Price Information and Duopolistic Competition with Seller Cost Uncertainty and Advertising

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Abstract

We build on Martinelli and Xiao (2024) by introducing a new model of duopolistic competition that incorporates seller cost uncertainty and advertising. Unlike Martinelli and Xiao (2024), our model explicitly includes advertising as both a parameters and a decision variable. This extension captures the impact of advertising expenditures by each firm, adding a crucial strategic dimension to the analysis. Although our model is formulated for an arbitrary positive integer number of firms, most of our analysis focuses on the case of two firms, placing it within the framework of a duopoly. This choice is driven by the algebraic complexity introduced by advertising, which significantly complicates the transition to a more general setting. In a certain way, we introduce advertising, an important element, but at the cost of sacrificing generality regarding the number of firms considered in Martinelli and Xiao (2024). We examine two scenarios: one where advertising is exogenous, limiting the optimization to pricing decisions, and another where advertising is treated as an endogenous strategic variable. For the exogenous case with $J = 2$, we derive analytical results, while for the more general cases, we rely on numerical analysis to explore the equilibrium outcomes.

JEL Codes: D43, L13, M37, C61.

Keywords: advertising, duopoly, cost uncertainty, numerical optimization, algebraic nonlinear equations.

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Introduction

The study of price dispersion and sales in markets with homogeneous goods has been a central focus in economic theory since the seminal works of Stigler (1961), Varian (1980), and Burdett and Judd (1983). These models established a foundation for understanding how consumer search frictions and firms' strategic pricing decisions lead to temporary price reductions. A key feature of these models is the reliance on mixed-strategy equilibria, where firms randomize their prices over a support to account for varying consumer behaviors and search costs.

Subsequent extensions have enriched this framework by introducing additional market frictions and heterogeneities. For example, Janssen and Moraga-Gonzalez (2004) and Stahl (1989) incorporate consumer awareness and sequential search processes into their analyses, while Shelegia and Wilson (2021) generalize the clearinghouse model to allow for firm-level heterogeneity and advertising as a strategic variable. These extensions have provided valuable insights into the interplay between consumer search behavior, advertising, and price competition, while also highlighting the complex dynamics that arise in asymmetric markets.

In particular, Shelegia and Wilson (2021) develop a generalized model of sales that emphasizes the role of advertising in shaping consumer consideration sets. Their framework introduces firm heterogeneity along multiple dimensions, such as advertising costs, shares of captive consumers, and profitability. Unlike earlier models that often assume symmetry among firms, their approach allows firms to differ significantly in their strategies and outcomes. Notably, they show that advertising expenditures can serve as a mechanism to soften price competition by increasing a firm's share of captive consumers, thereby reducing the intensity of competition for *shoppers* who compare prices.

Similarly, Armstrong and Vickers (2022) explore patterns of competitive interaction in oligopolistic markets where consumers consider only subsets of firms. Their analysis demonstrates how the structure of consumer consideration sets—whether symmetric, nested, or disjoint—affects equilibrium pricing and consumer welfare. For example, firms with larger *reach* (i.e., the fraction of consumers who consider them) tend to set higher prices on average, as they face less competitive pressure within their subsets.

Building on these contributions, our paper examines the role of advertising in duopolistic markets with seller cost uncertainty, as modeled by Martinelli and Xiao (2024). While existing models, such as Shelegia and Wilson (2021), treat advertising as a binary decision (advertise or not), and others, like Butters (1977), incorporate advertising as a stochastic allocation involving buyers, we take a more refined approach by allowing for continuous advertising choices. This framework captures the strategic interplay of firms' advertising expenditures more comprehensively. This allows us to capture the effects of varying advertising expenditures on firms' ability to attract consumer attention and influence consideration sets.

Specifically, we address the following key questions: How does advertising intensity influence the distribution of consumer consideration sets and, consequently, the degree of price competition in the market? What are the equilibrium implications of treating advertising as an endogenous decision variable rather than a fixed exogenous factor? How do firm heterogeneities—such as differences in advertising costs, consumer shares, and cost structures—shape equilibrium outcomes in terms of price dispersion, advertising intensity? We answer these questions from the perspective of the model we propose, which represents one possible approach to incorporating advertising.

