

# Immersed methods for fluid–structure interaction: Coupling strategies and resolution algorithms

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## Abstract

We present an initial formulation and comparative description of immersed methods for fluid–structure interaction (FSI). The focus is on the modelling and numerical treatment of the coupling between incompressible viscous fluids and deformable solids. Two coupling strategies are discussed in detail: the Immersed Boundary (IB) approach and the Immersed Domain (ID) / fictitious-domain approach. For each strategy we describe two numerical resolution algorithms: Lagrange multiplier enforcement and penalty-based enforcement. The manuscript emphasizes continuous and discrete weak formulations, choices of function spaces, and practical aspects such as stability, conditioning and implementation considerations. Numerical experiments and extensive benchmarking are deferred to future work.

## 1 Introduction

Fluid–structure interaction (FSI) problems arise across engineering and biological applications where a deformable solid interacts with a surrounding fluid. Immersed methods provide flexible, often Cartesian-grid-friendly frameworks that avoid body-fitted meshes by embedding the structure into a larger fluid domain. This flexibility simplifies mesh generation and remeshing for large structural deformations, at the expense of care required to impose interface conditions accurately.

This note sets out the mathematical formulation and algorithmic choices for two popular immersed frameworks: the Immersed Boundary (IB) method, rooted in work by Peskin, and the Immersed Domain (ID) or fictitious-domain family that enforces boundary/interface constraints through additional terms in the variational formulation. We examine two classes of algorithms to enforce the coupling: Lagrange multipliers, which impose interface constraints exactly at the continuous level, and penalty (or augmented Lagrangian) methods, which impose the constraints approximately. The goal is to clearly state the PDEs, weak forms, and discrete considerations so subsequent work can implement and compare them uniformly.

## 2 Problem formulation

We consider an incompressible viscous fluid occupying a time-dependent domain  $\Omega_f(t) \subset \mathbb{R}^d$  and an elastic solid occupying  $\Omega_s(t) \subset \mathbb{R}^d$  with  $d \in \{2, 3\}$ . The union defines the whole physical domain  $\Omega(t) = \Omega_f(t) \cup \Omega_s(t)$ . In immersed formulations we embed the problem in a fixed computational domain  $\widehat{\Omega} \supset \Omega(t)$  for all  $t \in [0, T]$ .

## 2.1 Governing PDEs

Fluid (Navier–Stokes) in Eulerian form on  $\Omega_f(t)$ :

$$\rho_f(\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) - \nabla \cdot \boldsymbol{\sigma}_f = \mathbf{f}_f, \quad (1)$$

$$\nabla \cdot \mathbf{u} = 0, \quad (2)$$

with  $\mathbf{u}$  the fluid velocity,  $\rho_f$  the density, and the Cauchy stress

$$\boldsymbol{\sigma}_f = -p\mathbf{I} + 2\mu_f \boldsymbol{\varepsilon}(\mathbf{u}), \quad \boldsymbol{\varepsilon}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T). \quad (3)$$

Solid mechanics is formulated in a Lagrangian frame on a reference configuration  $\Omega_{s,0}$ . Let  $\chi(\mathbf{X}, t) = \mathbf{X} + \mathbf{d}(\mathbf{X}, t)$  denote the motion and  $\mathbf{d}$  the displacement. The balance of linear momentum reads

$$\rho_s \partial_{tt} \mathbf{d} - \nabla_0 \cdot \mathbf{P}(\nabla_0 \chi) = \mathbf{f}_s \quad \text{in } \Omega_{s,0}, \quad (4)$$

where  $\mathbf{P}$  is the first Piola–Kirchhoff stress corresponding to a chosen constitutive law (hyperelastic or linearized elastic) and  $\nabla_0$  denotes derivatives w.r.t. reference coordinates  $\mathbf{X}$ .

