# About Brouwer fixed point theorem and its applications in general equilibrium

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December 31, 2023

We develop a path towards the proof of Brouwer's Fixed Point Theorem, although based on several sources, of our construction. We also present an application in economic theory. Specifically, we focus on general equilibrium theory. We aim to provide the simplest possible proof, the only requirements are real analysis and general topology. Besides one Lemma which is not proved in its most general case, we provide proofs for all the results building up to the main theorem. Furthermore, as far as we know, that the combination of elements that we present in this document has not been displayed in this exact form before. The applications that we provide in general equilibrium, besides classical definitions, are by our constructions, especially one last example that has been developed by our own. It is important to mention that this work does not seek to be original or innovative in the sense that it does not present any new results in the literature. Our objective is simply to develop an understandable approach to Brouwer's Fixed Point Theorem and its applications in general equilibrium.

# 1 Introduction

In economic theory, Brouwer Fixed Point Theorem is a very powerful tool, establishing very important results in, for instance, general equilibrium theory. This field of microeconomic theory has been one of the most captivating and significant topics in economic theory. Mathematical formalism elegantly intertwines with economics, which accounts for the numerous works, including both books and articles, that delve into this subject, for instance Echenique and Wierman (2012), Echenique (2023) or Ok (2007). There are numerous extensions and generalizations, as is the case with infinite consumption goods, the indivisibility of goods, etc Aliprantis et al. (1990). When we reach the applications part of this document, we mainly follow the classical scenario presented in Mas-Colell et al. (1995) or Ellickson (1993). We do not treat the general cases or extensions involving, for instance, elements of functional analysis.

The structure of this document is as follows. First, in Section 2 we present Brouwer Fixed Point Theorem (BFPT) and work our way up to the proof. For this, we require a strong Lemma known as «Borsuk Lemma». We provide a rigorous proof of this lemma, based on the sktech provided in Laczkovich and Sos (2017). Although it is not the most general statement of Borsuk lemma, it is one that avoids, for instance, passing through algebraic topology. Thereafter, we state and prove some additional results. Only equipped with this, we are ready to tackle the main result. Finally, in Section 3, we move to the application of BFPT in general equilibrium. We prove the existence of a Walrasian equilibrium and develop an example illustrating the theoretical results.

# 2 An elementary proof of Brouwer Fixed Point Theorem

The statement of Brouwer Fixed Point Theorem is very short and elegant. Nevertheless, its proof is not simple at all. Let us first announce this famous result.

**Theorem 1.** Let $X$ be a non empty convex and compact subset of $\mathbb{R}^n$ and $f : X \to X$ a continuous function. Then, there exists $x^* \in X$ such that

$f(x^*) = x^*$.

Theorem 1 establishes therefore the existence of a fixed point for a continuous function going from a convex and compact subset in $\mathbb{R}^n$ to itself.

To prove this result we start defining what a retraction is, in the context of general topology. After this, we state Borsuk lemma and provide a rigorous proof for a weaker case. Once this lemma is established, Theorem 5 arises as a direct application. We then present and prove several results that allow us to reach the BFPT.

**Definition 2.** Let $Y$ be a topological space and $S \subseteq Y$ be a subset of $Y$ equipped with the subspace topology. A continuous function $r : Y \to S$ is called a retraction if $r(x) = x$ for all $x \in S$ . In other words, a retraction is a continuous function $r : Y \to S$ that fixes $S$. When such a function exists we say that $S$ is a retract of $Y$.

**Lemma 3. Borsuk.** The $n$ dimensional closed unit ball $\overline{\mathbb{B}}^n$ does not retract to the $n-1$ dimensional unit sphere $\mathbb{S}^{n-1}$.

The proof of Lemma 3 involves, most of the time, passing through algebraic topology. See for instance Boothby (1971). In this work, we prove that there is no continuously differentiable retraction from $\overline{\mathbb{B}}^n$ to $\mathbb{S}^{n-1}$: a weaker statement since the general framework is for continuous maps.

**Proposition 4. No differentiable retraction.** There is no mapping $f$ such that

1. $f$ is continuously differentiable on an open set containing $\overline{\mathbb{B}}^n$
2. $f(\overline{\mathbb{B}}^n) = \mathbb{S}^{n-1}$
3. $f(x) = x$ for all $x \in \mathbb{S}^{n-1}$.

*Proof.* We proceed by contradiction. Take $f$ satisfying all these three conditions. For each $t \in [0, 1]$, define the mapping $f_t$ by

$$f_t(x) = (1 - t)f(x) + tx.$$

Clearly, $f_0 = f$, $f_1$ is the identity and each $f_t$ satisfies conditions (1) and (3). Furthermore, $f_t(\overline{\mathbb{B}}^n) \subset \overline{\mathbb{B}}^n$ due to the convexity of $\overline{\mathbb{B}}^n$. Consider the function $h : [0,1] \to \mathbb{R}$ given by

$$h(t) = \int_{\mathbb{B}^n} \det f_t'(x) dx.$$

We show that:

(i) $h(0) = 0$

(ii) $h(t)$ is a polynomial

(iii) $h(t) = m(\mathbb{B}^n)$ hold in an interval $(1-\delta, 1]$ for some $\delta > 0$. Here $m(\mathbb{B}^n)$ denotes the Lebesgue measure of the unit ball.

The contradiction is hard to miss: a polynomial which is constant in an open interval is a constant itself, and cannot take both the values of 0 and $m(\mathbb{B}^n) > 0$.

