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Generalized fuzzy rough sets

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Abstract

This paper presents a general framework for the study of fuzzy rough sets in which both constructive and axiomatic approaches are used. In constructive approach, a pair of lower and upper generalized approximation operators is defined. The connections between fuzzy relations and fuzzy rough approximation operators are examined. In axiomatic approach, various classes of fuzzy rough approximation operators are characterized by different sets of axioms. Axioms of fuzzy approximation operators guarantee the existence of certain types of fuzzy relations producing the same operators.

Keywords: Approximation operators; Fuzzy relations; Fuzzy rough sets; Rough sets

1. Introduction

The theory of rough sets, proposed by Pawlak [18], is an extension of set theory for the study of intelligent systems characterized by insufficient and incomplete information. Using the concepts of lower and upper approximations

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in rough set theory, knowledge hidden in information systems may be unravelled and expressed in the form of decision rules [5,6,12,19,23,31].

The basic operators in rough set theory are approximations. There are at least two approaches for the development of the rough set theory, the constructive and axiomatic approaches. In constructive approach, binary relations on the universe, partitions of the universe, neighborhood systems, and Boolean algebras are all the primitive notions. The lower and upper approximation operators are constructed by means of these notions [5,6,8,9,11,12,16,18–31]. The constructive approach is suitable for practical applications of rough sets. On the other hand, the axiomatic approach, which is appropriate for studying the structures of rough set algebras, takes the lower and upper approximation operators as primitive notions. In this approach, a set of axioms is used to characterize approximation operators that are the same as the ones produced by using constructive approach [10,27,28].

The initiations and majority of studies on rough sets have been concentrated on constructive approaches. In Pawlak's rough set model [19], an equivalence relation is a key and primitive notion. This equivalence relation, however, seems to be a very stringent condition that may limit the application domain of the rough set model. To solve this problem, several authors have generalized the notion of approximation operators by using nonequivalence binary relations [6,26,28,29]. This has lead to various other approximation operators [8–12,16,21,22,25–29,31]. On the other hand, by using an equivalence relation on U, one can introduce lower and upper approximations in fuzzy set theory to obtain an extended notion called rough fuzzy set [2,3]. Alternatively, a fuzzy similarity relation can be used to replace an equivalence relation. The result is a deviation of rough set theory called fuzzy rough sets [3,4,14]. More general frameworks can be obtained which involve the approximations of fuzzy sets based on fuzzy T-similarity relations [13], fuzzy similarity relations [30], weak fuzzy partitions on U [1,7], and Boolean subalgebras of $\mathscr{P}(U)$ [15] etc.

Comparing with the studies on constructive approach, there is less effort needed on axiomatic approach. The most important axiomatic studies for crisp rough sets were made by Yao [27,28], Yao and Lin [26] in which various classes of rough set algebras are characterized by different sets of axioms. Moris and Yakout [13] studied a set of axioms on fuzzy rough set, but their studies were restricted to fuzzy T-rough sets defined by fuzzy T-similarity relations which were ordinary (crisp) equivalence relations when they degenerated into crisp ones. So far, however, the axiomatic approach for the study of generalized fuzzy rough set is blank.

The present paper studies generalized fuzzy rough sets in which both the constructive and axiomatic approaches are used. In the constructive approach, based on an arbitrary fuzzy relation, a pair of dual generalized fuzzy approximation operators is defined. The connections between fuzzy binary relations and fuzzy approximation operators are examined. The resulting fuzzy rough sets are proper generalizations of generalized rough sets [26,28], rough fuzzy sets [3,17], and fuzzy rough set [3,4,30]. In the axiomatic approach, various classes of fuzzy rough sets are characterized by different sets of axioms, axioms of fuzzy approximation operators guarantee the existence of certain types of fuzzy relations producing the same operators.

2. Preliminaries

Let X be a finite and nonempty set called the universe. The class of all subsets (respectively, fuzzy subsets) of X will be denoted by $\mathscr{P}(X)$ (respectively, by $\mathscr{F}(X)$). For any $A \in \mathscr{F}(X)$, the α-level and the strong α-level of A will be denoted by A_α and $A_{\alpha+}$, respectively, that is, $A_\alpha = \{x \in X : A(x) \geqslant \alpha\}$ and $A_{\alpha+} = \{x \in X : A(x) > \alpha\}$, where $\alpha \in I = [0, 1]$, the unit interval, $A_0 = X$, and $A_{1+} = \emptyset$.

Let U and W be two finite and nonempty universes. A fuzzy subset $R \in \mathscr{F}(U \times W)$ is referred to as a fuzzy binary relation from U to W, $R(x, y)$ is the degree of relation between x and y, where $(x, y) \in U \times W$; If $U = W$, then R is referred to as a fuzzy relation on U.

Let R be a fuzzy relation from U to W, if for each $x \in U$, there exists $y \in W$ such that $R(x, y) = 1$, then R is referred to as a serial fuzzy relation from U to W.

Definition 2.1. Let R be a fuzzy binary relation on U. R is referred to as a reflexive fuzzy relation if $R(x, x) = 1$ for all $x \in U$; R is referred to as a symmetric fuzzy relation if $R(x, y) = R(y, x)$ for all x, $y \in U$; R is referred to as a transitive fuzzy relation if $R(x, z) \geqslant \vee_{y \in U} (R(x, y) \wedge R(y, z))$ for all x, $z \in U$; R is referred to as a similarity fuzzy relation if R is a reflexive, symmetric, and transitive fuzzy relation.

It is easy to see that R is a serial fuzzy relation iff R_α is a serial ordinary binary relation for all $\alpha \in I$; R is a reflexive fuzzy relation iff R_α is a reflexive ordinary binary relation for all $\alpha \in I$; R is a symmetric fuzzy relation iff R_α is a symmetric ordinary binary relation for all $\alpha \in I$; R is a transitive fuzzy relation iff R_α is a transitive ordinary binary relation for all $\alpha \in I$; R is a similarity fuzzy relation iff R_α is an equivalence ordinary binary relation for all $\alpha \in I$.

Definition 2.2. A set-valued mapping $N : I \to \mathscr{P}(U)$ is said to be nested if for all α, $\beta \in I$,

$$\alpha \leqslant \beta \Rightarrow N(\beta) \subseteq N(\alpha).$$

The class of all $\mathscr{P}(U)$-valued nested mappings on I will be denoted by $\mathscr{N}(U)$.

It is well-known that the following representation theorem holds.

