FACULTY OF SOCIAL SCIENCES
ECONOMICS MAJOR

Solutions to the 2nd Recitation

Microeconomics 2
Semester 2024-2

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1 Selected Exercises

Exercise 1.1. In each of the following cases, draw the Edgeworth box, some indifference curves for each consumer and find Walrasian (competitive) equilibrium in each case. **Later on, you should be able to find the Pareto set and the core (contract curve).**

a) $u_1(x_{11}, x_{21}) = 2x_{11}^2 x_{21}$, $u_2(x_{12}, x_{22}) = x_{12}x_{22}^3$, $\omega_1 = (2, 3)$ and $\omega_2 = (1, 2)$.

b) $u_1(x_{11}, x_{21}) = x_{11} + x_{21}$, $u_2(x_{12}, x_{22}) = \min\{x_{12}, x_{22}\}$, $\omega_1 = (1, 2)$ and $\omega_2 = (3, 4)$.

c) $u_1(x_{11}, x_{21}) = x_{11} + \ln x_{21}$, $u_2(x_{12}, x_{22}) = x_{12} + 2\ln x_{22}$, $\omega_1 = (2, 3)$ and $\omega_2 = (1, 2)$.

d) $u_1(x_{11}, x_{21}) = x_{11}x_{21}$, $u_2(x_{12}, x_{22}) = \min\{x_{12}, x_{22}\}$, $\omega_1 = (2, 6)$ and $\omega_2 = (4, 1)$.

e) $u_1(x_{11}, x_{21}) = \min\{2x_{11}, x_{21}\}$, $u_2(x_{12}, x_{22}) = \min\{x_{12}, 2x_{22}\}$, $\omega_1 = (1, 2)$ and $\omega_2 = (3, 4)$.

f) $u_1(x_{11}, x_{21}) = 3x_{11} + x_{21}$, $u_2(x_{12}, x_{22}) = x_{12} + 3x_{22}$, $\omega_1 = (2, 2)$ and $\omega_2 = (2, 2)$.

Identify whenever it is possible the type (Cobb-Douglas, CES, Leontief, linear...) of the utility function.

Solution: (a). We use $x_{11} = x_1, x_{21} = y_1, x_{12} = x_2$, and $x_{22} = y_2$. Below, we plot the initial endowments $\{(2,3),(1,2)\}$, some indifference curves:

$$x_{21} = \frac{\overline{U}_1}{2x_{11}^2},\ \overline{U}_1 \in \mathbb{R}_+$$

$$x_{22} = \sqrt[3]{\frac{\overline{U}_2}{x_{12}}},\ \overline{U}_2 \in \mathbb{R}_+$$

the curve Γ of Pareto optima (points of tangency between the marginal rates of substitution: $x_{21} = \frac{5x_{11}}{18-5x_{11}}$), the core (the intersection of Γ with the mutually beneficial zone), the equilibrium consumptions, and the corresponding budget line (see question 2 for the numerical values of the ratio and the demands):

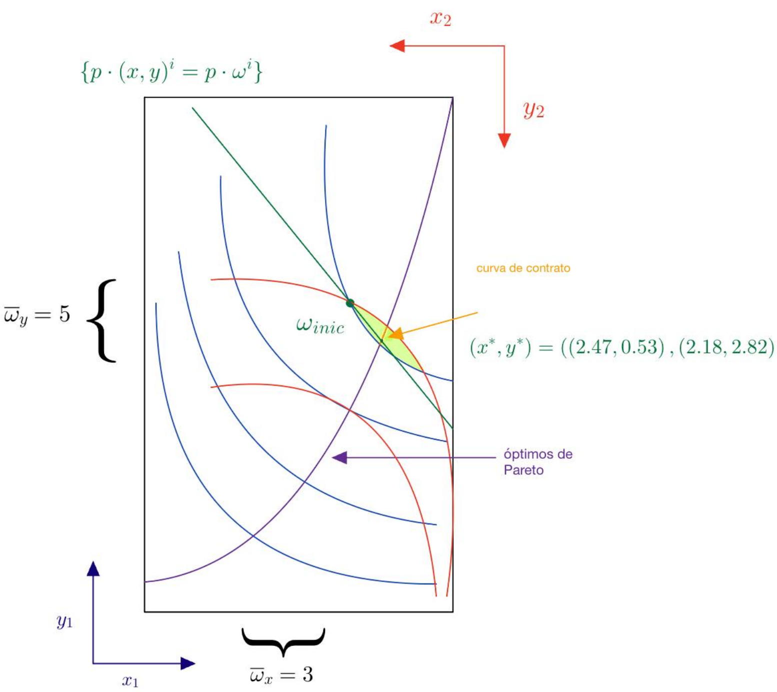


Figure 1: Complete situation.

Note that the indifference curves are asymptotic to their respective axes due to the specifications u^i. For the sake of precision, let us provide the same graph using Python:

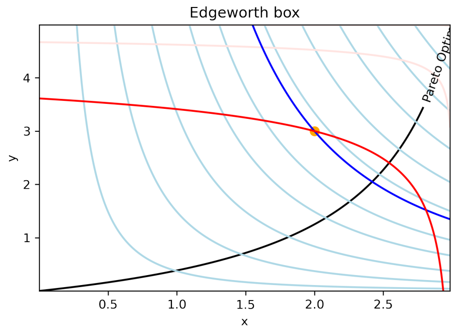


Figure 2: Indifference curves, Γ, and $\overline{\omega}$.

Given that the utility functions in question are differentiable and the solution can be on the boundary[1], the Pareto optima are characterized by the following two conditions:

$$\underbrace{\frac{\partial_{x_{11}} u^1}{\partial_{x_{21}} u^1} = \frac{\partial_{x_{12}} u^2}{\partial_{x_{22}} u^2}}_{\text{tangency condition}}$$

$$\begin{pmatrix} x_{11} + x_{12} \\ x_{21} + x_{22} \end{pmatrix} = \begin{pmatrix} \omega_x \\ \omega_y \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}. \tag{1}$$

Indeed, we need to solve:

$$\begin{aligned} &\max \ u_i(x_i, y^i) \\ &\text{s.t. } u_{-i}(x^{-i}, y^{-i}) \geq \overline{u}. \end{aligned}$$

We then compute the ratios of the marginal utilities:

$$\frac{4x_{11}x_{21}}{2x_{11}} = \frac{x_{22}^3}{3x_{12}x_{22}^2}.$$

Simplifying:

$$\frac{2x_{21}}{x_{11}} = \frac{x_{22}}{3x_{12}}.$$

Using (1)

$$\frac{2x_{21}}{x_{11}} = \frac{5 - x_{21}}{3(3 - x_{11})}.$$

Solving for x_{21} in terms of x_{11}, we obtain

$$x_{21} = \frac{5x_{11}}{18 - x_{11}}. \tag{2}$$

In Figure 3, we plot the Pareto optima (Equation 2) for (x_{11}, x_{21}) in the Edgeworth box $\square = [0, 3] \times [0, 5]$.