For (i), write $f = (f_1, \ldots, f_n)$ and note that condition (2) can be rephrased as

$$||f(x)||^2 = \sum_{i=1}^{n} f_i^2(x) = 1 \quad \forall x \in \mathbb{B}^n.$$

Differentiate to get

$$\sum_{i=1}^{n} 2 f_i(x) \nabla f_i(x) = 0 \quad \forall x \in \mathbb{B}^n.$$

That is, for each $x \in \mathbb{B}^n$ the row vectors $\{\nabla f_i(x)\}_{1 \le i \le n}$ of $f_t'(x)$ are linearly dependent and thus $\det f_t'(x) = 0$. Integrate on the unit ball to get $h(0) = 0$.

For (ii), observe that $f_t'(x) = (1-t)f'(x) + tI$, where $I \in \mathcal{M}_{n \times n}$ is the identity matrix. Then, the entries of $f_t'(x)$ are

$$(f_t'(x))_{ij} = \begin{cases} (1-t)\frac{\partial f_i}{\partial x_j}(x) + t, \ i = j \\ (1-t)\frac{\partial f_i}{\partial x_j}(x), \ i \neq j, \end{cases}$$

which are continuous in $x$ due to condition (1). It is not hard to see that $\det f_t'(x)$ has the form $\sum_{i=1}^{m} s_i(t) \cdot g_i(x)$, for some $s_i$ polynomials and $g_i$ continuous functions. In particular, for each $i$, $g_i$ is continuous in $\overline{\mathbb{B}}^n$ and hence bounded in the same region. Set $c_i = \int_{\mathbb{B}^n} g_i(x) dx \in \mathbb{R}$ for each $i$. Integrate $\det f_t'(x)$ on the unit ball finally to obtain that $h(t)$ is the polynomial $\sum_{i=1}^{m} c_i s_i(t)$.

For (iii), recall that, as each $f_t$ satisfies condition (1), a change of variables allows

$$\int_{\mathbb{B}^n} |\det f_t'(x)| dx = m(f_t(\mathbb{B}^n))$$

as long as $f_t$ is injective in $\mathbb{B}^n$ . Hence, it suffices to prove that, for $t$ sufficiently close to 1, $f_t$ is injective in $\mathbb{B}^n$, $\det f_t'(x) > 0$ for all $x \in \mathbb{B}^n$ and $f_t(\mathbb{B}^n) = \mathbb{B}^n$. We go in order.

As $f$ is continuously differentiable on the compact $\overline{\mathbb{B}^n}$, it is Lipschitz of constant $M > 0$. That is, $||f(x) - f(y)|| \leq M||x - y||$ for all $x, y \in \overline{\mathbb{B}^n}$. Let $x, y \in \mathbb{B}^n$ with $x \neq y$. For $t$ sufficiently close to 1 we get $t - (1 - t)M > 0$ and injectivity is then assured:

$$\begin{aligned} ||f_t(x) - f_t(y)|| &\geq t||x - y|| - (1 - t)||f(x) - f(y)|| \\ &\geq t||x - y|| - (1 - t)M||x - y|| \\ &= (t - (1 - t)M)||x - y|| > 0. \end{aligned}$$

Now consider the space $\mathcal{M}_{n\times n}$ with the norm $||\cdot||_\infty$. Given that the fuction $\det : \mathcal{M}_{n\times n} \to \mathbb{R}$ is a polynomial on its coordinates, it is continuous. The identity, $I$, has determinant equal to 1, then there exists $\xi > 0$ such that $||A - I||_\infty < \xi$ implies $|\det(A) - \det(I)| = |\det(A) - 1| < \frac{1}{2}$. In particular, it implies $\det(A) > 0$. Because $f'$ is continuous in $\overline{\mathbb{B}}^n$, we can define $C = \sup_{x\in\overline{\mathbb{B}}^n} ||f'(x) - I||_\infty < \infty$. For each $x \in \mathbb{B}^n$, we get:

$$\begin{aligned} ||f_t'(x) - I||_\infty &= ||(1 - t)f'(x) + (t - 1)I||_\infty \\ &= (1 - t)||f'(x) - I||_\infty \\ &\leq (1 - t)C. \end{aligned}$$

Thus, for $t$ sufficiently close to 1 we get $(1-t)C < \xi$ and this implies $\det(f_t'(x)) > 0$ for all $x \in \mathbb{B}^n$.

Lastly, we prove $f_t(\mathbb{B}^n) = \mathbb{B}^n$ for $t$ sufficiently close to 1. The two previous properties assure that $f_t(\mathbb{B}^n)$ is open for $t$ close enough to 1, due to the open mapping theorem. Let $x \in \mathbb{B}^n$, then $||f_t(x)|| \leq t||f(x)|| + (1 - t)||x|| < 1$. This settles $f_t(\mathbb{B}^n) \subset \mathbb{B}^n$. For the reverse inclusion, suppose there exists $p_t \in$

$\mathbb{B}^n \backslash f_t(\mathbb{B}^n)$. We have that $f_t(\overline{\mathbb{B}^n})$ is compact. Therefore, we can take $f_t(q_t) \in f_t(\overline{\mathbb{B}^n})$ that minimizes its distance to $p_t$. In particular,

$$||f(q_t) - p_t|| \leq ||f_t(p_t) - p_t|| = (1-t)||f(p_t) - p_t||.$$

If $q_t \in \mathbb{B}^n$ then there exists an open ball $V$ centered at $f_t(q_t) \in f_t(\mathbb{B}^n)$ such that $V \subset f_t(\mathbb{B}^n) \subset f_t(\overline{\mathbb{B}^n})$ and thus $f_t(q_t)$ cannot minimize the distance to $p_t$. Hence $q_t \in \mathbb{S}^{n-1}$ and, since $f_t$ fixes $\mathbb{S}^{n-1}$, $f_t(q_t) = q_t$. Then

$$||q_t - p_t|| \leq (1-t)||f(p_t) - p_t||$$

Recall that $f$ has Lipschitz constant $M$ in $\overline{\mathbb{B}}^n$. This allows

$$\begin{aligned} ||f(p_t) - p_t|| &\leq ||f(p_t) - f(q_t)|| + ||q_t - p_t|| \\ &\leq (M+1)||q_t - p_t|| \end{aligned}$$

and finally

$$\begin{aligned} ||q_t - p_t|| &\leq (1-t)(M+1)||q_t - p_t|| \\ 1 &\leq (1-t)(M+1). \end{aligned}$$

