Pontificia Universidad Católica del Perú
Economics Major

November 28, 2024

Solutions to Recitation 8
ECO 263

Professor: Pavel Coronado
TA's: Marcelo Gallardo, Fernanda Crousillat

Stochastic dominance

Exercise 1 (Mas-Colell et al.). Let $F(\cdot)$ and $G(\cdot)$ be lotteries defined over an amount of money x, and let $U : L \to \mathbb{R}$ be the expected utility function in the von Neumann-Morgenstern (vNM) framework, with a utility function $u : \mathbb{R} \to \mathbb{R}$ that is non-decreasing and twice differentiable.

1. Show that if $G(x) \geq F(x)\,\forall x$, then $U(F) \geq U(G)$ (First-Order Stochastic Dominance, FSD).

2. If, in addition, the utility function is strictly concave and the lotteries offer the same expected payoff, show that if

$$\int_0^x G(t)\,dt \geq \int_0^x F(t)\,dt \quad \forall x,$$

then $U(F) \geq U(G)$ (Second-Order Stochastic Dominance, SSD).

Solution: since $U : \mathcal{L} \to \mathbb{R}$ is an expected vNM utility function,

$$U(F) = \int u(x)dF(x) \text{ and } U(G) = \int u(x)dG(x).$$

This, if $F \succeq G$,

$$\begin{aligned}
\int u(x)dF(x) &\geq \int u(x)dG(x) \\
\int u(x)d[F(x) - G(x)] &\geq 0 \\
u(x)[F(x) - G(x)]_{-\infty}^{\infty} - \int u'(x)[F(x) - G(x)]dx &\geq 0 \\
\int u'(x)[G(x) - F(x)]dx &\geq 0 \\
G(x) &\geq F(x) \text{ a.e.}
\end{aligned}$$

This is because $u' > 0$. Now, with respect to the second part, applying once again integration by parts,

$$\int u'(x)[G(x) - F(x)]dx = u'(x)\int (G(s) - F(s))ds + \int -u''(x)\left(\int_0^x [G(t) - F(t)]dt\right) dx$$

If both lotteries provide the same expected payoff,

$$\int G(s)ds = \int F(s)ds.$$

Indeed,

$$\begin{aligned}
\int_{[a,b]} xdF &= \int_{[a,b]} xdG \\
bF(b) - aF(a) - \int_{[a,b]} F(x)dx &= bG(b) - aG(a) - \int_{[a,b]} G(x)dx \\
1 - 0 - \int_{[a,b]} F(x)dx &= 1 - 0 - \int_{[a,b]} G(x)dx \\
\int_{[a,b]} F(x)dx &= \int_{[a,b]} G(x)dx.
\end{aligned}$$

Thus,

$$\int u'(x)[G(x) - F(x)]dx = \int -u''(x)\left(\int_0^x [G(t) - F(t)]dt\right) dx$$

Since $u'' < 0$, if

$$\int_0^x G(t)dt \geq \int_0^x F(t),$$

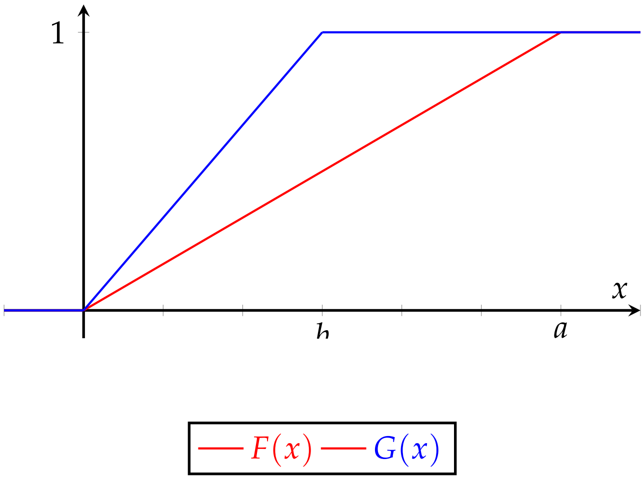
$$\int u'(x)[G(x) - F(x)]dx = \int u(x)dF(x) - \int u(x)dG(x) \geq 0.$$

Exercise 2 (T. Sarver, Duke). Suppose that F is the uniform distribution on the interval $[0, a]$:

$$F(x) = \begin{cases} 0, & \text{if } x < 0, \\ x/a, & \text{if } 0 \leq x \leq a, \\ 1, & \text{if } x > a. \end{cases}$$

Similarly, suppose that G is the uniform distribution on the interval $[0, b]$. If $a \geq b$, show that $F \geq_{FOSD} G$.

Solution: simply if $a \geq b$, then $F(x) \leq G(x)$; it follows immediately from Exercise 1.1.



Adverse selection

Exercise 3. Consider the following market for used cars, similar to Akerlof's, where the quality of these cars is given by $x \in [0, 1]$. A car of quality x is valued at x by the buyer and $v(x)$ by the seller, where $v(\cdot)$ is a continuous, strictly increasing function such that $v(x) \leq x$ for all x. If the density of qualities is $f(x)$, determine the equilibrium under the following conditions:

a) Buyers and sellers know the quality of each car.

b) Neither buyers nor sellers know the quality of each car.

c) If f is a uniform distribution on $[0, 1]$ and $v(x) = x^2$, and only the sellers know the quality of the car.

