FACULTY OF SOCIAL SCIENCES
ECONOMICS MAJOR

Exercises Session 1

Microeconomics 2
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1 Preliminaries

In this first exercise session, we will review much of the theory of static optimization in finite-dimensional spaces (specifically, $\mathbb{R}^n$). The main bibliographic sources for these topics are (Chavez and Gallardo, 2024), (de la Fuente, 2000), and (Sundaram, 1996). This is just a review; a thorough and detailed treatment can be found in these texts.

Consider a function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ and a non-empty set $\Omega \subset \mathbb{R}^n$. A static optimization problem is defined as:

$$\begin{aligned} &\text{opt}_x \ f(x) \\ &\text{s.t. } x \in \Omega \cap D. \end{aligned}$$

Suppose that, in particular, we are looking for a maximizer. In this case, change opt to max. How do we find $x^* \in \Omega \cap D$ such that $f(x^*) \geq f(x)$ for all $x \in \Omega \cap D$? Is this even possible? To address this, we assume that $f(x)$ is continuous. By Weierstrass theorem, whenever $\Omega \cap D$ is a compact set, we can ensure the existence of such an x^*.

Example 1.1. Consider $f(x) = u(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$, a utility function, and $\Omega = \{p_1x_1 + p_2x_2 \leq I\} \cap \mathbb{R}^2_+$, $p_1, p_2, I > 0$.

Example 1.2. Consider $f(x) = u(x)$, a utility function, with $D = \mathbb{R}^n$ and $\Omega = B(p, I)$, the budget set

$$B(p, I) = \{x \in \mathbb{R}^n_+ : p \cdot x \leq I\},$$

where $I > 0$ and $p \in \mathbb{R}^n_{++}$. Then, if u is continuous, there exists $x^* \in B(p, I)$ such that x^* solves

$$\begin{aligned} &\max \ u(x) \\ &\text{s.t. } x \in B(p, I). \end{aligned}$$

Now, recall that if x^* solves the static optimization problem and is an interior solution, meaning $x^* \in \text{int}(\Omega \cap D)$, and if f is differentiable, then

$$Df(x^*) = \nabla f(x^*) = \mathbf{0}.$$

On the other hand, if $\tilde{x}$ is such that $\nabla f(x^*) = 0$ and $Hf(x^*) \leq 0$ ($Hf(x^*) > 0$)[1], then x^* is a local maximum (minimum). Strict concavity (convexity) of f ensures the uniqueness of such x^*. If you don't feel comfortable with these topics, please consult the bibliography.

Example 1.3. Consider $f(x) = e^{-x^2}$. $f'(x) = -2xe^{-x^2}$, which is equal to zero only if $x = 0$. Then, you can check that $f''(0) < 0$. It follows that $x^* = 0$ is a global maximum. Now, take $f(x) = x^3$, even if $f'(x) = 0$ implies $x = 0$, zero is not a global, neither a local, maximum/minimum. What fails?

1.1 Lagrange

Consider the problem:

$$\begin{aligned} &\max_{x \in \mathbb{R}^n} f(x) \\ &\text{s.t. } h_j(x) = 0, \quad j = 1, \dots, p \end{aligned} \tag{1}$$

Suppose that the functions f and h_j for $j = 1, \dots, p$ are continuously differentiable. Let x^* be a local maximum and assume that the gradients of the equality constraints $h_j(x^*) = 0$ are linearly independent[2]. Then, there exist Lagrange multipliers μ_j for $j = 1, \dots, p$ such that:

$$\begin{aligned} \nabla f(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) &= \mathbf{0}, \\ h_j(x^*) = 0, \quad j &= 1, \dots, p. \end{aligned} \tag{2}$$

The Lagrangian for this problem is defined as:

$$L(x, \mu) = f(x) + \sum_{j=1}^{p} \mu_j h_j(x), \tag{3}$$

where μ_j are the Lagrange multipliers associated with the equality constraints.

The first-order necessary conditions, using the Lagrangian function, are

$$\begin{aligned} \nabla_x L(x^*, \mu^*) = \nabla f(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) &= \mathbf{0}, \\ h_j(x^*) = 0, \quad j &= 1, \dots, p. \end{aligned} \tag{4}$$

1.2 Karush-Kuhn-Tucker Theorem

Consider the nonlinear programming problem:

$$\begin{aligned} &\max_{x \in \mathbb{R}^n} f(x) \\ &\text{s.t. } g_i(x) \leq 0, \quad i = 1, \dots, m \\ &\quad\;\; h_j(x) = 0, \quad j = 1, \dots, p \end{aligned} \tag{5}$$

[1] Hf denotes the Hessian matrix of f and $Hf(x^*) \geq 0$ means that $Hf(x^*)$ is positive definite. Equivalently, all its eigenvalues are positive.

