Tech Know Learn (2017) 22:83–104
DOI 10.1007/s10758-016-9283-y

ORIGINAL RESEARCH

Intelligent Tutoring and the Development of Argumentative Competence

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Published online: 12 May 2016

Abstract This ethnographical study aims to interpret how an intelligent tutorial system, geogebraTUTOR, mediates to the student's argumentative processes. Data consisted of four geometrical problems proposed to a group of four students aged 16–17. Qualitative analysis of two selected cases led to the identification of the development of argumentative competences by the students, as well as the level of influence produced to them. As regards the influence of geogebraTUTOR on the students, the study revealed that the interactions of tutor–teacher–student produced a significant number of mathematical learning opportunities of 'thinking strategically' type; establishing figural inference conjectures and fostering the transition from empirical to deductive argumentations.

Keywords Intelligent tutorial systems · GeogebraTUTOR · Mathematical learning opportunities · Argumentative competence · Geometry problem solving

Abbreviations

DGS	Dynamic geometry software
MLO	Mathematical learning opportunities
ITS	Intelligent tutorial systems
ggbTUTOR	GeogebraTUTOR

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1 Introduction

In mathematics education, most of the curricular frameworks, such as the Principles and Standards for School Mathematics from the National Council of Teachers of Mathematics (NCTM 2000) establish three key aspects to consider in the teaching/learning process: (1) the acquisition of the *mathematical competence*, and within it the *reasoning and argument* capability; (2) the required *attention to diversity* both from capacities as well as the individual care, inside and outside the classroom. Last but not least, (3) the use of technology in everyday mathematics teaching.

The use of technological resources or *artefacts* such as dynamic geometry software (DGS) like GeoGebra can be very helpful. However, it requires an appropriate *instrumental orchestration* from the teacher (Drijvers et al. 2010) to maximize *mathematical learning opportunities* (MLO) (Yackel et al. 1991). Along this line of ideas, the *intelligent tutorial systems* (ITS) in geometry are tools that support the attention to diversity and personal orchestration.

Different studies have addressed social interactions as a MLO element both in problem solving with small groups (Hitt and Kieran 2009; Sfard and Kieran 2001; Yackel et al. 1991) as well as in a whole class (Ferrer et al. 2014; Morera 2013). However, the study of individual tutor–teacher–student interactions with geometry problems is relatively unexplored (Cobo et al. 2007). The ITS were born from the idea of having an artificial tutor accompanying the student during problem solving. GeogebraTUTOR (ggbTUTOR) is the most recent reference (Cobo et al. 2007; Richard et al. 2011) and the one we evaluated in our research. In this entire context, we considered the following research question:

> How does geogebraTUTOR mediate in the development of argumentative competence by solving geometry problems in students aged 16–17?

Our hypothesis is that the way in which ggbTUTOR interacts with the student during the problem solving would have a certain degree of influence on the development of argumentative competence. To approach the research question, we have established an objective: to *interpret the effect of the tutor–teacher–student interactions in the generation of argumentative MLOs*.

In particular, we aimed to reach our objective in order to elucidate how tutor–teacher interactions had an influence on the students in the following aspects: firstly, the generation of argumentative MLOs; secondly, the evolving of figural inferences; thirdly, the improvement in the quality of the argumentations; and finally, the type of elements required to produce valid demonstrations.

2 Theoretical Framework

In our study, we have intended that both the teacher and the ITS—our technological mediator—do not become the core of the teaching process but grant the role to the students, so they turn into active learners. These ideas are in line with the socio-cognitive constructivism theories from Vygostsky advocating that students must be the main actors of their learning and building of their own knowledge, which must be socially shared. In this sense, language, communication and more specifically argumentation as a means to structure the reasoning and knowledge must play a key role in the students' learning process.

Based on our epistemological considerations and research question and objectives, we have structured the theoretical framework in four main pillars: (1) the argumentative competence and the notion of MLO, the aspects and practices that characterizes and provokes it; (2) the technological mediation between artificial tutor and student's learning; (3) ggbTUTOR key features and technical architecture; and (4) characterize the types of instrumental and human-artificial orchestration.

2.1 Argumentative Competence and MLO's Notion

The arguments produced by the students can range from simple observations to formal and structured arguments. Marrades and Gutiérrez (2000), referring to Balacheff, identifies different types of *argumentations* classified into two groups: (1) empirical, characterized by the use of examples as the main element of conviction, differentiating three types within this group: *naïve empiricism, crucial experiment* and *generic example*; and (2) deductive, characterized by the decontextualization of the arguments used, distinguishing two more types: *thought experiment* and *formal deduction*.

More specifically, geometry in secondary education, due to its nature, leads to an *argumentative* process through problem solving, understanding argumentation such as "the capacity to produce oral or written statements that allows conclusions to be reached, both demonstrating a proposition or persuading or convincing someone" (Planas 2010, p. 116, our translation). In particular, the argumentation in geometry is a succession of several processes: (1) the semiotic and cognitive, whereby students interpret the information by coordinating discursive and figural registers; (2) the discovery with a reasonable conviction, by which students find or verify a conjecture; (3) the validation, by proving the conjecture; and (4) the dialectal process, by bringing possible refutations and adjustments. These four processes are aligned with the Spaces for Mathematical Work (ETM) concept from Kuzniak and Richard (2014).

The notion of MLO has been extensively studied in different studies (for example, Brewer and Stasz 1996; Cobb and Whitenack 1996; Cobo 1998; Morera 2013; Yackel et al. 1991). According to Ferrer et al. (2014), they consider MLOs are those relations between the aspects of mathematical learning (conceptual and procedural) together with the artefacts used (books, blackboard, computer, etc.) and the set of actions—systematic of preparation and orchestration—that potentially will facilitate its learning.

Beyond the fact of identifying MLOs, it is the analysis of the opportunities that have triggered actual learning in the student. In other words, we need to link MLOs with achieved learning by the student. According to Boukafri et al. (2015), an MLO is transformed into actual learning when evidence exists in terms of oral or written argumentative competences by the student that denote changes or *transitions* from an *empirical* to a *deductive argumentation* in his/her mathematical knowledge. Therefore, it is reasonable to think that the MLOs will evolve alongside the systematic preparation and the *particular* and *singular* orchestration, in our case through the interactions tutor–teacher–student.

2.2 Technological Mediation

The use of technology in a constructivist approach must be closely related to learning through problem solving, whose use is fully integrated with all subject content areas, playing the role of technological mediator in the sense of Vygotsky, who sees the tools and sign systems as mediators between the human action and learning. Specifically, technological mediation is based on the relationship between artefact and subject, being able to occur so that certain uses of the artefact transform the comprehension—cognition—of the subject completely. Rabardel and Bourmaud (2003) analysed the technological mediation that produced a DGS in the learning, particularly in the student's instrumental genesis. The notion of instrumental genesis refers to the process of considering a tool from an artefact to an instrument, which represents the combination of the artefact and the cognitive skills needed for its use. This aspect is especially relevant since ggbTUTOR—our intelligent DGS—plays an important role in the design of our didactical sequence.

2.3 Intelligent Tutorial Systems

The idea of a tutorial system accompanying the student in the problem solving process is not new. The first generation ITS started in the mid-eighties, but it is not until beginning of current century—coinciding with the widespread access to IT in all areas, including education—when research in this field becomes more active.

We distinguish two groups of ITS, depending on the paradigms of reference they are based (Richard et al. 2011): (1) formal geometry, that relies on an axiomatic approach, formal and deterministic for the development of competencies and (2) cognitive geometry, where the mathematical activity occurs mostly during the geometrical shape construction, allowing a proof and refutation dialectic during the problem solving process.

