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ORIGINAL ARTICLE

Didactic and theoretical-based perspectives in the experimental development of an intelligent tutorial system for the learning of geometry

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Abstract This paper aims at showing the didactic and theoretical-based perspectives in the experimental development of the geogebraTUTOR system (GGBT) in interaction with the students. As a research and technological realization developed in a convergent way between mathematical education and computer science, GGBT is an intelligent tutorial system, which supports the student in the solving of complex problems at a high school level by assuring the management of discursive messages as well as the management of problem situations. By situating the learning model upstream and the diagnostic model downstream, GGBT proposes to act on the development of mathematical competencies by controlling the acquisition of knowledge in the interaction between the student and the milieu, which allows for the adaptation of the instructional design (learning opportunities) according to the instrumented actions of the student. The inferential and construction graphs, a structured bridge (interface) between the contextualized world of didactical contracts and the formal computer science models, structure GGBT. This way allows for the tutorial action to adjust itself to the competential habits conveyed by a certain classroom of students and to be enriched by the research results in mathematical education.

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1 Preliminary point of view on the tutorial systems for the learning of geometry

Geometry at a high school level can be seen as a deductive science allowing the solving of problems in the mathematical field as well as a theoretical reference, which orients the wider process of extra mathematical modelling. This permits amongst other things the laying down of problems inspired by what is referred to as the real world or reality. Even if the solving of modelling or proof problems, along side of the curricular obligations, is a mathematical competence to prioritize in the education of young people, it remains difficult to develop in regards of the traditional relationship where the teacher who, alone in front of his class, gives insight into the mathematical reasoning, calculations and other problems to which are confronted students. Also, when it is the instrumental workings of the dynamic geometry tools that liven up his didactic interventions, the teacher may feel at loss when faced with the flow of interactions between each student and the computer device, and this in spite of the fact that these interactions

Table 1 Tutorial systems in high-school geometry

	Underlying paradigm of reference	Tutor's intervention
Advanced Geometry Tutor (Matsuda & Vanlehn, 2005)	Formal Geometry	Using the traditional writing of a two-column proof, the tutor teaches proof-writing with construction in two problem solving strategies, the forward chaining and the backward chaining. The system is directive and the pupil must follow the chaining envisaged. The tutor uses both proactive and reactive scaffolding, the first one occurs before the step it addresses, whereas the second (feedback) occurs after the step.
Baghera (Laboratoire Leibniz, 2003)	Formal Geometry	The student writes a solution and a demonstrator (first order logic deduction motor) verifies it. The return file contains an annotated proof, each reasoning step accompanied by an annotation attesting if it's correct or not. This file also contains a series of global annotations indicating if the proof is correct relatively to the problem's hypotheses.
Cabri-Euclide (Luengo, 2005)	Formal Geometry	For every statement written by the student, the system first verifies if the statement refers to an object from the constructed shape or if the property can be verified by one of the Cabri-géomètre's oracles. When the student invokes a definition or a theorem, the system verifies the coherence of the deductions or the consistency of the demonstration, automatically or at the student's request.
Geometry Explanation Tutor (Aleven et al., 2002)	Formal Geometry	Based on the standards of cognitive tutorials, the system considers a sort of gap with the proper course planned by the curricular architecture. For example, when the student writes the justification of an inference, the system returns a message according to an ensemble of «correct» or «partially correct» explanations.
AgentGeom (Cobo et al., 2007) and **Turing** (Richard & Fortuny, 2007)	Cognitive Geometry	Created according to the specifics of a problem situation and a given didactical contract, the system returns a message depending on the nature of the process under way (solving or writing of the solution) as well as on the student's action's heuristic or discursive value. The learner's iterative model is founded on inferential and expert construction graphs.

constitute a source of insight into the evolution and progression of mathematical competencies and knowledge.

The idea of a tutorial system accompanying the student in the solving of problems and thus completing the work of the teacher is undoubtedly not new. Amongst the recent technological realizations that relate best to **geogebraTUTOR** (Richard, Fortuny, Hohenwarter, & Gagnon, 2007), more precisely to the current notion of our tutorial system, it is suitable to mention,[1] to begin with, the **Advanced Geometry Tutor** (Matsuda & Vanlehn, 2005), the **Baghera** project (Laboratoire Leibniz, 2003), the **Cabri-Euclide** microworld (Luengo, 2005) and the **Geometry Explanation Tutor** (Aleven, Popescu, & Koedinger, 2002). All these systems essentially establish themselves according to formal geometry models that, in spite of clear Information Technology (IT) programming advantages, imply supporting an axiomatic approach for the student's development of geometric competencies, sometimes in a deterministic way (see Table 1). The paradigms of reference can be situated between Kuzniak's (2006) natural axiomatic geometry and formalistic axiomatic geometry.

There are the **AgentGeom** (Cobo, Fortuny, Puertas, & Richard, 2007) and the **Turing** (Richard & Fortuny, 2007) systems which are anterior to geogebraTUTOR (GGBT). Contrarily to previous systems, these devices are essentially based on cognitive geometry models: AgentGeom's tutorial action occurring mostly during the geometrical shape construction and Turing's occurring while the solving of a problem takes place. In regards of Kuzniak's (2006) geometric paradigms, the reference is situated in this case between the natural geometry and the natural

[1] Although they are relevant, we left out the first generation systems like **Geometry Proofs Tutor** (Anderson, Boyle, & Yost, 1986), **Mentoniezh** (Py, 1996) and **Géométrix** (http://geometrix.free.fr/ by Jacques Gressier) to avoid lengthening our paper.

axiomatic geometry. The matters of logical coherence and system of axioms, essential when it comes to formalistic axiomatic geometry, do not act as a framework for these tutorial systems. This allows for instance the integration of a proof and refutation dialectic in the student's discovery processes (in the sense intended by Lakatos, 1984), including the justification role of the counter-example within the instrumented reasoning steps (Hollebrands, Conner, & Smith, 2010).

2 Hypothesis of GeogebraTUTOR Project

Stemming from a multidisciplinary project between didactics of mathematics and computer science, GGBT defines itself as an intelligent tutorial system (ITS) that supports the student in the solving of complex problems by assuring the management of discursive messages as well as the management of problems. By *complex problems*, we mean the existence of many solving processes (heuristic requirement), the mobilization of a network of mathematical concepts and processes (cognitive requirement), the existence of an argumentative approach, of a multi-stage reasoning or non-routine calculations (discursive requirement) and the development of groups of competencies that go beyond simple reproduction (competential requirement). In this section, we present the assumptions that underlie the research and technological realization project before developing, in the next sections, the contextual framework and the elements of our adaptive research approach.

2.1 Focus around the student–milieu system

The notion of interaction between the student and the milieu formally appears as the basic concept in the theory of didactical situations in mathematics by Brousseau (1998). According to Margolinas (2009):

> Brousseau will consider the student-milieu interaction as being the smallest cognitive interaction unit. A state of balance for this interaction means a status of knowledge, the imbalance student-milieu generating new knowledge (pursuit of a new state of balance) (pp. 13–14, our translation).

