General Equilibrium
Under Uncertainty

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Note: I will closely follow Chapter 19 of Mas-Colell et al. Additionally, I will use lecture notes from Professors Pavel Coronado and José Gallardo. For the case in which there are only two goods, see Equilibrio, eficiencia e imperfecciones del mercado by Alejandro Lugon. Another reference is professor Federico Echenique's lecture notes General Equilibrium SS05. Finally, for the dynamic case, I will follow Ljyungqvist and Sargent, Recursive Macroeconomic Theory, 4th edition, Chapter 8.

A Market Economy with Contingent Commodities

In the following, we consider an economy with $n > 0$ goods (physical commodities), H consumers and J firms. The new element is that technologies, endowments and preferences are now uncertain. They depend on the state of the world. This concept was introduced in uncertainty theory. For simplicity, $S = \{1, \cdots, s, \cdots, S\}$ is taken finite (abusing notation).

Definition 1. For each commodity $\ell = 1, \cdots, n$ and state $s = 1, \cdots, S$, a unit of a state-contingent commodity ℓs is a title to receive a unit of the physical good ℓ if and only if s occurs. Accordingly, a state-contingent commodity vector is specified by

$$\underbrace{(x_1(1), ..., x_n(1))}_{x(1)}, ..., \underbrace{(x_1(s), ..., x_n(s))}_{x(s)}, ..., \underbrace{(x_1(S), ..., x_n(S))}_{x(S)} \in \mathbb{R}^{nS}.$$

Similarly, a price vector of contingent goods would be given by:

$$\underbrace{(p_1(1), ..., p_n(1))}_{p(1)}, ..., \underbrace{(p_1(s), ..., p_n(s))}_{p(s)}, ..., \underbrace{(p_1(S), ..., p_n(S))}_{p(S)}.$$

Remark. A contingent commodity vector $(x_1(s), \cdots, x_n(s))$ can be viewed as a collection of n random variables, $x_\ell : S \to \mathbb{R}$.

Similarly, we can define the endowments of consumers $h = 1, \cdots, H$ as the contingent vector

$$\omega^h = \underbrace{(\omega_1(1), ..., \omega_n(1))}_{\omega^h(1)}, ..., \underbrace{(\omega_1(s), ..., \omega_n(s))}_{\omega^h(s)}, ..., \underbrace{(\omega_1(S), ..., \omega_n(S))}_{\omega^h(S)}.$$

The preferences of consumers may also depend on the state of the world. For instance, the enjoyment of wine may depend on the health state of the individual. The preference of consumer h, $\succeq_h$ is defined over $X_h \subset \mathbb{R}^{nS}$.

Consumers evaluate contingent commodity vectors by first assigning to the state s a probability π_{sh}. Index h points out that the probability distribution

$$\pi^h = (\pi_1^h, \cdots, \pi_s^h, \cdots, \pi_S^h) \in \Delta(S)$$

depends on the individual. Then, evaluating the commodity vector at sate s, according to a Bernouilli state-dependent utility function

$$u_{sh}(x_1^h(s), \cdots, x_n^h(s)).$$

Hence, the preferences of consumer h over two contingent commodity vectors $x^h, x^{h'} \in X_h \subset \mathbb{R}^{nS}$ satisfies

$$x^h \succeq_h x^{h'} \Leftrightarrow \sum_{s \in \{1, \cdots, S\}} \pi^h(s) u_s^h(x^h(s)) \geq \sum_{s \in \{1, \cdots, S\}} \pi^h(s) u_s^h(x^{h'}(s))$$

We denote $U^h(x^h) = \sum_{s \in \{1, \cdots, S\}} \pi^h(s) u_s^h(x^h(s))$.

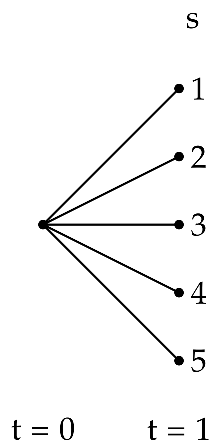
Example 1. Suppose there are only two states s_1 and s_2, representing good and bad weather. There are two commodities, seeds ($\ell = 1$) and crops ($\ell = 2$). Hence, the elements of Y_j are four dimensional vectors. Seeds must be planted before the resolution of uncertainty about the weather and a unity of seeds produce a unit of crops only if the weather is good. Thus,

$$y_j = (y_{11j}, y_{21j}, y_{12j}, y_{22j}) = (-1, 1, -1, 0),$$

where the convention is $y_{\ell s}$, is a feasible plan.

Remark. If we introduce production, we assume state independence on shares $\theta_{jh} \geq 0$, $\sum_j \theta_{jh} = 1$.

Remark. In the setting described previously in this section,, time plays no explicit formal role. In reality, nonetheless, states of the world unfold over time. Moreover, it is possible to consider $t = 0, ..., T$ periods (so $T + 1$ dates), S states and assume that the states emeerge gradually through a tree.



Arrow-Debreu Equilibrium

We now postulate the existence of a market for every contingent commodity ℓs. These market open before the resolution of uncertainty, at date 0 we could say. The price of the commodity is denoted $p_{\ell s}$. What is purchased or sold in the market for the contingent commodity ℓs is commitments to receiver or deliver amounts of the physical good ℓ, if and when state of the world s occurs. Note that, although commodities are contingent, payments are not. It is also mandatory that every agent is able to recognize the occurrence of state s. The model described is nothing but a particular (with a large number of goods) case of the economies studied in classical general equilibrium theory.

When dealing with contingent commodities, we refer to Arrow-Debreu equilibrium instead of Walrasian equilibrium.

Definition 2. An allocation

$$(x_1^*, \cdots, x_H^*, \cdots, y_1^*, \cdots, y_J^*) \in \prod_{h=1}^{H} X_h \times \prod_{j=1}^{J} Y_j \subset \mathbb{R}^{nS(H+J)}$$

and a system of prices for the contingent commodities $p = (p_{11}, \cdots, p_{nS}) \in \mathbb{R}^{nS}$ constitute an Arrow-Debreu equilibrium if

1. For very j, y_j^* satisfies $p \cdot y_j^* \geq p \cdot y_j$, $\forall\, y_j \in Y_j$.
2. For every h, x_h^* is a maximal element for $\succeq_h$ in the budget set
$$\left\{ x_h \in X_h : \; p \cdot x_h \leq p \cdot \omega_h + \sum_{j=1}^{J} \theta_{hj} p \cdot y_j^* \right\}$$
3. $\sum_{h=1}^{H} x_h^* = \sum_{j=1}^{J} y_j^* + \sum_{h=1}^{H} \omega_h \in \mathbb{R}^{nS}$.

Example 2. Let us consider the case with $H = 2$, $n = 1$ and $S = 2$. This lends itself into an Edgeworth box representation since there are precisely two commodities and two individuals.