[1] If any of the utility functions is evaluated at a vector with 0 units of one of the 2 goods, the utility equals 0, which is less than $u^i(\omega^i) > 0$.

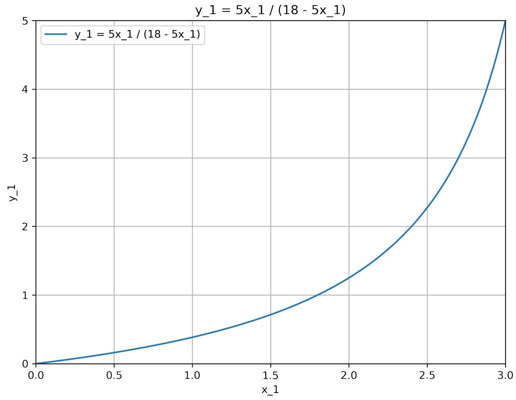


Figure 3: Pareto optima.

Now, to obtain the equilibrium prices and allocations, we analyze the problem from the market perspective, where each individual solves:

$$\mathcal{P}_i : \begin{cases} \max & u^i(x_{1i}, x_{2i}) \\ \text{s.t.} & p_1 x_{1i} + p_2 x_{2i} \leq \underbrace{p_1 \omega_{1i} + p_2 \omega_{2i}}_{\text{budget constraint}} \\ & x_{1i}, x_{2i} \geq 0. \end{cases}$$

Since the utility functions are increasing in both goods (first partial derivatives are positive), the constraint holds with equality and $x_{1i}, x_{2i} > 0$. We then apply the first-order conditions to the associated Lagrangian. For consumer 1, we have:

$$\mathscr{L}(x_{11}, x_{21}, \lambda) = \underbrace{2x_{11}^2 x_{21}}_{u_1(x_{11}, x_{21})} + \lambda(2p_1 + 3p_2 - p_1 x_{11} - p_2 x_{21}).$$

Then,

$$\frac{\partial \mathscr{L}}{\partial x_{11}} = 4x_{11}x_{21} - \lambda p_1 = 0$$
$$\frac{\partial \mathscr{L}}{\partial x_{21}} = 2x_{11}^2 - \lambda p_2 = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = 2p_1 + 3p_2 - p_1 x_{11} - p_2 x_{21} = 0.$$

Combining the first two equations, we obtain:

$$\frac{2x_{21}}{x_{11}} = \frac{p_1}{p_2}.$$

Thus,

$$x_{21} = \frac{x_{11}}{2} \frac{p_1}{p_2}.$$

Substituting into the budget constraint:

$$p_1 x_{11} + p_2 \left(\frac{x_{11}}{2} \frac{p_1}{p_2} \right) = 2p_1 + 3p_2$$

and solving for x_{11}, we finally obtain the Marshallian demands for consumer 1:

$$x_{11}(p_1, p_2) = \frac{4}{3} + 2\left(\frac{p_2}{p_1}\right) = \underbrace{\frac{2}{3}\left[\frac{2p_1 + 3p_2}{p_1}\right]}_{\frac{\alpha}{\alpha+\beta}\frac{I}{p_1}}$$

$$x_{21}(p_1, p_2) = 1 + \frac{2}{3}\left(\frac{p_1}{p_2}\right) = \underbrace{\frac{1}{3}\left[\frac{2p_1 + 3p_2}{p_2}\right]}_{=\frac{\beta}{\alpha+\beta}\frac{I}{p_2}}.$$

Solving similarly for consumer 2:

$$\mathscr{L}(x_{12}, x_{22}, \lambda) = \underbrace{x_{12}x_{22}^3}_{u_2(x_{12},x_{22})} + \lambda(p_1 + 2p_2 - p_1x_{12} - p_2x_{22})$$

$$\frac{\partial \mathscr{L}}{\partial x_{12}} = x_{22}^3 - \lambda p_1 = 0$$
$$\frac{\partial \mathscr{L}}{\partial x_{22}} = 3x_{12}x_{22}^2 - \lambda p_2 = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = p_1 + 2p_2 - p_1x_{12} - p_2x_{22} = 0.$$

Using the first two equations:

$$\frac{x_{22}}{3x_{12}} = \frac{p_1}{p_2}. \tag{3}$$

Substituting into $p_1 + 2p_2 = p_1x_{12} + p_2x_{22} = 0$:

$$p_1x_{12} + p_2\left(\frac{3p_1x_{12}}{p_2}\right) = p_1 + 2p_2.$$

Solving for x_{12} and substituting into 3

$$x_{12}(p_1, p_2) = \frac{1}{4}\left[\frac{p_1 + 2p_2}{p_1}\right]$$
$$x_{22}(p_1, p_2) = \frac{3}{4}\left[\frac{p_1 + 2p_2}{p_2}\right].$$

Note that, informally, by identifying the coefficients α, β, given the Cobb-Douglas structure: $u(x, y) = Ax^\alpha y^\beta$, we could directly recover the Marshallian demands: $\left(\frac{\alpha I}{(\alpha+\beta)p_1}, \frac{\beta I}{(\alpha+\beta)p_2}\right)$. These α and β are obtained by applying a monotonic transformation $g(\cdot)$ to u^i (e.g., $g(t) = t^{1/3}$ or $g(t) = t^{1/4}$).

To obtain the equilibrium price ratio, we must impose the clearing market condition. That is:

$$x_{11}(p) + x_{12}(p) - \overline{\omega}_x = \frac{2}{3}\left[\frac{2p_1 + 3p_2}{p_1}\right] + \frac{1}{4}\left[\frac{p_1 + 2p_2}{p_1}\right] - 3$$
$$x_{21}(p) + x_{22}(p) - \overline{\omega}_y = \frac{1}{3}\left[\frac{2p_1 + 3p_2}{p_2}\right] + \frac{3}{4}\left[\frac{p_1 + 2p_2}{p_1}\right] - 5.$$

Applying Walras' Law, it suffices to balance one of the markets:

$$\frac{4}{3} + \frac{2p_2}{p_1} + \frac{1}{4} + \frac{p_2}{2p_1} - 3 = 0.$$

This yields the ratio: $\frac{p_2}{p_1} = \frac{17}{30}$ (remember that in general equilibrium, what matters is the ratio, not the numerical value of each price; we can eventually normalize one to 1). Substituting into the demand functions, we obtain (numerically approximated to 10^{-2}):

$$\begin{aligned} x_{11} &\simeq 2.47 \\ x_{21} &\simeq 2.18 \\ x_{12} &\simeq 0.53 \\ x_{22} &\simeq 2.82. \end{aligned}$$

