Topological Spaces for Fuzzy Rough Sets Determined by Fuzzy Implication Operators

Wei-Zhi Wu
School of Mathematics, Physics and Information Science
Zhejiang Ocean University
Zhoushan, Zhejiang, 316004, P.R. China
Email: wuwz@zjou.edu.cn

Peirong Lin
Department of Computer Science and Engineering
Zhangzhou Teachers' College
Zhangzhou, Fujian, 363000, P.R. China
Email: prlin2007@163.com

***Abstract*—In this paper, a dual pair of lower and upper fuzzy rough approximation operators determined by a fuzzy implication operator are defined and their properties are presented. Topological structures of fuzzy rough sets are then investigated. It is shown that the lower and upper fuzzy rough approximation operators are, respectively, an interior operator and a closure operator in a fuzzy topological space if and only if the fuzzy binary relation is reflexive and transitive.**

***Keywords*-fuzzy implication operators; fuzzy rough sets; rough sets; topological spaces**

I. Introduction

The basic structure of rough set theory [12] is an approximation space consisting of a universe of discourse and a binary relation imposed on it. Using the concepts of lower and upper approximations in rough set theory, knowledge hidden in information systems may be unravelled and expressed in the form of decision rules.

In the Pawlak's rough set model, an equivalence relation is a key and primitive notion. This equivalence relation, however, seems to be a very stringent condition that may limit the application domain of the rough set model. Thus one of the main directions in the study of rough sets is the generalization of Pawlak's rough set approximations. Many authors have generalized the notion of approximation operators by using nonequivalence binary relations (see e.g. [23], [24]). Rough sets can also be extended to the fuzzy environment and the results are called rough fuzzy sets and fuzzy rough sets [1], [5], [20], [21], [22]. The lower and upper approximation operators in the fuzzy environment can be represented by fuzzy implication operators or triangular norms. It is well known that there are a lot of implicators [15] which have been widely used in fuzzy sets research and fuzzy inference results often depend upon the choice of the implicator and choice of the triangular norm. Therefore, for analyzing uncertainty in complicated fuzzy systems, lower and upper fuzzy rough approximations defined by various implicators in fuzzy set theory were examined in a number of studies [9], [10], [11], [14], [19], [25]. On the other hand, lower and upper approximation operators in many rough set models are dual with each other, the dual properties of lower and upper approximation operators are very important and have strong relationships with the interior and closure operators in topological space. The concept of topological structures and their generalizations are ones of the most powerful notions in data and system analysis. Comparing approximation operators with the interior and closure operators in topological spaces is an interesting and important research issue. In fact, many works have appeared for the study of topological structures of rough sets [1], [2], [3], [7], [13], [16], [17], [18], [26].

In this paper, by employing a general type of fuzzy implication operator, we introduce a dual pair of lower and upper fuzzy rough approximation operators and investigate the topological structures of the proposed fuzzy rough sets.

II. Fuzzy logical operators

A triangular norm, or t-norm in short, is an increasing, associative and commutative mapping $\mathcal{T}\ :\ [0,1]\times[0,1]\rightarrow[0,1]$, satisfying the boundary condition: for all $\alpha\in[0,1], \mathcal{T}(\alpha,1)=\alpha$.

The most popular continuous t-norms are:

• the standard min operator : $\mathcal{T}_{\mathrm{M}}(\alpha,\beta)=\min\{\alpha,\beta\}$ (the largest t-norm),

• the algebraic product : $\mathcal{T}_{\mathrm{P}}(\alpha,\beta)=\alpha*\beta$,

• the bold intersection (also called the Łukasiewicz t-norm): $\mathcal{T}_{\mathrm{L}}(\alpha,\beta)=\max\{0,\alpha+\beta-1\}$.

A triangular conorm (t-conorm in short) is an increasing, associative and commutative mapping $\mathcal{S}\ :\ [0,1]\times[0,1]\rightarrow[0,1]$ satisfying the boundary condition: for all $\alpha\in[0,1], \mathcal{S}(\alpha,0)=\alpha$.