1. Endowments: $\omega_1 = (1,0)$, $\omega_2 = (0,1)$, each entry for each state.
2. No aggregate risk: $\omega = \omega_1 + \omega_2 = (1,1)$. Uncertainty determines who gets the endowment.
3. $u_h(\cdot)$ state independent:

$$u_h(x_{1h}, x_{2h}) = \pi_{1h} u_h(x_{1h}) + \pi_{2h} u_h(x_{2h}).$$

4. Same probability distribution for each h. Hence, since at $x_{1h} = x_{2h}$

$$MRS_h = \frac{\pi_{1h}}{\pi_{2h}},$$

we have

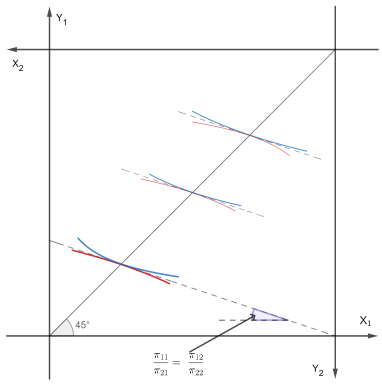


Figure 1: No aggregate risk and same subjective probabilities.

If we assume that $\pi_{11} < \pi_{12}$ (so the second consumer gives more probability so state 1), we would have

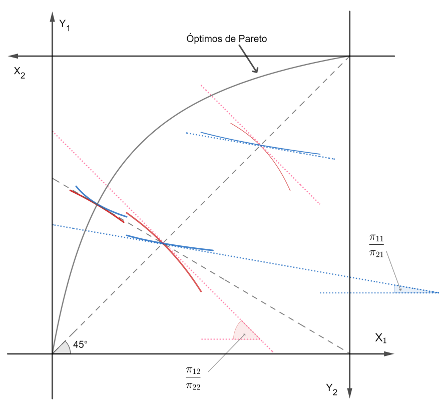


Figure 2: No aggregate risk and different subjective probabilities.

Each consumer's equilibrium consumption is higher in the state he thinks comparatively more likely (relative to the belief of the other consumer).

Example 3. Now we consider aggregate risk $\omega_1 + \omega_2 = (2, 1)$. Utilities and probabilities are the same, $u(\cdot)$ and (π_1, π_2). A graphical analysis of indifference curves yields

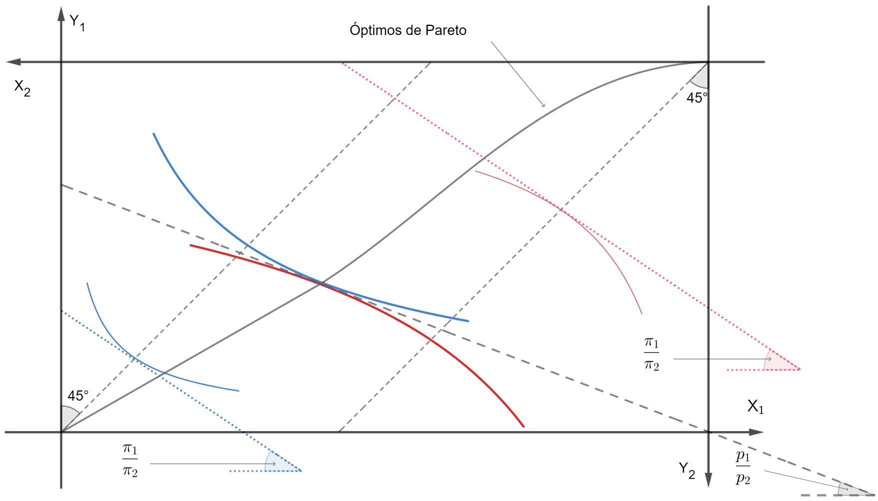


Figure 3: Aggregate risk and same subjective probabilities.

At the equilibrium, by a graphical analysis, we must have

$$\frac{p_1}{p_2} < \frac{\pi_1}{\pi_2}.$$

If, for instance, $\pi_1 = \pi_2 = 1/2$, then $p_1 < p_2$: the price of one contingent unit of consumption is larger for the state for which the consumption good is scarcer: contingent instruments are comparatively more valuable if their returns are negatively correlated with the market return.

Sequential trading

We introduce a model of se sequential trade and state that Arrow-Debreu equilibrium can be reinterpreted by means of trading processes that unfold through time.

We only consider pure exchange economies and $X_h = \mathbb{R}^{nS}_+$. We assume that there are two dates, $t = 0$ and $t = 1$. There is no consumption at $t = 0$. There are again nS possible contingent commodities which are set up at $t = 0$ and $(x_1^*, \cdots, x_I^*) \in \mathbb{R}^{nSH}$ is an Arrow-Debreu equilibrium allocation with prices $(p_{11}, \cdots, p_{nS}) \in \mathbb{R}^{nS}$. These markets are for delivery of goods at $t = 1$! When period $t = 1$ arrives, a state of the world s is revealed and contracts are executed, and every consumer receives

$$x_h^*(s) = (x_{1h}^*(s), \cdots, x_{nh}^*(s)) \in \mathbb{R}^n.$$

Suppose nonetheless that, after $t = 1$ and the resolution of the uncertainty, but before the consumption, markets for the n physical goods were open (spot markets). **Question:** would there be any incentive to trade in these markets? Awnser: **no**. Why? violation of Pareto optimality (first welfare theorem). Hence, at $t = 0$, the consumers can trade directly to an overall Pareto optimal allocation: there is nor reason for further trade to take place, ex ante Pareto implies ex post Pareto. **However**, if no all of the nS contingent commodity markets are available at $t = 0$, the business is complectly different.

Arrow (1953), noted that, even if not all the contingent goods are available at $t = 0$, it may still be the case, under some conditions, that at $t = 1$ Pareto optimality is reached. When is this? Well, if at least one of the physical commodities can be traded contingently at $t = 0$, spot markets occur at $t = 1$ and the spot equilibrium prices are correctly anticipated. **Intuition**: if spot trade can occur within each state, then the only task remaining at $t = 0$ is to transfer the consumer's overall purchasing power efficiently across states. This last can be accomplished using contingent trade in a **single** commodity. This allows to reduce the number of required forward markets from LS to S.

Synthesis: for the Arrow-Debreu Equilibrium to exist, it is necessary to have a large number of contingent goods markets (nS). This is a particularly strong requirement. It is however possible to relax this assumption by adopting a sequential structure in the economy and assuming the existence of only one contingent good for each state of nature (Arrow securities).

1. At $t = 0$, consumers have expectation regarding the spot prices at $t = 1$, for each $s \in \{1, \cdots, S\}$.

2. Let us denote the expected vector of prices to prevail in state s in the sport market by $p(s) \in \mathbb{R}^n$.

3. Let use denote the overall expectation vector by $p \in \mathbb{R}^{ns}$. We are implicitly assuming that expectation coincide between consumers.