Theorem 2.3. *Let $N \in \mathscr{N}(U)$. Define a function $f : \mathscr{N}(U) \to \mathscr{F}(U)$ by*:

$$A(x) := f(N)(x) = \vee_{\alpha\in I}(\alpha \wedge N(\alpha)(x)), \quad x \in U,$$

where $N(\alpha)(x)$ is the characteristic function of $N(\alpha)$. Then f is a surjective homomorphism, and the following properties hold:

(i) $A_{\alpha+} \subseteq N(\alpha) \subseteq A_\alpha$,
(ii) $A_\alpha = \bigcap_{\lambda<\alpha} N(\lambda)$,
(iii) $A_{\alpha+} = \bigcup_{\lambda>\alpha} N(\lambda)$,
(iv) $A = \vee_{\alpha\in I}(\alpha \wedge A_{\alpha+}) = \vee_{\alpha\in I}(\alpha \wedge A_\alpha)$.

3. Construction of generalized fuzzy rough approximation operators

3.1. Generalized rough sets

Definition 3.1. Let U and W be two finite universes. Suppose that R is an arbitrary relation from U to W. We can define a set-valued function $F : U \to \mathscr{P}(W)$ by:

$$F(x) = \{y \in W : (x,y) \in R\}, \quad x \in U.$$

Obviously, any set-valued function from U to W defines a binary relation from U to W by setting $R = \{(x,y) \in U \times W : y \in F(x)\}$. The triple (U,W,R) is referred to as a generalized approximation space. For any set $A \subseteq W$, a pair of lower and upper approximations, $\underline{R}(A)$ and $\overline{R}(A)$, are defined by

$$\underline{R}(A) = \{x \in U : F(x) \subseteq A\},$$
$$\overline{R}(A) = \{x \in U : F(x) \cap A \neq \emptyset\}.$$

The pair $(\underline{R}(A), \overline{R}(A))$ is referred to as a generalized rough set.

From the definition, the following theorem can easily be verified [28].

Theorem 3.2. *For any relation R from U to W, its lower and upper approximation operators satisfy the following properties*: *for all A, $B \in \mathscr{P}(W)$*, (L1) $\underline{R}(A) = \sim \overline{R}(\sim A)$,

(U1) $\overline{R}(A) = \sim \underline{R}(\sim A)$;

(L2) $\underline{R}(W) = U$,

(U2) $\overline{R}(\emptyset) = \emptyset$;

(L3) $\underline{R}(A \cap B) = \underline{R}(A) \cap \underline{R}(B)$,

(U3) $\overline{R}(A \cup B) = \overline{R}(A) \cup \overline{R}(B)$;

(L4) $A \subseteq B \Rightarrow \underline{R}(A) \subseteq \underline{R}(B)$,

(U4) $A \subseteq B \Rightarrow \overline{R}(A) \subseteq \overline{R}(B)$;

(L5) $\underline{R}(A \cup B) \supseteq \underline{R}(A) \cup \underline{R}(B)$,

(U5) $\overline{R}(A \cap B) \subseteq \overline{R}(A) \cap \overline{R}(B)$,

where $\sim A$ *is the complement of* A.

Properties (L1) and (U1) show that the approximation operators $\underline{R}$ and $\overline{R}$ are dual to each other. Properties with the same number may be considered as dual properties. With respect to certain special types, say, serial, reflexive, symmetric, and transitive binary relation on the universe U, the approximation operators have additional properties [28,29], say,

for serial relation	(L0)	$\underline{R}(\emptyset) = \emptyset$,
	(U0)	$\overline{R}(U) = U$,
	(LU0)	$\underline{R}(A) \subseteq \overline{R}(A)$;
for reflexive relation	(L6)	$\underline{R}(A) \subseteq A$,
	(U6)	$A \subseteq \overline{R}(A)$;
for symmetric relation	(L7)	$A \subseteq \underline{R}(\overline{R}(A))$,
	(U7)	$\overline{R}(\underline{R}(A)) \subseteq A$;
for transitive relation	(L8)	$\underline{R}(A) \subseteq \underline{R}(\underline{R}(A))$,
	(U8)	$\overline{R}(\overline{R}(A)) \subseteq \overline{R}(A)$.

If R is an equivalence relation on U, then the pair (U, R) is the Pawlak approximation space and more interesting properties of lower and upper approximation operators can be derived [19,28,29].

3.2. Generalized fuzzy rough sets

Let R be an arbitrary fuzzy relation from U to W. Define the mapping $F : U \to \mathscr{F}(W)$ by:

$$F(x)(y) = R(x, y), \quad (x, y) \in U \times W.$$

For any $\alpha \in I$, we further define $F_\alpha : U \to \mathscr{P}(W)$ by:

$$F_\alpha(x) = \{y \in W : F(x)(y) \geqslant \alpha\}, \quad x \in U.$$

Also for any $X \in \mathscr{P}(W)$, the lower and upper approximations of X with respect to the approximation space (U, W, F_α) are defined as follows:

$$\underline{F}_\alpha(X) = \{x \in U : F_\alpha(x) \subseteq X\},$$
$$\overline{F}_\alpha(X) = \{x \in U : F_\alpha(x) \cap X \neq \emptyset\}.$$

Lemma 3.3. *If R is an arbitrary fuzzy relation from U to W and $A \in \mathscr{F}(W)$, let $N(\alpha) = \underline{F}_{1-\alpha}(A_{\alpha+})$ and $H(\alpha) = \overline{F}_\alpha(A_\alpha)$, $\alpha \in I$. Then N and H are nested.*

Proof. Let $0 \leqslant \beta \leqslant \alpha \leqslant 1$, it is easy to see that $A_{\beta+} \supseteq A_{\alpha+}$, $A_\beta \supseteq A_\alpha$, $F_\beta(x) \supseteq F_\alpha(x)$, and $F_{1-\alpha}(x) \supseteq F_{1-\beta}(x)$.

(a) If $x \in N(\alpha)$, then by the definition of $N(\alpha)$ we have $F_{1-\alpha}(x) \subseteq A_{\alpha+}$. Hence

$$F_{1-\beta}(x) \subseteq F_{1-\alpha}(x) \subseteq A_{\alpha+} \subseteq A_{\beta+}.$$

Thus $x \in \underline{F}_{1-\beta}(A_{\beta+})$. Therefore

$$N(\alpha) = \underline{F}_{1-\alpha}(A_{\alpha+}) \subseteq \underline{F}_{1-\beta}(A_{\beta+}) = N(\beta).$$

It follows that N is nested.