Already situated in ggbTUTOR, it defines itself as an ITS that accompanies the student—complementing or replacing the teacher—in the resolution of problems with a high level of cognitive demands, by managing all the flow of discursive messages with the student. By *problems with high level of cognitive demands*, Stein and Smith (1998) it refers to those that (1) use *procedures with connections* in a way that encourages students to create connections between a network of processes and mathematical concepts; and (2) *doing mathematics*, which requires complex and non-algorithmic thinking, self-regulation and access to relevant knowledge and previous experience to make appropriate use in working through the resolution of the problem.

2.3.1 Technical Architecture

ggbTUTOR is a portable, web-based application, which is accessible from any Java enabled Internet browser. Its architecture is composed of three main components: (1) the ggbTUTOR interface, (2) the tutor agent and (3) the database, with the following characteristics:

1. The ggbTUTOR interface (Fig. 1), which provides the student all tools needed to solve a geometrical problem, it supports the reasoning process through figural inferences in the approach designed by Duval (1995) and Richard (2004). The ggbTUTOR interface is divided into the following areas: graphical construction (GeoGebra module), where the student creates the graph propositions (parallel, perpendicular lines, etc.); deductive area (deductions and justifications), where the student writes the argumentative propositions; and activity log, which includes the graphic and deductive record as well as the tutor messages.
2. The tutor agent, which has as its mission to supervise and help the student along the problem solving process. It consists of two modules (Richard et al. 2011): the construction tutor, which receives and processes all the graphical prepositions; and the deduction tutor, which is responsible for taking care of the deductive prepositions and

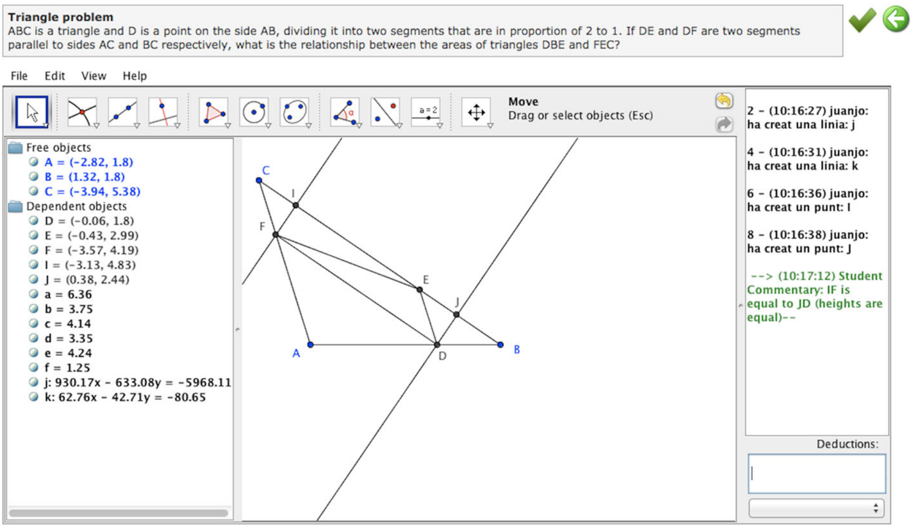


Fig. 1 GeogebraTUTOR interface

displaying certain messages to the student, according to the message selection algorithm (Fig. 2).

3. The database, located at the ggbTUTOR server, stores all the data of problems, users, etc. Therefore, ggbTUTOR requires two main elements: (1) the problem solution tree, built beforehand by the teacher, which is based on the basic space (solution tree) of the problem (Cobo 1998); and (2) the message selection algorithm. Both pieces provide the system with the ability to interact with the student during the problem solving process.

In this way, when the student starts a problem, ggbTUTOR is listening to his/her activity, by checking every action inside its preloaded solution tree, so ggbTUTOR can locate whether the student is within the resolution path accordingly. When ggbTUTOR realizes that a student is no longer following a valid path (e.g. when the student's last actions were not recognized, or when the student does not take any more actions), ggbTUTOR sends a message to him/her. The message sent to the student is the one linked to the last node of the actual strategy that he/she was following until then, i.e. the message associated with the last recognized action. If after two messages sent by ggbTUTOR since the last detected strategy the student either does not respond for 6' or makes five unrecognized actions, the system will then send a new message to suggest that the student changes his strategy (Fig. 2). In this case, we can say that an interaction tutor–teacher–student has occurred, which will have certain influence on the problem solving process and eventually resulting in a MLO.

2.4 Instrumental Characterization and Human-Artificial Orchestration

The systematic of preparation and orchestration based on certain resources and techniques is the key to promoting productive episodes. For this reason, the way in which the teacher analyses in advance all the aspects of the proposed problems in ggbTUTOR that will be later carried out during the orchestration of the tutor–teacher–student interactions, it

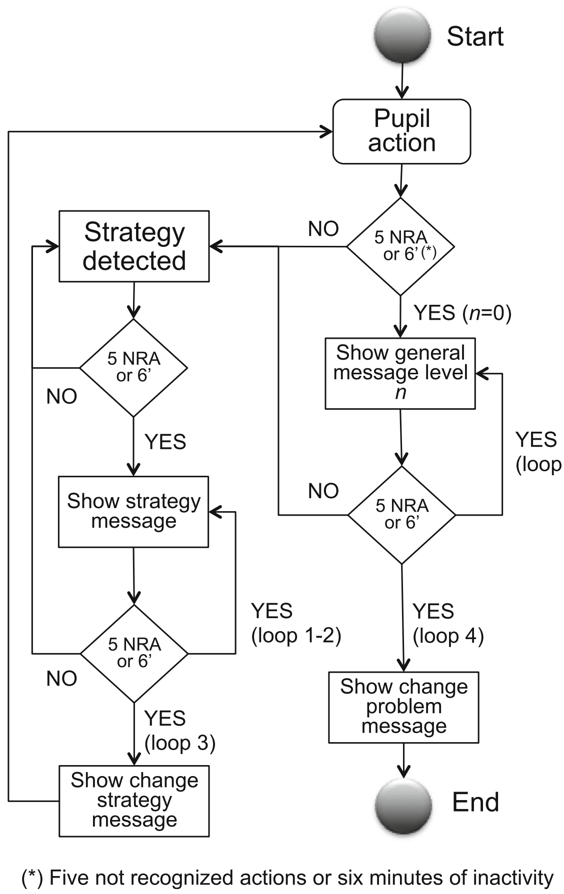


Fig. 2 Agent tutor message selection algorithm

follows a four phase systematic adapted from Morera (2013) and Smith and Stein (2011): (1) anticipation, (2) expanded didactical configuration, (3) mode of operation, and (4) monitoring.

The anticipation phase (1) consists of making a prior analysis on how the students can approach the problem, and anticipate their possible answers. This involves preparing a detailed study of all the possible ways to solve it, which possible messages may ggbTUTOR handle and when it would show them to the student, in case he/she encounters difficulties during the solving process. A key instrument during this phase is the basic space of the problem (Cobo 1998) consisting of a tree structure, in which its branches show the different strategies that a student could follow to solve the problem. For the wrong strategies, different tutor messages are included, with the objective of guiding the student during the solving process, providing only the minimum help, looking for the right balance between helping the student too much and leaving him/her blocked for too long. The expanded didactical configuration (2) describes the set of artefacts that the teacher decides to include in the didactical unit. The mode of operation (3) refers to the way in which the teacher interprets a didactical setup to meet his/her didactical intentions. Finally, the monitoring (4) is based on the follow-up of the mathematical thinking and the resolution strategies from the students while they are working on the problem.