Finally, we must verify that these allocations are Pareto optimal. This is consistent with the fact that the consumers' preferences $\preceq$, represented by the utility functions $u(\cdot)$, are increasing in their arguments (monotonic preferences[2] hence): this is the only necessary condition in the First Welfare Theorem. We verify that the Walrasian equilibrium belongs to Γ because:

$$\underbrace{\frac{5 \cdot 2.47}{18 - 5 \cdot 2.47}}_{\Gamma_{x_{11}^*}} \simeq \underbrace{2.18}_{=x_{21}^*}.$$

Let us conclude the question by corroborating all what has being done using the Python library Edgeworth:

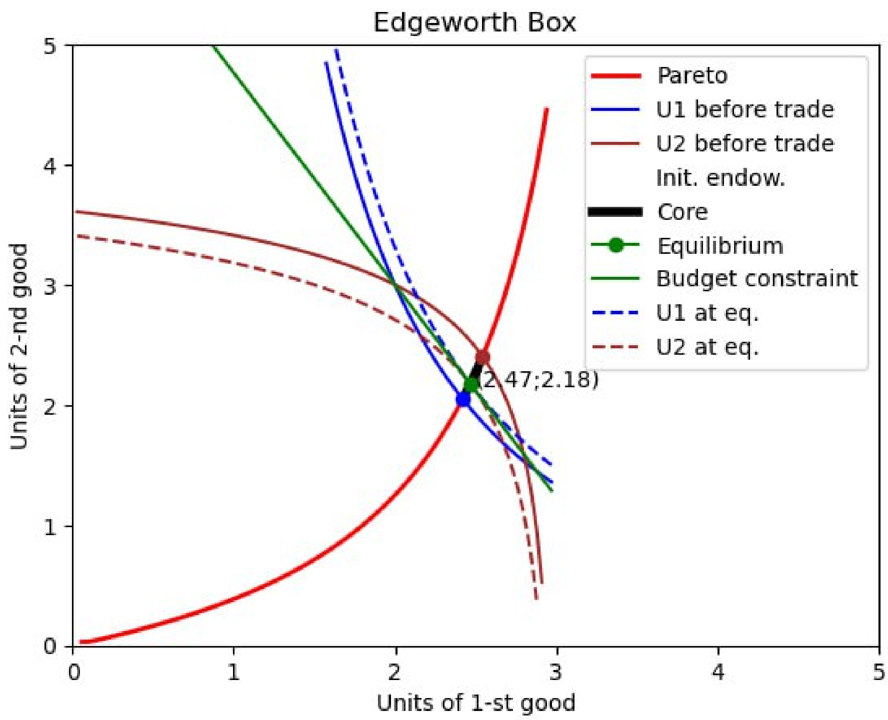
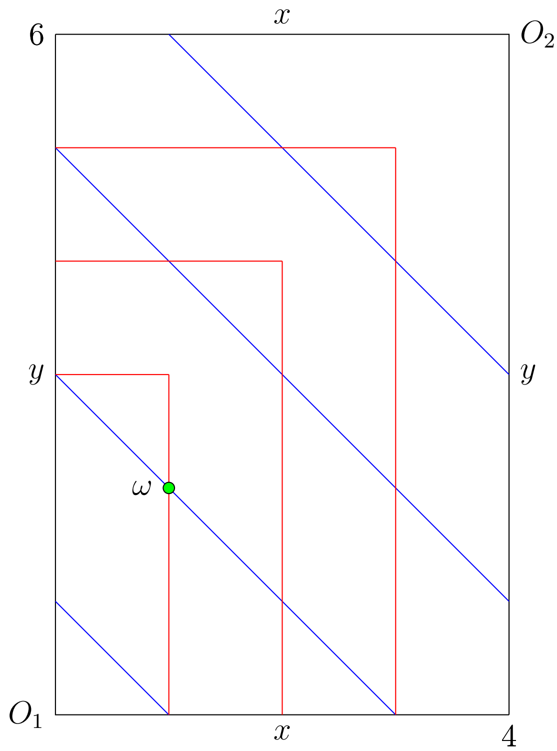


Figure 4: Summary of the Edgeworth box: $u_1(x_{11}, x_{21}) = 2x_{11}^2 x_{21}$ and $\omega^1 = (2, 3)$ $u_2(x_{12}, x_{22}) = x_{12}x_{22}^3$ and $\omega^2 = (1, 2)$.

[2] A.k.a. locally non-satiated preferences.

Solution: (b). We use $x_{11} = x_1, x_{21} = y_1, x_{12} = x_2$, and $x_{22} = y_2$. Below, we plot the initial endowments $\{(1,2),(3,4)\}$ and some indifference curves in the Edgeworth Box:



To obtain the equilibrium prices and allocations, we analyze the problem from the market perspective, where each individual solves:

$$\mathcal{P}_i : \begin{cases} \max & u^i(x_{1i}, x_{2i}) \\ \text{s.t.} & p_1 x_{1i} + p_2 x_{2i} \leq \underbrace{p_1\omega_{1i} + p_2\omega_{2i}}_{\text{budget constraint}} \\ & x_{1i}, x_{2i} \geq 0. \end{cases}$$

Note that consumer 1's utility function is linear, particularly that of perfect substitutes, so the Marshallian demands for consumer 1 are:

$$x_{11}(p_1,p_2) : \begin{cases} 0 & \text{if } 1 < \frac{p_1}{p_2} \\ [0, \frac{p_1+2p_2}{p_1}] & \text{if } 1 = \frac{p_1}{p_2} \\ \frac{p_1+2p_2}{p_1} & \text{if } 1 > \frac{p_1}{p_2} \end{cases}$$

$$x_{21}(p_1,p_2) : \begin{cases} 0 & \text{if } 1 > \frac{p_1}{p_2} \\ \frac{p_1+2p_2}{p_2} - \frac{p_1}{p_2}x_{11}(p_1,p_2) & \text{if } 1 = \frac{p_1}{p_2} \\ \frac{p_1+2p_2}{p_2} & \text{if } 1 < \frac{p_1}{p_2} \end{cases}$$

Consumer 2 has a Leontief utility function, so the Marshallian demands for consumer 2 are:

$$x_{12}(p_1, p_2) = \frac{3p_1 + 4p_2}{p_1 + p_2}$$
$$x_{22}(p_1, p_2) = \frac{3p_1 + 4p_2}{p_1 + p_2}$$

The equilibrium depends on the price ratio, let us impose the market cleaning condition and apply Walras' law for all cases. If $\frac{p_1}{p_2} < 1$:

$$\frac{3p_1 + 4p_2}{p_1 + p_2} - 6 = 0$$
$$3p_1 + 4p_2 = 6p_1 + 6p_2$$
$$3p_1 + 2p_2 = 0$$

Hence there would have to be a negative price, which is not possible, so this is not a scenario conducive to equilibrium.

