5. E : $0 \neq \# E < +\infty$, $E^{\mathbb{N}} = \{\omega = (\omega_i)_{i\in\mathbb{N}} : \omega_i \in E, \forall i \in \mathbb{N}\}$

$[\omega_1, \dots, \omega_m] := \{\omega' \in E^{\mathbb{N}} : \omega'_i = \omega_i, \forall i = 1, \dots, m\}$ / $\mu([\omega_1, \dots, \omega_m]) = \prod_{i=1}^{m} P_{\omega_i}$

$P = (P_e)_{e\in E}$ - PROBABILITY VECTOR $\iff \prod_{e\in E} P_e = 1$

Product Measure

Bernoulli Measure

$E = \{H, T\}$ (H = HEAD, T = TAIL)

$\Omega = E^{\mathbb{N}}$ $\qquad \Omega' := \{\omega \in \Omega : \exists!\, i, j,\ i \neq i,\ \omega_i = \omega_j = H\}$

$\mathcal{A} = \sigma\left(\bigcup_{m=0}^{+\infty} \mathcal{A}_m\right)$ $\qquad \mathcal{A}' := \mathcal{A}|_{\Omega'}$

$P_H = P_T = 1/2 \rightsquigarrow \mu_P$

$\Omega_m := \{\omega \in \{H, T\}^m : \exists!\, i, j,\ i \neq j, \text{ com } \omega_i = \omega_j = H\}$, $m \geq 2$

$\Omega := \bigcup_{m=2}^{+\infty} \Omega_m$

$\mathcal{A} := \sigma(\Omega)$

$\mathbb{P}(\Omega_2) = \mathbb{P}(\{(H, H)\}) = \frac{1}{2} \cdot \frac{1}{2} = 1/2^2$

$\mathbb{P}(\Omega_3) = \mathbb{P}(\{(T, H, H), (H, T, H)\}) = 2/2^3$

$\mathbb{P}(\Omega_4) = \mathbb{P}(\{(H, T, T, H), (T, H, T, H), (T, T, H, H)\}) = 3/2^4$

$\vdots$

$\mathbb{P}(\Omega_m) = \frac{m-1}{2^m}$

---

5. $E = \{H, T\}$ (H - HEAD, T - TAIL)

$\Omega_m := E^m = \{(\omega_1, \dots, \omega_m) : \omega_i \in E, \forall i = 1, \dots, m\}$, $m \geq 1$, $E^0 := \{\emptyset\}$

$\Omega = \bigcup_{m=0}^{+\infty} \Omega_m$, $\mathbb{P}(\{\omega\}) = 1/2^m \iff \omega \in \Omega_m, m \geq 1$

SAMPLE SPACE S:

$\Omega'_2 := \{(H, H)\}$ $\rightarrow \mathbb{P}(\Omega'_2) = 1 \cdot 1/2^2$

$\Omega'_3 := \{(T, H, H), (H, T, H)\}$ $\rightarrow \mathbb{P}(\Omega'_3) = 2 \cdot 1/2^3$

$\Omega'_4 := \{(T, T, H, H), (T, H, T, H), (H, T, T, H)\}$ $\rightarrow \mathbb{P}(\Omega'_4) = 3 \cdot 1/2^4$

$\vdots$

$\Omega'_m := \{(T, \dots, T, H, H), \dots, (H, T, \dots, T, H)\}$ $\rightarrow \mathbb{P}(\Omega'_m) = (m-1) \cdot 1/2^m$

$\vdots$

SAMPLE SPACE $S := \bigcup_{m=2}^{+\infty} \Omega'_m$

$$\boxed{\mathbb{P}(\Omega'_k) = \frac{k-1}{2^{k-1}}}$$

6. $\Omega = \{1, 2, \dots\}$

SUPPOSE THERE IS A UNIFORM PROBABILITY MEASURE $\mathbb{P}$ OVER $(\Omega, \mathcal{P}(\Omega))$