In its primary definition, the milieu presents itself as the antagonistical system to the taught system. However, this definition being very broad led Margolinas (2009) to propose a model for the structuring of the milieu which, from the professor's study point of view, allows the description, in a very thorough way, of the knowledge at stake in a didactic situation in the broad sense. If the building of the milieu characterizes each fragment of knowledge by the specific related situations, it is to allow the student's strategies to be motivated by the necessities of their relations to the milieu. According to Brousseau (1998):

> The modern conception of teaching will therefore ask of the teacher that he or she provoke in the student the sought adaptations by presenting him with a judicious choice of problems. These problems are chosen in order for the student to be able to accept them; they must drive him to react, to talk, to reflect and to evolve according to his free will. (…) The student knows that the problem has been chosen for him or her to acquire a new knowledge but he or she must also know that this knowledge is entirely justified by the internal logic of the situation and can be built without appealing to didactic reasons (…). Not only can it, but it must since the student won't truly acquire this knowledge unless he or she is capable of making use of it on his or her own (…) (p. 59, our translation).

This idea on necessity is important since it specifically establishes the notion of conception on the interaction between the student and the milieu (Balacheff & Margolinas, 2005). Moreover, it situates the question of the development of the student's autonomy in the acquisition of knowledge without him having to rely on the input of the teacher or classmates (Richard, 2004a). However, it supposes the existence of situations perfectly adapted to the coveted knowledge and it leaves to the student all the responsibility of managing the solving process, thus the very acquisition of new knowledge. Without going into a critique of this kind of situation and since the student is rarely alone when solving a problem—he also receives answers as he phrases his questions—we propose to introduce in the notion of milieu, the providential cognitive contribution which would possibly change the objective of a potential fundamental situation, but not the fundamental objective of the situation.[2]

2.2 The interactions "student–milieu" in GeogebraTUTOR

It should be stressed that or tutorial system is intended to supplement the teacher's work, not to replace this one. In other words, our system is fashioned to integrate the didactical contracts of real classes while benefitting from

[2] With his *fundamental situations*, Brousseau considers that for any mathematical knowledge there exists a family of situations likely to give it a correct sense. However, this existence is theoretical and is distinguished from the actual conceptualisation exercise (search of sense, formation of conceptions). Thus, the pupil who poses a problem in interaction with his or her companions or solves a problem that one proposed to him or her gradually discovers the objective during the devolution process. If the state of knowledge actually reached tends towards the sought knowledge, it will never have the scale of it since part of it will have been managed by the milieu.

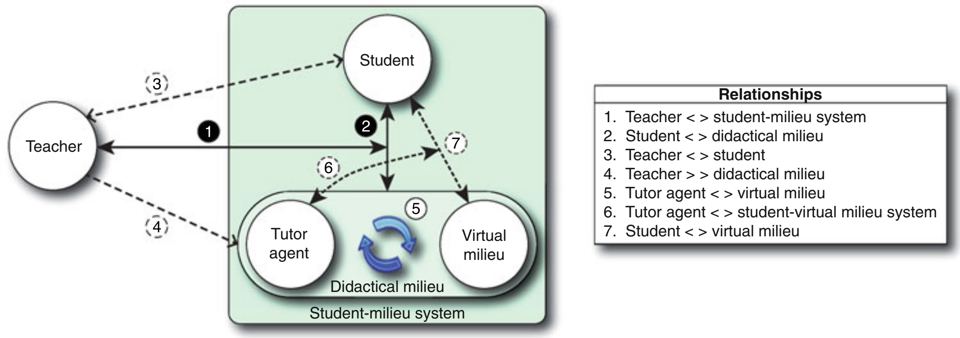

Fig. 1 Situational map

the dynamics of these for its development, thus justifying our interest towards didactical engineering to give account of pupil's mathematical competencies (see Sect. 4). Indeed, our research device gradually considers the effect of the didactical situations during the problem solving, of which the providential contribution of information from the tutor agent's messages. Normally, the didactical situations relate a teaching system to a taught system. One of the subtleties in our system is that this relationship also appears within the milieu, phenomenon shown in Fig. 1 with the concept of *didactical milieu*.

Regarding the student's conception, our ITS poses a simulated didactic relationship in which the tutorial agent plays, in spite of an individualized personality according to the iterative learner's model, a teacher role which is complementary to the one of the regular teacher. Even if he does not teach, the teacher supports the student in the devolution of the problems and, although indirectly, institutionalizes certain fragments of knowledge. This means, from the point of view of the theory of didactical situations, that the main act of the teacher (relationship 1 on 2 in Fig. 1) is transposed into the "student–milieu" system. Since the teacher's main intervention takes place with a system, which is itself composed of the student–milieu interaction (relationship 2), for the tutor agent the main goal is to intervene in the interaction with a more specific milieu (relationship 7). In fact, the notion of milieu needs a new distinction: the "didactical milieu" is antagonistic to the taught system (original definition of the author), in which the tutor agent appears as a sub-system, and the "virtual milieu", which competes with the student with whom it negotiates, thus establishing, conjointly, the main act of the tutor agent (relationship 6 on 7).

This corresponds rather well to what we know of the teacher's situation in the classroom: the teacher influences the student's milieu—e.g., he chooses problems for the student to solve and the conditions applying to the solution—and the student's interaction in this milieu—e.g., the providing by the students of answers to the problem or of the difficulties they encounter—supply information and feedback to the teacher. For the tutor agent, this feedback allows it to choose problems accordingly (relationship 5). However, the relationships that concern us are those that provide specific information with regard to mathematics and learning of mathematics. In other words, even if the teacher interacts directly with the student and impacts his milieu, his actions and interactions remain secondary (relationship 3 and 4). In the same sense, the tutor agent can return practical messages to the student, but these remain subordinate to the interactions raising mathematical knowledge.

