Recitation 5

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PC2

Question 1

An economy has two types of consumers and two goods. The agent type A has the following utility function:

$$u_A(x_{1A}, x_{2A}) = 4x_{1A} - \frac{x_{1A}^2}{2} + x_{2A}$$

and the agent type B has the following utility function:

$$u_B(x_{1B}, x_{2B}) = 3x_{1B} - \frac{x_{1B}^2}{2} + x_{2B}.$$

Good 2 is the numeraire, and each consumer has an income of 100. Additionally, the economy has N consumers of both type A and type B.

1. Identify the type of consumer with high demand and the type with low demand for good x_1. Compare the marginal willingness to pay for each type of consumer for good x_1.
2. The monopolist produces good 1 with the following cost function $C(x_1) = cx_1$ and cannot discriminate prices. Find the optimal price and quantity of good x_1 that the monopolist will choose. For which values of c will the monopolist choose to sell to both types of consumers?
3. The monopolist engages in second-degree price discrimination by offering a menu of prices and quantities to each type of consumer (r_A, x_A) and (r_B, x_B). Based on this, formulate the monopolist's optimization problem and find the optimal values (r_A^*, x_A^*) and (r_B^*, x_B^*).
4. If the monopolist engages in third-degree price discrimination, what will be the prices and quantities set by the monopolist in the markets for A-type and B-type consumers?
5. If the monopolist engages in first-degree price discrimination, find the quantity produced by the monopolist in the market for good x. Calculate the consumer surplus and the monopolist's surplus.

Solution:

- It follows that $u_A(x_1) > u_B(x_1)$ and $u'_A(x_1) > u'_A(x_1)$ for any x_1.
- We need consumer's demand. Consumer A solves

$$\max_{x_{1A}, x_{2A}} \underbrace{u_A(x_{1A})}_{=4x_{1A}-x_{1A}^2/2} + x_{2A}$$

$$\text{s.t } px_{1A} + x_{2A} = 100.$$

FOC leads to $4 - x_{1A} = p$, so $x_{1A}^d = 4 - p$. Thus, demand of agents of type A for good 1 is

$$\sum_{i=1}^{N} x_{1A_i}^d = \sum_{i=1}^{N} (4-p) = N(4-p).$$

Analogously, for type B consumer, he solves

$$\max_{x_{1A}, x_{2A}} \underbrace{u_B(x_{1B})}_{=3x_{1B}-x_{1B}^2/2} + x_{2B}$$

$$\text{s.t } px_{1B} + x_{2B} = 100.$$

FOC leads to $x_{1B} = 3 - p$. Therefore, aggregating

$$\sum_{i=1}^{N} x_{1B_i}^d = \sum_{i=1}^{N} (3-p) = N(3-p).$$

Full demand for good 1 is $N(7-2p)$.

Solution:

- Firm solves

$$\max_{x_1} \ pX_1 - c(x_1) = pX_1 - cX_1 = \left(\frac{7}{2} - \frac{X_1}{2N}\right) X_1 - cX_1.$$

FOC yields $X_1^* = N(3.5 - c)$ and $P^M 7/4 + c/2$.

- Under second degree discrimination, he solves (as explained before)

$$\begin{aligned} \max_{x_{1A}, x_{1B}} \ & N(t_A - cx_{1A}) + N(t_B - cx_{1B}) \\ \text{s.t.} \ & u_B(x_{1B}) = t_{1B} \\ & u_A(x_{1A}) - t_{1A} = u_A(x_{1B}) - t_{1B}. \end{aligned}$$

Replacing in the objective function, we need to solve

$$\max_{x_{1A}, x_{1B}} \ N\left[4x_{1A} - \frac{x_{1A}^2}{2} - 4x_{1B} + \frac{x_{1B}^2}{2} + 3x_{1B} - \frac{x_{1B}^2}{2} - cx_{1A} + 3x_{1B} - \frac{x_{1B}^2}{2} - cx_{1B}\right].$$

FOC yields

$$x_{1A}^* = 4 - c, \ \ x_{1B}^* = 2 - c, \ \ t_{1A}^* = 24 - \frac{c(c+16)}{2}, \ \ t_{1B}^* = \frac{(2-c)(4+c)}{2}.$$

Solution:

- Third degree price discrimination:

$$\max_{p_A, X_{1A}, p_B, X_{1B}} \quad p_A X_{1A} + p_B X_{1B} - c(X_{1A} + X_{1B}).$$

We already now that $p_A = 4 - \frac{X_{1A}}{N}$ and $p_B = 3 - \frac{X_{1B}}{N}$. Thus, the optimization problem becomes

$$\max_{X_{1A}, X_{1B}} \quad \Pi = \left(4 - \frac{X_{1A}}{N}\right) X_{1A} + \left(3 - \frac{X_{1B}}{N}\right) X_{1B} - c(X_{1A} + X_{1B}).$$

FOC lead to

$$X_{1A} = \frac{N(4-c)}{2} \implies p_A = \frac{4+c}{2}, \; X_{1B} = \frac{N(3-c)}{2} \implies p_B = \frac{3+c}{2}.$$

- The monopolist extracts all the consumer surplus. In this way, they charge a price equal to each consumer's maximum willingness to pay. Finally, they produce the competitive market quantity $X^* = N(7-2c)$, but only the last buyer pays the competitive market price $p = c$. Consumer surplus (CS) is 0, and producer surplus (PS) is $\frac{7}{2} - c$.

