1 The aim of the experiment

The aim of the experiment is [**??**]:

1. Learning about the Hooke's law and the oscillatory motion.

2. Learning and applying the linear least squares' method.

3. Learning and applying methods of determining uncertainties in measurements.

2 Measurement Method

We started by weighing each mass five times using a digital scale. To determine the spring constant k and the equivalent stiffness k_{eff} of system with series connected springs, we used the static method for both and the dynamic method for the spring constant k. For the static method, we measured the stiffness of the spring by attaching various weights of different masses m. We recorded the spring deformation for each weight combination using a meter, repeating each measurement five times. To determine k using the static method, we relied on the principle that the gravitational force Q acting on the spring is balanced by the spring's restoring force F.

By equating formulas:

$$Q = mg, \tag{1}$$

Q - the gravitational force acting on the spring, m - mass attached to one end of a spring hanging in the gravitational field, g - Earths' gravity

$$F = -kx. \tag{2}$$

F - force acting on the spring to restore it to its original form, k - the stiffness of the spring, x - the deformation of the spring

We obtain the following:

$$k = \frac{mg}{x}. \tag{3}$$

We repeated the measurements for the springs connected in a series configuration. The effective stiffness k_{eff} was calculated using the following formula:

$$\frac{1}{k_{eff}} = \sum_{i=1}^{N} \frac{1}{k_i} \tag{4}$$

In the dynamic method, we set the springs into oscillatory motion by attaching different combinations of weights of varying masses m and displacing them from their equilibrium position. Using a stopwatch, we recorded the time for 30 complete oscillations for each mass. The period T was calculated as the average time per oscillation. Using the formula for the period of oscillation:

$$T = 2\pi\sqrt{\frac{m}{k}} \tag{5}$$

We rearranged and squared the equation to obtain:

$$T^2 = \frac{4\pi^2}{k}m \tag{6}$$

By plotting T^2 versus m and analyzing the slope of the resulting line using the linear least squares' method, we calculated the stiffness k.

3   Results of the measurements and calculations

In Table 1 we collected the measured masses of 11 weights, and we also calculated their uncertainties from:

$$u_A(x) = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{N}(x_i - x)^2} \tag{7}$$

$$u_B(x) = \sqrt{\frac{(\Delta_p x)^2}{3} + \frac{(\Delta_e x)^2}{3}} \tag{8}$$

$$u(x) = \sqrt{u_A^2(x) + u_B^2(x)} \tag{9}$$

The type A uncertainty differs depending on the measurement, but the type B uncertainty is constant and equal to:

$$u_B(m) = \sqrt{\frac{(0.01g)^2}{3} + \frac{(0.02g)^2}{3}} = 0.012910\text{g} \tag{10}$$

Table 1: The measured masses of 11 weights with their uncertainties

Measurement Number	1	2	3	4	5	Mean
m_1 [g]	99.88	99.87	99.87	99.89	99.88	99.876
m_2 [g]	100.89	100.9	100.89	100.9	100.89	100.892
m_3 [g]	100.88	100.88	100.88	100.89	100.89	100.886
m_4 [g]	100.25	100.26	100.25	100.25	100.25	100.252
m_5 [g]	100.49	100.5	100.5	100.49	100.48	100.492
m_6 [g]	49.86	49.86	49.87	49.75	49.76	49.82
m_7 [g]	49.75	49.75	49.76	49.75	49.76	49.754
m_8 [g]	49.97	49.98	49.98	49.97	49.97	49.976
m_9 [g]	9.98	9.98	9.99	9.98	9.98	9.982
m_{10} [g]	9.99	9.99	9.99	9.98	9.98	9.988
m_{11} [g]	19.86	19.87	19.86	19.86	19.87	19.864
$u_A(m)$	0.002191	0.001789	0.003578	0.001789	0.003347	—
$u(m)$	0.02012	0.02008	0.02317	0.02008	0.020278	—

Table 2 shows the results of the deformation measurement for the first spring. Once again we calculated the uncertainties of the deformation using Eqs. (??),(??) and (??).

Table 2: The results of the deformation measurement for the first spring with uncertainties.

-	1 [cm]	2 [cm]	3 [cm]	4 [cm]	5 [cm]	Mean [cm]	$u_A(x)$	$u(x)$
$m_1 - m_{11}$	28.4	28.4	28.2	28.1	28.0	28.22	0.071554	0.072709
$m_1 - m_{10}$	27.4	27.2	27.2	27.2	27.1	27,22	0.043818	0.04568
$m_1 - m_8$	26.6	26.7	26.8	26.7	26.9	26,74	0.045607	0.047399
$m_1 - m_6$	22.7	22.6	22.7	22.7	22.8	22,7	0.028284	0.031091
$m_1 - m_5$	20.7	20.8	20.9	20.6	20.8	20,76	0.045607	0.047399
$m_1 - m_4$	16.8	16.9	16.9	16.7	16.7	16,8	0.04	0.042032
$m_1 - m_3$	12.7	12.5	12.6	12.7	12.8	12,66	0.045607	0.047399
$m_1 - m_2$	8.3	8.5	8.5	8.4	8.4	8,42	0.033466	0.03587

The type B uncertainty is once again constant and equal to:

$$u_B(x) = \sqrt{\frac{(0.01\text{mm})^2}{3} + \frac{(0.02\text{mm})^2}{3}} = 0.012910\text{mm} \tag{11}$$