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Our main contributions are as follows: (i) we formalize the model and establish its theoretical foundations, (ii) we demonstrate that, under mild assumptions, the main result of Martinelli and Xiao (2024) remains robust in the exogenous case, (iii) we highlight the algebraic complexity introduced by extending the analysis to an oligopoly setting, and (iv) we conduct comparative statics and numerical analysis, focusing on the duopoly case through the solution of nonlinear equations and finite-dimensional optimization.

1 The model

Our model incorporates advertising into the framework studied in Martinelli and Xiao (2024). Consider an economy with $J > 0$ risk-neutral firms, each selling an identical good. Each firm j has a cost structure represented by $c_j > 0$, which is private information known only to the firm itself. However, it is publicly known that $c_j \sim$ i.i.d. F, where F is a twice continuously differentiable cumulative distribution function ($F \in C^2$) with support $\text{supp}(F) = [\underline{c}, \overline{c}] \subset [0, 1]$. We define: $M(c) = \mathbb{P}\{c_j > c\} = 1 - F(c)$. Under the assumptions on the cumulative distribution function F, it follows that $M \in C^2$ and $M'(c) \in (-\infty, 0)$ for every $c \in \text{supp}(F)$.

On the side of consumers, each one is indexed by i and is aware of a specific subset of firms, denoted by K_i, with the cardinality of this set given by $k_i = |K_i|$. Each consumer will buy to the firm setting the lower price, less than 1, in K_i. Firms can purchase advertising $a \in A = [\underline{a}, \overline{a}]$ at a price p_a, which, without loss of generality, is normalized to 1. It is assumed that $\underline{a} > 0$[1]. Advertising decisions for all firms are summarized by the vector $\mathbf{a} \in \prod_{1 \leq j \leq J} A = A^J$. Each firm does not know the advertising levels chosen by its competitors. However, it is common knowledge that a_j follows a certain distribution φ_j, with support A.

Introducing advertising, although more realistic, adds complexity to the original model. We begin by defining the basic elements of the model in the presence of advertising. The probability that a firm belongs to the information set K_i of a consumer i, knowing that i is aware of k firms and $\mathbf{a} = \tilde{\mathbf{a}}$, is now:

$$\mathbb{P}\left\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\right\} = \frac{k}{J}\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right], \tag{1}$$

where $\theta_j = \tilde{a}_j / \sum_{\ell=1}^{J} \tilde{a}_\ell$. Equation 1 implies that all sellers have equal probability of being selected when they choose the same level of advertising. However, sellers who purchase a higher proportion of advertising benefit by having a higher probability of belonging to the information set, except when $k = J$.[2] With respect to the priors $\boldsymbol{\mu} \in \Delta \subset \mathbb{R}^J$, in presence of advertising, the fraction of consumers observing k prices is now conditioned to the advertising vector and modified as follows:

$$\mu_k \to \mu_k^{\tilde{\mathbf{a}}} = \mathbb{P}\{k_i = k | \mathbf{a} = \tilde{\mathbf{a}}\} = \frac{\mu_k w_k(\tilde{\mathbf{a}})}{\underbrace{\sum_{\ell=1}^{J} \mu_\ell w_\ell(\tilde{\mathbf{a}})}_{=\Upsilon(\mu_k, \tilde{\mathbf{a}}; \gamma, k)}}, \tag{2}$$

where

$$w_\ell(\tilde{\mathbf{a}}) = 1 + \gamma \ell \left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\overline{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}\right). \tag{3}$$

In (3), $\overline{\mathbf{a}} = (\overline{a}, \cdots, \overline{a}) \in \mathbb{R}^J$, $\underline{\mathbf{a}} = (\underline{a}, \cdots, \underline{a}) \in \mathbb{R}^J$ and $\gamma > 0$. By defining $\mu_k^{\tilde{\mathbf{a}}}$ as in (2), we ensure

[1] Firms must purchase a minimum level of advertising in order to *exist*. This can be interpreted as a fixed cost.

[2] If the consumer knows every firm, by overwhelming, advertising will play no role. Moreover, the coefficient $(J-k)/J$ is mandatory to ensure that $\mathbb{P}\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\} \in [0, 1]$, see Proposition 1.