2.3 Connections between reasoning and the operational dynamic figure

The notion of reasoning is closely linked to figural activity and the question of its expression arises in an obvious way during the practice of dynamic geometry. Traditionally, reasoning expresses itself with the help of discourse in order for it to be capable of managing (Duval, 2005) or accompanying (Richard, 2004b) the coordination of different types of reasoning or, in general, the functioning of the figural register. However, in spite of its dominating role in Duval's (1995) theory of language functions, discursive expansion is insufficient to convey the reasoning, which expresses itself through actions or by the use of certain technical tools. This can easily be observed in a paper–pencil environment when one uses a pantograph to create similar shapes or the Peaucellier mechanism to draw a straight line (Bryant & Sangwin, 2008). When it is the user who determines when and why he needs it, the technical tool essentially manages the representation in the movement, the dynamic coherence of the mechanism as well as the consummate logic of his construction.[3]

The idea of "consummate logic" readily projects itself to the usual production of a geometrical drawing. As a result on paper, the drawing constitutes the outcome of a reasoning process, which expressed itself through the use

[3] The design of a material tool, which is intended to produce a geometrical object in the paper–pencil environment, rests on a fully accomplished construction logic resulting itself from a geometrical model.

of a ruler and compass or simply freehand. Since the drawing conveys a reasoning process, any experienced reader even if stranger to the representation and construction processes is able, at least partially, to understand the underlying logic and to draw new properties from it, a part of the principle behind what are called visual mathematics (Alsina & Nelsen, 2006). Now, if the drawing appears at the interface of a dynamic geometry integrating tutorial system, the reader can also act on the object to test the limits of the representation or of the "consummate logic", renewing the classic link between reasoning and the geometrical shape in the student–milieu interaction. Since there are mathematical models underlying the functioning of these tools, some of these are not necessarily accessible to the student and their knowledge is generally neither a fact issue, nor a principle issue. However, their effects allow the student–milieu system to produce structured and functioning reasoning though the responsibility of controlling knowledge remains shared within the system. In Sect. 4.4, we illustrate three types of sharing in the use of GGBT depending on the inference used, including those justified with the help of oracles.

If we consider the geometrical figure's operational character, this brings us closer to the notion of geometrical space and of the didactic issue raised by Kuzniak (2006) when he proclaims that the teacher must "make sure the learner reorganises the diverse components of his geometrical working space in a coherent and operational whole" (our translation). By operational dynamical figure, we intend the figure which is constituted starting from the means of action or thought of the pupil in interaction with the milieu, in order to obtain a semiotic, cognitive and given situational result (Coutat, Laborde, & Richard, 2010). Based on the idea of an object-process, the reasoning intervenes as a treatment procedure as well as a control structure in the student–milieu interactions. This way, one same control structure may govern many different reasoning expressions like deductive reasoning, which controls the instrumented construction of a figure or the reading of invariant properties when animating a dynamic construction.

2.4 The instrumentation in the "student–milieu"

The integration of a tutor agent to the student–milieu system as well as the action's role in the constitution of the operational dynamic figure requires a review of the classic distinction between instrument and technical tool. Stemming from the instrumentation theory by Rabardel (1995), which proposes a cognitive approach to contemporary instruments:

> The instrumentation processes relate to the emergence and the evolution of the using patterns and of the instrumented action: their constitution, their functioning, their evolution by accommodation, coordination, combination, inclusion and reciprocal assimilation, assimilation of new tools to existing patterns, etc. The progressive discovery of the intrinsic properties of the tool by the subjects is accompanied by the accommodation of their patterns, but also by changes of significance of the instrument resulting from the association of the artefact to new patterns (pp. 111–116, our translation).

Either by the use of a simple construction primitive or from a set of significant actions, these actions or thinking modes can be interpreted with various levels of granularity according to the relations with the virtual milieu. The way by which the instrument is formed by a student is called instrumental genesis, phenomenon that can gladly be linked to the concept of devolution. If the genesis and devolution processes require an appropriation of the issues at stake in the problem or task, it is because the teacher or tutor agent has delegated, for some time, the responsibility of the mathematical work to the "student–milieu". The appropriation would in fact only make sense once the task is accomplished or once the problem is solved. Moreover, in the creative process by which we go from an initial state to a final state, Rabardel (2001) proposes the notion of instrumental mediation. That is to say that the teacher and the student are in a relationship mediated by the instrument, as are each of them in their relationship to the goal of the tasks and to the means of learning (knowledge or mathematical competency, see Sect. 3). Of course, the instrument is also mediator in the relationship to itself, as well as in the relationship with the intellectual milieu.

2.5 Some IT questions

Regarding the functioning of the system, the student constructs shapes, writes discursive propositions or calls upon mathematical properties in the solving of a root problem. The tutor agent returns to the student a discursive message based on his significant actions and, when the student stalls, may propose a related sub-problem to boost the initial solving process. We name the sub-problems *cognitive messages*, by analogy to the approach of Carnegie Learning's Cognitive Tutor.[4] However, in opposition to the approach by sub-problems equivalent to parts of the root problem, our cognitive messages are related to it thematically according to Neighbourhood criteria and anticipated difficulty levels. If the management of the discursive messages raises the IT issue of the recognition of the student's reasoning process from his significant actions, the

[4] See http://www.carnegielearning.com.

management of the cognitive messages raises the IT issue of the conceptual or procedural, heuristical, semiotic or metamathematical changes during the solving process (rupture point). Finally, the management of the problems raises the IT issue of the recognition of the similarity between problems while avoiding, by means of all the messages, to give at the same time answers that modify considerably the stakes in the devolution of the root problem. These questions are not quite hypotheses of the project; however, they develop the essential elements of the technological problematic regarding the hypotheses laid down in the previous sections.

3 Context

The notion of competency as general perspective, which covers traditional mathematical knowledge, has allowed us to drop the former curricular obligation of learning by pedagogical goals. In other words, we do not ask for as much attention to the goal itself, rather than to the way it can be met. Moreover, thinking in competency terms requires dropping the tendency to segment reference knowledge and consequently to evaluate the level of appropriation one after the other, heritage of deterministic conceptions stemming from the former goal corpus. Without getting into a debate between atomistic attitudes and holistic explications (emergent totalities), we can already advance that our tutorial system is primarily based on a structural approach of reasoning (see Sect. 4.4). Besides, if we have already introduced the notion of *complex problem* in Sect. 2, it was to distinguish ourselves from directive approaches adopted by certain recent tutorial systems. In this context, the student exercises or improves his knowledge while interacting with the milieu and he or she proceeds progressively throughout the problems he or she sets for him or herself or knows how to solve (Polya, 2007).

At the most general level of the Québec School Education Program, the mathematical competencies precisely concern problem solving as well as mathematical reasoning and communication using mathematical language (MÉLS, 2006 & 2007). Far from wanting to subordinate knowledge to a new systemic position (by competency), the program asks that we focus on the student's know-how: in his or her quest of sense according to the concepts and referential mathematical processes, his or her use of reasoning in the structuring of his or her knowledge and participating in situations representing an authentic challenge. As a backdrop, this means that the student does not study the subject of mathematics for itself, but rather train him or herself to practice them. If this disposition accentuates a definite relativism in comparison to the education content, it does not mean that the order in which is introduced mathematical knowledge is secondary. This arrangement continues to be part of the planning done by the teacher and integrates itself to the instructional design, at least in its intention (see Sect. 4.7). In other words, it is out of the question to prescribe a naturalistic evolution of the mathematical content engaged by the problem organization or to reproduce presupposed solution scenarios. It is rather a matter of making the acquisition of coherent and stable knowledge easier during the development of the student's problem solving mathematical competencies.

