Recitation 4

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Index

1. Schedule
2. References and some history
3. Contract Theory
4. Monopoly

Schedule

1. Motivation: contract theory.
2. Monopoly.
3. Second degree price discrimination.

Motivation

References and some history

1. UC Berkeley lecture notes: ECO206, Ilya Segal (now at Stanford) and Steven Tadelis (at Berkeley).
2. James Mirrlees (Nobel 1996), for his fundamental contributions to the economic theory of incentives under conditions of asymmetric information.
3. Michael Spence (Nobel 2001), asymmetric information, signaling.
4. Roger Myerson (Nobel 2007), mechanism design. Regulating a Monopolist with Unknown Costs, Econometrica 1982.
5. Jean Tirole (Nobel 2014), for his contributions in power market analysis and regulation.
6. Oliver Hart (Nobel 2016), (advisor of Ilya Segal, who was also advisor of Federico Echenique), for his fundamental contributions in contract theory.

Contract Theory

1. Seller chooses $x \in X \subset \mathbb{R}_+$, amount of some good produced.
2. Profits are $t - c(x)$, where $c(\cdot)$ is the cost function.
3. Consumers utility is $v(x, \theta) - t$, being $\theta \in \Theta \subset \mathbb{R}$ his type.
4. If $x = 0$, then $t = 0$.
5. θ unobserved.
6. Paiment

Definition

A tariff is a function $T : X \to \mathbb{R}$ that specifies a series of payments $T(x)$ that the agent has to make in order to receive different quantities of the good $x \in X$.

Hence, given a tariff $T(\cdot)$, an agent with type θ chooses

$$x \in \text{argmax}_{x \in X}[v(x, \theta) - T(x)].$$

Spence-Mirrelees condition

Definition

A function $\varphi : X \times \Theta \to \mathbb{R}$, with $X, \Theta \subset \mathbb{R}$, has the Single Crossing Property (SCP) if $\dfrac{\partial \varphi(x,\theta)}{\partial x}$ exists and is strictly increasing in θ for all x.

The SCP was initially suggested by [Mirrlees, 1971] and [Spence, 1973], applied to $v(x,\theta)$. Intuitively, $v(\cdot,\cdot)$ satisfies the SCP when the marginal utility of consumption v_x increases with the type θ. In this sense, higher types have steeper indifference curves in the $X - T$ space.

Definition

A function $\varphi : X \times \Theta \to \mathbb{R}$, with $X, \Theta \subset \mathbb{R}$, is increasing in differences if

$$\varphi(x'', \theta) - \varphi(x', \theta)$$

is increasing in θ for all $x', x'' \in X$ such that $x'' > x'$.

Lemma

If $\varphi(x, \theta)$ is C^1 and satisfies the SCP, and X is an interval, then φ is increasing in differences.

Proof.

For $\theta'' > \theta'$,

$$\begin{aligned}
\varphi(x'', \theta'') - \varphi(x', \theta'') &= \int_{x'}^{x''} \frac{\partial \varphi}{\partial x}(x, \theta'')\, dx \\
&> \int_{x'}^{x''} \frac{\partial \varphi}{\partial x}(x, \theta')\, dx \\
&= \varphi(x'', \theta') - \varphi(x', \theta').
\end{aligned}$$

□

If the agents' utility function $v(x, \theta)$ satisfies the property of increasing in differences, then the indifference curves for two types θ' and θ'' with $\theta' < \theta''$ do not cross more than once. Suppose, by contradiction, that there exist (x', t') and (x'', t'') with $x' < x''$ such that they intersect at these points. This implies that increasing consumption from x' to x'' is equivalent to $t' - t''$ for both agents:

$$\varphi(x'', \theta'') - \varphi(x', \theta'') = \varphi(x'', \theta') - \varphi(x', \theta'),$$

which is clearly a contradiction. This, to some extent, justifies the name of the SCP.