For $t$ close enough to 1 we arrive to a contradiction, settling $f_t(\mathbb{B}^n) = \mathbb{B}^n$.The aforementioned change of variables settles condition (iii). As argued, (i), (ii) and (iii) result in a contradiction, and thus there is no map $f$ satisfying the conditions (1), (2) and (3).

$\square$

As we will see, in the applications the maps involved are continuous but, eventually, not differentiable. Borsuk lemma can be extended to the most general case where, in the statement, the function is only continuous. For further details on this see Kannai (1981). To get an idea of how strong this lemma is, BFPT for the case $X = \overline{\mathbb{B}}^n$ is a direct implication.

**Theorem 5.** Let $f : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ be a continuous map. Then, there exists $x^* \in \overline{\mathbb{B}}^n$ such that $f(x^*) = x^*$.

*Proof.* Let us suppose by contradiction that the statement is false. Then, there exists $\phi : \overline{\mathbb{B}}^n \to \overline{\mathbb{B}}^n$ with no fixed point. Using $\phi$ let us define $r : \overline{\mathbb{B}}^n \to \mathbb{S}^{n-1}$ as the intersection point of the open ray that starts in $\phi(x)$ and passes through $x$ with $\mathbb{S}^{n-1}$. To define it explicitly as a formula we must find for each $x$ a $t > 0$ such that

$$||\phi(x) + t(x - \phi(x))||^2 = 1$$

Which is equivalent to

$$||\phi(x)||^2 + 2t\langle\phi(x), x - \phi(x)\rangle + t^2||x - \phi(x)||^2 = 1$$

Then this quadratic equation on $t$ has only one positive solution which is given by

$$t(x) = -\frac{\langle\phi(x), x - \phi(x)\rangle}{||x - \phi(x)||^2} + \left[\frac{\sqrt{\langle\phi(x), x - \phi(x)\rangle^2 + ||x - \phi(x)||^2(1 - ||\phi(x)||^2)}}{||x - \phi(x)||^2}\right]$$

Then $r$ must be

$$r(x) = \phi(x) + t(x)(x - \phi(x))$$

Of course, $t$ is continuous since it was defined explicitly, and thus $r$ is too. In addition, $r$ fixes $\mathbb{S}^{n-1}$ which can be easily seen using the geometric interpretation of $r$. Thus, $r$ is a retraction from $\overline{\mathbb{B}}^n$ to $\mathbb{S}^{n-1}$, and a contradiction arises due to Lemma 3. $\square$

Now we continue with the elements leading to the proof of the main result. The following propositions allow us to generalize Theorem 5 for $X$ a non-empty compact convex subset.

**Definition 6.** Given a compact and convex set $S$ in $\mathbb{R}^n$, denote the projection operator over $S$ as $\pi_S : \mathbb{R}^n \to S$. It is implicitly defined by the relation $d(x, \pi_S(x)) = d(x, S)$, where

$$d(x, S) = \inf\{d(x, s) = ||x - s|| : s \in S\}.$$

**Proposition 7.** $\pi_S$ is well defined.

*Proof.* Given a fixed $x \in \mathbb{R}^n$ there exists at least one $s$ such that $d(x, s) = d(x, S)$ since $S$ is compact. Now assume that there are two such points; let them be $s_1$ and $s_2$. Let $h$ be the projection of $x$ over the line that connects $s_1$ and $s_2$. It is not hard to see that $h$ lies in between $s_1$ and $s_2$ since the triangle with vertices

$x, s_1$ and $s_2$ is isosceles with $d(x, s_1) = d(x, s_2)$. Furthermore, $h$ belongs to $S$ due to its convexity. This is a contradiction since

$$d(x, h) < d(x, s_1) = d(x, S) \leq d(x, h).$$

Hence, there is a unique point that minimizes the distance between $x$ and $S$. $\square$

**Proposition 8.** $\pi_S$ is a continuous application.

*Proof.* Let $p$ and $q$ be two arbitrary points in $\mathbb{R}^n$. Firstly we realise that

$$\langle \pi_S(p) - \pi_S(q), q - \pi_S(q) \rangle \leq 0.$$

Otherwise denoting $v = \pi_S(p) - \pi_S(q)$ and $w = q - \pi_S(q)$. Clearly, by the previous proposition and the contradiction assumption $||v|| > 0$ and

$$d(\pi_S(q) + \varepsilon v, q)^2 = ||w||^2 + \varepsilon^2 ||v||^2 - 2\langle \varepsilon v, w \rangle$$
$$d(\pi_S(q) + \varepsilon v, q)^2 = ||w||^2 + \varepsilon ||v||^2 \left( \varepsilon - 2\frac{\langle v, w \rangle}{||v||^2} \right).$$