SUCH THAT IT SATISFY: $A, B \subseteq \Omega$, $|A| = |B| \Rightarrow \mathbb{P}(A) = \mathbb{P}(B)$.

$$\text{IF } \mathbb{P}(\{n\}) = 0 \rightarrow 1 = \mathbb{P}(\Omega) = \sum_{n=1}^{+\infty} \mathbb{P}(\{n\}) = 0 \quad (\rightarrow\leftarrow)$$

$$\text{IF } \mathbb{P}(\{n\}) > \delta > 0 \rightarrow 1 = \mathbb{P}(\Omega) = \sum_{n=1}^{+\infty} \mathbb{P}(\{n\}) > \lim_{n \to +\infty} \left[\sum_{i=1}^{n} \delta \right] = +\infty \quad (\rightarrow\leftarrow)$$

---

7. $A_1, A_2, \dots \in \mathcal{A}$ (EVENTS)

$$B_n := A_n \setminus \bigcup_{k=1}^{n-1} A_k, \; n > 1 \quad \text{AND} \quad B_1 := A_1$$

BY DEFINITION: IF $x \in B_n$, THEN $x \notin B_1, \dots, B_{n-1}$. THUS, IF $x \in B_n \cap B_m$ AND $n < m$, $x \in B_m \rightarrow x \notin B_n$ $(\rightarrow\leftarrow)$. THEN $B_i \cap B_j = \emptyset \; \forall i, j, i \neq j$.

$$\star \; \bigcup_{i=1}^{n} A_i = \bigsqcup_{i=1}^{n} B_i \quad \text{OR} \quad \bigcup_{i=1}^{+\infty} A_i = \bigsqcup_{i=1}^{+\infty} B_i$$

$$(\supseteq) \; x \in \bigsqcup_{i=1}^{n} B_i \rightarrow \exists j : x \in B_j \rightarrow x \in A_j \rightarrow x \in \bigcup_{i=1}^{n} A_i$$

$$(\subseteq) \; x \in \bigcup_{i=1}^{n} A_i \rightarrow i_0 := \min\{i \in \{1, \dots, n\} : x \in A_i\} \rightarrow x \in A_{i_0} \text{ AND } x \notin A_j : j = 1, \dots, i_0 - 1$$

$$\rightarrow x \in B_{j_0} \rightarrow x \in \bigsqcup_{i=1}^{n} B_i$$

$$\mathbb{P}\left(\bigcup_{n=1}^{+\infty} A_i\right) = \mathbb{P}\left(\bigsqcup_{n=1}^{+\infty} B_n\right) = \sum_{n=1}^{+\infty} \mathbb{P}(B_n) = \lim_{n \to +\infty} \left[\sum_{i=1}^{n} \mathbb{P}(B_i) \right] = \lim_{n \to +\infty} \left[\sum_{i=1}^{n} \left(\mathbb{P}(A_i) - \mathbb{P}\left(\bigcup_{k=1}^{i-1} A_i \right) \right) \right] =$$

$$= \lim_{n \to +\infty} \left[\sum_{i=1}^{n} \mathbb{P}(A_i) - \sum_{i=2}^{n} \mathbb{P}\left(\bigcup_{k=1}^{i-1} A_k \right) \right] \leq \lim_{n \to +\infty} \left[\sum_{i=1}^{n} \mathbb{P}(A_i) \right] = \sum_{i=1}^{+\infty} \mathbb{P}(A_i)$$

---

8. $\mathbb{P}(A_i) = 1, \forall i \in \mathbb{N} \rightarrow \mathbb{P}(A_i^c) = 0, \forall i \in \mathbb{N}$

$$\mathbb{P}\left(\bigcup_{n=1}^{+\infty} A_i^c\right) \leq \underbrace{\sum_{i=1}^{+\infty} \mathbb{P}(A_i^c)}_{\text{EXERCISE 7}} = 0 \quad \rightsquigarrow \quad \mathbb{P}\left(\bigcup_{n=1}^{+\infty} A_i^c\right) = 0$$

$$\Rightarrow 1 = \mathbb{P}\left(\left(\bigcup_{n=1}^{+\infty} A_i^c \right)^c \right) = \mathbb{P}\left(\bigcap_{n=1}^{+\infty} A_i \right)$$