4. At $t = 0$, there is trade in the S contingent commodities, $11, \cdots, 1s, \cdots, 1S$: there is contingent trade only in the good with label 1.

5. We denote the vector of prices of such contingent commodities traded at $t = 0$ by $q = (q(1), \cdots, q(S)) \in \mathbb{R}^S$.

6. Faced with prices $q \in \mathbb{R}^S$ at $t = 0$ and expected spot prices $(p(1), \cdots, p(S)) \in \mathbb{R}^{nS}$ at $t = 1$, the consumer h formulates a trading plan $(z^h(1), \cdots, z^h(s)) \in \mathbb{R}^S$ for contingent commodities at $t = 0$, as well as a set of spot market consumption plans $(x^h(1), \cdots, x^h(S)) \in \mathbb{R}^{nS}$.

7. The utility maximization problem faced by consumer h is therefore

$$\max_{\substack{\{x(1)^h,\ldots,x(S)^h\}\in\mathbb{R}^{nS}_{+},\\ \{z(1)^h,\ldots,z(S)^h\}\in\mathbb{R}^{S}}} \quad U_h(x(1)^h,\cdots,x(S)^h) \tag{1}$$

$$\text{s.t.} \sum_{s=1}^{S} q(s)z^h(z) \leq 0$$

$$p(s)\cdot x^h(s) \leq p(s)\omega^h(s) + p_1(s)z^h(s),\ \forall\, s\in\{1,\cdots,S\}.$$

The first restriction is the budget constraint corresponding to the trade at $t = 0$. The family of restrictions regarding the second line are the budget constraints for the different spot markets (one for each state). Note that $z^h(s)$ could be negative or positive. If $z^h(s) < -\omega^h_1(s)$, the one says that at $t = 0$ consumer h is selling good 1 for short. This is because he is selling at $t = 0$, contingent on state s ocurren, more than he has at $t = 1$ is s occurs.

Definition 3. A Radner equilibrium (Arrow securities) is an allocation

$$\{(x^{h*},z^{h*})\}_h \in \mathbb{R}^{H(nS+S)}$$

and a price vector $(p^*,q^*)\in\mathbb{R}^{nS+S}$ such that:

1. For each agent h, (x^{h*},z^{h*}) solves problem (1).
2. For each state s, $\sum_h z^{h*}(s) \leq 0$ and $\sum_h x^{h*}(s) \leq \sum_h \omega^h(s)$.

Proposition 1. *If the allocation $x^*\in\mathbb{R}^{nSH}$ and the contingent commodities price vectors $\{p(1),\cdots,p(S)\}\in\mathbb{R}^{nS}_{++}$ constitute an Arrow-Debreu equilibrium, then there are prices $q\in\mathbb{R}^S_{++}$ for contingent first good commodities and consumption plans for these commodities $\{z^{1*},\cdots,z^{H*}\}\in\mathbb{R}^{SH}$ such that the equilibrium plans x^*,z^* and the prices q and the spot prices $(p_1,\cdots,p_S)$ constitute a Radner equilibrium.*

Proposition 2. *If the consumption plans $x^*\in\mathbb{R}^{nHS}$, $z^*\in\mathbb{R}^{SH}$ and prices $q\in\mathbb{R}^S_{++}$, $(p(1),\cdots,p(S))\in\mathbb{R}^{nS}_{++}$ constitute a Radner equilibrium, then there are multipliers*

$$(\mu(1),\cdots,\mu(S))\in\mathbb{R}^S_{++}$$

such that the allocation x^* and the contingent commodities price vector

$$\{\mu(1)p(1),\cdots,\mu(S)p(S)\}\in\mathbb{R}^{nS}_{++}$$

constitute an Arrow-Debreu equilibrium.[1]

Proof. See Mas-Colell, Whinston and Green (1995), pp. 697. □

[1]The multiplier μ_s is interpreted as the value at $t = 0$ of a dollar at $t = 1$ and state s.

Asset Markets

The S contingent commodities studied in the previous section serve the purpose of transferring wealth across the states of the world that will be revealed in the future ($t = 1$). They are, however, theoretical constructs that rarely have exact counterparts in the real world.

Nonetheless, in the real world there are assets, or securities that, to some extent perform the wealth-transferring role which was assigned to the contingent commodities.

1. We consider again two periods, $t = 0$ and $t = 1$ (which is when the information is revealed).
2. Consumption takes only place at $t = 1$.
3. An asset (a unit of an asset), is a title to receive either physical goods or dollars at $t = 1$, and the amount depends on s.
4. The payoffs of an asset are called returns. If the returns are in physical goods, the asset is called real. Otherwise, they are called financial assets.
5. We are going to deal only with real assets.

Definition 4. A unit of an asset or security is a title to receive an amount r_s of good 1 at $t = 1$ if s occurs. An assert is therefore characterized by its return vector $r = (r_1, \cdots, r_S) \in \mathbb{R}^S$.

Example 4. Examples of assets include:

1. $r = (1, \cdots, 1)$, riskless asset.
2. $r = (0, \cdots, 0, 1, 0, \cdots, 0)$.
3. $r = (1, 2, 1, \cdots, 1, 2)$.

Example 5. **Options.** This is an example of a so-called derivative asset, i.e., an asset whose return are somehow derived from the returns of another asset. Hence, suppose there is an initial asset $r \in \mathbb{R}^S$. Then, en European call option on the primart at the strike price $c \in \mathbb{R}$ is itself an asset. A unit of this asset gives the option to buy, after the state is revealed, a unit of the primary asset at price c. Formally,

$$r(c) = (\max\{0, r_1 - c\}, \cdots, \max\{0, r_S - c\}).$$

1. The number of assets in the economy is K. The prices of the assets traded in period $t = 0$ are $q = (q_1, q_2, \ldots, q_K)$.
2. A vector of traded assets, denoted by $z = (z_1, z_2, \ldots, z_K) \in \mathbb{R}^K$, is called a portfolio.

Definition 5. A Radner equilibrium (Assets) is an allocation $\{(x^{h*}, z^{h*})\}_h \in \mathbb{R}^{H(nS+K)}$ and a price vector $(p^*, q^*) \in \mathbb{R}^{nS+K}$ such that

1. For each h, (x^{h*}, z^{h*}) maximizes expected utility
$$U^h(x(1)^h, \ldots, x(S)^h) = \sum_{s=1}^{S} \pi^h(s) u^h(x(s))$$
subject to $\sum_{k=1}^{K} q_k z_k^h \leq 0$ and $p(s)x(s)^h \leq p(s)\omega(s)^h + \sum_{k=1}^{K} p_1(s) z_k^h r_k(s)$ for all s.
2. $\sum_{h=1}^{H} z_k^{h*} \leq 0$ for all k.
3. $\sum_{h=1}^{H} x(s)^{h*} \leq \sum_h \omega(s)^h$ for all s.