Then for a positive $\varepsilon$ small enough,

$$d(\pi_S(q) + \varepsilon v, q) < ||w|| = d(q, \pi_S(q)) = d(q, S),$$

which is a contradiction since $\pi_S(q) + \varepsilon v$ belongs to $S$ due to its convexity. Now with this last result we can conclude the following:

$$d(q, \pi_S(p))^2 = d(q, \pi_S(q))^2 + d(\pi_S(p), \pi_S(q))^2 - 2\langle \pi_S(p) - \pi_S(q), q - \pi_S(q) \rangle.$$
$$d(q, \pi_S(p))^2 \geq d(q, \pi_S(q))^2 + d(\pi_S(p), \pi_S(q))^2. \tag{1}$$

On the other hand thanks to the triangular inequality,

$$d(p, q) + d(p, \pi_S(p)) \geq d(q, \pi_S(p)). \tag{2}$$

Then, with both (1) and (2) at our disposal we can conclude that

$$d(p, q)^2 + 2d(p, q)d(p, \pi_S(p)) + d(p, \pi_S(p))^2 \geq d(q, \pi_S(p))^2.$$
$$d(q, \pi_S(p))^2 \geq d(q, \pi_S(q))^2$$
$$+ d(\pi_S(p), \pi_S(q))^2.$$

$$d(p,q)^2 + 2d(p,q)d(p,\pi_S(p)) + d(p,\pi_S(p))^2 \geq d(q,\pi_S(q))^2 + d(\pi_S(p),\pi_S(q))^2.$$

$$d(p,q)^2 + 2d(p,q)d(p,\pi_S(p)) + d(p,\pi_S(p))^2 - d(q,\pi_S(q))^2 \geq d(\pi_S(p),\pi_S(q))^2.$$

Since $d(p,q)^2$, $2d(p,q)d(p,\pi_S(p))$ and $d(p,\pi_S(p))^2 - d(q,\pi_S(q))^2$ get closer to 0 as $q$ gets closer to $p$[1] we can conclude that $d(\pi_S(p),\pi_S(q))^2$ converges to 0 as $q$ converges to $p$, proving that $\pi_S$ is continuous. $\square$

We give now some last additional results, essential to prove BFPT.

**Proposition 9.** Let $S$ be a retract of $Y$. If $Y$ possesses the fixed point property, then $S$ does as well.

*Proof.* Let $f : S \to S$ be a continuous map and $r : Y \to S$ the retraction. First, $f \circ r : Y \to S \subset Y$ is continuous, and since $Y$ possesses the fixed point property, there exists $y \in Y$ such that $y = f(r(y)) \in S$. However, since $r(y) = y$, we obtain $y = f(y)$, which proves $S$ has the desired property. $\square$

**Proposition 10.** Let $S \subset \overline{\mathbb{B}}^n_\delta$ be a closed and convex set (therefore compact), where

$$\overline{\mathbb{B}}^n_\delta = \{x \in \mathbb{R}^n : \ ||x|| \leq \delta\}.$$

Then, there exists a retraction $r : \overline{\mathbb{B}}^n_\delta \to S$.

*Proof.* Let $r : \overline{\mathbb{B}}^n_\delta \to S$ be the restriction over $\overline{\mathbb{B}}^n_\delta$ of the projection operator $\pi_S$. By proposition 8 it is straightforward to see that $r$ is continuous. It is even clearer that $r$ fixes $S$. $\square$

**Proposition 11.** Let $A$ and $B$ be two homeomorphic spaces. If $A$ possesses the fixed point property, then $B$ does as well.

*Proof.* Let $f : B \to B$ be a continuous map and consider the homeomorphism $g : B \to A$. Then $g \circ f \circ g^{-1}$ is a continuous map from $A$ to $A$. There exists therefore $a^* \in A$ such that $a^* = (g \circ f \circ g^{-1})(a^*)$. Apply $g^{-1}$ on both sides to get $g^{-1}(a^*) = f(g^{-1}(a^*))$. Therefore $b^* = g^{-1}(a^*)$ is a fixed point for $f$. $\square$

[1] This arises from the fact that $x \to d(x, S)$ is a continuous function.

Hereafter the main proof of this document: Brouwer Fixed Point Theorem. As we shall see, it is a consequence of all previous propositions and theorems.

*Proof.* Since $X$ is convex and compact, there exists $\delta > 0$ such that $X \subset \overline{\mathbb{B}}^n_\delta$. Using the map $\varphi(x) = x/\delta$, $\varphi : \overline{\mathbb{B}}^n_\delta \to \overline{\mathbb{B}}^n$, we can establish that $\overline{\mathbb{B}}^n_\delta$ and $\overline{\mathbb{B}}^n$ are homeomorphic. By Theorem 5, we know that there is $x^* \in \overline{\mathbb{B}}^n$ such that $f(x^*) = x^*$, $f$ continuous. Then, by Proposition 11, $\overline{\mathbb{B}}^n_\delta$ possesses the fixed point property. Finally, by Proposition 10, $X$ is a retract of $\overline{\mathbb{B}}^n_\delta$ and, by Proposition 9, $X$ possesses the fixed point property too. □

We have proven BFPT using the stronger version of Borsuk's lemma. This lemma was proven in a weaker case, as the proof of the general case involves more sophisticated tools. Besides this lemma, our proof is relatively simple. Hereafter we present applications in general equilibrium theory. We start by situating ourselves in an economic context and then prove a highly significant result in economic theory that makes use of Brouwer's fixed-point theorem: the existence of Walrasian equilibrium.

# 3 Existence of the Walrasian equilibrium

The purpose of this section is to illustrate the importance of Brouwer Fixed Point Theorem in general equilibrium theory. Even if the content might be standard in the literature, we derive some results on our own.

For the following definitions and standard framework we mainly follow Ellickson (1993), Mas-Colell et al. (1995), and Echenique (2023).

