THE FUZZY DESCRIPTION OF THE BOUNDARY REGION IN ROUGH SETS AND ITS APPLICATIONS

BING-ZHEN SUN[1], ZENG-TAI GONG[2]

[1]College of Traffic and Transportation, Lanzhou Jiaotong University, Lanzhou 730070 Gansu China
[2]College of Mathematics and Information Science, Northwest Normal University, Lanzhou 730070 Gansu China
E-MAIL: bzh--sun@163.com

Abstract:

In this paper, the boundary region of a rough set in the universe U is divided into lower and upper boundaries, and a fuzzy set is defined in the boundary region of the rough set based on the theory of inclusion degree. It can descript the element in boundary region of any rough set more precision; therefore, it can characterize the process which the rough set from vagueness to crisp exactly, and it also shows the nature of the rough set. Based on this definition, we redefine the rough membership function by the fuzzy set on the boundary region, and the operation of the intersection and union of any two fuzzy set will be also hold under the new rough membership function. Hence, the deficiency of reference [3,4] is overcome. Finally, the measure of fuzziness of any rough set is discussed by the theory of fuzzy entropy.

Keywords:

Rough sets; Fuzzy sets; Inclusion degree; Boundary region; Rough membership function

1. Introduction

Rough sets[2], as a new mathematical tool to deal with vagueness, uncertainty and incomplete data, was first introduced by Pawlak in 1982. Fuzzy sets[1], a mathematical theory of describing the fuzzy phenomenon or fuzzy concepts which have not clearly border and not describe exactly for the same standard, was first introduced by Zadeh in 1965. The rough sets mainly method is to gain the decision rule of a problem or the classification rules which not change the ability of the classification with the knowledge reduction. The fuzzy sets and rough sets theories are all extensions of the classical sets when they deal to the uncertainty and incomplete problems. However, there are some differences in describing the uncertainty and incomplete knowledge. Fuzzy sets emphasize the vagueness of a set, but rough set theory base on the indiscernibility of objects in universe. The method of rough sets is the expression and reduction of the knowledge and fuzzy sets is focus on how to gain its membership functions.

In [3], the author studied the interrelations of fuzzy sets and rough sets by define the rough membership function of a rough set in the universe U. In [4], the author analyzed the drawbacks of the rough membership function defined in [3] and improved it by means of the approximation precision $\rho_R(X)$ of any rough set in U. However, they are not distinguishing the elements in the boundary region, that is to say, the membership degree of the elements in boundary region about X is the same value, and the difference is the membership value. Both of them are not characterize the process that the rough set from vagueness to crisp exactly.

In present paper, the boundary region of any rough set is divided into lower and upper boundaries according to the different interrelations of the element in boundary region with the set, and define a fuzzy set in the boundary region based on the membership value different about the similar degree of the element in boundary region with the set under the ability of the classification on the knowledge R. In the same time, we redefine the rough membership function by the fuzzy set which defined in the boundary region, and then the operation of the intersection and union of any two fuzzy set is also hold under the membership. Finally, the measure of fuzziness of any rough set is discussed by the theory of fuzzy entropy.

2. Definitions and Notations

We introduce in this section some definitions and notation used in the present paper.

Definition 2.1 Let U be a nonempty set which is called the universe. A fuzzy subset of U is a mapping from U to $[0,1]$: $\mu_A : U \to [0.1]$, denote as $A = \{(x, \mu_A(x)) | x \in U\}$, where μ_A stands for the

1-4244-0973-X/07/$25.00 ©2007 IEEE

membership function of A, A^C denote the complementation of A. The membership function of A^C defined as: $\mu_{A^C}(x) = 1 - \mu_A(x)$ for any $x \in U$.

Let X, Y be any two fuzzy subsets on universe U, the intersection and union of X and Y are defined, respectively, as following:

$$\mu_{X \cup Y}(x) = \max\{\mu_X(x), \mu_Y(x)\} \quad (1)$$

$$\mu_{X \cap Y}(x) = \min\{\mu_X(x), \mu_Y(x)\} \quad (2)$$

Definition2.2 Let U be a finite and nonempty set which is called the universe. Let R be an equivalence relation on U, we use $\frac{U}{R}$ to denote the family of all equivalence classes of R, and we use $[x]_R$ to denote an equivalence class in R containing element $x \in U$.

Definition2.3 Let U be the universe and R be an equivalence relation on U. For subset $X \subset U$, the pair $S = (U, R)$ is called an approximation space. The two subsets:

$$\underline{R}X = \{x \in U \mid [x]_R \subseteq U\}$$

$$\overline{R}X = \{x \in U \mid [x]_R \cap X \neq \phi\}$$

are called the R-lower and R-upper approximation of X, respectively.