[2] This is known as the rank condition and ensures that the set defined by $h(x) = 0$ is a surface. See Lima (2015).

Assume that the functions f, g_i for $i = 1, \ldots, m$, and h_j for $j = 1, \ldots, p$ are continuously differentiable. If x^* is a local minimum and the gradients of the active constraints $g_i(x^*) = 0$ and the equality constraints $h_j(x^*) = 0$ are linearly independent, then there exist Lagrange multipliers $\lambda_i \geq 0$ for $i = 1, \ldots, m$, and μ_j for $j = 1, \ldots, p$ such that:

$$\begin{aligned}
\nabla f(x^*) + \sum_{i=1}^{m} \lambda_i \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j \nabla h_j(x^*) &= \mathbf{0} \\
\lambda_i g_i(x^*) = 0, \quad i &= 1, \ldots, m \\
\lambda_i \geq 0, \quad g_i(x^*) \leq 0, \quad i &= 1, \ldots, m \\
h_j(x^*) = 0, \quad j &= 1, \ldots, p
\end{aligned} \tag{6}$$

1.3 Second-Order Conditions for Optimality

For a solution x^* to be a local maximum or minimum, the second-order conditions must hold as well. Conditions for a minimum are analogous.

1.3.1 Second-Order Sufficient Conditions (SOSC) for Lagrange

If x^* is a solution to the Lagrange problem and the first-order conditions hold, then x^* is a local maximum if:

$$d^T \nabla^2 L(x^*, \mu^*) d < 0 \tag{7}$$

for all $d \neq 0$ such that $\nabla h_j(x^*)^T d = 0$ for all j.

1.3.2 Second-Order Necessary Conditions (SONC) for KKT

If x^* is a local maximum for the KKT problem and the first-order conditions hold, then it is necessary that:

$$d^T \nabla^2 L(x^*, \lambda^*, \mu^*) d \leq 0 \tag{8}$$

for all d that satisfy the linearized constraints $\nabla g_i(x^*)^T d = 0$ for active constraints and $\nabla h_j(x^*)^T d = 0$ for all j.

For practical examples related to optimization problems with equality or inequality constraints, refer to Simon and Blume (1994). Next, we proceed with the exercises that are the focus of this problem session.

2 Exercises

Theoretical aspects of Microeconomic Theory which are not covered in this section can be found in (Echenique, 2005), Varian (1992), Mas-Colell et al. (1995) or Jehle and Reny (2011).

Exercise 2.1. Formulate the respective optimization problems and identify what type of problems they are (unconstrained, constrained, Lagrange, Kuhn-Tucker).

1. Utility maximization problem.

2. Expenditure minimization problem.

3. Profit maximization problem.
4. Cost minimization problem.
5. $\max_{x\in\mathbb{R}_{++}} \ \ln x - x$.
6. $\min_{x\in\mathbb{R}} \ x^2$.

Exercise 2.2. When the utility maximization problem (UMP) can be solved as a Lagrange problem? What conditions over u would you impose? What is the economic meaning?

Exercise 2.3. Solve the following optimization problems:

1. max x_1x_2 s.t. $x_1+x_2 \leq 1$, $x_1, x_2 \geq 0$.
2. max $\ln x_1 + \ln x_2$ s.t. $2x_1+3x_2 \leq 5$, $x_1, x_2 > 0$.

Exercise 2.4. Thomas Sargent (Tom) has the following utility function:

$$u(x) = \prod_{i=1}^{n} x_i^{\alpha_i}, \ 0 < \alpha_i < 1, \ \sum_{i=1}^{n} \alpha_i = 1.$$

Solve Tom's maximization problem considering $p \in \mathbb{R}^n_{++}$ and $I > 0$. Obtain the Marshallian demands for each good consumed by Tom and verify Roy's identity.

