TAREA-OPCIONAL

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1. Enunciado

Demostrar que $E(x) = \frac{1}{\lambda}$ y $V(x) = \frac{1}{\lambda^2}$:

Para demostrar que $E(x) = \frac{1}{\lambda}$:

$$\begin{aligned} E(x) &= \int_{-\infty}^{\infty} x f(x) dx = \int_{0}^{\infty} x \lambda e^{-\lambda x} dx \\ &= \lambda \int_{0}^{\infty} x e^{-\lambda x} dx \end{aligned}$$

Integral por partes, ILATE nos sugiere que $u = x$ y $dv = e^{-\lambda x}$ por lo tanto:

$$\begin{aligned} du &= dx \\ v &= -\frac{1}{\lambda} e^{-\lambda x} \\ \int u dv &= uv - \int v du \end{aligned}$$

Sustituyendo:

$$\begin{aligned} \int_{0}^{\infty} x e^{-\lambda x} dx &= -x \cdot \frac{1}{\lambda} e^{-\lambda x} \Big|_{0}^{\infty} - \int_{0}^{\infty} -\frac{1}{\lambda} e^{-\lambda x} dx \\ &= -\frac{x}{\lambda e^{\lambda x}} \Big|_{0}^{\infty} + \frac{1}{\lambda} \left[-\frac{1}{\lambda} e^{-\lambda x} \Big|_{0}^{\infty} \right] \\ &= -\frac{x}{\lambda e^{\lambda x}} \Big|_{0}^{\infty} - \frac{1}{\lambda^2 e^{\lambda x}} \Big|_{0}^{\infty} \\ &= -\lim_{x \to \infty} \frac{x}{\lambda e^{\lambda x}} - \lim_{x \to \infty} \frac{1}{\lambda^2 e^{\lambda x}} - \left[-\frac{0}{\lambda e^{\lambda 0}} - \frac{1}{\lambda^2 e^{\lambda 0}} \right] \\ &= -0 - 0 - \left[-\frac{0}{\lambda} - \frac{1}{\lambda^2} \right] = \frac{1}{\lambda^2} \end{aligned}$$

$$\begin{aligned} E(x) &= \lambda \left[\frac{1}{\lambda^2} \right] && \text{Remplazando} \\ &= \frac{1}{\lambda} \end{aligned}$$

Q.E.D. $\quad E(x) = \frac{1}{\lambda}$

Para demostrar que $V(x) = \dfrac{1}{\lambda^2}$:

$$V(x) = E(x^2) - [E(x)]^2$$

Es necesario calcular $E(x^2)$:

$$\begin{aligned}
E(x^2) &= \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{\infty} x^2 \lambda e^{-\lambda x} dx \\
&= \lambda \int_0^{\infty} x^2 e^{-\lambda x} dx \\
&= \lambda \left[x^2 \cdot \frac{-1}{\lambda e^{\lambda x}} \Big|_0^{\infty} - \int_0^{\infty} \frac{-1}{\lambda e^{\lambda x}} \cdot 2x dx \right] \\
&= \lambda \left[\frac{-x^2}{\lambda e^{\lambda x}} \Big|_0^{\infty} + \frac{2}{\lambda} \int_0^{\infty} \frac{x}{e^{\lambda x}} dx \right] \\
&= \lambda \left[-\lim_{x \to \infty} \frac{x^2}{\lambda e^{\lambda x}} + \frac{0^2}{\lambda e^{\lambda 0}} + \frac{2}{\lambda} \left(\frac{-x}{\lambda e^{\lambda x}} \Big|_0^{\infty} - \frac{1}{\lambda} \int_0^{\infty} \frac{-1}{\lambda e^{\lambda x}} dx \right) \right] \\
&= \lambda \left[-0 + 0 + \frac{2}{\lambda} \left(-\lim_{x \to \infty} \frac{x}{\lambda e^{\lambda x}} + \frac{0}{\lambda e^{\lambda 0}} + \frac{1}{\lambda^2} \int_0^{\infty} \frac{1}{e^{\lambda x}} dx \right) \right] \\
&= \lambda \left[\frac{2}{\lambda} \left(-0 + 0 + \frac{1}{\lambda^2} \left(\frac{-1}{\lambda e^{\lambda x}} \Big|_0^{\infty} \right) \right) \right] \\
&= \lambda \left[\frac{2}{\lambda} \left(\frac{1}{\lambda^2} \left(-\lim_{x \to \infty} \frac{1}{\lambda e^{\lambda x}} + \frac{1}{\lambda e^{\lambda 0}} \right) \right) \right] \\
&= \lambda \left[\frac{2}{\lambda} \left(\frac{1}{\lambda^2} (0 + 1) \right) \right] \\
&= \frac{2}{\lambda^2}
\end{aligned}$$

Luego:

$$\begin{aligned}
V(x) &= \frac{2}{\lambda^2} - \left[\frac{1}{\lambda} \right]^2 && \text{Remplazando} \\
&= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} \\
&= \frac{1}{\lambda^2}
\end{aligned}$$

Q.E.D. $\quad V(x) = \dfrac{1}{\lambda^2}$