Price Information and Duopolistic Competition with Seller Cost Uncertainty and Advertising

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Abstract

We build on Martinelli and Xiao (2024) by introducing a new model of duopolistic competition that incorporates seller cost uncertainty and advertising. Unlike Martinelli and Xiao (2024), our model explicitly includes advertising as a decision variable, leading to a more intricate yet realistic formulation of expected demand. This extension captures the impact of advertising expenditures by each firm, adding a crucial strategic dimension to the analysis.

Although our model is formulated for an arbitrary positive integer number of firms, most of our analysis focuses on the case of two firms, placing it within the framework of a duopoly. This choice is driven by the algebraic complexity introduced by advertising, which significantly complicates the transition to a more general setting.

We examine two scenarios: one where advertising is exogenous, limiting the optimization to pricing decisions, and another where advertising is treated as an endogenous strategic variable. For the exogenous case with $J = 2$, we derive analytical results, while for the more general cases, we rely on numerical analysis to explore the equilibrium outcomes.

Introduction

The study of price dispersion and sales in markets with homogeneous goods has been a central focus in economic theory since the seminal works of Stigler (1961), Varian (1980), and Burdett and Judd (1983). These models established a foundation for understanding how consumer search frictions and firms' strategic pricing decisions lead to temporary price reductions—commonly referred to as *sales*. A key feature of these models is the reliance on mixed-strategy equilibria, where firms randomize their prices over a support to account for varying consumer behaviors and search costs.

Subsequent extensions have enriched this framework by introducing additional market frictions and heterogeneities. For example, Janssen and Moraga-Gonzalez (2004) and Stahl (1989) incorporate consumer awareness and sequential search processes into their analyses, while Shelegia and Wilson (2021) generalize the clearinghouse model to allow for firm-level heterogeneity and advertising as a strategic variable. These extensions have provided valuable insights into the interplay between consumer search behavior, advertising, and price competition, while also highlighting the complex dynamics that arise in asymmetric markets.

In particular, Shelegia and Wilson (2021) develop a generalized model of sales that emphasizes the role of advertising in shaping consumer consideration sets. Their framework introduces firm heterogeneity along multiple dimensions, such as advertising costs, shares of captive consumers, and profitability. Unlike earlier models that often assume symmetry among firms, their approach allows firms to differ significantly in their strategies and outcomes. Notably, they show that advertising expenditures can serve as a mechanism to soften price competition by increasing a firm's share of captive consumers, thereby reducing the intensity of competition for *shoppers* who compare prices.

Similarly, Armstrong and Vickers (2022) explore patterns of competitive interaction in oligopolistic markets where consumers consider only subsets of firms. Their analysis demonstrates how the structure of consumer consideration sets—whether symmetric, nested, or disjoint—affects equilibrium pricing and consumer welfare. For example, firms with larger *reach* (i.e., the fraction of consumers who consider them) tend to set higher prices on average, as they face less competitive pressure within their subsets. These results underscore the importance of understanding how advertising and consumer awareness influence the competitive landscape.

Building on these contributions, our paper investigates the role of advertising in oligopolistic markets with seller cost uncertainty, as modeled by Martinelli and Xiao (2024). While existing models, such as those of Varian (1980) and Burdett and Judd (1983), treat advertising as a binary decision (advertise or not), we adopt a more nuanced approach that incorporates continuous advertising choices. This allows us to capture the effects of varying advertising expenditures on firms' ability to attract consumer attention and influence consideration sets. Specifically, we address the following questions:

- How does advertising intensity affect the distribution of consumer consideration sets and, consequently, the degree of price competition in the market?
- What are the equilibrium implications of treating advertising as an endogenous decision variable, as opposed to a fixed exogenous factor?
- How do firm heterogeneities—such as differences in advertising costs, consumer shares, and cost structures—shape the equilibrium outcomes in terms of price dispersion, advertising intensity, and firm profits?

We provide analytical results for the case of a duopoly and complement the analysis with simulations involving the solution of nonlinear equations and optimization in finite dimensions.