We will use $pos_R(X) = \underline{R}X$ to denote R-positive region of X; $Neg_R(X) = U - \overline{R}X$ to denote R-negative region of X; and $BN_R(X) = \overline{R}X - \underline{R}X$ to denote the R-borderline region of X.

Definition2.4 Let U be the universe and R be an equivalence relation on U. For $X \subset U$, a measure of roughness of the set X is defined as: $\rho_X = 1 - \frac{|\underline{R}X|}{|\overline{R}X|}$

where $|\cdot|$ denotes the cardinality of a set.

Remark2.1

(1) Since $\underline{R}X \subseteq X \subseteq \overline{R}X$, we have $0 \leq \rho_X \leq 1$;

(2) By convention, when $X = \phi$, $\underline{R}X = \phi = \overline{R}X$ and $\frac{|\underline{R}X|}{|\overline{R}X|} = 1$, i.e., $\rho_X = 0$;

(3) $\rho_X = 0$ if and only if X is an crisp set, i.e., $\underline{R}X = X = \overline{R}X$.

Definition2.5 Let $(X, \leq)$ be a partially ordered set, for any $x, y \in X$, there is a number $D(\frac{y}{x})$, and satisfied the following condition:

(1) $0 \leq D(\frac{y}{x}) \leq 1$,

(2) $x \leq y \Rightarrow D(\frac{y}{x}) = 1$,

(3) $x \leq y \leq z \Rightarrow D(\frac{x}{z}) \leq D(\frac{x}{y})$,

Then D is called a inclusion degree on X.

3. The Fuzzy Description of the Boundary Region in Rough Sets

Definition3.1 Let U be the universe and R be an equivalence relation on U. Let $X \subseteq U$, for any $x \in X$, the two sets

$$u_X(x) = [x]_R - X,$$

$$l_X(x) = [x]_R - u_X(x)$$

are called the basic factor of inducing rough and the correlation basic factor of inducing rough of X, respectively.

Obviously, we have $l_X(x) \cap u_X(x) = \phi$, and $l_X(x) \cup u_X(x) = [x]_R$, $u_X(x)$ is the collection of those objects which are in $[x]_R$ but not in the set X, $l_X(x)$ is the collection of those objects which are in $[x]_R$ and X.

Remark3.1 Obviously for $x \in X$, $u_X(x) = \phi$ if $x \in \underline{R}X$, i.e., the basic factor of inducing rough is the empty set and the correlation basic factor of inducing rough is the equivalence class containing the element x. Thus, $[x]_R = l_X(x)$ is called R-certain information of X.

Otherwise, $u_X(x) \neq \phi$ indicates that the equivalence class $[x]_R$ contains some elements which induce rough of the set X. So, in this case, $[x]_R$ is called R-uncertain information of X. The collection of R-certain information of X is the R-positive region of X, and the collection of R-uncertain information of X is the R-borderline region of X.

From the above definition, we give the following definition.

Definition3.2 Let U be the universe and R be an equivalence relation on U. For $X \subseteq U$ and $x \in X$, define two sets as:

$$\underline{BN}_R(X) = \bigcup \{ l_X(x) \mid x \in BN_R(X) \cap X \}$$

$$\overline{BN}_R(X) = \bigcup \{ u_X(x) \mid x \in BN_R(X) \cap X \}$$

are called the lower boundary region and upper boundary region of X, respectively.

Remark3.2 It is clearly that

$$BN_R(X) = \underline{BN}_R(X) \cup \overline{BN}_R(X),$$

$$\underline{BN}_R(X) \cap \overline{BN}_R(X) = \phi .$$

That is to say, for the set X, the boundary is divided into two part, where $\underline{BN}_R(X)$ is the collection of the objects which belong to X but not classified into set X correctly in current by the equivalence relation R, and $\overline{BN}_R(X)$ is the collection of the objects which not belong to X but not classified into the set X^C correctly in current by the equivalence relation R.

Definition 3.2 shows that the objects which we studied are not descript correctly since the limitation ability of the classification and the insufficient knowledge which we gained in currently. i.e., the roughness of a set be emerged as the reason of the boundary region existing. Thus, the roughness of any set X is not a static property. So, if we gained more knowledge about the object, we will characterize it more correctly. In particular, if we gained the whole knowledge about the studied object, then we will descript it completely, and so the other hands did.

From the above analyzed, we give the fuzzy characterized of the boundary region for any rough set.