Three well-known continuous t-conorms are:

• the standard max operator : $\mathcal{S}_{\mathrm{M}}(\alpha,\beta)=\max\{\alpha,\beta\}$ (the smallest t-conorm),

• the probabilistic sum : $\mathcal{S}_{\mathrm{P}}(\alpha,\beta)=\alpha+\beta-\alpha*\beta$,

• the bounded sum : $\mathcal{S}_{\mathrm{L}}(\alpha,\beta)=\min\{1,\alpha+\beta\}$.

A negator $\mathcal{N}$ is a decreasing $[0,1]\rightarrow[0,1]$ mapping satisfying $\mathcal{N}(0)=1$ and $\mathcal{N}(1)=0$. The negator $\mathcal{N}_{\mathrm{S}}(\alpha)=1-\alpha$ is usually referred to as the standard negator. A negator $\mathcal{N}$ is called involutive if and only if (iff) $\mathcal{N}(\mathcal{N}(\alpha))=\alpha$, for

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all $\alpha \in [0,1]$. It is well known that every involutive negator is continuous [6].

Given a negator $\mathcal{N}$, a t-norm $\mathcal{T}$ and a t-conorm $\mathcal{S}$ are called dual with respect to (w.r.t.) $\mathcal{N}$ iff the De Morgan's laws are satisfied, i.e.

$$\mathcal{S}(\mathcal{N}(\alpha),\mathcal{N}(\beta)) = \mathcal{N}(\mathcal{T}(\alpha,\beta)),\ \forall \alpha,\beta \in [0,1],$$
$$\mathcal{T}(\mathcal{N}(\alpha),\mathcal{N}(\beta)) = \mathcal{N}(\mathcal{S}(\alpha,\beta)),\ \forall \alpha,\beta \in [0,1].$$

It is well known [6] that for an involutive negator $\mathcal{N}$ and a t-conorm $\mathcal{S}$, the function $\mathcal{T}_\mathcal{S}$ defined by $\mathcal{T}_\mathcal{S}(\alpha,\beta) = \mathcal{N}(\mathcal{S}(\mathcal{N}(\alpha),\mathcal{N}(\beta))), \alpha,\beta \in [0,1]$, is a t-norm such that $\mathcal{T}_\mathcal{S}$ and $\mathcal{S}$ are dual w.r.t. $\mathcal{N}$. In what follows, it will be referred to as a t-norm dual to $\mathcal{S}$ w.r.t. $\mathcal{N}$.

Let X be a nonempty set called the universe of discourse. By a fuzzy set A in X, we mean a mapping $A : X \to [0,1]$. The class of all subsets of X (resp. all fuzzy sets in X) will be denoted by $\mathcal{P}(X)$ (resp. $\mathcal{F}(X)$). Zadeh's fuzzy union and fuzzy intersection will be denoted by $\cup$ and $\cap$ respectively, and $\sim_\mathcal{N}$ will be used to denote fuzzy complement determined by a negator $\mathcal{N}$, i.e. for every $A \in \mathcal{F}(X)$ and every $x \in X$, $(\sim_\mathcal{N} A)(x) = \mathcal{N}(A(x))$. If $\mathcal{N} = \mathcal{N}_s$, we then write $\sim A$ instead of $\sim_\mathcal{N} A$. For $\alpha \in [0,1]$, $\widehat{\alpha}$ will denote the constant fuzzy set: $\widehat{\alpha}(x) = \alpha$, for all $x \in X$.

By an implicator (fuzzy implication operator) we mean a function $\mathcal{I} : [0,1] \times [0,1] \to [0,1]$ satisfying $\mathcal{I}(1,0) = 0$ and $\mathcal{I}(1,1) = \mathcal{I}(0,1) = \mathcal{I}(0,0) = 1$. An implicator $\mathcal{I}$ is called left monotonic (resp. right monotonic) iff for every $\alpha \in [0,1], \mathcal{I}(\cdot,\alpha)$ is decreasing (resp. $\mathcal{I}(\alpha,\cdot)$ is increasing). If $\mathcal{I}$ is both left monotonic and right monotonic, then it is called hybrid monotonic. $\mathcal{I}$ is semicontinuous if

$$\mathcal{I}(\bigvee_j a_j, \bigwedge_k b_k) = \bigwedge_{j,k} \mathcal{I}(a_j, b_k), \qquad (1)$$

for all indexed families $\{a_j : j \in J\}$ and $\{b_k : k \in K\}$ of real numbers in $[0,1]$.