(b) If $y \in H(\alpha)$, then by the definition of $H(\alpha)$ we have $F_\alpha(y) \cap A_\alpha \neq \emptyset$. Since $F_\alpha(y) \cap A_\alpha \subseteq F_\beta(y) \cap A_\beta$, we obtain $F_\beta(y) \cap A_\beta \neq \emptyset$. Thus $y \in \overline{F}_\beta(A_\beta)$. Therefore $H(\alpha) \subseteq H(\beta)$. It follows that H is nested. $\square$

Based on Lemma 3.3 and Theorem 2.3, we can formulate the notion of generalized fuzzy rough set as follows.

Definition 3.4. Let R be an arbitrary fuzzy relation from U to W. The triple (U, W, R) is referred to as a generalized fuzzy approximation space. We define the lower and upper generalized fuzzy approximation operators $\underline{F}$ and $\overline{F}$ with respect to (U, W, R) by:

$$\underline{F}(A) = \vee_{\alpha\in I}(\alpha \wedge \underline{F}_{1-\alpha}(A_{\alpha+})), \quad A \in \mathscr{F}(W),$$
$$\overline{F}(A) = \vee_{\alpha\in I}(\alpha \wedge \overline{F}_\alpha(A_\alpha)), \quad A \in \mathscr{F}(W).$$

The pair $(\underline{F}(A), \overline{F}(A))$ is referred to as a generalized fuzzy rough set.

Remark. From Definition 3.4 and Theorem 2.3 we can immediately conclude that the lower approximation operator $\underline{F}$ satisfies:

$$(\underline{F}(A))_{\alpha+} \subseteq \underline{F}_{1-\alpha}(A_{\alpha+}) \subseteq (\underline{F}(A))_\alpha,$$

and furthermore,

$$(\underline{F}(A))_\alpha = \bigcap_{\lambda<\alpha} \underline{F}_{1-\lambda}(A_{\lambda+}),$$
$$(\underline{F}(A))_{\alpha+} = \bigcup_{\lambda>\alpha} \underline{F}_{1-\lambda}(A_{\lambda+}).$$

Likewise the upper approximation operator $\overline{F}$ satisfies:

$$(\overline{F}(A))_{\alpha+} \subseteq \overline{F}_\alpha(A_\alpha) \subseteq (\overline{F}(A))_\alpha,$$

and also

$$(\overline{F}(A))_\alpha = \bigcap_{\lambda<\alpha} \overline{F}_\lambda(A_\lambda),$$
$$(\overline{F}(A))_{\alpha+} = \bigcup_{\lambda>\alpha} \overline{F}_\lambda(A_\lambda).$$

Proposition 3.5. *Assume that R is an arbitrary fuzzy relation from U to W, let $A \in \mathscr{F}(W)$,*

(i) *If $0 \leqslant \beta \leqslant 1/2$, then*

$$\underline{F}_{1-\beta}(A_{\beta+}) \supseteq \underline{F}_\beta(A_{\beta+}) \cup \underline{F}_{1-\beta}(A_{(1-\beta)+}).$$

(ii) *If $1/2 \leqslant \beta \leqslant 1$, then*

$$\underline{F}_{1-\beta}(A_{\beta+}) \subseteq \underline{F}_{1-\beta}(A_{(1-\beta)+}) \cap \underline{F}_\beta(A_{\beta+}).$$

Proof

(i) When $0 \leqslant \beta \leqslant 1/2$, for any $y \in U$, we have

$$F_\beta(y) \supseteq F_{1-\beta}(y), \quad \text{and} \quad A_{\beta+} \supseteq A_{(1-\beta)+}.$$

If $y \in \underline{F}_\beta(A_{\beta+})$, by the definition of lower approximation we have $F_\beta(y) \subseteq A_{\beta+}$, then $F_{1-\beta}(y) \subseteq F_\beta(y) \subseteq A_{\beta+}$, in turn, $y \in \underline{F}_{1-\beta}(A_{\beta+})$, which implies that

$$\underline{F}_\beta(A_{\beta+}) \subseteq \underline{F}_{1-\beta}(A_{\beta+}).$$

If $y \in \underline{F}_{1-\beta}(A_{(1-\beta)+})$, then by the definition of lower approximation we have $F_{1-\beta}(y) \subseteq A_{(1-\beta)+} \subseteq A_{\beta+}$, that is, $y \in \underline{F}_{1-\beta}(A_{\beta+})$, which then implies

$$\underline{F}_{1-\beta}(A_{(1-\beta)+}) \subseteq \underline{F}_{1-\beta}(A_{\beta+}).$$

Hence

$$\underline{F}_{1-\beta}(A_{\beta+}) \supseteq \underline{F}_\beta(A_{\beta+}) \cup \underline{F}_{1-\beta}(A_{(1-\beta)+}).$$

(ii) It is similar to the proof of (i). □

Theorem 3.6. *If R is an arbitrary fuzzy relation from U to W. Then the pair of fuzzy approximation operators satisfies the following properties: for all A, $B \in \mathscr{F}(W)$, and for all $\alpha \in I$,*

(FL1) $\underline{F}(A) = \sim(\overline{F}(\sim A))$,

(FU1) $\overline{F}(A) = \sim(\underline{F}(\sim A))$;

(FL2) $\underline{F}(A \vee \hat{\alpha}) = \underline{F}(A) \vee \hat{\alpha}$,

(FU2) $\overline{F}(A \wedge \hat{\alpha}) = \overline{F}(A) \wedge \hat{\alpha}$;

(FL3) $\underline{F}(A \wedge B) = \underline{F}(A) \wedge \underline{F}(B)$,

(FU3) $\overline{F}(A \vee B) = \overline{F}(A) \vee \overline{F}(B)$;

(FL4) $A \subseteq B \Rightarrow \underline{F}(A) \subseteq \underline{F}(B)$,

(FU4) $A \subseteq B \Rightarrow \overline{F}(A) \subseteq \overline{F}(B)$;

(FL5) $\underline{F}(A \vee B) \supseteq \underline{F}(A) \vee \underline{F}(B)$,

(FU5) $\overline{F}(A \wedge B) \subseteq \overline{F}(A) \wedge \overline{F}(B)$,

where $\hat{\alpha}$ is the constant fuzzy set: $\hat{\alpha}(x) = \alpha$, for all $x \in U$ and $x \in W$.