1 The model

Our model incorporates advertising into the framework studied in Martinelli and Xiao (2024). Consider an economy with $J > 0$ risk-neutral firms, each selling an identical good. Each firm j has a cost structure represented by $c_j > 0$, which is private information known only to the firm itself. However, it is publicly known that $c_j \sim$ i.i.d. F, where F is a twice continuously differentiable cumulative distribution function ($F \in C^2$) with support $\operatorname{supp}(F) = [\underline{c}, \overline{c}] \subset [0, 1]$. We define:

$$M(c) = \mathbb{P}\{c_j > c\} = 1 - F(c). \tag{1}$$

Under the assumptions on the cumulative distribution function F, it follows that $M \in C^2$ and $M'(c) \in (-\infty, 0)$ for every $c \in \operatorname{supp}(F)$.

On the side of consumers, each one is indexed by i and is aware of a specific subset of firms, denoted by K_i, with the cardinality of this set given by $k_i = |K_i|$. Each consumer will buy to the firm setting the lower price, less than 1, in K_i. Now, firms can purchase advertising $a \in A = [\underline{a}, \overline{a}]$ at a price p_a, which, without loss of generality, is normalized to 1. It is assumed that $\underline{a} > 0$[1]. Advertising decisions for all firms are summarized by the vector $\mathbf{a} \in A^J$. Each firm does not know the advertising levels chosen by its competitors. However, it is common knowledge that a_j follows a certain distribution φ_j, with support A.

Introducing advertising, although more realistic, adds complexity to the original model. We begin by defining the basic elements of the model in the presence of advertising. The probability that a firm belongs to the information set K_i of a consumer i, knowing that i is aware of k firms and $\mathbf{a} = \tilde{\mathbf{a}}$, is now:

$$\mathbb{P}\left\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\right\} = \frac{k}{J}\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right], \tag{2}$$

where

$$\theta_j = \frac{\tilde{a}_j}{\sum_{\ell=1}^{J} \tilde{a}_\ell}.$$

Equation 2 implies that all sellers have equal probability of being selected when they choose the same level of advertising. However, sellers who purchase a higher proportion of advertising benefit by having a higher probability of belonging to the information set, except when $k = J$.[2]

With respect to the priors $\boldsymbol{\mu} \in \Delta \subset \mathbb{R}^J$, in presence of advertising, the fraction of consumers observing k prices is now conditioned to the advertising vector and modified as follows:

$$\mu_k \to \mu_k^{\tilde{\mathbf{a}}} = \mathbb{P}\{k_i = k | \mathbf{a} = \tilde{\mathbf{a}}\} = \frac{\mu_k w_k(\tilde{\mathbf{a}})}{\sum_{\ell=1}^{J} \mu_\ell w_\ell(\tilde{\mathbf{a}})}, \tag{3}$$

where

$$w_\ell(\tilde{\mathbf{a}}) = 1 + \gamma \ell \left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\overline{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}\right). \tag{4}$$

In (4), $\overline{\mathbf{a}} = (\overline{a}, \cdots, \overline{a}) \in \mathbb{R}^J$, $\underline{\mathbf{a}} = (\underline{a}, \cdots, \underline{a}) \in \mathbb{R}^J$ and $\gamma > 0$. By defining $\mu_k^{\tilde{\mathbf{a}}}$ as in (3), we ensure that for $\tilde{\mathbf{a}} \neq \underline{\mathbf{a}}$, $\mu_1 > \mu_1^{\tilde{\mathbf{a}}}$ and $\mu_J < \mu_J^{\tilde{\mathbf{a}}}$ (see Proposition 2). It follows by iterating Bayes rule and total probability rule that

$$\begin{aligned}
\mathbb{P}\{k_i = k | j \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{\mathbb{P}\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\}\mathbb{P}\{k_i = k | \mathbf{a} = \tilde{\mathbf{a}}\}}{\mathbb{P}\{j \in K_i | \mathbf{a} = \tilde{\mathbf{a}}\}} \\
&= \frac{\mathbb{P}\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\}\mathbb{P}\{k_i = k | \mathbf{a} = \tilde{\mathbf{a}}\}}{\sum_{\ell=1}^{J} \mathbb{P}\{j \in K_i | k_i = \ell, \mathbf{a} = \tilde{\mathbf{a}}\}\mathbb{P}\{k_i = \ell | \mathbf{a} = \tilde{\mathbf{a}}\}} \\
&= \frac{\frac{k}{J}\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right] \cdot \mu_k^{\tilde{\mathbf{a}}}}{\sum_{\ell=1}^{J} \frac{\ell}{J}\left[1 + \left(\frac{J-\ell}{J}\right)\theta_j\right]\mu_\ell^{\tilde{\mathbf{a}}}} \\
&= \frac{k\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right] \cdot \mu_k^{\tilde{\mathbf{a}}}}{\sum_{\ell=1}^{J} \ell\left[1 + \left(\frac{J-\ell}{J}\right)\theta_j\right]\mu_\ell^{\tilde{\mathbf{a}}}}.
\end{aligned}$$