It is easy to see that $\mathcal{I}(\alpha,1) = 1$, for all $\alpha \in [0,1]$, when $\mathcal{I}$ is a left monotonic implicator, and if $\mathcal{I}$ is right monotonic, then $\mathcal{I}(0,\alpha) = 1$, for all $\alpha \in [0,1]$.

For a left monotonic implicator $\mathcal{I}$, the function $\mathcal{N}$ defined by $\mathcal{N}(\alpha) = \mathcal{I}(\alpha,0), \alpha \in [0,1]$, is a negator, called a negator induced by $\mathcal{I}$. For example, the Łukasiewicz implicator $\mathcal{I}_\mathrm{L}$ defined by $\mathcal{I}_\mathrm{L}(\alpha,\beta) = \min\{1, 1-\alpha+\beta\}$ induces the standard negator $\mathcal{N}_\mathrm{S}$.

An implicator $\mathcal{I}$ is said to be a border implicator (or it satisfies the neutrality principle [4]) iff for every $\alpha \in [0,1], \mathcal{I}(1,\alpha) = \alpha$.

An implicator $\mathcal{I}$ is said to be an EP implicator (EP stands for exchange principle [15]) if it satisfies, for all $\alpha,\beta,\gamma \in [0,1]$,

$$\mathcal{I}(\alpha, \mathcal{I}(\beta,\gamma)) = \mathcal{I}(\beta, \mathcal{I}(\alpha,\gamma)). \qquad (2)$$

An implicator $\mathcal{I}$ is said to be a CP implicator (CP stands for confinement principle [4]) if it satisfies, for all $\alpha,\beta \in [0,1]$,

$$\alpha \le \beta \Longleftrightarrow \mathcal{I}(\alpha,\beta) = 1. \qquad (3)$$

Several classes of implicators have been studied in the literature. We recall here the definitions of two main classes of operators [4].

Let $\mathcal{T}, \mathcal{S}$, and $\mathcal{N}$ be a t-norm, a t-conorm, and a negator, respectively. An implicator $\mathcal{I}$ is called

- an S-implicator based on $\mathcal{S}$ and $\mathcal{N}$ iff

$$\mathcal{I}(\alpha,\beta) = \mathcal{S}(\mathcal{N}(\alpha),\beta) \text{ for all } \alpha,\beta \in [0,1]; \qquad (4)$$

- an R-implicator (residual implicator) based on a left-continuous t-norm $\mathcal{T}$ iff for all $\alpha,\beta \in [0,1]$,

$$\mathcal{I}(\alpha,\beta) = \sup\{\lambda \in [0,1] : \mathcal{T}(\alpha,\lambda) \le \beta\}. \qquad (5)$$

We will use $\theta_\mathcal{T}$ to denote the R-implicator, i.e.,

$$\theta_\mathcal{T}(\alpha,\beta) = \sup\{\lambda \in [0,1] : \mathcal{T}(\alpha,\lambda) \le \beta\}. \qquad (6)$$

From a t-conorm $\mathcal{S}$, we define a binary operation $\sigma_\mathcal{S}$ on $[0,1]$ as follows: for all $\alpha,\beta \in [0,1]$,

$$\sigma_\mathcal{S}(\alpha,\beta) = \inf\{\lambda \in [0,1] :\ \mathcal{S}(\alpha,\lambda) \ge \beta\}. \qquad (7)$$

If $\mathcal{S}$ is the dual t-conorm of a t-norm $\mathcal{T}$ w.r.t. $\mathcal{N}_s$, then $\sigma_\mathcal{S}$ is dual to $\theta_\mathcal{T}$ w.r.t. $\mathcal{N}_s$ [10], i.e. for all $\alpha,\beta \in [0,1]$,

$$\sigma_\mathcal{S}(1-\alpha, 1-\beta) = 1 - \theta_\mathcal{T}(\alpha,\beta). \qquad (8)$$