Proof. By Theorem 3.2 and using the fact that, for all $N \in \mathscr{N}(U)$,

$$\wedge_{\alpha\in I}(\alpha \vee N(\alpha)) = \vee_{\alpha\in I}(\alpha \wedge N(\alpha)),$$

we have

$$\begin{aligned}
\sim \overline{F}(\sim A) &= \sim \vee_{\alpha\in I}(\alpha \wedge (\overline{F}_\alpha(\sim A)_\alpha)) \\
&= \sim \vee_{\alpha\in I}(\alpha \wedge \overline{F}_\alpha(\sim A_{(1-\alpha)+})) \\
&= \sim \vee_{\alpha\in I}(\alpha \wedge (\sim \underline{F}_\alpha(A_{(1-\alpha)+}))) \\
&= \wedge_{\alpha\in I}(1 - \alpha \wedge (\sim \underline{F}_\alpha(A_{(1-\alpha)+}))) \\
&= \wedge_{\alpha\in I}((1 - \alpha) \vee \underline{F}_\alpha(A_{(1-\alpha)+})) \\
&= \wedge_{\alpha\in I}(\alpha \vee \underline{F}_{1-\alpha}(A_{\alpha+})) \\
&= \vee_{\alpha\in I}(\alpha \wedge \underline{F}_{1-\alpha}(A_{\alpha+})) = \underline{F}(A),
\end{aligned}$$

from which (FL1) follows. (FU1) can be directly induced by (FL1).

For any $x \in U$,

$$\begin{aligned}
\overline{F}(A \wedge \hat{\alpha})(x) &= \vee_{\beta\in I}(\beta \wedge \overline{F}_\beta(A \wedge \hat{\alpha}))(x) \\
&= \sup\{\beta \in I : F_\beta(x) \cap (A \wedge \hat{\alpha})_\beta \neq \emptyset\} \\
&= \sup\{\beta \in I : \exists y \in W[R(x,y) \geqslant \beta, A(y) \geqslant \beta, \alpha \geqslant \beta]\} \\
&= \alpha \wedge \sup\{\beta \in I : \exists y \in W[R(x,y) \geqslant \beta, A(y) \geqslant \beta]\} \\
&= (\hat{\alpha} \wedge \overline{F}(A))(x),
\end{aligned}$$

which implies (FU2). Similarly we can justify (FL2).

By (U3), we have

$$\begin{aligned}\overline{F}(A \vee B) &= \vee_{\alpha\in I}(\alpha \wedge \overline{F}_\alpha(A \vee B)_\alpha) \\ &= \vee_{\alpha\in I}(\alpha \wedge \overline{F}_\alpha(A_\alpha \cup B_\alpha)) \\ &= \vee_{\alpha\in I}(\alpha \wedge (\overline{F}_\alpha(A_\alpha) \cup \overline{F}_\alpha(B_\alpha))) \\ &= \vee_{\alpha\in I}((\alpha \wedge \overline{F}_\alpha(A_\alpha)) \vee (\alpha \wedge \overline{F}_\alpha(B_\alpha))) \\ &= (\vee_{\alpha\in I}(\alpha \wedge \overline{F}_\alpha(A_\alpha))) \vee (\vee_{\alpha\in I}(\alpha \wedge \overline{F}_\alpha(B_\alpha))) \\ &= \overline{F}(A) \vee \overline{F}(B).\end{aligned}$$

This implies (FU3).

Combining (FU3) and the dual properties (FL1) and (FU1), we can easily conclude (FL3). Furthermore, since

$$A \subseteq B \Rightarrow A_\alpha \subseteq B_\alpha, \quad \forall \alpha \in I \Rightarrow \alpha \wedge \overline{F}_\alpha(A_\alpha) \subseteq \alpha \wedge \overline{F}_\alpha(B_\alpha), \quad \forall \alpha \in I,$$

by the definition and the duality we conclude that (FU4) and (FL4) hold.

Properties (FL5) and (FU5) follow directly from the properties (FL4) and (FU4) respectively. □

Just in the case of generalized approximation operators, properties (FL1) and (FU1) show that fuzzy approximation operators $\underline{F}$ and $\overline{F}$ are dual to each other. Properties with the same number may be regarded as dual properties. The first three properties are independent. It can be checked that they imply the remaining properties. Properties (FL2) and (FU2) imply the following properties (FL2)′ and (FU2)′ which are formally similar to (L2) and (U2).

(FL2)′ $\underline{F}(W) = U$,

(FU2)′ $\overline{F}(\emptyset) = \emptyset$.

Properties (FL4) and (FU4) show that the fuzzy approximation operators are monotonic with respect to (fuzzy) set inclusion.

Additional properties of fuzzy approximation operators will be given in the next subsection.

3.3. Connections between special fuzzy relations and approximation operators

In this subsection, we show that some special fuzzy relations could be characterized by fuzzy approximation operators.

Proposition 3.7. *If R is an arbitrary fuzzy relation from U to W, then*

(i) $\overline{F}(1_y)(x) = R(x,y), \quad \forall (x,y) \in U \times W$,

(ii) $\underline{F}(1_{W\setminus\{y\}})(x) = 1 - R(x,y), \quad \forall (x,y) \in U \times W$;

(iii) $\overline{F}(1_X)(x) = \max\{R(x,y) : y \in X\}, \quad \forall x \in U, \quad X \in \mathscr{P}(W)$,
(iv) $\underline{F}(1_X)(x) = \min\{1 - R(x,y) : y \notin X\}, \quad \forall x \in U, \quad X \in \mathscr{P}(W)$,
where 1_y *denotes the fuzzy singleton with value* 1 *at* y *and* 0 *elsewhere*; 1_X *denotes the characteristic function of* X.

Proof

(i) When $\alpha > 0$, it is clear that $\overline{F}_\alpha(1_y)_\alpha = \overline{F}_\alpha(\{y\}) = \{u \in U : F_\alpha(u) \cap \{y\} \neq \emptyset\} = \{u \in U : y \in F_\alpha(u)\} = \{u \in U : R(u,y) \geqslant \alpha\}$. Then $\overline{F}_\alpha(1_y)_\alpha(x) = 1$ when $R(x,y) \geqslant \alpha$, otherwise, $\overline{F}_\alpha(1_y)_\alpha(x) = 0$. Hence, $\overline{F}(1_y)(x) = \vee_{\alpha \in I}(\alpha \wedge \overline{F}_\alpha(1_y)_\alpha)(x) = \sup\{\alpha \in I : R(x,y) \geqslant \alpha\} = R(x,y)$.

(ii) Follows immediately from (i) and the duality.