9. $B \in \mathcal{A}$, $\mathbb{P}(B) > 0 \rightsquigarrow \mu := \mathbb{P}(\cdot \mid B)$ IS A PROBABILITY MEASURE

(I) $\mu(A) = \mathbb{P}(A|B) = \dfrac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \geq 0$

(II) $\mu(\Omega) = \mathbb{P}(\Omega|B) = \dfrac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$

(III) $A_1, A_2, \dots$ DISJOINT EVENTS

$$\mu\left(\biguplus_{m=1}^{+\infty} A_m\right) = \mathbb{P}\left(\biguplus_{m=1}^{+\infty} A_m \,\Big|\, B\right) = \frac{\sum_{i=1}^{+\infty} \mathbb{P}(A_m \cap B)}{\mathbb{P}(B)} = \sum_{i=1}^{+\infty} \frac{\mathbb{P}(A_m \cap B)}{\mathbb{P}(B)} = \sum_{i=1}^{+\infty} \mu(A_m)$$

10. MONTY HALL PROBLEM (3 DOORS)

YOU ALWAYS PICK DOOR 1

$\Omega = \{(\omega_1, \omega_2) : \omega_1, \omega_2 \in \{1,2,3\}\}$, ω_1 - WHERE THE PRIZE IS, ω_2 - IS THE DOOR MONTY OPENS

SAMPLE SPACE

$$\Omega' := \begin{Bmatrix} (1,2), (1,3) \\ (2,3) \\ (3,2) \end{Bmatrix}$$

$\mathbb{P}(\{(1,2)\}) = \mathbb{P}(\{(1,3)\}) = \frac{1}{3} \cdot \frac{1}{2}$

$\mathbb{P}(\{(2,3)\}) = \frac{1}{3}$

$\mathbb{P}(\{(3,2)\}) = \frac{1}{3}$

* YOU CONTINUE WITH DOOR 1

$$\mathbb{P}(\omega_1 = 1 \mid \omega_2 = 2) = \frac{\mathbb{P}(\omega_1 = 1, \omega_2 = 2)}{\mathbb{P}(\omega_2 = 2)} = \frac{\mathbb{P}(\{(1,2)\})}{\mathbb{P}(\{(1,2),(3,2)\})} = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}} = \frac{1}{3}$$

$$\mathbb{P}(\omega_1 = 1 \mid \omega_2 = 3) = \frac{\mathbb{P}(\omega_1 = 1, \omega_2 = 3)}{\mathbb{P}(\omega_2 = 3)} = \frac{\mathbb{P}(\{(1,3)\})}{\mathbb{P}(\{(1,3),(2,3)\})} = \frac{\frac{1}{3}\cdot\frac{1}{2}}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}} = \frac{1}{3}$$

* YOU CHANGE THE DOOR

$$\mathbb{P}(\omega_1 = 2 \mid \omega_2 = 3) = \frac{\mathbb{P}(\omega_1 = 2, \omega_2 = 3)}{\mathbb{P}(\omega_2 = 3)} = \frac{\mathbb{P}(\{(2,3)\})}{\mathbb{P}(\{(1,3),(2,3)\})} = \frac{\frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}} = \frac{2}{3}$$

$$\mathbb{P}(\omega_1 = 3 \mid \omega_2 = 2) = \frac{\mathbb{P}(\omega_1 = 3, \omega_2 = 2)}{\mathbb{P}(\omega_2 = 2)} = \frac{\mathbb{P}(\{(3,2)\})}{\mathbb{P}(\{(1,2),(3,2)\})} = \frac{\frac{1}{3}}{\frac{1}{3}\cdot\frac{1}{2} + \frac{1}{3}} = \frac{2}{3}$$