Three most popular S-implicators are:

- the Łukasiewicz implicator : $\mathcal{I}_\mathrm{L}(\alpha,\beta) = \min\{1, 1-\alpha+\beta\}$, based on $\mathcal{S}_\mathrm{L}$ and $\mathcal{N}_\mathrm{S}$,
- the Kleene-Dienes implicator : $\mathcal{I}_\mathrm{KD}(\alpha,\beta) = \max\{1-\alpha,\beta\}$, based on $\mathcal{S}_\mathrm{M}$ and $\mathcal{N}_\mathrm{S}$,
- the Kleene-Dienes-Łukasiewicz implicator : $\mathcal{I}_\star(\alpha,\beta) = 1-\alpha+\alpha*\beta$, based on $\mathcal{S}_\mathrm{P}$ and $\mathcal{N}_\mathrm{S}$.

The most popular R-implicators are:

- the Łukasiewicz implicator $\mathcal{I}_\mathrm{L}$, based on $\mathcal{T}_\mathrm{L}$,
- the Gödel implicator : $\mathcal{I}_\mathrm{G}(\alpha,\beta) = 1$, for $\alpha \le \beta$ and $\mathcal{I}_\mathrm{G}(\alpha,\beta) = \beta$ elsewhere, based on $\mathcal{T}_\mathrm{M}$,
- the Gaines implicator : $\mathcal{I}_\Delta(\alpha,\beta) = 1$, for $\alpha \le \beta$ and $\mathcal{I}_\Delta(\alpha,\beta) = \beta$ elsewhere, based on $\mathcal{T}_\mathrm{P}$.

Proposition 1. [14] Every S-implicator and every R-implicator are a hybrid monotonic, border, and EP implicator. Every R-implicator is a CP implicator.

III. Fuzzy rough sets determined by fuzzy implicators

In this section, by employing a general type of implicator on $[0,1]$ we introduce concepts related to fuzzy rough sets, and examine basic properties of lower and upper fuzzy rough approximation operators.

A. Concepts of fuzzy rough sets

Definition 1. Let U and W be two non-empty universes of discourse. A fuzzy subset $R \in \mathcal{F}(U \times W)$ is called a fuzzy binary relation from U to W, $R(x,y)$ is the degree of relation between x and y, where $(x,y) \in U \times W$. If for each $x \in U$, there exists a $y \in W$ such that $R(x,y) = 1$, then R is a serial fuzzy relation from U to W. If $U = W$, then R is called a fuzzy relation on U; $R \in \mathcal{F}(U \times U)$ is a reflexive fuzzy relation if $R(x,x) = 1$ for all $x \in U$; $R \in \mathcal{F}(U \times U)$ is a $\mathcal{T}$-transitive fuzzy relation if $R(x,z) \geq \vee_{y \in U}(\mathcal{T}(R(x,y), R(y,z))$ for all $x, z \in U$, where $\mathcal{T}$ is a t-norm on $[0,1]$.

Definition 2. Let U and W be two non-empty universes of discourse and $R \in \mathcal{F}(U \times W)$, the triple (U, W, R) is called a fuzzy approximation space. When $U = W$ and R is a fuzzy relation on U, we also call (U, R) a fuzzy approximation space.

Definition 3. Let (U, W, R) be a fuzzy approximation space, $\mathcal{I}$ an implicator on $[0,1]$, and $\mathcal{N}$ a negator on $[0,1]$. For $A \in \mathcal{F}(W)$, the lower approximation of A w.r.t. (U, W, R), denoted as $\underline{R}_{\mathcal{I}}(A)$, is a fuzzy set of U whose membership function is defined by

$$\underline{R}_{\mathcal{I}}(A)(x) = \bigwedge_{y \in W} \mathcal{I}(R(x,y), A(y)), \quad x \in U. \tag{9}$$