(iii) Let $\overline{F}(1_X)(x) = a$ and $\max\{R(x,y) : y \in X\} = b$. Then

$$a = \vee_{\alpha \in I}(\alpha \wedge \overline{F}_\alpha(1_X)_\alpha)(x) = \sup\{\alpha \in I : x \in \overline{F}_\alpha(X)\} = \sup\{\alpha \in I : F_\alpha(x) \cap X \neq \emptyset\} = \sup\{\alpha \in I : \{z \in W : R(x,z) \geqslant \alpha)\} \cap X \neq \emptyset\}.$$

For any $\lambda < a$, there exists $\alpha \in (\lambda, a)$ such that $\{y \in W : R(x,y) \geqslant \alpha\} \cap X \neq \emptyset$. Hence $b \geqslant \alpha > \lambda$. Therefore $a \leqslant b$.
On the other hand, for any $\lambda > a$, we have $\{y \in W : R(x,y) \geqslant \lambda\} \cap X = \emptyset$, that is, $R(x,y) < \lambda$ for all $y \in X$. Hence $b < \lambda$. Therefore $b \leqslant a$. It follows that $a = b$.

(iv) Follows immediately from (iii) and the duality. □

Theorem 3.8. *If* R *is an arbitrary fuzzy relation from* U *to* W, *then* R *is serial iff one of the following properties holds*:

$$\begin{aligned} &(\mathrm{FL0}) \quad \underline{F}(\emptyset) = \emptyset, \\ &(\mathrm{FU0}) \quad \overline{F}(W) = U; \\ &(\mathrm{FLU0}) \quad \underline{F}(A) \subseteq \overline{F}(A), \quad \forall A \in \mathscr{F}(W). \end{aligned}$$

Proof. First we can deduce from the dual properties (FL1) and (FU1) that (FL0) and (FU0) are equivalent.

Second we are to prove that

$$R \text{ is serial} \Longleftrightarrow (\mathrm{FU0}).$$

In fact, we know from Proposition 3.7 that for any $x \in U$, $\overline{F}(W)(x) = \vee_{y \in W} R(x,y)$. Then

$$\begin{aligned} R \text{ is serial} &\Longleftrightarrow \exists y \in W \quad \text{such that } R(x,y) = 1 \\ &\Longleftrightarrow \vee_{y \in W} R(x,y) = 1 \\ &\Longleftrightarrow \overline{F}(W) = U. \end{aligned}$$

At last we are to prove that

R is serial $\Longleftrightarrow$ (FLU0).

If R is a serial fuzzy relation, from the definition of the approximation operators we only need to prove that for all $\alpha \in I$,

$$\underline{F}_{1-\alpha}(A_{\alpha+}) \subseteq \overline{F}_{\alpha}(A_{\alpha}).$$

In fact, if $x \in \underline{F}_{1-\alpha}(A_{\alpha+})$, then by the definition of lower approximation we have $F_{1-\alpha}(x) \subseteq A_{\alpha+}$. It means that $y \in F_{1-\alpha}(x)$ or equivalently $R(x,y) \geqslant 1-\alpha$ implies $A(y) > \alpha$. Since R is serial, there exists $y_0 \in W$ such that $R(x,y_0) = 1$. It follows that $R(x,y_0) \geqslant \alpha$, i.e., $y_0 \in F_{\alpha}(x)$. On the other hand, since $A(y_0) > \alpha$, we have $y_0 \in A_{\alpha+} \subseteq A_{\alpha}$, thus, $y_0 \in F_{\alpha}(x) \cap A_{\alpha}$, then by the definition of upper approximation we have $x \in \overline{F}_{\alpha}(A_{\alpha})$. Thus we conclude that (FLU0) holds.

Conversely, if we assume that (FLU0) holds, let $A = W$, then by Proposition 3.7 and (FL2)′ we have

$$\vee_{y \in W} R(x,y) = \overline{F}(W)(x) \geqslant \underline{F}(W)(x) = U(x) = 1,$$

which follows that R is serial. □

Theorem 3.9. *If R is an arbitrary fuzzy relation on U, then R is reflexive iff one of the following two properties holds*:

$$\text{(FL6)} \quad \underline{F}(A) \subseteq A, \quad \forall A \in \mathscr{F}(U),$$
$$\text{(FU6)} \quad A \subseteq \overline{F}(A), \quad \forall A \in \mathscr{F}(U).$$

Proof. (FL6) and (FU6) are equivalent because of (FL1) and (FU1). We only need to prove that the reflexivity of R is equivalent to (FU6).

Assume that R is reflexive. For any $A \in \mathscr{F}(U)$ and $x \in U$, let $A(x) = \alpha$, it is then clear that $x \in A_{\alpha}$. Since R is reflexive, R_{α} is an ordinary reflexive binary relation as well. Hence $x \in F_{\alpha}(x)$, and furthermore $x \in F_{\alpha}(x) \bigcap A_{\alpha}$, that is, $x \in \overline{F}_{\alpha}(A_{\alpha})$. Since $\overline{F}_{\alpha}(A_{\alpha}) \subseteq (\overline{F}(A))_{\alpha}$, we have that $x \in (\overline{F}(A))_{\alpha}$, that is, $\overline{F}(A)(x) \geqslant A(x)$ which implies (FU6).

Assume that (FU6) holds. For any $x \in U$, let $A = 1_x$. From Proposition 3.7 and by the assumption, we then have that $1 = 1_x(x) \leqslant \overline{F}(1_x)(x) = R(x,x)$. It follows that R is reflexive. □

Theorem 3.10. *If R is an arbitrary fuzzy relation on U, then R is symmetric iff one of the following two properties holds*:

$$\text{(FL7)} \quad \underline{F}(1_{U\setminus\{y\}})(x) = \underline{F}(1_{U\setminus\{x\}})(y), \quad \forall (x,y) \in U \times U,$$
$$\text{(FU7)} \quad \overline{F}(1_x)(y) = \overline{F}(1_y)(x), \quad \forall (x,y) \in U \times U.$$

Proof. It is immediately from Proposition 3.7. □

Lemma 3.11. *If R is an arbitrary fuzzy relation on U, then*

$$(\overline{F}(A))_{\alpha+} \subseteq \overline{F}_{\alpha}(A_{\alpha+}) \subseteq (\overline{F}(A))_{\alpha}, \quad A \in \mathscr{F}(U), \quad \alpha \in I.$$