Solution:

1. If $v(x) < x$, then each car is sold at $p = x$.

2. If $v(x) \leq x$, and it is not possible for either party to verify the quality of the cars,
$$p = \mathbb{E}[X] = \int_0^1 x f(x) dx \geq \int_0^1 v(x) f(x) dx = \mathbb{E}[v(x)].$$

3. In equilibrium,
$$p^* = \mathbb{E}[x \mid x \in \Theta^*], \; \Theta^* = \{x : v(x) \leq p\}.$$

Then, since the conditional expectation is computed as follows:

$$\mathbb{E}[X|B] = \frac{1}{\mathbb{P}(B)} \int_B X d\mathbb{P},$$

we have that

$$p^* = \frac{1}{\sqrt{p}} \int_0^{\sqrt{p}} \frac{x}{\sqrt{p}} dx = \left[\frac{x^2}{2\sqrt{p}} \right]_0^{\sqrt{p}}.$$

Thus, $p^* = 1/4$ and cars with $x > 1/2$ are sold. Another equilibrium is $p = 0$.

Moral hazard

Exercise 4. Prove that, if effort is not observable but the manager is risk-neutral, the **payment scheme** generates the same expected utility and profit than in the full (symmetric) information case. Do it for both the discrete and continuous model.

Solution: first of all, let us recall the optimization problems involved. When the **effort is observed** and $e \in \{e_L, e_H\}$, the optimization problem is

$$\begin{cases} \max_{e,w(\pi)} & \displaystyle\int_{\underline{\pi}}^{\overline{\pi}} (\pi - w(\pi)) f(\pi|e) d\pi \\ \text{s.t.} & \displaystyle\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \overline{u}. \end{cases}$$

The discrete version of this problem, i.e. when the support is $\{\pi_1, \cdots, \pi_n\}$ instead of $[\underline{\pi}, \overline{\pi}]$, is

$$\begin{cases} \max_{e,w(\pi_i)} & \sum_{i=1}^n (\pi_i - w(\pi_i)) p(\pi_i|e) \\ \text{s.t.} & \sum_{i=1}^n v(w(\pi_i)) p(\pi_i|e) - g(e) \geq \overline{u}. \end{cases}$$

Inequalities are actually equalities in the previous programs. FOC yields

$$\mathscr{L}(\{w(\pi_i)\}_{i=1}^n, \gamma) = \sum_{i=1}^n (\pi_i - w(\pi_i)) p(\pi_i|e) + \gamma \left(\sum_{i=1}^n v(w(\pi_i)) p(\pi_i|e) - g(e) - \overline{u} \right).$$

$$\frac{\partial \mathscr{L}}{\partial w(\pi_i)} = -p(w(\pi_i)|e) + \gamma v'(w(\pi_i)) p(w(\pi_i)|e) = 0$$

$$\frac{1}{v'(w(\pi_i))} = \gamma.$$

Then, since $v'(\cdot)$ is strictly decreasing, there exists a unique $w^* = w(\pi_i)$ for all i

$$\sum_{i=1}^{n} v(\underbrace{w(\pi_i)}_{=w^*})p(\pi_i|e) - g(e) - \overline{u} = 0,$$

$$\sum_{i=1}^{n} v(w^*)p(\pi_i|e) - g(e) - \overline{u} = 0,$$

$$v(w^*)\underbrace{\sum_{i=1}^{n} p(\pi_i|e)}_{=1} = g(e) + \overline{u},$$

$$w^* = v^{-1}(g(e) + \overline{u}).$$

Finally, e is chosen such that the principal maximizes their expected profits. That is:

$$\max_{e\in\{e_L,e_H\}} \underbrace{\sum_{i=1}^{n}[\pi_i - w^*(e)]p(\pi_i|e)}_{=\int_{\underline{\pi}}^{\overline{\pi}}[\pi-w(e)]f(\pi|e)d\pi}$$

$$= \sum_{i=1}^{n}[\pi_i - v^{-1}(g(e)+\overline{u})]p(\pi_i|e)$$

$$= \underbrace{\sum_{i=1}^{n}\pi_i p(\pi_i|e) - v^{-1}(g(e)+\overline{u})\sum_{i=1}^{n}p(\pi_i|e)}_{\int_{\underline{\pi}}^{\overline{\pi}}\pi(\pi|e)d\pi - v^{-1}(g(e)+\overline{u})\int_{\underline{\pi}}^{\overline{\pi}}\pi(\pi|e)d\pi}$$

$$= \sum_{i=1}^{n}\pi_i p(\pi_i|e) - v^{-1}(g(e)+\overline{u}).$$

When the **effort is not observed**, the manager solves the following problem:

$$\begin{cases} \min & \int_{\underline{\pi}}^{\overline{\pi}} w(\pi)f(\pi|e)d\pi \\ \text{s.t.} & \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi))f(\pi|e)d\pi - g(e) \geq \overline{u}, \\ & \underbrace{e \text{ solves } \max_{\tilde{e}} \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi))f(\pi|\tilde{e})d\pi - g(\tilde{e})}_{\text{Incentive compatibility.}} \end{cases}$$