Monty Hall Problem (4 Doors)

ω_1 - where the prize is

ω_2 - is the door Monty opens

Sample Space

$$\Omega' := \left\{ \begin{array}{c} (1,2),(1,3),(1,4) \\ (2,3),(2,4) \\ (3,2),(3,4) \\ (4,2),(4,3) \end{array} \right\} \rightsquigarrow \begin{array}{l} \mathbb{P}(\{(1,2)\}) = \mathbb{P}(\{(1,3)\}) = \mathbb{P}(\{(1,4)\}) = \frac{1}{4}\cdot\frac{1}{3} \\ \mathbb{P}(\{(2,3)\}) = \mathbb{P}(\{(2,4)\}) = \frac{1}{4}\cdot\frac{1}{2} \\ \mathbb{P}(\{(3,2)\}) = \mathbb{P}(\{(3,4)\}) = \frac{1}{4}\cdot\frac{1}{2} \\ \mathbb{P}(\{(4,2)\}) = \mathbb{P}(\{(4,3)\}) = \frac{1}{4}\cdot\frac{1}{2} \end{array}$$

* You continue with door 1

$$\mathbb{P}(\omega_1 = 1 \mid \omega_2 = 2) = \frac{\mathbb{P}(\omega_1 = 1, \omega_2 = 2)}{\mathbb{P}(\omega_2 = 2)} = \frac{\mathbb{P}(\{(1,2)\})}{\mathbb{P}(\{(1,2),(3,2),(4,2)\})} =$$

(Other cases are the same)

$$= \frac{\frac{1}{4}\cdot\frac{1}{3}}{\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}} = \frac{\frac{1}{3}}{\frac{1}{3}+1} = \frac{\frac{1}{3}}{\frac{4}{3}} = \frac{3}{4}\cdot\frac{1}{3} = \frac{1}{4} = 0.25$$

* You change the door

$$\mathbb{P}(\omega_1 = 2 \mid \omega_2 = 3) = \frac{\mathbb{P}(\omega_1 = 2, \omega_2 = 3)}{\mathbb{P}(\omega_2 = 3)} = \frac{\mathbb{P}(\{(2,3)\})}{\mathbb{P}(\{(1,3),(2,3),(4,3)\})} =$$

$$= \frac{\frac{1}{4}\cdot\frac{1}{2}}{\frac{1}{4}\cdot\frac{1}{3}+\frac{1}{4}\cdot\frac{1}{2}+\frac{1}{4}\cdot\frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{3}+1} = \frac{\frac{1}{2}}{\frac{4}{3}} = \frac{3}{4}\cdot\frac{1}{2} = \frac{3}{8} = 0.375$$

Monty Hall Problem (n Doors)

$$\Omega' = \left\{ \begin{array}{l} (1,2),(1,3),\dots,(1,n-1),(1,n) \\ (2,3),(2,4),\dots,(2,n) \\ \vdots \\ (n,2),(n,3),\dots,(n,n-1) \end{array} \right\} \quad \begin{array}{l} \mathbb{P}(\{(1,i)\}) = \frac{1}{n}\cdot\frac{1}{n-1}, \quad i = 2,3,\dots,n \\ \mathbb{P}(\{(j,k)\}) = \frac{1}{n}\cdot\frac{1}{n-2}, \quad j,k = 2,3,\dots,n \\ \qquad\qquad\qquad\qquad\qquad j \neq k \end{array}$$

$$\mathbb{P}(\omega_1 = 1 \mid \omega_2 = 2) = \frac{\mathbb{P}(\omega_1 = 1, \omega_2 = 2)}{\mathbb{P}(\omega_2 = 2)} = \frac{\mathbb{P}(\{(1,2)\})}{\mathbb{P}(\{(1,2),(3,2),\dots,(n,2)\})} = \frac{\frac{1}{n}\cdot\frac{1}{n-1}}{\frac{1}{n}\cdot\frac{1}{n-1}+(n-2)\cdot\frac{1}{n}\cdot\frac{1}{(n-2)}}$$