Let $i = 1, ..., I$ be the consumers of the economy, $X_i \subset \mathbb{R}^L$ their consumption sets, $\succeq_i$ their preferences over $X_i$ and $\omega_i \in \mathbb{R}^L$ their endowment. Assume furthermore that the preferences $\succeq_i$ can be represented through utility functions $u^i(\cdot)$.

A pure exchange economy will be fully characterized in this context by

$$\mathcal{E} = (\omega^i, u^i)_{i=1}^I.$$

**Definition 12.** An allocation for the pure exchange economy is a collection of consumption vectors

$$x = (x_1, ..., x_I) \in \prod_{i=1}^{I} X_i \subset \mathbb{R}^{IL}$$

Hereafter we define the notion of Walrasian equilibrium for this economy following Echenique (2023).

**Definition 13.** Given a pure exchange economy, an allocation $x^*$ and a price vector $\boldsymbol{p} = (p_1, ..., p_L)$ constitute a Walrasian equilibrium if

1. $\forall\ i$, $x_i^*(\mathbf{p}, \omega \cdot \mathbf{p}) \in X_i$ is maximal with respect to the choice for $\succeq_i$ over the set
$$B = \left\{ x_i \in X_i : \ \boldsymbol{p} \cdot x_i \leq \boldsymbol{p} \cdot \omega_i \right\}.$$
2. $\sum_i x_i^* = \sum_i \omega_i$.

In this framework, where the preferences are rational, convex and continuous preferences and $\sum_i \omega_i \gg \mathbf{0}$, an allocation $x^*$ and a price vector $\boldsymbol{p} = (p_1, ..., p_L)$ constitute a Walrasian equilibrium if

1. for each $i = 1, ..., I$, $x_i^* \in B(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i) = \{x_i \in \mathbb{R}_+^L : \boldsymbol{p} \cdot x_i \leq \boldsymbol{p} \cdot \omega_i\}$ and maximices $u_i(\cdot)$ over $B(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i)$.
2. $\sum_{i=1}^{I} x_i^*(\mathbf{p}, \mathbf{p} \cdot \omega_i) = \sum_{i=1}^{I} \omega_i$.

The difference with the previous statement is that , rationality and continuity of the preferences imply the existence of a utility function.

**Definition 14.** We define the aggregated demand excess function by

$$z(\boldsymbol{p}) = \sum_i z_i(\boldsymbol{p}) = \sum_{i=1} \underbrace{[x_i^*(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i) - \omega_i]}_{\text{individual excess of demand}} \quad .$$

Since $\boldsymbol{p} \gg \mathbf{0}$, $z : \mathbb{R}_{++}^L \to \mathbb{R}^L$.

Before addressing the main issue of this section, we present some properties of this function, very relevant in economic theory but also for the proof of the main result.

**Lemma 15. Maximum principle.** Let $\mathcal{X}$ and $\mathcal{Y}$ be two topological spaces and $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a continuous function with respect the product topology over $\mathcal{X} \times \mathcal{Y}$, and let $\Gamma : Y \rightrightarrows X$ be a compact-valued correspondence (see Lucas et al. (1988)) s.t. $\Gamma(y) \neq \emptyset \; \forall \, y \in \mathcal{Y}$. Let us define the value function $f^* : \mathcal{Y} \to \mathbb{R}$:

$$f^*(y) = \sup\{f(x,y) : \; x \in \Gamma(y)\}$$

and the set of maximizers $\Gamma^* : \mathcal{Y} \to \mathcal{X}$:

$$\Gamma^*(y) = \text{argmax}\{f(x,y) : \; x \in \Gamma(y)\}.$$

If $\Gamma$ is both upper and lower hemicontinuous at $y$, then $f^*$ is continuous and $\Gamma^*$ upper hemicontinuous, non empty and compact valued.

Lema 15 fits in the context of the utility maximization problem:

$$\begin{cases} \max & u(x) \\ \text{s.a.} & \boldsymbol{p} \cdot x \leq I \\ & x \in \mathbb{R}^L_+ . \end{cases}$$

Indeed, set $X = \mathbb{R}^L_+$ the space of commodities, $\mathcal{Y} = \mathbb{R}^L_{++} \times \mathbb{R}_{++}$ the space of prices: $(\boldsymbol{p}, I)$, $f(x,y) = u(x)$ the utility function and

$$\Gamma(y) = B(\boldsymbol{p}, I) = \{x \geq \mathbf{0} : \; \boldsymbol{p} \cdot x \leq I\}$$

the consumer budget set. Then:

1. The indirect utility function $v(\boldsymbol{p}, I)$ is continuous.
2. Marshallians demands $x^*(\boldsymbol{p}, I)$ are continuous.

**Proposition 16.** If $(\succeq_i, \omega_i)_{i=1}^I$ is a pure exchange economy s.t. $\overline{\omega} = \sum_{i=1}^I \omega_i \gg \mathbf{0}$ and $\succeq_i$ is continuous, strictly convex and strictly monotonic, then $z(\cdot)$ satisfies the following properties:

1. $z$ is continuous.
2. $z$ is homogeneous of degree zero: $z(\lambda \boldsymbol{p}) = z(\boldsymbol{p}) \; \forall \; \lambda > 0$.
3. $z$ satisfies Walras law: $\forall \; \boldsymbol{p} \in \mathbb{R}^L_{++} : \; \boldsymbol{p} \cdot z(\boldsymbol{p}) = 0$.

4. $\exists\ M > 0$ such that $\forall\ \ell = 1, ..., L$ and $\boldsymbol{p} \in \mathbb{R}^L_{++}$: $z_\ell(\boldsymbol{p}) > -M$.