The upper approximation of A w.r.t. (U, W, R), denoted as $\overline{R}_{\mathcal{I}}(A)$, is defined by

$$\overline{R}_{\mathcal{I}}(A) = \sim_{\mathcal{N}} \underline{R}_{\mathcal{I}}(\sim_{\mathcal{N}} A). \tag{10}$$

The pair $(\underline{R}_{\mathcal{I}}(A), \overline{R}_{\mathcal{I}}(A))$ is called the $\mathcal{I}$-fuzzy rough set of A w.r.t. (U, W, R). The operators $\underline{R}_{\mathcal{I}}$ and $\overline{R}_{\mathcal{I}}$ from $\mathcal{F}(W)$ to $\mathcal{F}(U)$ are, respectively, referred to as the lower and upper fuzzy $\mathcal{I}$-rough approximation operators of (U, W, R) and the system $(\mathcal{F}(U), \mathcal{F}(W), \cap, \cup, \sim, \underline{R}_{\mathcal{I}}, \overline{R}_{\mathcal{I}})$ is called a fuzzy rough set algebra.

Specially, if $\mathcal{I}$ is an S-implicator based on a t-conorm $\mathcal{S}$ and an involutive negator $\mathcal{N}$, and $\mathcal{T}$ and $\mathcal{S}$ are dual w.r.t. $\mathcal{N}$, then we call $(\underline{R}_{\mathcal{I}}(A), \overline{R}_{\mathcal{I}}(A))$ the S-fuzzy rough set of A w.r.t. (U, W, R). If $\mathcal{I}$ is an R-implicator based on a t-norm $\mathcal{T}$, then we call $(\underline{R}_{\mathcal{I}}(A), \overline{R}_{\mathcal{I}}(A))$ the R-fuzzy rough set of A w.r.t. (U, W, R).

Remark 1. (1) If $\mathcal{I}$ is an S-implicator based on a t-conorm $\mathcal{S}$ and a negator $\mathcal{N}$, $\mathcal{T}$ is the t-norm dual to $\mathcal{S}$ w.r.t. $\mathcal{N}$, then it can be verified that for $x \in U$,

$$\begin{aligned} \underline{R}_{\mathcal{I}}(A)(x) &= \bigwedge_{y \in W} \mathcal{S}(\mathcal{N}(R(x,y)), A(y)), \\ \overline{R}_{\mathcal{I}}(A)(x) &= \bigvee_{y \in W} \mathcal{T}(R(x,y), A(y)). \end{aligned} \tag{11}$$

More specially, if $\mathcal{T} = \min = \wedge$, $\mathcal{S} = \max = \vee$, and $\mathcal{N} = \mathcal{N}_s$, then for $x \in U$,

$$\begin{aligned} \underline{R}_{\mathcal{I}}(A)(x) &= \bigwedge_{y \in W} (1 - R(x,y)) \vee A(y), \\ \overline{R}_{\mathcal{I}}(A)(x) &= \bigvee_{y \in W} R(x,y) \wedge A(y). \end{aligned} \tag{12}$$

(2) If $\mathcal{I} = \theta_{\mathcal{T}}$ is an R-implicator based on a left-continuous t-norm $\mathcal{T}$, $\mathcal{S}$ is the t-conorm dual to $\mathcal{T}$ w.r.t. $\mathcal{N}_s$, then it can be verified that for $x \in U$,

$$\begin{aligned} \underline{R}_{\mathcal{I}}(A)(x) &= \bigwedge_{y \in W} \theta_{\mathcal{T}}(R(x,y), A(y)), \\ \overline{R}_{\mathcal{I}}(A)(x) &= \bigvee_{y \in W} \sigma_{\mathcal{S}}(1 - R(x,y), A(y)). \end{aligned} \tag{13}$$

B. Some properties of fuzzy rough approximation operators

In discussion to follow, we always take $\mathcal{N} = \mathcal{N}_s$ and assume that $\mathcal{I}$ is a semicontinuous, hybrid monotonic implicator on $[0,1]$. The next Proposition 2 gives some basic properties of fuzzy rough approximation operators.