Alternatively, if $\frac{p_1}{p_2} > 1$:

$$\frac{3p_1 + 4p_2}{p_1 + p_2} - 4 = 0$$
$$3p_1 + 4p_2 = 4p_1 + 4p_2$$
$$p_1 = 0$$

This scenario is also not conducive to equilibrium.

Finally, if $\frac{p_1}{p_2} = 1$, let's replace the ratio in the Marshallian demands and see if the market cleaning condition is met:

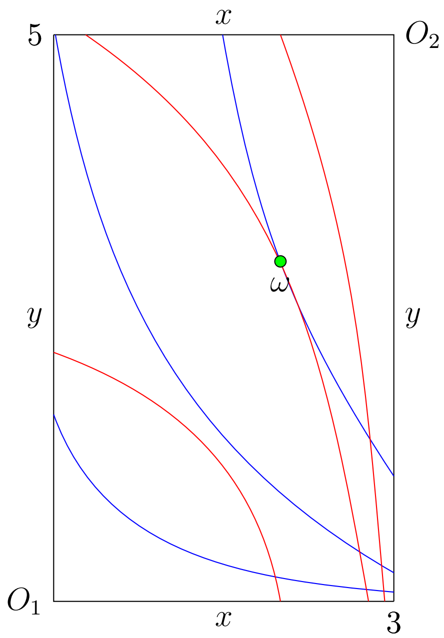
$$x_{12}(1, 1) = 3.5$$
$$x_{22}(1, 1) = 3.5$$

Therefore:

$$x_{11}(1, 1) = 0.5$$
$$x_{21}(1, 1) = 2.5$$

The market cleaning conditions are met, so we have an equilibrium when $p_1 = p_2$.

Solution: (c). We use $x_{11} = x_1, x_{21} = y_1, x_{12} = x_2$, and $x_{22} = y_2$. Below, we plot the initial endowments $\{(2,3),(1,2)\}$ and some indifference curves in the Edgeworth Box:



To obtain the equilibrium prices and allocations, we analyze the problem from the market perspective, where each individual solves:

$$\mathcal{P}_i : \begin{cases} \max & u^i(x_{1i}, x_{2i}) \\ \text{s.t.} & p_1 x_{1i} + p_2 x_{2i} \leq \underbrace{p_1\omega_{1i} + p_2\omega_{2i}}_{\text{budget constraint}} \\ & x_{1i}, x_{2i} \geq 0. \end{cases}$$

Since the utility functions are increasing in both goods (first partial derivatives are positive), the constraint holds with equality and $x_{1i}, x_{2i} > 0$. We then apply the first-order conditions to the associated Lagrangian. For consumer 1, we have:

$$\mathscr{L}(x_{11}, x_{21}, \lambda) = \underbrace{x_{11} + \ln x_{21}}_{u_1(x_{11}, x_{21})} + \lambda(2p_1 + 3p_2 - p_1 x_{11} - p_2 x_{21}).$$

Then,

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial x_{11}} &= 1 - \lambda p_1 = 0 \\ \frac{\partial \mathscr{L}}{\partial x_{21}} &= \frac{1}{x_{21}} - \lambda p_2 = 0 \\ \frac{\partial \mathscr{L}}{\partial \lambda} &= 2p_1 + 3p_2 - p_1 x_{11} - p_2 x_{21} = 0. \end{aligned}$$

Combining the first two equations, we obtain:

$$x_{21} = \frac{p_1}{p_2}.$$

Substituting into the budget constraint:

$$p_1 x_{11} + p_2 \left(\frac{p_1}{p_2}\right) = 2p_1 + 3p_2$$

and solving for x_{11}, we finally obtain the Marshallian demands for consumer 1:

$$x_{11}(p_1, p_2) = \frac{p_1 + 3p_2}{p_1}$$
$$x_{21}(p_1, p_2) = \frac{p_1}{p_2}.$$

Solving similarly for consumer 2:

$$\mathscr{L}(x_{12}, x_{22}, \lambda) = \underbrace{x_{12} + 2\ln x_{22}}_{u_2(x_{12}, x_{22})} + \lambda(p_1 + 2p_2 - p_1 x_{12} - p_2 x_{22})$$

$$\frac{\partial \mathscr{L}}{\partial x_{12}} = 1 - \lambda p_1 = 0$$
$$\frac{\partial \mathscr{L}}{\partial x_{22}} = \frac{2}{x_{22}} - \lambda p_2 = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = p_1 + 2p_2 - p_1 x_{12} - p_2 x_{22} = 0.$$

Using the first two equations:

$$\frac{x_{22}}{2} = \frac{p_1}{p_2}. \tag{4}$$
$$x_{22} = \frac{2p_1}{p_2}. \tag{5}$$

Substituting into the budget constraint:

$$p_1 x_{12} + p_2 \left(\frac{2p_1}{p_2}\right) = p_1 + 2p_2$$

and solving for x_{12}, we finally obtain the Marshallian demands for consumer 1:

$$x_{12}(p_1, p_2) = \frac{2p_2 - p_1}{p_1}$$
$$x_{22}(p_1, p_2) = \frac{2p_1}{p_2}.$$

To obtain the equilibrium price ratio, we must impose the clearing market condition. That is:

$$x_{11}(p) + x_{12}(p) - \overline{\omega}_x = \frac{p_1 + 3p_2}{p_1} + \frac{2p_2 - p_1}{p_1} - 3$$
$$x_{21}(p) + x_{22}(p) - \overline{\omega}_y = \frac{p_1}{p_2} + \frac{2p_1}{p_2} - 5.$$

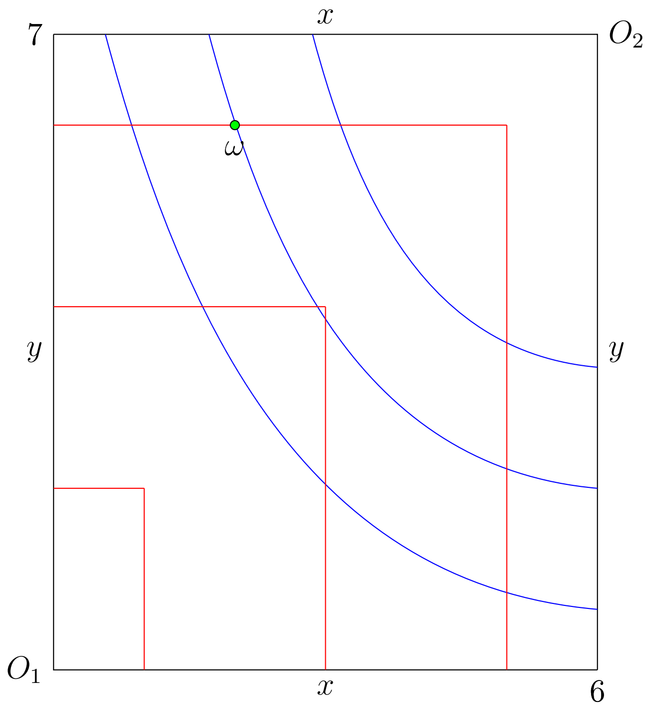
Applying Walras' Law, it suffices to balance one of the markets:

$$\frac{p_1}{p_2} + \frac{2p_1}{p_2} - 5 = 0$$

This yields the ratio: $\frac{p_2}{p_1} = \frac{3}{5}$ (remember that in general equilibrium, what matters is the ratio, not the numerical value of each price; we can eventually normalize one to 1). Substituting into the demand functions, we obtain:

$$x_{11} = \frac{14}{5}$$
$$x_{21} = \frac{5}{3}$$
$$x_{12} = \frac{1}{5}$$
$$x_{22} = \frac{10}{3}$$