Theorem

Topkis, Edlin-Shannon. *Let $\theta'' > \theta'$ and $x' \in \mathrm{argmax}_{x \in X} \varphi(x, \theta')$ and $x'' \in \mathrm{argmax}_{x \in X} \varphi(x, \theta'')$. Then,*

1. *If φ is increasing in differences, $x'' \geq x'$.*
2. *If φ satisfies the SCP and at least one of x' or x'' is interior to X, then $x'' > x'$.*

Proof.

By revealed preference,

$$\varphi(x', \theta') \geq \varphi(x'', \theta')$$
$$\varphi(x'', \theta'') \geq \varphi(x', \theta'').$$

Rearranging the terms,

$$\varphi(x'', \theta'') - \varphi(x', \theta'') \geq \varphi(x'', \theta') - \varphi(x', \theta').$$

This is only possible if $x'' \geq x'$. Now, by FOC, for interior solutions (w.l.o.g., suppose x' is interior),

$$\frac{\partial \varphi}{\partial x}(x', \theta') = 0.$$

Then,

$$\frac{\partial \varphi}{\partial x}(x', \theta'') > \frac{\partial \varphi}{\partial x}(x', \theta') = 0.$$

Hence, x' is not optimal for θ''. And since an increase in x (when $x = x'$) increases $\varphi(x, \theta'')$, it follows that $x'' > x'$. $\square$

Basically, three problems are studied: Complete Information:

$$\begin{cases} \max_{(x,t)\in X\times\mathbb{R}} & t - c(x) \\ \text{s. t.} & v(x,\theta) - t \geq v(0,\theta) = 0 \quad \text{Individual Rationality (IR).} \end{cases}$$

Incomplete Information (Discrete):

$$\max_{\{(t_i,x_i)\}_{i=1,\dots,n}} \sum_{i=1}^{n} \pi_i(t_i - c(x_i))$$

$$\text{s.t. } v(x_i,\theta_i) - t_i \geq 0, \ \forall\ i = 1, ..., n$$

$$v(x_i,\theta_i) - t_i \geq v(x_j,\theta_i) - t_j, \ \forall\ i \neq j.$$

Incomplete Information (Continuous):

$$\max_{\{x(\cdot),t(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} [t(\theta) - c(x(\theta))] f(\theta) d\theta$$

$$\text{s.a } v(x(\theta),\theta) - t(\theta) - v(x(\hat{\theta}),\theta) - t(\hat{\theta}), \ \forall\ \theta, \hat{\theta} \in \Theta$$

$$v(x(\theta),\theta) - t(\theta) \geq v(0,\theta).$$

For more details, please see the (working) lecture notes: Contract Theory for Financial Microeconomics.

Some properties lead to

$$\max_{\{x(\cdot),t(\cdot)\}} \int_{\underline{\theta}}^{\overline{\theta}} [t(\theta) - c(x(\theta))] f(\theta) d\theta$$

$$\text{s.a } x'(\cdot) \geq 0$$

$$v_x(x(\theta), \theta) x'(\theta) - t'(\theta) = 0, \ \forall \ \theta$$

$$v(x(\underline{\theta}), \underline{\theta}) - t(\underline{\theta}) \geq v(0, \underline{\theta}).$$

Considering almost every where with respect to Lebesgue measure FOC, one obtain

$$\max_{x(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} \left[v(x(\theta), \theta) - c(x(\theta)) - \int_{\underline{\theta}}^{\overline{\theta}} \frac{\partial v}{\partial \theta}(x(s), s) ds \right] f(\theta) d\theta.$$