Proposition 2. Let (U, W, R) be a fuzzy approximation space. Then, the fuzzy rough approximation operators $\underline{R}_{\mathcal{I}}$ and $\overline{R}_{\mathcal{I}}$ in Definition 3 have the following properties: For all $A, B \in \mathcal{F}(W)$, $A_j \in \mathcal{F}(W) (\forall j \in J)$, J is an index set,

(FL1) $\underline{R}_{\mathcal{I}}(W) = U$.
(FU1) $\overline{R}_{\mathcal{I}}(\emptyset) = \emptyset$.
(FL2) $\underline{R}_{\mathcal{I}}(\bigcap_{j \in J} A_j) = \bigcap_{j \in J} \underline{R}_{\mathcal{I}}(A_j)$.
(FU2) $\overline{R}_{\mathcal{I}}(\bigcup_{j \in J} A_j) = \bigcup_{j \in J} \overline{R}_{\mathcal{I}}(A_j)$.
(FL3) $A \subseteq B \Longrightarrow \underline{R}_{\mathcal{I}}(A) \subseteq \underline{R}_{\mathcal{I}}(B)$.
(FU3) $A \subseteq B \Longrightarrow \overline{R}_{\mathcal{I}}(A) \subseteq \overline{R}_{\mathcal{I}}(B)$.
(FL4) $\underline{R}_{\mathcal{I}}(\bigcup_{j \in J} A_j) \supseteq \bigcup_{j \in J} \underline{R}_{\mathcal{I}}(A_j)$.
(FU4) $\overline{R}_{\mathcal{I}}(\bigcap_{j \in J} A_j) \subseteq \bigcap_{j \in J} \overline{R}_{\mathcal{I}}(A_j)$.

Proof. The proof of (Fl1)-(FL4) are given in [19]. Properties (FU1)-(FU4) can be deduced from Eq. (10) and properties (FL1)-(FL4), respectively.

By using results in [19] and Eq. (10), we can obtain the following Propositions 3-5 which show that property of some special fuzzy relation can be characterized by the properties of its induced $\mathcal{I}$-fuzzy rough approximation operators.

Proposition 3. Let (U, W, R) be a fuzzy approximation space, and $\mathcal{I}$ a border implicator. If R is a serial fuzzy relation from U to W, then the $\mathcal{I}$-fuzzy rough approximation operators in Definition 3 satisfy the following properties:

(FLU0) $\underline{R}_{\mathcal{I}}(A) \subseteq \overline{R}_{\mathcal{I}}(A), \forall A \in \mathcal{F}(W)$.
(FL0) $\underline{R}_{\mathcal{I}}(\widehat{\alpha}) = \widehat{\alpha}, \ \forall \alpha \in [0,1]$.
(FU0) $\overline{R}_{\mathcal{I}}(\widehat{\alpha}) = \widehat{\alpha}, \ \forall \alpha \in [0,1]$.
(FL0)$'$ $\underline{R}_{\mathcal{I}}(\emptyset) = \emptyset$.
(FU0)$'$ $\overline{R}_{\mathcal{I}}(W) = U$.

Conversely, if $\mathcal{I}$ is a border and CP implicator, then one of the above properties implies that R is serial.

Proposition 4. Let (U, R) be a fuzzy approximation space and $\mathcal{I}$ a border implicator on $[0,1]$. If R is a reflexive fuzzy relation, then,

(FLR) $\underline{R}_{\mathcal{I}}(A) \subseteq A, \quad \forall A \in \mathcal{F}(U)$,
(FUR) $A \subseteq \overline{R}_{\mathcal{I}}(A), \quad \forall A \in \mathcal{F}(U)$.

Conversely, if $\mathcal{I}$ is a CP implicator, and (FLR) or (FUR) holds, then R is reflexive.

