Interest Rate Sensibility: Taylor Series Analysis of Convexity and Duration in Bonds

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Abstract

This project aims to quantify the impact of interest rate changes on bond prices using a second-order Taylor series approximation of the bond price equation. Incorporating fundamental elements such as convexity and duration, our goal is to provide a systematic understanding of the percentage changes in bond prices in response to varying interest rates.

1 Introduction

The price of a bond is determined by several factors, which are time to maturity, the coupon yield, and the discount rate. In fact, the price of a bond can be computed as the present value of all the cash flows that the bond owner will receive. While the coupon and the maturity are deterministic and well-known, the discount rate varies over time, changing the bond price constantly. Therefore, it is crucial to model how each bond will react to these variations in order to assess its possible price changes and, consequently, its risk.

The sensibility of a bond to interest rate (or the discount rate) variations can be easily modeled using two key factors: duration and convexity. The first measures the linearity of this relationship and is computed as the derivative of price concerning yield. Convexity is a measure of the curvature (or non-linearity) of how the price of a bond changes as the interest rate changes. This last measure is vital when we want to model larger variations of the discount rate because duration does not account for these. Convexity is calculated as the second derivative of price concerning yield.

Our approach to the modeling of bonds' price sensibility will be from the point of view of mathematical analysis. We will derive this key feature by using a Taylor series expansion of second order, as our derivatives will be duration and convexity. The plots and tables of this paper have been generated using a dashboard programmed in Python. The code is in the appendix section.

2 Bond Pricing

Our starting point will be the equation determining a bond's price. Using continuous compounding, the price of a bond is given by:

$$V = \sum_{i=1}^{n} V_i e^{-y t_i} \tag{1}$$

where V is the price of the bond, n is the time to maturity (in years), V_i are the bond's cash flows, t_i is the year and y is the discount rate. For simplicity, we will discount each cash flow at the same rate. However, a more accurate approach is to use a different rate for each cash flow.

The exponential term corresponds to the discount factor, which allows us to compute the present value of each cash flow V_i. As it is a decreasing exponential term, the farther away in time the cash flows are, the less they impact bond prices. It is important to note that each time the discount rate varies, all the cash flows change accordingly; therefore, if we have a bond with a longer time to maturity (TTM), we will have more terms changing than a bond with a shorter one. Consequently, bonds with a longer TTM will be more volatile (and, therefore, riskier).

The key takeaway of equation (1) is the inverse relationship between interest rate and bond price. The bond loses part of its value as the discount rate goes up. On the other hand, when interest rates go down, bonds appreciate. The following graph illustrates this key relationship for a given bond.

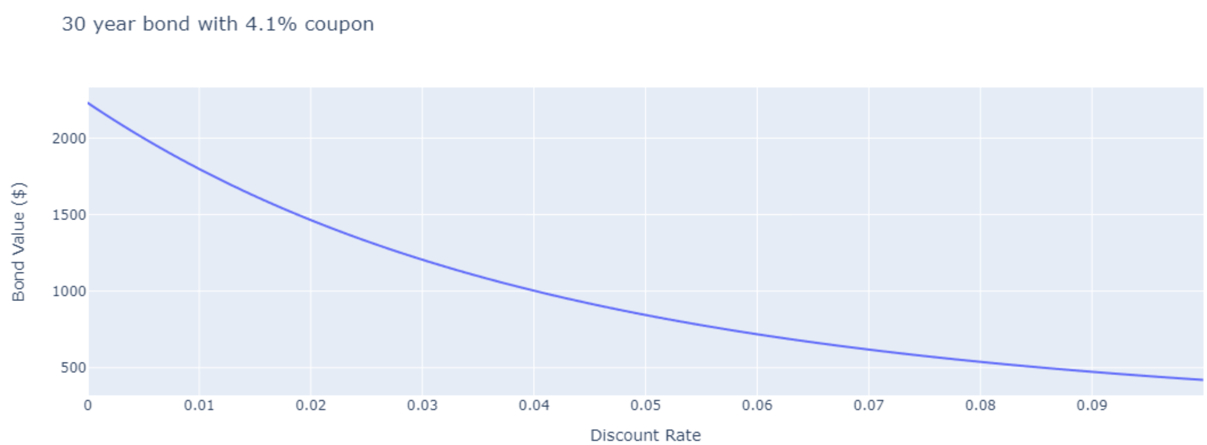

Figure 1: Bond Value vs Discount Rate

3 Duration

Duration determines the price sensibility of a bond to the changes in interest rates. It is, in fact, a weighted average of the times until those cash flows are received. It can be computed using the following expression:

$$D = \sum_{i=1}^{n} t_i \frac{V_i e^{-y t_i}}{V} \tag{2}$$

Note that if the bond is zero-coupon with a TTM of n, the duration will be n years. In any other case, the duration will be less than the TTM. Another way to compute the duration is by taking the first derivative of the equation (1). Therefore, we can express duration as:

$$D = -\frac{1}{V}\frac{\partial V}{\partial y} \tag{3}$$

Again, this illustrates the inverse relationship between bond prices and interest rates, as we can now see that the first derivative is negative. While duration is not exactly the first derivative of the price of a bond, it is proportional to it, so we can use it to assess the rate of change of a given bond regarding the variations of interest rates. Below are some examples of duration for different bonds:

Maturity	**Coupon Rate**	**Discount Rate**	**Duration**
30	1%	1%	26.06
30	5%	1%	20.32
30	5%	5%	16.01
20	5%	1%	14.85

Table 1: Examples of Duration for Different Bonds

Note that the lower the coupon rate, the higher the duration is. Also, the higher the discount rate is, the lower the duration. Therefore, bonds with a low coupon, emitted in times of low interest rates, are the most volatile, and on the other hand, bonds with higher coupons, emitted in times of high-interest rates, are the least volatile.

If we consider small variations in interest rates, it is approximately true that:

$$\Delta V = \frac{\partial V}{\partial y}\Delta y \tag{4}$$

Therefore, by plugging equation (3) into (4), we get the key duration relationship:

$$\frac{\Delta V}{V} = -D\Delta y \tag{5}$$

Which is an approximate relationship between percentage changes in a bond price and changes in its yield. Recall that this is only true for slight variations in interest rates.

4 Convexity

While duration quantifies the linear sensitivity of a bond's price to changes in yield, convexity captures the non-linear aspects of this relationship. We can compute the convexity of a given bond using the following expression:

$$C = \sum_{i=1}^{n} t_i^2 \frac{V_i e^{-y t_i}}{V} \tag{6}$$

The price function for a bond (as shown in Figure 1) is not linear, and therefore, we need a measure of the curvature of this function; in other words, we need convexity. If we take the second derivative of equation (1), we can define convexity as:

$$C = \frac{1}{V} \frac{\partial^2 V}{\partial y^2} \tag{7}$$

Again, convexity is not the second derivative of the price of a bond concerning the discount rate, but it is proportional to it. Note that in this case, the second derivative is positive, implying that the bond's price equation is always convex. Convexity is used similarly to the way gamma is used in derivatives risk management. Below are some examples of convexity for different bonds:

Maturity	**Coupon Rate**	**Discount Rate**	**Convexity**
30	1%	1%	743.05
30	5%	1%	513.98
30	5%	5%	365.82
20	5%	1%	261.92

Table 2: Examples of Convexity for Different Bonds

As we saw in the duration section, bonds with a low coupon, emitted in times of low-interest rates, are the most volatile (higher convexity), and bonds with higher coupons, emitted in times of high interest rates, are the least volatile (lower convexity).

In the previous section, we introduced equation (5), which shows the approximate relationship between percentage changes in a bond price and changes in its yield when these are small. However, if we account for convexity, we can derive an expression similar to equation (5), allowing us to measure percentage changes in bond prices for more considerable variations in interest rates. This will be our goal in the following section.

5 A Taylor Series Approach

In this section, we will take a mathematical analysis approach and try to derive an expression for the percentage changes in a bond price using the Taylor series, convexity, and duration.

Recall the Taylor series formula for a function $f(x)$ centered at the point a:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots \tag{8}$$

In our case, we will use a second-order approximation for the price equation of a bond. This leads us to the following expression:

$$V(y) = V(y_o) + V'(y_o)(y - y_o) + \frac{V''(y_o)}{2}(y - y_o)^2 \tag{9}$$

Setting $\Delta y = (y - y_o)$ and $\Delta V = V(y) - V(y_o)$ we get a more compact expression:

$$\Delta V = V'(y_o)\Delta y + \frac{V''(y_o)}{2}\Delta y^2 \tag{10}$$

Now is time to plug in convexity and duration; using equations (3) and (7), we get the following:

$$\frac{\Delta V}{V} = -D\Delta y + \frac{C}{2}\Delta y^2 \tag{11}$$

This is known as the second-order approximation of bond price movements due to rate changes. Therefore, we can model the percentage variations in bond prices using just duration and convexity. Below is a graph that illustrates this approximation.