Exercise 2.5. Let $f(z_1, z_2) = z_1^{\alpha} z_2^{\beta}$, with $\alpha, \beta \in [0, 1]$. Show that

$$c(w_1, w_2, q) = q^{\frac{1}{\alpha+\beta}} \theta \phi(w_1, w_2)$$

where $\phi(w_1, w_2) = w_1^{\frac{\alpha}{\alpha+\beta}} w_2^{\frac{\beta}{\alpha+\beta}}$, $q > 0$ is the production level and

$$\theta = \left(\frac{\alpha}{\beta}\right)^{\frac{\beta}{\alpha+\beta}} + \left(\frac{\beta}{\alpha}\right)^{\frac{\alpha}{\alpha+\beta}}.$$

Note that $c(w_1, w_2, q)$ is the cost function.

Exercise 2.6. Solve the utility maximization problem for

$$u(x_1, x_2) = x_1 + x_2$$

in terms of p_1, p_2 and I. Apply KKT theorem.

Exercise 2.7. Solve the following maximization problem,

$$\begin{aligned} &\max \ x_1x_2 \\ &\text{s. t. } x_1 + x_2^2 \leq 2 \\ &\qquad x_1, x_2 \geq 0. \end{aligned}$$

Exercise 2.8. Let $n \geq 2$ and consider the following optimization problem,

$$\begin{aligned} \min \quad & \sum_{j=1}^{n} \frac{c_j}{x_j} \\ \text{s. t.} \quad & \sum_{j=1}^{n} x_j = 1 \\ & x_j \geq \epsilon, \end{aligned}$$

where $c_j > 0$ and $\epsilon > 0$ are parameters. Solve this problem applying KKT.

Exercise 2.9. Consider the following optimization problem,

$$\begin{aligned} \min \quad & -\sum_{i=1}^{n} \ln(\alpha_i + x_i) \\ \text{s. t.} \quad & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \end{aligned}$$

where $\alpha_i > 0$ are parameters. Solve this problem applying KKT.

More exercises in then following links: Applications of Lagrangian: Kuhn Tucker Conditions and NMSA403 Optimization Theory – Exercises.

3 Additional problems

Exercise 3.1. Tirole's expenditure function is given by:

$$e(\mathbf{p}, \overline{u}) = \exp\left\{ \sum_{\ell=1}^{L} \alpha_\ell \ln(p_\ell) + \left(\prod_{\ell=1}^{L} p_\ell^{\beta_\ell} \right) \overline{u} \right\}, \ \mathbf{p} \in \mathbb{R}^L_{++}.$$

Assume (this is known as the duality theorem) that $e(\mathbf{p}, V(\mathbf{p}, I)) = I$, where I is the income in the utility maximization problem and V is the indirect utility function. Derive Tirole's indirect utility function and verify Roy's identity. Impose any conditions you deem appropriate on the parameter vector (α, β)[3].

Exercise 3.2. Daron Acemoglu has preferences represented by $u(x_1, x_2) = (x_1 + 1)(x_2 + 1)$. Prove that Acemoglu has convex preferences. Are they strictly convex? Perform the same analysis for the preferences of Robert Barro, represented by $v(x_1, x_2) = \min\left\{\frac{x_1}{3}, \frac{x_2}{10}\right\}$.

Exercise 3.3. Consider the utility maximization problem with $p_1, p_2, I > 0$ and $u \in C^2(\mathbb{R}^2)$. Additionally, assume that $\frac{\partial^2 u}{\partial x_i^2} < 0$, $\frac{\partial u}{\partial x_i} > 0$, and $\frac{\partial^2 u}{\partial x_1 \partial x_2} > 0, i = 1, 2$. Assume that $\mathbf{x}^* \in \mathbb{R}^2_{++}$ satisfies the Lagrange equations. Using the method of differentials (comparative statics), determine the effect (whether positive, negative, or inconclusive) of $\dfrac{\partial x_2^*}{\partial I}$, where (x_1^*, x_2^*) is the solution to the utility maximization problem considered. Provide an interpretation.

[3]Recall that expenditure functions are concave with respect to prices, non-decreasing in p_ℓ, and increasing in $\overline{u}$.

4 Berge Theorem

This section is only for the interested reader/student. We mainly follow Ok (2007). You might want to refresh yourself in some topics such as metric spaces, convergence of sequences and continuity. The following lecture notes might also be a good reference.

Definition 4.1. A correspondence from a nonempty set X into another nonempty set Y is a map from X into $2^Y - \{\emptyset\}$. This is, for each $x \in X$, $\Gamma(x)$ is a non empty subset of Y.

Definition 4.2. For any $S \subset X$, the domain of S is

$$\Gamma(S) = \bigcup \{\Gamma(x) : \; x \in S\}.$$

We often denote $\Gamma : X \rightrightarrows Y$.