Now, given $\tilde{\mathbf{a}} \in A$, let

$$G(p; \rho) = \int_{\underline{c}}^{\overline{c}} \mathbf{1}_{\{\rho(c,\ \tilde{\mathbf{a}}) > p\}} dF(c)$$

be the upper c.d.f. of price contingent on some pricing function ρ (see Martinelli and Xiao (2024)). Using independence, the probability that p is lower than $k-1$ prices is $G(p;\rho)^{k-1}$. Under the assumption of symmetry, the expected demand for a seller that sets the price p, given an advertising

[1]Firms must purchase a minimum level of advertising in order to *exist*. This can be interpreted as a fixed cost.

[2]If the consumer knows every firm, by overwhelming, advertising will play no role. Moreover, the coefficient $(J-k)/J$ is mandatory to ensure that $\mathbb{P}\left\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\right\} \in [0, 1]$, see Proposition 1.

vector $\tilde{\mathbf{a}}$, is[3]

$$Q(p;\boldsymbol{\mu}^{\tilde{\mathbf{a}}},\tilde{\mathbf{a}},\rho) = \underbrace{\sum_{\ell=1}^{J}\frac{\ell}{J}\mu_\ell^{\tilde{\mathbf{a}}}}_{\text{Mass of consumers.}} \left\{ \underbrace{\sum_{k=1}^{J}\left(\frac{k\left[1+\left(\frac{J-k}{J}\right)\theta_j\right]\mu_k^{\tilde{\mathbf{a}}}}{\sum_{\ell=1}^{J}\ell\left[1+\left(\frac{J-\ell}{J}\right)\theta_j\right]\mu_\ell^{\tilde{\mathbf{a}}}}\right)}_{\text{Probability of competing with } k-1 \text{ with } i \text{ fixed.}} \underbrace{G(p;\rho)^{k-1}}_{\text{Probability of being the cheaper.}} \right\}.$$

Finally, the expected profit for firm j, given the advertising level $\tilde{\mathbf{a}}$, is simply

$$\Pi_j(p;c_j,\boldsymbol{\mu}^{\tilde{\mathbf{a}}},\rho) = (p-c_j)Q(p;\boldsymbol{\mu}^{\tilde{\mathbf{a}}},\tilde{\mathbf{a}},\rho) - \tilde{a}_j.$$

Until now, we have considered the advertising vector $\mathbf{a}$ to be exogenous. However, it is reasonable and indeed more realistic to assume that firms simultaneously set both p and a_j. That is, $\mathbf{a}$ is endogenous. Thus, each firm j optimizes over p and a_j. Firm j does know is a prior distribution of $\mathbf{a}_{-j}$, denoted by $\boldsymbol{\Phi}_{-j}^{\mathbf{a}_j}$, with a corresponding density $\boldsymbol{\varphi}_{-j}^{\mathbf{a}_{-j}}$, and support $A_{-j} = [\underline{a},\overline{a}]^{J-1}$. Therefore, the expected profits for firm j, without knowing the exact realization of the advertising vector, are given by[4]

$$\mathbb{E}_{\mathbf{a}_{-j}}[\Pi_j(p;c_j,\boldsymbol{\mu}^{\mathbf{a}},\rho)] = \int_{\underline{a}}^{\overline{a}}\cdots\int_{\underline{a}}^{\overline{a}}((p-c_j)Q(p;\boldsymbol{\mu}^{\mathbf{a}},\mathbf{a},\rho)-a_j)\boldsymbol{\varphi}_{-j}^{\mathbf{a}_{-j}}\,da_1\cdots da_{j-1}da_{j+1}\cdots da_J,$$