By integration by parts,

$$\int_{\underline{\theta}}^{\overline{\theta}} \int_{\underline{\theta}}^{\theta} [v_\theta(x(s),s)ds] f(\theta)d\theta = \left[\int_{\underline{\theta}}^{\theta} v_\theta(x(s),s)ds\right] \cdot F(\theta)\Big|_{\underline{\theta}}^{\overline{\theta}} - \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta),\theta)F(\theta)d\theta$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta),\theta)d\theta - \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta),\theta)F(\theta)d\theta$$

$$= \int_{\underline{\theta}}^{\overline{\theta}} v_\theta(x(\theta),\theta)\left[\frac{1-F(\theta)}{f(\theta)}\right] f(\theta)d\theta.$$

Thus, the problem is re-written

$$\max_{x(\cdot)} \int_{\underline{\theta}}^{\overline{\theta}} \left[v(x(\theta),\theta) - c(x(\theta)) - v_\theta(x(\theta),\theta)\left(\frac{1-F(\theta)}{f(\theta)}\right)\right] f(\theta)d\theta.$$

Hence, by pointwise maximization, we must have

$$x(\theta) \in \operatorname{argmax}\, v(x,\theta) - c(x) - \left[\frac{1-F(\theta)}{f(\theta)}\right] v_\theta(x,\theta).$$

Thus:

$$\boxed{v_x(x(\theta),\theta) - c'(x(\theta)) - \frac{1}{h(\theta)} v_{x\theta}(x(\theta),\theta) = 0}.$$

Monopoly

Monopoly

We follow [Tirole, 1994].

Exercise 1: consider the classical problem of the monopolist. Prove that, if $q = D(p)$ is the demand for the good produced by the monopolist, then the optimal pricing for the monopolist satisfies

$$p^m - C'(D(p^m)) = -\frac{D(p^m)}{D'(p^m)}$$

where $C = C(q)$ is the cost function of the monopolist.

Solution: this follows (assuming differentiability and strict concavity) from FOC

$$\max_{p} \; pD(p) - C(D(p)).$$

Chain rule leads to

$$pD'(p) + D(p) - C'(D(p))D'(p)\Big|_{p^m} = 0. \tag{1}$$

Exercise 2: let ε be the demand elasticity at p^m. Then, prove that

$$\underbrace{\frac{p^m - C'}{p^m}}_{\text{relative markup}} = \underbrace{\frac{1}{\varepsilon}}_{\text{Lerner Index}} .$$

Solution: from (1), we derive by p^m and recall that

$$\varepsilon = \frac{\partial D}{\partial p}\frac{p}{D}.$$

Then,

$$\underbrace{\frac{p^m - C'}{p^m}}_{\text{relative markup}} = -\frac{D(p^m)}{p^m D'(p^m)} = -\underbrace{\frac{1}{\varepsilon}}_{\text{Lerner Index}} .$$

Exercise: prove that if $q(p) = kp^{-\epsilon}$ with $k, \epsilon > 0$, then Lerner index is constant.
Solution: let us compute the demand elasticity:

$$\frac{\partial q}{\partial p} = -\epsilon k p^{-\epsilon - 1}.$$

Thus,

$$\frac{\partial q}{\partial p}\frac{p}{q} = \left(-\epsilon k p^{-\epsilon-1}\right)\frac{p}{kp^{-\epsilon}} = -\epsilon.$$

Since price elasticity is constant (doesn't change with p), we conclude.

Exercise: prove that the monopoly price is a non-decreasing function of the marginal cost $C'(\cdot)$.
Solution: we want to prove that, if $C_2' > C_1'$ (as a function inequality), then $p_2^m > p_1^m$, where p_i^m is the monopolist price associated to the cost structure C_i. Thus, let us assume that $C_2'(q) > C_1'(q)$, $\forall\ q$. Let us denote (p_1^m, q_1^m) and (p_2^m, q_2^m) the optimal plans of pricing-consumption associated to each cost structure. Then, by definition:

$$p_1^m q_1^m - C_1(q_1^m) \geq p_2^m q_2^m - C_1(q_2^m)$$
$$p_2^m q_2^m - C_2(q_2^m) \geq p_1^m q_1^m - C_2(q_1^m).$$

