FACULTY OF SOCIAL SCIENCES
ECONOMICS MAJOR

Exercises Session 5

Microeconomics 2
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For more details see Varian (1992) or Mas-Colell et al. (1995).

1 Externalities

Exercise 1.1. The company S produces a certain amount of steel (s) and a certain amount of pollution (x), which it discharges into a river. The company F is a fish farm located downstream and is adversely affected by the pollution from company S. Suppose the cost function of S involves (s) and (x). Meanwhile, the company F involves (f) which represents the collection of fish and (x) which is the production of pollution. Furthermore, it should be considered that pollution increases the cost of fish production and reduces the cost of steel production.

a) Formulate the profit maximization problem for both companies.

b) What would be the conditions that characterize profit maximization? Remember that polluting has no price.

c) How would the efficient Pareto production plan for steel and fish look? What are the implications of this new scenario for pollution production?

Exercise 1.2. There are two firms in an economy called Berkeley. One of them produces good x and as a byproduct h, according to the cost function:

$$CT(x,h) = x^2 + \frac{1}{2}(h-4)^2.$$

The other firm produces good y according to the cost function:

$$CT(y,h) = y^2 + \frac{h^2}{4},$$

such that its costs are affected by the decision h made by the first firm.

(a) What type of externality does this model describe?

(b) What is the level h^* that the first firm will choose? Is this amount efficient? Explain.

(c) What is the efficient (Pareto optimal) level h^{ef}?

(d) What is the level of the Pigouvian tax?

(e) If all the rights over h are granted to the second firm, which produces y, could it charge the first firm for giving up these rights? Formulate the profit optimization problem for each firm and find the equilibrium level of h if the competitive price p_h is traded.

(f) Do your results change if now the first firm has the right to pollute up to a level $\hat{h}$?

Exercise 1.3. Consider an economy consisting of two agents $i = 1, 2$ and two goods: a numeraire m and an externality h. Each individual i has an initial endowment of the numeraire m_i. The initial endowments of the agents are $w_1 = (50, 0)$ and $w_2 = (30, 0)$. The externality can take values $h \in [0, 1]$, and its magnitude depends on agent 1, who incurs no cost in setting it. The utilities of the individuals can be written as:

$$U_1(m_1, h_1) = m_1 + \ln(1 + h_1)$$
$$U_2(m_2, h_2) = m_2 + b \ln(2 - h_1), \quad b \in \mathbb{R}.$$

(a) What is the optimal choice of h_1 for agent 1?

(b) Does this choice constitute a negative or positive externality for agent 2? If your answer is conditional, specify the conditions for each case. From now on, assume the value of b is such that the externality is negative.

(c) What is the socially efficient choice (Pareto Optimal) for h?

(d) Find the magnitude of the Pigouvian tax that corrects the inefficiency introduced by the externality.

(e) Suppose all property rights over the externality are assigned to agent 2, so that agent 1 can pay a price p_h for each unit of the externality. Find the equilibrium price p_h^e and the respective level h^e. Comment on your results.

Exercise 1.4. In a dairy town (producer of the best milk, good y, in the region), due to the problems caused by cows (good x) in the fields, the town mayor is trying to determine how much to charge for a field usage license. The milk production function is as follows:

$$y = f(x) = x^{0.8}.$$

Additionally, the cost of maintaining each cow is $w_x = 20$, while each glass of milk sells for 50 monetary units.

1. If the field is a communal good, how many cows will the company demand?

2. If the field were privately owned, what would be the number of cows that maximizes the owner's profits?

3. If they wanted to restrict the number of cows to those that maximize total benefits, how much should they charge per month for a field usage license?

2 Public Goods

Discrete Public Goods. A public good is a non-excludable and non-rival good. This means it is not possible to exclude others from consumption, and its consumption does not reduce the amount available. Suppose that the public good G takes the value 1 if it is provided and 0 otherwise: $G \in \{0, 1\}$.

1. $u_i(x_i, G)$ models the preferences of individuals, where G is the discrete public good and x_i is a numeraire good that can be used for consumption or for the provision of the public good.

