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Math for Machine Learning

Linear algebra - Week 4

W4 Lesson 1

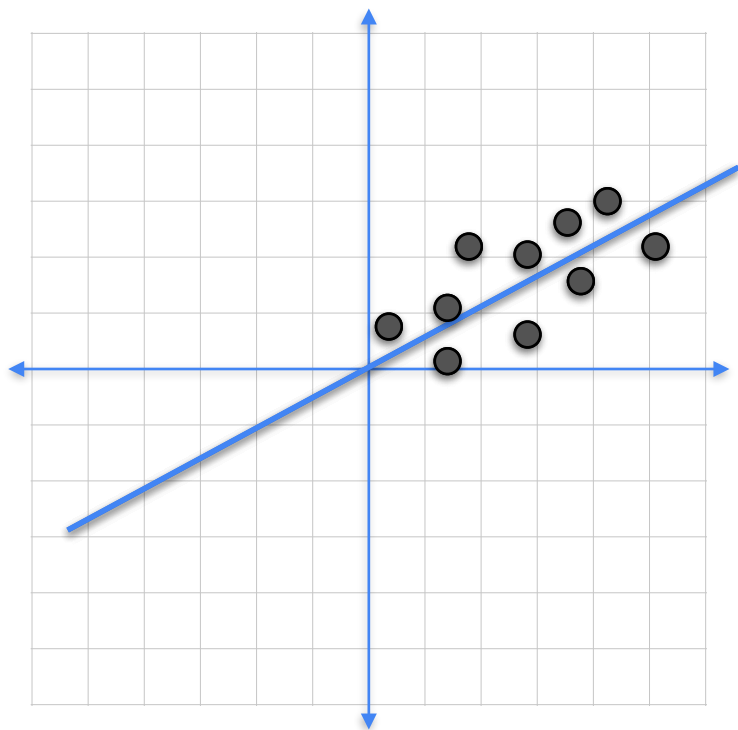


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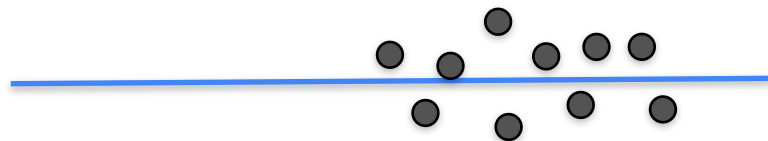
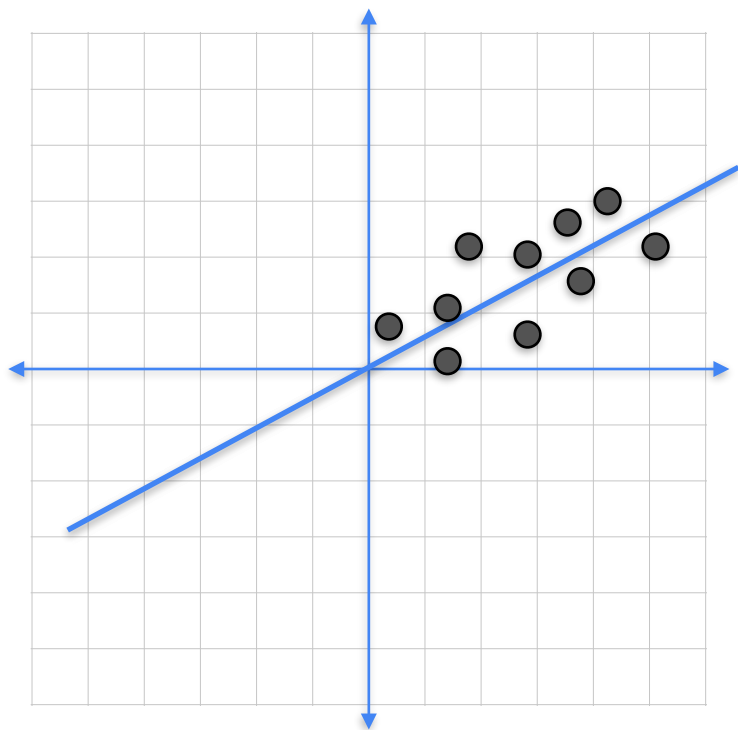
Determinants and Eigenvectors

Machine learning motivation

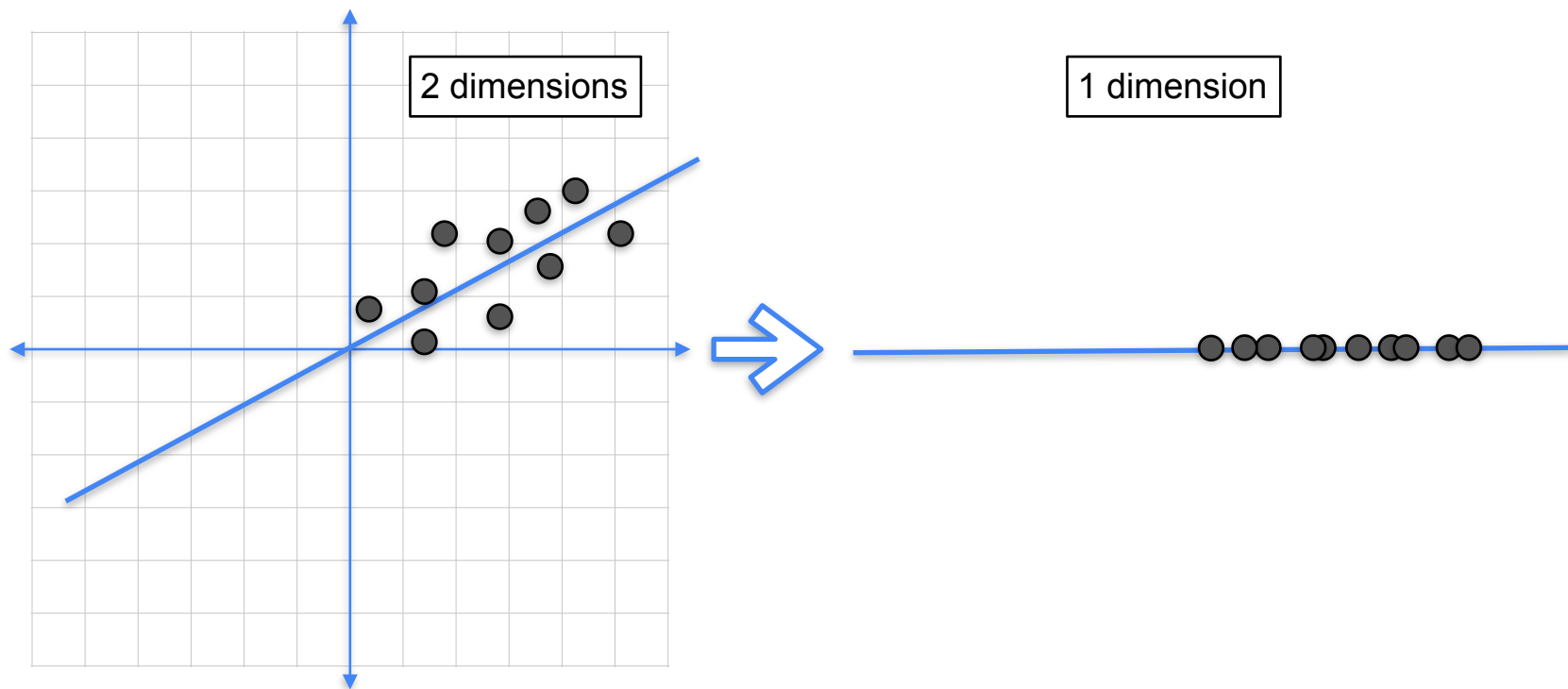
Principal Component Analysis



Principal Component Analysis

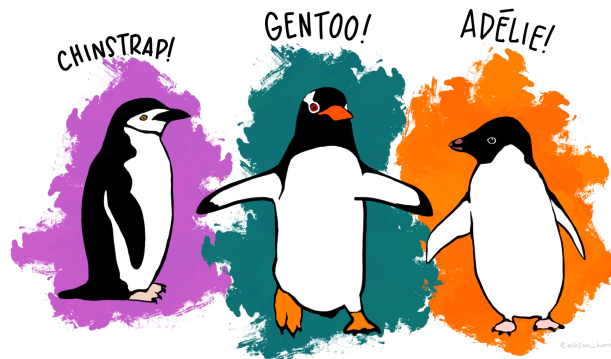


Principal Component Analysis



Principal Component Analysis

- Reduce dimensions (columns) of dataset
- Preserve as much information as possible

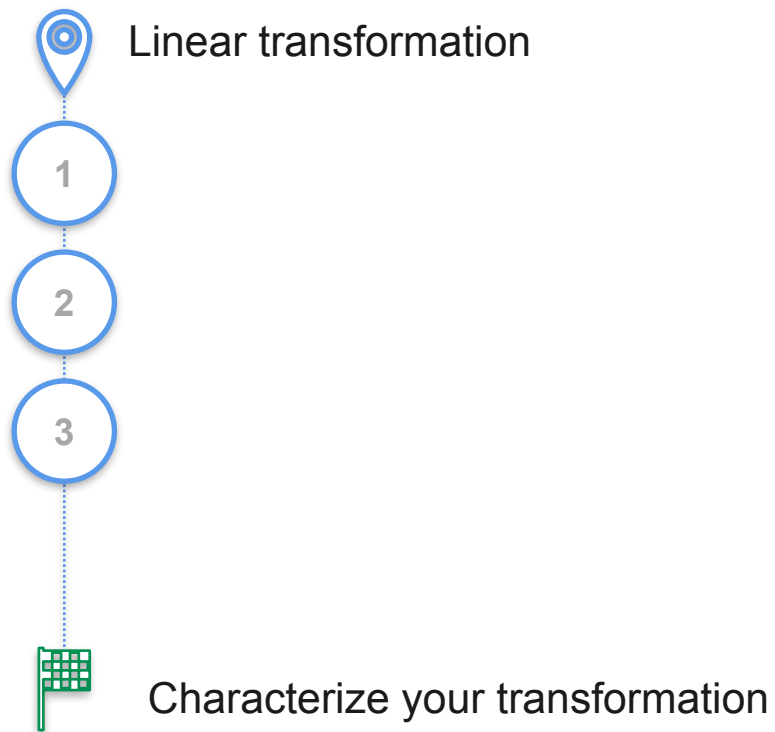


| species | culmen_length_mm | culmen_depth_mm | flipper_length_mm | body_mass_g |
|---------|------------------|-----------------|-------------------|-------------|
| Adelie | 40.6 | 17.2 | 187.0 | 3475.0 |
| Adelie | 38.9 | 17.8 | 181.0 | 3625.0 |
| Adelie | 35.7 | 16.9 | 185.0 | 3150.0 |
| Gentoo | 50.0 | 15.3 | 220.0 | 5550.0 |
| Adelie | 34.5 | 18.1 | 187.0 | 2900.0 |

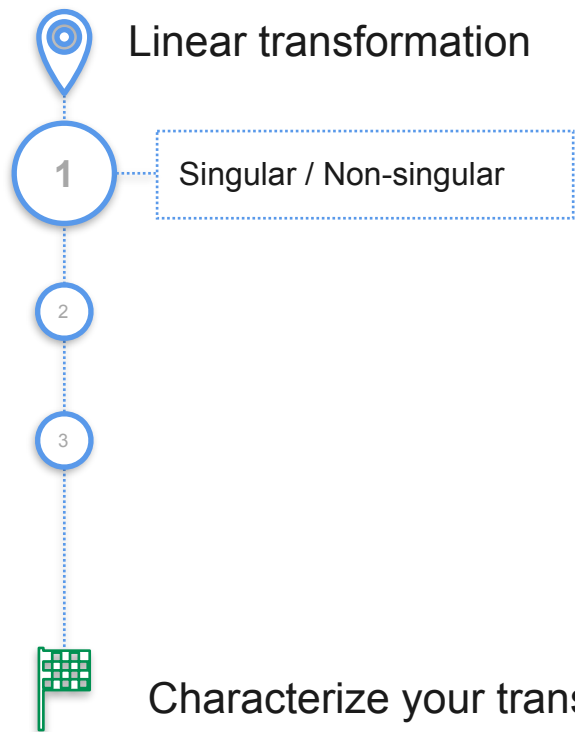


| PC1 | PC2 | species |
|-----------|-----------|---------|
| 1.353843 | -0.422253 | Adelie |
| 1.760446 | -0.350965 | Adelie |
| 2.005766 | -1.113797 | Adelie |
| -2.585758 | 0.061768 | Gentoo |
| 2.438111 | -0.786227 | Adelie |



What to expect?





What to expect?

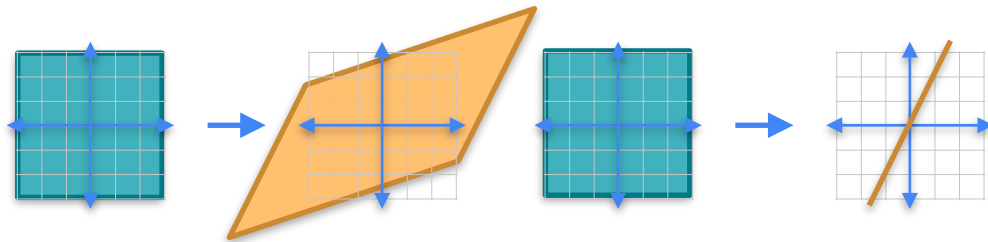


Non-singular

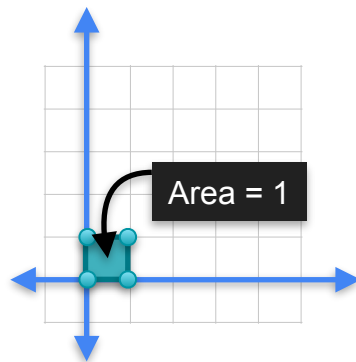
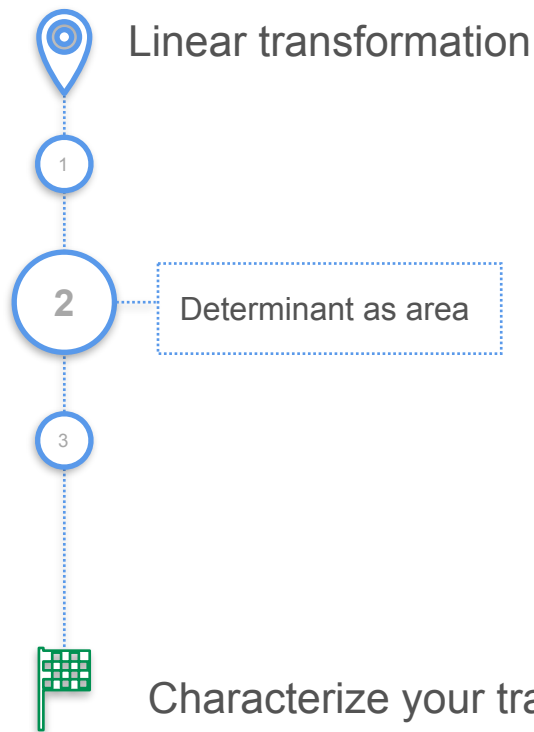
| | |
|---|---|
|  |  |
| 3 | 1 |
| 1 | 2 |



