SOLUCIÓN A:

$$\underline{\hat{x}} = \underset{\underline{x}}{\operatorname{argmin}} \; \|\underline{A}\,\underline{x}\|^2$$

$$\text{sujeto a} \quad \|\underline{x}\| = 1$$

$$\|\underbrace{\underline{A}}_{M\times N}\,\underbrace{\underline{x}}_{N\times 1}\|^2 = (\underline{A}\,\underline{x})^T(\underline{A}\,\underline{x})$$

$$= \underline{x}^T \underline{A}^T \underline{A}\,\underline{x}$$

Singular Value Decomposition (SVD) de $\underline{A}$

$$\underline{A} = \underline{U}\,\underline{\Sigma}\,\underline{V}^T$$

Propiedades de la SVD:

Si $\underline{A}$ es de $M\times N$ elementos:

$\underline{U}$ es de $M\times M$

$\underline{\Sigma}$ es de $M\times N$

$\underline{V}$ es de $N\times N$

$\underline{U}$, $\underline{V}$ son ortonormales:

$$\underline{U}^T\underline{U} = \underbrace{\underline{I}}_{(M\times M)} \qquad \underline{V}^T\underline{V} = \underbrace{\underline{I}}_{(N\times N)}$$

si $M \geq N$

$$\underline{\Sigma} = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & 0 \\ & & \ddots & \\ 0 & & & \sigma_N \\ \hline 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \quad M \times N$$

si $N \geq M$

$$\underline{\Sigma} = \left[\begin{array}{cccc|cc} \sigma_1 & & 0 & & 0 & 0 \\ & \ddots & & & \vdots & \vdots \\ 0 & & \sigma_M & & 0 & 0 \end{array}\right] \quad M \times N$$

En nuestro caso $M \geq N$. Los valores de σ_i son ordenados así:

$$\sigma_1 \geq \sigma_2 \geq \cdots \sigma_N \geq 0$$

Un poco de álgebra:

1) $$\underline{A}\,\underline{V} = \overbrace{\underline{U}\,\underline{\Sigma}\,\underline{V}^T}^{\underline{A}}\underline{V} = \underline{U}\,\underline{\Sigma}, \quad \underbrace{\underline{V}^T\underline{V}}_{I}$$

2) $$\underline{A}^T\underline{A} = \overbrace{(\underline{U}\,\underline{\Sigma}\,\underline{V}^T)^T}^{\underline{A}^T}\overbrace{(\underline{U}\,\underline{\Sigma}\,\underline{V}^T)}^{\underline{A}}$$

$$= \underline{V}\,\underline{\Sigma}^T\underbrace{\underline{U}^T\underline{U}}_{I}\underline{\Sigma}\,\underline{V}^T$$

$$= \underline{V}\,\underline{\Sigma}^T\underline{\Sigma}\,\underline{V}^T = \underline{V}\,\underline{\Sigma}^2\underline{V}^T$$

Para $M \geq N$

$$\underline{\Sigma}^T\underline{\Sigma} = \begin{bmatrix} \sigma_1^2 & & 0 \\ & \sigma_2^2 & \\ 0 & & \ddots \\ & & & \sigma_N^2 \end{bmatrix} = \underline{\Sigma}^2$$

3) $\underline{A}^T \underline{A}\,\underline{V} = \underline{V}\,\underline{\Sigma}^2 \underbrace{\underline{V}^T \underline{V}}_{\underline{I}} = \underline{V}\,\underline{\Sigma}^2$

Si $\underline{V} = [\underline{v}_1 \;\; \underline{v}_2 \; \dots \; \underline{v}_N]$ ← Columnas de $\underline{V}$

entonces

$$\underline{A}^T \underline{A} [\underline{v}_1 \dots \underline{v}_N] = [\underline{v}_1 \dots \underline{v}_N] \begin{bmatrix} \sigma_1^2 & & \\ & \ddots & \\ & & \sigma_N^2 \end{bmatrix}$$

$$= [\underline{v}_1 \sigma_1^2 \dots \underline{v}_N \sigma_N^2]$$

$$\Rightarrow \underline{A}^T \underline{A}\, \underline{v}_i = \underline{v}_i \sigma_i^2$$

$$\underline{v}_i^T \underline{A}^T \underline{A}\, \underline{v}_i = \underbrace{\underline{v}_i^T \underline{v}_i}_{1} \sigma_i^2 = \sigma_i^2$$

porque son ortonormales

El problema de optimización queda planteado como:

$$\|\underline{A}\,\underline{x}\|^2 = \|\underline{U}\,\underline{\Sigma}\,\underline{V}^T \underline{x}\|^2 \rightarrow \min \quad s.t. \; \|x\| = 1$$

Como $\underline{U}$ y $\underline{V}$ son ortonormales:

$$\|\underline{U}\,\underline{\Sigma}\,\underline{V}^T \underline{x}\|^2 = \|\underline{\Sigma}\,\underline{V}^T \underline{x}\|^2 \quad \text{y} \quad \|x\|^2 = \|\underline{V}\,\underline{x}\|^2$$

Se define $\underline{y} = \underline{V}^T \underline{x}$

entonces $\underline{x} = \underline{V}\,\underline{y}$

Así:

$$\|\underline{\Sigma}\,\underline{y}\|^2 \to \min \qquad s.t.\ \|y\|^2 = 1$$

Sabemos que

$$\Sigma = {}_M\left[\begin{array}{ccc} \sigma_1 & & \\ & \ddots & \sigma_N \\ \hline 0 & 0 & 0 \end{array}\right]_{N} \qquad \text{con } \sigma_1 \geq \sigma_2 \geq \dots \sigma_N$$

La solución entonces es

$$y = \underbrace{[0 \ 0 \dots 0 \ 1]}_{N}^T$$

Esto quiere decir que $\underline{x}$ es la última columna de $\underline{V}$!

$\boxed{\text{Resumen:}}$ En Matlab:

```
[U, S, V] = svd(A);
        x = V(:, end);
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