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that for $\tilde{\mathbf{a}} \neq \underline{\mathbf{a}}$, $\mu_1 > \mu_1^{\tilde{\mathbf{a}}}$ and $\mu_J < \mu_J^{\tilde{\mathbf{a}}}$ (see Proposition 2). It follows by iterating Bayes rule and total probability rule that

$$
\begin{aligned}
\mathbb{P}\{k_i = k | j \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{\mathbb{P}\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\}\mathbb{P}\{k_i = k | \mathbf{a} = \tilde{\mathbf{a}}\}}{\mathbb{P}\{j \in K_i | \mathbf{a} = \tilde{\mathbf{a}}\}} \\
&= \frac{\mathbb{P}\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\}\mathbb{P}\{k_i = k | \mathbf{a} = \tilde{\mathbf{a}}\}}{\sum_{\ell=1}^{J} \mathbb{P}\{j \in K_i | k_i = \ell, \mathbf{a} = \tilde{\mathbf{a}}\}\mathbb{P}\{k_i = \ell | \mathbf{a} = \tilde{\mathbf{a}}\}} \\
&= \frac{\frac{k}{J}\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right] \cdot \mu_k^{\tilde{\mathbf{a}}}}{\sum_{\ell=1}^{J} \frac{\ell}{J}\left[1 + \left(\frac{J-\ell}{J}\right)\theta_j\right]\mu_\ell^{\tilde{\mathbf{a}}}} \\
&= \frac{k\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right] \cdot \mu_k^{\tilde{\mathbf{a}}}}{\sum_{\ell=1}^{J} \ell\left[1 + \left(\frac{J-\ell}{J}\right)\theta_j\right]\mu_\ell^{\tilde{\mathbf{a}}}}.
\end{aligned}
$$

Now, given $\tilde{\mathbf{a}} \in A$, let

$$
G(p\,;\,\rho) = \int_{\underline{c}}^{\overline{c}} \mathbf{1}_{\{\rho(c,\,\tilde{\mathbf{a}})>p\}} dF(c)
$$

be the upper c.d.f. of price contingent on some pricing function ρ (see Martinelli and Xiao (2024)). Using independence, the probability that p is lower than $k-1$ prices is $G(p\,;\,\rho)^{k-1}$. Then, under the assumption of symmetry, the expected demand for a seller that sets the price p, given an advertising vector $\tilde{\mathbf{a}}$, is[3]

$$
Q(p; \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \tilde{\mathbf{a}}, \rho) = \underbrace{\sum_{\ell=1}^{J}\left(\frac{\ell}{J}\right)\mu_\ell^{\tilde{\mathbf{a}}}}_{\text{Mass of consumers.}} \left\{ \underbrace{\sum_{k=1}^{J}\left(\frac{k\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right]\mu_k^{\tilde{\mathbf{a}}}}{\sum_{\ell=1}^{J} \ell\left[1 + \left(\frac{J-\ell}{J}\right)\theta_j\right]\mu_\ell^{\tilde{\mathbf{a}}}}\right)}_{\text{Probability of competing with } k-1 \text{ with } i \text{ fixed.}} \underbrace{G(p\,;\,\rho)^{k-1}}_{\text{Proba. of being the cheaper of } k-1.} \right\}.
$$

Hence, the expected profit for firm j, given the advertising level $\tilde{\mathbf{a}}$, is simply

$$
\Pi_j(p\,;\,c_j, \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \rho) = (p - c_j)Q(p\,;\,\boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \tilde{\mathbf{a}}, \rho) - \tilde{a}_j. \tag{4}
$$

Until now, we have considered the advertising vector $\mathbf{a}$ to be exogenous, equal to $\tilde{\mathbf{a}}$. However, it is reasonable and indeed more realistic to assume that firms simultaneously set both p and a_j. That is, $\mathbf{a}$ is endogenous. Thus, each firm j optimizes over p and a_j. Firm j does know is a prior distribution of $\mathbf{a}_{-j}$, denoted by $\boldsymbol{\Phi}_{-j}^{\mathbf{a}_j}$, with a corresponding density $\boldsymbol{\varphi}_{-j}^{\mathbf{a}_{-j}}$, and support $A_{-j} = [\underline{a}, \overline{a}]^{J-1}$. Therefore, the expected profits for firm j, without knowing the exact realization of the advertising vector, and assuming the same pricing rule, are given by

$$
\mathbb{E}_{\mathbf{a}_{-j}}[\Pi_j(p\,;\,c_j, \boldsymbol{\mu}^{\mathbf{a}}, \rho)] = \int_{\underline{a}}^{\overline{a}} \cdots \int_{\underline{a}}^{\overline{a}} ((p - c_j)Q(p\,;\,\boldsymbol{\mu}^{\mathbf{a}}, \mathbf{a}, \rho) - a_j)\boldsymbol{\varphi}_{-j}^{\mathbf{a}_{-j}} da_1 \cdots da_{j-1} da_{j+1} \cdots da_J, \tag{5}
$$

