Yield Curve Construction Using Short Rate Models

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January 2025

Abstract

This paper aims to provide a straightforward and accessible introduction to modelling the yield curve, a key component in valuing interest rate derivatives. We utilize one-factor short-rate models, which allow the construction of the entire yield curve by simulating the interest rate for short-term borrowing. While this paper does not address the calibration of these models, it includes examples of simulated yield curves for illustrative purposes.

Introduction

The valuation of interest rate derivatives is more complex than equity or FOREX derivatives. This is mainly because the behaviour of the underlying, which in this case is the interest rates, is more difficult to model compared to a stock or an exchange rate. Additionally, the valuation of some products depends not only on one but several interest rates with different maturities; in other words, these products depend on the entire yield curve. As we will see later, modelling the yield curve is a challenging task, as this curve can present very diverse shapes, and the volatility on each point of the curve is different.

Our approach to modelling the yield curve involves several steps. First, we simulate the instantaneous short rate, which represents the risk-free rate applicable to an infinitesimally short time period. Next, we compute the bank account process, representing the value of one unit of currency invested in the money market and continuously reinvested at the short rate. Using the bank account process, we then calculate the prices of zero-coupon bonds for various maturities. Finally, we construct the yield curve by extracting yields from these zero-coupon bond prices, creating the zero-coupon yield curve.

The key component in this methodology is selecting an appropriate short-rate model and calibrating its parameters to fit current market prices. We will focus on several aspects, such as whether the model can produce negative interest rates, the distribution of the short rate, how suited the model is for Monte Carlo simulation, or if the model presents a mean-reverting behaviour. We will address these issues in the following sections.

Constructing the Yield Curve

We will start by defining the bank account process, $B(t)$, which represents a risk-less investment where profit is accrued continuously at the short rate (or risk-free) rate $R(t)$. Both the bank account and the short rate are stochastic processes. We will define $R(t)$ in the following section as it depends on our chosen model. We now give a more formal definition of the bank account process [1]:

Definition 1 ***Bank account process:*** *We define $B(t)$ to be the value of a bank account at time $t \geq 0$. We assume $B(0) = 1$ and that the bank account evolves according to the following differential equation:*

$$dB(t) = R(t)B(t)dt, \quad B(0) = 1 \tag{1}$$

Where $R(t)$ is the is the instantaneous rate at which the bank account accrues or the short rate. As a consequence,

$$B(t) = \exp\left(\int_0^t R(s)ds\right) \tag{2}$$

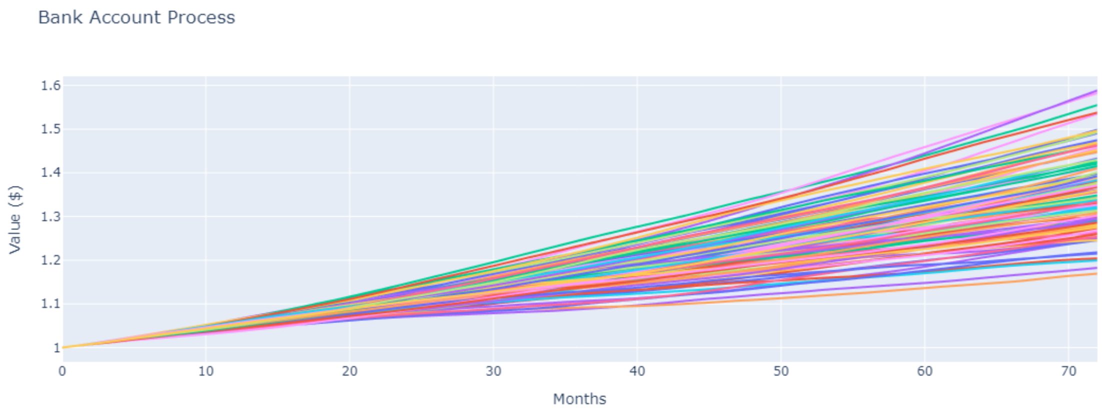

Figure 1: Bank Account Process

Additionally, we can compute the discount process, $D(t)$, from the bank account process as follows:

$$D(t) = \frac{1}{B(t)} = \exp\left(-\int_0^t R(s)ds\right) \tag{3}$$

This is also called the stochastic discount factor. When dealing with equities and exchange rates, the bank account and the discount factor are deterministic as $R(t)$ is assumed to be constant. However, in our case, we are forced to drop the deterministic setup as we will have to model $R(t)$ as a stochastic process.