Singular

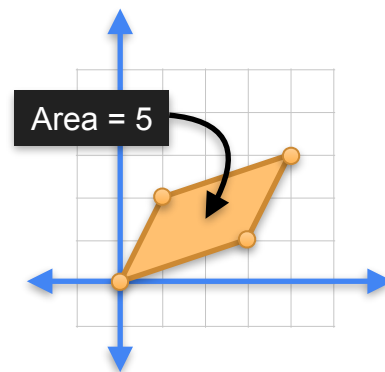
| | |
|---|---|
|  |  |
| 1 | 1 |
| 2 | 2 |



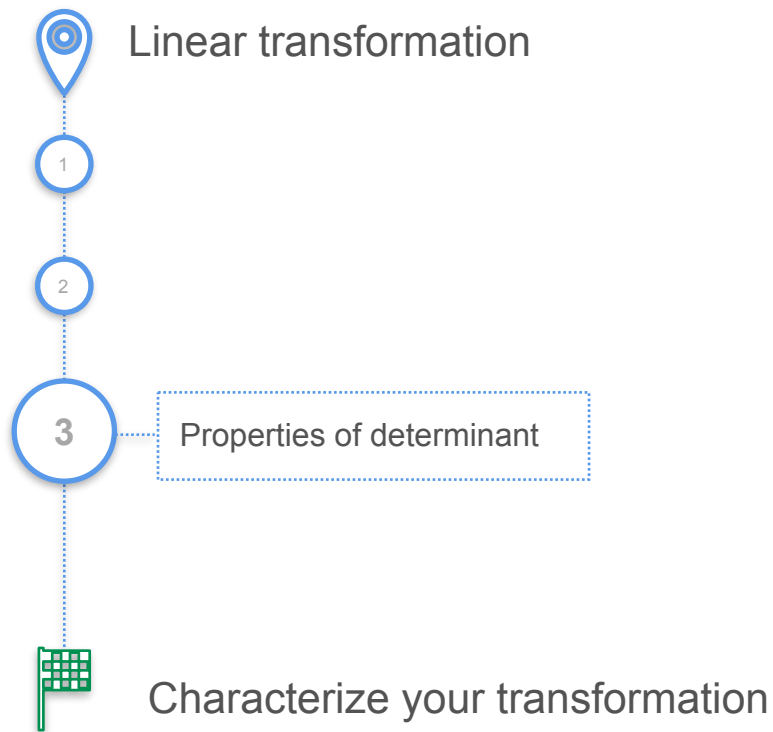
What to expect?



| | |
|---|---|
|  |  |
| 3 | 1 |
| 1 | 2 |



What to expect?



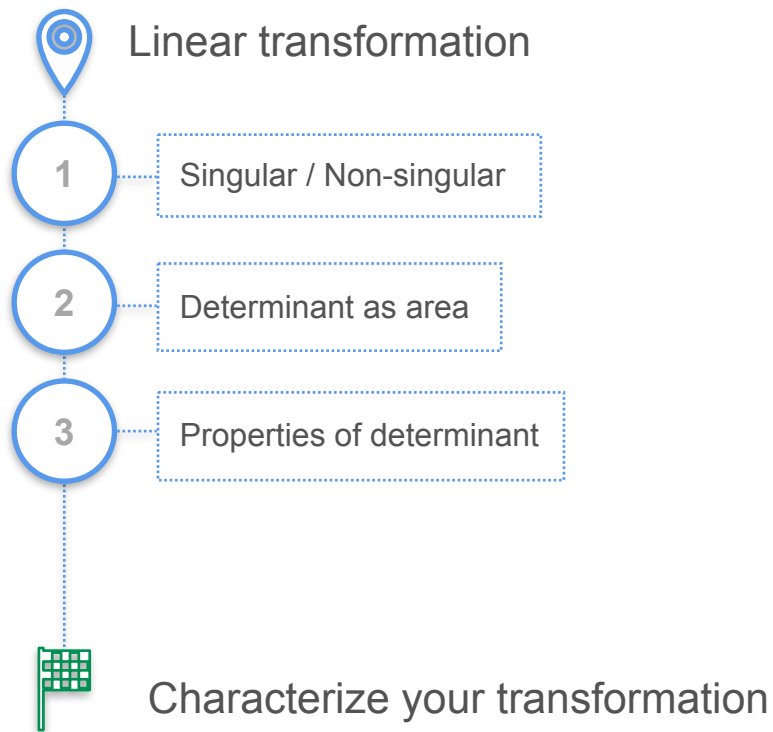
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -3 & 3 \end{bmatrix}$$

Det = 5 Det = 3 Det = 15
= 5 · 3

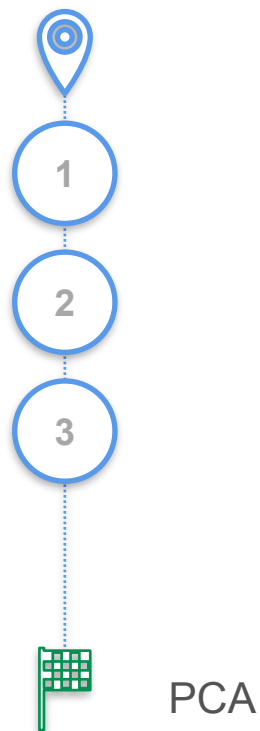
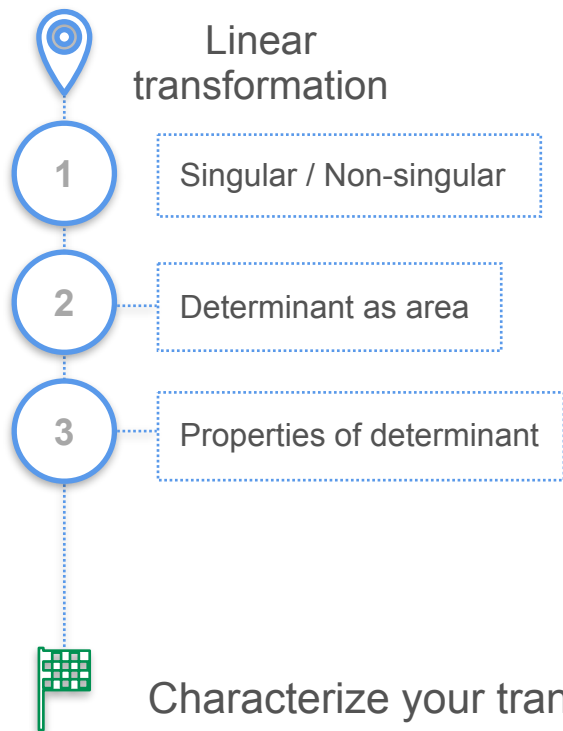
$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

Det = 5 Det = 0.2 = $\frac{1}{5}$

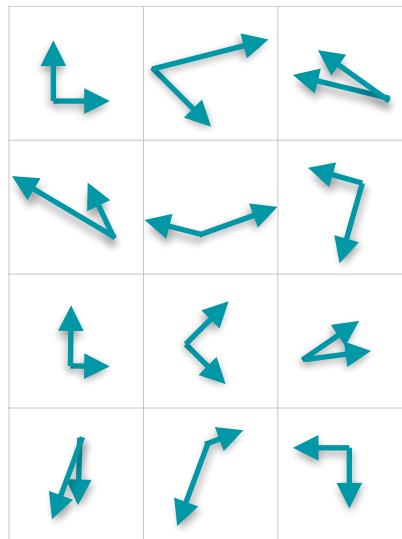
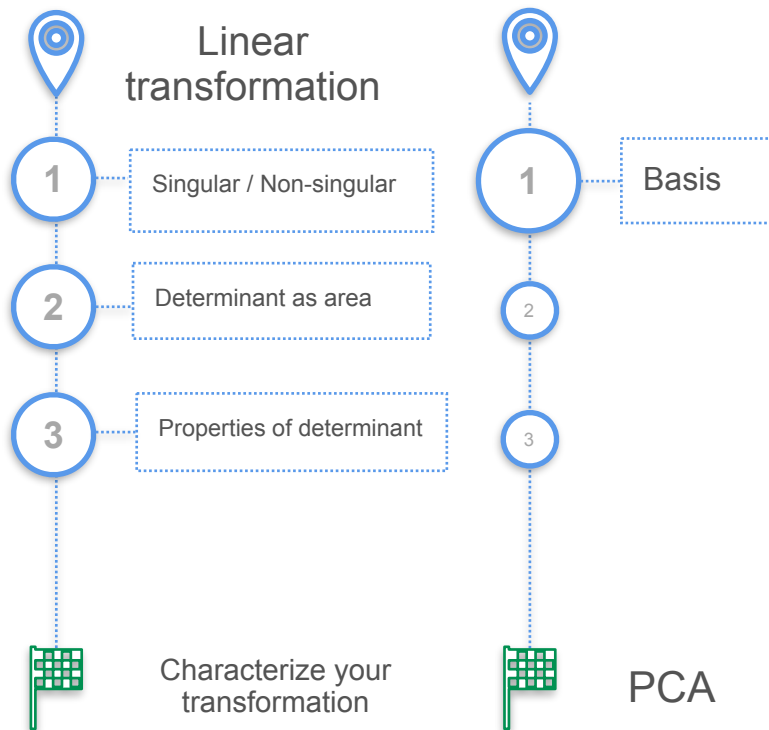
What to expect?



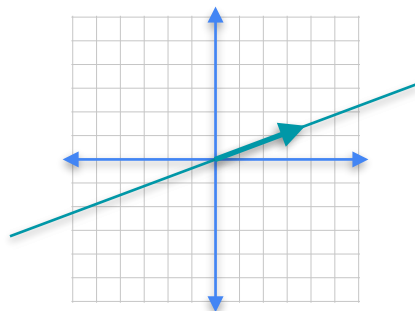
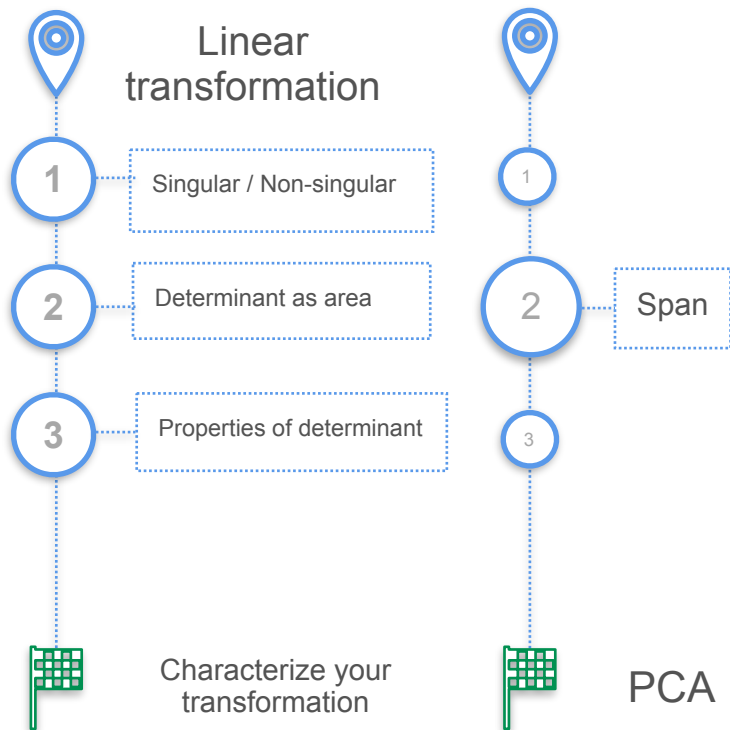
What to expect?



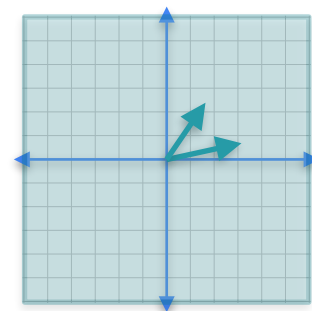
What to expect?



What to expect?

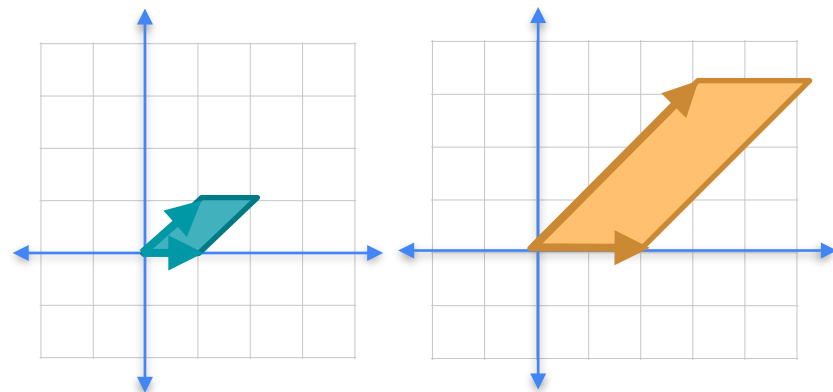
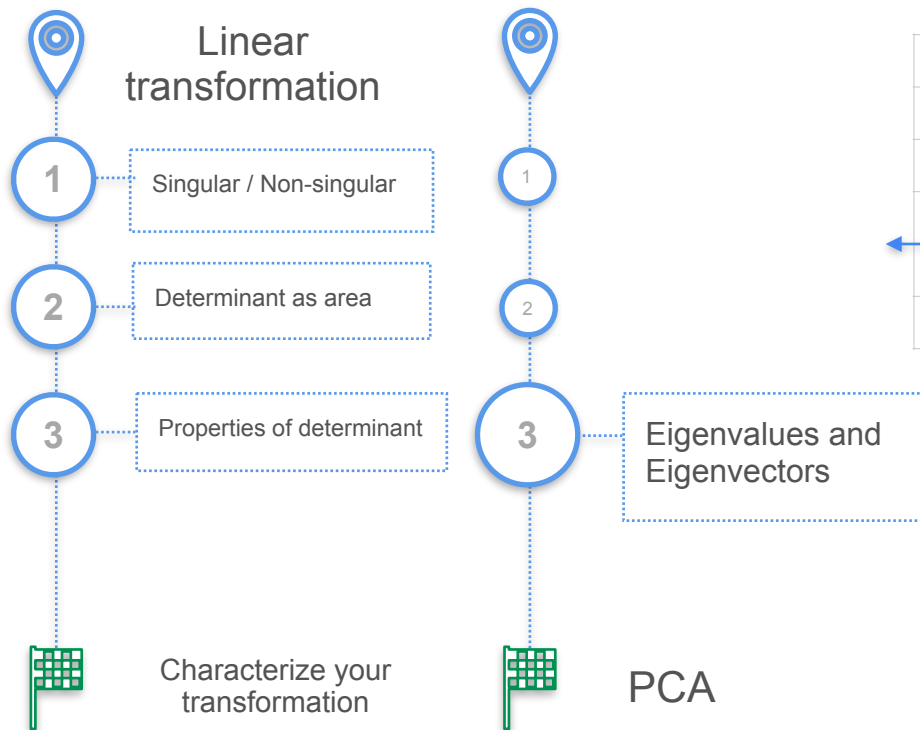


1 element
Dimensions: 1



2 elements
Dimensions: 2

What to expect?

