Pontificia Universidad Católica del Perú
Economics Major

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Test 3
ECO 263

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Exercise 1. **(8 points)** Consider a scenario where two individuals, with endowments w_1, w_2, are evaluating the provision of a public good x whose cost is $C = 100$ soles. The individuals have preferences that depend on the money m^i and on $x \in \{0, 1\}$:

$$u_i(m_i, x) = m_i + 40\left(\frac{7}{4}\right)^{i-1} x,\ i = 1, 2.$$

(a) Find the reservation price of each individual.

(b) Evaluate whether the provision of the public good is efficient or not.

(c) Find the net value of the provision of the public good if its financing is distributed equally (shares are equal to $s_i = 1/2$).

(d) Assume now that $s_i = r_i / \sum_i r_i$. Is the public good provided? What is the issue with this mechanism?

(e) Suppose the Groves-Clarke mechanism is applied, such that the public good is provided if the sum of the net values reported by each individual is greater than zero $(\sum_i \tilde{v}_i)$, and if it is provided, side payments are made to each individual equal to the sum of the valuations reported by the others $\left(\sum_{i \neq j} \tilde{v}_j\right)$. Express mathematically the profit function of each individual. Explain why the best option for each agent will be to report their true net value (v_i).

Solution:

a) Simply

$$w_i - r_i + 40\left(\frac{7}{4}\right)^{i-1} = w_i \implies r_i = 40\left(\frac{7}{4}\right)^{i-1}.$$

Hence, $r_1 = 40$ and $r_2 = 70$.

b) Since $\sum_{i=1,2} r_i = 110 > 100$, it is efficient to supply the public good.

c) If $s_i = 1/2$, then, since the net valuation is equal to $VN_i = r_i - s_i C$,

$$\begin{aligned} v_1 &= 4 - \frac{100}{2} = -10 \\ v_2 &= 70 - \frac{100}{2} = 20. \end{aligned}$$

d) If $s_i = \frac{r_i}{\sum_j r_j}$, then, $s_1 = \frac{40}{110}$ and $s_2 = \frac{70}{110}$. Hence, $VN_1 = 40/11$ and $VN_2 = \frac{70}{11}$, both positives. The problem with this mechanism is that consumer might under-report their valuation in order to pay less. Nonetheless, this can lead to an underprovisioning of the public good.

e) To solve this issue, we introduce Groves mechanism. Payoffs are

$$\begin{cases} v_i + \sum_{j \neq i} \tilde{v}_j & \text{if } \sum_j \tilde{v}_j \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

Exercise 2. **(5 points)** Consider a rural community that must decide the number of pine trees ($G \in \mathbb{R}_+$) to plant around the village. This is done to protect the community from potential landslides in the winter. Each villager allocates their wealth (w_i) to contribute to the financing of the public good as well as to the purchase of private goods (m_i). The preferences of the villagers are represented by the following utility functions:

$$u_i(m_i, G) = \frac{m_i}{3} + i\gamma \ln(G), \ \gamma > 0, \ i = 1, \dots, N.$$

The production function of the public good is $G = F(z) = 8z$. Assume that the community is composed of $N = 20$ villagers.

a) Formulate and solve the optimization problem that allows the socially efficient level of the public good to be reached.

b) Determine the Lindahl equilibrium taxes $\boldsymbol{\lambda} \in \overline{\Delta} = \{\mathbf{x} \geq \mathbf{0}, \ ||\mathbf{x}||_1 = 1\}$. Why do the agents pay different amounts?

a) We need to solve

$$\begin{cases} \max & \sum_{i=1}^{20} \alpha_i u_i(m_i, G) \\ \text{s.t.} & \sum_{i=1}^{20} m_i + z_i = \sum_{i=1}^{20} w_i \\ & F(z) = 8z = 8\sum_{i=1}^{20} z_i = G. \end{cases}$$

Solving this yields to the well-known Samuelson-Lindhal condition leads to

$$\sum_{i=1}^{20} MRS_i = \frac{1}{f'\left(\sum_{i=1}^{20} z_i\right)}.$$

Hence,

$$\sum_{i=1}^{20} \frac{\frac{i\gamma}{G}}{\frac{1}{3}} = \frac{1}{8}.$$

Thus,

$$\frac{3\gamma}{G} \underbrace{\sum_{i=1}^{20} i}_{=\frac{20(20+1)}{2}} = \frac{1}{8}$$

$$G^{ef} = 5040\gamma.$$

b) To find Lindahl-taxes λ_i, we must solve

$$\begin{cases} \max & u_i(m_i, G) = \frac{m_i}{3} + i\gamma \ln G \\ \text{s.t} & m_i + \lambda_i pG = w_i \end{cases}$$

Replacing the restriction into the objective function, this simplifies to

$$\max_{G \geq 0} \left(\frac{w_i - \lambda_i pG}{3} \right) + i\gamma \ln G.$$

Using the concavity of the objective function FOC are enough:

$$-\frac{\lambda_i p}{3} + \frac{i\gamma}{G} = 0.$$

Since Lindahl taxes allow to recover the efficient level of the public good,[1]

$$\lambda_i = \frac{3i\gamma}{pG^{ef}} = \frac{24i}{5040}.$$

Lindahl taxes differ since the individuals have different valuations for he public good. Note that, as expected, $\sum_i \lambda_i = 1$.