3 Method

In line with other studies on MLOs and student interactions, such as (Cobo et al. 2007; Doorman et al. 2012; Ferrer et al. 2014; Morera 2013), we followed an approach to qualitative methodology. Instead, based on our research question and objectives and inspired by Eisenhart (1988), we made an ethnographical study where from an *interpretivism* position we tried to obtain a holistic view of the student's argumentation processes. In particular, we looked to learn from the tutor–teacher–student interactions and find connections between argumentative MLOs to obtain a better understanding of the transition from an empirical to a deductive argumentation in a technological learning environment.

3.1 Design and Data

The study is a storyline research where we combined research with a teaching experiment to yield a creative mix of qualitative social research using an ethnographic approach. The study is based on a didactical sequence of Geometry where, over two sessions, we proposed four tasks to a group of four 16–17 year old students. The chosen students were selected in collaboration with the school board and shared the following profile: had a good academic performance in mathematics, were interested in technologies and had good communicative skills. These three aspects were important for our research, as we were interested in working with students engaged and motivated with the experiment, familiar with the use of technologies and capable of expressing the argumentative process full of richness. For this paper, we selected two students: Laura and Oriol.

The students had little previous experience in use of GeoGebra and DGS in general, so prior to the teaching experiment they received a brief GeoGebra training. It is also worth mentioning that students were not familiar in practicing argumentative competence, due to the lower cognitive demands of the daily classroom activities. Instead, the work methodology the students used in class combined the expositive sessions by the teacher with a variety of theory application activities, where students worked in pairs or groups of three with a large degree of freedom to comment on the activities, but they rarely worked individually.

In addition to the students, two key participants were part of the teaching experiment: ggbTUTOR—the artificial tutor—as well as a researcher member, who played the teacher role. Both of them—tutor and teacher—could eventually interact with the students through messages with certain mathematical content.

The four selected tasks were geometrical problems that compare areas of plane figures. Several reasons encouraged us to use this typology of problems: firstly, because they are tasks with a *high level of cognitive demands* in the sense described in the theoretical framework. Secondly, because they have been tested in multiple studies (Cobo et al. 2007; Cobo and Fortuny 2000; Richard et al. 2011, 2009). Finally, because they are suitable problems for students aged from 14 onwards, with an adequate complexity level to be considered by them as 'problems', but without being unsolvable. The design of the problems in ggbTUTOR—in line with the systematic of preparation and orchestration—

requires an in depth analysis of all the mathematical aspects and possible ways to solve them. To do this, we used the solution tree of the problem (Cobo 1998).

For this paper, we selected two problems, a *triangle problem* and a *quadrilateral problem* (see Figs. 3, 4).

The triangle problem can be solved through three main strategies: use of Thales's theorem, decomposition into unit triangles, and identifying similarity (Fig. 5), while the quadrilateral problem can be solved by using four different strategies: (1) dividing DECF quadrilateral by DC diagonal, then realizing that the resulting triangles share base length and D vertex with DBE and ADF triangles; (2) dividing the quadrilateral by the FE diagonal, then noticing that the shared base of resulting triangles is parallel to AB side; (3) drawing parallel lines to midpoints; (4) and with direct application of formulas.

To summarize we can say that both problems have equal difficulty and use similar mathematical concepts and procedures, of which we highlight: proportionality, parallelism, perpendicularity; similarity, equivalence and congruence of triangles; use of Tales' theorem, figure decomposition and search of particular cases.

Under these conditions, each of the students participating in the teaching experiment was asked to solve the problems with the eventual help of the human tutor (the teacher) and the artificial tutor (ggbTUTOR). We collected all data of the didactical sequence across four different sources: (1) the ggbTUTOR recording, which included the graphical construction and the tutor–student activity log; (2) the screen recording of each student's computer; (3) the classroom video recording that captured all the teacher–student activity; and (4) the field notes that took a research member in real-time during the course of the didactical sequence, in order to interpret the students' interactions. We transcribed all the data into a comprehensive worksheet format (one sheet for each pair student-problem).

3.2 Analysis

To analyse the transcriptions of the student problem solving process, we designed an instrument to enable us to characterize the tutor–teacher–student interactions, together with the different argumentative MLOs that could arise. The resulting instrument was triangulated with the research team members.

Firstly, we looked at the instrumental side of the tutor–teacher–student interactions, in order to understand how and in which way the artefacts were used along the didactical sequence. To achieve this we relied on the six orchestration types from Drijvers et al. (2010): *Technical-demo, Explain-the-screen, Link-screen-board, Discuss the-screen, Spot-and-show, Sherpa-at-work*. The first three orchestration types are dominated by the tutor–teacher actions, while students dominate more in the last three types.

Secondly, we characterized the tutor messages to analyse the type and level of influence on the student. To achieve this, we looked at two dimensions: (1) the type of message—the

Triangle problem

ABC is a triangle and D is a point on the side AB, dividing it into two segments that are in a ratio of 2 to 1. If DE and DF are two segments parallel to sides AC and BC respectively, what is the relationship between the areas of triangles DBE and FEC?

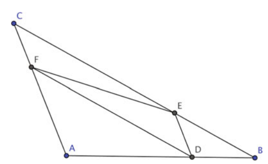


Fig. 3 Statement of the triangle problem

Quadrilateral problem

ABC is a triangle and E and F are the midpoints of sides BC and AC, respectively. If D is any point on the side AB, what is the relationship between the area of DECF quadrilateral and the sum of areas of DBE and ADF triangles?