4 The experimental development: an adaptive research approach

As it was already brought to the fore by Guin (1996), the efficiency of an IT environment for human learning depends on the modelling of human behaviour, which is anterior as well as posterior to the environment's conception. Since the student–milieu system breeds its own cognitive model, we consider that our tutorial system must allow a certain arrangement of its characteristics/components, such as the problem organization, the available strategies or the tutor's role, according to the student's instrumented behaviour. Such an expectation requires several back and forth motions between design and validation phases towards the convergence between the didactic and informatics models created this way. From a learning point of view, this supposes that the *appropriate geometrical working space* (in the French original sense «espace de travail géométrique idoine» of Kuzniak, 2006) reasonably exists following this convergence. However, from a research point of view, this requires the conciliation of two usually opposed approaches.

4.1 Development cycles: design by layer integration

Must we mention from the start that our approach by convergence is coherent with the idea of design in use developed by Rabardel and his colleagues. Regarding this, Rabardel (1995) already said:

> The approach in terms of instrumental genesis (…) allows to theoretically establish the articulation and continuity between institutional artefact design processes and the continuation of the design within usual activities. The instrumentation processes participate in the global design process by following a cycle: operational modes (envisaged by the originators), designs of use (elaborated by the users), new operational modes envisaged by the originators and stemming from designs in use. The instrumentalization

> processes follow a parallel cycle: constituent functions of the artefact (defined by the originators), made up functions (by the users), inscription of these made up functions in a new generation of artefacts (by the originators) (p. 5, our translation).

This means that not only is the final result a constituent of the system's tutorial design, it is responsible for its existence. This explains why a thorough didactical reflexion is needed for every stage of the development cycles. Thus, our system was conceived to produce a class of effects (support learning by messages, problems and controls) and its implementation, under the conditions which were envisioned with each cycle, allows the update of these effects following a use noted at the time of experimental phases.

A first back and forth motion (cycle 1 in the Fig. 2) stems from our initial system in which only the inferential and construction graphs for a group of problems are implemented. The validation phase aims at checking the relevance of these graphs and at studying the types of assistance messages given by a human tutor during the solving of these problems. The cycle is connected with the structural design of the messages and starts the second back and forth motion (management of the problems' cycle). This time, the validation phase aims at checking the discursive relevance of the tutor agent's messages and at studying the types of sub-problems (cognitive messages), which are returned by the human tutor when the student is stuck in the resolution of a problem. In the same way that the teachers had at their disposal a set of reference messages at the time of the first cycle, they could here have a group of reference sub-problems. This phase is currently at the experimental stage in real classes and it is about to engage the programming of the structure of the problems. The second cycle is connected with the structural design of the messages and starts a new back and forth motion (management of controls' cycle). The third phase of validation seeks the checking of the relevance of the tutor agent's integrated messages all while studying the students' instrumented behaviour towards the evaluation of their mathematical competencies by their teacher. In theory, this phase should lead to the organization of an evaluative structure for the tutorial system.

4.2 Merger of two paradigms from the point of view of didactics epistemology

The implementation of geogebraTUTOR is part of a research and developments project that begun with the AgentGeom and Turing projects: it partially borrows its conceptual melting pot and its methodological approach. In other terms, the learning models we claim to draw with the usage of our ITS are essentially based on the theory of didactical situations in mathematics (TDS), the theory of language functions (TLF) and the instrumentation theory (InsT), as far as introduced in Sect. 2. From a methodological point of view, we interweave the grounded theory analysis (GTA) by Glaser and Strauss and the didactics engineering (DE) by Artigue (1990) as source methods. Therefore, with our prior projects, we verified a first extension of the TDS + TLF + InsT theoretical framework when these theories are applied to an ITS for the learning of geometry at a high school level. However, the hypothetical-deductive paradigm, which had then allowed us to validate separately our research hypotheses regarding confined situations, has revealed itself to be insufficient in an integration perspective of AgentGeom + Turing, which allows the dynamic evolution of the student–milieu interactions. That is why the emergence of learning models adapted to the use of GGBT requires to add a comparative-inductive paradigm to our current approach, a paradigm we propose on the basis of an GTA + DE integration.

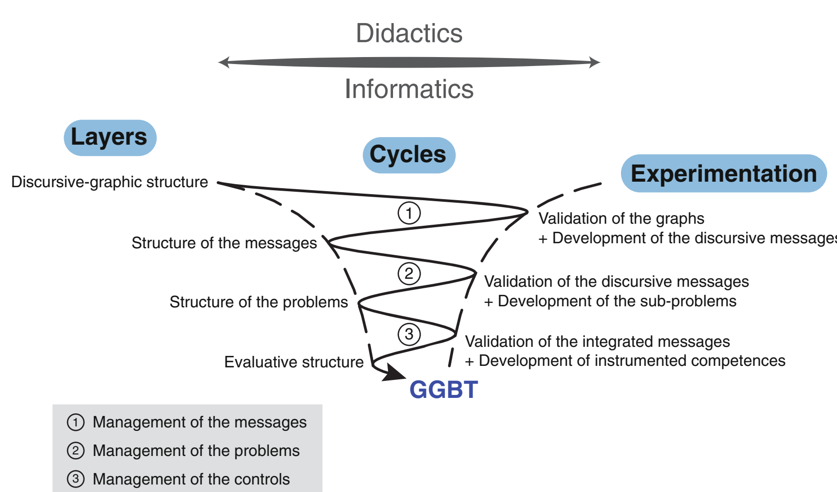

Fig. 2 The experimental development of GGBT

It is suitable to stress the fact that our approach differs from regular experimental methods, in the field of science of education, by the generating of hypotheses and by its validation mode. In their founding book, Glaser and Strauss (1967) made the following proposal:

> While verifying is the researcher's principal and vital task for existing theories, we suggest that his main goal in developing new theories is their purposeful systematic generation from the data of social research (p. 28).

Employed here in a rather broad meaning, the word theory refers to a more or less organized body of abstract ideas or concepts applied to the research field. In fact, what characterizes the systematic inductive approach of these authors is the way they look to phrase hypotheses from what goes on in the data as if the goal was to zoom into the incidents, that is to say the units of analysis in the grounded theory analysis (Glaser & Strauss 1967). In our project, we first and foremost explicit the problems or main themes drawn from the analysis of the student–milieu interactions in order to mutually link them ("each incident is compared with other incidents", Glaser & Strauss 1967, p. 114) or to interpret in the eyes of an emergent category ("with properties of a category, in terms of as many similarities and differences as possible"). However, our approach proceeds by a reasonable convergence of successive design and validation phases, which is susceptible of causing the variation of the likelihood or probability nature of the hypotheses during the research. Anyhow, according to the method favoured by the didactics engineering, our validation mode is internal and is based on the comparison between an a priori analysis, which relies on certain hypotheses, and an a posteriori analysis of the visible and significant student–milieu interactions. In addition, these interactions consider the integration of implicit models, amongst which the mathematical models implemented in the IT device, like those that underline the interface or the computations, added to the implicit key aspects of the instructional design or of specific didactical contracts (in regards of relationships 1 and 6 of Fig. 1).