Eventually, the incentive compatibility constraint can be written as:

$$\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi))f(\pi|e^*)d\pi - g(e^*) \geq \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi))f(\pi|\tilde{e})d\pi - g(\tilde{e}).$$

In this case, the solution is characterized by the following equation:

$$\frac{1}{v'(w(\pi))} = \gamma + \mu \left[1 - \frac{f(\pi|e_L)}{f(\pi|e_H)}\right].$$

Thus, since $\mu, \gamma > 0$ (this can be demonstrated), the payment scheme $w = w(\pi)$ is increasing if and only if the likelihood ratio $\frac{f(\pi|e_L)}{f(\pi|e_H)}$ is decreasing.
In the case where outcomes are discrete, the principal solves (a for high and b for low - effort):

$$\begin{cases} \max & \sum_{i=1}^n p_{ia}(x_i - w_i) \\ \text{s.t.} & \sum_{i=1}^n p_{ia}(v(w_i) - g(a)) \geq \overline{u}, \\ & \sum_{i=1}^n p_{ia}(v(w_i) - g(a)) \geq \sum_{i=1}^n p_{ib}(v(w_i) - g(b)). \end{cases}$$

We have already shown that the constraints hold with equality. Therefore, it suffices to apply first-order conditions to the Lagrangian:

$$\begin{aligned} \mathscr{L}(\{w_i\}_{i=1}^n, \gamma, \mu) = & \sum_{i=1}^n p_{ai}(x_i - w_i) \\ & + \gamma \left(\underbrace{\sum_{i=1}^n p_{ai}(v(w_i) - g(a))}_{=\sum_{i=1}^n p_{ai} v(w_i) - g(a)} - \overline{u} \right) \\ & + \mu \left(\underbrace{\sum_{i=1}^n p_{ia}(v(w_i) - g(a))}_{=\sum_{i=1}^n p_{ai} v(w_i) - g(a)} - \underbrace{\sum_{i=1}^n p_{ib}(v(w_i) - g(b))}_{=\sum_{i=1}^n p_{bi} v(w_i) - g(b)} \right). \end{aligned}$$

Then:

$$\frac{\partial \mathscr{L}(\{w_i\}_{i=1}^n, \gamma, \mu)}{\partial w_i} = -p_{ai} + \gamma p_{ai} v'(w_i) + \mu(p_{ai} - p_{bi}) v'(w_i) = 0.$$

Therefore:

$$v'(w_i) = \frac{p_{ai}}{\gamma p_{ai} + \mu(p_{ai} - p_{bi})} = \frac{1}{\gamma + \mu \left(1 - \frac{p_{bi}}{p_{ai}}\right)}.$$

Now, let us prove the statement.

Proof. The proof consists of showing that there exists a payment scheme that provides the principal with the same payoff as in the case where e is observable. Suppose the payment scheme is given by:

$$w(\pi) = \pi - \alpha, \ \alpha > 0.$$

In this sense, the principal *sells* the project to the agent. If the agent accepts the contract, they choose e such that it maximizes:

$$\int w(\pi)f(\pi|e)d\pi - g(e) = \int \pi f(\pi|e)d\pi - \alpha - g(e). \tag{1}$$

Now, recall that, when effort is observable, in the case $v(w) = w$, e^* solves:

$$\max_{e \in \{e_L, e_H\}} \int \pi f(\pi|e)d\pi - g(e) - \overline{u}.$$

This same e maximizes (1). Thus, the contract induces the optimal effort level (first best - full observability). Next, the agent accepts the contract if:

$$\int \pi f(\pi|e^*)d\pi - \alpha - g(e^*) \geq \overline{u}. \tag{2}$$

Let α^* be such that the constraint in (2) holds. Then, by setting:

$$\alpha^* = \int \pi f(\pi|e^*)d\pi - g(e^*) - \overline{u},$$

on one hand, the agent accepts the contract, and on the other hand, the principal obtains the same level of utility as in the case where e is observable:

$$\int \pi f(\pi|e^*)d\pi - v^{-1}(\overline{u} + g(e^*)).$$

□

Exercise 5. Suppose a principal-agent relationship where two outcomes are possible, valued at $50,000$ and $25,000$ dollars. The agent must choose between two possible effort levels. The probability distribution over outcomes as a function of effort levels is given in the following table:

	25,000	50,000
e_1	$1/4$	$3/4$
e_2	$1/2$	$1/2$

Assume the principal is risk-neutral and the agent is risk-averse, with preferences described by the following functions:

$$B(x, w) = x - w,$$

$$U(w, e) = 2\sqrt{w} - g(e),$$

where $g(e_1) = 40$ and $g(e_2) = 20$. The reservation utility level is $\overline{u} = 120$.

a) Write the optimal contracts under symmetric information for each effort level and the profits obtained by the principal in each case. Which effort level does the principal prefer?

b) Write the optimal contracts when there is a moral hazard problem. What is the effort level and the contract chosen by the principal? Where does the moral hazard problem manifest?