Finally, each firm maximizes its profit function (4) in the case of exogenous advertising or its expected profit function (5) in the endogenous advertising case. Concretely, each firm solves, for the exogenous case:

$$
\max_{p \geq 0}\ (p - c_j)Q(p\,;\,\boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \tilde{\mathbf{a}}, \rho) - \tilde{a}_j,
$$

and for the endogenous case[4]:

$$
\max_{p \geq 0,\ a_j \in A} \int_{\underline{a}}^{\overline{a}} \cdots \int_{\underline{a}}^{\overline{a}} ((p - c_j)Q(p\,;\,\boldsymbol{\mu}^{\mathbf{a}}, \mathbf{a}, \rho) - a_j)\boldsymbol{\varphi}_{-j}^{\mathbf{a}_{-j}} d\mathbf{a}_{-j}.
$$

[3]For the expression of the mass of consumers we follow Burdett and Judd (1983).

[4]Note that the pricing rule from Martinelli and Xiao (2024), for the endogenous case, could change. However, symmetry will be a global assumption in this work.

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2 Oligopolistic framework for exogenous advertising

The general case, as we will see in the next section, is challenging to address algebraically due to the complexity introduced by the modifications to prior and conditional probabilities. However, for the case of exogenous advertising, under the reasonable assumption that the pricing function is monotonically increasing in c, the main result in Martinelli and Xiao (2024) still holds.

Mathematically, for fixed $\tilde{\mathbf{a}}$, $\rho(c;\tilde{\mathbf{a}})$ is increasing in c when $\mu_1^{\tilde{\mathbf{a}}} < 1$, with $\rho(c\ ;\ \tilde{\mathbf{a}}) = 1$ for $\mu_1^{\tilde{\mathbf{a}}} = 1$, and $\rho(\bar{c}\ ;\ \tilde{\mathbf{a}}) = \bar{c}$ when $\mu_1^{\tilde{\mathbf{a}}} = 0$. This follows from the arguments provided in Martinelli and Xiao (2024). Furthermore, the profit function $\Pi(p\ ;\ c, \tilde{\mathbf{a}}, \rho)$ satisfies the single-crossing property introduced in Athey (2001), and the proof in Martinelli and Xiao (2024) applies directly. Additionally, monotonicity of $\rho(\cdot\ ;\ \tilde{\mathbf{a}})$ implies that:

$$
\begin{aligned}
G(\rho(c\ ;\ \tilde{\mathbf{a}})\ ;\ \rho) &= \int_{\underline{c}}^{\bar{c}} \mathbf{1}_{\{\rho(s\ ;\ \tilde{\mathbf{a}})>\rho(c\ ;\ \tilde{\mathbf{a}})\}} d(1-M(s)) \\
&= \int_{\underline{c}}^{\bar{c}} \mathbf{1}_{\{s>c\}} d(1-M(s)) \\
&= \int_{c}^{\bar{c}} d(1-M(s)) \\
&= M(c) - M(\bar{c}) \\
&= M(c) - \mathbb{P}\{\tilde{c} > \bar{c}\} \\
&= M(c),
\end{aligned}
$$

which implies, as in Martinelli and Xiao (2024), that

$$
\rho(c;\tilde{\mathbf{a}}) = c + \frac{\mathcal{Q}(\bar{c}\ ;\ \tilde{\mathbf{a}})}{\mathcal{Q}(c\ ;\ \tilde{\mathbf{a}})}(\rho(\bar{c}\ ;\ \tilde{\mathbf{a}}) - \bar{c}) + \int_{c}^{\bar{c}} \frac{\mathcal{Q}(x\ ;\ \tilde{\mathbf{a}})}{\mathcal{Q}(c\ ;\ \tilde{\mathbf{a}})} dx, \tag{6}
$$

where

$$
Q(\rho(c\ ;\ \tilde{\mathbf{a}}), \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \tilde{\mathbf{a}}\ ;\ \rho) = \mathcal{Q}(c\ ;\ \tilde{\mathbf{a}}, \boldsymbol{\mu}^{\tilde{\mathbf{a}}}).
$$

However, the results introduced after Theorem 1 in Martinelli and Xiao (2024) no longer apply directly to the current model, as the introduction of advertising alters the algebraic structure of the expected demand function. Specifically, the algebra of Proposition 1 in Martinelli and Xiao (2024) is what gets altered, and consequently, the resulting implications. To be precise, introducing $\mu_k^{\tilde{\mathbf{a}}} = \Upsilon(\mu_k, \tilde{\mathbf{a}}; \gamma, k)$ creates a more complex structure when evaluating the impact of μ_k on the different outcomes. This is why, from now on, we will focus on the case $J = 2$.