Proof. For any $\lambda > \alpha$, if $x \in \overline{F}_{\lambda}(A_{\lambda})$, then by the definition of upper approximation we have $F_{\lambda}(x) \cap A_{\lambda} \neq \emptyset$. Since $F_{\lambda}(x) \subseteq F_{\alpha}(x)$ and $A_{\lambda} \subseteq A_{\alpha+}$, we obtain that $F_{\alpha}(x) \cap A_{\alpha+} \supseteq F_{\lambda}(x) \cap A_{\lambda} \neq \emptyset$. Hence $x \in \overline{F}_{\alpha}(A_{\alpha+})$. Thus $\overline{F}_{\lambda}(A_{\lambda}) \subseteq \overline{F}_{\alpha}(A_{\alpha+})$ for all $\lambda > \alpha$. Since $(\overline{F}(A))_{\alpha+} = \bigcup_{\lambda>\alpha} \overline{F}_{\lambda}(A_{\lambda})$, we conclude that $(\overline{F}(A))_{\alpha+} \subseteq \overline{F}_{\alpha}(A_{\alpha+}) \subseteq \overline{F}_{\alpha}(A_{\alpha}) \subseteq (\overline{F}(A))_{\alpha}$. □

Theorem 3.12. *If R is an arbitrary fuzzy relation on U, then R is transitive iff one of the following two properties holds*:

$$\text{(FL8)} \quad \underline{F}(A) \subseteq \underline{F}(\underline{F}(A)), \quad \forall A \in \mathscr{F}(U),$$
$$\text{(FU8)} \quad \overline{F}(\overline{F}(A)) \subseteq \overline{F}(A), \quad \forall A \in \mathscr{F}(U).$$

Proof. (FL8) and (FU8) are equivalent because of (FL1) and (FU1). We are only to prove that the transitivity of R is equivalent to (FU8). To this end, let us assume that R is transitive. Then by Lemma 3.11 we have

$$\vee_{\alpha\in I}(\alpha \wedge (\overline{F}(A))_{\alpha+}) \subseteq \vee_{\alpha\in I}(\alpha \wedge (\overline{F}_{\alpha}(A_{\alpha+}))) \subseteq \vee_{\alpha\in I}(\alpha \wedge (\overline{F}(A))_{\alpha}).$$

This implies that for all $A \in \mathscr{F}(U)$,

$$\vee_{\alpha\in I}(\alpha \wedge \overline{F}_{\alpha}(A_{\alpha+})) = \overline{F}(A).$$

Using the above result and combining Lemma 3.11 and the property (U8), we then get

$$\begin{aligned}
\overline{F}(\overline{F}(A)) &= \vee_{\alpha\in I}(\alpha \wedge \overline{F}_{\alpha}((\overline{F}(A))_{\alpha+})) \\
&\subseteq \vee_{\alpha\in I}(\alpha \wedge \overline{F}_{\alpha}(\overline{F}(A_{\alpha+}))) \\
&\subseteq \vee_{\alpha\in I}(\alpha \wedge \overline{F}_{\alpha}(\overline{F}_{\alpha}(A_{\alpha}))) \\
&\subseteq \vee_{\alpha\in I}(\alpha \wedge \overline{F}_{\alpha}(A_{\alpha})) \\
&= \overline{F}(A).
\end{aligned}$$

Thus (FU8) holds.

Conversely, assume that (FU8) holds. Let $x, y, z \in U$ and $\lambda \in (0, 1]$ such that $R(x,y) \geqslant \lambda$ and $R(y,z) \geqslant \lambda$. Then on one hand,

$$\overline{F}(\overline{F}(1_z))(x) \leqslant \overline{F}(1_z)(x) = R(x,z),$$

and on the other hand,

$$\begin{aligned}\overline{F}(\overline{F}(1_z))(x) &= \sup\{\alpha : x \in \overline{F}_\alpha(\overline{F}(1_z))_\alpha\} \\ &= \sup\{\alpha : F_\alpha(x) \cap (\overline{F}(1_z))_\alpha \neq \emptyset\} \\ &= \sup\{\alpha : \exists u \in U[R(x,u) \geqslant \alpha, (\overline{F}(1_z))(u) = R(u,z) \geqslant \alpha]\} \\ &\geqslant \min\{R(x,y), R(y,z)\} \geqslant \lambda,\end{aligned}$$

thus $R(x,z) \geqslant \lambda$. It follows that R is transitive. □

Theorem 3.13. *If R is an arbitrary fuzzy relation on U, then R is a fuzzy similarity relation iff $\underline{F}$ satisfies properties* (FL6)–(FL8) *or equivalently, $\overline{F}$ satisfies properties* (FU6)–(FU8).

Proof. It follows immediately from Theorems 3.9, 3.10 and 3.12. □

Theorem 3.14. *If R is a reflexive and transitive fuzzy relation on U, then the following properties hold*:

$$\begin{aligned}&(\text{FL9}) \quad \underline{F}(A) = \underline{F}(\underline{F}(A)), \quad \forall A \in \mathscr{F}(U),\\ &(\text{FU9}) \quad \overline{F}(A) = \overline{F}(\overline{F}(A)), \quad \forall A \in \mathscr{F}(U).\end{aligned}$$

Proof. We are only to prove (FU9) because of the duality.

Since R is reflexive, by Theorem 3.9 we have $A \subseteq \overline{F}(A)$. Using Theorem 3.6 we then conclude that $\overline{F}(A) \subseteq \overline{F}(\overline{F}(A))$. Thus Theorem 3.14 follows from Theorem 3.12. □

Remark. Theorems 3.6, 3.8, 3.9 and 3.12 can be viewed as the counterparts of the generalized approximation operators. But when R is a symmetric fuzzy relation, the counterparts of properties (L7) and (U7) do not hold. Nevertheless, we can take properties (FL7) and (FU7) as the counterparts of properties (L7) and (U7).

Example 3.15. Let $U = \{1,2,3\}$, and let fuzzy relation R be given by the fuzzy set-valued function F defined by:

$$\begin{aligned}F(1) &= 0.1/1 + 1/2 + 0.6/3,\\ F(2) &= 1/1 + 0.1/2 + 0.4/3,\\ F(3) &= 0.6/1 + 0.4/2 + 0.1/3.\end{aligned}$$

It is easy to see that R is symmetric. Let $A = 0.1/1 + 1/2 + 0.9/3$. Then it can be checked that

$$\begin{aligned}\overline{F}(A) &= 1/1 + 0.4/2 + 0.4/3,\\ \underline{F}(\overline{F}(A)) &= 0.4/1 + 0.6/2 + 0.6/3,\end{aligned}$$

that is, the properties $A \subseteq \underline{F}(\overline{F}(A))$ and $\overline{F}(\underline{F}(A)) \subseteq A$ do not hold.