This quantity stays the same in the measurement of the deformation of the second spring and of the springs connected in series, so we will not repeat it. The notation $m_i - m_j$ indicates the sum of the masses from index i to index j. We can determine the formula for the stiffness coefficient k from:

$$x = \frac{mg}{k} \tag{12}$$

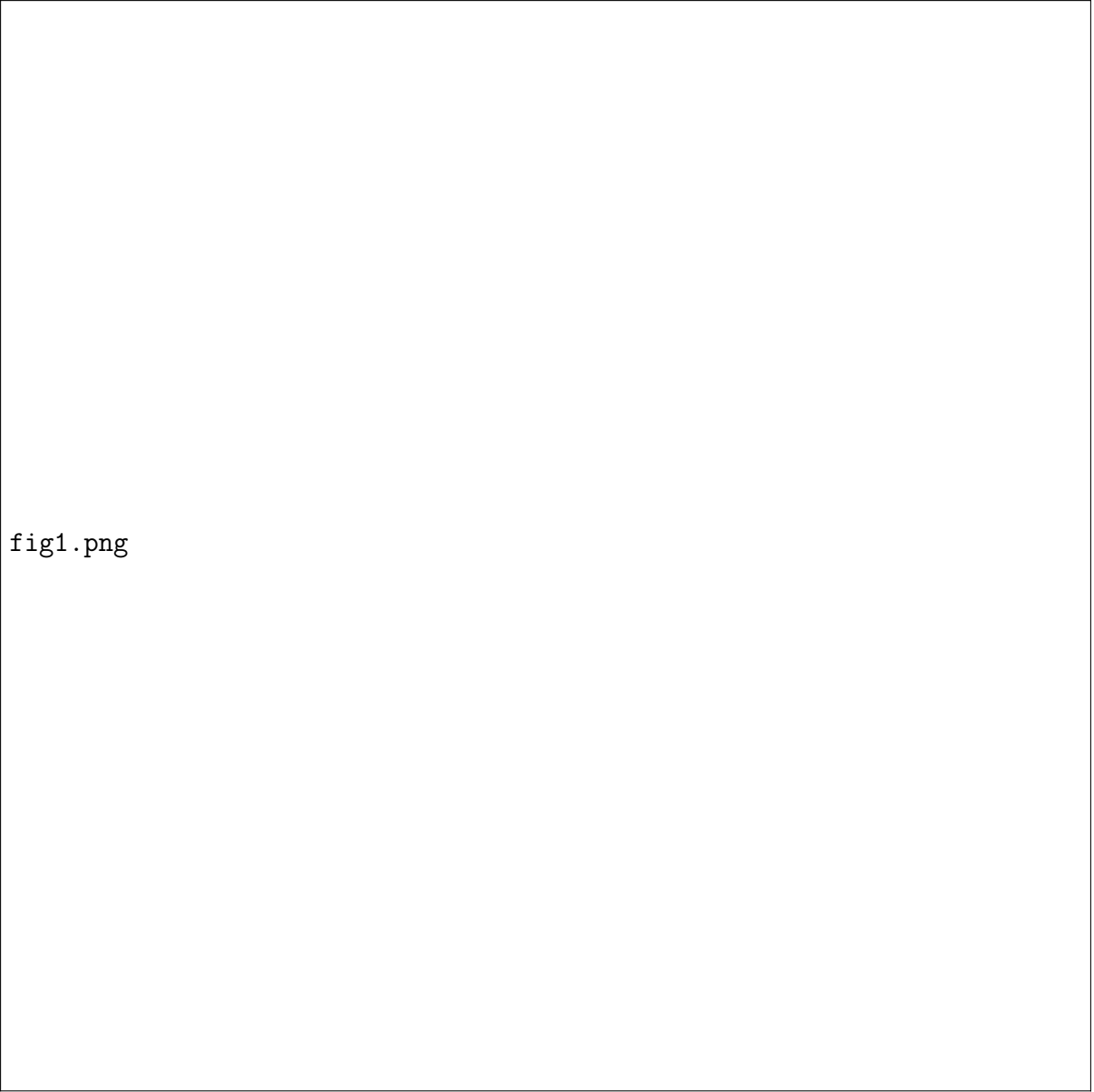Therefore, the stiffness coefficient can be obtained by calculating the slope coefficient of the $x(m)$ function.

Figure 1: The dependence of the deformation of the first spring on the mass.

$$a_1 = 0.401 \frac{\text{m}}{\text{kg}} \tag{13}$$

We can then solve for k by dividing the slope coefficient by the acceleration due to Earth's gravity.

$$k = \frac{g}{a} \tag{14}$$

So, the stiffness coefficient of the first spring is:

$$k_1 = \frac{9.81}{0.401} = 24.525 \frac{\text{kg}}{\text{s}^2} \tag{15}$$

Table 3 shows the result of the deformation measurements for the second spring. Once again we calculated the uncertainties of the deformation using Eqs. (**??**),(**??**) and (**??**).

The type B uncertainty is once again constant and equal to:

$$u_B(x) = \sqrt{\frac{(0.01\text{mm})^2}{3} + \frac{(0.02\text{mm})^2}{3}} = 0.012910\text{mm} \tag{16}$$

Table 3: The results of the deformation measurement for the second spring.