3 Duopoly

In this section, we study the case under the assumptions of an exogenous advertising vector and only two firms ($J = 2$). We begin by defining the elements corresponding to the case $J = 2$,

$$
\begin{aligned}
&\mathbb{P}\{1 \in K_i | k_i = 1; \mathbf{a} = \tilde{\mathbf{a}}\} = \frac{1}{2} + \frac{1}{4}\left(\frac{\tilde{a}_1}{\tilde{a}_1 + \tilde{a}_2}\right) \\
&\mathbb{P}\{1 \in K_i | k_i = 2; \mathbf{a} = \tilde{\mathbf{a}}\} = 1 \\
&\mathbb{P}\{2 \in K_i | k_i = 1; \mathbf{a} = \tilde{\mathbf{a}}\} = \frac{1}{2} + \frac{1}{4}\left(\frac{\tilde{a}_2}{\tilde{a}_1 + \tilde{a}_2}\right) \\
&\mathbb{P}\{2 \in K_i | k_i = 2; \mathbf{a} = \tilde{\mathbf{a}}\} = 1.
\end{aligned}
$$

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Then, setting $\beta = (\overline{a} - \underline{a})^2$ and $\eta(\tilde{\mathbf{a}}) = (\tilde{a}_1 - \underline{a})^2 + (\tilde{a}_2 - \underline{a})^2$

$$
\begin{aligned}
\mu_1^{\tilde{\mathbf{a}}} &= \frac{\mu_1 \left(1 + \gamma \frac{\eta(\tilde{\mathbf{a}})}{2\beta}\right)}{\mu_1 \left(1 + \gamma \frac{\eta(\tilde{\mathbf{a}})}{\beta}\right) + \mu_2 \left(1 + 2\gamma \frac{\eta(\tilde{\mathbf{a}})}{2\beta}\right)} \\
&= \frac{\mu_1 + \mu_1 \left(\frac{\gamma \eta(\tilde{\mathbf{a}})}{2\beta}\right)}{1 + (\mu_1 + 2\mu_2) \frac{\gamma \eta(\tilde{\mathbf{a}})}{2\beta}} \\
\mu_2^{\tilde{\mathbf{a}}} &= 1 - \mu_1^{\tilde{\mathbf{a}}} \\
&= \frac{1 + \mu_2 \left(\frac{\gamma \eta(\tilde{\mathbf{a}})}{\beta}\right) - \mu_1}{1 + (\mu_1 + 2\mu_2) \left(\frac{\gamma \eta(\tilde{\mathbf{a}})}{2\beta}\right)}.
\end{aligned}
$$

We now compute

$$
\begin{aligned}
\mathbb{P}\{k_i = 1 | 1 \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{\left(1 + \frac{\theta_1}{2}\right) \mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_1}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} \\
\mathbb{P}\{k_i = 1 | 2 \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{\left(1 + \frac{\theta_2}{2}\right) \mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_2}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} \\
\mathbb{P}\{k_i = 2 | 1 \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_1}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} \\
\mathbb{P}\{k_i = 2 | 2 \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\} &= \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_2}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}}.
\end{aligned}
$$

Hence,

$$
Q(p\ ;\ \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \tilde{\mathbf{a}}, \rho) = \left(1 - \frac{\mu_1^{\tilde{\mathbf{a}}}}{2}\right) \left\{ \frac{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} + \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} G(p\ ;\ \rho) \right\},
$$

and

$$
\Pi_j(p\ ;\ c_j, \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \rho) = (p - c_j) \left(1 - \frac{\mu_1^{\tilde{\mathbf{a}}}}{2}\right) \left\{ \frac{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} + \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} G(p\ ;\ \rho) \right\} - \tilde{a}_j.
$$

The advantage of considering the case $J = 2$ arises from the fact that we can use $\mu_1^{\tilde{\mathbf{a}}} + \mu_2^{\tilde{\mathbf{a}}} = 1$.