The next step will be computing the price of zero coupon bonds from the discount process. The zero coupon bonds represent the building blocks of the yield curve, as they provide a direct measure of spot rates in the market. These bonds pay a certain face amount, which we take to be 1, at a fixed maturity date of T. More formally, we can define a zero coupon bond as [1]:

Definition 2 ***Zero-coupon bond:*** *A T-maturity zero-coupon bond (pure discount bond) is a contract that guarantees its holder the payment of one unit of currency at time T, with no intermediate payments. The contract value at time $t < T$ is denoted by $P(t, T)$. Clearly, $P(T, T) = 1$ for all T.*

The price of a zero-coupon bond can be calculated by using the risk-neutral pricing formula, which states that the discounted price of the bond must be a martingale under the risk-neutral measure. Therefore, the price of a zero-coupon bond at time $t = 0$ should satisfy:

$$P(0, T) = \tilde{\mathbb{E}}[D(T)\, P(T, T)] = \tilde{\mathbb{E}}\left[\exp\left(-\int_0^T R(s)ds\right)\right] \tag{4}$$

We can generalize this expression for any time t such that $0 \leq t \leq T$, getting the following:

$$P(t, T) = \tilde{\mathbb{E}}\left[\exp\left(-\int_t^T R(s)ds\right)\right] \tag{5}$$

Intuitively, the price of a zero-coupon bond is the average discounted value across all possible future interest rate paths over the bond's life (until T) at any time t. We can also express the price of a zero coupon bond as:

$$P(t, T) = e^{-Y(t,T)(T-t)} \tag{6}$$

Where $Y(t, T)$ is the constant rate of continuously compounding interest rate between times t and T, or, in short, the yield between t and T. From the previous expression, we can deduce the following:

$$Y(t, T) = -\frac{1}{T-t} \log P(t, T) \tag{7}$$

This, in fact, is the function of the yield curve (also known as the zero-cupon curve), as we can compute the yield at any time t for several bond maturities T. For example, the 30-year rate at time t is $Y(t, t + 30)$. This is an example of a long rate. If we make T approach t, in other words, if we take a very short maturity bond, we get:

$$\lim_{T \to t^+} Y(t, T) = R(t) \tag{8}$$

This implies that the beginning of the yield curve is the short rate $R(t)$, and the following points of the curve are then calculated with (7). Note that once we adopt a model for the short rate $R(t)$, we can determine the long rates $Y(t, T)$. We have accomplished our goal of constructing a yield curve using an arbitrary short-rate model.

Our next step will be choosing an appropriate model for $R(t)$, which we will discuss in the following section.

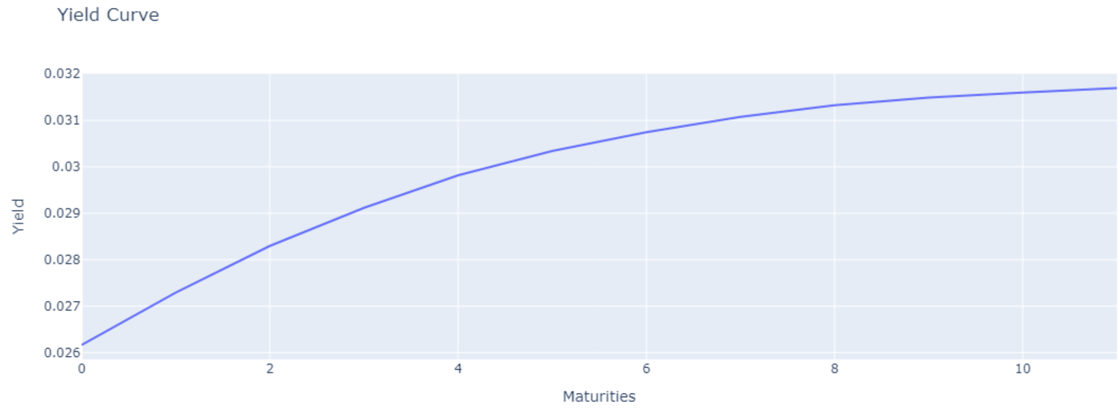

Figure 2: Yield curve constructed with a Vasicek model

Short-Rate Models

A short-rate model can be described with the following stochastic differential equation:

$$dR(t) = \mu\left(t, R(t)\right)\, dt + \sigma\left(t, R(t)\right)\, d\tilde{W}(t) \tag{9}$$

where $\tilde{W}(t)$ is a Brownian Motion under the risk-neutral probability measure $\tilde{\mathbb{P}}$ and μ and σ are known functions representing the drift and the diffusion coefficients respectively. In the equity and FOREX world, we assume an evolution of the price of the underlying under the actual measure, and then we switch to a risk-neutral measure once we have determined the market price of risk. However, interest rates are not the price of an asset. Thus, we can not compute the market price of risk, therefore we build these models using a risk-neutral measure from the outset. For a more detailed explanation, refer to Chapter 10 of [3].

We will focus on one-factor short-rate models, which rely on a single stochastic differential equation to describe the behaviour of interest rates. In such models, all rates move in the same direction over short time intervals, though not necessarily by the same amount. This allows us to model parallel shifts in the yield curve. However, these models cannot capture more complex changes in the yield curve, such as variations in its slope or curvature over time.

For the task of choosing a suitable model, it is important to consider the following key aspects:

- Does the model allow negative interests rates, $R(t) < 0$?
- What is the distribution of the short-rate? Is it a fat-tailed distribution?
- Are bond prices $P(t, T)$ computable from the dynamics of the model?
- How suitable is the model for Monte Carlo simulation?
- Is the model mean-reverting (as interest rates tend to behave in this way)?

In the context of this paper, we will watch closely if the yield curve calculated from the chosen model can fit the actual term structure of interest rates, in other words, if $Y(0,T)$ can fit the current spot yield curve.

The model's mean-reverting behaviour is a crucial aspect. Without incorporating this dynamic for interest rates, the model may produce unrealistic outcomes. When interest rates are high, the economy typically slows down, leading to reduced demand for funds from borrowers, which causes rates to decrease. Conversely, when interest rates are low, borrower demand for funds tends to increase, driving rates upward. We can confirm this behaviour empirically by taking a look on the evolution of any US Treasury Yield.

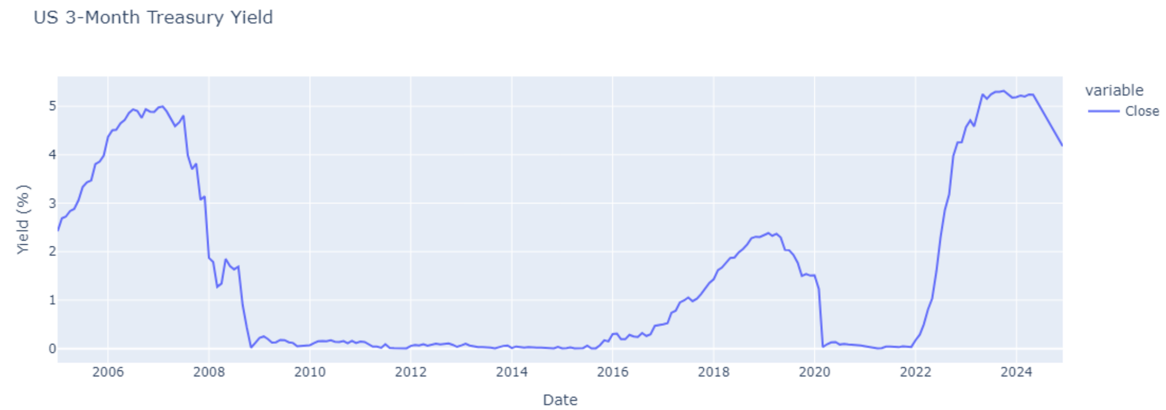

Figure 3: Interest rates usually present a mean reverting behaviour

This paper focuses on three one-factor short-rate models: Vasicek, CIR, and Hull-White. Although numerous other models exist, including multi-factor models (which are more realistic), our analysis will be confined to these three models.