*Proof.* Item by item:

1. The continuity of $u^i(\cdot)$ and properties of $B(\mathbf{p}, I)$ ensures by Lemma 15 the continuity of $x_i^*(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i)$ and therefore the continuity of $z$.

2. For each consumer $i = 1, ..., I$ its budget set

$$B(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i) = \{x_i \geq 0 : \boldsymbol{p} \cdot x_i \leq \boldsymbol{p} \cdot \omega_i\}$$

clearly remains unchanged if $\boldsymbol{p} \to \lambda \boldsymbol{p}$.

3. Since $\succeq_i$ is strictly monotonic for each consumer,

$$\forall\ i = 1, ..., I :\ \underbrace{\boldsymbol{p} \cdot x_i^*(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i)}_{\text{expenditure}} = \underbrace{\boldsymbol{p} \cdot \omega_i}_{\text{income from endowment}}.$$

Hence:

$$\begin{aligned} \sum_{i=1}^{I} \boldsymbol{p} \cdot x_i^*(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i) &= \sum_{i=1}^{I} \boldsymbol{p} \cdot \omega_i \\ \boldsymbol{p} \cdot \left( \sum_{i=1}^{I} x_i^*(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i) - \omega_i \right) &= 0 \\ \boldsymbol{p} \cdot z(\boldsymbol{p}) &= 0. \end{aligned}$$

4. Since $x_{\ell i}^*(\boldsymbol{p}, \boldsymbol{p} \cdot \omega_i)$ is positive for each consumer $i = 1, ..., I$ and good $\ell = 1, ..., L$:

$$z_\ell(\boldsymbol{p}) \geq -\overline{\omega}_\ell.$$

Let $M > \max_{\ell=1,...,L}\{\overline{\omega}_\ell\}$. Then, $z_\ell(\boldsymbol{p}) > -M$ for all $\ell$ and $\boldsymbol{p} \in \mathbb{R}^L_{++}$.

□

Another property that $z(\cdot)$ satisfies is that if $\{\boldsymbol{p}_n\}_{n \in \mathbb{N}} \subset \mathbb{R}^L_{++}$ converges to $\overline{\boldsymbol{p}} \neq \mathbf{0}$ such that there exists $\ell :\ \overline{p}_\ell = 0$, then

$$\max\{z_1(\boldsymbol{p}_n), ..., z_L(\boldsymbol{p}_n)\} \to \infty.$$

The proof can be found in Echenique (2023), we now focus on the main topic of this section: how the Brouwer fixed point theorem is applied in order to proof Walrasian equilibrium existence.

**Theorem 17. Existence of Walrasian equilibrium.** In the context of Proposition 16, for $z : \mathbb{R}^L_+ \to \mathbb{R}^L$ there exists $\boldsymbol{p}^*$ such that $z(\boldsymbol{p}^*) \leq \mathbf{0}$. Furthermore, if $z : \mathbb{R}^L_{++} \to \mathbb{R}^L$, there exists $\boldsymbol{p}^*$ such that $z(\boldsymbol{p}^*) = \mathbf{0}$.

The proof of this theorem relies on Theorem 1 (Brouwer). Nowadays the following is well known and can be found (following similar or very different approaches) in, for example, Mas-Colell et al. (1995), Varian (1992) or Ellickson (1993).

*Proof.* First, since $z$ is homogeneous of degree zero, we can restrict $\boldsymbol{p}$ to the $\Delta$ (also known as $n-$dimensional simplex), defined as follows:

$$\Delta = \left\{ \boldsymbol{p} \in \mathbb{R}^L_+ : \sum_{\ell=1}^{L} p_\ell = 1 \right\}.$$

This set is clearly convex and compact. Indeed, given $\boldsymbol{p}_1, \boldsymbol{p}_2 \in \Delta$,

$$\boldsymbol{p}_3 = \theta \boldsymbol{p}_1 + (1-\theta)\boldsymbol{p}_2 \in \Delta :$$

$$\begin{aligned} \sum_{\ell=1}^{L} p^3_\ell &= \sum_{\ell=1}^{L} \theta p^1_\ell + (1-\theta) p^2_\ell \\ &= \theta \sum_{\ell=1}^{L} p^1_\ell + (1-\theta) \sum_{\ell=1}^{L} p^2_\ell \\ &= \theta + (1-\theta) = 1. \end{aligned}$$

With respect to the compactness, $\Delta$ is closed since it is the intersection of the orthant $\mathbb{R}^L_+$ and the hyperplane $H\left((1,...,1),1\right)$. It is bounded since $\Delta \subset [0,1]^L$. Hence, since all of this occurs in $\mathbb{R}^L$, $\Delta$ is a compact set. It is therefore possible to apply Brouwer fixed point over $\Delta$. We would only need to prove that $z(\boldsymbol{p}) + \boldsymbol{p}$ maps $\Delta$ onto $\Delta$. However, this is not the case in general. This is where the following trick is employed, which allows us to conclude the matter using Brouwer's Fixed Point Theorem. Let us define $\Psi : \Delta \to \mathbb{R}^L$ defined as follows:

$$\Psi_\ell = \frac{p_\ell + \max\{0, z_\ell(\boldsymbol{p})\}}{1 + \sum_{\ell=1}^{L} \max\{0, z_\ell(\boldsymbol{p})\}}, \ \forall \ \ell = 1, ..., L.$$