Solution: (d). We use $x_{11} = x_1, x_{21} = y_1, x_{12} = x_2$, and $x_{22} = y_2$. Below, we plot the initial endowments $\{(2,6),(4,1)\}$ and some indifference curves in the Edgeworth Box:



To obtain the equilibrium prices and allocations, we analyze the problem from the market perspective, where each individual solves:

$$\mathcal{P}_i : \begin{cases} \max & u^i(x_{1i}, x_{2i}) \\ \text{s.t.} & p_1 x_{1i} + p_2 x_{2i} \leq \underbrace{p_1\omega_{1i} + p_2\omega_{2i}}_{\text{budget constraint}} \\ & x_{1i}, x_{2i} \geq 0. \end{cases}$$

Since the utility function for consumer 1 is increasing in both goods (first partial derivatives are positive), the constraint holds with equality and $x_{11}, x_{21} > 0$. We then apply the first-order conditions to the associated Lagrangian. For consumer 1, we have:

$$\mathscr{L}(x_{11}, x_{21}, \lambda) = \underbrace{x_{11}x_{21}}_{u_1(x_{11},x_{21})} + \lambda(2p_1 + 6p_2 - p_1x_{11} - p_2x_{21}).$$

Then,

$$\frac{\partial \mathscr{L}}{\partial x_{11}} = x_{21} - \lambda p_1 = 0$$
$$\frac{\partial \mathscr{L}}{\partial x_{21}} = x_{11} - \lambda p_2 = 0$$
$$\frac{\partial \mathscr{L}}{\partial \lambda} = 2p_1 + 6p_2 - p_1 x_{11} - p_2 x_{21} = 0.$$

Combining the first two equations, we obtain:

$$\frac{x_{21}}{x_{11}} = \frac{p_1}{p_2}.$$

Thus,

$$x_{21} = \frac{p_1}{p_2} x_{11}.$$

Substituting into the budget constraint:

$$p_1 x_{11} + p_2 \left(\frac{p_1}{p_2} x_{11} \right) = 2p_1 + 6p_2$$
$$p_1 x_{11} + p_1 x_{11} = 2p_1 + 6p_2$$
$$2p_1 x_{11} = 2p_1 + 6p_2$$

and solving for x_{11}, we finally obtain the Marshallian demands for consumer 1:

$$x_{11}(p_1, p_2) = \frac{2p_1 + 6p_2}{2p_1}$$
$$x_{21}(p_1, p_2) = \frac{2p_1 + 6p_2}{2p_2}.$$

Consumer 2 has a Leontief utility function, so the Marshallian demands for consumer 2 are:

$$x_{12}(p_1, p_2) = \frac{4p_1 + p_2}{p_1 + p_2}$$
$$x_{22}(p_1, p_2) = \frac{4p_1 + p_2}{p_1 + p_2}$$

To obtain the equilibrium price ratio, we must impose the clearing market condition. That is:

$$x_{11}(p) + x_{12}(p) - \overline{\omega}_x = \frac{2p_1 + 6p_2}{2p_1} + \frac{4p_1 + p_2}{p_1 + p_2} - 6$$
$$x_{21}(p) + x_{22}(p) - \overline{\omega}_y = \frac{2p_1 + 6p_2}{2p_2} + \frac{4p_1 + p_2}{p_1 + p_2} - 7.$$

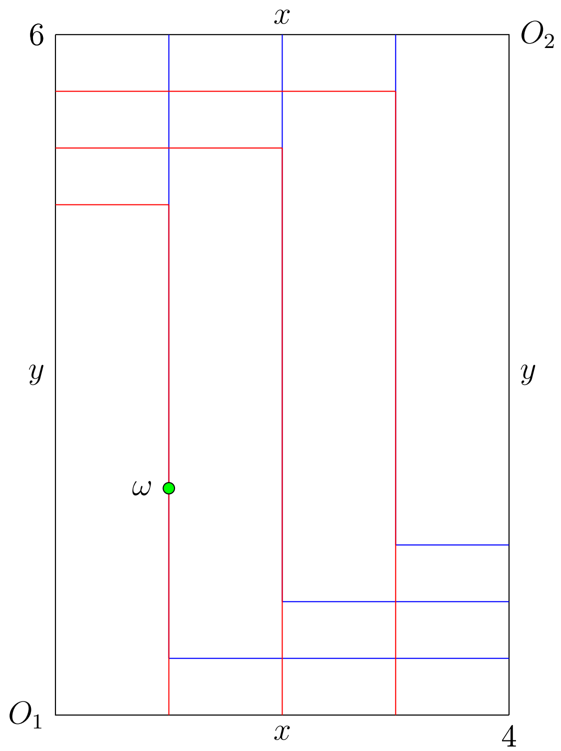
Applying Walras' Law, it suffices to balance one of the markets:

$$\frac{2p_1 + 6p_2}{2p_1} + \frac{4p_1 + p_2}{p_1 + p_2} - 6 = 0.$$

This yields the ratio: $\frac{p_2}{p_1} \simeq 0.768$ (remember that in general equilibrium, what matters is the ratio, not the numerical value of each price; we can eventually normalize one to 1). Substituting into the demand functions, we obtain (numerically approximated to 10^{-3}):

$$\begin{aligned} x_{11} &\simeq 3.304 \\ x_{21} &\simeq 4.302 \\ x_{12} &\simeq 2.697 \\ x_{22} &\simeq 2.697. \end{aligned}$$

Solution: (e). We use $x_{11} = x_1, x_{21} = y_1, x_{12} = x_2$, and $x_{22} = y_2$. Below, we plot the initial endowments $\{(1,2),(3,4)\}$ and some indifference curves in the Edgeworth Box:



To obtain the equilibrium prices and allocations, we analyze the problem from the market perspective, where each individual solves:

$$\mathcal{P}_i : \begin{cases} \max & u^i(x_{1i}, x_{2i}) \\ \text{s.t.} & p_1 x_{1i} + p_2 x_{2i} \leq \underbrace{p_1 \omega_{1i} + p_2 \omega_{2i}}_{\text{budget constraint}} \\ & x_{1i}, x_{2i} \geq 0. \end{cases}$$

Both consumers have Leontief utility functions, so the Marshallian demands for consumer 1 are:

$$\begin{aligned} x_{11}(p_1, p_2) &= \frac{p_1 + 2p_2}{p_1 + 2p_2} = 1 \\ x_{21}(p_1, p_2) &= \frac{2p_1 + 4p_2}{p_1 + 2p_2} \end{aligned}$$

And, similarly, for consumer 2:

$$x_{12}(p_1,p_2) = \frac{6p_1+8p_2}{2p_1+p_2}$$
$$x_{22}(p_1,p_2) = \frac{3p_1+4p_2}{2p_1+p_2}$$

To obtain the equilibrium price ratio, we must impose the clearing market condition. That is:

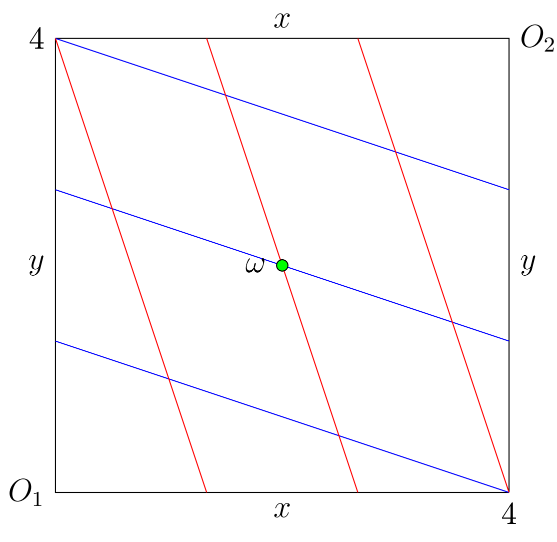
$$x_{11}(p)+x_{12}(p)-\overline{\omega}_x = 1+\frac{6p_1+8p_2}{2p_1+p_2}-4$$
$$x_{21}(p)+x_{22}(p)-\overline{\omega}_y = \frac{2p_1+4p_2}{p_1+2p_2}+\frac{3p_1+4p_2}{2p_1+p_2}-6.$$

Applying Walras' Law, it suffices to balance one of the markets:

$$1+\frac{6p_1+8p_2}{2p_1+p_2}-4=0$$
$$\frac{6p_1+8p_2}{2p_1+p_2}=3$$
$$6p_1+8p_2=6p_1+3p_2$$

Since p_2 would have to equal 0 (and p_1 would also equal 0 if we verify in the other market), we conclude that there is no equilibrium.

Solution: (f). We use $x_{11}=x_1, x_{21}=y_1, x_{12}=x_2$, and $x_{22}=y_2$. Below, we plot the initial endowments $\{(2,2),(2,2)\}$ and some indifference curves in the Edgeworth Box:



To obtain the equilibrium prices and allocations, we analyze the problem from the market perspective, where each individual solves:

$$\mathcal{P}_i : \begin{cases} \max & u^i(x_{1i},x_{2i}) \\ \text{s.t.} & p_1x_{1i}+p_2x_{2i} \leq \underbrace{p_1\omega_{1i}+p_2\omega_{2i}}_{\text{budget constraint}} \\ & x_{1i},x_{2i}\geq 0. \end{cases}$$

Note that both consumers have linear utility functions, so the Marshallian demands for consumer 1 are:

$$x_{11}(p_1,p_2) : \begin{cases} 0 & \text{if } 3 < \frac{p_1}{p_2} \\ [0, \frac{2p_1+2p_2}{p_1}] & \text{if } 3 = \frac{p_1}{p_2} \\ \frac{2p_1+2p_2}{p_1} & \text{if } 3 > \frac{p_1}{p_2} \end{cases}$$

$$x_{21}(p_1,p_2) : \begin{cases} 0 & \text{if } 3 > \frac{p_1}{p_2} \\ \frac{2p_1+2p_2}{p_2} - x_{11}(p_1,p_2) & \text{if } 3 = \frac{p_1}{p_2} \\ \frac{2p_1+2p_2}{p_2} & \text{if } 3 < \frac{p_1}{p_2} \end{cases}$$

Similarly, the Marshallian demands for consumer 2 are:

$$x_{12}(p_1,p_2) : \begin{cases} 0 & \text{if } \frac{1}{3} < \frac{p_1}{p_2} \\ [0, \frac{2p_1+2p_2}{p_1}] & \text{if } \frac{1}{3} = \frac{p_1}{p_2} \\ \frac{2p_1+2p_2}{p_1} & \text{if } \frac{1}{3} > \frac{p_1}{p_2} \end{cases}$$

$$x_{22}(p_1,p_2) : \begin{cases} 0 & \text{if } \frac{1}{3} > \frac{p_1}{p_2} \\ \frac{2p_1+2p_2}{p_2} - x_{11}(p_1,p_2) & \text{if } \frac{1}{3} = \frac{p_1}{p_2} \\ \frac{2p_1+2p_2}{p_2} & \text{if } \frac{1}{3} < \frac{p_1}{p_2} \end{cases}$$

Here, the equilibrium depends on the price ratio, particularly, if $\frac{1}{3} < \frac{p_1}{p_2} < 3$:

$$\begin{aligned} x_{11} &= 0 \\ x_{21} &= \frac{2p_1+2p_2}{p_1} \\ x_{12} &= \frac{2p_1+2p_2}{p_2} \\ x_{21} &= 0 \end{aligned}$$

This means in equilibrium consumer 1 demands the total amount of good x_2 in the economy and consumer 2 demands the total amount of good x_1. Particularly, for both x_{21} and x_{12} to equal 4 (the total endowment in the economy), the prices would have to be equal $\left(\frac{p_1}{p_2} = 1\right)$.