Theorem 3.16. *If R is a reflexive fuzzy relation on U, then the following properties hold*:

$$\begin{aligned} &(\mathrm{FLU10}) \quad \underline{F}(\hat{a}) = \overline{F}(\hat{a}) = \hat{a}, \quad a \in I; \\ &(\mathrm{FL11}) \quad \inf\{\underline{F}(A)(x) : x \in U\} = \inf\{A(x) : x \in U\}, \quad A \in \mathscr{F}(U), \\ &(\mathrm{FU11}) \quad \sup\{\overline{F}(A)(x) : x \in U\} = \sup\{A(x) : x \in U\}, \quad A \in \mathscr{F}(U). \end{aligned}$$

Proof

(a) It is easy to see that

$$(\hat{a})_\alpha \neq \emptyset \Longleftrightarrow a \geqslant \alpha \Longleftrightarrow (\hat{a})_\alpha = U.$$

Since R is reflexive, for all $\alpha \in I$, R_α is reflexive as well. Hence $x \in F_\alpha(x)$ for all $x \in U$, i.e., $F_\alpha(x) \neq \emptyset$ for all $\alpha \in I$. Then for all $x \in U$, we have

$$\begin{aligned} \overline{F}(\hat{a})(x) &= \vee_{\alpha \in I}(\alpha \wedge \overline{F}_\alpha(\hat{a})_\alpha)(x) \\ &= \sup\{\alpha \in I : x \in \overline{F}_\alpha(\hat{a})_\alpha\} \\ &= \sup\{\alpha \in I : F_\alpha(x) \cap (\hat{a})_\alpha \neq \emptyset\} \\ &= \sup\{\alpha \leqslant a : F_\alpha(x) \neq \emptyset\} = a. \end{aligned}$$

This implies $\overline{F}(\hat{a}) = \hat{a}$.
Property $\underline{F}(\hat{a}) = \hat{a}$ follows from $\overline{F}(\hat{a}) = \hat{a}$ and the duality of the approximation operators.

(b) Let $\inf\{A(x) : x \in U\} = a$. Obviously, $\hat{a} \subseteq A$. Since R is reflexive, we have

$$\hat{a} = \underline{F}(\hat{a}) \subseteq \underline{F}(A) \subseteq A.$$

Hence

$$a \leqslant \inf\{\underline{F}(A)(x) : x \in U\} \leqslant \inf\{A(x) : x \in U\} = a.$$

Thus (FL11) holds.

(c) (FU11) follows from (FL11) and the duality of the approximation operators. □

Theorem 3.17. *If R is an ordinary binary relation from U to W, and let $F(x) = \{y \in W : (x, y) \in R\}$. Then*
(i) $\underline{F}(1_A) = \{x \in U : F(x) \subseteq A\}, \quad A \in \mathscr{P}(W)$,
(ii) $\overline{F}(1_A) = \{x \in U : F(x) \cap A \neq \emptyset\}, \quad A \in \mathscr{P}(W)$.

Proof. Straightforward. □

Remark. The above theorem shows that the pair of generalized fuzzy approximation operators is a generalization of generalized approximation operators.

Theorem 3.18. *If R is an ordinary equivalence relation on U, and $[x]$ is the R-equivalent class of x, then*
(i) $\underline{F}(A)(x) = \inf\{A(y) : y \in [x]\}, \quad \forall A \in \mathscr{F}(U)$,
(ii) $\overline{F}(A)(x) = \sup\{A(y) : y \in [x]\}, \quad \forall A \in \mathscr{F}(U)$.

Proof. We only prove (ii) as an example. (i) can be justified similarly.

For $A \in \mathscr{F}(U)$ and $x \in U$, let $\overline{F}(A)(x) = a$ and $\sup\{A(y) : y \in [x]\} = b$. Then

$$\begin{aligned} a &= \vee_{\alpha \in I}(\alpha \wedge \overline{F}_\alpha(A_\alpha))(x) \\ &= \sup\{\alpha \in I : x \in \overline{F}_\alpha(A_\alpha)\} \\ &= \sup\{\alpha \in I : F_\alpha(x) \cap A_\alpha \neq \emptyset\} \\ &= \sup\{\alpha \in I : [x] \cap A_\alpha \neq \emptyset\}. \end{aligned}$$

Since $[x] \cap A_\alpha \neq \emptyset$ implies that $\sup\{A(y) : y \in [x]\} \geqslant \alpha$, we have $b \geqslant a$. On the other hand, if $\lambda > a$, then $[x] \cap A_\lambda = \emptyset$, that is, $A(y) < \lambda$ for all $y \in [x]$. Hence $b = \sup\{A(y) : y \in [x]\} < \lambda$. Therefore $b \leqslant a$. It follows that $a = b$. $\square$

Remark. The above theorem shows that the pair of generalized fuzzy approximation operators is a generalization of rough fuzzy approximation operators [3,17].

4. Axioms of generalized fuzzy approximation operators

In an axiomatic approach, rough sets are axiomatized by abstract operators. For the case of fuzzy rough sets, the primitive notion is a system $(\mathscr{F}(U), \mathscr{F}(W), \wedge, \vee, \sim, L, H)$, where $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ are operators from $\mathscr{F}(W)$ to $\mathscr{F}(U)$. In this subsection, we show that fuzzy approximation can be characterized by axioms, the results may be viewed as the generalization counterparts of Yao [27,28].

Definition 4.1. Let $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ be two operators. They are referred to as dual operators if for all $A \in \mathscr{F}(W)$,

$$\text{(fl1)} \quad L(A) = \sim H(\sim A),$$
$$\text{(fu1)} \quad H(A) = \sim L(\sim A).$$

Theorem 4.2. *Suppose that $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ are two dual operators. Then there exists a fuzzy relation R from U to W such that for all $A \in \mathscr{F}(W)$, $L(A) = \underline{F}(A)$ and $H(A) = \overline{F}(A)$ iff L and H satisfy the axioms: for all $A, B \in \mathscr{F}(W)$ and $\alpha \in I$,* (fu2) $H(\hat{\alpha} \wedge A) = \hat{\alpha} \wedge H(A)$,

(fl2) $L(\hat{\alpha} \vee A) = \hat{\alpha} \vee L(A)$;

(fu3) $H(A \vee B) = H(A) \vee H(B)$,

(fl3) $L(A \wedge B) = L(A) \wedge L(B)$.

Proof. " $\Rightarrow$ " follows immediately from Theorem 3.6.