1. Vasicek Model

The Vasicek model is a one-factor short-rate model that assumes a mean-reverting behaviour, ensuring that interest rates do not drift to extreme values. The following stochastic differential equation defines the model:

$$dR(t) = \kappa(\theta - R(t))dt + \sigma d\tilde{W}(t) \tag{10}$$

where:

- $R(t)$ is the short rate at time t
- $\kappa > 0$ is the speed of mean reversion
- θ is the long-term mean level of the rate
- $\sigma > 0$ is the volatility
- $\tilde{W}(t)$ is a standard Brownian motion under the risk-neutral measure

Above is a plot of the probability density function of $R(T)$ (at maturity) and several yield curves constructed with the previously outlined methodology for different time instants.

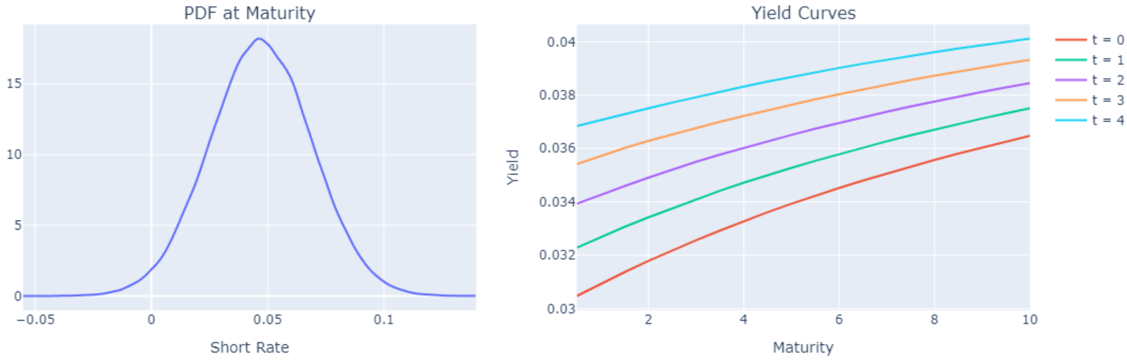

Figure 4: Vasicek model with parameters $R(0) = 0.03$, $\theta = 0.05$, $\kappa = 0.1$ and $\sigma = 0.01$

Note that the PDF allows for negative values of the short rate. This is one major drawback of the Vasicek model. For some parameter values (especially if σ is large enough), the model produces negative values for $R(t)$. This is, in general, a not-so-desirable feature as interest rates, under normal market conditions, should not be negative.

If we look at the yield curves constructed with the model, we can see that they vary over time, although they tend to produce just parallel shifts, failing to reproduce changes on the slope or the curvature. This means that the Vasicek model will fail to reproduce some shapes of the yield curve no matter how the parameters are chosen, making the model useless in some market scenarios.

2. CIR Model

The Cox-Ingersoll-Ross (CIR) model is a one-factor short-rate model that ensures mean-reverting behaviour while preventing negative interest rates by using a square root diffusion process. The following stochastic differential equation defines the model:

$$dR(t) = \kappa(\theta - R(t))dt + \sigma\sqrt{R(t)}d\tilde{W}(t) \tag{11}$$

which is very similar to (10), but incorporates the square root term in the diffusion part of the equation. This fact makes the model more realistic as the CIR model will not produce negative values for $R(t)$ (if the parameters don not take extreme values). We can confirm this by observing the PDF of the model (as shown below).

Like the Vasicek model, the CIR model cannot reproduce certain yield curve shapes. This can be improved by using exogenous term structure models, which incorporate the current spot yield curve into the model by adding a time-varying parameter so the model can reproduce any yield curve shape at $t = 0$ and allows for more realistic modelling of the yield curve for other time instants. In the next section, we study the Hull-White model, which is an example of an exogenous model.