Exercise 3. **(4 points)** Discuss whether the following statements are true or flase. **Justifiy** your awnser.

a) If preferences are quasilinear, in the context of discrete public goods, the provision of the public good—whether it is provided or not—does not depend on the wealth distribution $w_1, \cdots, w_I$.

b) For continuous public goods, Samuelson-Lindhal conditions states that $\sum_{h=1}^{N} MRS_h = f'(z)$, where $f(z)$ is the production function.

c) In an economy, there are only two individuals, Manuel and Carlos, and their marginal valuations for the tenth unit consumed of a public good are 12 and 18 soles, respectively. Explain the condition required for the optimal production level to be 10 units.

[1] $p = 1/f'(z) = 1/8$.

d) In an industry, there are N firms competing in quantities and facing a linear demand $p(Q) = a - bQ$ and constant marginal costs c, where $a > c > 0$ and $b > 0$. Assume that M firms merge, where $M \leq N$. Find the number of firms that will have profits after merging that are greater than before merging.

1. True since $\sum_i r_i = \sum_i v_i(1) \neq f(w_1, \cdots, w_I)$.

2. False, Samuelson-Lindahl condition states that $\sum_h MRS_h = \frac{1}{f'(z)} = MgC$.

3. To find the optimal provision of the public good, we use the Samuelson rule. Therefore, the sum of the marginal rate of substitution for each individual for the tenth unit (MRS_{10}^{M} and MRS_{10}^{C}) must satisfy the following:

$$\sum_{i=1}^{2} \text{MRS}_i = \text{MC} \Rightarrow 12 + 18 = 30$$

Thus, the condition is that the marginal cost of the tenth unit should be 30.

4. Benefits functions are given by $\pi_i = (a - bQ_{-i} - bq_i)q_i - cq_i$, where $Q = q_i + Q_{-i}$. Hence, FOC leads to $a - bQ_{-i} - 2bq_i - c = 0$. Symmetry allows us to write $Q_{-i} = (N-1)q_i$. Hence, $q_i^* = \frac{a-c}{b(1+N)}$, $Q = \frac{N(a-c)}{b(N+1)}$, $p = \frac{a+cN}{N+1}$ and $\pi_i = \frac{(a-c)^2}{b(N+1)^2}$. If M firms merge, $N - M + 1$ are alone and the benefits of the merged firm is $\frac{(a-c)^2}{b(2+N-M)^2}$. Firms will merge only if

$$\underbrace{\frac{(a-c)^2}{b(N-M+2)^2}}_{\text{profits merging}} \geq \frac{M(a-c)^2}{b(1+N)^2}.$$

This occurs only if $M \geq \left\lfloor \frac{1}{N}\frac{3+2N-\sqrt{5+4N}}{2} \right\rfloor$.

Exercise 4 (**3 points**). Let $X = \{0, 100, 400, 1000\}$ be a set of monetary rewards. Fernando has strongly monotonic preferences over these rewards (his elementary utility function $v(x)$ is strictly increasing). Fernando also declares that he is an expected utility maximizer. Cristina presents him with the following lotteries:

$$L = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}\right) \text{ and } L' = \left(0, \frac{1}{4}, \frac{10}{24}, \frac{8}{24}\right).$$

Fernando decides to choose L over L'. Is Fernando an expected utility maximizer?

Solution: we have that

$$U(L) = \sum_{n=1}^{N} v(x_n)p_n = \frac{1}{4}v(0) + \frac{1}{4}v(100) + \frac{1}{3}v(400) + \frac{1}{6}v(1000)$$

while

$$U(L') = \sum_{n=1}^{N} v(x_n)p'_n = 0 \cdot v(0) + \frac{1}{4}v(100) + \frac{11}{24}v(400) + \frac{7}{24}v(1000).$$

Then,

$$U(L') - U^e(L) = \frac{1}{8}v(1000) + \frac{1}{8}v(400) - \frac{1}{4}v(0).$$

Since v is strictly increasing (strong monotonic preferences), $v(1000) > v(400) > v(0)$. Therefore, $U(L') - U(L) > 0$. Thus, Fernando **DOES NOT** maximize his expected utility.