2 Exogenous advertising

For fixed $\tilde{\mathbf{a}}$, $\rho(c;\tilde{\mathbf{a}})$ is increasing in c when $\mu_1^{\tilde{\mathbf{a}}} < 1$, with $\rho(c;\tilde{\mathbf{a}}) = 1$ for $\mu_1^{\tilde{\mathbf{a}}} = 1$, and $\rho(\overline{c};\tilde{\mathbf{a}}) = \overline{c}$ when $\mu_1^{\tilde{\mathbf{a}}} = 0$. This follows from the arguments provided in Martinelli and Xiao (2024). Furthermore, the profit function $\Pi(p;c,\tilde{\mathbf{a}},\rho)$ satisfies the single-crossing property introduced in Athey (2001), and the proof in Martinelli and Xiao (2024) applies directly. Additionally, monotonicity of ρ implies that:

$$\begin{aligned} G(\rho(c;\tilde{\mathbf{a}});p) &= \int_{\underline{c}}^{\overline{c}} \mathbf{1}_{\{\rho(s;\ \tilde{\mathbf{a}})>\rho(c;\ \tilde{\mathbf{a}})\}} d(1-M(s)) \\ &= \int_{\underline{c}}^{\overline{c}} \mathbf{1}_{\{s>c\}} d(1-M(s)) \\ &= \int_{c}^{\overline{c}} d(1-M(s)) \\ &= M(c) - M(\overline{c}) \\ &= M(c) - \mathbb{P}\{\tilde{c} > \overline{c}\} \\ &= M(c), \end{aligned}$$

which implies, as in Martinelli and Xiao (2024), that

$$\rho(c;\tilde{\mathbf{a}}) = c + \frac{\mathcal{Q}(\overline{c};\tilde{\mathbf{a}})}{\mathcal{Q}(c;\tilde{\mathbf{a}})}\left(\rho(\overline{c};\tilde{\mathbf{a}}) - \overline{c}\right) + \int_{c}^{\overline{c}} \frac{\mathcal{Q}(x;\tilde{\mathbf{a}})}{\mathcal{Q}(c;\tilde{\mathbf{a}})}dx, \tag{5}$$

where

$$Q(\rho(c;\tilde{\mathbf{a}}),\boldsymbol{\mu}^{\tilde{\mathbf{a}}},\tilde{\mathbf{a}};\rho) = \mathcal{Q}(c;\tilde{\mathbf{a}},\boldsymbol{\mu}^{\tilde{\mathbf{a}}}).$$

However, the results introduced after Theorem 1 in Martinelli and Xiao (2024) no longer apply directly to the current model, as the introduction of advertising alters the algebraic structure of the

[3]For the expression of the mass of consumers we follow Burdett and Judd (1983).
[4]Each firm solves $\max_{p,a_j}\ \left\{\mathbb{E}_{\mathbf{a}_{-j}}[\Pi_j(p;c_j,\boldsymbol{\mu}^{\mathbf{a}},\rho)]\right\}$.

expected demand function. The complexity of this issue is evident, for example, when attempting to derive ρ with respect to a_j. Therefore, the following section focuses on the case $J = 2$.

3 Duopoly

In this section, we study the case under the assumptions of an exogenous advertising vector and only two firms ($J = 2$). We begin by defining the elements corresponding to the case $J = 2$,

$$
\begin{aligned}
\mathbb{P}\{1 \in K_i | k_i = 1; \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{1}{2} + \frac{1}{4}\left(\frac{\tilde{a}_1}{\tilde{a}_1 + \tilde{a}_2}\right) \\
\mathbb{P}\{1 \in K_i | k_i = 2; \mathbf{a} = \tilde{\mathbf{a}}\} &= 1 \\
\mathbb{P}\{2 \in K_i | k_i = 1; \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{1}{2} + \frac{1}{4}\left(\frac{\tilde{a}_2}{\tilde{a}_1 + \tilde{a}_2}\right) \\
\mathbb{P}\{2 \in K_i | k_i = 2; \mathbf{a} = \tilde{\mathbf{a}}\} &= 1.
\end{aligned}
$$