Question 2

Consider $q(p) = p^{-\epsilon}$ and assume constant marginal cost. Prove that the social welfare in competitive equilibrium is

$$\mathcal{W}^s = \frac{c^{1-\epsilon}}{\epsilon - 1}.$$

Then, compute the welfare loss. Hint: recall that the total surplus is in the competitive case

$$\int_{p=c'}^{\infty} q(s)ds.$$

So, in this case, $\int_{p=c}^{\infty} s^{-\epsilon} ds = \frac{c^{1-\epsilon}}{\epsilon - 1}$. For the monopolist case, apply FOC

$$p^m = \frac{c}{1 - \frac{1}{\epsilon}}.$$

Thus, you can conclude that

$$\mathcal{W}^s - \mathcal{W}^m = \left(\frac{c^{1-\epsilon}}{\epsilon - 1}\right) \left[1 - \left(\frac{2\epsilon - 1}{\epsilon - 1}\right) \left(\frac{\epsilon}{\epsilon - 1}\right)^{-\epsilon}\right].$$

FOC leads to

$$p^m = \frac{c}{1 - \frac{1}{\epsilon}}.$$

Thus,

$$\mathcal{W}^s - \mathcal{W}^m = \left(\frac{c^{1-\epsilon}}{\epsilon - 1}\right)\left[1 - \left(\frac{2\epsilon - 1}{\epsilon - 1}\right)\left(\frac{\epsilon}{\epsilon - 1}\right)^{-\epsilon}\right].$$

Indeed,

$$\mathcal{W}^s = \frac{c^{1-\epsilon}}{\epsilon - 1}$$

$$\Pi^m = p^m q^m - cq^m = \left[\frac{c\epsilon}{\epsilon - 1}\right]\left[\frac{c\epsilon}{\epsilon - 1}\right]^{-\epsilon} - c\left[\frac{c\epsilon}{\epsilon - 1}\right]^{-\epsilon} = c^{1-\epsilon}\underbrace{\left[\frac{\epsilon}{\epsilon - 1} - 1\right]}_{\frac{1}{\epsilon - 1}}\left[\frac{\epsilon}{\epsilon - 1}\right]^{-\epsilon}.$$

and the new consumer surplus is

$$\int_{\frac{c}{1-1/\epsilon}}^{\infty} s^{-\epsilon} ds = (\epsilon - 1)^{\epsilon - 2}\epsilon^{1-\epsilon}c^{1-\epsilon}.$$

Adding,

$$\underbrace{\int_{\frac{c}{1-1/\epsilon}}^{\infty} s^{-\epsilon} ds}_{\text{nuevo excedente}} + \frac{c^{1-\epsilon}}{\epsilon - 1}\left[\frac{\epsilon}{\epsilon - 1}\right]^{-\epsilon} = \frac{2\epsilon - 1}{\epsilon - 1}\left(\frac{\epsilon}{\epsilon - 1}\right)^{-\epsilon}.$$

Indeed

$$\int_{\frac{c}{1-1/\epsilon}}^{\infty} s^{-\epsilon} ds + \frac{c^{1-\epsilon}}{\epsilon-1}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon} = (\epsilon-1)^{\epsilon-2}\epsilon^{1-\epsilon}c^{1-\epsilon} + \frac{c^{1-\epsilon}}{\epsilon-1}\frac{\epsilon^{-\epsilon}}{(\epsilon-1)^{-\epsilon}}$$

$$= c^{1-\epsilon}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}\left[\frac{\epsilon}{(\epsilon-1)^2} + \frac{1}{\epsilon-1}\right]$$

$$= c^{1-\epsilon}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}\left[\frac{\epsilon(\epsilon-1)+(\epsilon-1)^2}{(\epsilon-1)^3}\right]$$

$$= c^{1-\epsilon}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}\left(\frac{2\epsilon-1}{(\epsilon-1)^2}\right)$$

$$= c^{1-\epsilon}\left[\frac{\epsilon}{\epsilon-1}\right]^{-\epsilon}\frac{1}{\epsilon-1}\left(\frac{2\epsilon-1}{\epsilon-1}\right).$$