Solution:

a) In the observable case, there is always a constant wage scheme:

$$w^* = v^{-1}(\overline{u} + g(e)).$$

For high effort (e_1),

$$w_1 = \frac{(\overline{u} + g(e_1))^2}{4} = \frac{(120+40)^2}{4} = 6400.$$

Then, in the low-effort case (e_2),

$$w_2 = \frac{(\overline{u} + g(e_2))^2}{4} = \frac{(120+20)^2}{4} = 4900.$$

Next, we evaluate which effort maximizes the principal's profits. For high effort e_1:

$$\Pi_1 = \frac{1}{4} \cdot 25000 + \frac{3}{4} \cdot 50000 - 6400 = 37350.$$

For low effort e_2:

$$\Pi_2 = \frac{1}{2} \cdot 25000 + \frac{1}{2} \cdot 25000 - 4900 = 32600.$$

Thus, the principal prefers to implement e_1.

b) In the case of unobservable e, i.e., moral hazard, we solve:

$$\begin{cases} \max_{e,w(\pi)} & \sum_{i=1}^{2} [\pi_i - w_i] p(\pi_i|e) \\ \text{s.t.} & v(w(\pi_1))p(\pi_1|e) + v(w(\pi_2))p(\pi_2|e) - g(e) \geq \overline{u}, \\ & e \text{ maximizes } v(w(\pi_1))p(\pi_1|e) + v(w(\pi_2))p(\pi_2|e) - g(e). \end{cases}$$

First, we find the wage scheme as a function of the generated profits. Note that the multipliers are positive, so the constraints hold with equality. Working with e_2, the low-effort scenario offers a fixed wage equal to 4900 (low effort induces the same payment as the full-information case):

$$2\sqrt{w(\pi_2)}\frac{1}{4} + 2\sqrt{w(\pi_1)}\frac{3}{4} - 40 = 120,$$
$$2\sqrt{w(\pi_2)}\frac{1}{4} + 2\sqrt{w(\pi_1)}\frac{3}{4} - 40 = 2\sqrt{w(\pi_2)}\frac{1}{2} + 2\sqrt{w(\pi_1)}\frac{1}{2} - 20.$$

These equations form a linear system:

$$\begin{pmatrix} 1/2 & 3/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} \sqrt{w(\pi_2)} \\ \sqrt{w(\pi_1)} \end{pmatrix} = \begin{pmatrix} 160 \\ 20 \end{pmatrix}.$$

Solving, we get $w(\pi_2) = 2500$ and $w(\pi_1) = 8100$. Finally, we evaluate the principal's preference. The principal minimizes:

$$\sum_{i=1}^{2} w_i p(\pi_i|e).$$

The profits obtained under e_1 are:

$$\frac{1}{4} \cdot (25000 - 2500) + \frac{3}{4} \cdot (50000 - 8100) = 37050.$$

Thus, since under this contract implementing e_1 yields higher profits than the $e = e_2$ case (32600), the principal will choose to implement high effort.

Exercise 6 (I. Segal and S. Tadelis, ECON 206 UCB). Consider a standard moral-hazard problem with the following features:

- The principal (p) and the agent (a) are both risk neutral. Let x be the verifiable output, e the agent's unobserved effort, and $w(x)$ the payment to the agent. The two parties' final utility levels are given by:

  $$u_p = x - w(x), \quad u_a = w(x) - v(e),$$

  where $v(\cdot)$ is a strictly increasing function of effort.

- The agent has finite wealth, which constrains the principal to offer incentive schemes $w(x) > 0$ for all x. This constraint guarantees that the agent is willing to work for the principal (no additional participation constraint is needed).

- Output can take three values: $x_1 = 1$, $x_2 = 2$, and $x_3 = 3$. Effort can take two values: $e_0 = 0$ and $e_1 = 1$. Normalize $v(0) = 0$.

- The probability of x given e, denoted $\pi(x \mid e)$, satisfies the Monotone Likelihood Ratio Property:

  $$\frac{\pi(x_j \mid e = 1)}{\pi(x_j \mid e = 0)} > \frac{\pi(x_{j-1} \mid e = 1)}{\pi(x_{j-1} \mid e = 0)} \quad \text{for } j = 2, 3.$$

Assume that a second-best solution (moral hazard) to this problem induces the agent to choose e_1. Show that in such a solution $w(x_1) = w(x_2) = 0$, and $w(x_3) > 0$.

Solution: the statement points out that the principal wishes to implement $e = 1$ at the lowest, subject to the participation and compatibility constraints. Hence, he solves

$$\begin{cases} \min_{w(i), i\in\{1,2,3\}} & \sum_{i=1}^{3} \pi(i|e=1)w(i) \\ \text{s.t.} & \sum_{i=1}^{3} \pi(i|e=1)w(i) - v(1) \geq \sum_{i=1}^{3} \pi(i|e=0)w(i) \\ & w(i) \geq 0,\ \forall\ i \in \{1,2,3\}. \end{cases}$$

The associated Lagrangian is

$$\mathscr{L} = \sum_{i=1}^{3} \pi(i|e=1)w(i) - \lambda \left[\sum_{i=1}^{3} (\pi(i|e=1) - \pi(i|e=0))w(i) - v(1) \right] - \sum_{i=1}^{3} \mu_i w(i).$$