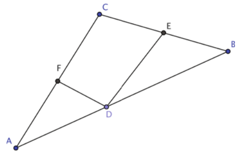


Fig. 4 Statement of the quadrilateral problem

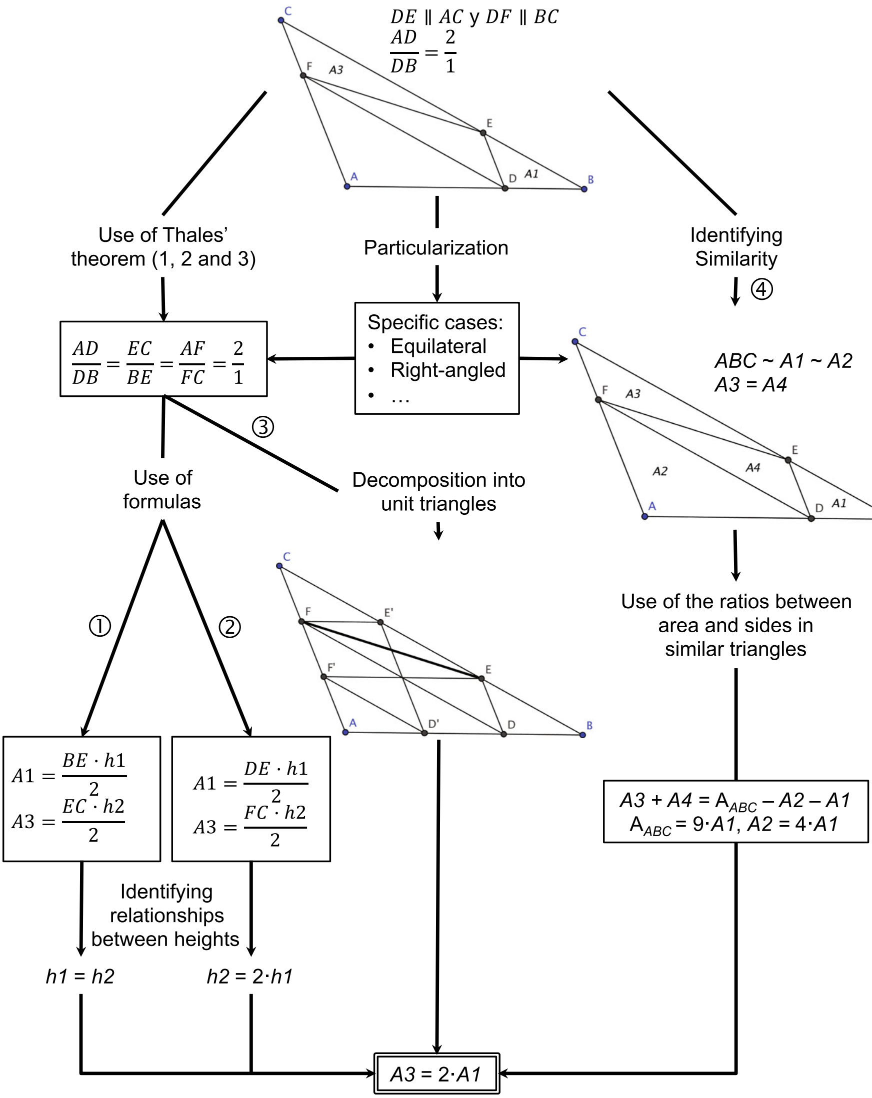


Fig. 5 Solution tree of the triangle problem

cognitive dimension—according to the communicative processes that emerged between tutor and student during the problem solving (Cobo and Fortuny 2007): *conceptual, heuristic, metacognitive* and *semiotic*; and (2) the level of message, depending on the level of information they contained (Cobo et al. 2007; Cobo and Fortuny 2007): *level 0* were general messages without mathematical content, *level 1*, which contained little relevant information, and *level 2* that contained more detailed information, but without being complete to non-transform the problem to a merely simple activity. Those messages are sent by the system depending on the stage of the resolution process where the student stands (familiarization, planning-execution and review).

Finally, we characterized the types of MLO according to the main groups defined by Morera et al. (2013): *mathematical contents, thinking strategies,* and *self-regulating activities*. As we were especially interested in the argumentative processes, we looked in more detail on at those MLOs within the *thinking strategies* group, together with the type of argumentation produced by the student, by using the classification made by Marrades and Gutiérrez (2000), as detailed in the theoretical framework.

To analyse the problem solving process, we divided the transcript into episodes, which are time periods where the student completes a phase of the process followed, as defined by Cobo and Fortuny (2000).

During the first session and prior to starting with the first problem, the teacher introduced ggbTUTOR to the students through a *technical-demo* orchestration (Drijvers et al. 2010), where he highlighted the main differences with GeoGebra, stressing the fact that ggbTUTOR is continuously listening to the student mathematical thinking, in the sense of Leatham et al. (2015), and based on that it makes decisions on whether to proceed or not with the guided resolution, by showing different types of messages. Additionally, as the students were not familiar in practicing the argumentative competence in the daily classroom activities, the teacher placed special emphasis on this subject by reviewing the meaning of argumentation in mathematical problem solving and by showing examples of what is a valid argument in secondary education. For example, the teacher presented examples to the students to illustrate what is a valid argument, i.e. "*due to being perpendicular they form a 90° angle*" is a valid argument, versus "*because it can be clearly seen in the figure*" which is not valid, because—as we mentioned earlier—there is not a direct relation between the discursive and figural registers of the semiotic processes and the cognitive ones linked to the argumentation in geometry.

4 Results and Discussion

The results of the study are presented in the form of a storyline of the student learning process. This storyline is illustrated with two examples of student work and is empirically supported by qualitative findings. As we detailed in the method section above, the data of the experiment was collected from four different sources and transcribed into a worksheet format. Once data was transcribed, we designed an instrument to categorize the data and facilitate the actual analysis. Due to the fact that discourse often contained imprecise expressions and statements, the interpretation of the analysis of discursive data was triangulated across the different data sources, including the researcher's field notes, in order to assure the required internal validity and reliability (Denzin 1970). All data was analysed and reviewed by at least two members of the research team. Among the achieved results of the study, we evidenced four main aspects: (1) both ggbTUTOR and the teacher had an influence on the student, producing a number of MLOs of *thinking strategies type*, in which students meditated and established a strategic line to solve the problem, allowing the students to issue deductive conjectures and (2) meditating and internalizing about the type of elements that were required to produce valid demonstrations. We experienced an overall progress in the students' development of the argumentative competence by (3) establishing figural inference argumentations and visualizing in their minds; and (4) a qualitative improvement in students' arguments, transitioning from empirical to deductive argumentations.

4.1 Thinking Strategies Analysis

The results of the initial open-ended group activities of the first teaching experiment show a variety of strategic lines and conjectures. When students became familiar with the problem, they started looking for relationships between the elements of the geometrical figure, starting by comparing angles and writing down their thinking in the ggbTUTOR deductions area (for example the case of Laura in the action 6):

6.	Laura:	At a first glance I don't see any relationship between the angles.

Even if it is not very common that students start looking at angles, and although her reasoning was not on the right path, we consider that Laura initiated the work of the argumentative competence. However, before Laura confirmed the observation she made, ggbTUTOR sent her a *metacognitive* message of *level 0*:

7.	ggbTUTOR:	*The concepts associated with the figure or the problem statement suggests any new information?*

The system decided to send this message as a consequence of Laura not making any recognized action for 6 min (we recall that action 6 of comparing angles is not in the solution tree of the problem). The fact that ggbTUTOR was sending a message to the student after five unrecognized actions or 6 min of inactivity is based on a didactical decision and our broad experience with ggbTUTOR. In any case, the teacher can adjust this parameter if required, according to the complexity of the problem and the characteristics of his group. The message made Laura discard the strategy of angles comparison and influenced her seen by the fact that she stopped writing, re-read the problem statement and meditated on the possible ways to approach the problem in order to establish the strategic line to try to solve it, starting by looking for relationships between segments as the problem statement suggested. For this reason, we consider that the ggbTUTOR message had an influence on the student, producing a MLO of *thinking strategies* type, acting both as the start and generator of the whole episode.

After the ggbTUTOR message, Laura established a *deductive conjecture* (actions 8 and 9).

8.	Laura:	*Student Commentary: Taking into account that the AB segment is divided into a ratio of 2 to 1; I have the impression that the parallels to AC and AB have divided these segments into the same proportion. To prove this I will use the Thales's theorem.*
9.	Laura:	*Student has added an Argument: Thales's theorem.*

Laura formalized the mathematical procedures that would be used to prove it, by using previous knowledge (Thales's theorem). Laura tried to divide the BC segment into three equal parts by using the Thales's theorem; however she did not follow the process properly, as she drew an auxiliary segment CI, but started the process of drawing auxiliary circumferences from point I, which is just the other way around. This is a common mistake that—based on our teaching experience—we have seen in many students when dividing segments into equal parts. In any case, Laura divided the BC segment properly, as she managed it by drawing the auxiliary segment with a known length (Fig. 6).

At this point, ggbTUTOR recognized the graphical actions made by Laura previously and established that she was following the strategy "*use of Thales's theorem*" (Fig. 5).

Finally, Laura argued that the BC side was following the same proportion rules as AB, even though it said incorrectly a ratio of 2 to 1, when it should be 3 to 1 when referring to BC, and concluded—without proving—that the same should happen with AC (action 24).

24.	Laura:	*Student Commentary: Given that it coincides in the BC segment, which is in a ratio of 2 to 1, I can conclude that the three segments are in the same ratio.*

From the analysis of the episode, we can conclude that it commenced with the beginning of the argumentative competence by Laura, which was influenced by ggbTUTOR by producing a MLO of *thinking strategically* type, in which Laura meditated and established a strategic line to solve the problem, which was to find the ratio between the sides of the triangles. She started with the sides shared with the main triangle (DB, BE from DBE and EC, CF from FEC) by establishing a deductive conjecture and demonstrating it in actions 8 and 24, respectively.