Then, setting $\beta = (\overline{a} - \underline{a})^2$ and $\eta(\tilde{\mathbf{a}}) = (\tilde{a}_1 - \underline{a})^2 + (\tilde{a}_2 - \underline{a})^2$

$$
\begin{aligned}
\mu_1^{\tilde{\mathbf{a}}} &= \frac{\mu_1\left(1 + \gamma\frac{\eta(\tilde{\mathbf{a}})}{2\beta}\right)}{\mu_1\left(1 + \gamma\frac{\eta(\tilde{\mathbf{a}})}{\beta}\right) + \mu_2\left(1 + 2\gamma\frac{\eta(\tilde{\mathbf{a}})}{2\beta}\right)} \\
&= \frac{\mu_1 + \mu_1\left(\frac{\gamma\eta(\tilde{\mathbf{a}})}{2\beta}\right)}{1 + (\mu_1 + 2\mu_2)\frac{\gamma\eta(\tilde{\mathbf{a}})}{2\beta}} \\
\mu_2^{\tilde{\mathbf{a}}} &= 1 - \mu_1^{\tilde{\mathbf{a}}} \\
&= \frac{1 + \mu_2\left(\frac{\gamma\eta(\tilde{\mathbf{a}})}{\beta}\right) - \mu_1}{1 + (\mu_1 + 2\mu_2)\left(\frac{\gamma\eta(\tilde{\mathbf{a}})}{2\beta}\right)}.
\end{aligned}
$$

We now compute

$$
\begin{aligned}
\mathbb{P}\{k_i = 1 | 1 \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{\left(1 + \frac{\theta_1}{2}\right)\mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_1}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} \\
\mathbb{P}\{k_i = 1 | 2 \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{\left(1 + \frac{\theta_2}{2}\right)\mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_2}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} \\
\mathbb{P}\{k_i = 2 | 1 \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_1}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} \\
\mathbb{P}\{k_i = 2 | 2 \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_2}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}}.
\end{aligned}
$$

Hence,

$$
Q(p; \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \tilde{\mathbf{a}}, \rho) = \left(1 - \frac{\mu_1^{\tilde{\mathbf{a}}}}{2}\right)\left\{\frac{\left(1 + \frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} + \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} G(p; \rho)\right\},
$$

and

$$
\Pi_j(p; c_j, \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \rho) = (p - c_j)\left(1 - \frac{\mu_1^{\tilde{\mathbf{a}}}}{2}\right)\left\{\frac{\left(1 + \frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} + \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} G(p; \rho)\right\} - \tilde{a}_j.
$$

3.1 Comparative statics

We are now interested in performing comparative statics with respect to $\tilde{a}_j$, analyzing how advertising impacts the different defined elements. In particular, we would like to compute its effect on demand, and for this, it is necessary to examine its effect on priors and the probability of competition.

$$
\begin{aligned}
\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} &= \frac{\frac{\mu_1 \gamma}{\beta}(\tilde{a}_j - \underline{a})D - 2N\frac{\gamma}{\beta}(\tilde{a}_j - \underline{a})}{D^2} \\
&= \underbrace{\frac{\frac{\gamma}{\beta}(\tilde{a}_j - \underline{a})}{D^2}}_{\geq 0}(D - 2N)
\end{aligned}
$$

where

$$
D = \mu_1 \left(1 + \gamma \frac{\eta(\tilde{\mathbf{a}})}{\beta}\right) + \mu_2 \left(1 + 2\gamma \frac{\eta(\tilde{\mathbf{a}})}{2\beta}\right)
$$

and

$$
N = \mu_1 \left(1 + \frac{\gamma \eta(\tilde{\mathbf{a}})}{2\beta}\right).
$$

Hence, the sign of $\partial \mu_1^{\tilde{\mathbf{a}}}/\partial a_j$ depends on the sign of $D - 2N$.

$$
\begin{aligned}
D - 2N &= \mu_1 \left[\mu_1 + \mu_2 + \frac{\mu_1 \gamma}{\beta} + \frac{\mu_2 \gamma \eta}{\beta}\right] - 2N \\
&= -\mu_1 < 0.
\end{aligned}
$$

Thus, $\partial \mu_1^{\tilde{\mathbf{a}}}/\partial a_j > 0$. Since $\mu_2^{\tilde{\mathbf{a}}} = 1 - \mu_1^{\tilde{\mathbf{a}}}$, it follows that

$$
\frac{\partial \mu_2^{\tilde{\mathbf{a}}}}{\partial a_j} = -\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} > 0.
$$