Adding both inequalities:

$$p_1^m q_1^m - C_1(q_1^m) + p_2^m q_2^m - C_2(q_2^m) \geq p_2^m q_2^m - C_1(q_2^m) + p_1^m q_1^m - C_2(q_1^m).$$
$$C_2(q_1^m) - C_2(q_2^m) \geq C_1(q_1^m) - C_1(q_2^m)$$
$$\int_{q_2^m}^{q_1^m} C_2'(q)dq \geq \int_{q_2^m}^{q_1^m} C_1'(q)dq.$$

Since $C_2' \geq C_1'$, we must have $q_1^m \geq q_2^m$ (so the integration is done in the correct order and positivity holds). Finally, since demand is down-sloped with respect to price, $p_2^m \geq p_1^m$.

Exercise (conceptual): analyze the following statements:

1. Monopolies usually have large fixed costs.
2. Monopolist do not exhibit (often) cost subadditivity.
3. The monopolist operates in the inelastic part of the demand, $|\varepsilon| < 1$.

Solution:

1. True.
2. False, they exhibit: $C\left(\sum_k q_k\right) < \sum_k C(q_k)$.
3. False, in the inelastic part of the demand, a decrease in the quantity demanded would increase the price, but total revenue would decrease, which would not maximize the monopolist's profits. Therefore, the monopolist avoids operating in the inelastic region of the demand curve.

1. **Demand Function and Elasticity:**

$$\varepsilon = \underbrace{\frac{dQ}{dP}}_{<0} \cdot \frac{P}{Q} < 0, \quad |\varepsilon| < 1.$$

2. **Total Revenue (TR) and Marginal Revenue (MR):**

$$TR = P \cdot Q, \quad MR = P + Q \cdot \frac{dP}{dQ}.$$

3. **Relationship with Elasticity:**

$$\frac{dP}{dQ} = \frac{P}{\varepsilon \cdot Q}, \quad MR = P\left(1 + \frac{1}{\varepsilon}\right).$$

Exercise: suppose that good q is produced by a monopolistic firm in the short run. If the market demand is given by $q^D = 100 - 0.5p$ and the firm's cost curve is $C(q) = 2q^2 + 10q + 4$,

1. Find the quantity produced, the price of the good, and the monopolist's profits.
2. Find the quantity produced, the price of the good, and the monopolist's profits if we consider a new cost curve given by $C(q) = 2q^2 + 10q$.

Solution: we set the maximization problem. This is,

$$\max_{q\geq 0} \; p(q)q - C(q).$$

In the first case, this is

$$\max_{q\geq 0} \; \underbrace{(200-2q)q - (2q^2+10q+4)}_{=-4+190q-4q^2.}.$$

Apply FOC (since the function is strictly concave),

$$q^* = 23.75, \; p^* = 152.5.$$

Solution is the same for the second cost structure: **why?** Homework:

1. Consider $C(q) = q^2 + 10q + a$, $a > 0$.
2. Consider $D(p) = a - bp$, $a, b > 0$.

Price discrimination

Theory

Participation constraint:

$$
\begin{aligned}
u_1(x_1) - t_1 &\geq 0 \\
u_2(x_2) - t_2 &\geq 0.
\end{aligned}
$$

Incentive compatibility:

$$
\begin{aligned}
u_1(x_1) - t_1 &\geq u_1(x_2) - t_2 \\
u_2(x_2) - t_2 &\geq u_2(x_1) - t_1.
\end{aligned}
$$

Combining this:

$$
\begin{cases}
u_1(x_1) & \geq t_1 \\
u_1(x_1) - u_1(x_2) - t_2 & \geq t_1 \\
u_2(x_2) & \geq t_2 \\
u_2(x_2) - u_2(x_1) + t_1 & \geq t_2.
\end{cases}
$$

Assumptions: $u_2 > u_1$ and $u_2' > u_1'$ for all $x \in X \subset \mathbb{R}_+$.