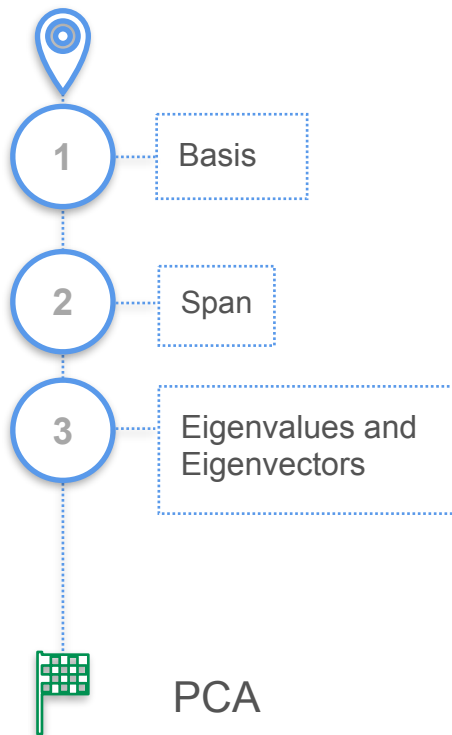
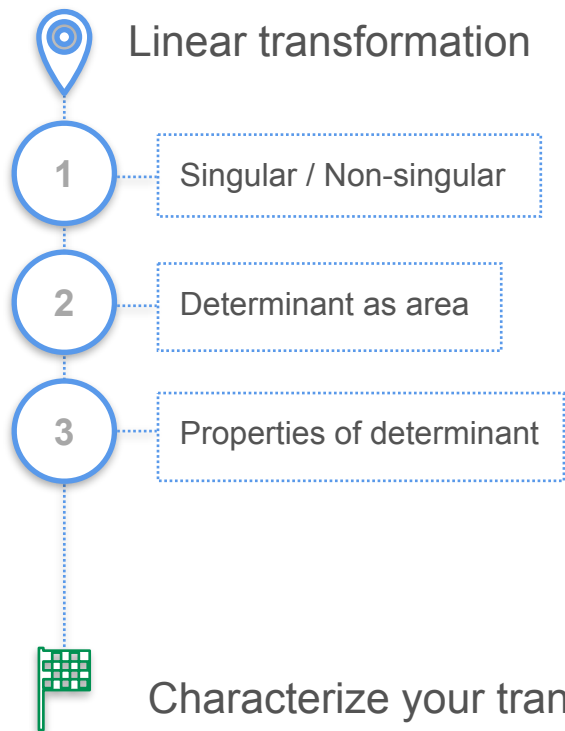


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$(1,0) \rightarrow (2,0)$$

$$A v_1 = \lambda_1 v_1$$

What to expect?





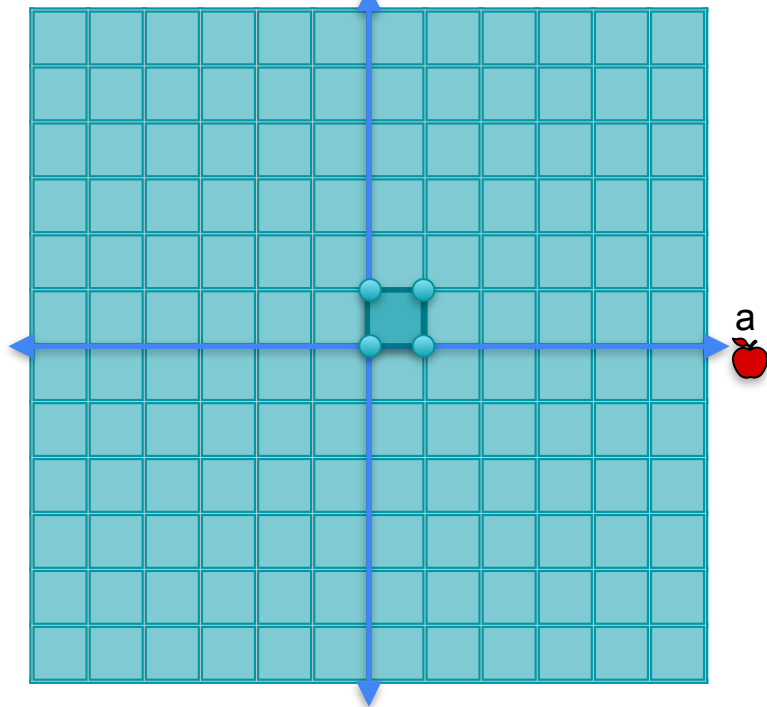
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Determinants and Eigenvectors

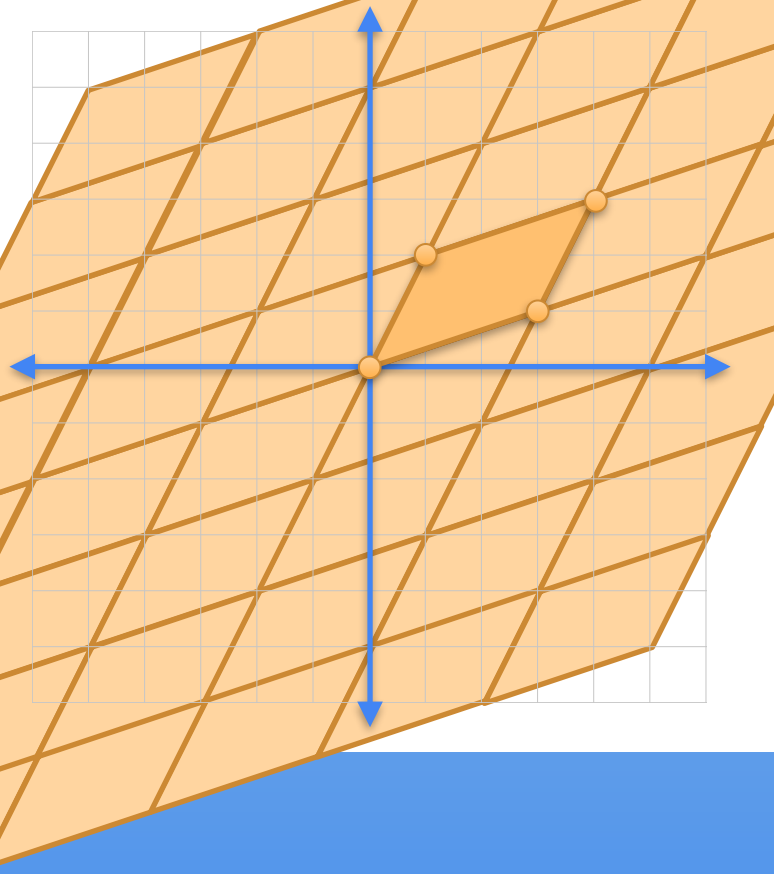
Singularity and rank of linear transformations

Non-singular transformation

b



| | |
|---|---|
| 3 | 1 |
| 1 | 2 |



Singular transformation

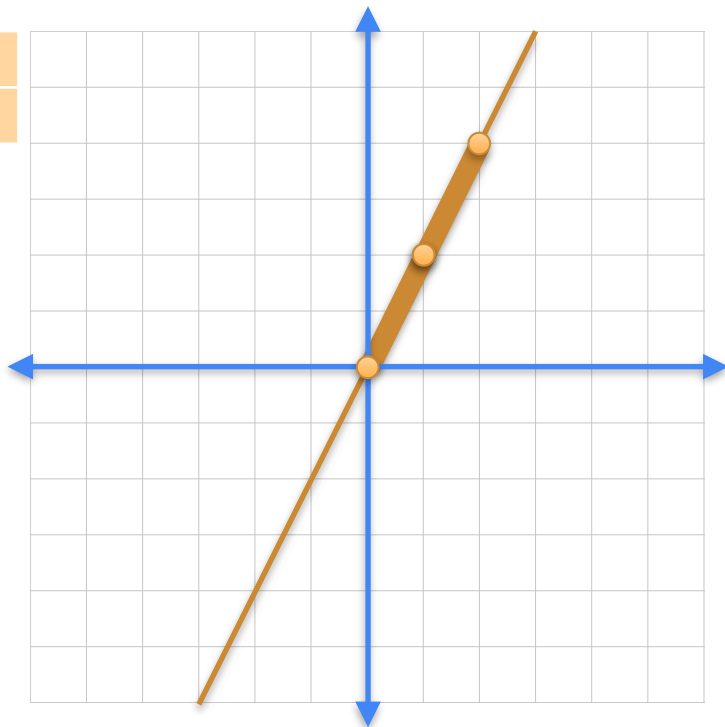
b

🍎 🍌

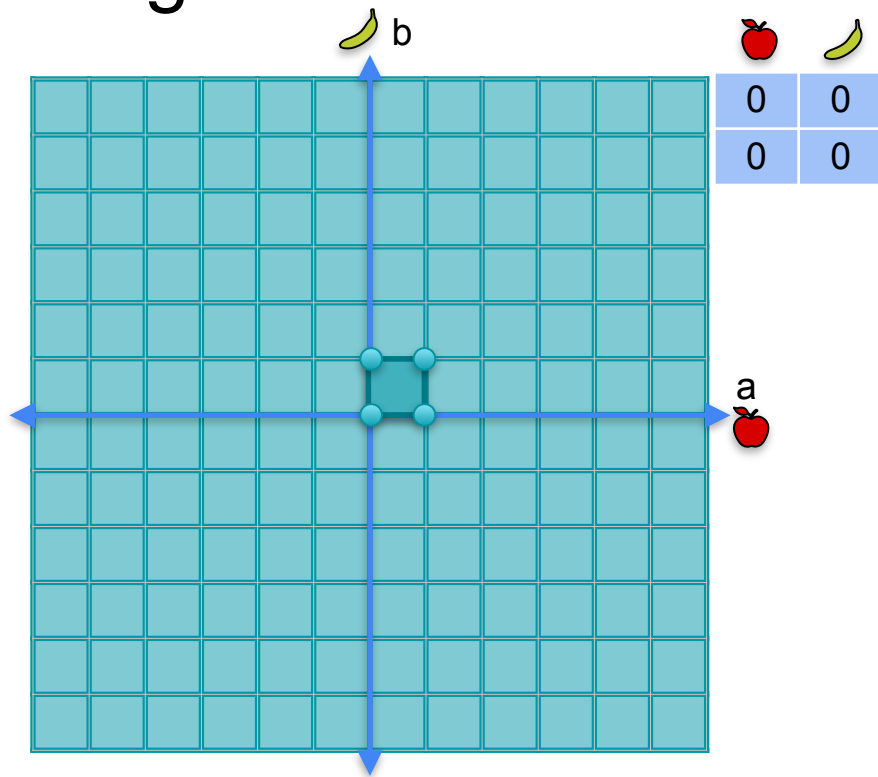
| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | = | 2 |
| 2 | 2 | 1 | | 4 |

a

(0,0) → (0,0)
(1,0) → (1,2)
(0,1) → (1,2)
(1,1) → (2,4)



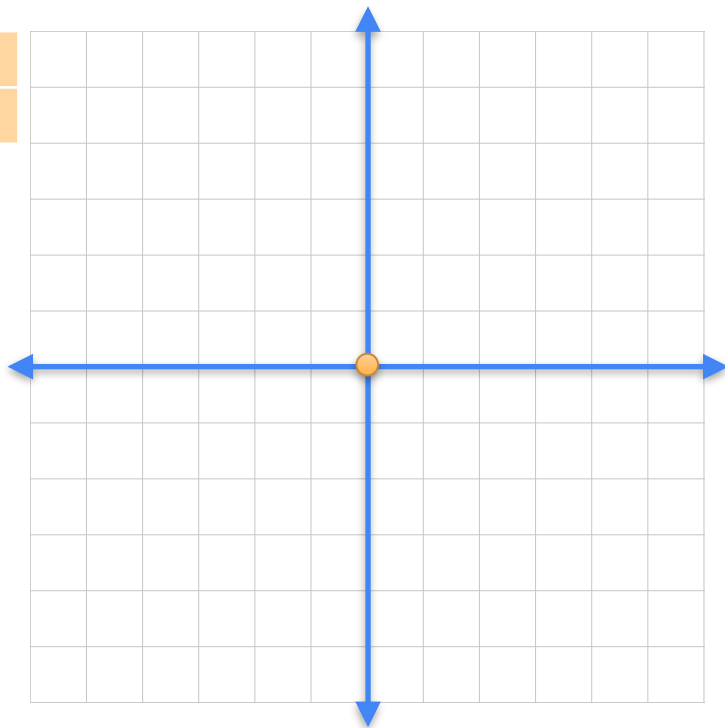
Singular transformation



| | | |
|---|---|---|
| 0 | 0 | a |
| 0 | 0 | b |



 $=$

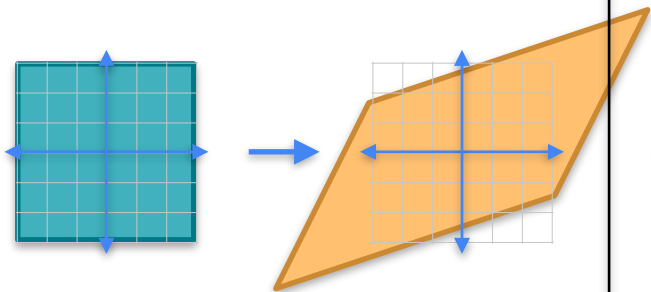
| |
|---|
| 0 |
| 0 |





Singular and non-singular transformations

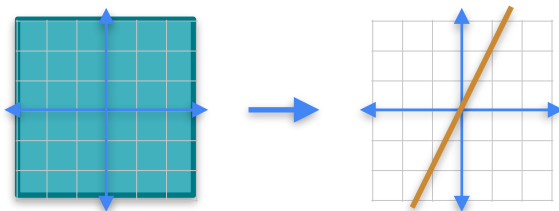
Non-singular

| | |
|---|---|
|  |  |
| 3 | 1 |
| 1 | 2 |





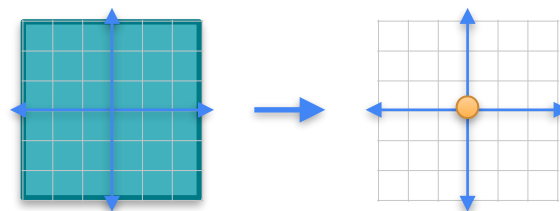
Singular

| | |
|---|--|
|  |  |
| 1 | 1 |
| 2 | 2 |





Singular

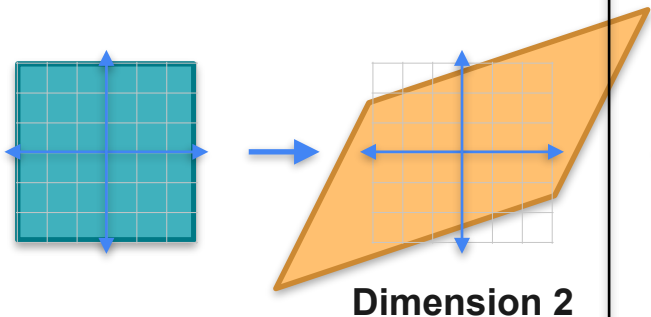
| | |
|---|---|
|  |  |
| 0 | 0 |
| 0 | 0 |