FOC yields

$$\pi(i|e=1)w(i) - \lambda[\pi(i|e=1) - \pi(i|e=0)] - \mu_i = 0,\ \forall\ i \in \{1,2,3\},$$

together with complementary slackness

$$w(i)\mu_i = 0.$$

For some i^* we must have $w(i^*) = 0$. Thus, $\mu_{i^*} = 0$ and

$$\lambda = \frac{1}{1 - \frac{\pi(i^*|e=0)}{\pi(i^*|1)}} > 1.$$

For any other i,

$$\frac{\mu_i}{\pi(i|e=0)} = \lambda - (\lambda - 1)\frac{\pi(i|e=1)}{\pi(i|e=0)}.$$

Since $\frac{\pi(i|e=1)}{\pi(i|e=0)}$ is increasing in i and $\lambda > 1$,

$$\frac{\mu_1}{\pi(1|e=0)} > \frac{\mu_2}{\pi(2|e=0)} > \frac{\mu_3}{\pi(3|e=0)} \geq 0.$$

It follows since $\mu_3 = 0$ that $\mu_1, \mu_2 > 0$. Thus $w(1) = w(2) = 0$ and $w(3) > 0$.

Suggested exercises

Stochastic Dominance

Exercise 7. Consider two random variables X and Y defined on a probability space $(\Omega, \mathscr{F}, P)$.

- Prove that if $X \geq Y$, then $F_X \geq_{FOSD} F_Y$. **Hint:** note that

$$\{\omega \in \Omega : X(\omega) \leq x\} \subset \{\omega \in \Omega : Y(\omega) \leq x\}.$$

- Show that the converse of the previous statement is not true. **Hint:** consider $\Omega = [0, 1]$, P as the uniform distribution, and $X(\omega) = 1 + \omega$, $Y = 2 - 2\omega$.

Exercise 8. Let X and Y be real-valued random variables, and let $g : \mathbb{R} \to \mathbb{R}$ be an increasing function. Discuss the truth of the following statements:

a) If X stochastically dominates Y in the first order, then $g(X)$ stochastically dominates $g(Y)$ in the first order.

b) If X stochastically dominates Y in the second order, then $g(X)$ stochastically dominates $g(Y)$ in the second order.

Solution:

1. True: $\mathbb{P}\{g(X) \leq t\} = \mathbb{P}\{X \leq g^{-1}(t)\} \leq \mathbb{P}\{Y \leq g^{-1}(t)\} = \mathbb{P}\{g(Y) \leq t\}$.

2. False: consider $X = 1/2$ and Y uniformly distributed over $[0, 1]$. Then,

$$\int_{-\infty}^{t} F_X(s)ds = t - 1/2 \leq \int_{-\infty}^{t} F_Y(s)ds = t^2/2.$$

   Now, consider $g(t) = t^2$ and $u(t) = t$. Then,

$$\mathbb{E}[u(g(Y))] = \mathbb{E}[g(Y)] > g(\mathbb{E}[Y]) = g(X) = \mathbb{E}[u(g(X))].$$

Adverse selection

Exercise 9. Consider a car market where the maximum available quality is $Q = 1.9$, and the distribution of q is uniform with $a = 0$ and $b = Q$. A car of quality q is valued at most by the buyer as q and by the seller as q/Q. It is assumed that there are sufficiently many buyers to drive the gains from trade on the seller's side. Since $q/Q < q$, the buyer assigns a higher value to the car than the seller, allowing both parties to trade the car at a price p between q/Q and q, generating a profit for the seller and a surplus for the buyer. In particular, if a car of quality q is traded at price p, the buyer obtains a utility

$$u(p, q) = q - p,$$

while the seller obtains a profit of

$$\pi(p, q, Q) = p - \frac{q}{Q}.$$

a) Calculate the buyer's expected value assuming there is asymmetric information.

b) Based on the previous part, calculate the cutoff point for cars offered by the seller, the associated price, and their profits.

c) Calculate the valuation of cars by the buyer and seller under conditions of perfect information.

d) If a buyer anticipates the interval where cars will not be offered, determine the new expected value of offered cars and the seller's decision. What happens to the new prices and the seller's profits?

Solution:

a) The expected value of quality, as perceived by the consumer, is

$$\mathbb{E}[q] = \int_0^Q \frac{x}{Q-0} dx = \frac{Q}{2} = 0.95.$$

b) Thus, the consumer will be willing to pay $p = 0.95$. On the other hand, the quality offered will be $q < 0.95 \cdot Q = \frac{Q^2}{2} = 1.805$. Finally, the seller's profits will be $\pi = p - \frac{q}{Q} = 0.95 - \frac{q}{1.9}$.

c) In the case of *perfect information*, the buyer values exactly q, and the seller $q/Q = 0.52q$.

d) If the buyer anticipates that $q \in [0, 1.805 = Q^2/2)$, they will be willing to pay

$$p = \mathbb{E}[q | q \in [0, 1.805 = Q^2/2)] = \int_0^{Q^2/2} \frac{x}{Q^2/2} dx = \frac{Q^2}{4} = \frac{1.805}{2}.$$

Finally, this implies that $q/Q < p = \frac{Q^2}{4}$. That is, $q < Q^3/3$.
Note: the solution in its most general form is given by:

$$\mathbb{E}[q | p - q/Q \geq 0] = \mathbb{E}[q | q \leq Qp] = \begin{cases} \frac{pQ}{2}, & \text{if } p < 1, \\ \frac{Q}{2}, & \text{if } p \geq 1. \end{cases}$$

Exercise 10. Consider a competitive labor market with many firms seeking to hire a worker for a specific position. The worker (seller of labor services) privately observes their own productivity θ, but firms (buyers of labor) cannot observe it. Firms offer a wage based on the worker's expected productivity:

$$\mathbb{E}[\theta] = \frac{1}{2}, \; \theta \in U[0, 1].$$

At this wage level, only workers with a productivity $\theta < 1/2$ would be interested in accepting the position, while those with $\theta > 1/2$ will remain unemployed. It is assumed that the cost of not working for the firm is working at home, where they receive a reservation wage of $1/2$.

a) What wage will a rational employer offer?

b) Who will accept the job at that wage level?