Exercise 1.2. From Mas-Colell et al. (1995). Consider a 2×2 economy in which consumers preferences are monotonic. Prove that (here below $\omega_\ell = \omega_{1\ell} + \omega_{2\ell}$)

$$p_1 \left(\sum_{i=1}^{2} x_{1i}(p_1,p_2) - \omega_1 \right) + p_2 \left(\sum_{i=1}^{2} x_{2i}(p_1,p_2) - \omega_2 \right) = 0.$$

Use this to explain Walras law, *if one market clears the other too.*

Solution: The budget constraints of each consumer are

$$p_1 x_{i1}(p_1,p_2) + p_2 x_{i2}(p_1,p_2) \leq p_1\omega_{i1} + p_2\omega_{i2}.$$

Now, assume that the inequality is strict for some i. That is,

$$p_1 x_{i1}(p_1,p_2) + p_2 x_{i2}(p_1,p_2) < p_1\omega_{i1} + p_2\omega_{i2}.$$

Since preferences are monotonic, they are also locally non-satiated. Therefore, given $\epsilon > 0$, we can find $(z_{i1}, z_{i2}) \in B((x_{i1}(p_1,p_2), x_{i2}(p_1,p_2)), \epsilon)$ such that

$$(z_{i1}, z_{i2}) \succ_i (x_{i1}(p_1,p_2), x_{i2}(p_1,p_2)).$$

and

$$(p_1,p_2)\cdot(z_{i1},z_{i2}) < (p_1,p_2)\cdot(\omega_{i1},\omega_{i2}).$$

This is a contradiction since, by definition,

$$x_i(p_1,p_2) \succeq_i z_i,\ \forall\ z_i \in B_i(p).$$

Therefore,

$$p_1 x_{i1}(p_1,p_2) + p_2 x_{i2}(p_1,p_2) = p_1\omega_{i1} + p_2\omega_{i2}.$$

Summing over i,

$$\sum_{i=1}^{2} p_1 x_{i1}(p_1,p_2) + p_2 x_{i2}(p_1,p_2) = \sum_{i=1}^{2} p_1\omega_{i1} + p_2\omega_{i2}.$$

Re-arranging the terms, we conclude. Finally, assume, without loss of generality, that market one clears:

$$p_1\left(\sum_{i=1}^{2} x_{1i}(p_1,p_2) - \omega_1\right) = 0.$$

Then,

$$\underbrace{p_1\left(\sum_{i=1}^{2} x_{1i}(p_1,p_2) - \omega_1\right)}_{=0} + p_2\left(\sum_{i=1}^{2} x_{2i}(p_1,p_2) - \omega_2\right) = 0$$

implies

$$p_2\left(\sum_{i=1}^{2} x_{2i}(p_1,p_2) - \omega_2\right) = 0.$$

This shows that when preferences are locally non-satiated, Walras' Law holds, and only one market needs to be cleared.

Exercise 1.3. From Mas-Colell et al. (1995). Consider and Edgeworth box economy in which each consumer has Cobb-Douglas preferences

$$\begin{aligned} u_1(x_{11}, x_{21}) &= x_{11}^{\alpha} x_{21}^{1-\alpha} \\ u_2(x_{12}, x_{22}) &= x_{12}^{\beta} x_{22}^{1-\beta}, \end{aligned}$$

with $\alpha, \beta \in (0,1)$. Consider endowments $(\omega_{1i}, \omega_{2i}) > 0$ for $i = 1, 2$. Solve for the equilibrium price ratio and allocation.

Solution: let us proceed step by step. First, we compute the demands given a price vector. These are

$$\begin{aligned} x_1(p_1,p_2) &= \left(\frac{\alpha p\cdot\omega_1}{p_1}, \frac{(1-\alpha)p\cdot\omega_1}{p_2}\right) \\ x_2(p_1,p_2) &= \left(\frac{\beta p\cdot\omega_2}{p_1}, \frac{(1-\beta)p\cdot\omega_2}{p_2}\right) \end{aligned}$$

where $p\cdot\omega_1 = p_1\omega_{11}+p_2\omega_{21}$ and $p\cdot\omega_2 = p_1\omega_{12}+p_2\omega_{22}$. Then, by Walras Law (preferences are monotone)

$$x_{21}^* + x_{22}^* = \frac{(1-\alpha)(p_1\omega_{11}+p_2\omega_{21})}{p_2} + \frac{(1-\beta)p_1\omega_{12}+p_2\omega_{22}}{p_2}$$

$$= \frac{p_1}{p_2}((1-\alpha)\omega_{11}+(1-\beta)\omega_{12}) + (1-\alpha)\omega_{21}+(1-\beta)\omega_{22} = \omega_{21}+\omega_{22}.$$

Thus,

$$\frac{p_1^*}{p_2^*} = \frac{\alpha\omega_{21}+\beta\omega_{22}}{(1-\alpha)\omega_{11}+(1-\beta)\omega_{12}}.$$

Finally,

$$x_1^*(p_1^*,p_2^*) = (\omega_{11}\omega_{21}+\beta\omega_{11}\omega_{22}+(1-\beta)\omega_{21}\omega_{12})\left(\frac{\alpha}{\alpha\omega_{21}+\beta\omega_{22}}, \frac{1-\alpha}{(1-\alpha)\omega_{11}+(1-\beta)\omega_{12}}\right)$$

and

$$x_2^*(p_1^*,p_2^*) = (\omega_{12}\omega_{22}+(1-\alpha)\omega_{11}\omega_{22}+\alpha\omega_{21}\omega_{12})\left(\frac{\beta}{\alpha\omega_{21}+\beta\omega_{22}}, \frac{1-\beta}{(1-\alpha)\omega_{11}+(1-\beta)\omega_{12}}\right).$$

Exercise 1.4. There are two consumers, A and B, with the following utility functions,

$$u_A(x_A^1,x_A^2) = a\ln x_A^1 + (1-a)\ln x_A^2,\ \omega_1 = (0,1)$$
$$u_B(x_B^1,x_B^2) = \min\{x_B^1,x_B^2\},\ \omega_2 = (1,0).$$

Compute the prices and quantities that clear the market. Interpret. Hibt: u_A is actually a Cobb-Douglas.

Exercise 1.5. Consider two individuals in a pure exchange (2×2) economy whose indirect utilities are

$$v_1(p_1,p_2,w) = \frac{w}{p_1+p_2}$$
$$v_2(p_1,p_2,w) = \frac{abw}{bp_1+ap_2},\ \ a,b>0.$$

Endowments are $\omega_1 = (1,1)$ and $\omega_2 = (1,1)$. Obtain the equation that prices which clear the market must satisfy. *Hint*: apply Roy's identity. Note (prove) that $u_1(x,y) = \min\{x,y\}$, $u_2(x,y) = \min\{ax,by\}$.