Rank of linear transformations

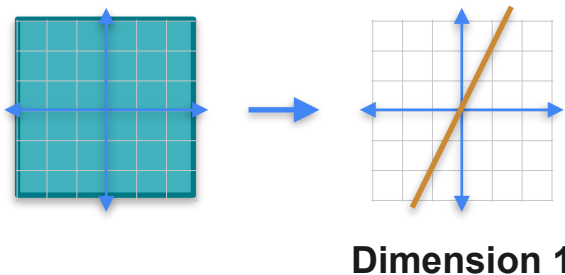
Rank 2

| | |
|---|---|
|  |  |
| 3 | 1 |
| 1 | 2 |





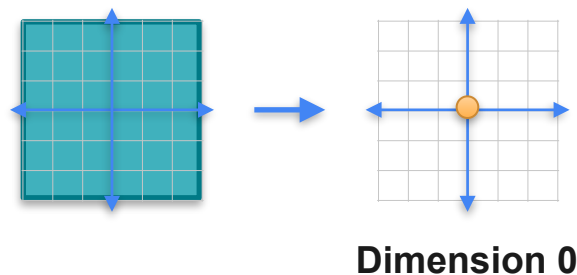
Rank 1

| | |
|---|--|
|  |  |
| 1 | 1 |
| 2 | 2 |



Rank 0

| | |
|---|---|
|  |  |
| 0 | 0 |
| 0 | 0 |



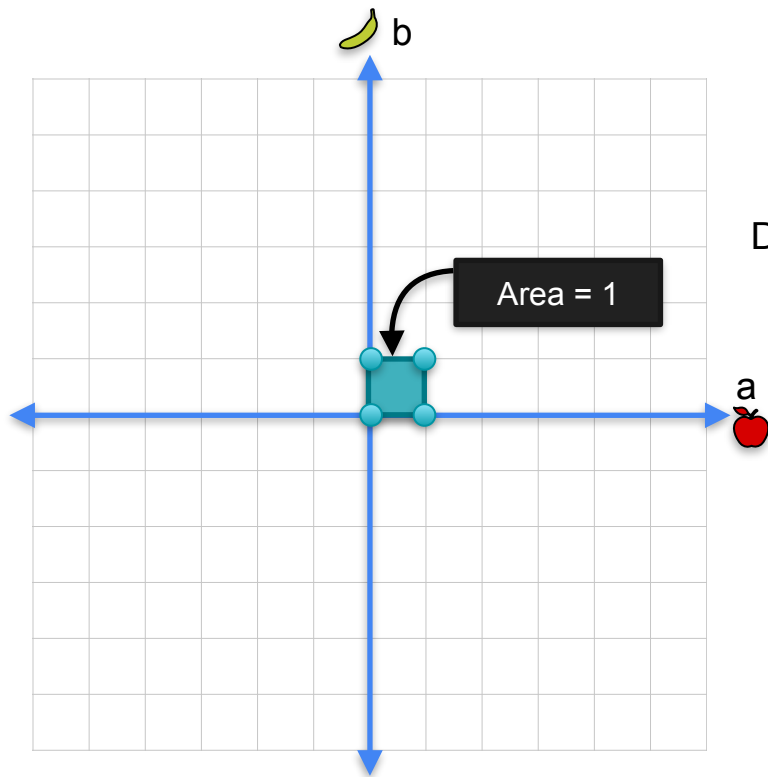




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Determinants and Eigenvectors

Determinant as an area

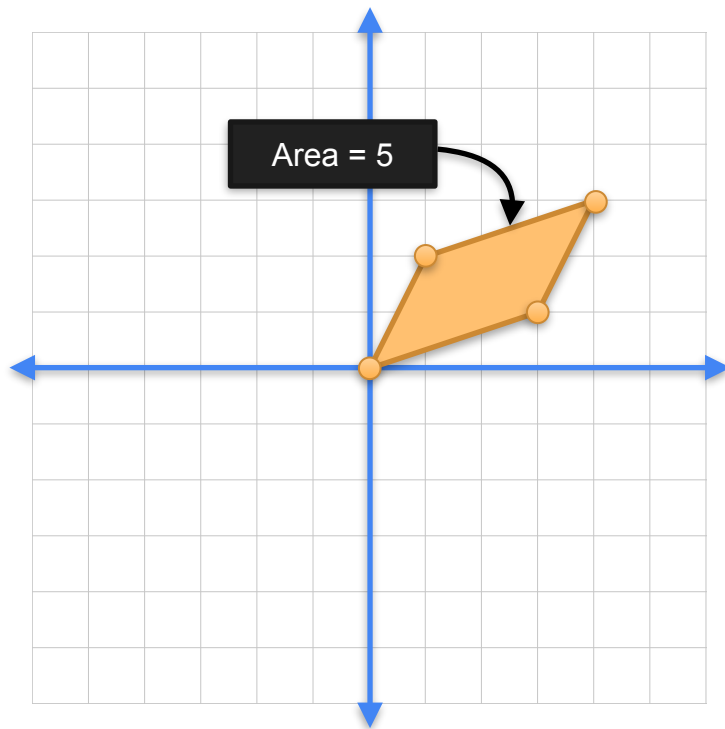
Determinant as an area



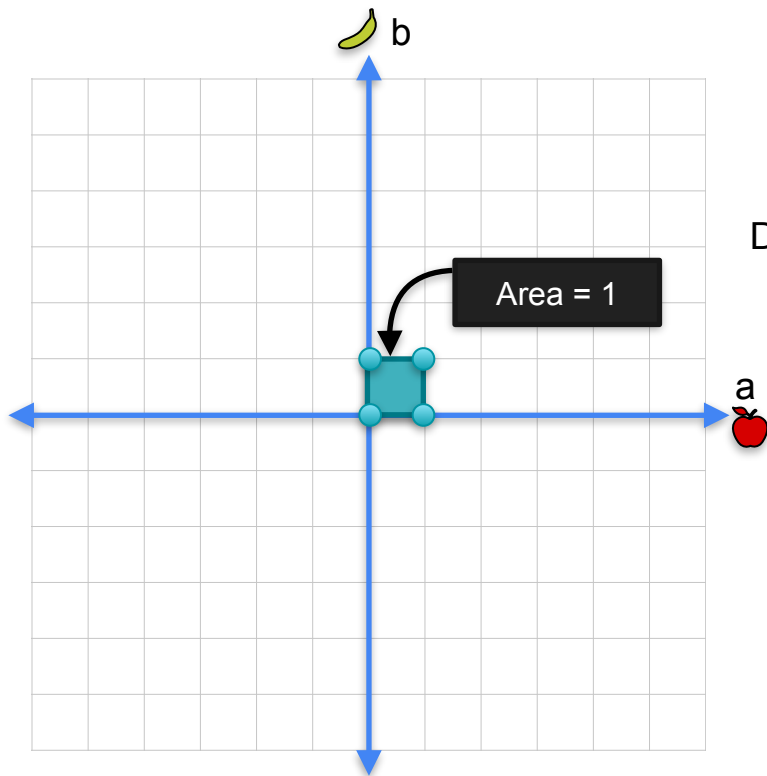
| | |
|---|--|
|  3 |  1 |
| 1 | 2 |



$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$



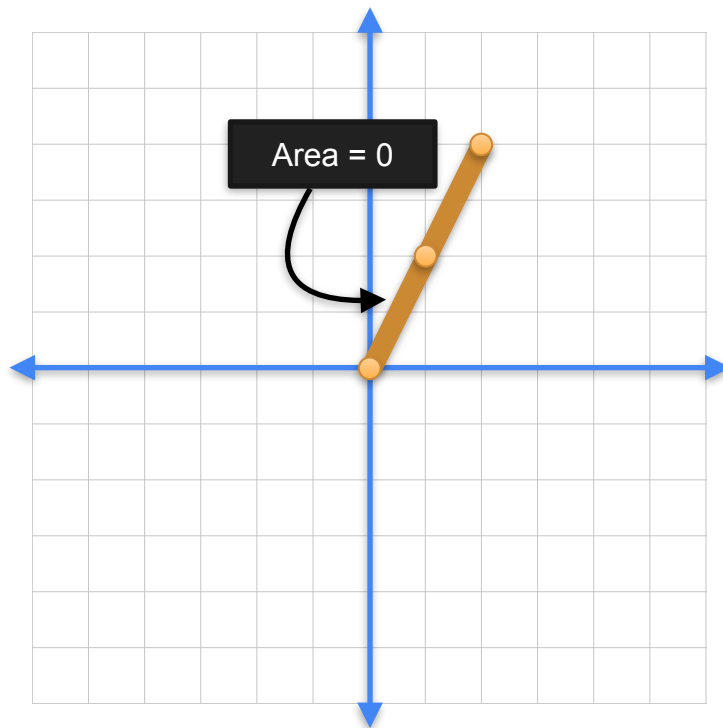
Determinant as an area



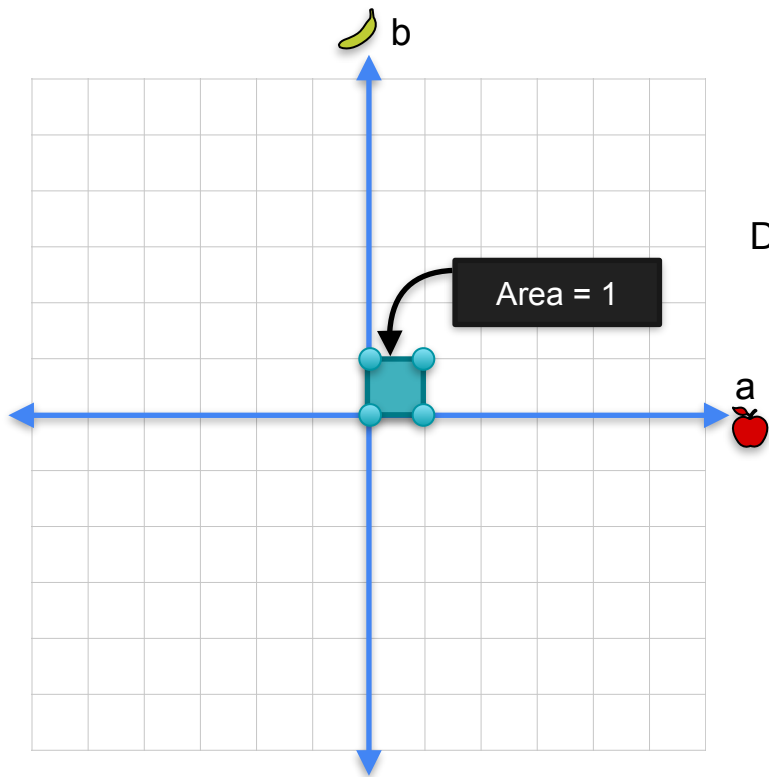
| | |
|---|--|
|  1 |  1 |
| 2 | 2 |



$$\text{Det} = 1 \cdot 2 - 1 \cdot 2$$

$$\text{Det} = 0$$



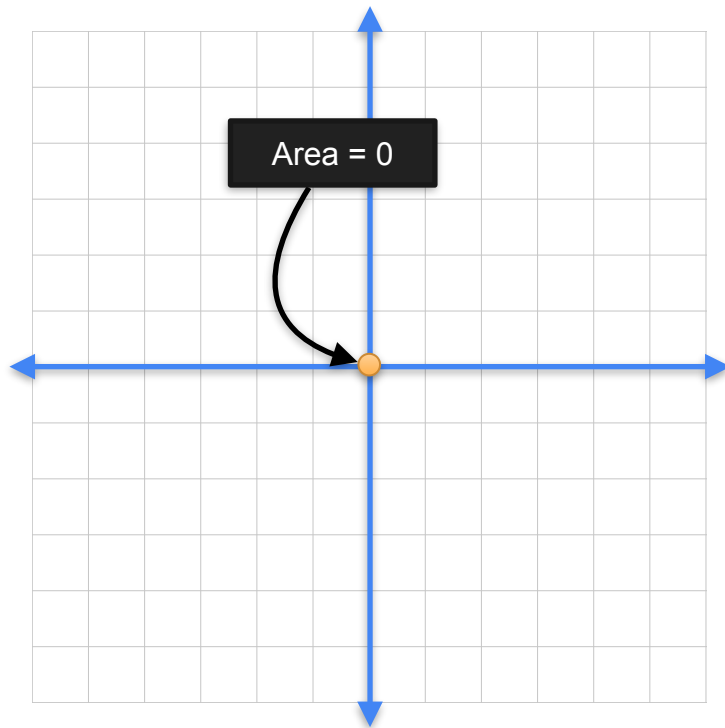
Determinant as an area



| | |
|---|--|
|  |  |
| 0 | 0 |
| 0 | 0 |



$$\text{Det} = 0 \cdot 0 - 0 \cdot 0$$

$$\text{Det} = 0$$

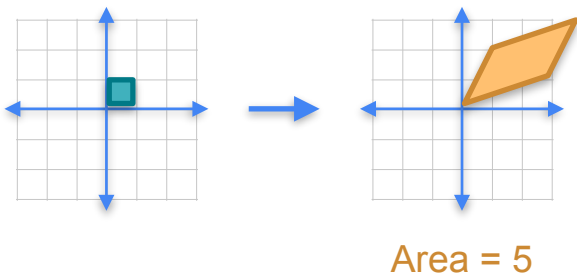


Determinant as an area



Non-singular

| | |
|---|---|
|  |  |
| 3 | 1 |
| 1 | 2 |

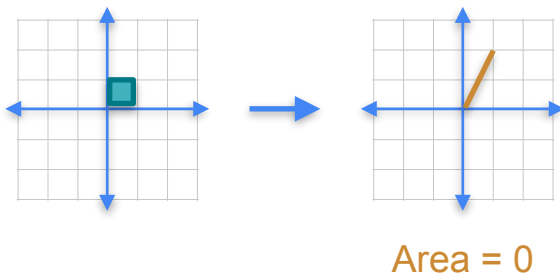
Determinant = 5





Singular

| | |
|---|--|
|  |  |
| 1 | 1 |
| 2 | 2 |

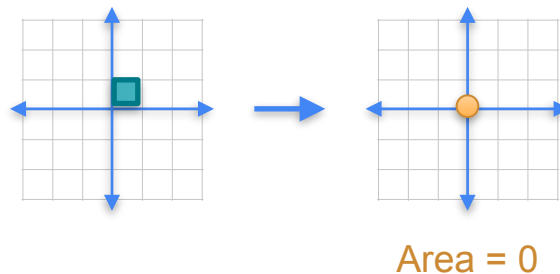
Determinant = 0





Singular

| | |
|---|---|
|  |  |
| 0 | 0 |
| 0 | 0 |

Determinant = 0





Negative determinants?



| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

$$\text{Det} = 3 \cdot 2 - 1 \cdot 1$$

$$\text{Det} = 5$$

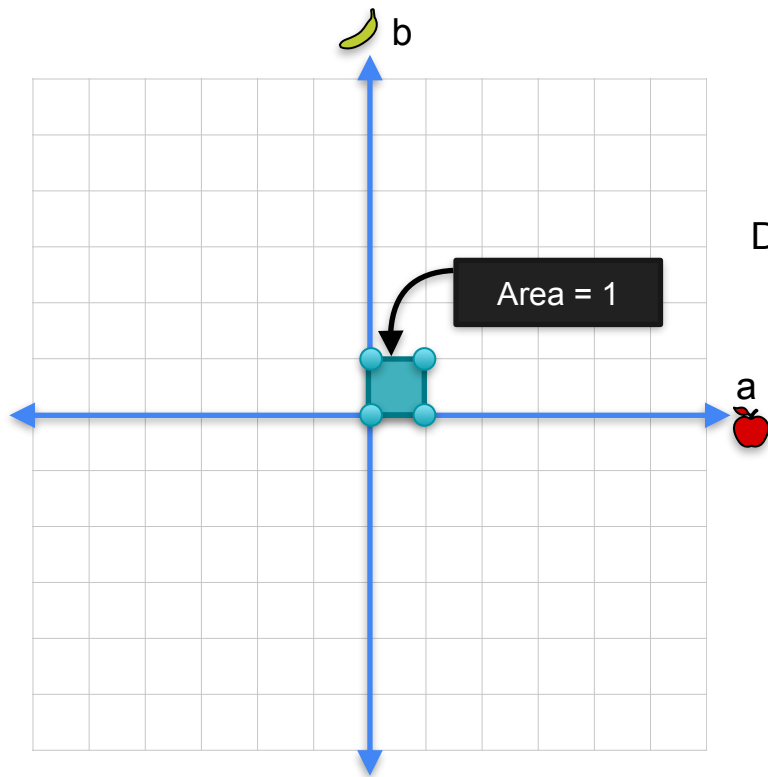




| | |
|---|---|
| 1 | 3 |
| 2 | 1 |

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

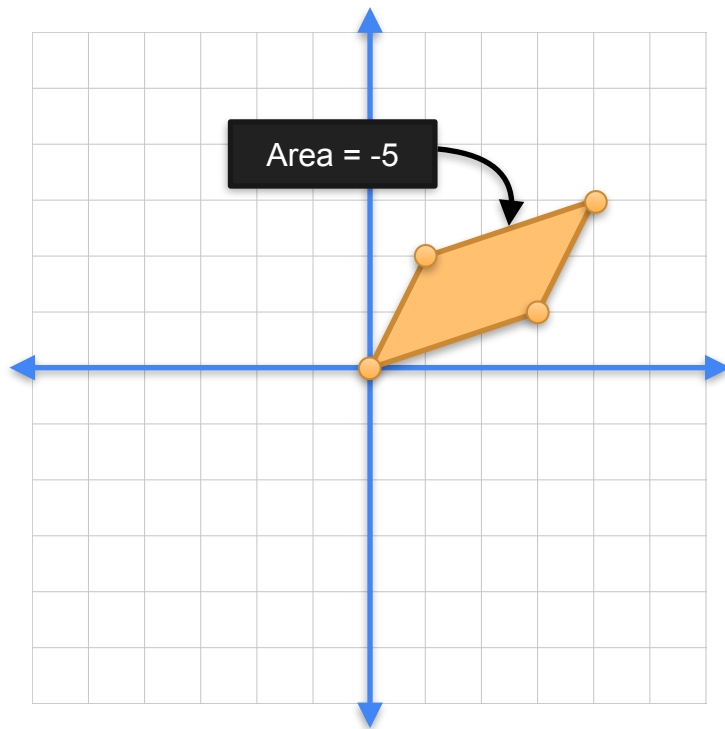
Determinant as an area



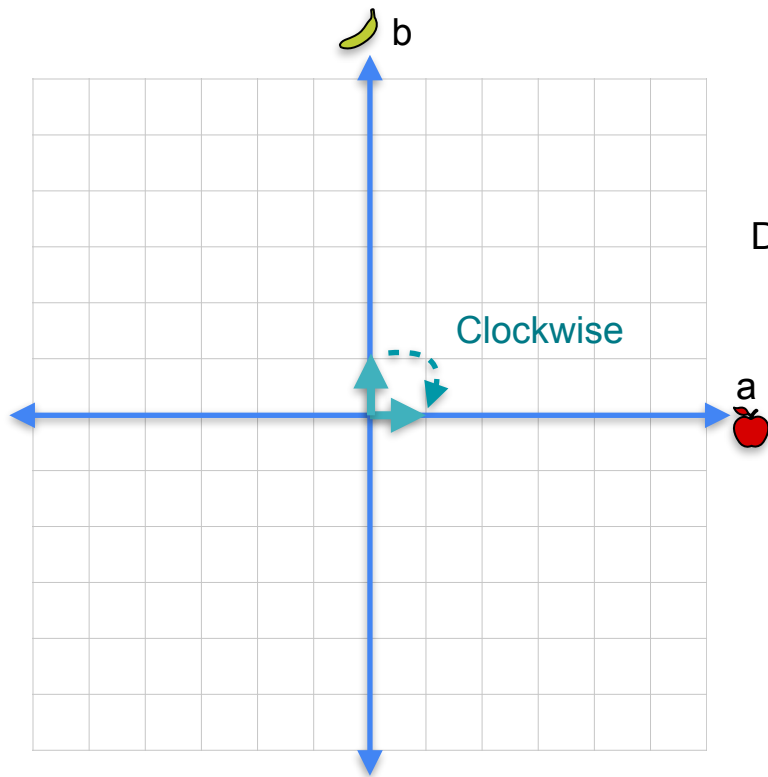
| | |
|---|--|
|  1 |  3 |
| 2 | 1 |



$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$



Determinant as an area

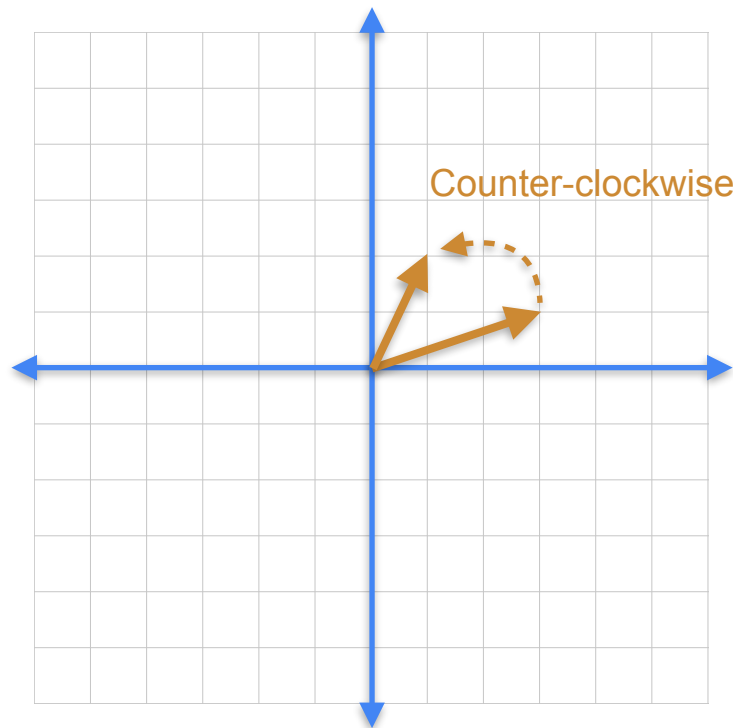


| | |
|---|--|
|  1 |  3 |
| 2 | 1 |

$$\text{Det} = 1 \cdot 1 - 3 \cdot 2$$

$$\text{Det} = -5$$

Negative





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Determinants and Eigenvectors

Determinant of a product

Determinant of a product

| | | | | | | | | | | | | | | | |
|---|---------------------------------------|---|---|---|---|---|---|---|---|---|--|----|---|---|---|
| <table><tr><td>3</td><td>1</td></tr><tr><td>1</td><td>2</td></tr></table> | 3 | 1 | 1 | 2 | <table><tr><td>5</td><td>2</td></tr><tr><td>1</td><td>2</td></tr></table> | 5 | 2 | 1 | 2 | = | <table><tr><td>16</td><td>8</td></tr><tr><td>7</td><td>6</td></tr></table> | 16 | 8 | 7 | 6 |
| 3 | 1 | | | | | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | | | | | |
| 5 | 2 | | | | | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | | | | | |
| 16 | 8 | | | | | | | | | | | | | | |
| 7 | 6 | | | | | | | | | | | | | | |
| $\det = 5$ $3 \cdot 2 - 1 \cdot 1$ | $\det = 8$ $5 \cdot 2 - 2 \cdot 1$ | | $\det = 40$ $16 \cdot 6 - 8 \cdot 7$ | | | | | | | | | | | | |

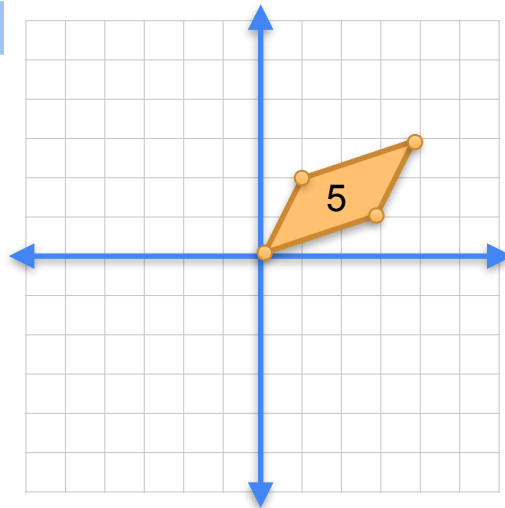
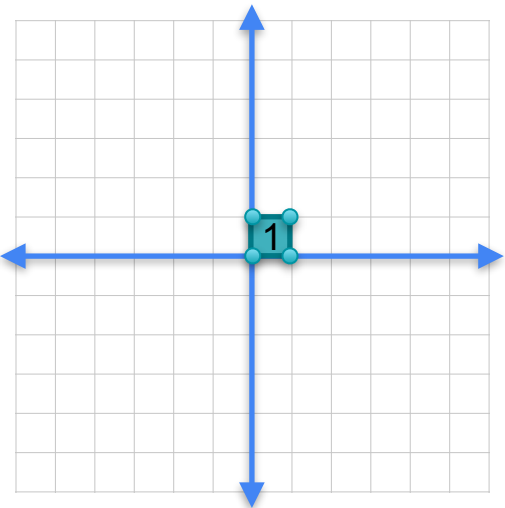
Determinant of a product

$$\det(AB) = \det(A) \det(B)$$

Determinant of a product

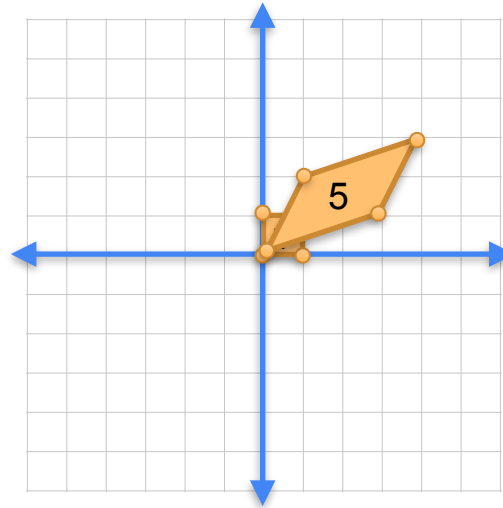
| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

Det = 5



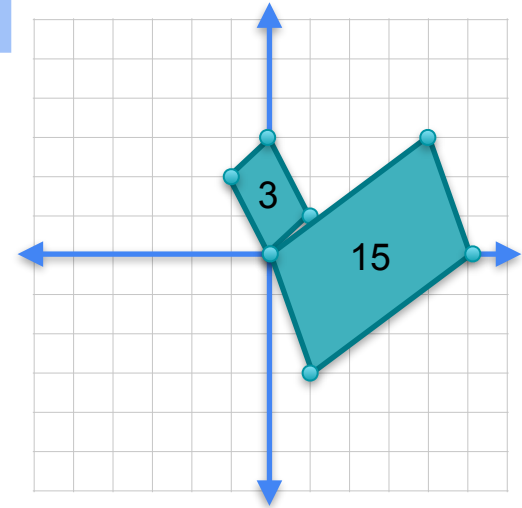
Area blows up by 5

Determinant of a product



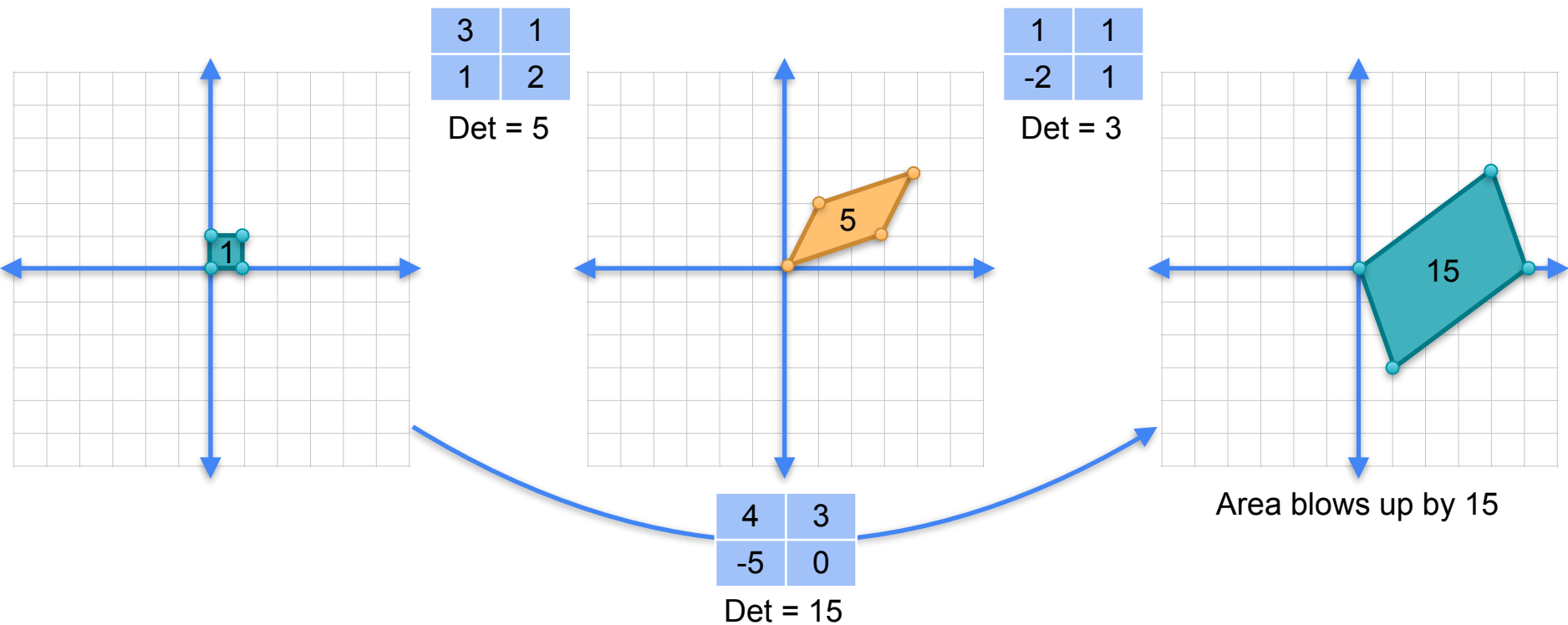
| | |
|----|---|
| 1 | 1 |
| -2 | 1 |

Det = 3



Area blows up by 3

Determinant of a product



Quiz

- The product of a singular and a non-singular matrix (in any order) is:
 - Singular
 - Non-singular
 - Could be either one

Solution

- If A is non-singular and B is singular, then $\det(AB) = \det(A) \times \det(B) = 0$, since $\det(B) = 0$. Therefore $\det(AB) = 0$, so AB is **singular**.