Since $\sum_{\ell=1}^{L} p_\ell = 1$,

$$\sum_{\ell=1}^{L} \Psi_\ell = \sum_{\ell=1}^{L} \left\{ \frac{p_\ell + \max\{0, z_\ell(\boldsymbol{p})\}}{1 + \sum_{\ell=1}^{L} \max\{0, z_\ell(\boldsymbol{p})\}} \right\} = 1,$$

i.e., $\Psi(\Delta) \subset \Delta$. Hence, by Theorem 1, there exists $\boldsymbol{p}^*$ such that $\Psi(\boldsymbol{p}^*) = \boldsymbol{p}^*$. This yields to: $\forall\ \ell = 1, ..., L$

$$\begin{aligned}
p_\ell^* &= \frac{p_\ell^* + \max\{0, z_\ell(\boldsymbol{p}^*)\}}{1 + \sum_{\ell=1}^{L} \max\{0, z_\ell(\boldsymbol{p}^*)\}} \\
p_\ell^* \left(1 + \sum_{\ell=1}^{L} \max\{0, z_\ell(\boldsymbol{p}^*)\}\right) &= p_\ell^* + \max\{0, z_\ell(\boldsymbol{p}^*)\} \\
p_\ell^* \sum_{\ell=1}^{L} \max\{0, z_\ell(\boldsymbol{p}^*)\} &= \max\{0, z_\ell(\boldsymbol{p}^*)\} \\
z_\ell(\boldsymbol{p}^*) p_\ell^* \sum_{\ell=1}^{L} \max\{0, z_\ell(\boldsymbol{p}^*)\} &= z_\ell(\boldsymbol{p}^*) \max\{0, z_\ell(\boldsymbol{p}^*)\} \\
\underbrace{\sum_{\ell=1}^{L} z_\ell(\boldsymbol{p}^*) p_\ell^*}_{=0} \left[\sum_{\ell=1}^{L} \max\{0, z_\ell(\boldsymbol{p}^*)\}\right] &= \sum_{\ell=1}^{L} z_\ell(\boldsymbol{p}^*) \max\{0, z_\ell(\boldsymbol{p}^*)\}
\end{aligned}$$

Therefore,

$$\sum_{\ell=1}^{L} z_\ell(\boldsymbol{p}^*) \max\{0, z_\ell(\boldsymbol{p}^*)\} = 0. \tag{3}$$

Equation 3 points out that $z_\ell(\boldsymbol{p}^*) \leq 0,\ \forall\ \ell = 1, ..., L$. Finally, once again by Walras Law (Proposition 16) since we must have

$$\sum_{\ell=1}^{L} p_\ell^* z_\ell(\boldsymbol{p}^*) = 0 \tag{4}$$

with $p_\ell \geq 0$, combining (4) with Equation 3 we must have $p_\ell z_\ell(\boldsymbol{p}^*) = 0$ for all $\ell = 1, ..., L$. Finally, for $p_\ell > 0$, necessarily $z_\ell(\boldsymbol{p}^*) = 0$ for all $\ell = 1, ..., L$, concluding therefore the proof. □

Theorem 17 allows us to appreciate the power of Brouwer's fixed-point argument: it has been proven under very reasonable assumptions about consumer preferences that there exists a price vector that clears the market. The applications of general equilibrium theory are vast, as mentioned in Echenique and Wierman (2012). One of the main challenges is for instance to compute this equilibrium, which is of great interest in macroeconomics. To conclude this work, by way of an example, we will compute the vector of prices for Walrasian equilibrium when all consumers share the same preferences: Cobb-Douglas, with $I = L - 1$.

**Example 18.** Let us consider a pure exchange economy where all consumers have the same preferences: Cobb-Douglas

$$u^i(x_{1i}, ..., x_{Li}) = \prod_{i=1}^{L} x_{\ell i}^{\alpha_{\ell i}},$$

such that $\sum_{\ell=1}^{L} \alpha_{\ell i} = 1$. Let us denote, as usual, $\omega_{\ell i}$ the endowment of $\ell$ good of consumer $i$. The maximization problem faced by all consumers is

$$\mathcal{P}_i \begin{cases} \max & u^i(x_i) = \prod_{\ell=1}^{L} x_\ell^{\alpha_{\ell i}} \\ \text{s.a.} & \sum_{\ell=1}^{L} p_\ell x_{\ell i} = \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} \\ & x_{\ell i} \geq 0. \end{cases}$$

If we consider that the endowments are such that all consumers have a positive amount of all goods, there is $\epsilon > 0$ such that the problem might be rewritten as

$$\mathcal{P}_i \begin{cases} \max & u^i(x_i) = \prod_{\ell=1}^{L} x_\ell^{\alpha_{\ell i}} \\ \text{s.a.} & \sum_{\ell=1}^{L} p_\ell x_{\ell i} = \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} \\ & x_{\ell i} \geq \tilde{\epsilon}. \end{cases}$$

Since preferences are continuous, strictly convex (strict quasi-concavity of the utility function), and strictly monotonic

$$u^i\left(x_i + \epsilon \underbrace{\mathbf{v}}_{\in \mathbb{R}^L_{++}}\right) > u(x_i), \ \forall \ \epsilon > 0,$$

whenever $\overline{\omega} >> \mathbf{0}$, Theorem 17 ensures the existence of a Walrasian equilibrium for this particular economy. Let us find the equilibrium vector price.