Remark. The budget constraint is given by

$$\left\{ x \in \mathbb{R}_+^{nS} : \exists\, z_h \in \mathbb{R}^K,\ q \cdot z_h \leq 0 \ \wedge \ \begin{bmatrix} p(1)(x^h(1) - \omega^h(1)) \\ \vdots \\ p(S)(x^h(S) - \omega^h(S)) \end{bmatrix} \leq \begin{bmatrix} r_1(1) & \cdots & r_K(1) \\ \vdots & \ddots & \vdots \\ r_1(S) & \cdots & r_K(S) \end{bmatrix} \begin{bmatrix} z^h(1) \\ \vdots \\ z^h(S) \end{bmatrix} \right\}.$$

Proposition 3. *Assume that every return vector is nonnegative and nonzero, that is, $r_k \geq 0$ and $r_k \neq 0$ for all $k = 1, \cdots, K$. Then, for every $q \in \mathbb{R}^K$ of asset prices arising in a Radner equilibrium, we can find $\mu \in \mathbb{R}_+^S$ such that $q_k = \sum_{s=1}^{S} \mu(s) r_k(s)$.*

Dynamics

In each period $t \geq 0$, there is a realization of a stochastic event $s_t \in S$. We denote the history of events up and until time t be denoted $s^t = [s_0, s_1, \cdots, s_t]$. The unconditional probability of observing a particular sequence of events s^t will be denoted $\pi_t(s^t)$, a probability measure. For $t > \tau$, we write the probability of observing s^t conditional on the realization of s^τ as $\pi_t(s^t|s^\tau)$. Now, we assume that trade occurs after observing s_0 (hence, $\pi_0(s_0) = 1$, an initial state.

The model considers $h = 1, \cdots, H$ consumers. Consumer h owns a stochastic endowment of one good $y_t^h(s^t)$ that depends on the history s^t. The history is observed by every one. Consumer h purchases a history-dependent consumption plan $c^h = \{c_t^h(s^t)\}_{t=0}^\infty$ and orders (in terms of an implicity preference) these consumption streams by

$$\underbrace{U^h(c^h) = \sum_{t=0}^\infty \sum_{s^t} \beta^t u^h(c_t^h(s^t))\pi_t(s^t)}_{Expected \quad utility.}, \quad \beta \in (0,1). \tag{2}$$

Note that this is equivalente to

$$\mathbb{E}_0 \left[\sum_{t=0}^\infty \beta^t u^h(c_t^h) \right],$$

where $\mathbb{E}_0$ denotes the mathematical expectation operator conditioned on s_0. We assume, as it is natural and usual, that $u^h(c)$ is twice differentiable, increasing and strictly concave for $c \geq 0$. Moreover, $\lim_{c \downarrow 0} \frac{d}{dc} u^h = \infty$ (Inada condition). A feasible allocation satisfies

$$\sum_{h=1}^H c_t^h(s^t) \leq \sum_{h=1}^H y_t^h(s^t), \ \forall\ t, s^t. \tag{3}$$

Pareto problem

As a benchmark against which ti measure allocations attained by a market economy, we seek efficient allocations. In particular, we will be interested in Pareto optimal/efficient allocations. In this situation, the central planner attaches nonegative Pareto weights λ_h to the consumer's utilities and chooses allocations c^h to maximize

$$W = \sum_{h=1}^H \lambda_h U^h(c^h) = \sum_{t=0}^\infty \sum_{s_t} \left\{ \sum_{h=1}^H \lambda_h \beta^t u_h(c_t^h(s^t))\pi_t(s^t) \right\}.$$

Hence, if $\theta_t(s^t)$ is the Lagrange multiplier associated feasibility constraint (3),

$$\mathcal{L} = \sum_{t=0}^\infty \sum_{s_t} \left\{ \sum_{h=1}^H \lambda_h \beta^t u_h(c_t^h(s^t))\pi_t(s^t) + \theta_t(s^t) \sum_{h=1}^H (y_t^h(s^t) - c_t^h(s^t)) \right\}.$$

FOC with respect to $c_t^h(s^t)$ yields

$$\beta^t u_h'(c_t^h(s^t))\pi_t(s^t) = \lambda_h^{-1}\theta_t(s^t),$$

for each h, t, s^t. Hence,

$$\frac{u_h'(c_t^h(s^t))}{u_1'(c_t^1(s_t))} = \frac{\lambda_1}{\lambda_h}$$
$$c_t^h(s^t) = [u_h']^{-1}(\lambda_h^{-1}\lambda_1 u_1'(c_t^1(s^t))).$$

Thus, replacing in the feasibility condition,

$$\sum_h [u_h']^{-1}(\lambda_h^{-1}\lambda_1 u_1'(c_t^1(s^t))) = \sum_h y_t^h(s^t).$$

Time 0 trading: Arrow-Debreu securities

Consumers trade a complete set of dated history-contingent claims to consumption. Trades occur at time 0, after s_0 is realized. At $t = 0$, consumers can exchange claims on time t consumption, contingent on history s^t at price $q_t^0(s^t)$. The consumer's budget constraint is

$$\sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)c_t^h(s^t) \leq \sum_{t=0}^{\infty}\sum_{s^t} q_t^0(s^t)y_t^h(s^t).$$

The consumer's problem is to chose c^h in order to maximize (2). In this model, all trades occur at $t = 0$. After time 0 trades that were agreed to at time 0 are executed. FOC conditions lead to

$$\frac{\partial U_h(c^h)}{\partial c_t^h(s^t)} = \beta^t u_h'(c_t^h(s^t))\pi_t(s^t) = \mu_h q_t^0(s^t),\ \forall\ h, t, s^t.$$

Thus, for any $h_1, h_2 \in \{1, \cdots, H\}$,

$$\frac{u_{h_1}'(c_t^{h_1}(s^t))}{u_{h_2}'(c_t^{h_2}(s^t))} = \frac{\mu_{h_1}}{\mu_{h_2}}.$$

Therefore,

$$c_t^h(s^t) = [u_h']^{-1}\left(u_1'(c_t^1(s^t))\frac{\mu_h}{\mu_1}\right)$$
$$\sum_h [u_h']^{-1}\left(u_1'(c_t^1(s^t))\frac{\mu_h}{\mu_1}\right) = \sum_h y_t^h(s^t).$$

For $u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}$, $\gamma > 0$, the CRRA Bernouilli's utility function,

$$c_t^h(s^t) = c_t^{\tilde{h}}(s^t) \left(\frac{\mu_h}{\mu_{\tilde{h}}}\right)^{-1/\gamma}.$$

Remark. No aggregate risk: assume there are only two consumers with endowments, $y_t^1(s^t) = s_t$ and $y_2(s^t) = 1 - s_t$. Then, $\sum_h y_t^h(s^t) = 1$. Hence,

$$c_t^h(s^t) = \bar{c}^h = (1-\beta) \sum_{t \geq 0} \sum_{s^t} \beta^t \pi_t(s^t) y_t^h(s^t).$$

See Ljyungqvist and Sargent Chapter 8 for more details.