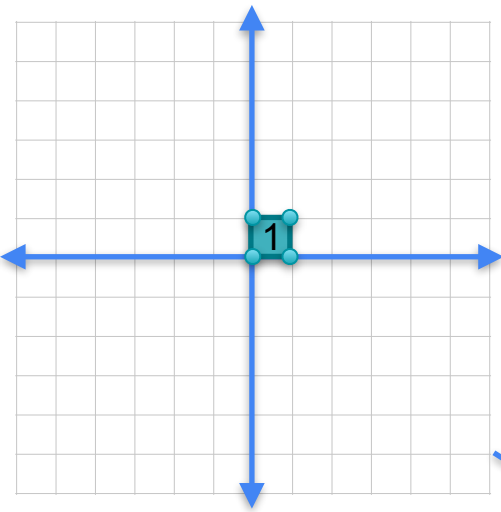
When one factor is zero

$$5 \cdot 0 = 0$$

When one factor is singular...

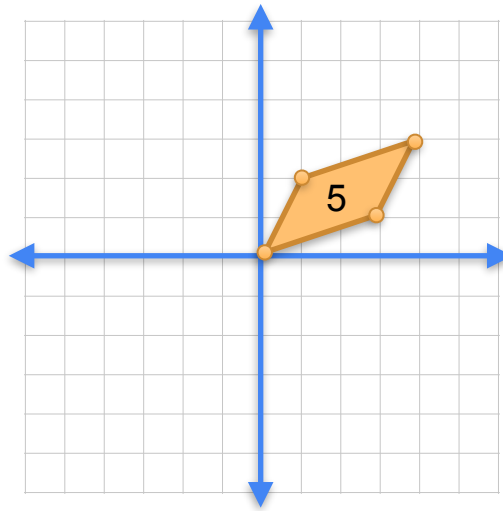
| Non-singular | | Singular | | Singular | | | | | | | | | | | | |
|---|---|----------|---|----------|--|---|---|---|---|---|---|---|---|---|---|---|
| <table><tr><td>3</td><td>1</td></tr><tr><td>1</td><td>2</td></tr></table> | 3 | 1 | 1 | 2 | | <table><tr><td>1</td><td>2</td></tr><tr><td>1</td><td>2</td></tr></table> | 1 | 2 | 1 | 2 | = | <table><tr><td>4</td><td>8</td></tr><tr><td>3</td><td>6</td></tr></table> | 4 | 8 | 3 | 6 |
| 3 | 1 | | | | | | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | | | | | | |
| 1 | 2 | | | | | | | | | | | | | | | |
| 4 | 8 | | | | | | | | | | | | | | | |
| 3 | 6 | | | | | | | | | | | | | | | |
| Det = 5 | | Det = 0 | | Det = 0 | | | | | | | | | | | | |

If one factor is singular...



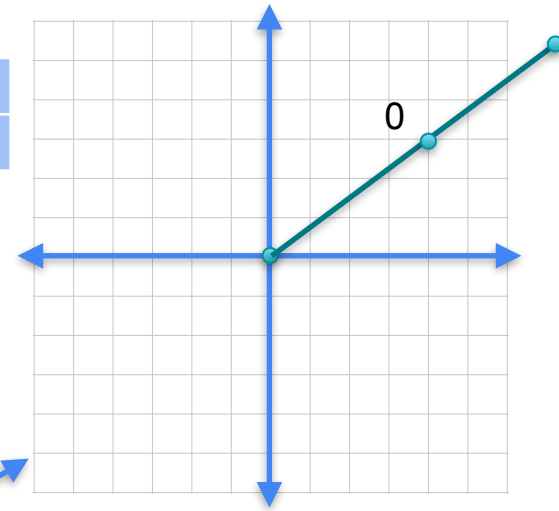
| | |
|---|---|
| 3 | 1 |
| 1 | 2 |

Det = 5



| | |
|---|---|
| 1 | 2 |
| 1 | 2 |

Det = 0



Area blows up by 0

| | |
|---|---|
| 4 | 8 |
| 3 | 6 |

Det = 0



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Determinants and Eigenvectors

Determinant of inverse

Quiz

- Find the determinants of the following matrices

| | |
|------|------|
| 0.4 | -0.2 |
| -0.2 | 0.6 |

| | |
|--------|-------|
| 0.25 | -0.25 |
| -0.125 | 0.625 |

Solution

$$\text{Det} \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix} = (0.4)(0.6) - (-0.2)(-0.2) = 0.2$$

$$\text{Det} \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix} = (0.25)(0.625) - (-0.125)(-0.25) = 0.125$$

Determinant of an inverse

$$\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.4 & -0.2 \\ -0.2 & 0.6 \end{bmatrix}$$

det = 5

det = 0.2

$$5^{-1} = 0.2$$

$$\begin{bmatrix} 5 & 2 \\ 1 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 0.25 & -0.25 \\ -0.125 & 0.625 \end{bmatrix}$$

det = 8

det = 0.125

$$8^{-1} = 0.125$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} ? & ? \\ ? & ? \end{bmatrix}$$

det = 0

det = ???

$$0^{-1} = ???$$

Determinant of an inverse

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Why?

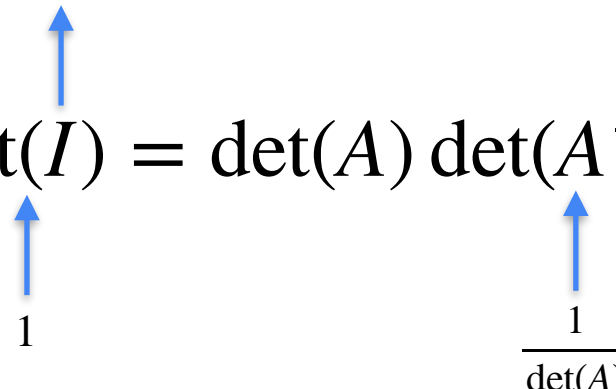
Why is this?

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

$$\det(AB) = \det(A) \det(B)$$

$$\det(AA^{-1}) = \det(A) \det(A^{-1})$$

$$\det(I) = \det(A) \det(A^{-1})$$



1 $\frac{1}{\det(A)}$

Determinant of the identity matrix

$$\det \begin{array}{|c|c|} \hline 1 & 0 \\ \hline 0 & 1 \\ \hline \end{array} = 1 \cdot 1 - 0 \cdot 0 = 1$$

$$\det(I) = 1$$

W4 Lesson 2

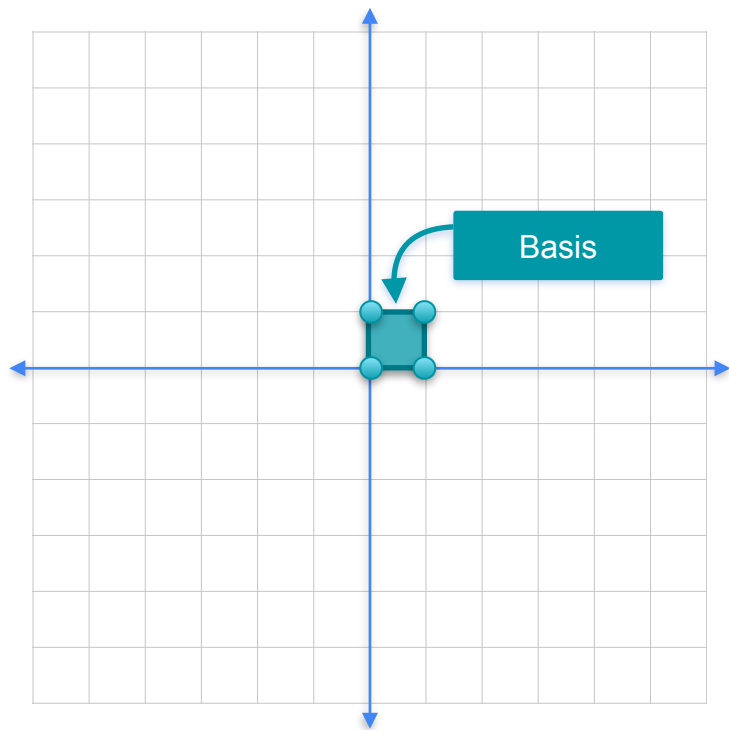


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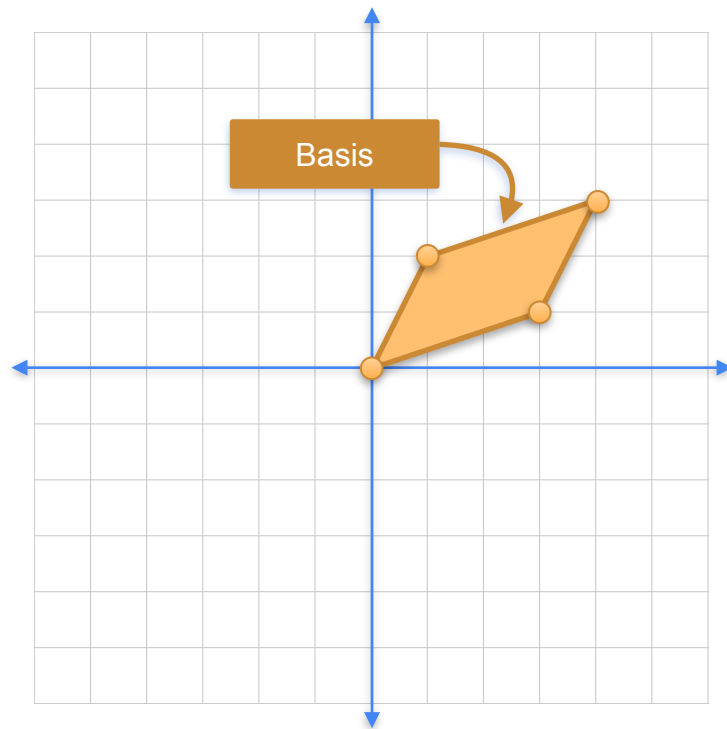
Determinants and Eigenvectors