" $\Leftarrow$ " Suppose that the operator H obeys the axioms (fu2) and (fu3). Using H, we can define a fuzzy relation from U to W by:

$$R(x,y) = H(1_y)(x), \quad (x,y) \in U \times W.$$

It is evident that for all $A \in \mathscr{F}(W)$,

$$A = \vee_{y \in W}(1_y \wedge \widehat{A(y)}).$$

For any $x \in U$, by (fu2) and (fu3) we have

$$\begin{aligned}
\overline{F}(A)(x) &= \overline{F}(\vee_{y \in W}(1_y \wedge \widehat{A(y)}))(x) \\
&= \vee_{y \in W}\overline{F}(1_y \wedge \widehat{A(y)})(x) \\
&= \vee_{y \in W}(\overline{F}(1_y) \wedge \widehat{A(y)})(x) \\
&= \vee_{y \in W}(\overline{F}(1_y)(x) \wedge A(y)) \\
&= \vee_{y \in W}(R(x,y) \wedge A(y)) \\
&= \vee_{y \in W}(H(1_y)(x) \wedge A(y)) \\
&= H(\vee_{y \in W}(1_y \wedge \widehat{A(y)}))(x) \\
&= H(A)(x),
\end{aligned}$$

which implies that $H(A) = \overline{F}(A)$.

$L(A) = \underline{F}(A)$ follows immediately from the conclusion $H(A) = \overline{F}(A)$ and the assumption. $\square$

Definition 4.3. Let $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ be a pair of dual operators. If L satisfies axioms (fl2) and (fl3) or equivalently, H satisfies axioms (fu2) and (fu3), then the system ($\mathscr{F}(U)$, $\mathscr{F}(W)$, $\wedge$, $\vee$, $\sim$, L, H) is referred to as a fuzzy rough set algebra, and L and H are referred to as fuzzy approximation operators.

Theorem 4.4. *Suppose that $L, H : \mathscr{F}(W) \to \mathscr{F}(U)$ is a pair of dual fuzzy approximation operators, i.e., L satisfies axioms* (fl2) *and* (fl3), *and H satisfies* (fu2) *and* (fu3). *Then there exists a serial fuzzy relation R from U to W such that $L(A) = \underline{F}(A)$ and $H(A) = \overline{F}(A)$ for all $A \in \mathscr{F}(W)$ iff L and H satisfy axioms*:

(fl0) $L(\emptyset) = \emptyset$,
(fu0) $H(W) = U$,
(flu0) $L(A) \subseteq H(A), \quad \forall A \in \mathscr{F}(W)$.

Proof. " $\Rightarrow$ " follows immediately from Theorem 3.8, and " $\Leftarrow$ " follows immediately from Theorems 4.2 and 3.8. $\square$

Theorem 4.5. *Suppose that $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ is a pair of dual fuzzy approximation operators, i.e., L satisfies axioms* (fl2) *and* (fl3), *and H satisfies* (fu2) *and* (fu3). *Then there exists a reflexive fuzzy relation R on U such that $L(A) = \underline{F}(A)$ and $H(A) = \overline{F}(A)$ for all $A \in \mathscr{F}(U)$ iff L and H satisfy axioms*:

(fl4) $L(A) \subseteq A, \quad \forall A \in \mathscr{F}(U)$,
(fu4) $A \subseteq H(A), \quad \forall A \in \mathscr{F}(U)$.

Proof. " $\Rightarrow$ " follows immediately from Theorem 3.9, and " $\Leftarrow$ " follows immediately from Theorems 4.2 and 3.9. $\square$

Theorem 4.6. *Suppose that $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ is a pair of dual fuzzy approximation operators. Then there exists a symmetric fuzzy relation R on U such that $L(A) = \underline{F}(A)$ and $H(A) = \overline{F}(A)$ for all $A \in \mathscr{F}(U)$ iff L and H satisfy axioms*:

(fl5) $L(1_{U\setminus\{y\}})(x) = L(1_{U\setminus\{x\}})(y), \quad \forall (x, y) \in U \times U$,
(fu5) $H(1_y)(x) = H(1_x)(y), \quad \forall (x, y) \in U \times U$.

Proof. " $\Rightarrow$ " follows immediately from Theorem 3.10, and " $\Leftarrow$ " follows immediately from Theorems 4.2 and 3.10. $\square$

Theorem 4.7. *Suppose that $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ is a pair of dual fuzzy approximation operators. Then there exists a transitive fuzzy relation R on U such that $L(A) = \underline{F}(A)$ and $H(A) = \overline{F}(A)$ for all $A \in \mathscr{F}(U)$ iff L and H satisfy axioms*:

(fl6) $L(A) \subseteq L(L(A)), \quad \forall A \in \mathscr{F}(U)$,
(fu6) $H(H(A)) \subseteq H(A), \quad \forall A \in \mathscr{F}(U)$.

Proof. " $\Rightarrow$ " follows immediately from Theorem 3.12, and " $\Leftarrow$ " follows immediately from Theorems 4.2 and 3.12. $\square$

Theorem 4.8. *Suppose that* $L, H : \mathscr{F}(U) \to \mathscr{F}(U)$ *is a pair of dual fuzzy approximation operators. Then there exists a similarity fuzzy relation R on U such that* $L(A) = \underline{F}(A)$ *and* $H(A) = \overline{F}(A)$ *for all* $A \in \mathscr{F}(U)$ *iff L satisfies axioms* (fl4)–(fl6) *and H satisfies axioms* (fu4)–(fu6).

Proof. It follows immediately from Theorems 4.5–4.7. □

5. Conclusion

As a suitable mathematical model to handle partial knowledge in data bases, rough set theory is emerging as a powerful theory and has been found its successive applications in the fields of artificial intelligence such as pattern recognition, machine learning, and automated knowledge acquisition.

There are at least two aspects in the study of rough set theory: constructive and axiomatic approaches. In constructive approaches, the lower and upper approximation operators are defined in terms of binary relations, partitions of the universe, neighborhood systems or Boolean subalgebras of $\mathscr{P}(U)$. The axiomatic approaches consider the reverse problem, namely, the lower and upper approximation operators are taken as primitive notions. A set of axioms is used to characterize approximation operators that are the same as those derived by using constructive approaches.

In this paper, we have developed a general framework for the study of generalized fuzzy rough set models in which both constructive and axiomatic approaches are considered. This work may be viewed as the extension of Yao [26–28] to the fuzzy environment. We believe that the constructive approaches we offer here will turn out to be more useful for practical applications of the rough set theory while the axiomatic approaches will help us to gain much more insights into the mathematical structures of fuzzy approximation operators.

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