Asset Pricing

This section is based in Federico Echenique's lecture notes General Equilibrium SS05. We now turn to a two-period model where consumption occurs at both $t = 0$ and $t = 1$. We assume a single physical good. Time is indexed by $t = 0, 1$, and uncertainty at $t = 0$ is resolved at $t = 1$. This is the second model studied before.

Basic Definitions and Notation

1. Let $S = \{s_1, s_2, \dots, s_m\}$ be a set of states.
2. A column vector $c \in \mathbb{R}^{1+m}$ is called a cash flow.
3. A column vector $a = (a_1, \dots, a_m)' \in \mathbb{R}^m$ is called an asset, where a_k is the payment of asset a in period 1 under state s_k.
4. Let $\{a^1, \dots, a^J\}$ be a collection of J assets. Collect them in a matrix A with the j-th column equal to a^j. That is,

$$A = \begin{bmatrix} a^1 & a^2 & \cdots & a^J \end{bmatrix}_{m \times J}.$$

Examples of Assets

Example 6. A risk-free asset is given by

$$a^{rf} = \begin{bmatrix} 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathbb{R}^m.$$

Example 7. An Arrow-Debreu security $a_k^{AD} = e_k$ delivers a unit of the good if and only if the realized state is s_k.

Example 8. An option. Suppose $s_j = j$, where the state represents the value of a stock market index, and consider an option to buy the index at a fixed strike price p. The asset can be written as:

$$a = (0, \dots, 0, j - p, (j+1) - p, \dots, m - p),$$

where $j - p \geq 0$ and $j - 1 - p < 0$. This option is exercised only when the price s exceeds p, providing a payoff of $s - p$.

Cash Flows and Portfolios

Let $q_j \in \mathbb{R}_+$ be the price of asset a^j. Purchasing one unit of asset a^j generates the following cash flow:

$$(-q_j, a_1^j, \dots, a_m^j) \in \mathbb{R}^{1+m}.$$

Analogously, selling one unit of a^j at price q_j generates the cash flow:

$$(q_j, -a_1^j, \dots, -a_m^j) \in \mathbb{R}^{1+m}.$$

Define $q = (q_1, \ldots, q_J) \in \mathbb{R}^J_+$ as the vector of asset prices, where q_j is the price of asset a^j. Define W as:

$$W = \begin{bmatrix} -q \\ A \end{bmatrix}_{(1+m)\times J}.$$

The j-th column of W represents the cash flow generated by purchasing one unit of asset a^j.

Notation 1. We denote:

$$\langle W \rangle = \{Wz : z \in \mathbb{R}^J\}$$

as the set of cash flows that can be achieved through a portfolio of assets $\{a^1, \ldots, a^J\}$. Note that $\langle W \rangle$ is a linear subspace of $\mathbb{R}^{1+m}$.

Definition 6. A **market** is a pair (A, q).

Definition 7. An **arbitrage opportunity** is a cash flow $c \in \mathbb{R}^{1+m}$ such that $c > 0$. A market (A, q) is free of arbitrage opportunities if there is no arbitrage opportunity in $\langle W \rangle$.

Fundamental Theorem of Arbitrage Pricing

Theorem 1. *Let (A, q) be a market. The following statements are equivalent:*

1. *For any continuous and strictly monotonic utility function $u(\cdot) : \mathbb{R}^{1+m}_+ \to \mathbb{R}$, the following maximization problem has a solution:*

   $$\max_x u(x) \quad s.t. \quad x \in B(\omega, A, q),$$

   where $B(\omega, A, q) = \{x \in \mathbb{R}^{1+m} : \exists z \in \mathbb{R}^J \text{ s.t. } x \leq \omega + Wz\}$.

2. *The market (A, q) is free of arbitrage opportunities.*

3. *There exists $\pi \in \mathbb{R}^{1+m}_{++}$ such that $\pi W = 0$.*

4. *$B(\omega, A, q)$ is compact, and there exists $\pi \in \mathbb{R}^{1+m}_{++}$ such that:*

   $$B(\omega, A, q) \subseteq \{x \in \mathbb{R}^{1+m}_+ : \pi \cdot x \leq \pi \cdot \omega\}.$$

Proof. See General Equilibrium Theory: SS205. □

Farkas' Lemma

Lemma 1. *Let $W \in \mathcal{M}_{n\times m}$. Then, exactly one of the following statements is true:*

1. *There exists $z \in \mathbb{R}^m$ such that $Wz > 0$.*

2. *There exists $\pi \in \mathbb{R}^n_{++}$ such that $\pi W = 0$.*

About Vector π

Let us discuss a little bit more about vector π which is such that

$$\begin{aligned}\pi W &= \pi \begin{bmatrix} -q \\ A \end{bmatrix} \\ &= \pi \begin{bmatrix} -q_1 & \cdots & -q_j & \cdots & -q_J \\ a_1^1 & \cdots & a_1^j & \cdots & a_1^J \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ a_m^1 & \cdots & a_m^j & \cdots & a_m^J \end{bmatrix} \\ &= [0]_{(1+m)\times J}.\end{aligned}$$

Therefore, $\forall\ j = 1, \ldots, J$,

$$\begin{bmatrix} \pi_0 & \pi_1 & \ldots & \pi_m \end{bmatrix} \begin{bmatrix} -q_j \\ a_1^j \\ \vdots \\ a_m^j \end{bmatrix} = \pi_0(-q_j) + \sum_{i=1}^m \pi_i a_i^j = 0.$$

We can write:

$$q_j = \sum_{i=1}^m \left(\frac{\pi_i}{\pi_0}\right) a_i^j,\ \forall\ j = 1, \ldots, J.$$

Thus, the price of asset j, q_j, at time $t = 0$ is a weighted sum of its future payments under the different states $a_1^j, \ldots, a_m^j$. The weight on the payoff a_i^j in state s_i is $\frac{\pi_i}{\pi_0}$.