Definition3.3 Let U be the universe and R be an equivalence relation on U. For $X \subseteq U$ and $x \in X$, $0 \le \rho_R(X) \le 1$, define $\mu_X(x)$

$$= \begin{cases} (1-\rho_R(X)) \max\{D(\frac{l_X(x)}{BN_R(X)}), D(\frac{u_X(x)}{BN_R(X)})\} & \\ \quad + \rho_R(X) D(\frac{\overline{BN}_R(x)}{BN_R(X)}), & x \in \underline{BN}_R(X) \\ \min\{\rho_R(X), \alpha_R(X)\} D[\frac{[x]_R \cap \overline{BN}_R(X)}{\overline{R}X}], & x \in \overline{BN}_R(X) \end{cases}$$

Where $D(\cdot)$ is the inclusion degree of U.

Remark3.3 If $\rho_R(X) = 1$, then $BN_R(X) = \phi$, and the set X be a crisp set of U, therefore, $\mu_X(x) = 1$ for any $x \in X$.

Obviously, under the definition3.3, the element in boundary region has the different relation with X by the knowledge R on U.

In reference [3], the author defines the rough membership function of a rough set as following:

Let $X \subseteq U$, for any $x \in U$,

$$\mu_X{}^r(x) = \begin{cases} 1 & x \in \underline{R}X \\ \frac{1}{2} & x \in BN_R(X) \\ 0 & x \in -\overline{R}X \end{cases} \qquad (3)$$

Meanwhile, the author points out that the equations (1)and (2) are not hold under the formula(3).

In reference [4], the author analyzed the drawback of (3) and improved it as following:

Let $X \subseteq U$, $0 < \rho_R(X) \le 1$, for any $x \in U$,

$$\mu_X{}^r(x) = \begin{cases} 1 & x \in \underline{R}X \\ \frac{\rho_R(X)}{2-\rho_R(X)} & x \in BN_R(X) \\ 0 & x \in -\overline{R}X \end{cases} \qquad (4)$$

Similar to the reference [3], the author point out that the equations (1) and (2) are not hold under the formula (4).

The definition in [4] improved the definition in [3], however, both of them are not characterized the nature for there are no difference of the elements in the boundary region. According to the definition, we redefine the rough membership function as following:

For any $x \in U$, define

$$\mu_X{}^r(x) = \begin{cases} 1 & x \in \underline{R}X \\ \mu_X(x) & x \in BN_R(X) \\ 0 & x \in -\overline{R}X \end{cases} \quad (5)$$

From the above definition we know that $\mu_X(x)$ is a fuzzy set on the boundary region of X in U. Therefore, the formula (5) defined a fuzzy subsets on U.

From the formula(5), we easily to know

$$\mu_{X \cup Y}(x) = \max\{\mu_X{}^r(x), \mu_Y{}^r(x)\}$$

$$\mu_{X \cap Y}(x) = \min\{\mu_X{}^r(x), \mu_Y{}^r(x)\}$$

are hold.

Example3.1 Let $U=\{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$, R be an equivalence classes:

$E_1 = \{x_1, x_4, x_8\}$ $\quad E_3 = \{x_3\}$

$E_2 = \{x_2, x_5, x_6, x_7\}$

Let $X = \{x_1, x_3, x_7\}$, we have

$\underline{R}X = E_3 = \{x_3\}$,

$\overline{R}X = E_1 \cup E_2 \cup E_3 = \{x_1,x_2,x_3,x_4,x_5,x_6,x_7,x_8\}$,

$BN_R(X) = \{x_1, x_2, x_4, x_5, x_6, x_7, x_8\}$,

$\rho_R(X) = \dfrac{|\underline{R}X|}{|\overline{R}X|} = \dfrac{1}{8}$,

$\underline{BN}_R(X) = \{x_1, x_7\}$,

$\overline{BN}_R(X) = \{x_2, x_4, x_5, x_6, x_8\}$,

$l_X(x_1) = \{x_1\}$,

$l_X(x_7) = \{x_7\}$, $\quad u_X(x_1) = \{x_4, x_8\}$,

$u_X(x_1) = \{x_2, x_5, x_6\}$,

Using the formula(3), there are

$\mu_X{}^r(x_1) = \mu_X{}^r(x_2) = \mu_X{}^r(x_4) = \mu_X{}^r(x_5) = \mu_X{}^r(x_6) = \mu_X{}^r(x_7) = \mu_X{}^r(x_8) = \dfrac{1}{2}$;

Using the formula(4), there are

$\mu_X{}^r(x_1) = \mu_X{}^r(x_2) = \mu_X{}^r(x_4) = \mu_X{}^r(x_5) = \mu_X{}^r(x_6) = \mu_X{}^r(x_7) = \mu_X{}^r(x_8) = \dfrac{1}{15}$;

Using the formula(5),we have

$\mu_X{}^r(x_1) = \dfrac{19}{56}$;

$\mu_X{}^r(x_2) = \mu_X{}^r(x_5) = \mu_X{}^r(x_6) = \dfrac{3}{64}$;

$\mu_X{}^r(x_7) = \dfrac{13}{28}$; $\quad \mu_X{}^r(x_4) = \mu_X{}^r(x_8) = \dfrac{1}{32}$;

According to the above example, we know that the boundary region is a special fuzzy set which the membership function value of every element in boundary region is a constant, thus, the uncertainty information of the boundary region is not characterized correctly, but under the rough membership function(5), the uncertainty information is characterized well.