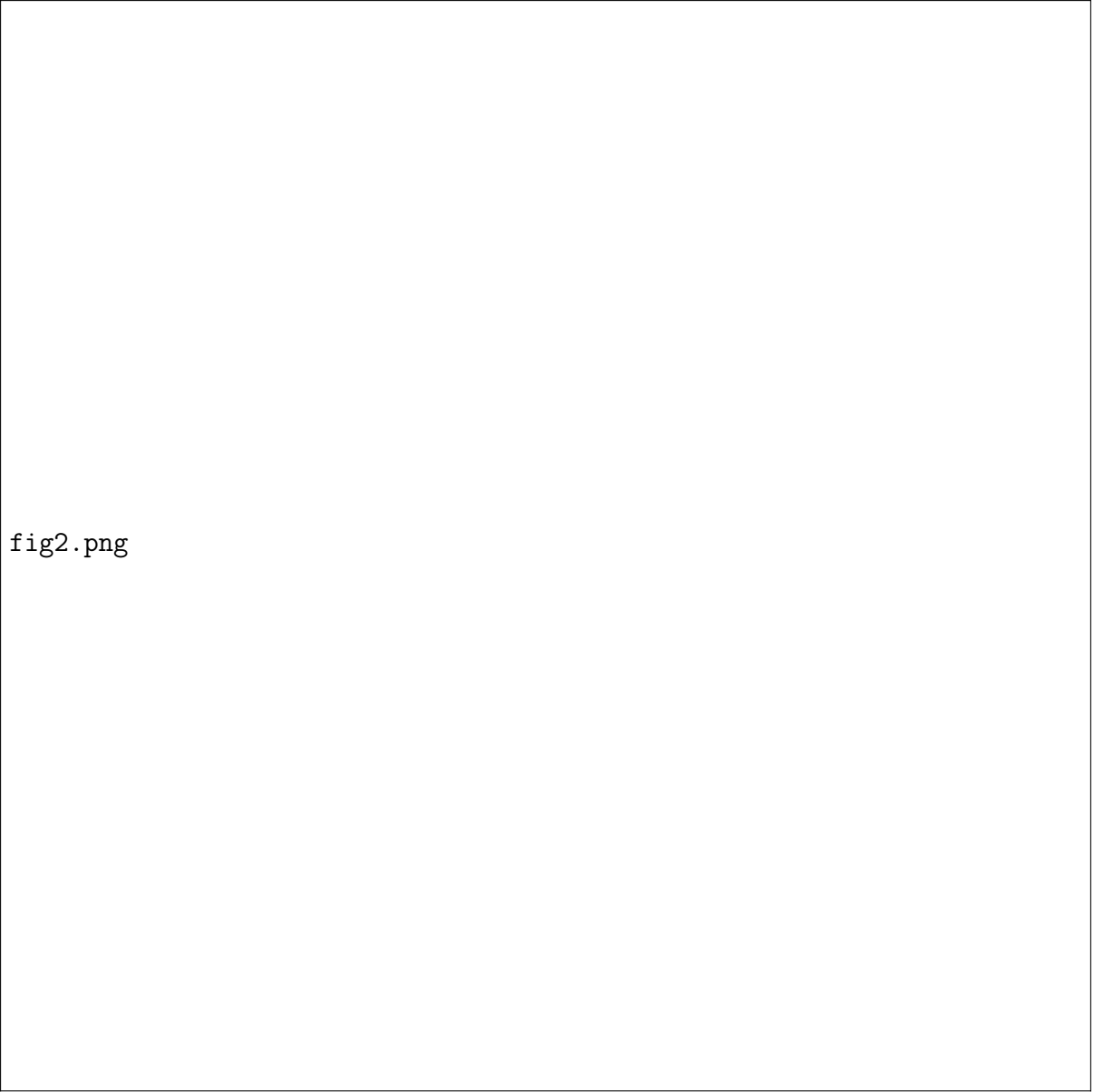| -	1 [cm]	2 [cm]	3 [cm]	4 [cm]	5 [cm]	Mean [cm]	$u_A(\mathbf{x})$	$u(x)$
$m_1 - m_{11}$	26.5	26.6	26.5	26.5	26.4	26.5	0.028284	0.031091
$m_1 - m_{10}$	25.8	25.7	25.7	25.7	25.6	25.7	0.028284	0.031091
$m_1 - m_8$	25.1	24.7	24.8	24.9	24.9	24.88	0.05933	0.060718
$m_1 - m_6$	20.8	20.7	20.8	20.9	21.0	20.84	0.045607	0.047399
$m_1 - m_5$	18.8	19.0	18.9	18.9	18.9	18.92	0.033466	0.03587
$m_1 - m_4$	15.1	15.0	15.0	15.1	15.2	15.08	0.033466	0.03587
$m_1 - m_3$	10.9	11.0	11.0	10.9	10.9	10.96	0.021909	0.02543
$m_1 - m_2$	7.9	7.2	7.2	7.1	7.2	7.32	0.130843	0.131479

Figure 2: The dependence of the deformation of the first spring on the mass.

We can see that the slope coefficient of the function is equal to

$$a_2 = 0.394 \frac{\text{m}}{\text{kg}}. \tag{17}$$

Using Eq. (**??**) we can calculate the stiffness coefficient of the second spring:

$$k_2 = \frac{9.81}{0.394} = 24.898 \frac{\text{kg}}{\text{s}^2} \tag{18}$$

Table 4: The results of the deformation measurement for the system of two springs connected in series.

-	1 [cm]	2 [cm]	3 [cm]	4 [cm]	5 [cm]	Mean	$u_A(x)$	$u(x)$
$m_1 - m_1$	25.7	25.9	25.7	25.8	25.9	25.8	0.04	0.042032
$m_1 - m_2$	33.9	34.2	34.0	34.0	34.2	34.06	0.053666	0.055197
$m_1 - m_3$	42.1	42.2	42.3	42.3	42.3	42.2	0.04	0.042032
$m_1 - m_4$	50.4	50.4	50.4	50.3	50.4	50.38	0.017889	0.022061
$m_1 - m_5$	58.3	58.5	58.4	58.3	58.5	58.4	0.04	0.042032
$m_1 - m_6$	59.4	59.6	59.5	59.4	59.6	59.5	0.04	0.042032
$m_1 - m_7$	62.8	63.0	63.2	63.2	63.2	63.06	0.066933	0.068166
$m_1 - m_8$	66.8	66.9	66.8	66.8	66.7	66.8	0.028284	0.031091

Figure 3: The dependence of the deformation of the spring system on the mass.