Question 3

Consider an economy where there are two goods (y_1, y_2), two consumers, and a firm. Each consumer has an initial endowment of four units of good 1 and nothing of good 2. Good y_1 is not producible, and good y_2 is produced by the firm using good 1 as input (the quantities of good 1 not directly consumed by individuals) based on the following production function:

$$y_2 = f(z) = z_1^{1/2},$$

where y_2 is the quantity of good 2 produced and z_1 is the amount of good 1 used as input. The profits earned by the firm are equally distributed between the two consumers, $\theta_j = 1/2$. Both consumers derive utility from the consumption of the two goods. However, the production of good 2 generates noise and pollution, which negatively affects their well-being. As a result, the utility function of the consumers is given by the following expression:

$$U_i(y_{i1}, y_{i2}, y_2) = y_{i1} + \ln y_{i2} - \frac{1}{2} \ln y_2^s, \quad i = 1, 2.$$

The superscript refers to the consumer.

1. Calculate the quantities of good 2 produced and consumed by the two individuals in the general equilibrium, assuming the price of good 1 as the numéraire, whether used as a consumption good or as an input ($p_1 = w_1 = 1$).
2. Calculate the quantities of good 2 produced and consumed by the two individuals in the efficient allocation, and comment on the results in comparison to the previous part. Assume a total endowment of 8 for good 1.

The first step is to solve firm's maximization problem

$$\max \ \Pi = p\sqrt{z_1} - wz_1.$$

The, FOC provide

$$\frac{\partial \Pi}{\partial z_1} = \frac{p}{2\sqrt{z_1}} - w_1 = 0.$$

Thus,

$$z_1^d = \frac{p^2}{4}$$

and

$$y_2^s = \frac{p}{2}.$$

Next step: max U for each consumer:

$$\begin{cases} \max_{y_{i1}, y_{i2}} & U_i(y_{i1}, y_{i2}, y_2) = y_{i1} + \ln y_{i2} - \frac{1}{2} \ln y_2 \\ s.a.: & \underbrace{w_1 y_{i1} + p y_{i2}}_{= y_{i1} + p y_{i2}} = 4 w_1 + \underbrace{\theta_j \Pi}_{= p^2/4}. \end{cases}$$

The Lagrangian is

$$\mathscr{L}(y_{i1}, y_{i2}, y_2, \lambda) = y_{i1} + \ln y_{i2} - \frac{1}{2} \ln y_2 + \lambda \left(4 + \frac{p^2}{4} - y_{i1} - p y_{i2} \right).$$

$$\frac{\partial \mathscr{L}}{\partial y_{i1}} = 1 - \lambda = 0$$

$$\frac{\partial \mathscr{L}}{\partial y_{i2}} = \frac{1}{y_{i2}} - \lambda p = 0 \implies \frac{1}{y_{i2}} = p.$$

To clear the market we must have

$$
\begin{aligned}
y_{12} + y_{22} &= \frac{2}{p} \\
&= y_2^s \\
&= \frac{p}{2}.
\end{aligned}
$$

Thus, $p = 2$. Finally,

$$y_{12}^* = y_{22}^* = \frac{1}{2}, \; z_1^* = 1, \; y_2^* = 1.$$

With respect to Pareto Optimal Allocations, we must solve

$$\begin{cases} \max & U^1 = y_{11} + \ln y_{12} - \frac{1}{2}\ln y_2 \\ s.a.: & U^2 = y_{21} + \ln y_{22} - \frac{1}{2}\ln y_2 = \overline{U} \\ & \sqrt{z_1} = y_2 \\ & z_1 + y_{11} + y_{21} = 8 \\ & y_{12} + y_{22} = y_2. \end{cases}$$

Replacing the restrictions,

$$\begin{cases} \max\limits_{y_{21}, y_{22}, y_2, z_1} & (8 - y_{21} - z_1) + \ln(y_2 - y_{22}) - \frac{1}{2}\ln y_2 \\ s.a.: & U^2 = y_{21} + \ln y_{22} - \frac{1}{2}\ln y_2 = \overline{U} \\ & \sqrt{z_1} = y_2. \end{cases}$$

The Lagrangian is

$$\mathscr{L}(y_{21}, y_{22}, y_2, z_1, \lambda, \mu) = (8 - y_{21} - z_1) + \ln(y_2 - y_{22}) - \frac{1}{2}\ln y_2 + \lambda(\overline{U} - y_{21} - \ln y_{22} + \frac{1}{2}\ln y_2) + \mu(\sqrt{z_1} - y_2).$$