## 2.2 Interface conditions

Denote the current fluid–solid interface by  $\Gamma(t) = \partial\Omega_s(t) \cap \partial\Omega_f(t)$ . The strong coupling conditions are

$$\mathbf{u}|_\Gamma = \partial_t \chi|_\Gamma = \partial_t \mathbf{d}|_\Gamma, \quad (5)$$

$$\boldsymbol{\sigma}_f \mathbf{n}_f + \boldsymbol{\sigma}_s \mathbf{n}_s = \mathbf{0} \quad \text{on } \Gamma(t), \quad (6)$$

where  $\mathbf{n}_f$  and  $\mathbf{n}_s$  are outward normals for the fluid and solid subdomains respectively, and  $\boldsymbol{\sigma}_s$  is the Cauchy stress in the solid (push-forward of  $\mathbf{P}$  when needed).

## 2.3 Function spaces and variational setting

Let the following Hilbert spaces denote the natural energy spaces for each field:

$$V_f = \{\mathbf{v} \in H^1(\widehat{\Omega})^d : \mathbf{v} = \mathbf{0} \text{ on } \partial\widehat{\Omega}_D\},$$

$$Q_f = L^2(\widehat{\Omega}),$$

$$V_s = H^1(\Omega_{s,0})^d \quad (\text{Lagrangian solid space}),$$

$$\Lambda = H^{-1/2}(\Gamma)^d \text{ or a suitable trace space for multipliers.}$$

We write the fluid bilinear (and semilinear) forms in a general notation: for velocity test function  $\mathbf{v}$  and pressure test  $q$ ,

$$a_f(\mathbf{u}, \mathbf{v}) = \int_{\widehat{\Omega}} 2\mu_f \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) dx + \int_{\widehat{\Omega}} \rho_f(\mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} dx, \quad (7)$$

$$b(\mathbf{v}, q) = - \int_{\widehat{\Omega}} q \nabla \cdot \mathbf{v} dx. \quad (8)$$

For the solid, the elasticity form reads

$$a_s(\mathbf{d}, \mathbf{w}) = \int_{\Omega_{s,0}} \mathbf{P}(\nabla_0 \chi) : \nabla_0 \mathbf{w} d\mathbf{X}. \quad (9)$$

The coupling between domains will be written using interface integrals (or volumetric integrals over  $\Omega_s$  in volumetric immersed methods). For a multiplier  $\boldsymbol{\lambda} \in \Lambda$  and a trace  $\mathbf{v}|_\Gamma$ , the duality pairing is denoted  $\langle \boldsymbol{\lambda}, \mathbf{v}|_\Gamma \rangle_\Gamma$ .

### 3 Immersed coupling strategies

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We outline two immersed approaches for representing the coupled problem on a fixed computational mesh.

#### 4.1 Immersed Boundary (IB) approach

The IB approach (Peskin-type) represents the structure by a lower-dimensional force distribution acting on the fluid. The solid is typically tracked via Lagrangian markers with positions  $\mathbf{X}(s, t)$  parametrized by  $s$  (material coordinate). The structural force density  $\mathbf{F}(s, t)$  is spread to the Eulerian fluid grid via a regularized Dirac kernel  $\delta_h$ :

$$\mathbf{f}_f(\mathbf{x}, t) = \int_{\Omega_{s,0}} \mathbf{F}(s, t) \delta_h(\mathbf{x} - \mathbf{X}(s, t)) ds. \quad (10)$$

The structure moves with the local fluid velocity (no-slip):

$$\partial_t \mathbf{X}(s, t) = \int_{\widehat{\Omega}} \mathbf{u}(\mathbf{x}, t) \delta_h(\mathbf{x} - \mathbf{X}(s, t)) d\mathbf{x}. \quad (11)$$

Remarks: the IB method enforces kinematic coupling through spreading/interpolation operators. The interface traction is not imposed explicitly — it arises from the structural force spread onto the fluid. The method is well-suited for thin or lower-dimensional structures and offers simplicity and robustness for large deformations, but accurate resolution of boundary layers and stress transmission can require careful kernel choice and grid refinement.