The slope coefficient is equal to

$$a_3 = 0.739 \frac{m}{kg}. \tag{19}$$

Once again we derive the stiffness coefficient of the system of springs using Eq. (**??**):

$$k_3 = \frac{9.81}{0.739} = 13.275 \frac{kg}{s^2} \tag{20}$$

The result is close to the one that can be derived using Eq (7). If we solve for k_{eff} we obtain that:

$$k_{eff} = \frac{k_1 \cdot k_2}{k_1 + k_2} \tag{21}$$

So, in this case

$$k_3 = \frac{k_1 \cdot k_2}{k_1 + k_2} = \frac{24.525 \times 24.898}{24.525 + 24.898} = 12.355 \frac{kg}{s^2} \tag{22}$$

The result is close to the one obtained from the slope coefficient.

Then we calculate the stiffness coefficient of the second spring using the dynamic method. Table 5 shows the result of the measurement of 30 oscillations for different masses. The table also includes the period, type A uncertainty, as well as the standard combined uncertainty of each time measurement, which are obtained from Eqs. (**??**) and (**??**). The type B uncertainty is not in the table as it is constant and can be calculated by using Eq. (**??**):

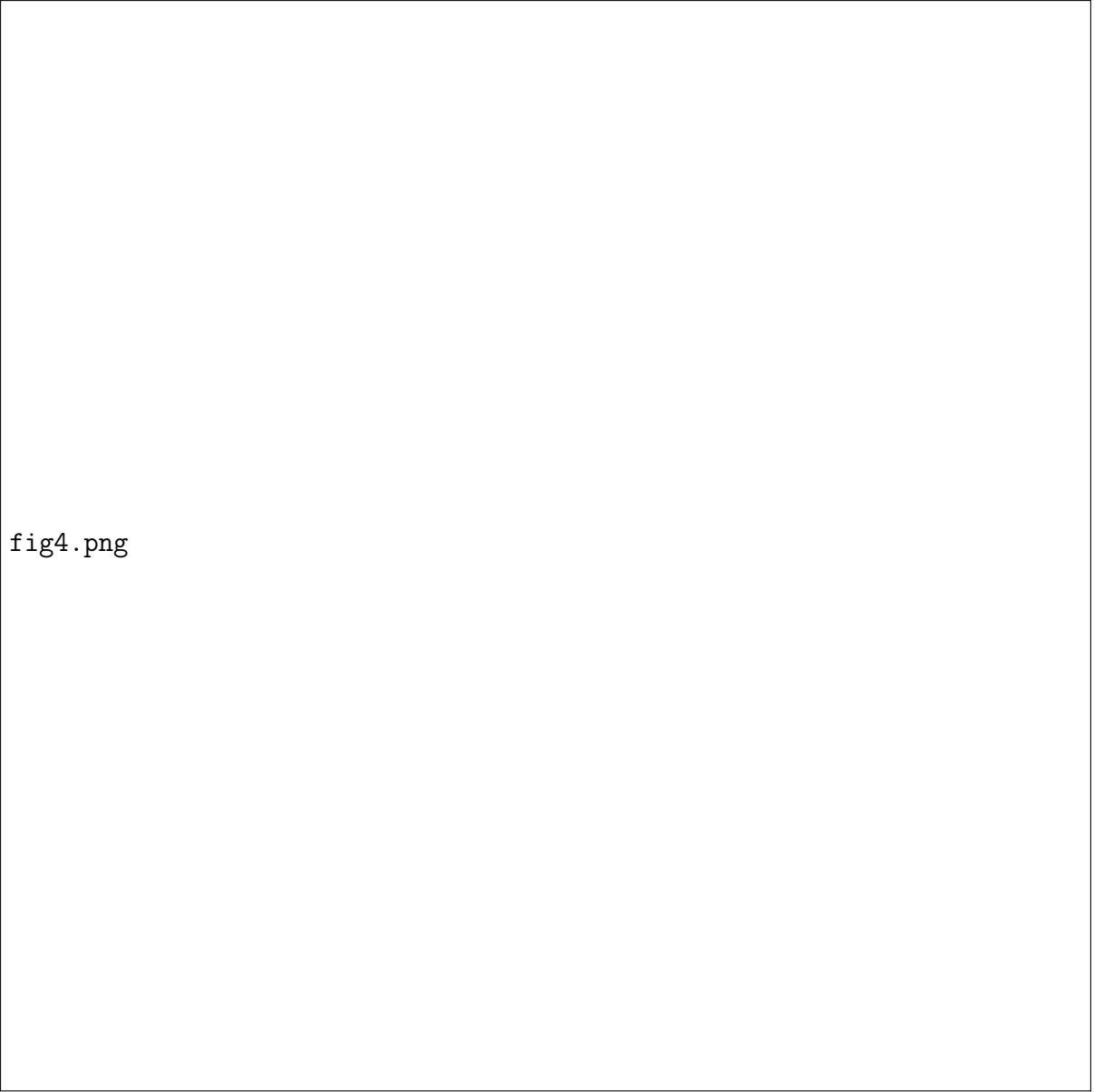$$u_B(t) = \sqrt{\frac{(0.01s)^2}{3} + \frac{(0.5s)^2}{3}} = 0.28873s \tag{23}$$

Figure 4: The dependence of the square of the period on the mass.