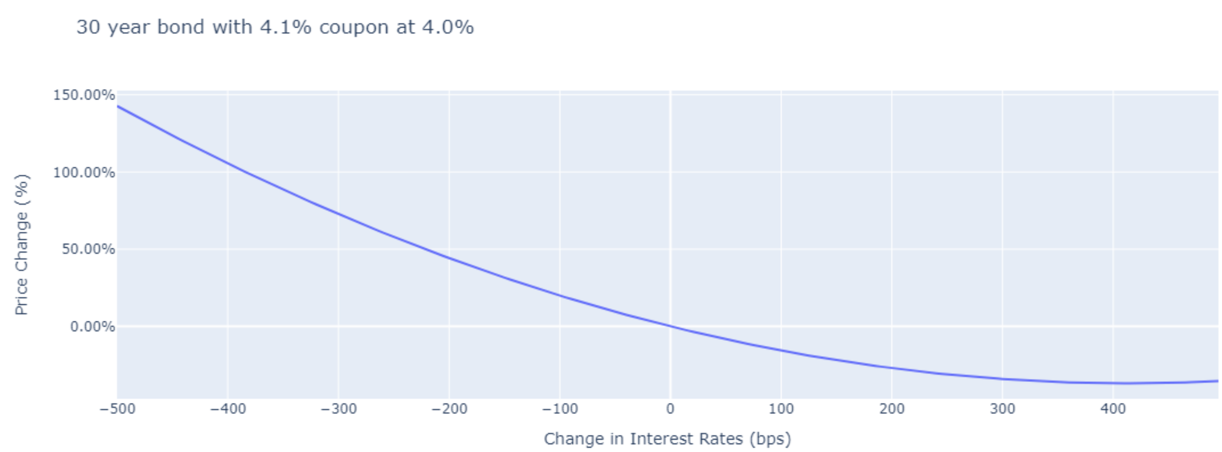

Figure 2: Percentage Change due to Variations on Interest Rates

6 Conclusions

We have successfully derived an analytical expression for the percentage change in bond prices regarding changes in yields. The only requirement will be computing the duration and convexity of a given bond. The first accounts for the linear relationship, and the latter for the non-linear one.

References

[1] John C.Hull, (2014). Options, futures and other derivatives (9th edition).

7 Appendix: Dashboard

Here is the code that displays an interactive dashboard that has been programmed in Python using Plotly and Dash libraries. All the tables and plots of the paper have been obtained from the dashboard. In addition, the functions used to compute duration, convexity, bond prices... are also included.

7.1 Functions

```

import numpy as np
import plotly.express as px

#BOND PRICING
def price(n,c,A,y):
    """
    Calculate the present value of a bond.

    Parameters:
    - n: Maturity (years)
    - c: Coupon Rate (%)
    - A: Principal ($)
    - y: Discount Rate (%)

    Returns:
    - B: Present value of the bond
    """

    C = A*c
    B = 0
    for t in range(1,n+1):
        d =  np.exp(-y*t)
        B = B + C*d

    B = B + A*np.exp(-y*n)

```

```
    return B

# DURATION
def duration(n,c,A,y):
    """
    Calculate the duration of a given bond.

    Parameters:
    - n: Maturity (years)
    - c: Coupon Rate (%)
    - A: Principal ($)
    - y: Discount Rate (%)

    Returns:
    - D: Duration of the bond (years)
    """
    B = price(n,c,A,y)
    C = A*c
    D = 0
    for t in range(1,n+1):
        d_w =  t*np.exp(-y*t)
        D = D + C*d_w

    D = (D + A*np.exp(-y*n)*n)/B
    return D

# CONVEXITY
def convexity(n,c,A,y):
    """
    Calculate the convexity of a given bond.

    Parameters:
    - n: Maturity (years)
    - c: Coupon Rate (%)
    - A: Principal ($)
    - y: Discount Rate (%)

    Returns:
    - C: Convexity of the bond
    """
    B = price(n,c,A,y)
    C = A*c
    Cvx = 0
    for t in range(1,n+1):
        d_w =  (t**2)*np.exp(-y*t)
        Cvx = Cvx + C*d_w

    Cvx = (Cvx + A*np.exp(-y*n)*(n**2))/B
    return Cvx

```

```python
# SECOND ORDER APPROX. FOR BOND PRICES
def sensibility(n,c,A,y,delta_y):
    """
    Calculate the second order approximation for bond price
       movements

    Parameters:
    - n: Maturity (years)
    - c: Coupon Rate (%)
    - A: Principal ($)
    - delta_y: Change on interest rates

    Returns:
    - risk_B: percentage changes in bond price (%)
    """
    D = duration(n,c,A,y)
    C = convexity(n,c,A,y)
    risk_B = -D*delta_y + 0.5*C*(delta_y)**2
    return risk_B
```