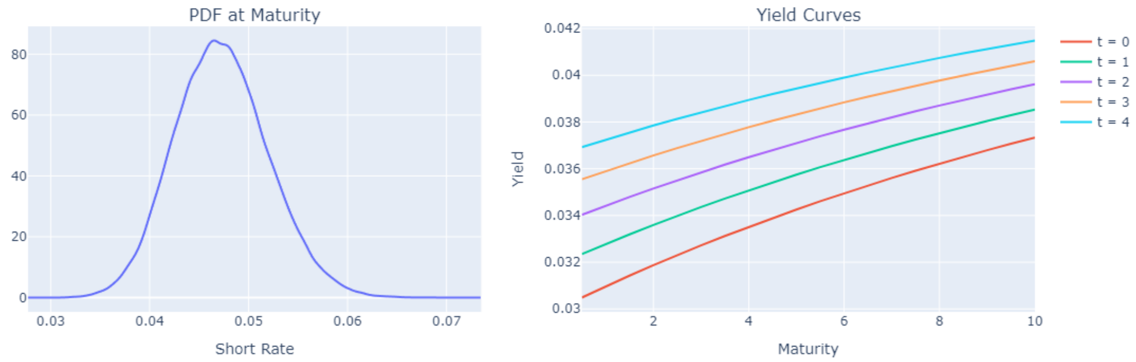

Figure 5: CIR model with parameters $R(0) = 0.03$, $\theta = 0.05$, $\kappa = 0.1$ and $\sigma = 0.01$

3. Hull-White Model

The Hull-White model is an extension of the Vasicek model that allows for time-dependent mean reversion and volatility parameters, making it more flexible for fitting the initial term structure of interest rates. It is defined by the following stochastic differential equation:

$$dR(t) = \kappa(\theta(t) - R(t))dt + \sigma d\tilde{W}(t) \tag{12}$$

Note that the long-term mean parameter θ now is time-dependent, making the model more flexible and able to produce more complex shapes of the yield curve. $\theta(t)$ must be chosen so as to exactly fit the term structure of interest rates currently observed in the market. The analytic expression for this parameter is the following:

$$\theta(t) = \frac{\partial f^M(0,t)}{\partial T} + \kappa f^M(0,t) + \frac{\sigma^2}{2\kappa}\left(1 - e^{-2\kappa t}\right) \tag{13}$$

Where $f^M(0,T)$ is the market instantaneous forward rate at time 0 for the maturity T, which can be computed from the actual spot yield curve observed in the market, $Y^M(0,T)$:

$$f^M(0,T) = Y^M(0,T) + T\,\frac{\partial Y^M(0,T)}{\partial T} \tag{14}$$

For a more detailed explanation, refer to [1] chapter 3. One of the drawbacks of the Hull-White model is that it allows negative interest rates (as can be seen in the PDF below). Note that despite computing the expression of $\theta(t)$, we are not able to reproduce the current spot curve as we would need to calibrate the model for the other parameters κ and σ.

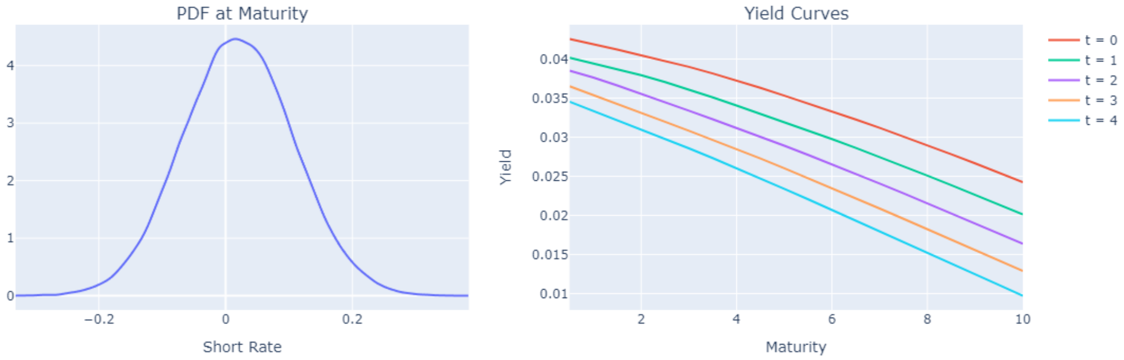

Figure 6: Hull-White model with parameters $R(0) = 0.04$, $\theta = \theta(t)$, $\kappa = 0.05$ and $\sigma = 0.03$