Example 4.1. Let $n \in \mathbb{N}$ and $p \in \mathbb{R}^n_{++}$, $I > 0$. Then, $B : \mathbb{R}^{n+1}_{++} \to \mathbb{R}^n_+$,

$$B(p, I) = \left\{ x \in \mathbb{R}^n_+ : \; \sum_{i=1}^{n} p_i x_i \leq I \right\}$$

is a correspondence, known as the budget correspondence.

Definition 4.3. For any two metric spaces X and Y, a correspondence $\Gamma : X \rightrightarrows Y$ is said to be upper hemicontinuous at $x \in X$ if for every open set $V \subset Y$, with $\Gamma(x) \subset U$, there existe $\delta > 0$ such that $\Gamma(N_{\delta,X}(x)) \subset U$. Here $N_{\delta,X}(x)$ is a neighborhood of x in X delimited by $\mathcal{B}(x, \delta)$.

Definition 4.4. For any two metric spaces X and Y, a correspondence $\Gamma : X \rightrightarrows Y$ is said to be compact-valued if $\Gamma(x)$ is a compact subset of Y for each $x \in X$.

Exercise 4.1. If $\Gamma : X \rightrightarrows Y$ is compact-valued and upper hemicontinuous, then $\Gamma(S)$ is compact in Y for any compact subset S of X.

Theorem 1. Let X and Y be two metric spaces and $\Gamma : X \rightrightarrows Y$ a correspondence. Then, Γ is upper hemicontinuous at $x \in X$ if for any $x_n \in X$ and $y_n \in Y$ with $x_n \to x$ and $y_n \in \Gamma(x_n)$ for each n, there exists $x_{n_k} \to y \in \Gamma(x)$. If Γ is compact-valued, the converse is also true.

Exercise 4.2. Prove that the budget correspondence $B : \mathbb{R}^{n+1}_{++} \to \mathbb{R}$ is upper-hemicontinuous.

Definition 4.5. For any two metric spaces X and Y, $\Gamma : X \rightrightarrows Y$ is said to be lower hemicontinuous at $x \in X$ if $\forall \; U \subset Y$ open with $\Gamma(x) \cap U \neq \emptyset$, there exists $\delta > 0$ such that $\Gamma(\tilde{x}) \cap U \neq \emptyset$, $\forall \; \tilde{x} \in N_{\delta,X}(x)$.

Theorem 2. Let X and Y be two metric spaces and $\Gamma : X \rightrightarrows Y$ a correspondence. Γ is lower hemicontinuous at $x \in X$ if and only if for any $x_n \in X$ with $x_n \to x$ and $y \in \Gamma(x)$, there is $y_n \in Y$ such that $y_n \to y$ and $y_n \in \Gamma(x_n)$ for each $n \in \mathbb{N}$.

Definition 4.6. Let X and Y be two metric spaces and $\Gamma : X \rightrightarrows Y$ a correspondence. We say that Γ is continuous at $x \in X$ if it is both upper and lower hemicontinuous.

Exercise 4.3. Prove that the budget correspondence is continuous.

Now, with all these definitions in place, we are ready to state the famous Berge's theorem.

Theorem 3. Berge. Let Θ and X be two metric spaces, $\Gamma : \Theta \rightrightarrows X$ a compact valued correspondence and $\varphi \in C^0(X \times \Theta, \mathbb{R})$. Define

$$\psi(\theta) = \operatorname{argmax}\{\varphi(x, \theta) : x \in \Gamma(\theta)\}$$
$$\varphi^*(\theta) \max\{\varphi(x, \theta) : x \in \Gamma(\theta)\}, \ \forall\, \theta \in \Theta.$$

Assume that Γ is continuous at $\theta \in \Theta$. Then,

(a) $\psi : \Theta \rightrightarrows X$ is compact valued, upper hemicontinuous and closed at θ

(b) $\varphi^* : \Theta \to \mathbb{R}$ is continuous at θ.

The proof can be found in Ok (2007).

Example 4.2. By Berge's theorem, Walrasian demand and indirect utility function are continuous whenever the utility function $u(\cdot)$ is continuous. Indeed, let $\Theta = \mathbb{R}^{n+1}_{++}$, $\theta = (p, I)$, $X = \mathbb{R}^n_+$, $u(x) = \varphi(x, \theta)$ and $\Gamma(\theta) = B(p, I)$, the budget correspondence. Then, $\psi(\theta) = x^*(p, I)$ and $\varphi^*(\theta) = V(p, I)$, the indirect utility function.

Lima, August 19, 2024.

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