FOC lead to

$$\frac{\partial \mathscr{L}}{\partial y_{21}} = -1 - \lambda = 0$$
$$\frac{\partial \mathscr{L}}{\partial y_{22}} = -\frac{1}{y_2 - y_{22}} - \frac{\lambda}{y_{22}} = 0$$
$$\frac{\partial \mathscr{L}}{\partial y_2} = \frac{1}{y_2 - y_{22}} - \frac{1}{2y_2} + \frac{\lambda}{2y_2} + \mu = 0$$
$$\frac{\partial \mathscr{L}}{\partial z_1} = 1 - \frac{\mu}{2\sqrt{z_1}} = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = \overline{U} - y_{21} - \ln y_{22} + \frac{1}{2}\ln y_2 = 0$$
$$\frac{\partial \mathscr{L}}{\partial \mu} = y_2 - \sqrt{z_1} = 0$$

From the first equation, $\lambda = -1$. Then,

$$\frac{1}{y_2 - y_{22}} = \frac{1}{y_{22}} \implies y_2 = 2y_{22}.$$

Next, from the third equation,

$$\frac{1}{y_{22}} - \frac{1}{2y_2} - \frac{1}{2y_2} + \mu = 0 \implies \mu = -\frac{1}{2y_{22}} = -\frac{1}{y_2}.$$

Substituting this into the fourth equation, and using that $y_2 = \sqrt{z_1}$,

$$1 = \frac{1}{2\sqrt{z_1}y_2} = \frac{1}{2z_1} \implies z_1^* = \frac{1}{2},\ y_2^* = \frac{1}{\sqrt{2}}.$$

Finally, since $y_{12} + y_{22} = y_2$,

$$\begin{cases} y_{12}^* = \frac{1}{2\sqrt{2}} \\ y_{22}^* = \frac{1}{2\sqrt{2}} \end{cases}$$

Question 4

The company S produces a certain amount of steel (s) and a certain amount of pollution (x), which is discharged into a river. The company F is a fish farm located downstream and is negatively affected by the pollution from company S. Suppose that the cost function of S involves both s and x. Meanwhile, the company F depends on f, representing the collection of fish, and x, which represents the production of pollution. Additionally, it must be considered that pollution increases the cost of fish production and reduces the cost of steel production.

1. Formulate the profit maximization problem for both companies.
2. What are the conditions that characterize profit maximization? Remember that polluting has no price.
3. How would the efficient production plan of steel and fish in the Pareto sense look? What are the implications of this new scenario for pollution production?

1) The steel firm's maximization problem is

$$\max_{x,s} \; \Pi_S = p_S \cdot s - C_S(x, s),$$

while, for the fishing firm, the problem is

$$\max_{f} \; \Pi_F = p_F \cdot f - C_F(x, f).$$

b) Applying first-order conditions, we obtain

$$\frac{\partial \Pi_S}{\partial s} = p_S - \frac{\partial C_S}{\partial s} = 0$$
$$\frac{\partial \Pi_S}{\partial x} = -\frac{\partial C_S}{\partial x} = 0$$
$$\frac{\partial \Pi_F}{\partial f} = p_F - \frac{\partial C_F}{\partial f} = 0.$$

This indicates that the steel firm will pollute until the marginal cost of polluting (which is negative because they save by polluting) is zero. On the other hand, both the steel and fishing firms will equalize the marginal cost of production, whether for steel or fish, with the market price. Firm S does not take into account the social cost of its pollution. Therefore, firm S is expected to produce too much pollution from a social perspective. In this case, the question arises: **What would the efficient (Pareto) production levels be?** Let's see.

c) Recall that Pareto efficiency implies that: No one's welfare can be improved without worsening someone else's situation. To account for this welfare concept, we assume that a single owner possesses both firms, and that the two are merged into one. In other words, to account for this welfare idea, both firms are integrated into a single entity:

$$\max_{x,f,s} \Pi = p_s s + p_f f - C_S(s,x) - C_F(f,x).$$

The first-order conditions are

$$\frac{\partial \Pi}{\partial s} = p_s - \frac{\partial C_s}{\partial s} = 0$$
$$\frac{\partial \Pi}{\partial f} = p_f - \frac{\partial C_f}{\partial f} = 0$$
$$\frac{\partial \Pi}{\partial x} = -\frac{\partial C_s}{\partial x} - \frac{\partial C_f}{\partial x} = 0$$

1. Compared to the individual situations, the only difference is the last equation.
2. The relevant condition is
$$-\frac{\partial C_s}{\partial x} = \frac{\partial C_f}{\partial x}.$$

Overview Midterm

1. General Equilibrium
    - 2×2: Edgeworth box, Pareto set, contract curve.
    - **Pareto optimal allocations, Walrasian equilibrium.**
    - Robinson Crusoe.
    - Pure Exchange Economy.
    - Private Ownsership Economy.
2. Monpoly (classical)
    - First degree price discrimination.
    - **Second degree price discrimination.**
    - Third degree price discrimination.
3. **Externalities.**