From Eq. (??) we can observe that the slope coefficient of the dependence is equal to

$$a = \frac{4\pi^2}{k}. \tag{24}$$

By solving for k we arrive at the equation:

$$k = \frac{4\pi^2}{a} \tag{25}$$

We can observe on the figure that the slope coefficient of the function is: $a_4 = 1.503 \frac{\text{s}^2}{\text{kg}}$
We then calculate the spring coefficient k of the second spring using Eq. (??):

$$k_4 = \frac{4\pi^2}{1.503} = 26.266 \frac{\text{kg}}{\text{s}^2} \tag{26}$$

We can observe that the obtained coefficient is close in value to k_1, which is equal to $24.898\frac{\text{kg}}{\text{s}^2}$. The difference is affected mainly by the time uncertainty of each measurement.

Now we can focus on calculation of the uncertainties of obtained stiffness coefficient. In the static method, where the stiffness coefficient is expressed by Eq. (??), we can calculate $u(k)$ using propagation of uncertainty.
We define the sensitivity coefficients:

$$c_g = \frac{1}{a} \tag{27}$$

$$c_a = -\frac{g}{a^2} \tag{28}$$

For the calculations we use the uncertainty of g $u(g) = 0.01\frac{\text{m}}{\text{s}^2}$. The uncertainty of the slope coefficients was obtained using the LINEST function in Microsoft Excel.

Table 5: Summary results of measurement uncertainty values for the deformation measurement of the first spring

x_j	c_{x_j}	$u(x_j)$
$g\frac{\text{m}}{\text{s}^2}$	2.4938 $\frac{\text{kg}}{\text{m}}$	0.01 $\frac{\text{m}}{\text{s}^2}$
$a\frac{\text{m}}{\text{kg}}$	-61.007 $\frac{\text{kg}^2}{\text{s}^2\cdot\text{m}}$	0.004977 $\frac{\text{m}}{\text{kg}}$

We then repeat the same calculations for the measurements of the deformations of the second spring, as well as the series of two springs connected in series.
Table ?? shows the values of the sensitivity coefficients and the uncertainties used to determine the uncertainty of the stiffness coefficient for the measurement of the deformation of the second spring.

Table 6: Summary results of measurement uncertainty values for the deformation measurement of the second spring

x_j	c_{x_j}	$u(x_j)$
$g\frac{\text{m}}{\text{s}^2}$	2.5381 $\frac{\text{kg}}{\text{m}}$	0.01 $\frac{\text{m}}{\text{s}^2}$
$a\frac{\text{m}}{\text{kg}}$	-63.194 $\frac{\text{kg}^2}{\text{s}^2\cdot\text{m}}$	0.00414 $\frac{\text{m}}{\text{kg}}$

Table ?? shows the values of the sensitivity coefficients and the uncertainties used to determine the uncertainty of the stiffness coefficient for the measurement of the deformation of the system of two springs connected in series.

Table 7: Summary results of measurement uncertainty values for the deformation measurement of the system of two springs connected in series

x_j	c_{x_j}	$u(x_j)$
$g\frac{\text{m}}{\text{s}^2}$	1.3532 $\frac{\text{kg}}{\text{m}}$	0.01 $\frac{\text{m}}{\text{s}^2}$
$a\frac{\text{m}}{\text{kg}}$	-17.963 $\frac{\text{kg}^2}{\text{s}^2\cdot\text{m}}$	0.011677 $\frac{\text{m}}{\text{kg}}$

The formula for the uncertainty of the stiffness coefficient is:

$$u(k) = \sqrt{c_g^2 u^2(g) + c_a^2 u^2(a)} \tag{29}$$

Now we can determine the combined uncertainties of the stiffness coefficients of the first spring, the second spring, and the system of two springs connected in series.

$$u_c(k_1) = \sqrt{2.4938^2 \times 0.01^2 + (-61.007)^2 \times 0.004977^2} = 0.30465 \frac{\text{kg}}{\text{s}^2} \tag{30}$$

$$u_c(k_2) = \sqrt{2.5381^2 \times 0.01^2 + (-63.194)^2 \times 0.00414^2} = 0.26285 \frac{\text{kg}}{\text{s}^2} \tag{31}$$

$$u_c(k_3) = \sqrt{1.3532^2 \times 0.01^2 + (-17.963)^2 \times 0.011677^2} = 0.21019 \frac{\text{kg}}{\text{s}^2} \tag{32}$$

When it comes to the dynamic method, we also use propagation of uncertainty, but for Eq. (??)
The sensitivity coefficient is:

$$c_a = -\frac{4\pi^2}{a^2} \tag{33}$$

We calculate the uncertainty of the spring coefficient of the dynamic method using Eq. (??).

$$u_c(k_4) = \sqrt{(-\frac{4\pi^2}{1.503^2})^2 \times 0.004159^2} = 0.072683 \frac{\text{kg}}{\text{s}^2} \tag{34}$$