Since $\ln(\cdot)$ is strictly increasing and concave, we might re-write $\mathcal{P}_i$ as follows:

$$\mathcal{P}_i \begin{cases} \max & \ln[u^i(x_i)] = \ln\left(\prod_{\ell=1}^{L} x_{\ell i}^{\alpha_{\ell i}}\right) = \sum_{\ell=1}^{L} \alpha_{\ell i} \ln x_{\ell i} \\ \text{s.a.} & \sum_{\ell=1}^{L} p_\ell x_{\ell i} = \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} \\ & x_{\ell i} \geq \epsilon. \end{cases}$$

In the optimization procedure, it can be shown that restrictions $x_{\ell i} \geq \epsilon$ will play no fundamental role. The associated Lagrangian is

$$\mathscr{L}(\{x_{\ell i}\}_{\ell=1}^{L}, \lambda) = \sum_{\ell=1}^{L} \alpha_{\ell i} \ln x_{\ell i} + \lambda \left[ \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} - \sum_{\ell=1}^{L} p_\ell x_{\ell i} \right].$$

First-order conditions will be enough to characterize the equilibrium in reason of the differentiability and strict concavity of the utility function:

$$\forall\ \ell = 1, ..., L: \ \frac{\partial \mathscr{L}}{\partial x_{\ell i}} = \frac{\alpha_{\ell i}}{x_{\ell i}} - \lambda p_\ell = 0.$$

Summing over $\ell$

$$\underbrace{\sum_{\ell=1}^{L} \alpha_{\ell i}}_{=1} = \sum_{\ell=1}^{L} \lambda p_\ell x_{\ell i} \implies \lambda = \frac{1}{\sum_{\ell=1}^{L} p_\ell x_{\ell i}} = \frac{1}{\sum_{\ell=1}^{L} p_\ell \omega_{\ell i j}}.$$

From this,

$$\forall\ i, \ell: \ x_{\ell i} = \frac{\alpha_{\ell i}}{p_\ell} \left( \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} \right).$$

Since the clearing marker conditions impose

$$\sum_{i=1}^{I} x_{\ell i} = \sum_{i=1}^{I} \omega_{\ell i}, \ \forall\ \ell = 1, ..., L$$

we have

$$\sum_{i=1}^{I} \alpha_{\ell i} \left( \sum_{\ell=1}^{L} p_\ell \omega_{\ell i} \right) = p_\ell \sum_{i=1}^{I} \omega_{\ell i}, \ \forall\ \ell = 1, ..., L. \tag{5}$$

After normalizing without loss of generality $p_1 = 1$, the right side of Equation 5 can be written as follows in a compact way

$$\underbrace{\begin{bmatrix} \sum_{i=1}^{I} \omega_{1i} & 0 & \cdots & 0 \\ 0 & \sum_{i=1}^{I} \omega_{2i} & & \vdots \\ \vdots & & \ddots & \\ 0 & & & \sum_{i=1}^{I} \omega_{Li} \end{bmatrix}}_{\boldsymbol{W}} \underbrace{\begin{bmatrix} \underbrace{p_1}_{=1} \\ p_2 \\ \vdots \\ p_L \end{bmatrix}}_{\tilde{\boldsymbol{p}}}.$$

On the other hand, the left side of Equation 5 can be put under matrix form too as follows (writing $\omega_{i\ell}$ and $\alpha_{i\ell}$)

$$\underbrace{\begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1L} \\ \alpha_{21} & \alpha_{22} & & \vdots \\ \vdots & & \ddots & \\ \alpha_{I1} & & & \alpha_{IL} \end{bmatrix}^T \begin{bmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1L} \\ \omega_{21} & \omega_{22} & & \vdots \\ \vdots & & \ddots & \\ \omega_{I1} & & & \omega_{IL} \end{bmatrix}} \begin{bmatrix} \underbrace{p_1}_{=1} \\ p_2 \\ \vdots \\ p_L \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \sum_{i=1}^{I} \alpha_{i1}\omega_{i1} & \sum_{i=1}^{I} \alpha_{i1}\omega_{i2} & \cdots & \sum_{i=1}^{I} \alpha_{i1}\omega_{iL} \\ \sum_{i=1}^{N} \alpha_{i2}\omega_{i1} & \ddots & & \vdots \\ \vdots & & \ddots & \\ \sum_{i=1}^{I} \alpha_{iL}\omega_{i1} & & & \sum_{j=1}^{L} \alpha_{iL}\omega_{iL} \end{bmatrix}}_{=\mathbf{\Sigma}}$$

If we denote $\mathbf{b}$ as the first column vector of matrix $\mathbf{\Sigma}$, excluding the first entry, $\mathbf{A}$ as the sub-matrix $I \times (L-1)$, which includes all columns except the first column of $\mathbf{\Sigma}$, $\overline{\mathbf{p}}$ as the truncated price vector, excluding $p_1 = 1$, and $\overline{\mathbf{W}}$ as the $(L-1) \times (L-1)$ matrix (excluding the first row and first column of $\mathbf{W}$), then:

$$\overline{\mathbf{p}} = (\overline{\mathbf{W}} - \mathbf{A})^{-1}\mathbf{b}, \; p_1 = 1.$$

It is easy to check that the conditions of Theorem 17 are satisfied, backing up our conclusion.

## 4 Conclusion

In this document, we have provided a proof of Borsuk's lemma for continuously differentiable retractions, as outlined in Laczkovich and Sos (2017) (which provides a sketch of the general argument). Theorem 5 follows as a direct result. We then presented and proved several results that allowed us to prove the general case of Brouwer Fixed Point Theorem. We followed the statements according to Ok (2007), providing our own constructions

After completing the proof of the BFPT, we delved into General Equilibrium Theory. The Walrasian existence theorem for pure exchange economies was proven, and we also presented an example which is not of standard character, illustrating the power of Theorem 17.

We hope the reader will find this document highly interesting and useful, especially for better understanding and applying how Brouwer's Fixed Point Theorem, after being proven in a slightly more restrictive case, is utilized in general equilibrium to derive one of the main results of the theory.

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