Once Laura proved the relationship of the shared sides with the ABC triangle, she deleted all the auxiliary elements that she used in the previous demonstration. This fact coincided with a *metacognitive* message of *level 0* that the teacher sent to all the students in which he gave them an advice that they would probably need drawing auxiliary lines to solve the problem. This message made Laura think about it, asking a question back to the teacher for advice on her initial decision to remove the auxiliary drawings (see actions 42, 43 and 44).

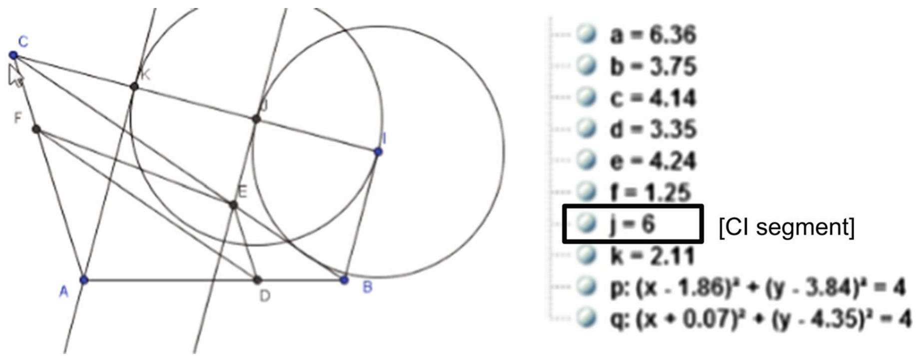


Fig. 6 Auxiliary lines drawn by Laura during the episode of analysis and execution

42.	Teacher:	*To solve a geometry problem, the drawing of auxiliary lines to the given figure is usually needed, mainly to justify it.*
43.	Laura:	*So, do we have to leave the auxiliary drawings?*
44.	Teacher:	*Sure, yes, yes. Well as you prefer. As it is all recorded, we will see it anyway*

For this reason, we consider that the teacher's message had an influence on Laura, by making her meditate on the fact that she possibly took a wrong decision, and by understanding that keeping auxiliary elements used in the argumentation is key in geometrical demonstrations. For all of that, we consider that the teacher message led to a MLO of *thinking strategically* that would potentially be leveraged in subsequent episodes.

Continuing with the line strategy of finding the relationship between the triangle sides, Laura tried to find the relationship between the remaining sides of DBE and FEC triangles, which are the DE and FE sides, respectively. To do this, Laura drew three auxiliary circumferences, together with three segments, in order to compare the length of the DE, CF and CE, FE sides, respectively.

Laura made the previous comparison empirically, instead of noticing that FDEC formed a parallelogram (Fig. 7), and she then argued her findings through an empirical argumentation (action 57), and therefore weakly.

57.	Laura:	*Student Commentary: By drawing two circumferences, I have been able to check, as it looked at first glance, that the CF and DE sides are equal*

We have noted that in the arguments produced by Laura, that she did not make any explicit relationships: "*in a ratio of 1 [or 2] as regards their respective original segments*" (actions 58 and 62), hence we have interpreted that she referred to ratios of 1 to 3 and 2 to 3, respectively.

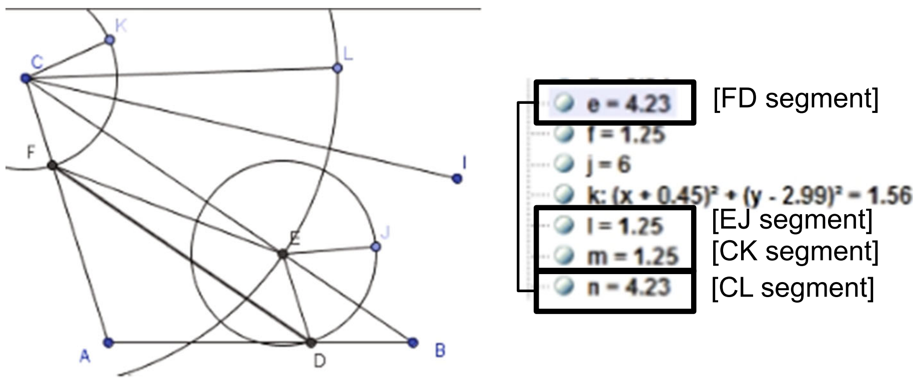


Fig. 7 Auxiliary lines drawn by Laura

58.	Laura:	*Student Commentary: In addition to this, I know that the three segments that form the DBE triangle are in a ratio of 1 in respect to their respective original segments (DE from AC, BE from BC and DB from AB)*
…	…	…
62.	Laura:	*Student Commentary: While FEC triangle is formed by one [side] in ratio of 1 (CF) and the others (CE and FE) are in ratio of 2 in respect to their originals*

Also, Laura made a mistake in stating that FE side would also follow a 2 to 3 ratio with its "*original*", while in this case we cannot be sure which segment she was pointing out, we suspect that she was referring—incorrectly—to the BC side.

From the analysis of the episode, we conclude that Laura completed the finding of the relationship between the sides of DBE, FEC and ABC triangles through an empirical argumentation with two main inaccuracies: she did not make the relationships explicit and made a mistake with the relationship of the FE side.

As a result, we can affirm that the didactical sequence we proposed to Laura, together with the teacher and the technological mediation of ggbTUTOR provided the scaffolding (in form of MLOs) that she required to familiarize herself with the problem, to initiate the development of the argumentative competence and to establish the strategic line to solve it, which she started to follow as she had planned.

4.2 Figural Inference

Laura tried guessing the possible relationship between the area of triangles, by establishing a figural inference argumentation and visualizing in her mind the possible result without making it public. We interpret that Laura visualized the proportions of the heights of the triangles. The teacher reminded her that she had to argue everything appropriately, so Laura explained to him the strategy she was following, which he finally validated (actions 74 to 77).

74.	Laura:	*If I have some relationships because I have done it graphically, but I have done the proportions mentally?*
75.	Teacher:	*Ok, but you will have to explain it. You have to justify it*
76.	Laura:	*So, what should I do? As I have the proportions of the sides, then should I put the relationship?*
77.	Teacher:	*Well, that's it, but you have to write it…*

Laura started looking at the heights of the triangles, with the idea of using the formula for the area of the triangle to obtain the final result, once she had found the required relationship. Based on that, Laura drew the heights of DBE and FEC triangles from E vertex to DB and FC sides, respectively, to compare the measurements (Fig. 8).

However, she did not choose the appropriate heights, as she should have opted for those on the parallel sides. In addition, instead of making the comparison between the heights, Laura compared empirically lengths of heights and sides of the triangles; hence she did not reach any satisfactory conclusion.

From the analysis of the episode, we conclude that Laura started with a mental visualization of the possible result by establishing a figural inference conjecture, as she suspected that the heights of the triangles could follow similar proportion rules as the corresponding triangle sides. To prove her conjecture, she started drawing the heights of the requested triangles to check the measurements, but she made the wrong decision twice: drawing the heights on the non-parallel sides, and comparing the lengths of heights and sides. Laura failed in her strategy, so she had to think of a new plan in subsequent episodes.

As a result, we can affirm that Laura tried to continue with the strategic plan that she defined initially and made some progress in the development of the argumentative competence, especially the figural aspect.

4.3 From Empirical to Deductive Argumentation

Laura abandoned the strategy line she followed during the previous episode, by deleting all the auxiliary lines that she had drawn. Being observed by the teacher, Laura was asked by him with a *conceptual level 0* message: "*Have you finished?*" (action 93), while she answered: "*No, because I have realized that formula for the area is not this one…*" (action 94), referring to the impossibility of making the comparison that she tried before.