Bases

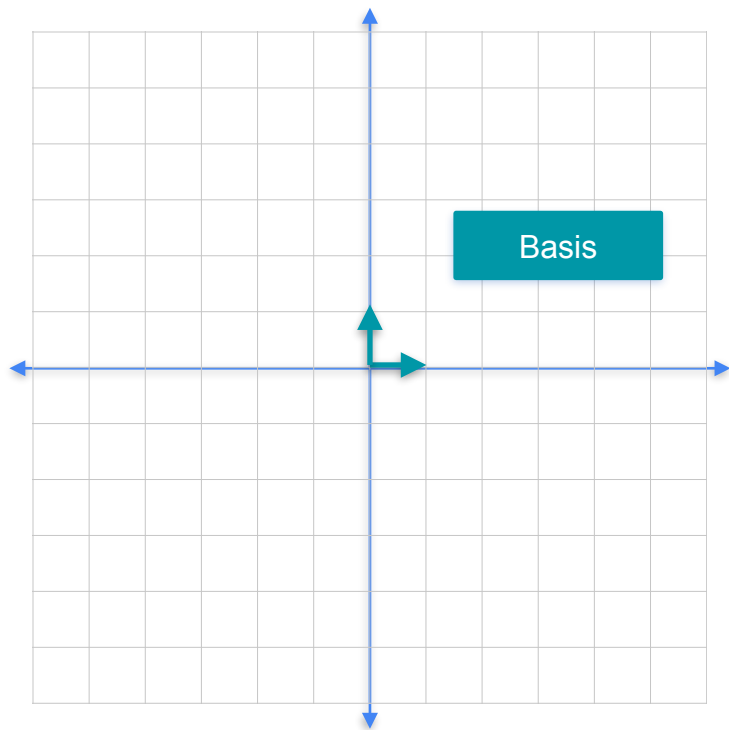
Bases



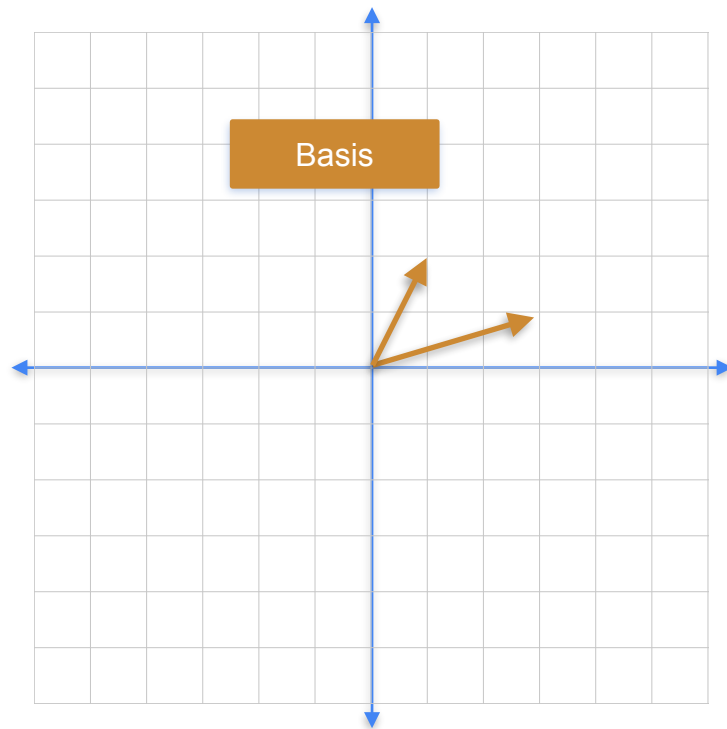
| | |
|---|---|
| 3 | 1 |
| 1 | 2 |



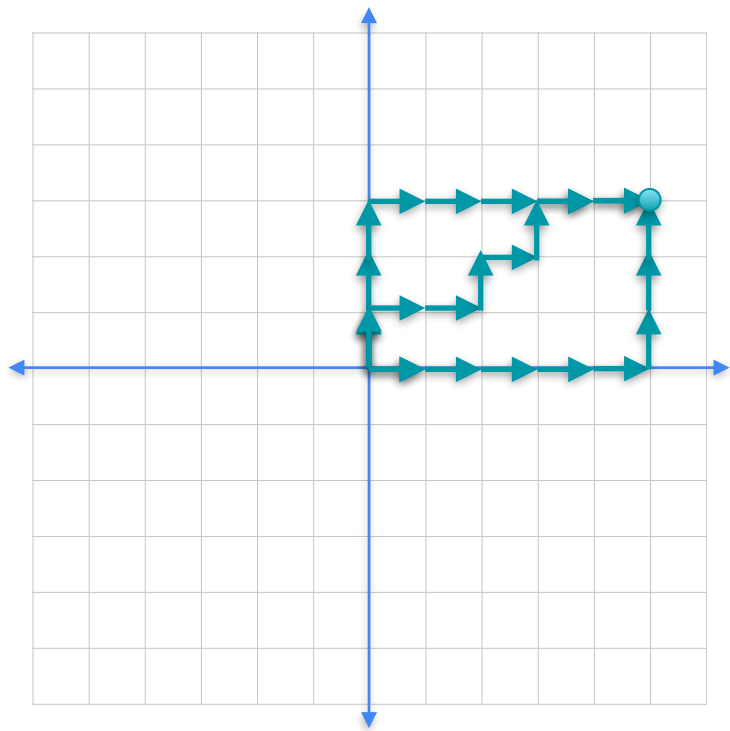
Bases



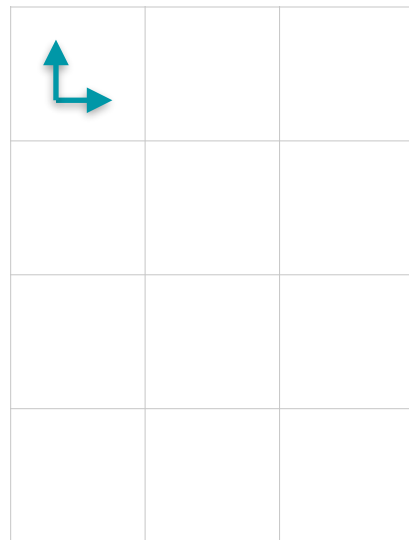
| | |
|---|---|
| 3 | 1 |
| 1 | 2 |



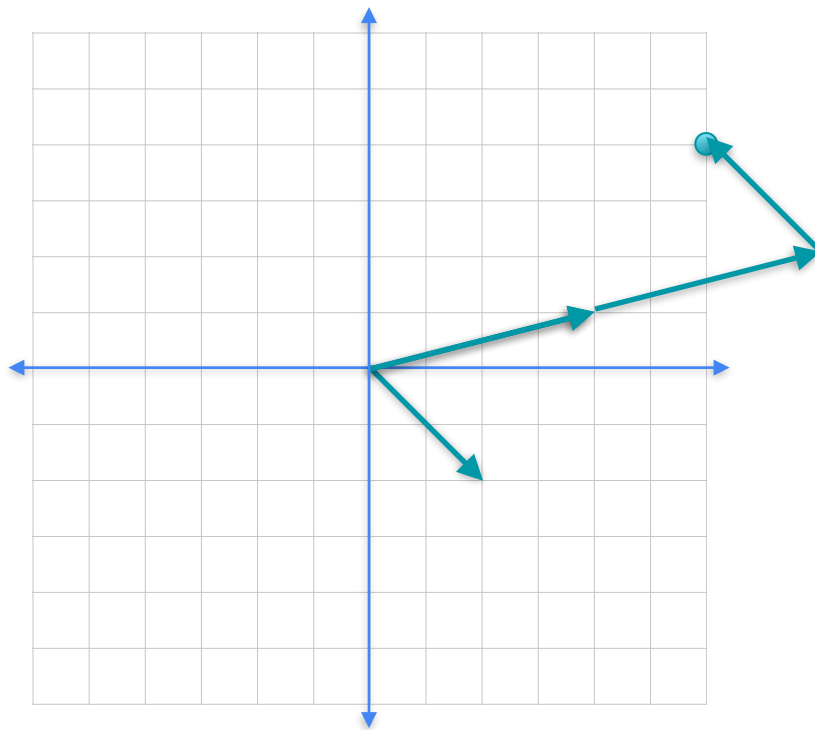
Bases



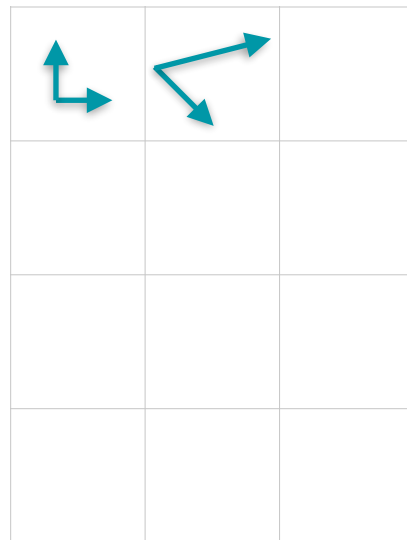
Bases



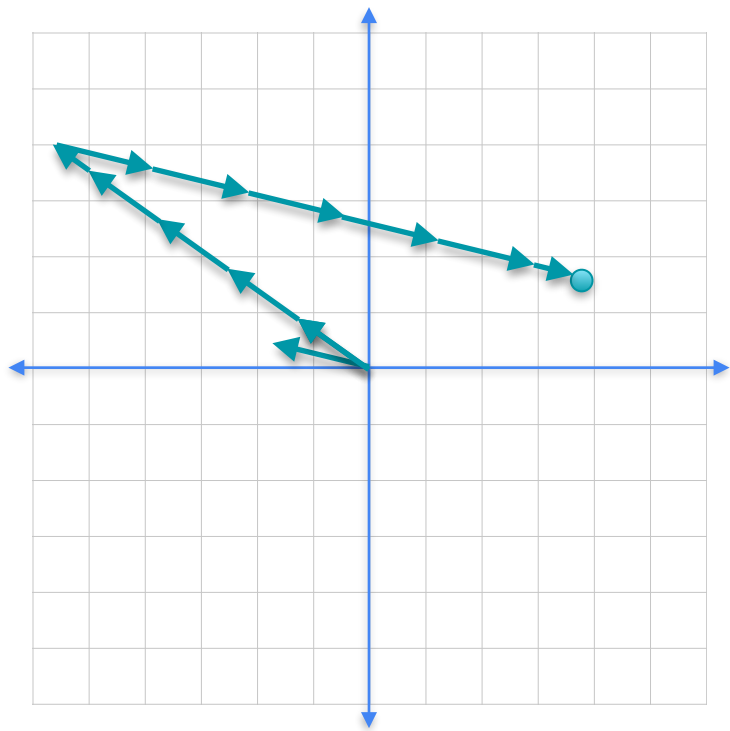
Bases



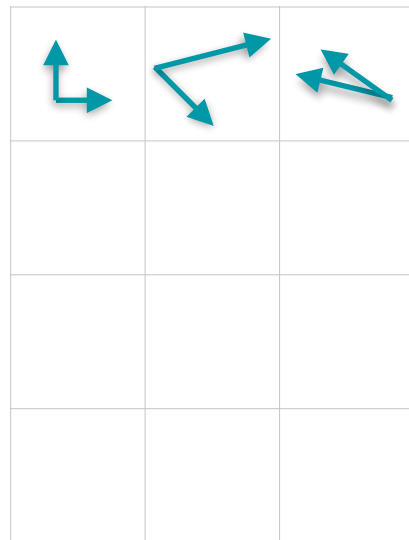
Bases



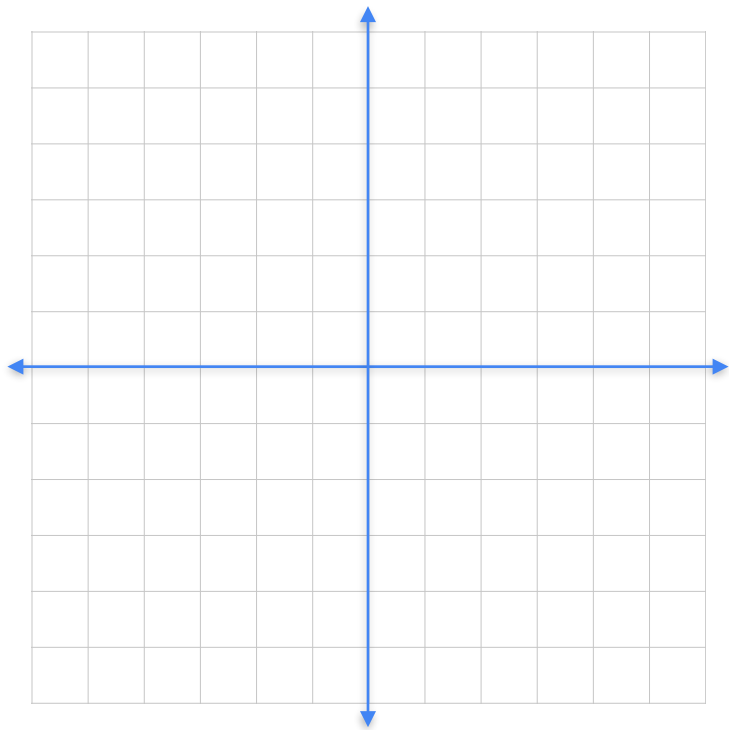
Bases



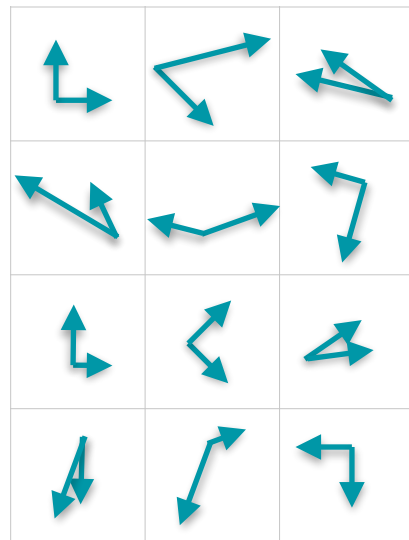
Bases



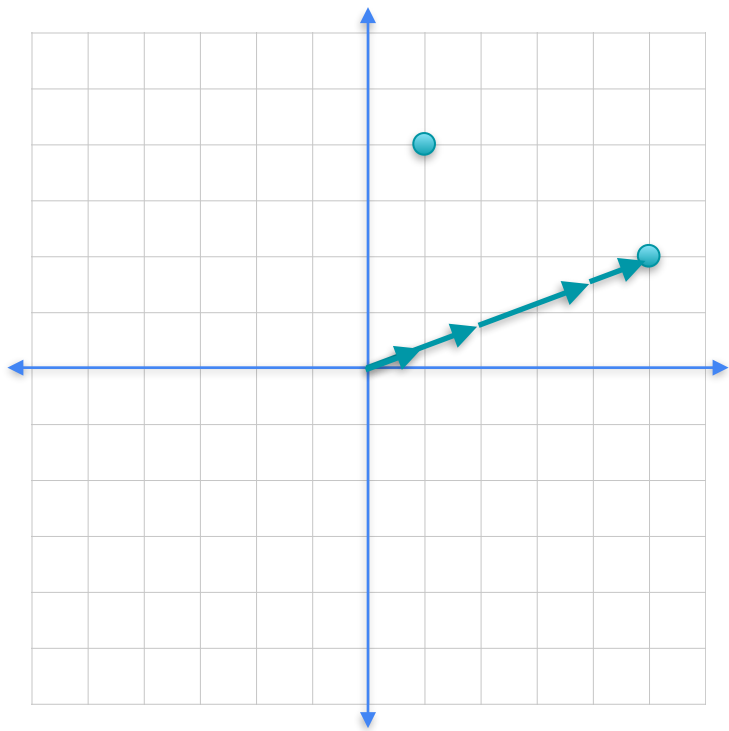
Bases



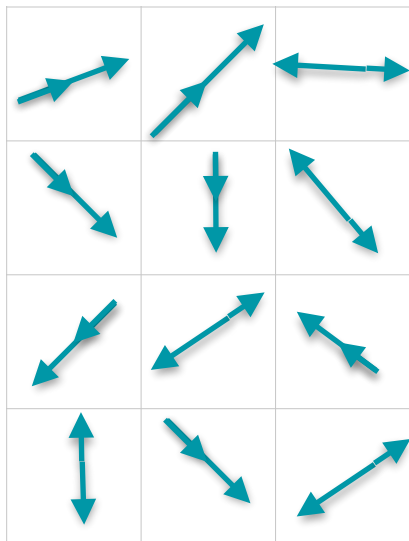
Bases



What is not a basis?