Case 1: $u_2(x_2) = t_2$.

$$\begin{aligned} t_2 - u_2(x_1) + t_1 &\geq t_2 \\ t_1 &\geq u_2(x_1) \\ &> u_1(x_1) \\ &\geq t_1 \end{aligned}$$

a contradiction. Thus

$$t_2 = u_2(x_2) - u_2(x_1) + t_1.$$

Case 2: $t_1 = u_1(x_1) - u_2(x_2) + t_2$

$$\begin{aligned} t_1 &= u_1(x_1) - u_1(x_2) + t_2 \\ &= u_1(x_1 0) - u_1(x_2) + [u_2(x_2) - u_2(x_1) + t_1] \\ u_1(x_2) - u_1(x_1) &= u_2(x_2) - u_2(x_1) \\ \int_{x_1}^{x_2} u_1'(s)ds &= \int_{x_1}^{x_2} u_2'(s)ds. \end{aligned}$$

Thus

$$t_1 = u_1(x_1).$$

Hence, the firms maximization problem is

$$
\begin{aligned}
\max_{x_1, x_2} \Pi = {} & [t_1 - c_1 x_1] + [t_2 - cx_2] \\
& [u_1(x_1) - cx_1] + [u_2(x_2) - u_2(x_1) + u_1(x_1) - cx_2].
\end{aligned}
$$

FOC are:

$$
\begin{aligned}
u_1'(x_1) - cu_1'(x_1) - u_2'(x_1) &= 0 \\
u_2'(x_2) &= 0.
\end{aligned}
$$

Intuition

1. Low type: indifferent between participate or no. Plane vs bus. Universal Studios vs no holidays.
2. High type: indifferent versus high service and low service. First class vs economy. Fast pass vs regular pass in Universal Studios.
3. Further research: aspirations and inequality from Garance Genicot and Debraj Ray, Econometrica.

An economy has two types of consumers and two goods. The agent type Anakin has the following utility function:

$$u_A(x_{1A}, x_{2A}) = 4x_{1A} - \frac{x_{1A}^2}{3} + x_{2A}$$

and the agent type Ben Kenobi has the following utility function:

$$u_B(x_{1B}, x_{2B}) = 3x_{1B} - \frac{x_{1B}^2}{2} + x_{2B}.$$

Good 2 is the numeraire, and each consumer has an income of 100. Additionally, the economy has N consumers of both type Anakin and type Ben Kenobi.

1. Identify the type of consumer with high demand and the type with low demand for good x_1. Compare the marginal willingness to pay for each type of consumer for good x_1.
2. The monopolist produces good 1 with the following cost function $C(x_1) = cx_1$ and cannot discriminate prices. Find the optimal price and quantity of good x_1 that the monopolist will choose. For which values of c will the monopolist choose to sell to both types of consumers?
3. The monopolist engages in second-degree price discrimination by offering a menu of prices and quantities to each type of consumer (r_A, x_A) and (r_B, x_B). Based on this, formulate the monopolist's optimization problem and find the optimal values (r_A^*, x_A^*) and (r_B^*, x_B^*).
4. If the monopolist engages in third-degree price discrimination, what will be the prices and quantities set by the monopolist in the markets for Anakin-type and Ben Kenobi-type consumers?
5. If the monopolist engages in first-degree price discrimination, find the quantity produced by the monopolist in the market for good x. Calculate the consumer surplus and the monopolist's surplus.