3.1 Comparative statics

We are now interested in performing comparative statics with respect to $\tilde{a}_j$, analyzing how advertising impacts the different defined elements. In particular, we would like to compute its effect on demand, and for this, it is necessary to examine its effect on priors and the probability of competition.

$$
\begin{aligned}
\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} &= \frac{\frac{\mu_1 \gamma}{\beta} (\tilde{a}_j - \underline{a}) D - 2N \frac{\gamma}{\beta} (\tilde{a}_j - \underline{a})}{D^2} \\
&= \underbrace{\frac{\frac{\gamma}{\beta} (\tilde{a}_j - \underline{a})}{D^2}}_{\geq 0} (D - 2N)
\end{aligned}
$$

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where

$$D = \mu_1 \left(1 + \gamma \frac{\eta(\tilde{\mathbf{a}})}{\beta}\right) + \mu_2 \left(1 + 2\gamma \frac{\eta(\tilde{\mathbf{a}})}{2\beta}\right)$$

and

$$N = \mu_1 \left(1 + \frac{\gamma \eta(\tilde{\mathbf{a}})}{2\beta}\right).$$

Hence, the sign of $\partial \mu_1^{\tilde{\mathbf{a}}} / \partial a_j$ depends on the sign of $D - 2N$.

$$\begin{aligned} D - 2N &= \mu_1 \left[\mu_1 + \mu_2 + \frac{\mu_1 \gamma}{\beta} + \frac{\mu_2 \gamma \eta}{\beta}\right] - 2N \\ &= -\mu_1 < 0. \end{aligned}$$

Thus, $\partial \mu_1^{\tilde{\mathbf{a}}} / \partial a_j < 0$. Since $\mu_2^{\tilde{\mathbf{a}}} = 1 - \mu_1^{\tilde{\mathbf{a}}}$, it follows that

$$\frac{\partial \mu_2^{\tilde{\mathbf{a}}}}{\partial a_j} = -\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} > 0.$$

Using these results, we now compute

$$\begin{aligned} \frac{\partial \mathbb{P}\{k_i = 1 | j \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\}}{\partial a_j} &= \frac{\partial}{\partial a_j} \left[\frac{C \mu_1^{\tilde{\mathbf{a}}}}{C \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}}\right] \\ &= \frac{C \frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} \tilde{D} - C \mu_1^{\tilde{\mathbf{a}}} \left[C \frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} - 2 \frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}\right]}{\tilde{D}^2} \\ &= \frac{2C \frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}}{\tilde{D}^2} < 0, \end{aligned}$$

with $C = 1 + \theta_j / 2$ and $\tilde{D} = C \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}$. Analogously,

$$\begin{aligned} \frac{\partial \mathbb{P}\{k_i = 2 | j \in K_i, \mathbf{a} = \tilde{\mathbf{a}}\}}{\partial a_j} &= \frac{\partial}{\partial a_j} \left[\frac{2\mu_2^{\tilde{\mathbf{a}}}}{C \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}}\right] \\ &= \frac{-2(C + \mu_1)}{\tilde{D}^2} \frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial \mu_1} > 0, \end{aligned}$$

where $\tilde{D} = C \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}$. Finally, for $p = \rho(c)$, fixing $\tilde{\mathbf{a}}$:

$$\begin{aligned} \frac{\partial Q(p; \boldsymbol{\mu}^{\tilde{\mathbf{a}}}, \tilde{\mathbf{a}}, \rho)}{\partial a_j} = &-\frac{1}{2} \frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} \left\{ \frac{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} + \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1 + \frac{\theta_j}{2}\right) \mu_1^{\tilde{\mathbf{a}}} + 2\mu_2^{\tilde{\mathbf{a}}}} M(c) \right\} \\ &+ \left(1 - \frac{\mu_1^{\tilde{\mathbf{a}}}}{2}\right) \left\{ \frac{2C \frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} - 2 \frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} (C + \mu_1^{\tilde{\mathbf{a}}}) M(c)}{\tilde{D}^2} \right\}. \end{aligned}$$

In this way, as expected, when advertising increases, it is more likely to become aware of all the firms rather than just one, and for the information sets to include both firms instead of just one.

Finally, the sign of how demand changes is not trivial to analyze. However, if we want to ensure that an increase in advertising raises demand, the following condition must hold:

$$2C - (C + \mu_1) M(c) < 0.$$

This can occur, for example, if $\theta_j \ll \mu_1$ and $\mu_1 > 1/2$.