State Price Vector

Let us normalize the vector π by defining:

$$\overline{\pi} = \left[1,\ \ \frac{\pi_1}{\pi_0},\ \ \cdots,\ \ \frac{\pi_m}{\pi_0}\right].$$

Since $\pi W = 0$, we have $\frac{\pi}{\pi_0} W = \overline{\pi} W = 0$. Therefore:

$$q_j = \sum_{i=1}^m \overline{\pi}_i a_i^j,\ \forall\ j = 1, \ldots, J.$$

Hence, $\overline{\pi} \in \Delta \subset \mathbb{R}_{++}^m$ is a price vector. The price of a unit good at time $t = 0$ is 1, and the price of a unit good at time $t = 1$ and state s_i is $\overline{\pi}_i$.

The state price vector $\overline{\pi}$ allows us to price assets. For example, let $y_i \in \mathbb{R}$ for every state $i = 1, \ldots, m$, and define the column vector $\tilde{y} = (y_1, y_2, \ldots, y_m)' \in \mathbb{R}^m$, so that $\tilde{y}$ is an asset. The price of asset $\tilde{y}$ can be calculated as:

$$\sum_{i=1}^m \overline{\pi}_i y_i.$$

Risk-Free Asset and Risk-Free Rate

A risk-free asset is $a^{rf} = (1, \ldots, 1) \in \mathbb{R}^m$. In a market with no arbitrage, the price of the risk-free asset is:

$$q^{rf} = \sum_{i=1}^{m} \overline{\pi}_i.$$

Define the rate of return of the risk-free asset as:

$$R^{rf} = \frac{1}{q^{rf}} = \frac{1}{\sum_{i=1}^{m} \overline{\pi}_i}.$$

This rate, R^{rf}, is known as the **risk-free rate**.

Risk-Neutral Probability Measure

Definition 8. Risk-Neutral Probability Measure. For any asset j, the expected rate of return on j is the expected value of the random variable $s \mapsto a_s^j / q^j$, which we denote $(R_s^j)_{s \in S}$ or $\tilde{R}^j$.

The expectation of $\tilde{R}^j$ depends on the probability measure used. A particularly useful probability measure can be defined as:

$$p_i = \frac{\overline{\pi}_i}{\sum_{h=1}^{m} \overline{\pi}_h},$$

so that $p = (p_1, \ldots, p_m) \in \Delta$ is a probability distribution over S. This distribution p is termed the **risk-neutral probability measure**. Its name comes from the following calculation:

$$\frac{q^j}{q^{rf}} = \sum_{h=1}^{m} p_h a_h^j = \mathbb{E}_p[\tilde{a}^j],$$

and:

$$R^{rf} = \frac{1}{q^{rf}} = \sum_{i=1}^{m} p_i \frac{a_s^j}{q^j} = \mathbb{E}_p[\tilde{R}_s^j].$$

The Rate of Return and Risk-Neutral Probability Measure

The rate R^{rf} is known as the risk-free rate.

Definition 9. Risk-neutral probability measure. For any asset j, the expected rate of return on j is the expected value of the random variable $s \mapsto a_s^j / q^j$, which we denote $(R_s^j)_{s \in S}$ or by $\tilde{R}^j$.

The expectation of $\tilde{R}^j$ depends on the probability measure used. A particularly useful probability measure is given by:

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and:

$$R^{rf} = \frac{1}{q^{rf}} = \sum_{i=1}^{m} p_i \frac{a_s^j}{q^j} = \mathbb{E}_p[\tilde{R}_s^j].$$

Market Incompleteness

Let (q, A) be a financial market, and define the matrix W as before. Suppose the market is free of arbitrage.

Definition 10. The market (A, q) is **complete** if $\dim(\langle W \rangle) = |S| = m$. Otherwise, the market (q, A) is **incomplete**.

When a market is incomplete, agents can use the assets to carry out transfers of the good across states.

Proposition 4. *The market (q, A) is complete if and only if dim$(\langle A \rangle) = m$.*

Proof. When (q, A) is free of arbitrage, $q = \overline{\pi} \cdot A$ (equivalence 3 in Theorem **??**). Thus, q is a linear combination of the rows of A. Therefore, A and W have the same rank. $\square$

Observation 1. *When the market is free of arbitrage, then $\pi W = 0$ implies that $\pi \in \langle W \rangle^{\perp}$ (orthogonal complement of $\langle W \rangle$), so that $\langle W \rangle^{\perp} \neq \emptyset$. Since $\pi \neq 0$, dim$(\langle W \rangle) \geq 1$.*

The Capital Asset Pricing Model (CAPM)

Traditional CAPM

The traditional CAPM is summarized by the following equation:

$$\mathbb{E}[\tilde{R}_j] = R^{rf} + \beta(\mathbb{E}[\tilde{R}^m] - R^{rf}),$$

where R^m is the market rate of return. In practice, R^m is taken as the return of some market index such as the S&P500. The CAPM equation is a linear regression:

$$\beta = \frac{\mathrm{Cov}(\tilde{R}^j, \tilde{R}^m)}{\mathrm{Var}(\tilde{R}^m)}.$$

The CAPM indicates that the expected return of an asset j is given by the risk-free rate plus a risk premium $\beta(\mathbb{E}[R^m] - R^{rf})$ that depends on the β assigned to j.

An asset that varies closely with the market returns (high β) has high systemic risk and commands a larger risk premium. Optimally choosing a portfolio allows an agent to fully diversify idiosyncratic risk, leaving systemic risk reflected in the expected return of the asset.

CAPM and No-Arbitrage

The expected returns in the CAPM are calculated according to a probability distribution over states. Let $\hat{p} \in \Delta$, a probability measure on $\{1, \ldots, m\}$. Then:

$$q^j = \sum_{i=1}^{n} \overline{\pi}_i a_i^j = \sum_{i=1}^{n} \frac{\hat{\pi}_i}{\hat{p}_i} a_i^j = \sum_{i=1}^{n} \theta_i \hat{p}_i a_i^j = \mathbb{E}_{\hat{p}}[\tilde{\theta}\tilde{a}^j],$$

where $\tilde{\theta}$ is a **stochastic discount factor**.

For any two random variables $\tilde{X}$ and $\tilde{Y}$, we have:

$$\mathrm{Cov}_{\hat{p}}(\tilde{X}, \tilde{Y}) = \mathbb{E}_{\hat{p}}[\tilde{X}\tilde{Y}] - \mathbb{E}_{\hat{p}}[\tilde{X}]\mathbb{E}_{\hat{p}}[\tilde{Y}].$$

Thus:

$$q^j = \mathbb{E}_{\hat{p}}[\tilde{\theta}\tilde{a}^j] = \mathrm{Cov}_{\hat{p}}(\tilde{\theta}, \tilde{a}^j) + \mathbb{E}_{\hat{p}}[\tilde{\theta}]\mathbb{E}_{\hat{p}}[\tilde{a}^j].$$

Since:

$$\mathbb{E}_{\hat{p}}[\tilde{\theta}] = \sum_i \overline{\pi}_i = q^{rf} = \frac{1}{R^{rf}},$$

we obtain:

$$\mathbb{E}_{\hat{p}}[\tilde{R}^j] = R^{rf} - \mathrm{Cov}_{\hat{p}}(\tilde{v}, \tilde{R}^j),$$

where $\tilde{v} = R^{rf}\tilde{\theta}$.