$$= \frac{1}{1+(n-1)} = \frac{1}{n}$$

$$\mathbb{P}(\omega_1 = 2 \mid \omega_2 = 3) = \frac{\mathbb{P}(\omega_1 = 2, \omega_2 = 3)}{\mathbb{P}(\omega_2 = 3)} = \frac{\mathbb{P}(\{(2,3)\})}{\mathbb{P}(\{(1,3),(2,3),(4,3),\dots,(n,3)\})} = \frac{\frac{1}{n}\cdot\frac{1}{n-2}}{\frac{1}{n}\cdot\frac{1}{n-1}+(n-2)\cdot\frac{1}{n}\cdot\frac{1}{n-2}} =$$

$$= \frac{\frac{1}{n-2}}{\frac{1}{n-1}+1} = \frac{\frac{n-1}{n-2}}{1+(n-1)} = \frac{n-1}{n(n-2)}$$

$$\boxed{\frac{n-1}{n-2} > 1 \rightsquigarrow \frac{1}{n} < \frac{1}{n}\cdot\frac{n-1}{n-2}}$$

11. $(\Omega, \mathcal{A}, \mathbb{P})$ $\quad A, B \in \mathcal{A}$ ARE INDEPENDENTS

LETS SHOW THAT A^c, B ARE INDEPENDENTS

$$\mathbb{P}(B) = \mathbb{P}(B \cap (A \sqcup A^c)) = \mathbb{P}(B \cap A) + \mathbb{P}(B \cap A^c) = \mathbb{P}(B) \cdot \mathbb{P}(A) + \mathbb{P}(B \cap A^c)$$

$$\hookrightarrow \mathbb{P}(B) - \mathbb{P}(B) \cdot \mathbb{P}(A) = \mathbb{P}(B \cap A^c) \longrightarrow \mathbb{P}(B) \cdot (1 - \mathbb{P}(A)) = \mathbb{P}(B \cap A^c)$$

$$\rightarrow \mathbb{P}(B) \cdot \mathbb{P}(A^c) = \mathbb{P}(B \cap A^c)$$

SAME ARGUMENTS WE CONCLUDE THAT A^c, B^c ARE INDEPENDENTS.

12.

	FRONT	BACK
C_1	G	G
C_2	R	R
C_3	G	R
	R	G

$$\mathbb{P}(\{(C_1, G, G)\}) = \frac{1}{3}$$

$$\mathbb{P}(\{(C_2, R, R)\}) = \frac{1}{3}$$

$$\mathbb{P}(\{(C_3, G, R)\}) = \mathbb{P}(\{(C_3, R, G)\}) = \frac{1}{3} \cdot \frac{1}{2}$$

$$\mathbb{P}(\text{BACK} = G \mid \text{FRONT} = G) = \frac{\mathbb{P}(\text{FRONT} = G, \text{BACK} = G)}{\mathbb{P}(\text{BACK} = G)} = \frac{\mathbb{P}(\{(C_1, G, G)\})}{\mathbb{P}(\{(C_1, G, G)\}) + \mathbb{P}(\{(C_3, G, R)\})} =$$

$$= \frac{\frac{1}{3}}{\frac{1}{3} + \frac{1}{3} \cdot \frac{1}{2}} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$$

13.