Proposition 5. Let (U, R) be a fuzzy approximation space,$\mathcal{T}$ a lower semicontinuous t-norm and $\mathcal{I}$ an implicator satisfying, $\forall \alpha, \beta, \gamma \in [0, 1]$,

$$\mathcal{I}(\alpha, \mathcal{I}(\beta, \gamma)) = \mathcal{I}(\mathcal{T}(\alpha, \beta), \gamma). \tag{14}$$

If R is $\mathcal{T}$-transitive, then

(FLT) $\underline{R_{\mathcal{I}}}(A) \subseteq \underline{R_{\mathcal{I}}}(\underline{R_{\mathcal{I}}}(A)), \quad \forall A \in \mathcal{F}(U)$,

(FUT) $\overline{R}_{\mathcal{I}}(\overline{R}_{\mathcal{I}}(A)) \subseteq \overline{R}_{\mathcal{I}}(A), \quad \forall A \in \mathcal{F}(U)$.

Conversely, when $\mathcal{I}$ is a CP and border implicator, if (FLT) or (FUT) holds, then R is $\mathcal{T}$-transitive.

C. Fuzzy topological structures of fuzzy rough set algebras

Definition 5. [8] A subset τ of $\mathcal{F}(U)$ is called a fuzzy topology on U iff it satisfies:

(T1) For any constant fuzzy set $\widehat{\alpha} \in \mathcal{F}(U)$, $\widehat{\alpha} \in \tau$,

(T2) $A, B \in \tau \Longrightarrow A \cap B \in \tau$,

(T3) $\mathcal{A} \subseteq \tau \Longrightarrow \cup\{A : A \in \mathcal{A}\} \in \tau$.

Definition 6. [8] A mapping $\Psi : \mathcal{F}(U) \to \mathcal{F}(U)$ is called a fuzzy interior operator iff for all $A, B \in \mathcal{F}(U)$ it satisfies:

(1) $\Psi(A) \subseteq A$,

(2) $\Psi(A \cap B) = \Psi(A) \cap \Psi(B)$,

(3) $\Psi^2(A) = \Psi(A)$,

(4) $\Psi(\widehat{\alpha}) = \widehat{\alpha}, \quad \forall \alpha \in [0, 1]$.

Definition 7. [8] A mapping $\Phi : \mathcal{F}(U) \to \mathcal{F}(U)$ is called a fuzzy closure operator iff for all $A, B \in \mathcal{F}(U)$ it satisfies:

(1) $A \subseteq \Phi(A)$,

(2) $\Phi(A \cup B) = \Phi(A) \cup \Phi(B)$,

(3) $\Phi^2(A) = \Phi(A)$,

(4) $\Phi(\widehat{\alpha}) = \widehat{\alpha}, \quad \forall \alpha \in [0, 1]$.

The elements of a fuzzy topology τ are referred to as open fuzzy sets, and it is easy to show that a fuzzy interior operator Ψ defines a fuzzy topology

$$\tau_{\Psi} = \{A \in \mathcal{F}(U) : \Psi(A) = A\}. \tag{15}$$

So, the open fuzzy sets are the fixed points of Ψ.

In the sequel, we assume that $\mathcal{T}$ is a lower semicontinuous t-norm on $[0, 1]$, $\mathcal{S}$ is the t-conorm dual to $\mathcal{T}$.

Theorem 1. Let (U, R) be a fuzzy approximation space and $\mathcal{I}$ a CP and border implicator satisfying Eq. (14). Then

(1) The operator $\Phi = \overline{R}_{\mathcal{I}} : \mathcal{F}(U) \to \mathcal{F}(U)$ is a fuzzy closure operator iff R is a reflexive and $\mathcal{T}$-transitive fuzzy relation.

(2) The operator $\Psi = \underline{R_{\mathcal{I}}} : \mathcal{F}(U) \to \mathcal{F}(U)$ is a fuzzy interior operator iff R is a reflexive and $\mathcal{T}$-transitive fuzzy relation.