2. Endowments $w_i \geq 0$. These can be allocated to x_i or to the provision of the public good g_i. Thus, $w_i = x_i + g_i$.

3. The cost of the public good is $c \geq 0$. Therefore,

$$G = \begin{cases} 1, & \text{if } \sum_i g_i \geq c, \\ 0, & \text{otherwise.} \end{cases}$$

4. In the case of 2 individuals, providing the public good is Pareto efficient if $g_1 + g_2 \geq c$, $u_i(w_i - g_i, 1) > u_i(w_i, 0)$.

5. The reservation price r_i is the maximum amount of the private good that agent i is willing to give up to obtain the public good: $u_i(w_i - r_i, 1) = u_i(w_i, 0)$.

6. The public good is then provided if $r_1 + r_2 > g_1 + g_2 \geq c$.

Groves-Clarke Mechanism. Consider the following mechanism: each agent reports their net value for the public good ($\tilde{v}_i$). This value may or may not be their true value (v_i). The public good is provided if $\sum_i \tilde{v}_i \geq 0$ and not provided if $\sum_i \tilde{v}_i < 0$. Each agent i receives a **side payment** equal to the sum of the net values of the other agents $\sum_{j \neq i} \tilde{v}_j$ if the public good is provided (this payment can be either positive or negative). The profits of agent i take the following form:

$$\Pi_i = \text{profits}_i = \begin{cases} v_i + \sum_{j \neq i} \tilde{v}_j, & \text{if } \tilde{v}_i + \sum_{j \neq i} \tilde{v}_j \geq 0 \\ 0, & \text{if } \tilde{v}_i + \sum_{j \neq i} \tilde{v}_j < 0. \end{cases}$$

1. If $v_i + \sum_{j \neq i} \tilde{v}_j > 0$, then the agent can ensure that the public good is provided by declaring $\tilde{v}_i = v_i$.

2. If $v_i + \sum_{j \neq i} \tilde{v}_j < 0$, then the agent can ensure that the public good is not provided by declaring $\tilde{v}_i = v_i$.

Clarke Taxes. While the Groves-Clarke mechanism leads to truthful revelation, it can be excessively expensive to implement. It is also generally not possible to design a mechanism where the side payments sum to zero. What is possible, however, is to ensure that they are never positive. This implies that agents will pay a tax. In this sense, the allocation will not be Pareto efficient, but at least the public good will be provided if and only if it is efficient to do so. The function h_i, known as the Clarke tax, is

$$h_i(\tilde{v}_{-i}) = \begin{cases} -\sum_{j\neq i} \tilde{v}_j & \text{if } \sum_{j\neq i} \tilde{v}_j \geq 0, \\ 0 & \text{if } \sum_{j\neq i} \tilde{v}_j < 0. \end{cases}$$

Thus,

$$\Pi_i = \text{profits}_i = \begin{cases} v_i & \text{if } \tilde{v}_i + \sum_{j\neq i} \tilde{v}_j \geq 0 \ \wedge \ \sum_{j\neq i} \tilde{v}_j \geq 0, \\ v_i + \sum_{j\neq i} \tilde{v}_j & \text{if } \tilde{v}_i + \sum_{j\neq i} \tilde{v}_j < 0 \ \wedge \ \sum_{j\neq i} \tilde{v}_j < 0, \\ -\sum_{j\neq i} \tilde{v}_j & \text{if } \tilde{v}_i + \sum_{j\neq i} \tilde{v}_j < 0 \ \wedge \ \sum_{j\neq i} \tilde{v}_j \geq 0, \\ 0 & \text{if } \tilde{v}_i + \sum_{j\neq i} \tilde{v}_j < 0 \ \wedge \ \sum_{j\neq i} \tilde{v}_j < 0. \end{cases}$$

Exercise 2.1. Let us consider a scenario where three individuals living in a building are considering the construction of an elevator, which costs $C = 2100$ units of money. The individuals have preferences that depend on the money m^i they have to consume other goods, and on x, which takes the value of 1 if there is an elevator and 0 if there is not. The initial wealth that each individual has is w^i for $i = 1, 2, 3$.