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Finally, from, (6), and knowing that $\rho(\overline{c}\ ;\ \tilde{\mathbf{a}})$ is fixed, we can compute

$$\frac{\partial \rho(c;\tilde{\mathbf{a}})}{\partial a_j} = \frac{\frac{\partial \mathcal{Q}(\overline{c};\tilde{\mathbf{a}})}{\partial a_j}\mathcal{Q}(c;\tilde{\mathbf{a}}) - \mathcal{Q}(\overline{c};\tilde{\mathbf{a}})\frac{\partial \mathcal{Q}(c;\tilde{\mathbf{a}})}{\partial a_j}}{\mathcal{Q}(c;\tilde{\mathbf{a}})^2} + \int_c^{\overline{c}} \left(\frac{\frac{\partial \mathcal{Q}(x;\tilde{\mathbf{a}})}{\partial a_j}\mathcal{Q}(c;\tilde{\mathbf{a}}) - \mathcal{Q}(x;\tilde{\mathbf{a}})\frac{\partial \mathcal{Q}(c;\tilde{\mathbf{a}})}{\partial a_j}}{\mathcal{Q}(c,\tilde{\mathbf{a}})^2} \right) dx.$$

The first term is equal to

$$\frac{C\left(2M(c)\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j}\left(C^2(\mu_1^{\tilde{\mathbf{a}}}-2)(\mu_1^{\tilde{\mathbf{a}}})^2 + C(\mu_1^{\tilde{\mathbf{a}}}-2)\mu_1^{\tilde{\mathbf{a}}}\left((\mu_1^{\tilde{\mathbf{a}}})^2+4\mu_2^{\tilde{\mathbf{a}}}\right) + 2(\mu_1^{\tilde{\mathbf{a}}}-2)\mu_2^{\tilde{\mathbf{a}}}\left((\mu_1^{\tilde{\mathbf{a}}})^2+2\mu_2^{\tilde{\mathbf{a}}}\right) + \mu_1^{\tilde{\mathbf{a}}}\mu_2^{\tilde{\mathbf{a}}}\tilde{D}^2\right)\right)}{(\mu_1^{\tilde{\mathbf{a}}}-2)\tilde{D}^2\left(2\mu_2^{\tilde{\mathbf{a}}}M(c)+C\mu_1^{\tilde{\mathbf{a}}}\right)^2}$$

$$+\frac{C\left(-2(\mu_1^{\tilde{\mathbf{a}}})^2\mu_2^{\tilde{\mathbf{a}}}\tilde{D}^2M(c) + C(\mu_1^{\tilde{\mathbf{a}}})^2\tilde{D}^2\left(\frac{\partial \mu_1^{\tilde{\mathbf{a}}}}{\partial a_j} - \mu_1^{\tilde{\mathbf{a}}}\right)\right)}{(\mu_1^{\tilde{\mathbf{a}}}-2)\tilde{D}^2\left(2\mu_2^{\tilde{\mathbf{a}}}M(c)+C\mu_1^{\tilde{\mathbf{a}}}\right)^2}.$$

As can be seen from the previous expression, there is little hope of analytically determining the sign of $\partial\rho(c\ ;\ \tilde{\mathbf{a}})/\partial a_j$, especially considering that $\partial\mathcal{Q}/\partial a_j(c\ ;\ \cdot)$ and $\mathcal{Q}(c\ ;\ \cdot)$ exhibit opposite monotonicity.

In the case of exogenous advertising, as discussed in Section 2, we analyzed $\rho(\cdot\ ;\ \mathbf{a})$ with properties inherited from $\rho(\cdot)$, (Martinelli and Xiao, 2024). However, when considering endogenous advertising, we can no longer assume such straightforward behavior. While it is reasonable to maintain the assumption of monotonicity in c, the dependence on a is more complex. Specifically, the sign of $\partial\rho(c\ ;\ \tilde{\mathbf{a}})/\partial a_j$ may vary depending on the level of c. For instance, at lower cost levels, increased advertising could lead to a decrease in price due to heightened competitiveness, whereas at higher cost levels, the effect might differ. To derive feasible conclusions, we explore various specifications for $\rho(c\ ,\ a)$:

$$\rho(c,a) = h(c) + g(a,c), \tag{7}$$

where $h(c)$ is a monotonically increasing function of c, and $g(a,c)$ is defined as:

$$g(a,c) = \left(\lambda_1 \mathbf{1}_{\{c>c^*\}} + \lambda_2 \mathbf{1}_{\{c\leq c^*\}}\right)\psi(a), \tag{8}$$

with $\psi(a)$ being a monotonic function of a, and $\lambda_1\lambda_2 < 0$. This formulation allows $\rho(c,a)$ to exhibit different monotonic behaviors with respect to a depending on whether c is above or below the threshold c^*.