Consumption CAPM

Consider a consumer with income I who can invest in J assets. The random return on asset j is $\tilde{R}^j$. Asset 1 is a risk-free asset.

Let x_0 denote the consumption (of money) on date 0. The agent uses $I - x_0$ to invest in assets for consumption on date 1. Investment in asset j is:

$$z_j = (I - x_0)\eta^j,$$

where $\eta = (\eta^j)_{j=1}^J \in \mathbb{R}_+^J$ with $\sum_j \eta^j = 1$.

The random payoff of a portfolio defined by η is:

$$\begin{aligned}\tilde{x}_1 &= (I - x_0)\sum_{j=1}^J \eta^j \tilde{R}^j,\\ &= (I - x_0)\left[R^1 + \sum_{j=2}^J \eta^j(\tilde{R}^j - R^1)\right].\end{aligned}$$

The consumer's problem is:

$$\begin{aligned}&\max u(x_0) + \delta\mathbb{E}\left[u\left((I - x_0)\left[R^1 + \sum_{j=2}^J \eta^j(\tilde{R}^j - R^1)\right]\right)\right],\\ &\text{s.t. } 0 \leq x_0 \leq I,\\ &\quad 0 \leq \eta_j,\\ &\quad \sum_{j=2}^J \eta^j \leq 1.\end{aligned}$$

Suppose $u : \mathbb{R}_+ \to \mathbb{R}$ is smooth, monotonic, and concave. Also assume interior solutions. The first-order conditions characterizing the solution are:

$$u'(x_0) = \delta\mathbb{E}[u'(\tilde{x}_1)\tilde{R}], \quad \text{(Euler equation)},$$

where $\tilde{R} = R^1 + \sum_j \eta^j(\tilde{R}^j - R^1)$, and:

$$\delta\mathbb{E}[u'(\tilde{x}_1)(I - x_0)(\tilde{R}^j - R^1)] = 0, \quad \forall j = 2, \ldots, J.$$

Since $I - x_0 > 0$,

$$\mathbb{E}[u'(\tilde{x}_1)(\tilde{R}^j - R^1)] = 0, \quad \forall j = 2, \ldots, J.$$

Expanding this:

$$\begin{aligned}
0 &= \mathbb{E}[u'(\tilde{x}_1)(\tilde{R}^j - R^1)], \\
&= \text{Cov}(u'(\tilde{x}_1), \tilde{R}^j - R^1) + \mathbb{E}[u'(\tilde{x}_1)]\mathbb{E}[\tilde{R}^j - R^1], \\
&= \text{Cov}(u'(\tilde{x}_1), \tilde{R}^j) + \mathbb{E}[u'(\tilde{x}_1)]\mathbb{E}[\tilde{R}^j - R^1].
\end{aligned}$$

Thus:

$$\mathbb{E}[\tilde{R}^j] = R^1 - \frac{1}{\mathbb{E}[u'(\tilde{x}_1)]}\text{Cov}(u'(\tilde{x}_1), \tilde{R}^j).$$

Lucas Tree Model

Now we study the case of a single agent and many goods. Specifically, consumption occurs over time, and it is uncertain. Endowments are stochastic and arrive over time. The problem is to characterize prices that support the autarky equilibrium where the agent consumes their endowment.

There is a single good, "fruit," in each period, and a single asset, "a tree." The tree pays off "dividends," a random production of fruit in every period. Time is infinite, ranging from $t = 0, 1, \ldots$. In period t, the production of fruit is realized, and a spot market opens in fruit. The consumer can sell and purchase fruit in the spot market and can buy trees. We normalize the price of fruit in each spot market to 1 and determine the price q_t of trees in period t.

In period t, the consumer's income derives from holding trees and their production of fruit. If each tree produces a dividend d_t and the agent holds s_t trees, their income at period t is:

$$w_t = s_t(q_t + d_t).$$

This income can be used to purchase fruit for consumption c_t and trees. If the agent buys s_{t+1} trees, they spend $a_t = s_{t+1}q_t$. The budget constraint for period t is:

$$c_t + a_t \leq w_t.$$

The rate of return on trees is:

$$R_{t+1} = \frac{q_{t+1} + d_{t+1}}{q_t},$$

composed of a capital gain and a dividend payoff. Note that:

$$R_{t+1}a_t = s_{t+1}(q_{t+1} + d_{t+1}) = w_{t+1}.$$

The consumer seeks to maximize the expected discounted sum of period utility. Let $u : \mathbb{R}_+ \to \mathbb{R}$ be the utility function, assumed C^1, strictly increasing, and concave.

The discount factor is $\delta \in (0,1)$. The maximization problem is:

$$\begin{aligned} &\max \mathbb{E}\left[\sum_{t=0}^{\infty} \delta^t u(c_t)\right], \\ &\text{s.t. } c_t + a_t \leq w_t, \\ &\quad w_{t+1} = R_{t+1} a_t. \end{aligned}$$

Assume $\{R_t\}$ follows a Markov process. The Bellman equation is:

$$v(w,R) = \sup\left\{u(c) + \delta \mathbb{E}[v(\tilde{R}a, \tilde{R}) \mid R]\right\}.$$

Consider:

$$\max_{0 \leq a \leq w} \left\{u(w-a) + \delta \mathbb{E}[v(\tilde{R}a, \tilde{R}) \mid R] : c + a = w\right\}.$$

Assuming the solution is interior and v is differentiable, let $\frac{\partial v}{\partial w}$ be denoted v_1'. The first-order condition (FOC) provides:

$$u'(w-a) = \delta \mathbb{E}[v_1'(\tilde{R}a, \tilde{R}) \mid R].$$

By the envelope theorem:

$$v_1'(w,R) = u'(w-a).$$

Thus:

$$u'(c_t) = \delta \mathbb{E}_t[u'(c_{t+1}) R_{t+1}].$$

This implies:

$$1 = \mathbb{E}_t\left[\underbrace{\delta \frac{u'(c_{t+1})}{u'(c_t)}}_{\text{stochastic discount factor } M_t^{t+1}} R_{t+1}\right].$$

Using the definition of R_{t+1}:

$$q_t = \mathbb{E}_t\left[\delta \frac{u'(c_{t+1})}{u'(c_t)} (q_{t+1} + d_{t+1})\right].$$

By the law of iterated expectations:

$$\begin{aligned} \mathbb{E}_t[q_{t+1}] &= \mathbb{E}_t\left[\mathbb{E}_{t+1}[M_{t+1}^{t+2}(q_{t+2} + d_{t+2})]\right], \\ &= \mathbb{E}_t[M_{t+1}^{t+2}(q_{t+2} + d_{t+2})]. \end{aligned}$$