4.3 Student's interface and the system's structure

From a computer science point of view, it is important to present the user with an interface paradigm with which he

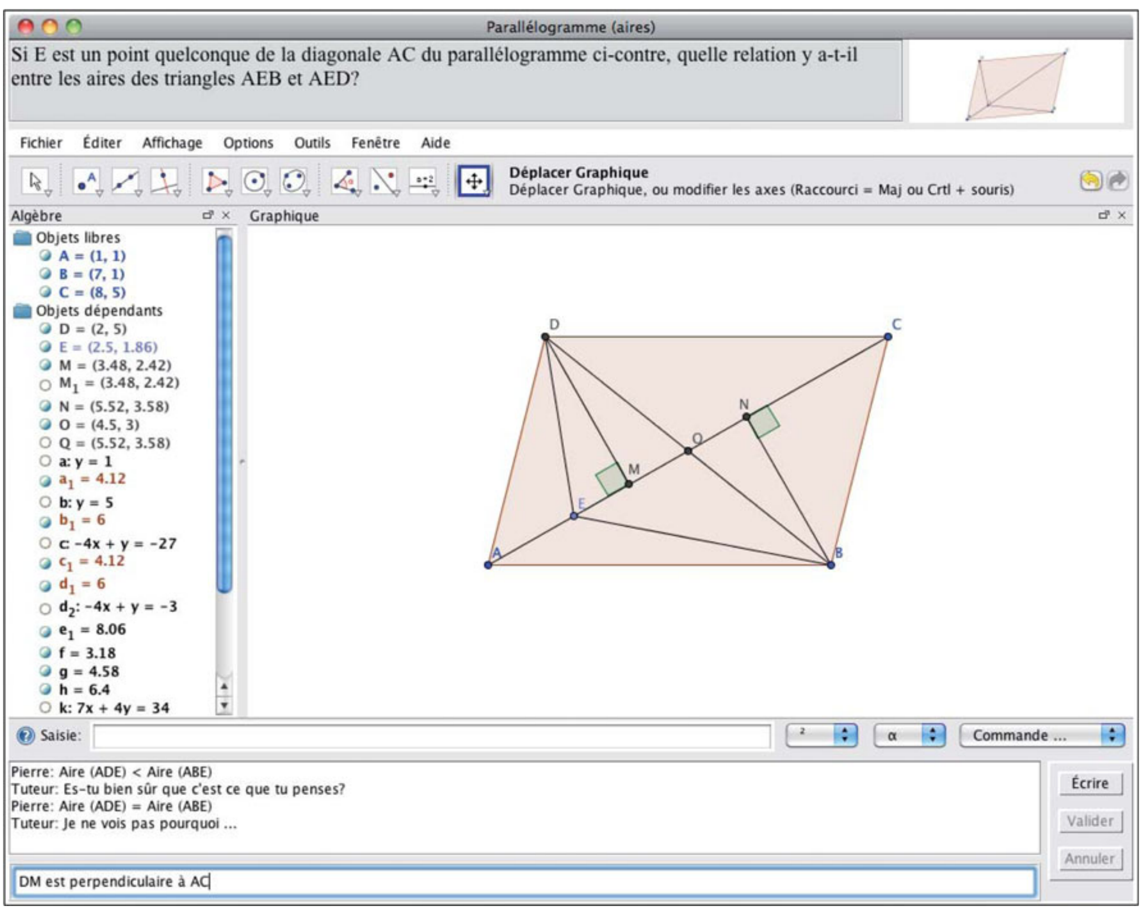

Fig. 3 Student's interface

is familiar in order to facilitate the instrumental genesis or to mechanize, in a certain way, the schemes of use (in the sense of Rabardel, 1995). The interface we propose to the student is composed of three distinct areas (Fig. 3). At the top of the window, over the menus, can be found the problem statement to which we can link a drawing. These elements are continuously shown and cannot be modified by the student. In the middle, the interface of GeoGebra (GGB), without modifications, is found. It can propose an initial shape, matching or differing from the drawing linked to the problem statement, on which the student is susceptible of launching his work (construction, movement, etc.). The tutor agent follows closely every action of the student in the GGB window and tries to identify what seems to be the student's solving strategy, which then allows the system to assist or advise the student.

Under the GGB module, we exploit the paradigm of a chat window to simulate a dialogue between the student and the ITS. This one is disposed to respond to the student's actions by a message that appears at the bottom of the window. The actions can be discursive (lower module), graphic or symbolic (GGB module), while the messages of the tutor agent can be discursive (propositions in the lower module) or cognitive (hyperlink towards a sub-problem, see Sect. 4.6). Like in regular chat systems, the student (and the system) has access to previous messages. Nevertheless, the student's interface underwent some modifications according to the experimental development. Thus, at the time of the management of the messages' cycle (Fig. 2), the regular teacher could recognize the student's progression according to the completion degree of a path in the graph (see Fig. 4, the list of numbers following «Tuteur» represents the percentage of completeness for each path).

As we write this article, we have chosen GeoGebra as our dynamic geometry interface because it is free and open source and it implements what is needed for our tutor. However, in the future, we aim at adding the possibility of using any application that implements the Intergeo Project (I2GEO) Application Programming Interface (API) as long as the API provides what is crucial to the running of our tutor. For the moment, the I2GEO API lacks two behaviours that our tutor needs. Because every interaction contains a cognitive message, the first behaviour needed is an event system that notifies the tutorial application when a student interacts with the dynamic geometry application. The second specification missing in the API is a way to get an application panel that can easily be embedded into our tutor. Consequently, as soon as the I2GEO API suits our needs and applications implement it, we will be capable and eager to experience the association of any dynamic geometry application to our tutorial system.

In GGBT, the data used to describe a problem are saved in text and extensible mark-up language files. However, for the geometrical figure, which is shown in the dynamic geometry window of our application, we plan on generating I2GEO format files. Since this format is universal amongst I2GEO capable applications, the user will be able to choose his preferred application and be assured that it will transmit the starting figure. Consequently, this represents an asset since any given student will be able to use a geometry software interface to which he or she is already accustomed, minimizing the cognitive load and flattening the learning slope.

Figure 5 illustrates the internal part of the system. Our ITS is composed of two main subsystems containing six elements each: the GGBT on the left and the tutor agent on the right. This partition aims at granting independence from the student's interface to our tutor agent, to facilitate, amongst other things, the test and integration of the tutor with the different systems and to grant/allow the possibility to create a client–server application. However, for the moment, the system is a standalone Java application, which requires no network besides the downloading of the application.