Solution:
a) Under the assumptions of this model, we have that

$$w = \mathbb{E}[\theta \mid \theta \leq 1/2] = \int_0^{1/2} \left(\frac{\theta}{\frac{1}{2} - 0} \right) d\theta = \frac{1}{4}.$$

b) Since $r(\theta) = 1/2 > 1/4$, no one works at the firm.

Moral Hazard

Exercise 11. Suppose Manuel wants to hire a research assistant. It is known that the probability of the assistant doing good or bad work $(x_i, i = 1, 2)$ depends on the effort they put in, meaning everything is expressed as $\mathbb{P}(x \mid e)$. Two levels of effort are identified: high effort e_H and low effort e_L. On the other hand, Manuel is risk-neutral, so he maximizes the expected value of profits:

$$\pi = x_i - w_i,$$

where w_i is the salary that Manuel pays the research assistant, who is risk-averse. Therefore, the assistant's preferences are represented by:

$$U(w, e) = v(w) - g(e),$$

where v is strictly concave and increasing, and g is strictly convex and also increasing. Based on this, assuming the high effort level executed by the agent maximizes the principal's expected profits:

a) Formulate and solve the principal's optimization problem under symmetric information. What will be the payment scheme w_i?

b) Formulate and solve the principal's optimization problem under asymmetric information. Explain the relationship between the salary and a higher value of $\mathbb{P}(x \mid e = e_H)$.

Exercise 12. Consider the following Principal-Agent problem, where the Agent chooses between two actions, $a \in \{a_1, a_2\} = \{0, 1\}$. The Principal pays the Agent a wage w_s, with $s \in \{1, 2, 3, 4\}$, contingent on observing the production $Y \in \{y_1, y_2, y_3, y_4\}$, where $y_1 < y_2 < y_3 < y_4$. The probability distribution of each level of production is as follows, $p_s(a_i) := \Pr\{Y = y_s \mid a = a_i\}$, $i \in \{1,2\}$:

a_i	$p_1(a_i)$	$p_2(a_i)$	$p_3(a_i)$	$p_4(a_i)$
a_1	3/8	3/8	1/8	1/8
a_2	1/4	1/4	1/4	1/4

The Agent's utility function is:

$$U(w,a) = \sqrt{w} - a$$

Assume that the Agent's reservation utility is $u = 2$. The Principal is risk-neutral.

a) If the action were observable, characterize the optimal contract to incentivize $a_2 = 1$.

b) Suppose the action a_i is not observable. Characterize the optimal contract that implements a_2. Use the following rule to determine salaries: $\mathrm{LR}_k = \frac{p_k(a_2)}{p_k(a_1)}$, $w_s > (=) w_r \iff \mathrm{LR}_s > (=) \mathrm{LR}_r$. *LR* denotes likelihood ratio.

c) Indicate the efficiency loss due to information asymmetry.

1 Optional exercises

Moral Hazard: continuous effort

Exercise 13 (Mas-Colell et al.). Consider a hidden-action model where the principal is risk-neutral, while the manager's preferences are determined by:

$$UE(w, e \mid \phi) = \mathbb{E}[w] - \phi \text{Var}(w) - g(e),$$

with $g'(0) = 0$, $g^{(k)}(\cdot) > 0$ for $k = 1, 2, 3$, $e > 0$, and $\lim_{e \to \infty} g'(e) = \infty$. We consider $e \in \mathbb{R}_+$. The benefits subject to effort are distributed according to $\mathcal{N}(e, \sigma^2)$.

a) If the payment scheme is linear $w(\pi) = \alpha + \beta\pi$, prove that the manager's expected utility given $w(\pi), e$, and σ^2 is:

$$\alpha + \beta e - \phi \beta^2 \sigma^2 - g(e).$$

b) Derive the optimal contract when e is observable.

c) Derive the optimal linear payment scheme when effort is not observable. Analyze the effects of β and σ^2.