We can now calculate the expanded uncertainties of the stiffness coefficients for each measurement. We will be using the formula:

$$U(k) = k \cdot u_c(k) = 2 \cdot u_c(k) \tag{35}$$

We calculate:

$$U(k_1) = 2 \times 0.30465 = 0.60930 \frac{\text{kg}}{\text{s}^2} \tag{36}$$

$$U(k_2) = 2 \times 0.26285 = 0.52570 \frac{\text{kg}}{\text{s}^2} \tag{37}$$

$$U(k_3) = 2 \times 0.21019 = 0.42038 \frac{\text{kg}}{\text{s}^2} \tag{38}$$

$$U(k_4) = 2 \times 0.072683 = 0.145366 \frac{\text{kg}}{\text{s}^2} \tag{39}$$

4 Results

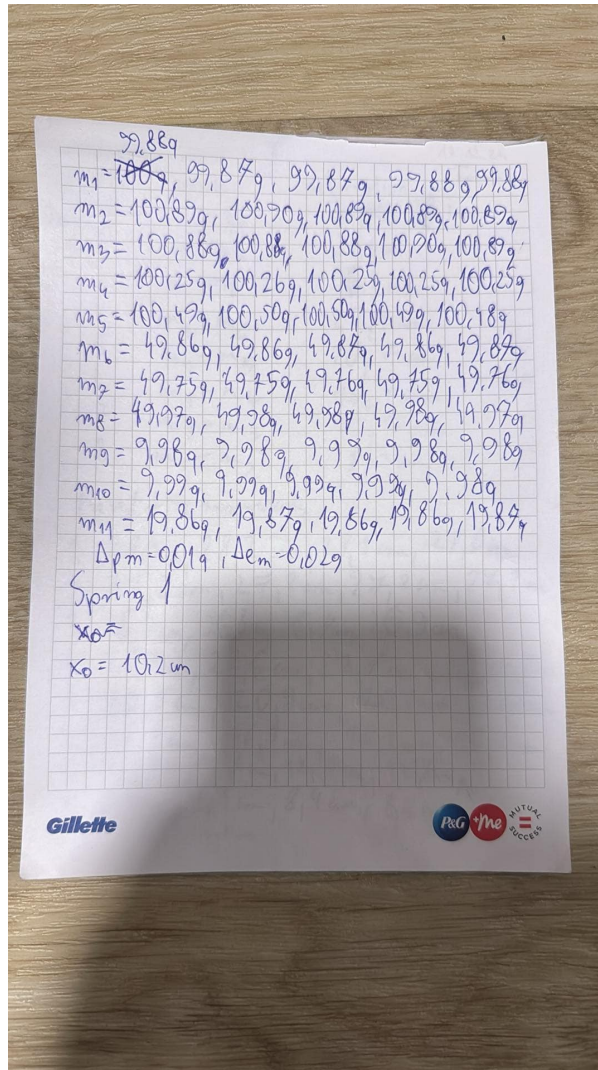Finally we arrive at the obtained stiffness coefficients and their uncertainties:

$$k_1 = 25(0.30) \frac{\text{kg}}{\text{s}^2}, k_1 = (25 \pm 0.61) \frac{\text{kg}}{\text{s}^2} \tag{40}$$

$$k_2 = 25(0.26) \frac{\text{kg}}{\text{s}^2}, k_1 = (25 \pm 0.53) \frac{\text{kg}}{\text{s}^2} \tag{41}$$

$$k_3 = 12(0.21) \frac{\text{kg}}{\text{s}^2}, k_1 = (12 \pm 0.42) \frac{\text{kg}}{\text{s}^2} \tag{42}$$

$$k_4 = 26(0.072) \frac{\text{kg}}{\text{s}^2}, k_1 = (26 \pm 0.15) \frac{\text{kg}}{\text{s}^2} \tag{43}$$

The stiffness coefficient obtained from the dynamic method has substantially lower uncertainties than the stiffness coefficients obtained from the static method, which shows its superior precision.

$m_1 = 100g, 99.87g, 99.87g, 99.88g, 99.88g$
$m_2 = 100.89g, 100.90g, 100.89g, 100.8\%, 100.89g$
$m_3 = 100.88g, 100.88g, 100.88g, 100.90g, 100.89g$
$m_4 = 100.25g, 100.26g, 100.25g, 100.25g, 100.25g$
$m_5 = 100.49g, 100.50g, 100.50g, 100.49g, 100.48g$
$m_6 = 49.86g, 49.86g, 49.87g, 49.86g, 49.89g$
$m_7 = 49.75g, 49.75g, 19.76g, 49.75g, 49.76g$
$m_8 = 49.97g, 49.98g, 49.98g, 49.98g, 49.97g$
$m_9 = 9.98g, 9.98g, 9.99g, 9.98g, 9.98g$
$m_{10} = 9.99g, 9.99g, 9.99g, 9.99g, 9.98g$
$m_{11} = 19.86g, 19.87g, 19.86g, 19.86g, 19.84g$
$\Delta pm = 0.01g; \Delta em = 0.02g$

Spring 1

$X_0 =$

$X_0 = 10.2$ cm

5 Conclusion

6 References

1. https://ftims.edu.p.lodz.pl/mod/resource/view.php?id=120359

7 Our Notes

	masses	deformation	homework
1	$m_1 - m_{11}$	~~30,2 cm~~ 28,4 cm	
2	$m_1 - m_{10}$	27,4 cm	
3	$m_1 - m_8$	25,6 cm	
4	$m_1 - m_6$	22,7 cm	
5	$m_1 - m_5$	20,7 cm	
6	$m_1 - m_4$	16,7 cm	~10,2
7	$m_1 - m_3$	12,7 cm	
8	$m_1 - m_2$	8,3 cm	