Solution:

- It follows that $u_A(x_1) > u_B(x_1)$ and $u'_A(x_1) > u'_A(x_1)$ for any x_1.
- We need consumer's demand. Consumer A solves

$$\max_{x_{1A},x_{2A}} \quad \underbrace{u_A(x_{1A})}_{=4x_{1A}-x_{1A}^2/2} \quad +x_{2A}$$

$$\text{s.t } px_{1A} + x_{2A} = 100.$$

FOC leads to $4 - x_{1A} = p$, so $x_{1A}^d = 4 - p$. Thus, demand of agents of type A for good 1 is

$$\sum_{i=1}^{N} x_{1A_i^d} = \sum_{i=1}^{N}(4-p) = N(4-p).$$

Analogously, for type B consumer, he solves

$$\max_{x_{1A},x_{2A}} \quad \underbrace{u_B(x_{1B})}_{=3x_{1B}-x_{1B}^2/2} \quad +x_{2B}$$

$$\text{s.t } px_{1B} + x_{2B} = 100.$$

FOC leads to $x_{1B} = 3 - p$. Therefore, aggregating

$$\sum_{i=1}^{N} x_{1B_i}^d = \sum_{i=1}^{N}(3-p) = N(3-p).$$

Full demand for good 1 is $N(7-2p)$.

Solution:

- Firm solves

$$\max_{x_1} \; pX_1 - c(x_1) = pX_1 - cX_1 = \left(\frac{7}{2} - \frac{X_1}{2N}\right) X_1 - cX_1.$$

  FOC yields $X_1^* = N(3.5 - c)$ and $P^M 7/4 + c/2$.

- Under second degree discrimination, he solves (as explained before)

$$\begin{aligned} \max_{x_{1A}, x_{1B}} \;& N(t_A - cx_{1A}) + N(t_B - cx_{1B}) \\ \text{s.t. } & u_B(x_{1B}) = t_{1B} \\ & u_A(x_{1A}) - t_{1A} = u_A(x_{1B}) - t_{1B}. \end{aligned}$$

  Replacing in the objective function, we need to solve

$$\max_{x_{1A}, x_{1B}} \; N\left[4x_{1A} - \frac{x_{1A}^2}{2} - 4x_{1B} + \frac{x_{1B}^2}{2} + 3x_{1B} - \frac{x_{1B}^2}{2} - cx_{1A} + 3x_{1B} - \frac{x_{1B}^2}{2} - cx_{1B}\right].$$

  FOC yields

$$x_{1A}^* = 4 - c, \; x_{1B}^* = 2 - c, \; t_{1A}^* = 24 - \frac{c(c+16)}{2}, \; t_B^* = \frac{(2-c)(4+c)}{2}.$$

Solution:

- Third degree price discrimination:

$$\max_{p_A, X_{\mathbf{1}A}, p_B, X_{\mathbf{1}B}} \quad p_A X_{1A} + p_B X_{1B} - c(X_{1A} + X_{1B}).$$

We already now that $P_A = 4 - \frac{X_{\mathbf{1}A}}{N}$ and $P_B = 3 - \frac{X_{\mathbf{1}B}}{N}$. Thus, the optimization problem becomes

$$\max_{X_{\mathbf{1}A}, X_{\mathbf{1}B}} \quad \Pi = \left(4 - \frac{X_{1A}}{N}\right) X_{1A} + \left(3 - \frac{X_{1B}}{N}\right) X_{1B} - c(X_{1A} + X_{1B}).$$

FOC lead to

$$X_{1A} = \frac{N(4-c)}{2} \implies p_A = \frac{4+c}{2}, \; X_{1B} = \frac{N(3-c)}{2} \implies p_B = \frac{3+c}{2}.$$

- The monopolist extracts all the consumer surplus. In this way, they charge a price equal to each consumer's maximum willingness to pay. Finally, they produce the competitive market quantity $X^* = N(7-2c)$, but only the last buyer pays the competitive market price $p = c$. Consumer surplus (CS) is 0, and producer surplus (PS) is $\frac{7}{2} - c$.