Using these results, we now compute

$$
\begin{aligned}
\frac{\partial \mathbb{P}\{k_i = 1 | j \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\}}{\partial a_j} &= \frac{\partial}{\partial a_j}\left[\frac{C\mu_1^{\tilde{\mathbf{a}}}}{C\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}}\right] \\
&= \frac{C\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}\tilde{D} - C\mu_1^{\tilde{\mathbf{a}}}\left[C\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} - 2\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}\right]}{\tilde{D}^2} \\
&= \frac{2C\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}}{\tilde{D}^2} < 0,
\end{aligned}
$$

with $C = 1 + \theta_j/2$ and $\tilde{D} = C\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}$. Analogously,

$$
\begin{aligned}
\frac{\partial \mathbb{P}\{k_i = 2 | j \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\}}{\partial a_j} &= \frac{\partial}{\partial a_j}\left[\frac{2\mu_2^{\tilde{\mathbf{a}}}}{C\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}}\right] \\
&= \frac{-2(C + \mu_1)}{\tilde{D}^2}\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial \mu_1} > 0,
\end{aligned}
$$

where $\tilde{D} = C\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}$. Finally, for $p = \rho(c)$

$$\begin{aligned}\frac{\partial Q(p; \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \tilde{\mathbf{a}}, \rho)}{\partial a_j} &= -\frac{1}{2}\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}\left\{\frac{\left(1+\frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}}}{\left(1+\frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} + \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1+\frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} M(c)\right\} \\ &\quad + \left(1 - \frac{\mu_1^{\tilde{\mathbf{a}}}}{2}\right)\left\{\frac{2C\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} - 2\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}(C + \mu_1^{\tilde{\mathbf{a}}})M(c)}{\tilde{D}^2}\right\}\end{aligned}$$

In this way, as expected, when advertising increases, it is more likely to become aware of all the firms rather than just one, and for the information sets to include both firms instead of just one.

Finally, the sign of how demand changes is not trivial to analyze. However, if we want to ensure that an increase in advertising raises demand, the following condition must hold:

$$2C - (C + \mu_1)M(c) < 0.$$

This can occur, for example, if $\theta_j \ll \mu_1$ and $\mu_1 > 1/2$. Finally, from, (5), and knowing that $\rho(\overline{c}; \tilde{\mathbf{a}}) = \text{cte}$, we can compute

$$\begin{aligned}\frac{\partial \rho(c; \tilde{\mathbf{a}})}{\partial a_j} &= \frac{\frac{\partial \mathcal{Q}(\overline{c};\tilde{\mathbf{a}})}{\partial a_j}\mathcal{Q}(c; \tilde{\mathbf{a}}) - \mathcal{Q}(\overline{c}; \tilde{\mathbf{a}})\frac{\partial \mathcal{Q}(c;\tilde{\mathbf{a}})}{\partial a_j}}{\mathcal{Q}(c; \tilde{\mathbf{a}})^2} \\ &\quad + \int_{\underline{c}}^{\overline{c}} \frac{\frac{\partial \mathcal{Q}(x;\tilde{\mathbf{a}})}{\partial a_j}\mathcal{Q}(c; \tilde{\mathbf{a}}) - \mathcal{Q}(x; \tilde{\mathbf{a}})\frac{\partial \mathcal{Q}(c;\tilde{\mathbf{a}})}{\partial a_j}}{\mathcal{Q}(c, \tilde{\mathbf{a}})^2} dx.\end{aligned}$$

The first term is equal to

$$\frac{C\left(2M(c)\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}\left(C^2(\mu_1^{\tilde{\mathbf{a}}} - 2)(\mu_1^{\tilde{\mathbf{a}}})^2 + C(\mu_1^{\tilde{\mathbf{a}}} - 2)\mu_1^{\tilde{\mathbf{a}}}\left((\mu_1^{\tilde{\mathbf{a}}})^2 + 4\mu_2^{\tilde{\mathbf{a}}}\right) + 2(\mu_1^{\tilde{\mathbf{a}}} - 2)\mu_2^{\tilde{\mathbf{a}}}\left((\mu_1^{\tilde{\mathbf{a}}})^2 + 2\mu_2^{\tilde{\mathbf{a}}}\right) + \mu_1^{\tilde{\mathbf{a}}}\mu_2^{\tilde{\mathbf{a}}}\tilde{D}^2\right)\right)}{(\mu_1^{\tilde{\mathbf{a}}} - 2)\tilde{D}^2\left(2\mu_2^{\tilde{\mathbf{a}}}M(c) + C\mu_1^{\tilde{\mathbf{a}}}\right)^2}$$