The GGBT subsystem essentially contains the modules that are related to the user's interface. The GGBT and GGB modules contain the code for the interface. The GGB module supports a computer algebra system (CAS) that can be used by the student to resolve some selected problems and an Automatic Deduction system (AD in Fig. 5) that can be used by the ITS to execute some verifications on the student's input. Furthermore, this module generates an event log, which can possibly be analysed for the interpretation of the student–milieu interactions. To these are added the graphic and discursive analyses, which allow the transformation of raw actions at the interface into significant actions for the system. In other words, the tutor can

Fig. 4 Appearance of the student's interface dialog module during cycle 1

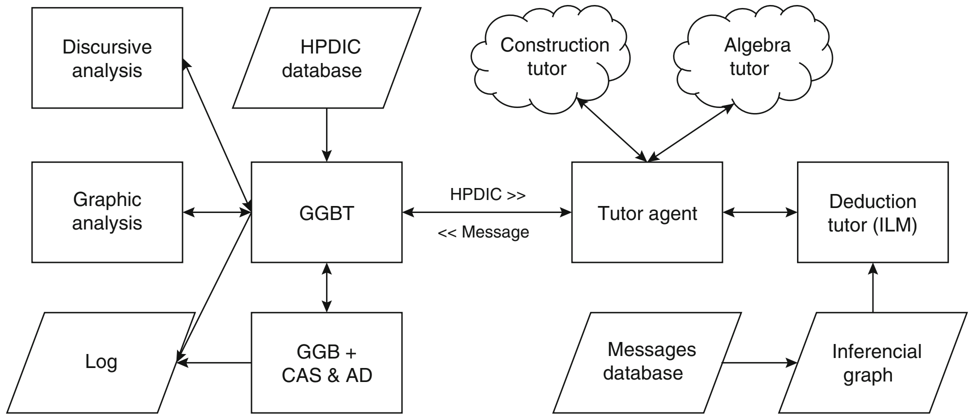

Fig. 5 System's structure

treat the student's input as HPDIC messages (acronym of Hypotheses, Propositions, Definitions, Intermediate results and Conclusions in Fig. 5). The valid messages for the tutor are stored in the HPDIC database.

Each valid HPDIC message is sent to the tutor subsystem, which is a multi-agent module that aspires to help the student throughout the resolution, from problem solving to composition, i.e., the writing of a solution (see Fig. 6). According to the message type, the tutor asks for help from the construction, algebra or discursive tutor. The construction tutor essentially uses the AgentGeom approach (Cobo et al., 2007). The algebra tutor is still at its conception phase since it depends on the recent development of the GGB CAS module. The deduction tutor, based on the Turing approach (Richard & Fortuny, 2007), is called when the student enters a discursive proposition at the interface of GGBT. We then consider that he or she is explicitly trying to produce an inference in order to generate a proof. This tutor contains the inferential graph as well as a bank of discursive messages that could be used to assist the student. The output of the discursive messages is based on the iterative learner's model (ILM in Fig. 5), which is defined as an emergent support model, characteristic of GGBT, which creates the illusion that a communication is taking place with the student (see Sect. 4.5). Finally, the tutor agent can reach a state in which the message addresses different aspects of the problem, where help from more than one tutor at a time is necessary (intersection of construction tutor and deduction tutor in Fig. 6). In this case, the tutor must decide what message is more useful to the student.

4.4 Reasoning, graphs and inferences

Even if we suppose that in their schooling students must one day adhere to pre-existent mathematics (an expert mathematical model), the stakes related to the development of their mathematical competencies result in a progressive and mostly contextualized learning. In fact, about the didactical issue, we are reminded by Kuzniak (2006) that:

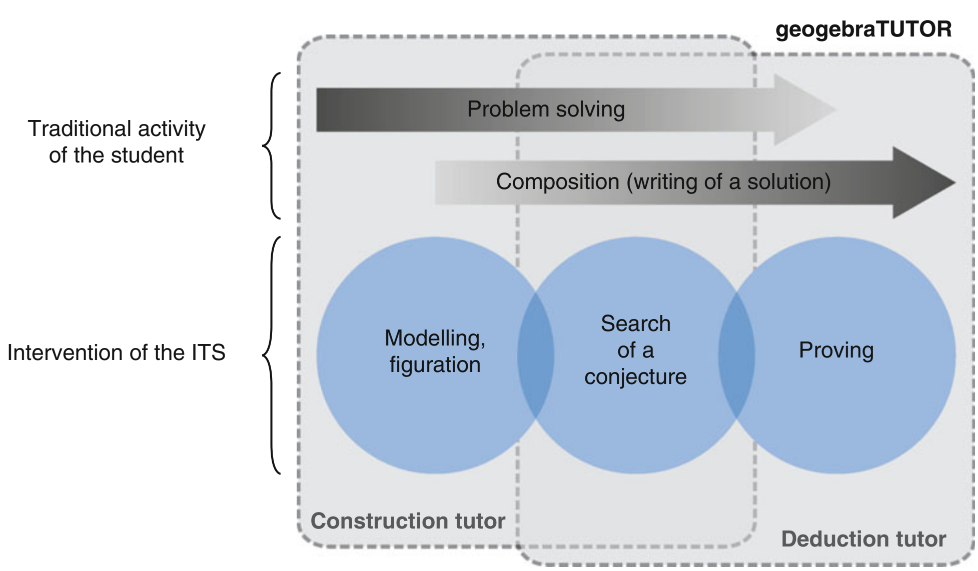

Fig. 6 GGBT's rendering with regard to the steps of the student

> In a given school environment, the solving of a geometrical problem supposes that it was possible to organize a geometrical working space to allow the solving of the problem by the student (p. 178, our translation).

In other words, the particularity of the didactical contract in which the students evolve or even the specificity of each problem situation seem to make difficult the practice within an expert model. One of the reasons that lead us to the idea of cognitive geometry lies in our commitment to introduce these contextual aspects into the development of our ITS. Since cognitive geometry can be locally coherent without necessarily being globally coherent—contrarily to formal geometry—we adopted a structural approach to reasoning, which not only combines these aspects of geometry, but also first and foremost allows the IT programming.

Inspired at first of Duval (1995) and Richard's (2004a) works, the concept of inference expressed by dialogue, figural representations and instrumented action, considered as a voluntary reasoning step, intervenes at the heart of our system. Each inference respects an "antecedent → consequent" structure with a justification that "controls" the reasoning step, allowing it or participating in it in order to allow the student to produce a sequence of inferences that remain coherent according to the logic of the problem or significant in its context. The question of significance that is here introduced does not aim only at satisfying the possibilities of cognitive geometry (e.g., acceptance of certain discursive inferences, in the sense of Duval (1995), or of inferential shortcuts depending on the habits of a didactical contract), but also to deal with the inductive nature of inferences allowed by the practice of dynamic geometry or by the modelling activity.

To illustrate our idea, we introduce three types of inferences in Table 2, regarding control of the justification in the student–milieu interactions, taking charge of the inferences by the system as well as the discursive-graphic effect of these on the student's reasoning. If the two first inference types, respectively, relate to deduction and induction, the third calls to mind the abduction in Pierce's semiotic in the sense that in the student's reasoning, the produced conclusion is only cognitively probable although it seems necessary to the student–milieu system.