The rest of the measurements

1. 28,4 cm, 28,2 cm, 28,1 cm, 28,0 cm
2. 27,2 cm, 27,2 cm, 27,2 cm, 27,1 cm
3. 25,7 cm, 25,8 cm, 25,7 cm, 25,9 cm
4. 22,7 cm, 22,6 cm, 22,7 cm, 22,8 cm
5. 20,8 cm, 20,9 cm, 20,6 cm, 20,8 cm
6. 16,8 cm, 16,9 cm, 16,9 cm, 16,7 cm
7. 12,5 cm, 12,6 cm, 12,7 cm, 12,8 cm
8. 8,5 cm, 8,5 cm, 8,4 cm, 8,4 cm

$\Delta p_{km} = 0,1$ cm

1 $m_1 - m_{11}$ 26,5cm, 26,6 cm, 26,5 cm, 26,5 cm, 26,4cm

2 $m_1 - m_{10}$ 25,8cm, 25,7 cm, 25,7 cm, 25,7 cm, 25,6 cm

3 $m_1 - m_8$ 25,1 cm, 24,7 cm, 24,8 cm, 24,9 cm, 24,9 cm

4 $m_1 - m_6$ 20,8 cm, 20,7 cm, 20,8 cm, 20,9 cm, 21 cm

5 $m_1 - m_5$ 18,8 cm, 19 cm, 18,9 cm, 19 cm, 18,9 cm

6 $m_1 - m_4$ 15,1 cm, 15,0 cm, 15 cm, 15,1 cm, 15,2 cm
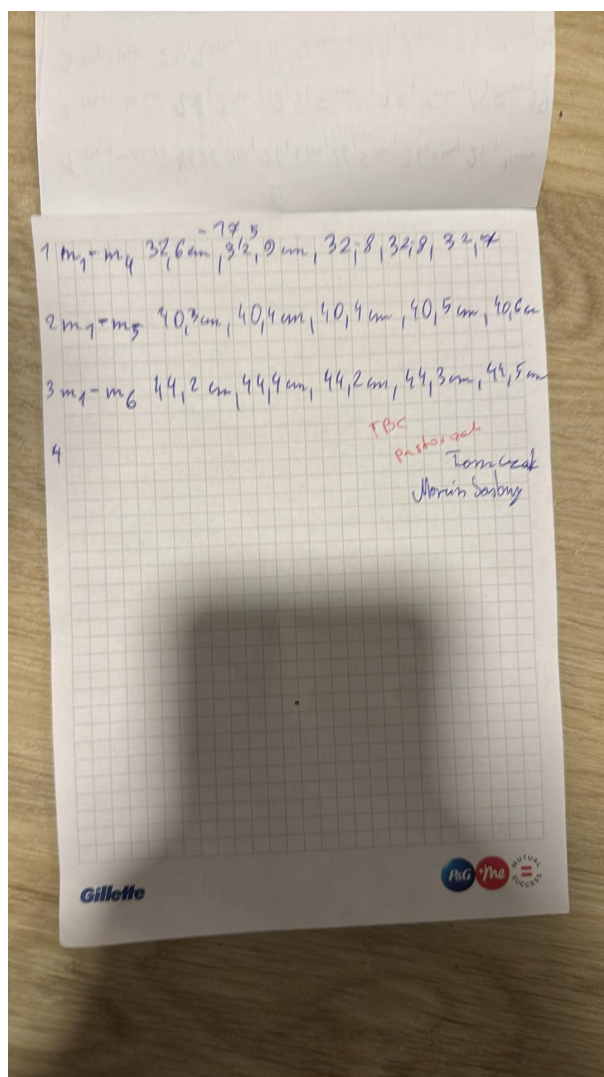
7 $m_1 - m_3$ 10,9 cm, 11 cm, 11 cm, 11 cm, 10,9 cm

8 $m_1 - m_2$ 7,9 cm, 7,2 cm, 7,2 cm, 7,1 cm, 7,2 cm

17,3 cm
17,4 cm

Gillette

P&G +me MUTUAL SUCCESS

$$= 77,5$$

1 $m_1 - m_4$ 32,6 cm, 32,9 cm, 32,8, 32,8, 32,4

2 $m_1 - m_5$ 40,3 cm, 40,4 cm, 40,4 cm, 40,5 cm, 40,6 cm

3 $m_1 - m_6$ 44,2 cm, 44,4 cm, 44,2 cm, 44,3 cm, 44,5 cm

4

TBC
Pastorczak
Tomiczak
Marcin Sajow

System of springs \serial

Mikołaj
Tomerde
Marcin
Sadowy

7.01.2025 c

1. $m_1 - m_4$, without $x_0 = 18.0$ cm
   x: 39,8 cm
   40,0 cm
   39,8 cm $\}$ ~14,1 cm
   39,9 cm
   40,0 cm

2. $m_1 - m_2$
   x: 48,0 cm
   48,3 cm
   48,1 cm $\}$ ~14,1 cm
   48,1 cm
   48,3 cm

3. $m_1 - m_3$
   x: 56,2 cm   56,2 cm
   56,3 cm   56,4 cm $\}$ ~14,1 cm
   56,4 cm

4. $m_1 - m_4$
   X: 64,5 cm
   64,5 cm
   64,4 cm   ⎫ –14,1 cm
   64,4 cm
   64,5 cm

7. $m_1 - m_7$
   X: 62,8 cm
   63,0 cm
   63,2 cm   ⎫ –2,5 cm
   63,1 cm
   63,2 cm

5. $m_1 - m_5$
   X: 72,4 cm
   72,6 cm
   72,5 cm   ⎫ –14,1 cm
   72,4 cm
   72,6 cm