#### Discrete IB ingredients

- Discrete kernel: typical choices include the 4-point Peskin kernel or other compact, smooth approximations of the delta distribution. The kernel width is usually a small multiple of the Eulerian mesh size  $h$ .
- Marker spacing: Lagrangian markers are spaced to resolve the structure; a common heuristic is marker spacing  $\Delta s \approx h/2$  to avoid leaks.
- Interpolation/spreading operators: denote by  $\mathcal{S}_h$  the spreading and  $\mathcal{J}_h$  the interpolation operators; they should satisfy adjointness properties to preserve energy in certain schemes.

#### 4.2 Immersed Domain (ID) / Fictitious-domain approach

In ID/fictitious-domain methods the structure occupies an explicit subregion inside a fixed Eulerian mesh. The fluid equations are extended to the whole computational domain  $\widehat{\Omega}$ , and the presence of the solid is enforced by constraints or additional terms in the variational formulation. A typical continuous formulation adds a constraint that the fluid velocity equals the solid velocity inside the solid region (or on the interface), and the solid constitutive forces appear as body forces or through interface tractions. This family includes penalty formulations, Lagrange multiplier methods, and Nitsche-type weak imposition techniques.

#### Unified variational embedding

One convenient monolithic embedding for the ID approach is to write the fluid equations on  $\widehat{\Omega}$  and add the solid weak form transformed to Eulerian coordinates (or keep it in Lagrangian coordinates and couple with appropriate pull/push mappings). The kinematic constraint is enforced either on the interface or inside  $\Omega_s$  by adding suitable terms to the global variational statement.

Remarks: ID approaches provide a unified variational framework and work naturally with finite element discretizations. They enable direct control of traction and kinematic conditions, and extend to thick solids with volumetric representation.

## 5 Resolution algorithms: Lagrange multipliers and penalty methods

We now describe two algorithmic strategies for enforcing the kinematic coupling between fluid and solid within the ID framework (the IB approach uses spreading/interpolation operators instead).

### 5.1 Lagrange multiplier formulation

Introduce a multiplier field  $\lambda$  (defined on the interface or in the solid region) that enforces the kinematic constraint  $\mathbf{u} = \dot{\mathbf{d}}$  on the interface (or in  $\Omega_s$ ). The continuous variational problem reads: find  $(\mathbf{u}, p, \mathbf{d}, \lambda)$  such that

(12)

with the signs chosen consistently with the duality pairing conventions. Here  $c_\Gamma(\cdot, \cdot)$  denotes the duality pairing between velocities and multipliers on the interface (or integral over  $\Omega_s$  if multipliers are volumetric). The fluid and solid bilinear forms  $a_f$  and  $a_s$  encode inertia, viscosity, and elasticity respectively;  $b(\cdot, \cdot)$  is the divergence constraint. for all test functions  $(\mathbf{v}, q, \mathbf{w}, \mu)$ . Here  $c_\Gamma(\cdot, \cdot)$  denotes the duality pairing between velocities and multipliers on the interface (or integral over  $\Omega_s$  if multipliers are volumetric). The fluid and solid bilinear forms  $a_f$  and  $a_s$  encode inertia, viscosity, and elasticity respectively;  $b(\cdot, \cdot)$  is the divergence constraint.

Advantages: exact enforcement of constraints at the continuous level, flexibility in choosing multiplier spaces to satisfy inf-sup conditions. Drawbacks: introduces additional saddle-point structure, leading to larger algebraic systems and the need to choose compatible discrete spaces to avoid spurious modes.

#### Discrete multiplier choices

- Interface multipliers: define  $\Lambda_h$  on a discrete representation of  $\Gamma$  (e.g. trace mesh or mortar spaces). Stability requires an interface inf-sup condition between  $\Lambda_h$  and the trace of the discrete velocities.
- Volumetric multipliers: multipliers defined on  $\Omega_s$  can be used to impose volumetric constraints; these lead to larger systems but simpler coupling operators.