At first, for each problem situation, we associate construction graphs (reference shapes) and for each

Table 2 Inference types treated by GeogebraTUTOR

Typical inference	Control of the justification	Taking charge by the ITS	Discursive-graphic effect
Since « (PM) ‖ (OA) » by hypothesis, so, according to «the Thales' Theorem», « $\frac{PM}{4} = \frac{3-OM}{3}$ ».	Managed by the student	Comparison with the inferential graph	Transformation of states recognised by the student, the consequent is obtained by deduction (cognitive or formal)
Confrontation between a satisfying configuration and a non-satisfying configuration by dynamism of the shape or by construction of it, letting emerge «maximum area is obtained when P is the middle of [AB]».	Shared management by the instantaneous actions-feedbacks	Comparison to the collection of construction graphs and to the inferential graph related to the presumed model	Transformation of significant actions into a state by induction, the consequent is obtained by interpretation of the satisfying configuration (figural inference)
Since « $PMNO = 4 \cdot OM - \frac{4 \cdot OM^2}{3}$ », so, according to « $fMax\left(4 \cdot x - \frac{4 \cdot x^2}{3}, x\right)$ », «the maximum area is obtained when $OM = \frac{3}{2}$ ».	Chosen by the student but essentially managed by the milieu	Use of the CAS or of a GeoGebra oracle, formal comparison of symbolic expressions	Transformation of states by CAS function or oracle, the consequent is produced by the system or chosen by the student according to a list of possibilities («fill the blank» expressions)

The mathematical expressions found in the table come from the solution to a problem, which does not appear in the text. They play an illustrative role in the propositions in quotation marks construction graph, an inferential graph. Each construction constitutes a modelling of the problem, which summarizes the figural result of a satisfying configuration. These configurations are a result of the basic space of the problem (Cobo & Fortuny, 2000), meaning they origin from the gathering of the solving strategies that can be collected in a given classroom by a teacher or an expert. This way, with the AgentGeom approach, we can compare the student's actions to the collection of satisfying configurations in order to anticipate the model in which the student seems to be evolving. Then, for each satisfying configuration, the correspondent inferential graph must be formed. This graph, inspired by the Turing approach, reveals the different proof strategies that follow from the reading of a satisfying configuration in the shape of a series of well-structured inferences from discursive, symbolic and figural propositions (in the sense of Richard, 2004b).

4.5 System's intelligence and iterative learner's model

In order to enable the system to state the stage of advancement of a student in his or her solution, we coded the construction and the inferential graphs. Firstly, the construction graph is a hierarchical list of actions which, when executed, produce a group of different geometrical figures that a student is susceptible of using when solving a problem. One of the main tutor agent's tasks is to determine which figure the student is drawing in order to help him or her when he or she encounters a difficulty while working on the construction. To achieve this, each path of the construction graph is extracted as a construction protocol, which consists of a series of actions, which, in the end, produce a certain geometrical figure, or more precisely, a construction showing a number of geometrical attributes. Moreover, one or more actions, called nodes, are grouped as milestones and help messages are associated to each milestone. Therefore, when a student starts drawing something, the tutor agent attempts to model what the user is doing and, using heuristics, compare the student's actions to the ones from the expert construction graph. Then, accordingly, the tutor agent activates the corresponding action's nodes in every construction protocol in which it appears.

When the student runs into an obstacle and asks for help, the system refers itself to the construction protocol and to the milestone that the student is currently working on in order to respond with an appropriate message. The milestone considered as the one the student is working on is the milestone that is not completed yet but has the highest completion ratio within a construction protocol identified in accordance with the same criteria. If the tutor is unable to find a milestone that is partly completed, it gives one or more hints to jump-start the student's construction process with the first milestone he or she has not yet worked on from the ongoing construction protocol the student has not yet worked on. Consequently, the student works on his geometrical figure, which, after some time and effort, results in something that can be referred to in order to reason towards the solving of the problem.

Each action on the figure can be significant for the construction graph, but, furthermore, it can also be meaningful for the inferential graph. In fact, it depends on the current construction the student is working on. The inferential graph, which is the second type of expert graph allowing the system to evaluate the stage of advancement of a student in his or her solution, is chosen to serve as template for the tutorial actions. Each construction having an associated inference graph, only after identifying which inference graph is appropriate, can the tutor proceed to activating hypothesis nodes according to the construction steps taken by the student. Some hypothesis inferences are purely discursive. However, some graphical inferences resulting from the construction process correspond to hypotheses form the inferential graph. These hypotheses are graphical inasmuch as they need to be graphically constructed by the student to be recognized and activated in the inferential graph. For example, if a student adds (constructs) a perpendicular line to an existing diagonal from the initial geometrical figure, the hypothesis node stating a perpendicular line to this diagonal can be activated in the inferential graph. Moreover, this action is necessary for the solving paths stating or based on this graphical element. In the end, the construction tutor has a double role: first it must help student construct a valid geometrical figure and second, it must activate hypotheses nodes according to the student's graphical interventions in the inferential graph.

An example of an inferential graph is shown in Fig. 7. In these graphs, the nodes P and D[5] are inference justifications, while the nodes prior to them (H or I) or posterior (I or C) are, respectively, antecedents or consequents of the justifications, the I node playing a double role depending on the considered justification. From a logical point of view, the entrances of each justification node are conjunctions, while all the other links are disjunctions. This means that all the antecedents and a justification are needed to legitimize a consequent. However, there may be many paths to legitimize a consequent, but only one is necessary to complete a proof, which can be considered complete when we obtain a path joining the hypotheses to the conclusion and when all antecedents of the justifications that

[5] The distinction between P and D is on the status of the node (property vs. definition) and not of their function in the inference. In fact, the definitions are always logical equivalences, while the properties are implications for which reciprocal logic is not necessarily true.

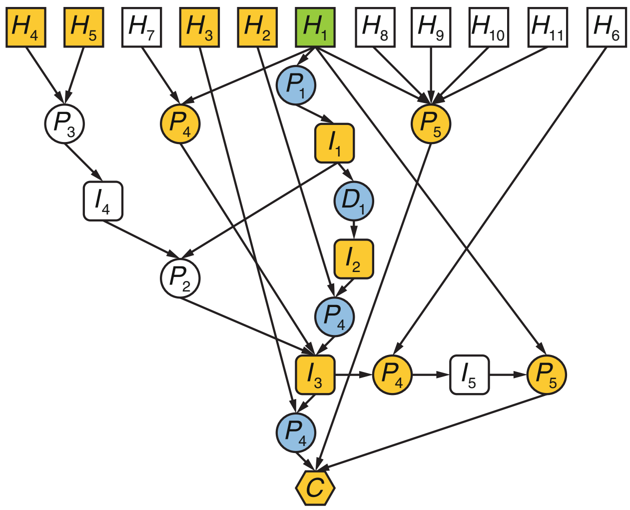

Fig. 7 Inferential graph AND/OR

appear in the path have been activated by the student's actions.