Continuing this fashion:

$$q_t = \mathbb{E}_t\left[\sum_{\tau=1}^{T} M_t^{t+\tau} d_{t+\tau}\right] + \mathbb{E}_t[M_t^{t+T} q_{t+T}].$$

Ruling out bubbles and assuming the monotone convergence theorem:

$$q_t = \lim_{T\to\infty} \mathbb{E}_t\left[\sum_{\tau=1}^{T} M_t^{t+\tau} d_{t+\tau}\right] = \mathbb{E}_t\left[\sum_{\tau=1}^{\infty} M_t^{t+\tau} d_{t+\tau}\right].$$

Hence:

$$q_t = \mathbb{E}_t\left[\sum_{\tau=1}^{\infty} \delta^{\tau} \frac{u'(d_{t+\tau})}{u'(d_t)} d_{t+\tau}\right].$$

Appendix: Financial Market Modeling

For this appendix, we make use of basic measure theory and discrete martingales.

The market in this context is defined as the triple

$$[\Omega \text{ finite}, \ \{\mathscr{G}_n\}, \ M_n = (1, S_n), \ n = 0, 1, 2, \ldots, N],$$

with:

1. Ω: the set of all possible states.
2. $\mathscr{G}_n$: filtration.
3. M_n: the vector of assets.

Definition 11. A strategy (portfolio) is defined as

$$\Gamma_n : \Omega \to \mathbb{R}^2, \ n = 1, 2, \ldots, N,$$

such that Γ_n is $\mathscr{G}_{n-1}$ measurable for all $n = 1, 2, \ldots, N$.

Definition 12. A strategy Γ_n is said to be self-financing if

$$\Gamma_{n-1} \cdot M_{n-1} = \Gamma_n \cdot M_{n-1}, \quad \forall n.$$

The sample space is

$$\Omega = \{\omega_1, \ldots, \omega_k\}.$$

In the general case, we can have ℓ assets $S_n^i : \Omega \to \mathbb{R}$, $i = 1, \ldots, \ell$, and the market state is given by:

$$M_n : \Omega \to \mathbb{R}^{\ell+1},$$

with:

$$M_n(\omega) = (1, S_n^1(\omega), \ldots, S_n^\ell(\omega)).$$

Alternatively, we can write:

$$M_n(\omega) = (B_n, S_n^1(\omega), \ldots, S_n^\ell(\omega)),$$

where B_n is a non-stochastic asset representing, for example, cash in a bank account. Given the market, the **position** of the investor is represented by the vector

$$\Gamma = (\beta, \alpha_1, \ldots, \alpha_\ell),$$

where the first coordinate β corresponds to the cash position (or the position in the reference asset B_n, often normalized to $B_n = 1$), and the remaining ℓ coordinates represent the position in each risky asset. The sign convention is:

- $\beta > 0$: the investor holds $|\beta| B_n$ units in cash.
- $\beta < 0$: the investor owes $|\beta| B_n$ units to the bank.

- $\alpha_i > 0$: the investor holds $|\alpha_i|$ units of asset i.
- $\alpha_i < 0$: the investor owes $|\alpha_i|$ units of asset i.

The value of the portfolio at time $n \in \{0, \ldots, N\}$ is denoted by V_n^Γ and calculated as:

$$\begin{cases} V_0^\Gamma &= \Gamma_1 \cdot M_0, \quad \text{the cost of initiating the strategy,} \\ V_n^\Gamma &= \Gamma_n \cdot M_n, \quad \text{the value of the portfolio at time } n. \end{cases}$$

Proposition 5. *If the strategy is self-financing:*

$$\begin{aligned} V_n^\Gamma &= V_0^\Gamma + \sum_{j=1}^{n} \Gamma_j \cdot (M_j - M_{j-1}), \\ &= \int_0^n \Gamma \, dM + \underbrace{\int_0^n M \, d\Gamma}_{=0}. \end{aligned}$$

Remark. The following properties hold:

$$\begin{aligned} V_n^{\Gamma_1+\Gamma_2} &= V_n^{\Gamma_1} + V_n^{\Gamma_2}, \\ V_n^{\lambda\Gamma} &= \lambda V_n^\Gamma. \end{aligned}$$

Definition 13. A self-financing strategy Γ_n is said to be an arbitrage opportunity in the market M_n if:

1. $V_0^\Gamma \leq 0$ (no initial investment).
2. $V_N^\Gamma \geq 0$ (risk-free).
3. $V_N^\Gamma(\omega) > 0$ for some $\omega \in \Omega$.

Definition 14. A market (M_n) is free of arbitrage if there exists no Γ_n that constitutes an arbitrage opportunity.

Example 9. **Binomial model** (s, a, b) **with** $-1 < a$**.** Define $\Omega = \{a, b\}^N$ and $\rho_j \to \{a, b\}$, $j = 1, \ldots, N$. The price process $S_n : \Omega \to \mathbb{R}$ is given by:

$$\begin{cases} S_0 &= s, \\ S_n &= S_{n-1}(1 + \rho_n), \quad n \geq 1. \end{cases}$$

Thus, $M_n = (1, S_n)$.

Proposition 6. *In the binomial model, the market is free of arbitrage if and only if $a < 0 < b$.*

Proof. If the market is arbitrage-free, it must hold that $a < 0 < b$. Conversely, if $a < 0 < b$, we can construct a risk-neutral probability measure Q. □

Proposition 7. *Given a market $[\Omega, \mathscr{G}_n, M_n]$, a probability Q on Ω is called a* ***risk-neutral probability*** *or* ***equivalent martingale measure (EMM)*** *if $(M_n, \mathscr{G}_n)$ is a Q-martingale and $Q(\omega) > 0$ for all $\omega \in \Omega$.*

Remark. If Q is a risk-neutral probability, then $(V_n^\Gamma, \mathscr{G}_n)$ is a martingale for any self-financing strategy Γ_n. Consequently, if Γ_n is an arbitrage opportunity:

1. $V_0^\Gamma \leq 0$,
2. $V_N^\Gamma \geq 0$,
3. $V_N^\Gamma(\omega) > 0$ for some ω,

then, since $V_N^\Gamma \geq 0$,

$$\mathbb{E}_Q[V_N^\Gamma] = \mathbb{E}_Q[V_0^\Gamma] \leq 0.$$

It follows that $V_N^\Gamma = 0$ Q-a.s., leading to a contradiction. Hence, if Q exists, M_n is arbitrage-free.

Theorem 2. *If $|\Omega| < \infty$ and $n = 0, 1, 2, \ldots, N$, then:*

$$M_n \textit{ is arbitrage-free } \Leftrightarrow \exists\, Q \textit{ EMM}.$$

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