FACULTY OF SOCIAL SCIENCES
ECONOMICS MAJOR

PUCP

Exercises Session 3

Microeconomics 2
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Some exercises have been extracted and adapted from Professor Alejandro Lugón's material.

1 Pure Exchange Economies

Exercise 1.1 (Adapted from Aliprantis et al. (1990))**.** Consider an economy with 3 consumers and 2 goods. Utilities and endowments are given by

$$\begin{aligned}
u_1(x_{11}, x_{21}) &= x_{11}^{1/2} + x_{21}^{1/2},\ (\omega_{11}, \omega_{21}) = (1, 2)\\
u_2(x_{12}, x_{22}) &= \min\{x_{12}, x_{22}\},\ (\omega_{12}, \omega_{22}) = (3, 4)\\
u_3(x_{13}, x_{23}) &= x_{23}e^{x_{13}},\ (\omega_{13}, \omega_{23}) = (1, 1).
\end{aligned}$$

Prove that the optimal demands are given by

$$\begin{aligned}
x_{11} &= \frac{p_2p_1 + 2p_2^2}{p_1^2 + p_2p_1},\ x_{21} = \frac{p_1^2 + 2p_2p_1}{p_2p_1 + p_2^2}\\
x_{12} = x_{22} &= \frac{3p_1 + 4p_2}{p_1 + p + 2}\\
x_{13} &= \frac{p_2}{p_1},\ x_{23} = \frac{p_1}{p_2}.
\end{aligned}$$

Exercise 1.2. Find the optimal demands in a pure exchange economy with L consumption goods, N consumers, where each consumer $k = 1, ..., N$ has preferences represented by

$$u_k(x_k) = \prod_{\ell=1}^{L} x_{\ell k}^{\alpha_{\ell k}},$$

$\sum_{\ell=1}^{L} \alpha_{\ell k} = 1,\ \alpha_{\ell k} \in (0, 1)$, and endowments $\omega_k > 0$. Do not seek to find the Walrasian equilibrium.

Before passing to the following two exercises, check Mas-Colell et al. (1995) Chapter 17 or Varian (1992) Chapter 17 in order to define and analyze the properties of the excess of demand function. This is only for the interested student since it is not going to be evaluated.

Exercise 1.3. Let $z(p_1, p_2) = \left(\frac{Bp_2}{p_1}, \frac{Ap_1}{p_2}\right) - (A, B)$. Prove that z satisfies the five properties of an excess demand function.

Exercise 1.4. Consider a 2×2 economy where the first consumer has preferences represented by a Cobb-Douglas utility function

$$u_1(x_{11}, x_{21}) = x_{11}x_{21}$$

and initial endowment $\omega_1 = (2, 6)$. The second consumer has preferences

$$u_2(x_{12}, x_{22}) = \min\{x_{12}, x_{22}\}$$

and initial endowment $\omega_2 = (4, 1)$. Let $p = (p_1, p_2) \in \mathbb{R}^2_{++}$.

a) Find the demand (correspondence) of each consumer.

b) Find the excess demand (correspondence) of each consumer.

c) Verify if z satisfies the usual properties of excess demand functions.

d) ¿Is there an equilibrium in this economy?

e) Find the Pareto optimal allocations.

2 Economies with production

Exercise 2.1. State welfare theorems for production economies. Prove the first one.

Exercise 2.2. Study the properties of the technology

$$Y = \left\{(x, y) \in \mathbb{R}^2 : x < 1, y \leq \frac{x}{x-1}\right\}.$$

In particular: closeness, convexity and free disposal[1].

Exercise 2.3. Consider a firm with technology $Y = \{(-x, z) \in \mathbb{R}^2 : x \geq 0, z \leq f(x)\}$. Prove that if Y possess the free disposal property, then f is non decreasing.

Exercise 2.4. Suppose that Acemoglu is a castaway living on an island and can gather apples with a technology given by $y = 4L^{1/2}$. His utility function is $u(y, \ell) = 2y\ell$, where L is the number of hours worked and ℓ is the number of hours of leisure. We know that he has 30 hours available each day for either gathering apples or resting.

a) Formulate the consumer's problem specifying the control variables.

[1]These definitions can be found in Mas-Colell et al. (1995) or Jehle and Reny (2011).

b) Find Acemoglu's optimal consumption (apples and leisure). What is the profit that Acemoglu obtains as a producer, and what is the level of welfare he achieves as a consumer?

c) What is the shadow relative price of both goods?

d) Is the equilibrium found in (b) a Pareto optimum?

e) Plot the solution.