Solution:

a) If $w(\pi) = \alpha + \beta\pi$,

$$\begin{aligned} UE(w, e|\phi) &= \mathbb{E}[w] - \phi \text{Var}(w) - g(e) \\ &= \mathbb{E}[\alpha + \beta\pi] - \phi \text{Var}(\alpha + \beta\pi) - g(e) \\ &= \alpha + \beta \mathbb{E}[\pi] - \phi\beta^2 \text{Var}(\pi) - g(e) \\ &= \alpha + \beta e - \phi\beta^2\sigma^2 - g(e). \end{aligned}$$

b) Since w is constant ($UE = w$), we solve

$$\max \ \mathbb{E}[\pi|e] - g(e) = e - g(e).$$

Applying first-order conditions, we obtain $g'(e^*) = 1$ and $w^* = g(e^*)$.

c) The problem to be solved is

$$\begin{cases} \max & \int_{\underline{\pi}}^{\overline{\pi}} (\pi - w(\pi)) f(\pi|e) d\pi \\ s.a. & \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \overline{u}, \\ & \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi)) f(\pi|e) d\pi - g(e) \geq \int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi)) f(\pi|\tilde{e}) d\pi - g(\tilde{e}), \ \forall \ \tilde{e}. \end{cases}$$

In this case,

$$\int_{\underline{\pi}}^{\overline{\pi}} v(w(\pi)) f(\pi|e) d\pi - g(e) = \alpha + \beta e - \phi\beta^2\sigma^2 - g(e).$$

Thus, we only need to solve

$$\max_{e}\ \alpha + \beta e - \phi\beta^2\sigma^2 - g(e).$$

Then, by FOC $g'(e) = \beta$,

$$\alpha = \phi g'(e)^2\sigma^2 - g'(e)e + g(e).$$

Finally, substituting into the principal's problem,

$$\max\ \left\{ \int_{\underline{\pi}}^{\overline{\pi}} \pi f(\pi|e)d\pi - \int_{\underline{\pi}}^{\overline{\pi}} [\phi g'(e)^2\sigma^2 - g'(e)e + g(e) + g'(e)\pi] f(\pi|e)d\pi \right\}.$$

Since $\mathbb{E}[\pi] = e$, this becomes

$$\max\ e - g(e) - \phi g'(e)^2\sigma^2.$$

Exercise 14. Suppose an employer (E) hires a lawyer (A) for representation. For an agent of type θ, the time required to produce legal services in the amount x is denoted as a, where $a = \frac{x}{2\theta}$. The agent can either be of low productivity (θ_1) or high productivity (θ_2), with $\theta_2 > \theta_1 > 0$. Let y be the payment made to the lawyer. The agent has a reservation utility of 0, and their utility function is $U(y,a) = \sqrt{y} - a$. The principal E is risk-neutral, and legal services have a unit price of 1 monetary unit.

a) If you observe the type of lawyer you are hiring, what contract would you offer? Is this contract efficient? Graph the equilibrium, including the optimal quantities, indifference curves, and isoprofit lines, with y on the vertical axis and x on the horizontal axis.

b) Suppose now that you believe $\pi_1 = \Pr(\theta = \theta_1)$, $\pi_2 = \Pr(\theta = \theta_2)$, and assume $\theta_1 \geq (1-\pi_1)\theta_2$. What would the set of contracts you offer look like now? Graph the equilibria, clearly indicating the optimal quantities, indifference curves, and isoprofit lines.

Advanced exercises

Exercise 15 (Mas-Colell et al.). Suppose the reservation price, opportunity cost of working, or production at home $r(\theta)$ is a continuous and strictly increasing function. Additionally, assume there exists a productivity value $\hat{\theta} \in [\underline{\theta}, \overline{\theta}]$ such that $r(\theta) > \theta$ for $\theta > \hat{\theta}$ and $r(\theta) < \theta$ for $\theta < \hat{\theta}$. The density of workers of type θ is $f(\theta)$ with $f(\theta)$, $\forall\ \theta \in [\underline{\theta}, \overline{\theta}]$. Based on this, answer the following:

a) Find the Pareto-efficient allocation. What will be the aggregate production? Graph $r(\theta)$ and θ.

b) Find the competitive equilibrium in the labor market under information asymmetry, and prove that it is not Pareto efficient. Analyze it as a function of $\hat{\theta}$.

c) Suppose $r(\theta) = \frac{1}{\theta}$, $\theta \sim U[1/2, 2]$, $r(\underline{\theta}) < \underline{\theta}$, and $r(\overline{\theta}) < \overline{\theta}$. Find the socially efficient labor allocation and the corresponding aggregate production.

d) In the context of information asymmetry, find the conditional expectation of productivity.

Solution:
a) In the Pareto efficient scenario, the wage is equal to the worker's productivity, that is, $w = \theta$. Then, if $\theta < \hat{\theta}$, since $r(\theta) < \theta = w$, employment is offered in the firm. Otherwise, i.e., if $\theta > \hat{\theta}$, since $r(\theta) > \theta = w$, employment is not offered in the firm. Thus, recalling that

$$I(\theta) = \begin{cases} 0, & \text{if } \theta < r(\theta), \\ 1, & \text{if } \theta \geq r(\theta). \end{cases}$$

$$\begin{aligned} N\int_{\underline{\theta}}^{\overline{\theta}} (\theta I(\theta) + (1 - I(\theta))r(\theta))dF(\theta) &= N\int_{\underline{\theta}}^{\hat{\theta}} (\theta I(\theta) + (1 - I(\theta))r(\theta))dF(\theta) \\ &+ N\int_{\hat{\theta}}^{\overline{\theta}} (\theta I(\theta) + (1 - I(\theta))r(\theta))dF(\theta) \\ &= N\left(\int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) + \int_{\hat{\theta}}^{\overline{\theta}} r(\theta)dF(\theta)\right). \end{aligned}$$