4. Fuzziness of Rough Set

From the above analyzed, we know that there is no uncertainty in the positive region and negative region for a set $X \subseteq U$, and the roughness is mainly brought from the boundary region, therefore the uncertainty of the element in boundary region determined the precision degree of any set $X \subseteq U$. Then we only to measure the uncertainty of element in boundary region for any set X in the universe U. In section 3, we define a fuzzy set on the boundary region, and then we give the measure of fuzziness of any rough set on U using the definition 3.3 as following.

Firstly, we introduce the definition of the fuzzy entropy.

Let U be the universe, $F(U)$ be all of the fuzzy sets on U. $\mu_A(x)$ be the membership function of $A \in F(U)$, $\varphi(U)$ be all of the crisp sets on U; $[\frac{1}{2}]_U$ be the collection of fuzzy sets on U which satisfied $\mu_{[\frac{1}{2}]_U}(x) = \dfrac{1}{2}$ for any $x \in U$; F be the sub-classes and satisfied the following condition:

(1) $\varphi(U) \subseteq F$,

(2) $[\frac{1}{2}]_U \in F$,

(3) $A, B \in F \Rightarrow A \cup B \in F, A^C \in F$, where A^C is the complement of A.

Definition4.1 Let $e: F \to [0,1]$ be a real function, if

e satisfied the following properties:

(1) $e(D)=0$, for any $D \in \varphi(U)$,

(2) $e([\frac{1}{2}]_U)=\max_{A \in F} e(A)$,

(3) For any $A, B \in F$, $\mu_B(x) \geq \mu_A(x)$ if $\mu_A(x) \geq \frac{1}{2}$, and $\mu_B(x) \leq \mu_A(x)$ if $\mu_A(x) \leq \frac{1}{2}$, then $e(A) \geq e(B)$,

(4) $e(A^C)=e(A)$ for any $A \in F$, then e is called a entropy on F.

Definition4.2 Let U be universe and R be equivalence relation on U. For any $X \subseteq U$, the measure of fuzziness of the rough set X is defined as:

$$E(X)=\sum_{i=1}^{|BN_R(X)|} \mu_X{}^r(x_i)(1-\mu_X{}^r(x_i))$$

for any $x_i \in BN_R(X)$.

It is easily to proof that $E(X)$ satisfied the condition of definition4.1.

From the definition 4.2, the following theorem is obviously.

Theorem4.1 For any universe U, the fuzziness of any crisp set on universe U will be 0.

Theorem4.2 For any universe U, there is the same fuzziness of any rough set with its complement set on the universe U.

Acknowledgements

This wok was funded by the Scientific Fund of Gansu Educational Office (0601-20) and the Natural Science Fund of China (40671076).

References

[1] L.A.Zadeh, Fuzzy sets, information and control, vol. 8, pp. 338-353, 1965.

[2] Z.Pawlak, Rough sets, International Journal of Information science, vol. 11, No. 5, pp.341-356, 1982.

[3] Z.Pawlak, Rough sets and fuzzy sets, Fuzzy Sets and Systems, vol. 17, pp. 99-102, 1985.

[4] Sun Bing zhen, Gong Zeng tai, The fuzzy membership representations of rough sets and its interrelations, Journal of Northwest Normal University(Natural Science), vol. 42, No. 4, pp.23-26, 2006.

[5] Wygralak. M, Rough sets and fuzzy sets----Some remarks on interrelations, Fuzzy Sets and Systems, vol. 29, pp.241-241, 1989.

[6] Huaguang Zhang, Hongli Liang, Derong Liu, Two new operations in rough set theory with application to fuzzy sets, Information Sciences, vol. 166, pp.147-165, 2004.

[7] M.Banerjee, S.K.Pal, Roughness of a fuzzy sets, Information and computation Sciences, pp.235-246, 1996.

[8] K.Chakrakarty, R.Biswas, S.Nanda, Fuzziness in rough sets, Fuzzy sets and systems, Vol.110, No.2, pp.247-251, 2000.

[9] Liu X.C, Entropy, distance measure and similarity measure of fuzzy sets and their relations, Fuzzy Sets and Systems ,Vol. 52, pp.305-318, 1992.