(a) SAMPLE SPACE

$$\Omega'_2 := \{(H, T), (T, H)\}$$

$$\Omega'_3 := \{(H, H, T), (T, T, H)\}$$

$$\Omega'_4 := \{(H, H, H, T), (T, T, T, H)\}$$

$$\vdots$$

$$\Omega'_n := \{(H, \dots, H, T), (T, \dots, T, H)\}$$

$$\rightsquigarrow \Omega' := \bigcup_{n=2}^{+\infty} \Omega'_n \quad \text{SAMPLE SPACE}$$

(b) PROBABILITY

$$\rightsquigarrow \mathbb{P}(\Omega'_2) = 2 \cdot \frac{1}{2^2}$$

$$\rightarrow \mathbb{P}(\Omega'_3) = 2 \cdot \frac{1}{2^3}$$

$$\rightsquigarrow \mathbb{P}(\Omega'_4) = 2 \cdot \frac{1}{2^4}$$

$$\rightsquigarrow \mathbb{P}(\Omega'_n) = 2 \cdot \frac{1}{2^n} = \frac{1}{2^{n-1}}$$

$$\mathbb{P}(\Omega'_3) = \frac{1}{2^2}$$

14. $\mathbb{P}(A) \in \{0,1\} \rightarrow$ A IS INDENPENDENT OF EVERY OTHER EVENT

$E \in \mathcal{A}$ - EVEN SPACE

IF $\mathbb{P}(A)=0$, THEN $\mathbb{P}(A\cap E) \leq \mathbb{P}(A) = 0 \rightarrow \mathbb{P}(A\cap E) = 0 = 0 \cdot \mathbb{P}(E) = \mathbb{P}(A) \cdot \mathbb{P}(E)$ (i)

IF $\mathbb{P}(A)=1$, THEN $\mathbb{P}(A^c) = 0$ AND

$$\mathbb{P}(E) \cdot \mathbb{P}(A) = \mathbb{P}(E) \cdot 1 = \mathbb{P}(E) = \mathbb{P}(E \cap (A \cup A^c)) = \mathbb{P}(E\cap A) + \mathbb{P}(E \cap A^c)$$
$$\overset{(i)}{=} \mathbb{P}(E\cap A) + 0 = \mathbb{P}(E\cap A)$$

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15. BE = "BLUE EYES" NBE = "NOT BLUE EYES" / $\mathbb{P}(BE) = 1/4$ $\mathbb{P}(NBE) = 3/4$

(a) $\Omega = \{(x,y,z) : x,y,z \in \{BE, NBE\}\}$ SAMPLE SPACE

$\mathcal{A} = \mathcal{P}(\Omega) = \{X : X \subseteq \Omega\}$ EVENT SPACE

X = NUMBER OF CHILDREN WITH BLUE EYES

$$\mathbb{P}(X \geq 2 \mid X \geq 1) = ?$$

$$\mathbb{P}(X \geq 1) = 1 - \mathbb{P}(X=0) = 1 - (3/4)^3 = \frac{64-27}{4^3} = \frac{37}{4^3}$$

$$\mathbb{P}(X \geq 2) = \mathbb{P}(X=2) + \mathbb{P}(X=3) = 3 \cdot (1/4)^2 \cdot (3/4)^1 + (1/4)^3 = (1/4)^3 (3 \cdot 1 \cdot 3 + 1)$$
$$= (1/4)^3 (9+1) = 10/4^3$$

$$\mathbb{P}(X \geq 2 \mid X \geq 1) = \frac{\mathbb{P}(X\geq 2, X \geq 1)}{\mathbb{P}(X \geq 1)} = \frac{\mathbb{P}(X \geq 2)}{\mathbb{P}(X \geq 1)} = \frac{10/4^3}{37/4^3} = \frac{10}{37}$$

(b) $Z = \{(x, y, BE) : x, y \in \{BE, NBE\}\}$

$$\mathbb{P}(X \geq 2 \mid Z) = \frac{\mathbb{P}(X \geq 2, Z)}{\mathbb{P}(Z)} = \frac{\mathbb{P}(\{(BE,NBE,BE),(NBE,BE,BE),(BE,BE,BE)\})}{\mathbb{P}(Z)}$$
$$= \frac{(1/4)^3 (2 \cdot 1^2 \cdot 3^1 + 1)}{1/4} = \frac{7}{4^2} = \frac{7}{16}$$

16.