Exercise 2.5. Consider an economy called Courcsant, consisting of two consumers, two goods, and a firm. The agents consume two goods: blue milk (x) and lightsabers (y). However, the agents only have initial endowments of kyber cristals, $\omega_1 = (3,0)$ and $\omega_2 = (2,0)$ respectively. On the other hand, the only firm produces lightsabers with the following technology $Y = \{(x,y) \in \mathbb{R}^2 | x \leq 0,\ y \leq \sqrt{-x}\}$. Additionally, the preferences of the consumers are represented by $u_1(x_1,y_1) = \sqrt{x_1 y_1}$ and $u_2(x_2,y_2) = 2\ln x_2 + \ln y_2$, respectively.

a) With the information provided, does the economy reach a Walrasian equilibrium, or is an additional condition on the distribution of property rights required? If so, propose a distribution.

b) Find the firm's demand function for the input kyber cristals (x^d), the firm's supply function (y^s), and the profits π^*.

c) Find the demand function for each consumer.

Exercise 2.6. Consider an economy with two goods, two consumers (Obi-Wan and Palpatine) and one firm. Obi-Wan has preferences represented by

$$u_1(x_{11}, x_{21}) = \sqrt{x_{11} x_{21}},$$

with initial endowment $\omega_1 = (1,0)$ and $\theta_1 = 0.3$. Palpatine has quasilinear preferences

$$u_2(x_{12}, x_{22}) = x_{12} + \ln(x_{22}),$$

with initial endowment $\omega_2 = (2,0)$ and $\theta_2 = 0.7$. On the other hand, the firms technology is

$$Y = \left\{ (x,y) \in \mathbb{R}^2 : x \leq 0, y \leq \frac{Ax}{x-1} \right\}$$

where $A > 0$ is a productivity factor.

1. Find the offer function of the firm.

2. Find Obi-Wan and Palpatine correspondence demand.

3. Study the effect of the productivity factor A over the equilibrium (prices and allocation). In other terms, do some comparative statics focusing on the parameter A.

Exercise 2.7. Consider and economy with two consumers, D. Acemoglu (A) and R. Barro (B):

$$u_A(x_{A1}, x_{A2}) = \min\left\{x_{A1}, \frac{x_{A2}}{4}\right\},\ \omega_A = (a, 1), \theta_A = 1/3$$
$$u_B(x_{B1}, x_{B2}) = (x_{B1})^{1/3}(x_{B2})^{2/3},\ \omega_B = (1, b),\ \theta_B = 2/3,$$

with $a, b > 0$. Let

$$Y = \{(x_1, x_2) : 4x_2 + x_1 \leq 0,\ 4x_1 + x_2 \leq 0\}$$

be the firms technology.

a) Set the firm problem and solve it; specify all the price vector $p \in \Lambda$ for which the problem has a solution. Obtain the offer and profit correspondences.

b) Consider a specific $p \in \Lambda$. Set and solve the consumers problem (for each one).

c) Obtain the excess demand function and analyze if it satisfies the basic properties[2].

Exercise 2.8. Consider an economy with two goods, two consumers and a firm. Consumers have quasilinear utilities:

$$u_1(m_1, x_1) = m_1 + 4\ln x_1$$
$$u_2(m_2, x_2) = m_2 + \ln x_2.$$

Initial endowments are $\omega_1 = (100, 0)$ and $\omega_2 = (100, 0)$. Each one owns a fraction θ_i of a firm whose technology is given by

$$Y = \{(-m_e, x_e) : x_e = \sqrt{m_e},\ x_e \geq 0, m_e \geq 0\}.$$

We take $x_i \geq 0$ but $m_i \in \mathbb{R}$. This is, consumers can consume a negative amount of m. Let p_m be the price of good m and p_x the price of good x.

1. Find the firm's offer.
2. Find each consumers demand.
3. Find the aggregated excess demand function.
4. There is a property which is not satisfied[3], which one? Why?
5. Can you normalize $p_m = 1$? Justify.
6. Prove that, in this economy, the equilibrium prices do not depend on the initial wealth[4] distribution.

[2] They are analogous to the Pure Exchange Economies case. See Mas-Colell et al. (1995) for a more detailed discussion.

[3] From the properties that aggregated demand function satisfy.

[4] Endowments and shares.

3 Additional exercises

The following exercises are mathematically more advanced. They remain for the interested reader/student since they are not going to be evaluated.