Practical approaches to ensure stability include mortar discretizations, enrichment of traces, or stabilized Lagrange multiplier techniques (Galerkin least-squares-like stabilizations).

### 5.2 Penalty and augmented Lagrangian methods

A penalty method removes the multiplier by adding a penalization term to the variational formulation that weakly enforces  $\mathbf{u} = \dot{\mathbf{d}}$ :

(13)

where  $\varepsilon > 0$  is the penalty parameter (small  $\varepsilon$  yields a stronger enforcement). The augmented Lagrangian combines both ideas, adding a penalty term while retaining a multiplier to mitigate

ill-conditioning:

$$a_f(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) + c_\Gamma(\mathbf{v}, \boldsymbol{\lambda}) + \frac{1}{\varepsilon} \langle \mathbf{u} - \dot{\mathbf{d}}, \mathbf{v} \rangle_\Gamma = \langle \mathbf{F}_f, \mathbf{v} \rangle, \quad (14)$$

$$a_s(\mathbf{d}, \mathbf{w}) - c_\Gamma(\mathbf{w}, \boldsymbol{\lambda}) + \frac{1}{\varepsilon} \langle \mathbf{w}, \mathbf{u} - \dot{\mathbf{d}} \rangle_\Gamma = \langle \mathbf{F}_s, \mathbf{w} \rangle, \quad (15)$$

$$c_\Gamma(\mathbf{u} - \dot{\mathbf{d}}, \boldsymbol{\mu}) = 0. \quad (16)$$

Choice of  $\varepsilon$ : to balance consistency and conditioning, one typically scales  $\varepsilon$  with the mesh size. For an  $L^2(\Gamma)$ -type penalization a common scaling is  $\varepsilon = Ch^\alpha$  with  $\alpha \in [1, 2]$  depending on the discrete norms used; selecting  $\alpha$  and  $C$  should account for the desired accuracy and available linear solvers.

Augmented Lagrangian strategies update  $\boldsymbol{\lambda}$  iteratively while solving a better-conditioned penalized problem at each step; these methods combine robustness with relatively mild conditioning compared to pure penalties. where  $\varepsilon > 0$  is the penalty parameter (small  $\varepsilon$  yields a stronger enforcement). The augmented Lagrangian combines both ideas, adding a penalty term while retaining a multiplier to mitigate ill-conditioning.

Advantages: simpler implementation (no extra multiplier DOFs) and symmetric positive-definite-like structures in some choices; Drawbacks: conditioning deteriorates as  $\varepsilon \rightarrow 0$ , and the constraint is only approximate unless  $\varepsilon$  is taken extremely small. Careful preconditioning or augmentation is often required.

### 5.3 Discrete considerations

Key considerations for discretization include: - Choice of finite element spaces for the fluid velocity/pressure to satisfy inf-sup stability. - Discrete representation of the solid: Lagrangian mesh (finite elements) or reduced models; mapping between solid and Eulerian mesh for IB methods. - For Lagrange multipliers: selection of multiplier spaces and stable coupling operators (Mortar methods, matched meshes, or stabilized choices). - For penalty methods: scaling of  $\varepsilon$  with mesh size to balance consistency and conditioning; when  $\varepsilon = O(h)$  or  $O(h^2)$  is chosen depends on theoretical error estimates and practice.

Implementation notes: - Matrix structure: multiplier methods lead to saddle-point systems; - Solvers: block preconditioners, Schur complement approaches, and iterative solvers (GMRES, MINRES) with tailored preconditioners are common. - Time discretization: monolithic (fully coupled) vs partitioned (staggered) schemes; immersed methods often pair well with monolithic solves to preserve stability when strong added-mass effects occur.

#### Recommended finite elements and solvers

- Fluid: stable mixed pairs such as Taylor–Hood (P2–P1) or stabilized equal-order interpolations when necessary. - Solid: continuous Lagrangian elements of polynomial degree matching the kinematic variables (P1 or P2 depending on accuracy needs). - Multipliers: low-order polynomial spaces on the interface or mortar-type spaces; stabilization if the inf-sup condition is not satisfied.