7.2 Dashboard

```python
import numpy as np
import plotly.express as px
import dash
from dash import dcc, html
from dash.dependencies import Input, Output
from bond_funcs import price, sensibility, convexity, duration

# Set initial bond parameters
n = 20
c = 0.04
A = 1000
y = 0.05
delta_y = np.arange(-0.05, 0.05, 0.0005)
y_range = np.arange(0, 0.1, 0.00001)

# Initial calculation
delta = sensibility(n, c, A, y, delta_y)
B = price(n, c, A, y_range)
C = convexity(n,c,A,y)
D = duration(n,c,A,y)

# Create Dash app
app = dash.Dash(__name__)

# Define layout
app.layout = html.Div(style={'backgroundColor': '#f5f5f5', '
   color': '#333', 'padding': '20px'}, children=[
```

```
    html.H1("Bond Sensitivity Dashboard", style={'fontSize': '50px'}),

    # Input sliders
    html.Label("Select Bond Parameters: ", style={'fontSize': '30px', 'marginBottom': '1px'}),

    html.Div([
        html.Label("Maturity (years): ", style={'fontSize': '16px'}),
        dcc.Input(id='maturity-input', type='number', value=n, step=1, min=1, max=50, placeholder="Enter Maturity"),
    ]),
    html.Div([
        html.Label("Coupon Rate (%): ", style={'fontSize': '16px'}),
        dcc.Input(id='coupon-input', type='number', value=c, step=0.001, min=0, max=1, placeholder="Enter Coupon Rate"),
    ]),
    html.Div([
        html.Label("Principal ($): ", style={'fontSize': '16px'}),
        dcc.Input(id='principal-input', type='number', value=A, step=10, min=0, max=100000, placeholder="Enter Principal"),
    ]),
    html.Div([
        html.Label("Discount Rate (%): ", style={'fontSize': '16px'}),
        dcc.Input(id='rate-input', type='number', value=y, step=0.001, min=0, max=1, placeholder="Enter Discount Rate"),
    ]),

    # Calculation boxes for the first graph
    html.Div([
        html.P(id='duration-box', children=f"Duration: {D}"),
    ], style={'marginBottom': '10px', 'fontSize': '18px', 'fontWeight': 'bold'}),

    html.Div([
        html.P(id='convexity-box', children=f"Convexity: {C}"),
    ], style={'marginBottom': '5px','fontSize': '18px', 'fontWeight': 'bold'}),

    # Graphs
    html.Label("Second-Order Approximation of Bond Price Movements", style={'fontSize': '30px', 'marginBottom': '1
```

```python
        px'}),
    dcc.Graph(id='bond-sensitivity-chart', style={'margin': 'auto', 'textAlign': 'center'}),
    html.Label("Bond Price for Different Interest Rates", style={'fontSize': '30px', 'marginBottom': '1px'}),
    dcc.Graph(id='bond-value-chart'),
])


# Define callback to update the first graph
@app.callback(
    Output('bond-sensitivity-chart', 'figure'),
    [Input('maturity-input', 'value'),
     Input('coupon-input', 'value'),
     Input('principal-input', 'value'),
     Input('rate-input', 'value')]
)
def update_graph_1(maturity, coupon, principal, rate):
    delta_y_range = np.arange(-0.05, 0.05, 0.0005)
    risk = sensibility(maturity, coupon, principal, rate, delta_y_range)

    fig = px.line(x= 10000*delta_y_range, y=risk, labels={'x': 'Change in Interest Rates (bps)', 'y': 'Price Change (%)'},
                  title=f'{maturity} year bond with {round(100 * coupon, 2)}% coupon at {round(100 * rate, 2)}% interest rate')

    fig.update_layout(yaxis_tickformat='%')
    fig.update_yaxes(tickformat=".2%")
    fig.update_layout(paper_bgcolor='#f5f5f5')
    return fig

# Define callback to update the second graph
@app.callback(
    Output('bond-value-chart', 'figure'),
    [Input('maturity-input', 'value'),
     Input('coupon-input', 'value'),
     Input('principal-input', 'value')]
)
def update_graph_2(maturity, coupon, principal):
    y_range = np.arange(0, 0.1, 0.00001)
    bond_values = price(maturity, coupon, principal, y_range)

    fig = px.line(x=y_range, y=bond_values, labels={'x': 'Interest Rate', 'y': 'Bond Value ($)'},
                  title=f'{maturity} year bond with {round(100 * coupon, 2)}% coupon')

```

```
    fig.update_layout(xaxis_tickformat='%')
    fig.update_xaxes(tickformat=".2%")
    fig.update_layout(paper_bgcolor='#f5f5f5')
    return fig

# Define callback to update the calculation boxes
@app.callback(
    [Output('duration-box', 'children'),
     Output('convexity-box', 'children')],
    [Input('maturity-input', 'value'),
     Input('coupon-input', 'value'),
     Input('principal-input', 'value'),
     Input('rate-input', 'value')]
)

def update_calculation_boxes(maturity, coupon, principal, rate):
    C = convexity(maturity,coupon,principal,rate)
    D = duration(maturity,coupon,principal,rate)
    duration_text = f"Duration: {round(D,2)}"
    convexity_text = f"Convexity: {round(C,2)}"
    return duration_text, convexity_text

# Run the app
if __name__ == '__main__':
    app.run_server(debug=True)
```