b) If $w^* = \hat{\theta}$ and given that r is strictly increasing, $w^* > r(\hat{\theta}) > r(\theta)$, $\forall\theta \in [\underline{\theta}, \hat{\theta}]$. On the other hand, since $r(\theta) > \theta > \hat{\theta}$ for all $\theta \in (\hat{\theta}, \overline{\theta})$, it follows that

$$\Theta = [\underline{\theta}, \hat{\theta}].$$

As a consequence, given the continuity of $f(\cdot)$,

$$\mathbb{E}[\theta | \theta \in \Theta] < \hat{\theta} = w^*.$$

Therefore, the firm will not offer employment (as it would incur losses), and thus, profits would be lower than in the full-information scenario:

$$N\int_{\underline{\theta}}^{\overline{\theta}} r(\theta)dF(\theta) < N\left(\int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) + \int_{\hat{\theta}}^{\overline{\theta}} r(\theta)dF(\theta)\right). \tag{3}$$

Now consider the case where $w^* > \hat{\theta}$. By continuity and monotonicity, there must exist some θ^* such that $r(\theta^*) = w^*$. Then,

$$r(\theta^*) > \theta^* > \hat{\theta}.$$

$$\Theta = [\underline{\theta}, \theta^*].$$

Hence,

$$\mathbb{E}[\theta | \theta \in \Theta] < \theta^* < r(\theta^*) = w^*.$$

As a result, no employment is offered, and the outcome is again represented by equation (3). Finally, if $w^* < \hat{\theta}$, again by continuity and monotonicity, there must exist some θ^* such that $r(\theta^*) = w^*$. In this case, since $r(\theta^*) < \theta^*$,

$$\mathbb{E}[\theta|\theta \in \Theta] = w^*,$$

where $\Theta = [\underline{\theta}, \theta^*]$. This leads to equilibrium in the labor market, but the allocation of labor is inefficient. Fewer individuals work for the firm compared to the socially optimal level. Again, production is lower since

$$N\left(\int_{\underline{\theta}}^{\theta^*} \theta dF(\theta) + \int_{\theta^*}^{\overline{\theta}} \theta dF(\theta)\right) < N\left(\int_{\underline{\theta}}^{\hat{\theta}} \theta dF(\theta) + \int_{\hat{\theta}}^{\overline{\theta}} r(\theta) dF(\theta)\right).$$

In fact,

$$\int_{\theta^*}^{\hat{\theta}} r(\theta) dF(\theta) < \int_{\theta^*}^{\hat{\theta}} \theta dF(\theta).$$

c) Now, suppose $r(\theta) = 1/\theta$ and $\theta \sim U([1/2, 2])$. In the full-information scenario, $w = \theta$. Thus, if $r(\theta) > w$, i.e.,

$$r(\theta) = \frac{1}{\theta} > w = \theta \implies \theta^2 < 1 \implies \theta \in [1/2, 1),$$

employment is not offered. Otherwise, the individual works for the firm. This configuration yields the following level of production:

$$N \int_{1/2}^{1} r(\theta) dF(\theta) + N \int_{1}^{2} \theta dF(\theta).$$

d) Finally, in the incomplete-information scenario,

$$\begin{aligned}\Theta &= \{\theta : r(\theta) \leq w\} \\ &= \{\theta : 1/\theta \leq w\} \\ &= \{\theta : w \leq \theta\}.\end{aligned}$$

Then,

$$\mathbb{E}[\theta|\theta \in \Theta] = \int_{1/w}^{2} \frac{\theta}{2 - \frac{1}{w}} d\theta = \frac{1}{2w} + 1.$$

Exercise 16 (I. Segal, ECON 206 UCB). An entrepreneur (E) has a project requiring an initial investment of k. The project's random output, $\tilde{x} > 0$, depends on E's choice of effort as follows: $\tilde{x} = e\tilde{y}$, where $e > 0$ is E's choice of effort and $\tilde{y}$ is a random variable distributed uniformly on $[0, 2]$. E's private cost of effort is $g(e) = \frac{1}{2}e^2$, and this effort is unobservable, while the output is observable and verifiable. E makes a take-it-or-leave-it offer to an investor (I). Assume that E has no starting capital, so I must pay the start-up cost k. The contract specifies the sharing rule of output such that when output is x, E keeps $w(x)$ for himself and the rest goes to I. E has limited liability, which constrains $w(x) > 0$ for all x. Both parties are risk-neutral, and the market interest rate is normalized to zero.

a) Suppose that E's effort is observable and verifiable. Solve for the first-best level of effort. For what values of k would the project be worth undertaking if there were no moral hazard?

b) Assume that the project is worth undertaking. Show that despite the unobservability of effort by E, an optimal contract implements the first-best level of effort. *Hint: Consider contracts of the form*

$$w(x) = \begin{cases} 0 & \text{if } x < a, \\ w & \text{if } x \geq a. \end{cases}$$

c) Now suppose that before output is observed by outsiders (including the principal), E can destroy or borrow output at no cost. What restrictions does this impose on the contract? Solve for the form of the optimal contract in this case. *Hint: You can use integration by parts to express the objective function and constraints through $w(0)$ and $w'(\cdot)$.*

d) Interpret the resulting contract in financial terms.