8. $m_1 - m_8$
   X: 66,8 cm
   66,9 cm
   66,8 cm
   66,8 cm
   66,7 cm

6. $m_1 - m_6$
   X: 59,4 cm
   59,6 cm
   59,5 cm   ⎫ –2,5 cm
   59,4 cm
   59,6 cm

Oscillations $\Delta_t = 0.01s$ Det $= 0.5$ s

1. $m_1 - m_3$ 2. $m_1 - m_4$ 07.2025

t: 20,78s   t: 23,66s   Mikołaj
   20,73s   23,80s   Tomczak
   20,72s   23,85s   Marun
   20,70s   23,83s   Sadowi
   20,75s   23.77s

3. $m_1 - m_5$ 4. $m_1 - m_6$

t: 26,49s   t: 27,87s
   26,56s   28,08s
   26,65s   27,62s
   26,53s   27,97s
   26,51s   27,91s
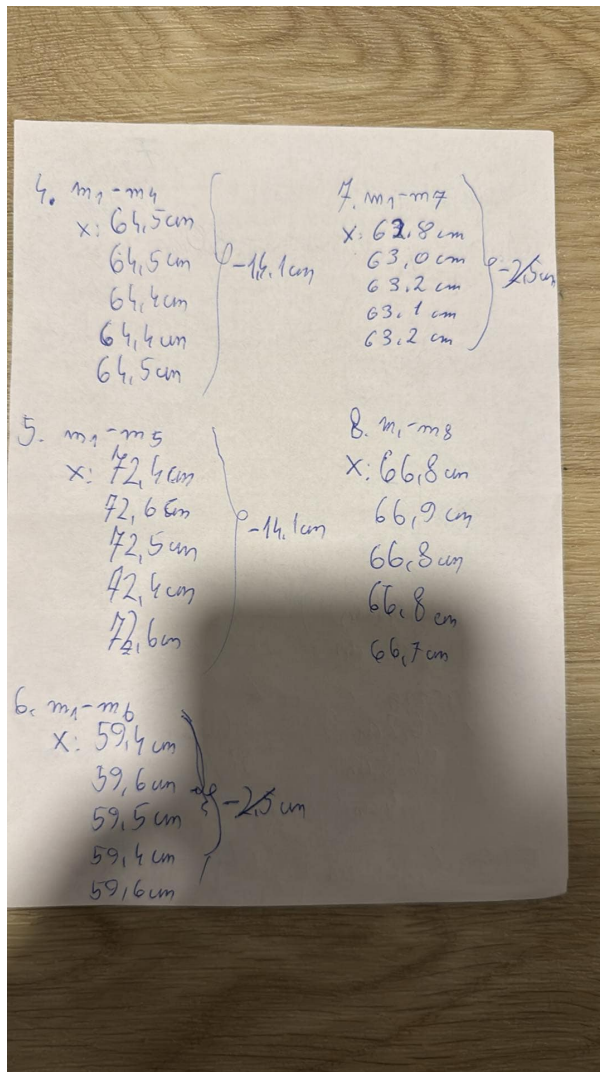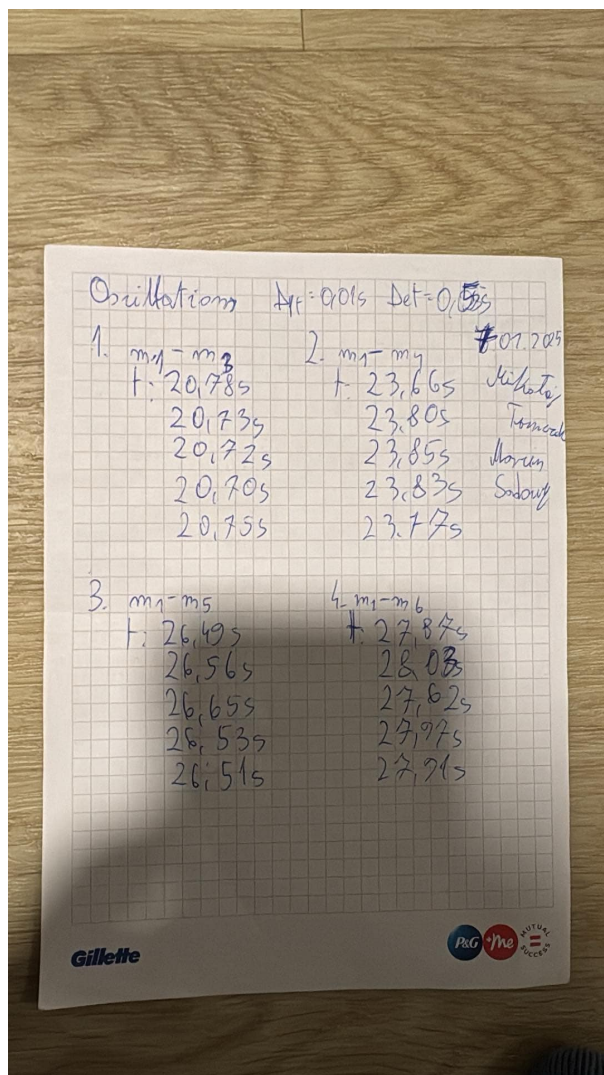
5. $m_1 - m_7$
t: 28,75s
28,93s
28,95s
28,84s
28,97s

6. $m_1 - m_8$
t: 29,99s
29,99s
30,04s
30,01s
30,04s

7. $m_1 - m_9$
t: 30,45s
30,20s
30,24s
30,30s
30,35s

8. $m_1 - m_{10}$
t: 30,45s
30,53s
30,60s
30,68s
30,49s

Pastornok