $k \in \mathbb{N}$, and $y \in Y$, $ky \in Y$.

11. Convexity: Y is convex.

1. Suppose that $f(\cdot)$ is the production associated with a single-output technology and let Y be the production set of this technology. Show that Y satisfies constant returns to scale iff $f(\cdot)$ is homogeneous of degree one.

2. Show that for a single output technology, Y is convex iff the production function $f(z)$ is concave.

3. Prove that the production set Y is additive and satisfies nonincreasing returns conditions iff it is a convex cone.

4. Prove that if Y is convex, additive, closed, and $-\mathbb{R}^L_+ \subset Y$, then Y exhibit the property of free disposal.

Hint: for any $y' \leq y$, you can write $y' = y + v$ with $v \in -\mathbb{R}^L_+$. Then, you can take

$$nv\frac{1}{n} + \left(1 - \frac{1}{n}\right) y \in Y.$$

Now, let us study the **profit maximization** and **cost minimization problem**.

The profit maximization problem is the following: given a price vector $p > 0$ and a production vector $y \in Y$, the profit generated by implementing y is $p \cdot y = \sum_{\ell=1}^{L} p_\ell y_\ell$. By the sign convention, this is precisely the total revenue minus the total cost. Given the technological constraints represented by its production set Y, the firm solves

$$\begin{aligned} &\max\, p \cdot y \\ &\text{s.t. } y \in Y. \end{aligned}$$

Eventually, when possible, using a transformation function, this is

$$\begin{aligned} &\max\, p \cdot y \\ &\text{s.t. } F(y) \leq 0. \end{aligned}$$

The optimum

5. Prove that, if Y exhibits nondecreasing returns to scale, then either $\pi(p) \leq 0$ or $\pi(p) = \infty$.

If the transformation function is differentiable, then the FOC provides

$$p_\ell = \lambda \frac{\partial F(y^*)}{\partial y_\ell}, \ \ell = 1, ..., L,$$

or equivalently,

$$p = \lambda \nabla F(y^*).$$

Remark. When there is a single output, the firm solves,

$$\max_{z \geq 0} pf(z) - w \cdot z.$$

Hence, if z^* is optimal, by FOC (Karush-Kuhn-Tucker),

$$p \frac{\partial f(z^*)}{\partial z_\ell} \leq w_\ell,$$

with equality if $z_\ell^* \geq 0$.

Proposition 36. Suppose that $\pi(\cdot)$ if the profit function[10] and that $y(\cdot)$ is the associated supply correspondence. Assume also that Y is closed and satisfies free disposal property. Then,

1. $\pi(\cdot)$ is homogeneous of degree one.
2. $\pi(\cdot)$ is convex.
3. If Y is convex then $Y = \{y \in \mathbb{R}^L : p \cdot y \leq \pi(p),\ \forall\ p > 0\}$.
4. $y(\cdot)$ is homogeneous of degree zero.
5. If Y is convex, then $y(p)$ is a convex set for all p. Moreover, if Y is strictly convex, then $y(p)$ is single-valued, if non-empty.
6. If $y(p)$ consistes of a single point, then $\pi(\cdot)$ is differentiable at p and
$$\nabla \pi(p) = y(p).$$
7. If $y(\cdot)$ is a differentiable function at p, then $Dy(p) = D^2\pi(p)$ is a symmetric and positive semi-definite matrix with $Dy(p)p = 0$.

6. Prove Proposition 36.

Now we move to the cost minimization problem. Given a price of inputs $w > 0$, a production level $q > 0$ and a production function $f(\cdot)$, the firm solves

$$\begin{aligned} &\min\ w \cdot z \\ &\text{s. t. } f(z) \geq q. \end{aligned}$$

The optimized value of the CMP is given by the cost function $c(w, q)$.

First order conditions provide

$$w_\ell \geq \lambda \frac{\partial f(z^*)}{\partial z_\ell},\ z^*_\ell > 0,$$

for some $\lambda \geq 0$ and $\ell = 1, ..., L$.

Proposition 37. Suppose that $c(w, q)$ is the cost function of the single-output technology Y with production function $f(\cdot)$ and that $z(w, q)$ is the associated conditional factor demand correspondence. Assume also that Y is closed and satisfies the free disposal property. Then,

[10] $\max_{y \in Y} p \cdot y$.

1. $c(\cdot)$ is homogeneous of degree one in w and nondecreasing in q.

2. $c(\cdot)$ is a concave function of w.

3. If the sets $\{z \geq 0 : f(z) \geq q\}$ are convex for every q, then $Y = \{(-z, q) : w \cdot z \geq c(w, q),\ \forall\ w > 0\}$.

4. If the set $\{z \geq 0, f(z) \geq q\}$ is convex, then $z(w, q)$ is a convex set. Moreover, if $\{z \geq 0 : f(z) \geq q\}$ is a strictly convex set, then $z(w, q)$ is single valued.

5. Shepard's Lema: if $z(w, q)$ consists of a single point, then $c(\cdot)$ is differentiable with respect to w and $\nabla_w c(w, q) = z(w, q)$.

6. If $z(\cdot)$ is differentiable at w, then $D_w z(w, q) = D_w^2 z(w, q) w = 0$.

7. If $f(\cdot)$ is homogeneous of degree one, then $c(\cdot)$ and $z(\cdot)$ are homogeneous of degree one in q.

8. If $f(\cdot)$ is concave, then $c(\cdot)$ is a convex function of q.

7. Prove Proposition 37.

Using the cost function, it is possible to restate the firm's problem of determining its profit maximizing production level as

$$\max_{q \geq 0} pq - c(w, q).$$

FOC are

$$p - \frac{\partial c(w, q^*)}{\partial q} \leq 0,$$

with equality if $q^* > 0$. When $c(w, q)$ is convex, then FOC are sufficient.

8. Let $f(z_1, z_2) = z_1^\alpha z_2^\beta$, $\alpha, \beta \in [0, 1]$. Solve the cost minimization problem. Prove that

$$c(w_1, w_2, q) = q^{\frac{1}{\alpha+\beta}} \theta \phi(w_1, w_2)$$

with $\phi(w_1, w_2) = w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}}$ and

$$\theta = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\alpha}{\beta}\right)^{\frac{-\alpha}{\alpha+\beta}}.$$

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