$\{A,B\}$ ARE INDEPENDENT EVENTS $\iff \mathbb{P}(A\cap B) = \mathbb{P}(A)\cap\mathbb{P}(B)$

$\{A,B\}$ ARE INDEPENDENT EVENTS $\implies \mathbb{P}(A|B) = \dfrac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} = \dfrac{\mathbb{P}(A)\cdot\mathbb{P}(B)}{\mathbb{P}(B)} = \mathbb{P}(A)$ (if $\mathbb{P}(B)>0$)

$$\left.\begin{aligned} \mathbb{P}(A|B) = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(B)} &\rightarrow \mathbb{P}(A\cap B) = \mathbb{P}(A|B)\cdot\mathbb{P}(A) \\ \mathbb{P}(B|A) = \frac{\mathbb{P}(B\cap A)}{\mathbb{P}(A)} &\rightarrow \mathbb{P}(B\cap A) = \mathbb{P}(B|A)\cdot\mathbb{P}(A) \end{aligned}\right\} \quad \begin{aligned} \mathbb{P}(A\cap B) &= \mathbb{P}(B\cap A) = \\ &= \mathbb{P}(A|B)\cdot\mathbb{P}(B) \\ &= \mathbb{P}(B|A)\cdot\mathbb{P}(A) \end{aligned}$$

17.

$$\mathbb{P}(A\cap B\cap C) = \mathbb{P}(A\cap(B\cap C)) \overset{16.}{=} \mathbb{P}(A|B\cap C)\cdot\mathbb{P}(B\cap C) \overset{16.}{=} \mathbb{P}(A|B\cap C)\cdot\mathbb{P}(B|C)\cdot\mathbb{P}(C)$$

18.

LET'S PROVE BY CONTRADICTION. SUPPOSE $\mathbb{P}(A_i|B) < \mathbb{P}(A_i), \forall i = 2,\dots,k$, WE ALSO HAVE $\mathbb{P}(A_1|B) < \mathbb{P}(A_1)$ BY HYPOTHESIS.

$$\mathbb{P}(A_i|B) < \mathbb{P}(A_i), \forall i = 1,\dots,k \qquad (*)$$

$$\rightsquigarrow \mathbb{P}(B) = \mathbb{P}\left(B\cap\left(\biguplus_{i=1}^{k} A_i\right)\right) = \sum_{i=1}^{k}\mathbb{P}(B\cap A_i) = \sum_{i=1}^{k}\mathbb{P}(A_i|B)\cdot\mathbb{P}(B)$$

$$= \left(\sum_{i=1}^{k}\mathbb{P}(A_i|B)\right)\cdot\mathbb{P}(B) \overset{*}{<} \left(\sum_{i=1}^{k}\mathbb{P}(A_i)\right)\cdot\mathbb{P}(B) = \mathbb{P}\left(\biguplus_{i=1}^{k} A_i\right)\cdot\mathbb{P}(B) = 1\cdot\mathbb{P}(B)$$

$$\rightsquigarrow \mathbb{P}(B) < \mathbb{P}(B) \quad (\rightarrow\leftarrow)$$

19.

MAC	WIN	LIN
30%	50%	20%

65% MAC GOT VIRUS
82% WIN GOT VIRUS
50% LIN GOT VIRUS

$\mathbb{P}(PC = WIN \mid VIRUS = TRUE) = ?$

$$\mathbb{P}(PC = WIN \mid VIRUS = TRUE) = \frac{\mathbb{P}(VIRUS = TRUE \mid PC = WIN)\cdot\mathbb{P}(PC = WIN)}{\sum_{PC = X}\mathbb{P}(VIRUS = TRUE \mid \mathbb{P}(PC = x))\cdot\mathbb{P}(PC = x)}$$

$$= \frac{(0.82)\cdot(0.5)}{(0.65)\cdot(0.3)+(0.82)\cdot(0.5)+(0.5)\cdot(0.2)} \simeq 0.5815 = 58.15\%$$