Proof. (1) " $\Rightarrow$ " Assume that $\Phi = \overline{R}_{\mathcal{I}} : \mathcal{F}(U) \to \mathcal{F}(U)$ is a fuzzy closure operator. Since $A \subseteq \Phi(A)$ for all $A \in \mathcal{F}(U)$, by Proposition 4, we know that R is a reflexive fuzzy relation. Since $\Phi^2(A) = \Phi(A)$, i.e.

$$\overline{R}_{\mathcal{I}}(\overline{R}_{\mathcal{I}}(A)) = \overline{R}_{\mathcal{I}}(A), \quad \forall A \in \mathcal{F}(U), \tag{16}$$

by the reflexivity of R and (FUR), we must have

$$\overline{R}_{\mathcal{I}}(\overline{R}_{\mathcal{I}}(A)) \subseteq \overline{R}_{\mathcal{I}}(A), \quad \forall A \in \mathcal{F}(U). \tag{17}$$

Hence, by Proposition 5, we conclude that R is a $\mathcal{T}$-transitive fuzzy relation.

" $\Leftarrow$ " If R is a reflexive and $\mathcal{T}$-transitive fuzzy relation, by Proposition 4, we have

$$A \subseteq \Phi(A), \quad \forall A \in \mathcal{F}(U). \tag{18}$$

It should be noted that the reflexivity of R implies that R is serial, it follows from Proposition 3 that

$$\Phi(\widehat{\alpha}) = \widehat{\alpha}, \quad \forall \alpha \in [0, 1]. \tag{19}$$

Since R is $\mathcal{T}$-transitive, by Proposition 5, we have that

$$\Phi^2(A) \subseteq \Phi(A), \quad \forall A \in \mathcal{F}(U). \tag{20}$$

On the other hand, by the reflexivity and (FUR), it is easy to see that

$$\Phi^2(A) \supseteq \Phi(A), \quad \forall A \in \mathcal{F}(U). \tag{21}$$

Hence, by combining Eqs. (20) and (21), we conclude

$$\Phi^2(A) = \Phi(A), \quad \forall A \in \mathcal{F}(U). \tag{22}$$

From property (FU2) in Proposition 2 we have

$$\Phi(A \cup B) = \Phi(A) \cup \Phi(B), \quad \forall A, B \in \mathcal{F}(U).$$

Thus we have proved that $\Phi = \overline{R}_{\mathcal{I}} : \mathcal{F}(U) \to \mathcal{F}(U)$ is a fuzzy closure operator.

(2) It is similar to the proof of (1).

Theorem 2. Assume that R is a reflexive and $\mathcal{T}$-transitive fuzzy relation on U. Then there exists a fuzzy topology $\tau_{_R}$ on U such that $\Psi = \underline{R_{\mathcal{I}}} : \mathcal{F}(U) \to \mathcal{F}(U)$ and $\Phi = \overline{R}_{\mathcal{I}} : \mathcal{F}(U) \to \mathcal{F}(U)$ are, respectively, the fuzzy interior and closure operators induced by $\tau_{_R}$.

Proof. Since R is a reflexive and $\mathcal{T}$-transitive fuzzy relation on U, by Theorem 1, we conclude that $\Psi = \underline{R_{\mathcal{I}}} : \mathcal{F}(U) \to \mathcal{F}(U)$ and $\Phi = \overline{R}_{\mathcal{I}} : \mathcal{F}(U) \to \mathcal{F}(U)$ are the fuzzy interior and closure operators, respectively. Define

$$\tau_{_R} = \{A \in \mathcal{F}(U) : \underline{R_{\mathcal{I}}}(A) = A\}.$$

Then it is easy to prove that $\tau_{_R}$ is a fuzzy topology on U.

IV. Conclusion

By employing fuzzy implication operators, we have introduced a general type of fuzzy rough set model. Many existing fuzzy rough set models in the literature can be treated as special cases of our definitions by taking specific fuzzy implication operators. We have presented some basic properties of the $\mathcal{I}$-fuzzy rough approximation operators. We have shown that the lower and upper $\mathcal{I}$-fuzzy rough approximation operators are, respectively, an interior operator and a closure operator in a fuzzy topological space if and only if the fuzzy relation in the approximation space is reflexive and transitive. The proposed fuzzy rough set model may provide a potentially useful tool for reasoning and knowledge acquisition in fuzzy systems and fuzzy decision systems by employing different types of fuzzy implication operators.

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