Linear solvers: - Monolithic saddle-point systems require block preconditioners. A common approach is to build an approximate Schur complement for the velocity-pressure block and use iterative Krylov solvers (GMRES, MINRES) with block triangular preconditioners. - For penalty/augmented formulations, preconditioners that treat the penalized constraint block explicitly improve convergence. Multigrid on the fluid block combined with robust coarse solvers for the solid block is effective in practice.

Time discretization: - Backward Euler or BDF2 for fluid; Newmark-type or generalized- $\alpha$  schemes for the solid. Monolithic time-stepping treats fluid and solid terms consistently and avoids splitting instabilities, notably the added-mass effect.

Implementation remarks: - Mapping Lagrangian-to-Eulerian integrals (and viceversa) requires accurate quadrature when the solid cuts Eulerian elements. Immersed finite element, cutFEM, or fictitious domain quadrature rules are commonly used. - For IB methods, ensure adjointness of interpolation/spreading to preserve energy properties in discrete time-stepping.

## 6 Comparison and discussion

High-level trade-offs: - IB methods: simple coupling via kernel-based spreading and interpolation; excellent for thin or filamentary structures and large deformations; may smear interface tractions and require fine grids or corrections to capture stresses accurately. - ID/fictitious-domain with Lagrange multipliers: precise enforcement of kinematic constraints and traction continuity; requires stable multiplier discretizations and more complex linear algebra. - Penalty methods: easier to implement but require careful parameter tuning and robust linear solvers to handle ill-conditioning when enforcing strong constraints.

Choice recommendations (qualitative): for problems prioritizing geometric flexibility and ease of implementation (e.g., biofluid applications with complex moving boundaries), IB variants are attractive. For problems demanding accurate stress transmission and strong enforcement of interface conditions (e.g., contact or thin gaps), ID with Lagrange multipliers or augmented Lagrangian schemes is preferable.

## 7 Conclusions and future work

This document presented an initial formulation and comparison of immersed coupling strategies and resolution algorithms for FSI. Future directions include: (i) rigorous error and stability analysis for representative model problems; (ii) consistent discretization choices and implementation of solvers; and (iii) a comprehensive numerical study comparing accuracy, robustness, and computational cost across representative benchmarks.

## Appendix: representative weak forms

For reference, we provide a concise monolithic weak form for the Lagrange multiplier ID formulation. Find  $(\mathbf{u}, p, \mathbf{d}, \boldsymbol{\lambda})$  such that for all test functions  $(\mathbf{v}, q, \mathbf{w}, \boldsymbol{\mu})$ :

$$\int_{\widehat{\Omega}} \rho_f (\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u}) \cdot \mathbf{v} + 2\mu_f \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) - p \nabla \cdot \mathbf{v} \, d\mathbf{x} + \langle \boldsymbol{\lambda}, \mathbf{v}|_\Gamma \rangle_\Gamma = \langle \mathbf{f}_f, \mathbf{v} \rangle, \quad (17)$$

$$\int_{\widehat{\Omega}} q \nabla \cdot \mathbf{u} \, d\mathbf{x} = 0, \quad (18)$$

$$\int_{\Omega_{s,0}} \rho_s \partial_{tt} \mathbf{d} \cdot \mathbf{w} + \mathbf{P}(\nabla_0 \chi) : \nabla_0 \mathbf{w} \, d\mathbf{X} - \langle \boldsymbol{\lambda}, \mathbf{w}|_\Gamma \rangle_\Gamma = \langle \mathbf{f}_s, \mathbf{w} \rangle, \quad (19)$$

$$\langle \boldsymbol{\mu}, \mathbf{u}|_\Gamma - \partial_t \mathbf{d}|_\Gamma \rangle_\Gamma = 0. \quad (20)$$

This weak form highlights the saddle-point structure and the natural placement of multipliers on the interface.

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Placeholder for acknowledgements.

## References

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