(a) Suppose that preferences can be represented by the following Cobb-Douglas type utility functions:

$$U^i(m^i, x) = \sqrt{m^i}(x+i)^{0.5}, \ \ i = 1, 2, 3.$$

Find the reservation price of each individual. Then, assuming an endowment of 5000 soles, show that whether the provision of the public good is efficient or not will depend on the distribution of wealth.

(b) Now suppose that preferences can be represented by quasilinear utility functions:

$$U^i(m^i, x) = m^i + 300(i+1)(x)^{0.5}, \ \ i = 1, 2, 3.$$

Find the reservation price of each individual. Assess whether the provision of the public good is efficient or not, and show that, unlike the previous case, your answer does not depend on the distribution of wealth.

(c) What is the individual contribution that maximizes the welfare of agent i given the contributions made by the rest of the community members?

(d) Assuming the costs are shared equally, calculate the net value v_i of the elevator installation for each individual. If majority voting is used as the decision mechanism for the provision of the public good, who votes in favor and who votes against? Is the public good installed?

Exercise 2.2. In the McGregor building, there are 9 residents who are considering installing free Wi-Fi service for everyone. Installing this service costs C money units.

Each resident $i = 1, 2, \ldots, 9$ has preferences expressed through the following utility function:

$$U_i(m_i, G) = m_i + \alpha_i G^{1/2},$$

where m_i is the money that agent i spends on private consumption goods, and G is a dichotomous variable that takes the value 1 when the Wi-Fi is installed and 0 otherwise. Finally, each resident has an initial endowment of w_i soles.

(a) Calculate the reservation price for each resident.

(b) Write the condition that must be met for $G = 1$ to be efficient.

(c) If the residents must contribute with equal payments, what is the condition for a resident to vote in favor of installing the Wi-Fi?

(d) If the installation is approved by majority vote, does this mean that $G = 1$ is efficient?

(e) Suppose the Groves-Clarke mechanism is applied, such that the public good is provided if the sum of the net values reported by each individual is greater than zero $(\sum_i \tilde{v}_i)$, and if it is provided, side payments are made to each individual equal to the sum of the valuations reported by the others, $(\sum_{i \neq j} \tilde{v}_i)$. Express mathematically the profit function of each individual. Explain why each agent's best option will be to report their true net value (v_i).

Exercise 2.3. Consider a scenario where three individuals are evaluating the provision of a public good whose cost is $C = 3000$ units of money. The individuals have preferences that depend on the money m^i they have to consume other goods and on G, which takes the value 1 if the public good is provided and 0 if it is not:

$$U^i(m^i, G) = m^i + 50(2^{i-1})\sqrt{G}, \; i = 1, 2, 3.$$

(a) Find the reservation price of each individual.

(b) Suppose that if it is decided to provide the public good, the cost is distributed equally, $s_i = 1/3$. Find the corresponding net value for each individual.

(c) Suppose that each agent contributes with payments in proportion to how they value the public good $s_i = \frac{r_i}{\sum r_i}$. Who votes in favor and who votes against? If the decision is made by majority vote, is the public good provided? What is the problem with this mechanism?

Exercise 2.4. Consider a scenario where two individuals are evaluating the provision of a public good whose cost is $C = 100$ soles. The individuals have preferences that depend on the money m^i they have to consume private goods and on $x \in \{0, 1\}$:

$$U^i(m^i, x) = m^i + 40\left(\frac{7}{4}\right)^{i-1}\sqrt{x}, \; i = 1, 2.$$

(a) Find the reservation price of each individual. Evaluate whether the provision of the public good is efficient or not.

(b) Find the net value of the provision of the public good if its financing is distributed equally.

(c) Suppose that individual 2 is willing to assume a larger share of the cost so that the other individual does not have a negative net value. Find the minimum fraction of the cost (s_2) that individual 2 must assume and the respective net values.

(d) Suppose the Groves-Clarke mechanism is applied, such that the public good is provided if the sum of the net values reported by each individual is greater than zero ($\sum_i \tilde{v}_i$), and if it is provided, side payments are made to each individual equal to the sum of the valuations reported by the others $\left(\sum_{i \neq j} \tilde{v}_j\right)$. Express mathematically the profit function of each individual. Explain why the best option for each agent will be to report their true net value (v_i).