When the student tries to solve a problem, the tutor accordingly activates different nodes in the inferential graph. Nevertheless, since the nodes will generally not be activated in a purely descendent or ascendant order, as is presumed in the Matsuda & VanLehn (2003) approach, the iterative learner's model supports the student using the historic of his or her significant actions, respecting the heuristic aspects of his or her solving method. Theoretically, the system tries to develop a path around the last action of the student, but if he or she happens to be stuck, the learner's model is led to revive the solving process according to a previous significant action. Consequently, our objective is not to force the student into a deterministic mould, but rather to incite him or her to follow his or her intuition or solution until he or she obtains a complete solution that can be structured depending on the logic of the problem.

4.6 Identity cards and neighbourhood of the problems

As in an ordinary class, the organization of the problems is an important question for the development of mathematical competencies. In the classic contract, the teacher gives to the whole class the same group of problems to solve, these generally growing increasingly difficult. However, the stakes of the problems undergo little adaptation according to the evolution of the competencies of each student and even less to their instrumented behaviour. In Sect. 2.5, we introduced the notion of cognitive message; we also evoked the question of neighbourhood of the problems. Really, the criteria for neighbouring is based on a prerequisite characterization of each problem, which we call identity card, based on the categories "processes and concepts", "heuristics", "semiotic" or "metamathematical". Without going into details, we can stress that each category develops according to the shapes or values made possible by the realization of a given didactical contract. We can, however, mention that a part of the considered values are inspired by the I2GEO project's multilingual and multicurricular ontology, especially for the category "processes and concepts". Two problems are then neighbours if they share a same subset of values. The information supplied by the identity cards, according to the instrumented behaviours of the students, is susceptible to lead to learning itineraries adapted to the whole class.

The identity cards constitute a global description of the problem. But, when a student is stuck, it is usually because he or she is stumbled on a particular difficulty. In order to identify this difficulty, we can compare the student's effective method to the information, which, combined to the iterative learner's model, can be found in the inferential graphs. So, by identifying the steps the student is unable to deduce, we can suppose that the difficulty lies in part in the knowledge he or she should have used. This point allows the identifying and refining of the difficult steps into simpler tasks; it reproduces the functional behaviour of the teacher who seeks to help the pupil by returning sub-problems (cycle 2 in Fig. 2). A second criterion consists in comparing the graphs of chosen sub-problems to keep only the one that sticks the best with the presumed difficulty. Consequently, we consider it to be advantageous to address the issue of the neighbourhood of the problems, complementing the other issues: the reasoning, the graphs and the learner's model.

4.7 Learning models and diagnostic model

In spite of any promise of a diagnostic of instrumented behaviour, the tutor agent claims to act on mathematical conceptions, which reveal themselves locally by the student–milieu interactions. In reference to the knowledge model of Balacheff and Margolinas (2005), the action of this agent relies on the problems, the solving operators, the representation systems, as well as the control structures, these being the four components of a conception. However, the development of mathematical competencies is a long-term project. If our system undoubtedly influences their evolution, it is not so much because of the tutor's interventions, but mostly due to the choice of problems, which are the elements that globally create learning opportunities. This point of view leads us to consider that the learning model that ensues of the use of GGBT is bigger than its diagnostic model, this last one relying first on the student's conceptions. In other words, our system proposes to act on the development of mathematical competencies by offering a control on the acquisition of knowledge and an adaptation of the instructional design according to the student's instrumented behaviour.

With the cognitive messages' dimension, GGBT's instructional design is open and leads to learning routes far from deterministic script development. In fact, our tutorial system joins to the problem's internal logic, a transversal logic based on the instrumental genesis and the problem's organization in a curricular perspective. After our original research with AgentGeom and Turing, we have chosen to adapt a fine "automation" of a paradigmatic ensemble of seven complex problems. Then, to this, we added a set of neighbouring problems, which in certain cases claim to favour instrumental genesis (Iranzo, Fortuny, Richard, & Tessier-Baillargeon, 2009) and cover fully all of the geometrical concepts and processes from the official academic program (see Sect. 3). The diagnostic model is then founded on:

- The milestoning of the processes in problem solving and composition (Fig. 6) using markers to evaluate the local mathematical competencies' development as well as independence improvement, by means of the categories "processes and concepts", "heuristics", "semiotic" and "metamathematical".
- The profile definition using selected markers according to the construction and inferential graphs, which allows to locally create neighbouring problem trees and to globally plan adapted itineraries (set of root problems).
- The generation of forests (set of trees consisting of neighbouring problems) which are related to the itineraries and are susceptible of producing a curricular coverage of the geometrical aspects from the official academic program. Each sub-problem to solve within a tree becomes a cognitive message for which its resolution, with its markers, also integrates itself to the profile definition.

Of course, this is an internal model to the tutorial system. If in our research approach certain hypotheses must be phrased according to the incidents in the analysis of the student–milieu interactions, the validation process can use research techniques, which are external to the diagnostic model implemented during the cycle of the management of the controls (cycle 3 in the Fig. 2).

5 Conclusion

If the instructional design is undoubtedly profitable for the learning of high school geometry, the tutorial systems that we have inventoried were mostly developed on the basis of epistemological models. Even though by their formal characteristics these models favour the exercise of computer programming and, in some way, the programming of learning scenarios, they suppose that formalistic axiomatic geometry is in itself the pedagogical objective. As it was mentioned earlier in the hypotheses and context of our article, the perspective we propose is opposite and is founded on the idea of mathematical competencies' development. Our starting point is first and foremost didactical and our research approach claims, on the basis of student–milieu interactions, to respect the habits of the existing didactical contracts including, as part of these, the actual use of the tutorial system. This explains why we chose to bring closer informatics and didactical models in an approach proceeding by convergence of successive design and validation phases, adapting along the way with a suited methodology.

From a technological point of view, even if the backcloth of our ITS obviously is weaved around human learning and that we try to create a space of dialogue in the student–milieu exchanges, the discourse of the tutor agent overlaps with the discourse of the student without there being any real communication. Nevertheless, as a wink to Vygotsky, our experimental approach assures the existence of a sort of zone of proximal development, in spite of the tutor agent's essentially reactive character, which surfs on graphs built a priori. If certain IT issues remain unanswered, like the ones evoked in Sect. 2.5, we have to mention that the complexity of our approach forced us to defer the analysis, by the system, of the sentences written in natural language. For the moment, we work mostly on the other discursive aspects of the system and on the iterative learner's model, as well as on the link between these and figural modelling. The integration of a CAS, which depends on the current development of the GeoGebraCAS, and the organization of the cognitive messages are not in our priorities. Despite everything, the enrichment of the problem organization and of the graphs by instrumented solving with real students are in progress.

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