After discarding the previous strategy, Laura wrote an argumentation: "*Given that ED side is parallel to AC, I can conclude that the DBE triangle is three times smaller than ABC (…). With the same respect, we can say that the FEC triangle has a ratio of 5/9 with ABC*" (actions 95 and 96), trying to establish a relationship between the area of the DBE, FEC triangles and ABC.

From the analysis of the episode, we highlight that Laura established a correct *deductive argumentation*, as she tried to compare the areas between DBE and ABC triangles, and between the FEC with ABC ones, and thereafter established the relationship between the DBE and FEC areas. In addition we perceived that Laura made an important qualitative step forward in her reasoning as she transitioned from making relationships between sides of triangles to relationships between areas of triangles. However, the numerical comparisons that Laura made to specify her arguments were incorrect. Firstly, due to thinking that the area ratio would be also be linear. This is a common error in secondary school students (Hart et al. 1981). And secondly, dragging the mistake that Laura made earlier with the ratio of the FE side of the FEC triangle, she established a ratio of 5/9 that we did not manage to interpret, although we presume that she came to it by making a kind of sum with the proportions of the sides of the triangles.

Once Laura found the relationship between the area of the DBE, FEC triangles and ABC, she had to make the last step to obtain the final problem result. During this process, ggbTUTOR detected the errors that Laura made in the areas comparison and as a

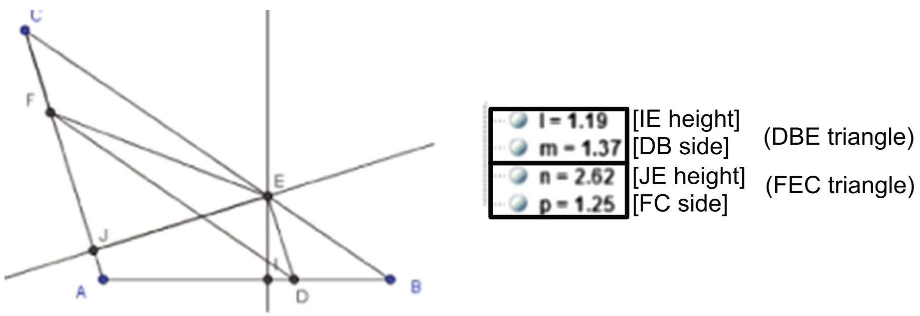


Fig. 8 Auxiliary lines drawn by Laura during episode 5 of execution

consequence, it sent a *conceptual* message of *level 1* (see action 97) related with the strategy of identifying relationships between bases and heights, which was the last form of comparison that Laura tried. The message from ggbTUTOR was the result of five unrecognized actions performed by Laura, related with the drawing of unsuitable heights and with the incorrect relationships between areas. As a result, the message gave Laura an important hint that made her think, and thereafter to start a brief discussion with the teacher (see actions 98 to 104).

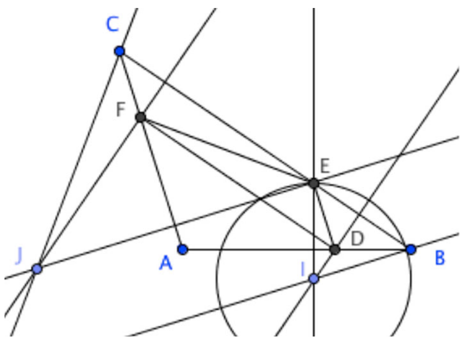
97.	ggbTUTOR:	*Remember the number of heights of a triangle and how to draw them*
98.	Laura:	*What does this mean [pointing out the ggbTUTOR message]? It's giving me a hint?*
99.	Teacher:	*Yes. It is a message that the tutor is sending you.*
100.	Laura:	*I think I know the proportion, but I don't know [how to prove it]*
101.	Teacher:	*Then write it down. That's a conjecture.*
102.	Laura:	*I know the ratios of the sides of the triangles, all of them. So I think I know the [triangle area] proportion*
103.	Laura:	*But I don't know the relationship between these two [pointing out DBE and FEC triangles]*
104.	Teacher:	*This is the problem. Write down everything you know about.*

Laura conjectured through a *mental visualization* what could be the relationship between the area of the DBE and FEC triangles through the ratios of its sides with the ABC triangle, however she did not know how to make the final step.

The message from ggbTUTOR was very appropriate, as Laura did not choose the right heights before. For this reason, we consider that the message from ggbTUTOR influenced how the episode was developed, as it made Laura go back to the strategy that she had previously abandoned of looking for relationships between heights, and caused a new MLO of *thinking strategically*. This MLO had an immediate effect in the subsequent actions in which she drew the three heights of DBE and FEC triangles (Fig. 9).

However, Laura kept comparing empirically, and incorrectly, heights and sides of the same triangles instead of looking at the heights of the two triangles. After few attempts to elucidate the remaining geometrical connection, she asked the teacher to try to find out more about how she could obtain the required relationship. Finally, the time allowed to solve the problem was over and Laura did not get to complete the resolution.

Fig. 9 Auxiliary lines drawn by Laura



From the analysis of the episode, we conclude that the ggbTUTOR message sent to Laura influenced completely in how the episode was developed, as established a continuity in the solving process, linking with the strategy line that Laura discarded previously and triggering a MLO of *thinking strategically* with immediate effect to her, which caused a continuity in the solving process and drawing the three heights of the two triangles. However Laura didn't change the comparison elements and kept looking at the heights and respective sides of the corresponding triangles, so she did not manage to solve the problem completely.

As a result, we can affirm that after Laura abandoned her initial strategy and due to the teacher and ggbTUTOR mediation, it evolved several MLOs that assisted Laura to define a new strategic line to solve the problem, establishing a qualitative step forward enabling her to continue with her development of the argumentative competence, from which we noted a first attempt to move from empirical to deductive argumentation.

4.4 Tutor–Teacher–Student Interactions

In terms of tutor–teacher–student interactions, there were several key moments that produced MLOs of *thinking strategically*: firstly, the one produced by the teacher when he made Laura meditate and internalize about what type of elements were required to produce valid demonstrations (actions 42 to 44). Secondly, another key moment produced by ggbTUTOR that made Laura return to her initial plan (actions 97 to 104).

Finally, to highlight a moment propitiated by the teacher when he explained the 'dragging' functionality of DGS artefacts Marrades and Gutiérrez (2000), before the start of the quadrilateral problem (Fig. 4), which influenced completely the development of the whole problem solving process. This is the case of Oriol, a student, who—using his geometrical view—commenced the quadrilateral problem by dragging the figure and by visualizing a triangle with a regular shape, so he based his strategy on that. Oriol achieved a particularization *empirically* (equilateral triangle, Fig. 10).

Oriol obtained the problem answer directly, however he based his proof on a misconception of the demonstration in dynamic geometry, as he thought—wrongly—that if he was able to find the answer in a specific particularization, the relationship would remain when dragging the figure back to its initial shape.

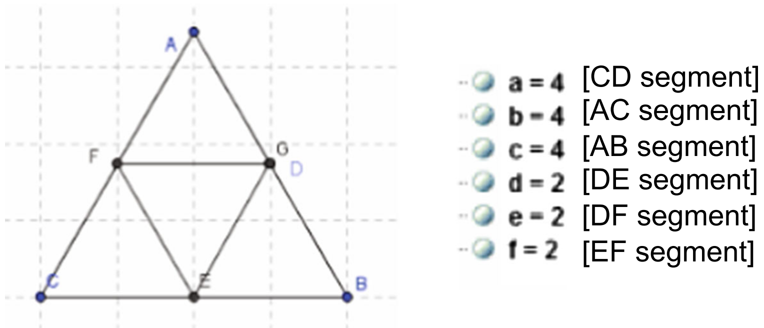


Fig. 10 Figure dragging and auxiliary lines made by Oriol

Later, during Oriol's search for the path to the generalization, he came back to his initial idea, by asking the teacher about the validity of his misconception of the demonstration in dynamic geometry (action 32).