Continuous Public Goods: Pareto Efficient Solution. Let G be a continuous public good and x a private good. Suppose the preferences $\preceq$ are given by $u^h(G, x^h)$, $h = 1, \ldots, n$, and the individuals have an endowment $w^h = x^h + z^h$ (x^h is consumed, and z^h is allocated to the *production* of G). To obtain the Pareto efficient levels of public good production, we solve

$$\begin{cases} \max_{G, \{x^h\}_{h=1}^n} & \sum_{h=1}^n \alpha_h u^h(G, x^h) \\ s.t.: & \sum_{h=1}^n x^h + \sum_{h=1}^n z^h = \sum_{h=1}^n w^h, \\ & f\left(\sum_{h=1}^n z^h\right) = G. \end{cases}$$

$\alpha_h \geq 0$, which yields the Samuelson-Lindahl condition:

$$\sum_{h=1}^{n} \frac{\partial u^h / \partial G}{\partial u^h / \partial x^h} = \sum_{h=1}^{n} MRS_h = \frac{1}{f'(\underbrace{z}_{=\sum_{h=1}^n z^h})}.$$

Exercise 2.5. Let G be a public good and x a private good. The utility of the individuals can be expressed as follows:

$$u^h(G, x^h) = \alpha \ln G + \ln x^h, \;\; h = 1, 2, \;\; \alpha \in [0, 1].$$

The production function for the public good is expressed as:

$$G = f(z) = z.$$

Finally, the endowments are expressed as w^h.

a) Find the Pareto optimal level of provision for the public good G.

b) Determine the optimal level of production for the public good. Assume that the price of the good is equal to 1.

c) Will the provision of the public good be efficient in competitive equilibrium? Justify your answer.

d) Does the provision of the public good change when a Lindahl tax is introduced? How does this result compare to what is described in part (c)?

Continuous Public Goods: Lindahl Taxes or Equilibrium. Weighting coefficients λ^h are introduced such that $g^h = \lambda^h G$ and $\sum_h \lambda^h = 1$. Thus, one solves for all $h = 1, \ldots, n$

$$\begin{cases} \max & u^h(\underbrace{g^h + G^{-h}}_{=G}, x^h) \\ s.t. & x^h + p\lambda^h G = w^h. \end{cases}$$

From there,

$$MRS_h = \frac{\partial u^h/\partial G}{\partial u^h/\partial x^h} = p\lambda^h.$$

Summing over h,

$$\sum_h MRS_h = \sum_h p\lambda^h = p = \frac{1}{f'(z)}.$$

This recovers the Pareto optimality condition.

Exercise 2.6. Consider a rural community that must decide the number of pine trees (G) to plant around the village. This is done to protect the community from potential landslides in the winter. Each villager allocates their wealth (w_i) to contribute to the financing of the public good as well as to the purchase of private goods (m_i). The preferences of the villagers are represented by the following utility functions:

$$U_i(m_i, G) = \frac{m_i}{3} + i\theta \ln(G),\ \theta > 0,\ i = 1, \ldots, n.$$

The production function of the public good is:

$$G = F(g) = 4g.$$

Assume that the community is composed of 20 villagers.

a) Formulate and solve the optimization problem that allows the socially efficient level of the public good to be reached.

b) Determine the Lindahl equilibrium prices. Why do the agents pay different amounts?

Exercise 2.7. Suppose two individuals have different endowments (wealth) but share the same preferences, represented by the following utility function:

$$u_i(x_i, G) = x_i^{1-\alpha} G^\alpha.$$

What must be the wealth difference between consumer 1 and consumer 2 to explain why consumer 2 contributes no money to the public good?

Exercise 2.8. Assume there are n consumers with the same preferences, represented by Cobb-Douglas utility functions:

$$u_i(x_i, G) = x_i^{1-\alpha} G^\alpha.$$

There is an initial wealth $w > 0$ that is divided among $k \leq n$ consumers. What is the provided amount of the public good? How does this change as k increases?

Lima, September 30, 2024.

References

Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995). *Microeconomic Theory*. Oxford University Press, New York.

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