Some interesting exercises

From [Varian, 1992]. What shape must the demand curve have for $\frac{dp}{dc} = 1$? $MC = c$. FOC provide

$$\begin{aligned}
p'(y)y + p(y) - c'(y) &= 0 \\
p'(y)y + p(y) - c &= 0 \\
\frac{d}{dc}[p'(y)y + p(y) - c] &= 0 \\
p''(y)\frac{dy}{dc} + 2p'(y)\frac{dy}{dc} - 1 &= 0 \\
\frac{dy}{dc}[p''(y) + 2p'(y)] &= 1 \\
\frac{dy}{dc} &= \frac{1}{p''(y) + 2p'(y)} \\
\frac{dp}{dy}\frac{dy}{dc} &= \frac{p'(y)}{p''(y) + 2p'(y)} \\
\frac{dp}{dc} &= \frac{p'(y)}{p''(y) + 2p'(y)} \\
1 &= \frac{1}{\frac{p''(y)y}{p'(y)} + 2} \\
p'(y) &= yp''(y) + 2p'(y).
\end{aligned}$$

Doing $x = p'$ and $y = t$,

$$
\begin{aligned}
x'(t) &= -\frac{1}{t}x(t) \\
\frac{dx}{x} &= -\frac{1}{t} \\
\ln|x| &= C - \ln|t| \\
x(t) &= Ae^{-\ln|t|} = \frac{A}{t} \\
p(y) &= A\ln|y| + B.
\end{aligned}
$$

Learning by doing

From [Tirole, 1994]. In some industries, cost reductions are achieved over time simply because of learning. Learning by doing is especially apparent in industrial activity. This is for instance the case of the military aircraft production. Consider a single-good monopolist producing at dates $t = 1, 2$. Assume that $q_t = D(p_t)$. The total cost at $t = 1$ is $C_1(q_1)$ and at $t = 2$, $C_2(q_1, q_2)$ where $\dfrac{\partial C_2}{\partial q_1} < 0$ (why?).

You will use dynamic optimization

Adapted from [Tirole, 1994]. Assume that a monopolist has a unit-cost function such that $c = c(\omega(t))$ where $\omega(t)$ is the firm's experience at time t.

1. Explain why it is logical to assume that, denoting by $q = q(t)$ the output at time t, $\frac{d\omega}{dt} = q$.
2. Consider the monopolist maximization problem:

$$\begin{aligned} \max_{q(t)\in\mathbb{R}_+} \; & \int_0^\infty [R(q(t)) - c(\omega(t))q(t)]e^{-rt}dt \\ \text{s. t. } & w'(t) = q(t) \\ & w(0) = w_0. \end{aligned}$$

Prove that

$$R'(q(t)) = c(\omega(t)) + \int_t^\infty c'(\omega(s))q(s)e^{-(s-t)}ds.$$

Hint: apply the Maximum Principle, see [Acemoglu, 2009] or [Cerdá, 2012]. Note that the current value Hamiltonian is

$$\mathcal{H}(\omega(t), q(t), \psi(t), t) = R(q(t)) - c(\omega(t))q(t) + \psi(t)q(t).$$

Pontryaguin Maximum Principle leads to

$$
\begin{aligned}
R'(q(t)) - c(\omega(t)) + \psi = 0 \\
\psi' &= -\frac{\partial \mathcal{H}}{\partial \omega} + r\psi \\
\psi'(t) - r\psi(t) &== -\frac{\partial \mathcal{H}}{\partial \omega} = c'(\omega(t))q(t) \\
[e^{-rt}\psi(t)] &= c'(\omega(t))q(t)e{-}rt \\
\psi(t) &= \int^{t} c'(\omega(s))q(s)e{-}rse^{rt}ds \\
\psi(t) &= \int^{t} c'(\omega(s))q(s)e^{-r(s-t)}ds \\
R'(q(t)) &= c(\omega(t)) - \psi(t) \\
&= c(\omega(t)) + \int_{t}^{\infty} c'(\omega(s))q(s)e^{-(s-t)}ds.
\end{aligned}
$$

Thank you

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