Roy's Identity leads to

$$x_{11}^* = \frac{p_1\omega_{11}+p_2\omega_{21}}{p_1+p_2} = 1$$
$$x_{12}^* = \frac{b(p_1+p_2)}{bp_1+ap_2}.$$

Market only clears if $a=b$. Recall that, when preferences are not strictly monotonic or convex, existence of W.E. may fail. When $a=b$, $p_1=p_2$ in equilibrium and the assignment of the W.E is

$$x^* = ((1,1),(1,1)).$$

2 Hints to additional exercises

Exercise 2.1. Suppose that in a 2×2 economy consumer i has Cobb-Douglas preferences $u_i(x_{1i}, x_{2i}) = x_{1i}^{\alpha} x_{2i}^{1-\alpha}$. Furthermore, assume that endowments are $\omega_1 = (1, 2)$ and $\omega_2 = (2, 1)$. Find the (a)[3] Walrasian equilibrium. **Later on, you should be able to find the optimal Pareto assignments.**

Exercise 2.2. For when you've seen Pareto Optimality in class. Under some conditions over the preferences, in a 2×2 economy, every Pareto Optimal allocation can be characterized as the solution of the following maximization problem (you should try to prove it), $\mathcal{P}_k$:

$$\begin{aligned} &\max\ u_1(\mathbf{x}_1) \\ &\text{s. t. } u_2(\mathbf{x}_2) \geq k \\ &\qquad \mathbf{x}_1 + \mathbf{x}_2 = \boldsymbol{\omega}_1 + \boldsymbol{\omega}_2 \\ &\qquad \mathbf{x}_i \geq \mathbf{0} \end{aligned}$$

where $k \in \mathbb{R}$. Find the aforementioned conditions over the preferences.

Solution: it is not difficult to prove by definition that, $\mathbf{x}^*$ solves this maximization problem if and only if $\mathbf{x}^*$ is P.O.Now, the conditions over the preferences are:

1. Continuous (both).
2. Strictly monotone (both).
3. For $k > 0$, $u_i(\mathbf{0}) = 0$ for $i = 1, 2$.

Using this conditions, you prove that if $\mathbf{x}^*$ solves the problem, then it is a P.O. allocation. Let us prove the converse (which is a little bit easier). Let $\mathbf{x}^* = (\mathbf{x}_1^*, \mathbf{x}_2^*)$ be P.O. Then, you cannot find $(\mathbf{x}_1, \mathbf{x}_2)$ such that

$$\begin{aligned} \mathbf{x}_1 &\succeq_1 \mathbf{x}_1^* \\ \mathbf{x}_2 &\succeq_2 \mathbf{x}_2^* \end{aligned}$$

and $\mathbf{x}_1 \succ \mathbf{x}_1^*$ or $\mathbf{x}_2 \succ \mathbf{x}_2^*$. Let $k = u_2(\mathbf{x}_2^*)$. Then, $(\mathbf{x}_1^*, \mathbf{x}_2^*)$ solves $\mathcal{P}_k$ since

$$\begin{aligned} \mathbf{x}_1 \succeq \mathbf{x}_1^* &\Leftrightarrow u_1(\mathbf{x}_1) \geq u_1(\mathbf{x}_1^*) \\ \mathbf{x}_2 \succeq \mathbf{x}_2^* &\Leftrightarrow u_2(\mathbf{x}_2) \geq u_2(\mathbf{x}_2^*). \end{aligned}$$

Exercise 2.3. Medium-difficulty. From Chavez and Gallardo (2024). Consider an economy with N consumers, two goods, and preferences given by

$$u_i(x_{1i}, x_{2i}) = x_{1i}^2 + x_{2i}^2.$$

Endowments are $\omega_i = (1, 1)$. If N is even, find, if it exists, a Walrasian equilibrium. What if N is odd?

Exercise 2.4. Mandatory to know. Prove that if $\succeq$ is monotone, then it is locally

[3] We don't know if it is unique or no! However, under some conditions over the preferences, which are

non satiated. Here $\succeq$ represents a preference relation over $\mathbb{R}_+^L$.

Recall that, $\succeq$ is locally non satiated over $X = \mathbb{R}^L$ if for every $x \in X$ and $\epsilon > 0$, there exists $y \in \mathcal{B}(x, \epsilon) = \{z \in \mathbb{R}^L : ||x - z|| = \sqrt{\sum_{i=1}^{L}(x_i - z_i)^2} < \epsilon\}$ such that $y \succ x$.

Solution: consider $\mathbf{z} = \mathbf{x} + \sqrt{\frac{\epsilon}{2L}}\mathbf{1}_L$. Then, $\mathbf{z} \succ \mathbf{x}$ and $||\mathbf{z} - \mathbf{x}||_2 < \epsilon$.

Exercise 2.5. Medium-difficulty. Prove 1st Welfare theorem for a 2×2 economy. This is, if preferences are locally non satiated, then, every Walrasian equilibrium is Pareto optimal. Can you generalize this for a pure exchange economy with N consumers and L goods? You can guide yourself from Echenique (2015).

Solution: if preferences are locally non-satiated, every W.E. is P.O. The proof is as seen in the course: let (x^*, p^*) be a W.E. Proceeding by contradiction, suppose that the allocation x^* is not P.O. In this case, there must exist a feasible allocation $x = \{x_i\}_{i=1}^I$, such that for each $i = 1, \ldots, I$, $x_i \succeq_i x_i^*$, and at least for some $i_0 \in \{1, \ldots, I\}$, $x_{i_0} \succ_{i_0} x_{i_0}^*$. We will prove that for such an allocation, the inequality

$$\sum_{i=1}^{I} x_i > \overline{\omega},$$

holds, which contradicts the fact that x is feasible. First, the condition $x_i \succeq_i x_i^*$ implies that $p^* \cdot x_i \geq p^* \cdot \omega_i$. Indeed, if $p^* \cdot x_i < p^* \cdot \omega_i$, then there exists $\epsilon > 0$, such that for all $z \in \mathcal{B}(x_i; \epsilon)$, $p^* \cdot z < p^* \cdot \omega_i$; and since preferences are locally non-satiated, $\exists\ z_0 \in \mathcal{B}(x_i; \epsilon)$ such that $z_0 \succ_i x_i \succeq_i x_i^*$. However, this contradicts the maximality of x_i^*. On the other hand, the condition $x_{i_0} \succ x_{i_0}^*$ implies that $p^* \cdot x_{i_0} > p^* \cdot \omega_{i_0}$. Indeed, the contrary inequality, that is, $p^* \cdot x_{i_0} \leq p^* \cdot \omega_{i_0}$, contradicts the maximality of $x_{i_0}^*$. Thus, we conclude that

$$\sum_{i=1}^{I} p^* \cdot x_i = \sum_{i \neq i_0} p^* \cdot x_i + p^* \cdot x_{i_0} > \sum_{i \neq i_0} p^* \cdot \omega_i + p^* \cdot \omega_{i_0} = \sum_{i=1}^{I} p^* \cdot \omega_i.$$

Since $p^* \in \mathbb{R}_+^L - \{0\}$, this equation implies that we cannot have $\sum_{i=1}^{I} x_i \leq \overline{\omega}$. That is, it must hold that $\sum_{i=1}^{I} x_i > \overline{\omega}$, as we wanted to show.

Lima, September 2, 2024.

satisfied in this exercise, existence is ensured.

References

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Echenique, F. (2015). Lecture notes general equilibrium theory.

Mas-Colell, A., Whinston, M. D., and Green, J. R. (1995). *Microeconomic Theory*. Oxford University Press, New York.