29.	Oriol:	*Student Commentary: From this point, independently, as we move [drag] the triangle, the base of the CEF triangle is equal to the base of the EDB triangle, and the CF side of the CEF triangle is equal to the AF side of the FGA triangle*
…	…	…
32.	Oriol:	*If we say that due to the fact that the problem asks us for the relationship between the quadrilateral and the sum of triangles and we state that it is always the same as I found it for one case, then it will be the same for all… It's enough, or do I have to justify more?*
33.	Teacher:	*The reason that you have given is met in all cases? (…) If you are able to prove that it keeps fulfilling, then it's fine, otherwise not (…) Why have you noticed that it worked?*
34.	Oriol:	*Well, because I have used a very particular case where all the triangles are equal*
35.	Teacher:	*And this is still met?*
36.	Oriol:	*Yes*
37.	Teacher:	*And why it fulfils? That's what you have to think about*

The teacher replied to Oriol with a *metacognitive* message of *level 1*, trying to explain to him that he needed to provide a more deductive kind of demonstration to be an acceptable answer (action 33). Oriol argued that the condition should be met for all cases with a conjecture of *crucial experiment* type, as it was based on his carefully selected example (equilateral triangle) and his misconception (actions 34 and 36). The teacher reemphasized to Oriol that the argument he had provided was not enough, as he had to find out why the condition was always met. Given this fact, Oriol—during few minutes—tried to think about alternative ways to be provided with a complete demonstration. Finally, the time allowed to solve the problem was over, and Oriol did not manage to prove the general case.

Oriol tried to make a qualitative step forward to the generalization, but kept providing empirical argumentations and maintained in his mind the misconception of demonstration in GDS environments. As stated by Gutiérrez (2005) "it is an obstacle for the students to understand the need of the deductive demonstration and to learn how to produce this type of demonstration (…) in other studies, when getting to this blocking point, the teacher has introduced the deductive demonstrations to the students in a way as to understand why the conjectures are true" (p. 43, our translation).

As a final result, we can affirm that during the course of the didactical sequence, we observed in Laura and Oriol a positive trend in the development of the argumentative competence, in both its linguistic quality and its attempt to generalize the result. During this process, it was relevant both teacher and ggbTUTOR mediation—in form of tutor–teacher–student interactions—that motivated a significant number of MLOs.

5 Conclusions

In this study we proposed a research question with the main aim of interpreting the effect of the tutor–teacher–student interactions in the generation of the argumentative MLOs. The analysis of the results of the cases of Laura and Oriol has revealed the development of the argumentative competences they produced and the level of influence of ggbTUTOR to the students. We also showed in detail how the students used figural elements to appropriate ways of seeing an empirical and deductive proof, how they reacted to the ggbTUTOR messages and how they changed the focus during the solving process.

Regarding the influence of ggbTUTOR on the students, the study revealed that the tutor produced a significant number of MLOs of *thinking strategically type*, establishing figural inference (Richard 2004) conjectures and fostering the transition from empirical to deductive argumentations, which had a positive effect on both Laura and Oriol, as it guided them through the path of the problem resolution. This is undoubtedly a clear advantage of the ITS artefacts that considerably help in the attention to diversity, since unlike the teacher; they are able to continuously listen to the student thinking and, based on that, they can take the appropriate decision, sending different types of messages to the student. The study also revealed some negative influences of ggbTUTOR, which is a legacy from the DGS environments: the misconception of demonstration, and the obstacle in understanding the need of the deductive demonstrations in dynamic geometry. This latter aspect has also been discussed by Gutiérrez (2005). These two obstacles should be considered when designing didactical sequences that use DGS artefacts.

Regarding the development of the argumentative competence, we have noticed a trend, both from Laura and Oriol, in the use of empirical argumentative competences. Although Laura always tried to think deductively, she often relied on empirical data. This is a known effect on DGS environments (Gutiérrez 2005) that we also experienced with ggbTUTOR. In terms of the quality of the arguments produced, we have seen a positive change on the evolution of Laura and Oriol's progress of argumentative competences during the problem solving process, and in two main aspects: in the lexical and semantic quality and in the qualitative, with a step forward from particularization to generalization.

The messages from ggbTUTOR followed the interactions model from Cobo et al. (2007) and Kieran (2001), as well as the idea of *appropriation* from Moschkovich (2004), in the sense that it allowed to describe how learning was mediated by the tutor–teacher interactions and how the students learned from them. The interaction model allowed evaluating the progress of cognitive and heuristic abilities in a collaborative problem-solving environment.

In our study about how ggbTUTOR mediated in the development of argumentative competence, we considered different aspects in the geometrical activity (epistemological and cognitive), genesis (instrumental, discursive and semiotic) and cognitive math competences (reasoning, communication and discovery). This granted us the possibility to proceed with the Mathematical Working Spaces model, which allowed the design and the organization of the environmental thought process and enabled the work of individuals solving mathematical problems (Kuzniak and Richard 2014).

We consider that the study of the ggbTUTOR has contributed to the exploitation of the argumentative MLOs and how the development of the argumentative competence evolves. This deserves further investigation that could be continued by introducing didactical sequences with problem itineraries, followed by whole class discussions and by studying the ggbTUTOR interactions with students working in pairs.

GeogebraTUTOR posed a simulated didactical relationship in which the intelligent tutor played, in spite of personalized attention according to the iterative learner's model, a teacher role that was complementary to the role of the regular teacher that only accompanied the student. Even if the teacher did not teach as such, he supported the student in the review and return of the problems and, although indirectly, institutionalized certain fragments of knowledge. The teacher acted as a peer with ggbTUTOR, so even if he often interacted with the student and influenced his milieu, the teacher's actions and interactions remained secondary. In the same way, ggbTUTOR could send certain messages to the student, but these remained subordinate to the interactions, raising mathematical knowledge. GeogebraTUTOR provided each student with a different experience by offering a just-in-time feedback oriented towards helping the student development argumentative competences.

Finally, as a main contribution of the study to the field, we noted that during the course of the interactions between artificial tutor (ggbTUTOR), human tutor (teacher) and student, there was an interactive discourse where each of the communicative exchanges brought a series of arguments that led to what is called a *dialogal talk*, which following Rowland and Turner (2008) and Flecha (2000) produced a change in the communicative register in the sense of the social constructivism in which we based our study. This resulted in an overall learning and improvement of students' argumentative competence. In this sense, the student's discourse evolved, both in the content area related with the progress of the argumentative competences, and in the interlocutive dimension together with the appropriation of the need to demonstrate, as it can be seen during the course of the respective student actions. It is evidenced that the change of the student's discourse was influenced by ggbTUTOR and teacher help messages and with the different concretion levels, following the possible specific strategies of the problem solution tree, which were previously foreseen by the teacher and loaded into the system together with the corresponding pedagogical messages, which were promptly triggered based on the problem solution tree. Hence it was not relevant to evaluate whether the student had successfully resolved the problem or not, but rather the problematic and interactive situation that ggbTUTOR had influenced the student to progress on his level of acquisition of the argumentative competence, leveraging the MLO's that the system enabled in an efficient and effective manner.