3.2 Endogenous advertising

Under the assumptions specified in (7) and (8), each firm maximizes j with respect to p and a:

$$\int_A \left[(p-c_j)\left(1-\frac{\mu_1^{\tilde{\mathbf{a}}}}{2}\right)\left\{\frac{\left(1+\frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}}}{\left(1+\frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}}+2\mu_2^{\tilde{\mathbf{a}}}} + \frac{2\mu_2^{\tilde{\mathbf{a}}}}{\left(1+\frac{\theta_j}{2}\right)\mu_1^{\tilde{\mathbf{a}}}+2\mu_2^{\tilde{\mathbf{a}}}}G(p\ ;\ \rho)\right\} - a_j\right] da_{-j}$$

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where

$$
\begin{aligned}
G(p,\rho) &= \int_{\underline{a}}^{\overline{a}} \int_{\underline{c}}^{\overline{c}} \mathbf{1}_{\{h(c)+g(a,c)>p\}} dF d\Phi \\
&= \int_{\underline{a}}^{\overline{a}} \int_{\underline{c}}^{\overline{c}} \mathbf{1}_{\{h(c)+g(a,c)>p\}} f(c)\varphi(a) dc da.
\end{aligned}
$$

3.3 Numerical solution

Conclusions

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A Proofs

Proposition 1. For any combination of k, j, i and $\tilde{\mathbf{a}}$, (1) is a probability.

Proof. Certainly,

$$\mathbb{P}\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\} \geq 0.$$

It remains to prove that

$$\mathbb{P}\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\} = \frac{k}{J}\left[1 + \left(\frac{J-k}{J}\right)\theta_j\right] \leq 1.$$

First, $\theta_j, (k/J) \in [0,1]$. Then, we denote $x = k/J$. Hence,

$$\mathbb{P}\{j \in K_i | k_i = k, \mathbf{a} = \tilde{\mathbf{a}}\} = x(1+\theta_j) - \theta_j x^2 = f(x;\theta_j).$$

The function is increasing from 0 to $x^\circ = (1+\theta_j)/(2\theta_j) > 1$. Since $f(0;\theta_j) = 0$ and f is increasing from 0 to 1, we only need to prove that $f(1;\theta_j) \leq 1$. This follows directly replacing $x = 1$: $f(1;\theta_j) = 1$. □

Proposition 2. The priors defined in (2) satisfy that

$$\mu_1 > \mu_1^{\mathbf{a}} \text{ and } \mu_J < \mu_J^{\mathbf{a}}, \tag{9}$$

for every $\boldsymbol{\mu} \in \Delta \subset \mathbb{R}^J$ non degenerate[5] Moreover, for any $\boldsymbol{\mu} \in \Delta$, (9) holds with weak inequality.

Proof. For $\mu_1 \neq 1$, $1 < \sum_{\ell=1}^J \mu_\ell \ell$. Then,

$$\mu_1 + \mu_1\gamma\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right) < \mu_1 + \mu_1 \sum_{\ell=1}^J \mu_\ell \ell\gamma\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right).$$

Dividing by $1 + \sum_{\ell=1}^J \mu_\ell \ell\gamma\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right)$, we conclude. Analogously, since

$$\sum_{\ell=1}^J \mu_\ell \ell < J \sum_{\ell=1}^J \mu_\ell = J,$$

we have that

$$\mu_J + \mu_J \sum_{\ell=1}^J \mu_\ell \ell\gamma\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right) < \mu_J\left[1 + \gamma J\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right)\right]$$

$$\mu_J < \frac{\mu_J\left[1 + \gamma J\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right)\right]}{\sum_{\ell=1}^J \mu_\ell\left[1 + \gamma\ell\left(\frac{||\tilde{\mathbf{a}} - \underline{\mathbf{a}}||_2^2}{||\bar{\mathbf{a}} - \underline{\mathbf{a}}||}\right)\right]}.$$

□

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[5] This means that $\mu_1 \neq 1$.

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