$$+\frac{C\left(-2(\mu_1^{\tilde{\mathbf{a}}})^2\mu_2^{\tilde{\mathbf{a}}}\tilde{D}^2 M(c) + C(\mu_1^{\tilde{\mathbf{a}}})^2\tilde{D}^2\left(\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} - \mu_1^{\tilde{\mathbf{a}}}\right)\right)}{(\mu_1^{\tilde{\mathbf{a}}} - 2)\tilde{D}^2\left(2\mu_2^{\tilde{\mathbf{a}}}M(c) + C\mu_1^{\tilde{\mathbf{a}}}\right)^2}$$

As can be seen from the previous expression, there is little hope of analytically determining the sign of $\partial\rho(c; \tilde{\mathbf{a}})/\partial a_j$. The same issue would arise when considering $J \geq 3$.[5]

3.2 Endogenous advertising

Leibniz rule and first order condition yields

$$\int_{\underline{a}}^{\overline{a}} \left\{(p - c_j)\left(\frac{\partial Q(p; \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, a_{-j}, a_j, \rho)}{\partial a_j}\right) - 1\right\} da_{-j} = 0.$$

3.3 Numerical solution

Conclusions

[5]The advantage of the case $J = 2$ arises from the fact that we can use $\mu_1^{\tilde{\mathbf{a}}} + \mu_2^{\tilde{\mathbf{a}}} = 1$.

A Proofs

Proposition 1. For any combination of k, j, i and $\tilde{\mathbf{a}}$, (2) is a probability.

Proof. Certainly,

$$\mathbb{P}\left\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\right\} \geq 0.$$

It remains to prove that

$$\mathbb{P}\left\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\right\} = \frac{k}{J}\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right] \leq 1.$$

First, $\theta_j, (k/J) \in [0,1]$. Then, we denote $x = k/J$. Hence,

$$\mathbb{P}\left\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\right\} = x(1+\theta_j) - \theta_j x^2 = f(x;\theta_j).$$

The function is increasing from 0 to $x^\circ = (1+\theta_j)/(2\theta_j) > 1$. Since $f(0;\theta_j) = 0$ and f is increasing from 0 to 1, we only need to prove that $f(1;\theta_j) \leq 1$. This follows directly replacing $x = 1$: $f(1;\theta_j) = 1$. □

Proposition 2. The priors defined in (3) satisfy that

$$\mu_1 > \mu_1^{\mathbf{a}} \text{ and } \mu_J < \mu_J^{\mathbf{a}}, \tag{6}$$

for every $\boldsymbol{\mu} \in \Delta \subset \mathbb{R}^J$ non degenerate[6] Moreover, for any $\boldsymbol{\mu} \in \Delta$, (6) holds with weak inequality.

Proof. For $\mu_1 \neq 1$, $1 < \sum_{\ell=1}^J \mu_\ell \ell$. Then,

$$\mu_1 + \mu_1 \gamma\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right) < \mu_1 + \mu_1 \sum_{\ell=1}^J \mu_\ell \ell \gamma\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right).$$

Dividing by $1 + \sum_{\ell=1}^J \mu_\ell \ell \gamma\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right)$, we conclude. Analogously, since

$$\sum_{\ell=1}^J \mu_\ell \ell < J \sum_{\ell=1}^J \mu_\ell = J,$$

we have that

$$\mu_J + \mu_J \sum_{\ell=1}^J \mu_\ell \ell \gamma\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right) < \mu_J\left[1 + \gamma J\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right)\right]$$

$$\mu_J < \frac{\mu_J\left[1 + \gamma J\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right)\right]}{\sum_{\ell=1}^J \mu_\ell\left[1 + \gamma \ell\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right)\right]}.$$

□

[6] This means that $\mu_1 \neq 1$.

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