Implementation

In this section, we will briefly comment on the implementation of the models and the computation of the yield curves. We have taken a Monte Carlo approach for the short-rate calculation by discretizing the stochastic differential equations and calculating the values for $R(t)$ in an iterative way. For example, in the case of the Vasicek model, the equation (10) becomes:

$$R_{t+\Delta t} = R_t + \kappa(\theta - R_t)\Delta t + \sigma\sqrt{\Delta t}Z \tag{15}$$

where $Z \sim N(0,1)$ is a standard normal random variable. Regarding the yield curve, once the short rate is calculated, we have made use of expressions (5) and (7) in order to come up with the values of $Y(t,T)$.

The PDFs of the short-rate models have been calculated empirically using a Gaussian kernel density estimator, which has the following expression:

$$\hat{f}(x) = \frac{1}{nh\sqrt{2\pi}} \sum_{i=1}^{n} \exp\left(-\frac{(x - x_i)^2}{2h^2}\right) \tag{16}$$

where:

- n is the number of data points,
- h is the bandwidth (smoothing parameter),
- x_i are the observed data points.

For the Hull-White model, we have computed $\theta(t)$ using expressions (13) and (14) and inputting the current spot yield curve observed from the market.

For a more detailed explanation of the implementation in Python, please click here.

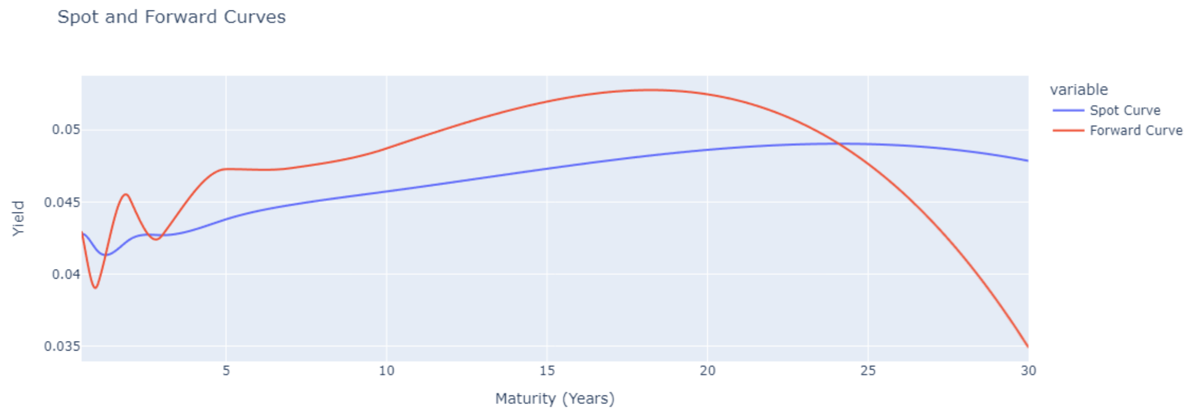

Figure 7: Current spot and forward curves (at $t = 0$)

Conclusion

In conclusion, we successfully developed a model to construct interest rate curves from an arbitrary short-rate model. However, the models we studied fall short in capturing the complexities of the yield curve. Some produce unrealistic scenarios, such as negative interest rates, while others fail to fully explain the curve's behaviour.

To address these shortcomings, future work could explore alternative short-rate models, particularly exogenous models with time-varying coefficients or multi-factor models that better capture the dynamics of the yield curve.

References

[1] Damiano Brigo, Fabio Mercurio, et al. *Interest rate models: theory and practice*, volume 2. Springer, 2001.

[2] John C Hull and Sankarshan Basu. *Options, futures, and other derivatives*. Pearson Education India, 2016.

[3] Steven E Shreve et al. *Stochastic calculus for finance II: Continuous-time models*, volume 11. Springer, 2004.