Complementary to the contribution of the study to the field, it can also be seen as an added value, as the study allows evaluating the implications of the use of an IT tool like ggbTUTOR as a classroom support to the students' learning.

Acknowledgments The work reported herein is part of the research project EDU2011-23240, funded by the Spanish Ministry of Education. The authors wish to thank the teachers and students of the Pius Font i Quer secondary school from Manresa for their collaboration and the anonymous reviewers for their useful comments on earlier versions of the manuscript.

References

Boukafri, K., Ferrer, M., & Planas, N. (2015). Whole class discussion in the context of mathematics problem solving with manipulatives. In *Proceedings of the 9th congress of European research in mathematics education*. Prague.

Brewer, D. J., & Stasz, C. (1996). Enhancing opportunity to learn measures in NCES data. RAND Corporation.

Cobb, P., & Whitenack, J. (1996). A method for conducting longitudinal analyses of classroom videorecordings and transcripts. *Educational Studies in Mathematics, 30*(3), 213–228.

Cobo, P. (1998). *Análisis de los procesos cognitivos y de las interacciones sociales entre alumnos (16-17) en la resolución de problemas que comparan áreas de superficies planas. Un estudio de casos*. (Unpublished doctoral thesis). Universitat Autònoma de Barcelona.

Cobo, P., & Fortuny, J. M. (2000). Social interactions and cognitive effects in contexts of area-comparison problem solving. *Educational Studies in Mathematics, 42*, 115–140.

Cobo, P., & Fortuny, J. M. (2007). AgentGeom: un sistema tutorial para el desarrollo de competencias argumentativas de los alumnos a través de la resolución de problemas. *Matematicalia: Revista Digital* *de Divulgación Matemática de La Real Sociedad Matemática Española*, *3*(3). Retrieved from http://www.matematicalia.net/index.php?option=com_content&task=view&id=407&Itemid=242

Cobo, P., Fortuny, J. M., Puertas, E., & Richard, P. R. (2007). AgentGeom: A multiagent system for pedagogical support in geometric proof problems. *International Journal of Computers for Mathematical Learning, 12*(1), 57–79. doi:10.1007/s10758-007-9111-5.

Denzin, N. K. (1970). *The research act: A theoretical introduction to sociological methods*. Chicago: Aldine.

Doorman, M., Drijvers, P., & Gravemeijer, K. (2012). Tool use and the development of the function concept: from repeated calculations to functional thinking. *International Journal of Science and Mathematics Education, 10*(6), 1243–1267.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234. doi:10.1007/s10649-010-9254-5.

Duval, R. (1995). *Sémiosis et pensée humaine: registres sémiotiques et apprentissages intellectuels*. Berne: Peter Lang.

Eisenhart, M. A. (1988). The ethnographic research tradition and mathematics education research. *Journal for Research in Mathematics Education, 19*(2), 99–114. doi:10.2307/749405.

Ferrer, M., Fortuny, J. M., & Morera, L. (2014). Efectos de la actuación docente en la generación de oportunidades de aprendizaje matemático. *Enseñanza de las Ciencias, 32*(3), 385–405. doi:10.5565/rev/ensciencias.1231.

Flecha, R. (2000). *Sharing words: Theory and practice of dialogic learning*. Lanham: Rowman & Littlefield.

Gutiérrez, Á. (2005). Aspectos de la investigación sobre aprendizaje de la demostración mediante exploración con software de geometría dinámica. In A. Maz, B. Gómez, & M. Torralbo (Eds.), *Proceedings of the Noveno Simposio de la Sociedad Española de Educación Matemática SEIEM* (pp. 27–44). Córdoba.

Hart, K., Brown, M., & Kuchemann, D. (1981). *Children's understanding of mathematics: 11–16*. London: John Murray.

Hitt, F., & Kieran, C. (2009). Constructing knowledge via a peer interaction in a CAS environment with tasks designed from a task-technique-theory perspective. *International Journal of Computers for Mathematical Learning, 14*(2), 121–152. doi:10.1007/s10758-009-9151-0.

Kieran, C. (2001). The mathematical discourse of 13-year-old partnered problem solving and its relation to the mathematics that emerges. *An International Journal, 46*(1), 187–228. doi:10.1023/A:1014040725558.

Kuzniak, A., & Richard, P. R. (2014). Spaces for mathematical work: Viewpoints and perspectives. *Revista Latinoamericana de Investigación En Matemática Educativa (RELIME), 17*(4-I), 5–15. doi:10.12802/relime.13.1741b.

Leatham, K. R., Peterson, B. E., Stockero, S. L., & Van Zoest, L. R. (2015). Conceptualizing mathematically significant pedagogical opportunities to build on student thinking. *Journal for Research in Mathematics Education, 46*(1), 88–124. doi:10.5951/jresematheduc.46.1.0088.

Marrades, R., & Gutiérrez, Á. (2000). Proofs produced by secondary school students learning geometry in a dynamic computer environment. *Educational Studies in Mathematics, 44*(1–2), 87–125. doi:10.1023/A:1012785106627.

Morera, L. (2013). *Contribución al estudio de la enseñanza y del aprendizaje de las isometrías mediante discusiones en gran grupo con el uso de tecnología*. (Unpublished doctoral thesis). Universitat Autònoma de Barcelona.

Morera, L., Planas, N., & Fortuny, J. M. (2013). Design and validation of a tool for the analysis of whole group discussions in the mathematics classroom. In B. Uhuz (Ed.), *Proceedings of the 8th congress of the European society for research in mathematics education*. ERME: Antalya, Turkey.

Moschkovich, J. N. (2004). Appropriating mathematical practices: A case study of learning to use and explore functions through interaction with a tutor. *Educational Studies in Mathematics, 55*, 49–80. doi:10.1023/B:EDUC.0000017691.13428.b9.

NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Planas, N. (Ed.). (2010). *Pensar i comunicar matemàtiques*. Barcelona: Fundació Propedagògic.

Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. *Interacting with Computers, 15*(5), 665–691. doi:10.1016/S0953-5438(03)00058-4.

Richard, P. R. (2004). *Raisonnement et stratégies de preuve dans l'enseignement des mathématiques*. Berne: Peter Lang.

Richard, P. R., Fortuny, J. M., Gagnon, M., Leduc, N., Puertas, E., & Tessier-Baillargeon, M. (2011). Didactic and theoretical-based perspectives in the experimental development of an intelligent tutorial system for the learning of geometry. *ZDM—The International Journal on Mathematics Education, 43*(3), 425–439. doi:10.1007/s11858-011-0320-y.

Richard, P. R., Iranzo, N., Fortuny, J. M., & Puertas, E. (2009). Influence of dynamic geometry and problem solving strategies toward an interactive tutorial system. In *World conference on e-learning in corporate, government, healthcare, and higher education* (Vol. 2009, pp. 649–658).

Rowland, T., & Turner, F. (2008). How shall we talk about "subject knowledge"for mathematics teaching? In M. Joubert (Ed.), *Proceedings of the British Society for Research into Learning Mathematics (BSRLM)* (Vol. 28, pp. 91–96). Retrieved from http://www.bsrlm.org.uk/IPs/ip28-2/BSRLM-IP-28-2-16.pdf

Sfard, A., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity, 8*(1), 42–76. doi:10.1207/S15327884MCA0801-04.

Smith, M. S., & Stein, M. K. (2011). *5 practices for orchestrating productive mathematics discussions*. Reston, VA: NCTM.

Stein, M. K., & Smith, M. S. (1998). Mathematical tasks as a framework for reflection: From research to practice. *Mathematics Teaching in the Middle School, 3*(4), 268–275.

Yackel, E., Cobb, P., & Wood, T. (1991). Small-group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education, 22*(5), 390–408.