Proposition 1. If $(\succeq_i, \omega_i)_{i=1}^I$ is an exchange economy in which $\overline{\omega} = \sum_{i=1}^I \omega_i > 0$ and each $\succeq_i$ is continuous, strictly convex and strictly monotone, then the aggregate excess demand function satisfies:

1. z is continuous.
2. z is homogeneous of degree zero.
3. Walras Law: $\forall\ p \in \mathbb{R}^L_{++}$: $p \cdot z(p) = 0$.
4. Bounded below: $\exists\ M > 0$ such that $\forall\ \ell, p \in \mathbb{R}^L_{++}$, $z_\ell(p) > -M$.
5. Boundary condition: if $\{p^n\}$ is a sequence in $\mathbb{R}^L_{++}$ and $\overline{p} = \lim_n p^n$ where $\overline{p} \in \mathbb{R}^L_0 \backslash \mathbb{R}^L_{++}$ and $\overline{p} \neq 0$, then there is $\ell \in \{1, ..., L\}$ such that $\{z_\ell(p^n)\}_n$ is unbounded.

Theorem 1. Existence of Walrasian equilibrium. In the context of Proposition 1, for $z : \mathbb{R}^L_+ \to \mathbb{R}^L$, there exists $p^* \in \mathbb{R}^L_+$ such that $z(p^*) \leq 0$. Furthermore, if $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$, there exists p^* such that $z(p^*) = 0$.

Exercise 3.1. Prove Proposition 1.

Exercise 3.2. Prove Theorem 1. *Hint: apply Brouwer fixed point theorem to*

$$\Delta = \left\{ p \in \mathbb{R}^L_+ : \sum_{\ell=1}^L p_\ell = 1 \right\}$$

and $\Psi : \Delta \to \mathbb{R}^L$ *defined as follows:*

$$\Psi_\ell = \frac{p_\ell + \max\{0, z_\ell(p)\}}{1 + \sum_{\ell=1}^L \max\{0, z_\ell(p)\}},\ \forall\ \ell = 1, ..., L.$$

Let $\mathcal{E} = \{\omega^i, \preceq_i\!: \ i = 1, ..., I\}$ be a pure exchange economy. Following Echenique (2005) notation:

- An allocation $\leq$ in E is a vector $x = (x_i)_{i=1}^I \in \mathbb{R}^{IL}_+$, such that $\sum_{i=1}^I x_i \leq \sum_{i=1}^I \omega_i = \overline{\omega}$.
- An allocation $=$ in E is a vector $x = (x_i)_{i=1}^I \in \mathbb{R}^{IL}_+$, such that $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i = \overline{\omega}$.
- A nonempty subset $S \subseteq \{1, \ldots, I\}$ of agents is called a coalition.
- Let S be a coalition. A vector $(y_i)_{i \in S}$ is an S-allocation $\leq$ if $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$.
- Let S be a coalition. A vector $(y_i)_{i \in S}$ is an S-allocation $=$ if $\sum_{i \in S} y_i = \sum_{i \in S} \omega_i$.

Definition 3.1. We say that

- A coalition S *blocks* the allocation $\leq\ x$ in E if there exists an S-allocation$\leq (y_i)_{i \in S}$ such that $y_i > x_i$ for all $i \in S$.

- An allocation$\leq$ is *weakly Pareto optimal* if it is not blocked by the coalition I of all consumers.
- It is *individually rational* if no coalition consisting of a single consumer blocks it.
- It is a *core allocation* if there is no coalition that blocks it.

Let $C(\mathcal{E})$ be the set of core allocations$\leq$ of $\mathcal{E}$. We refer to $C(\mathcal{E})$ as the core of the economy $\mathcal{E}$. Let $\mathcal{P}(\mathcal{E})$ be the set of Pareto Optimal allocations$\leq$ of the economy $\mathcal{E}$, and let $\mathcal{W}(\mathcal{E})$ be the set of Walrasian Equilibrium allocations$\leq$. Note that $C(\mathcal{E})$, $\mathcal{W}(\mathcal{E})$, and $\mathcal{P}(\mathcal{E})$ are subsets of $\mathbb{R}_+^{IL}$.

Definition 3.2. A coalition S *weakly blocks* the allocation $\mathbf{x}$ if there exists an S-allocation$\leq$ $(y_i)_{i\in S}$ such that $y_i \geq x_i$ for all $i \in S$, and $y_j > x_j$ for some $j \in S$.

Exercise 3.3. Prove that, if each $\succeq_i$ is continuous and strictly monotonic, then a coalition blocks an allocation if and only if it weakly blocks it. *Hint: the surplus can be divided. Consider* $z_i = (1-\delta)y_i$ *for* δ *small enough, and* $z_j = \frac{\delta y_i}{|S|-1} + y_i$.

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