Not bases



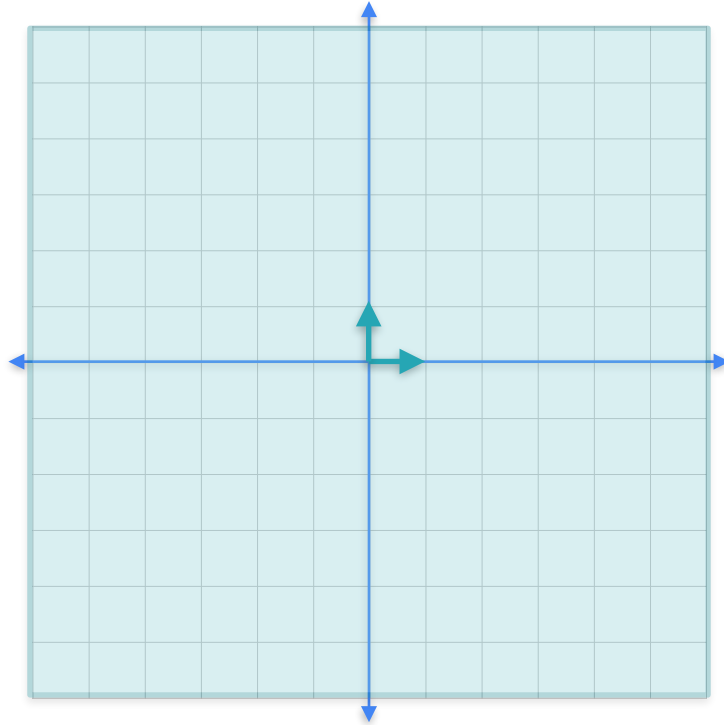


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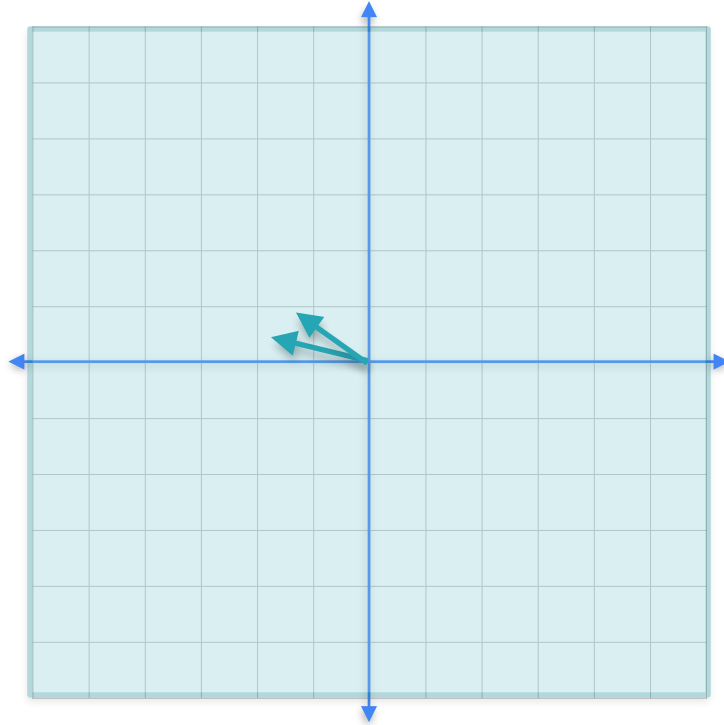
Determinants and Eigenvectors

Span

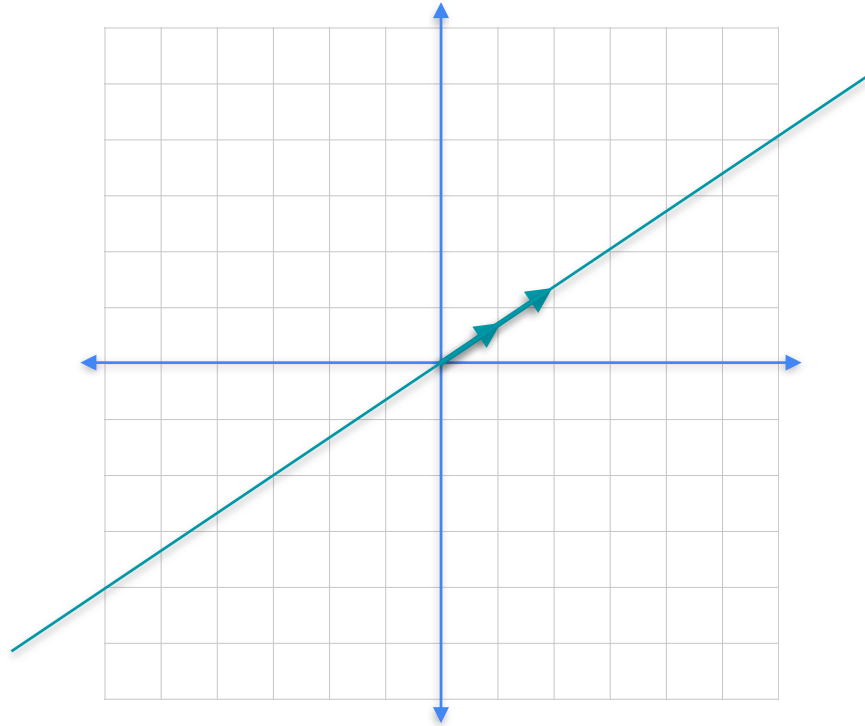
Span



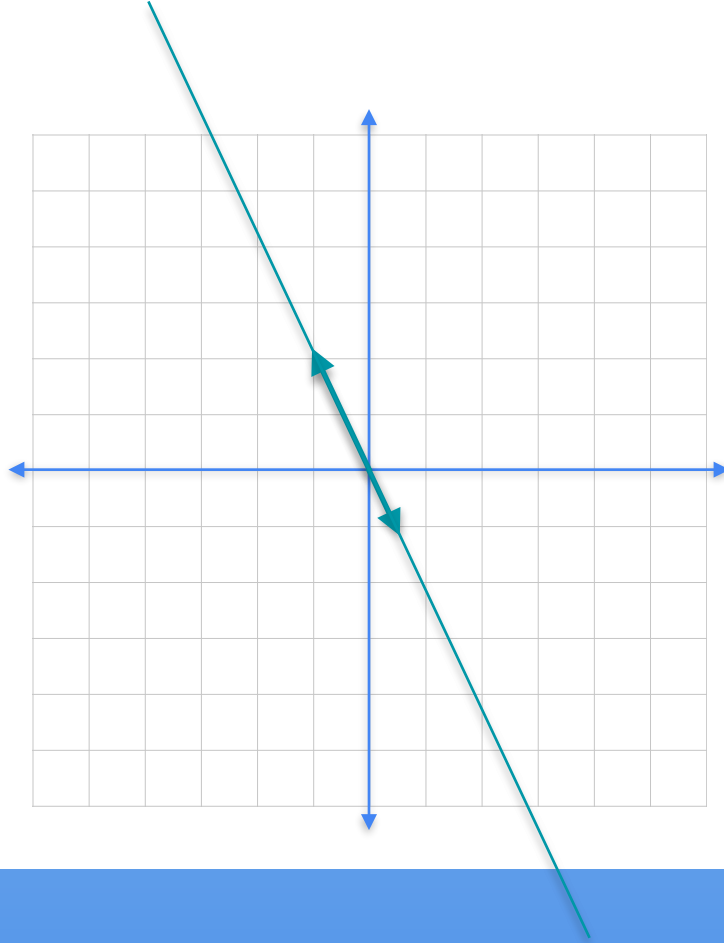
Span



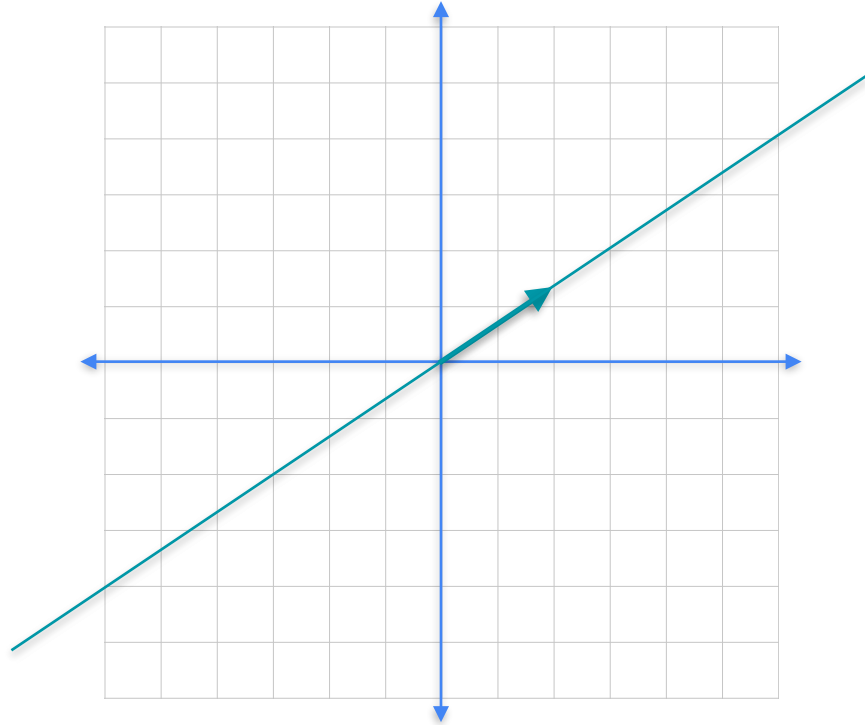
Span



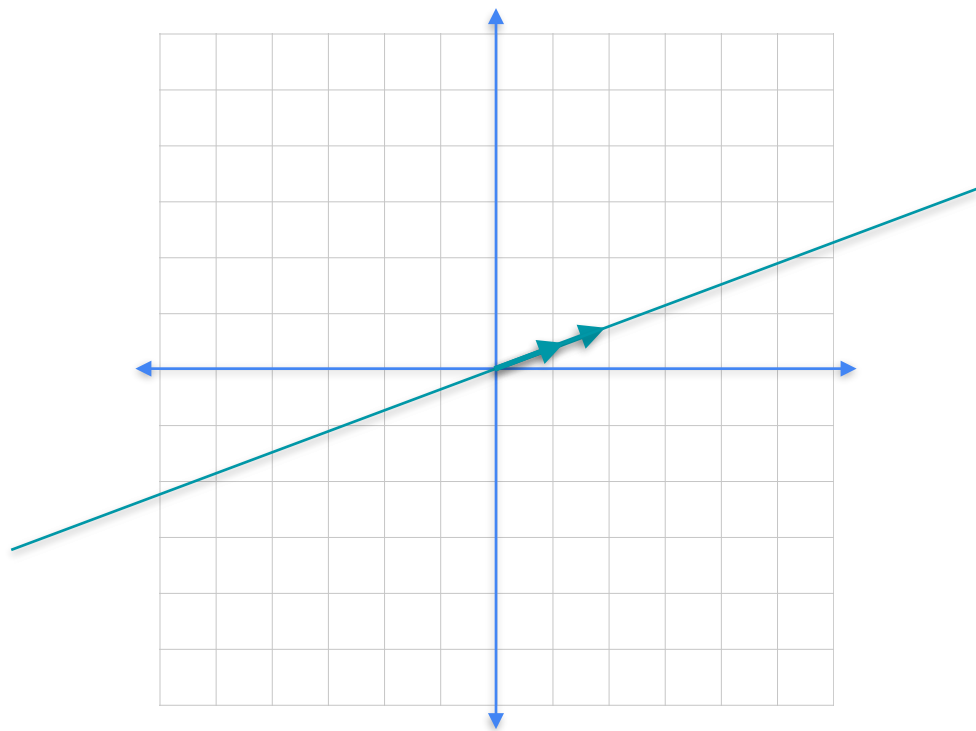
Span



Span

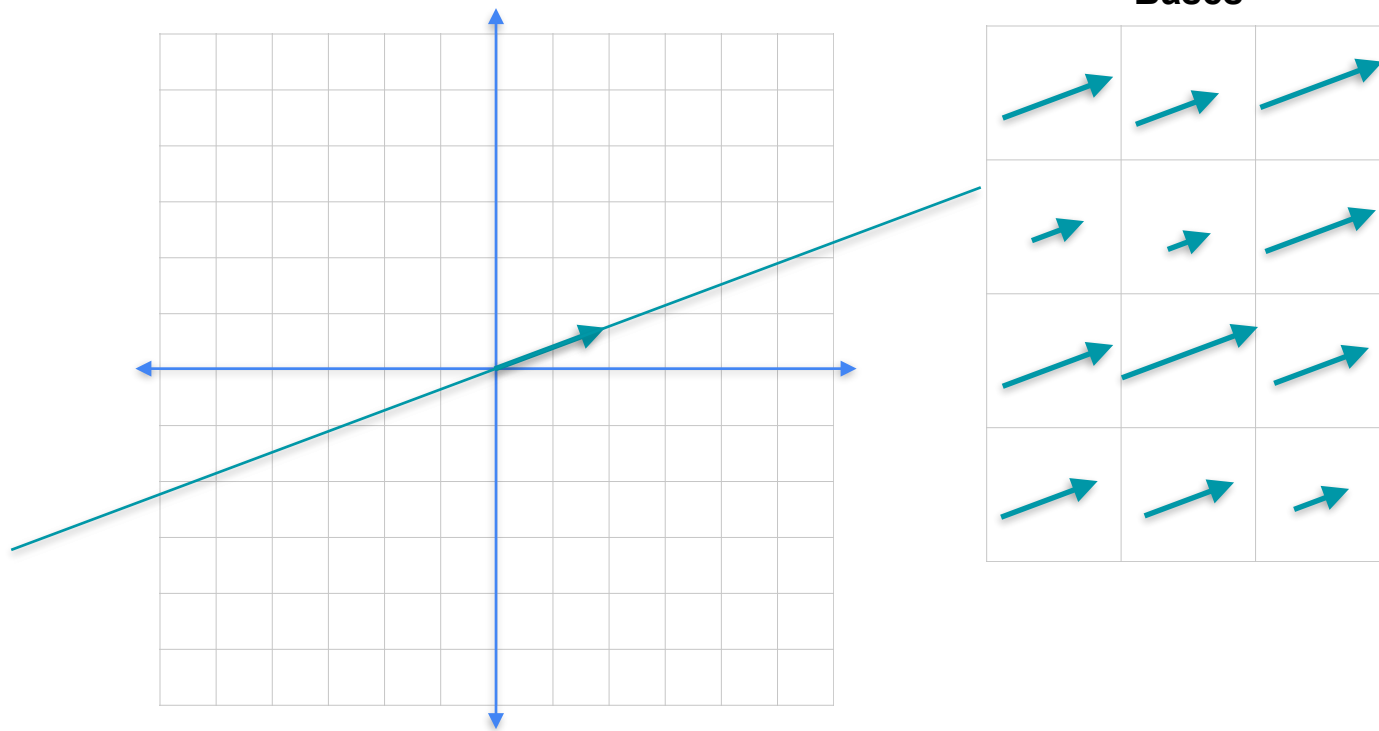


Is this a basis?

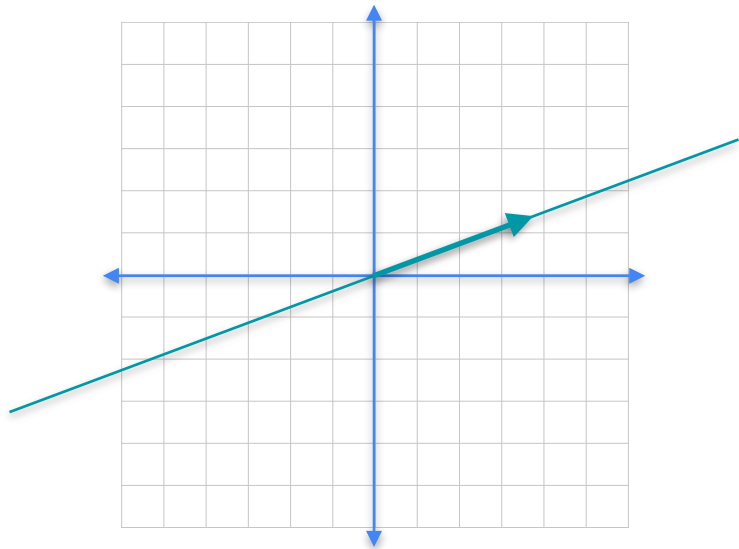


No

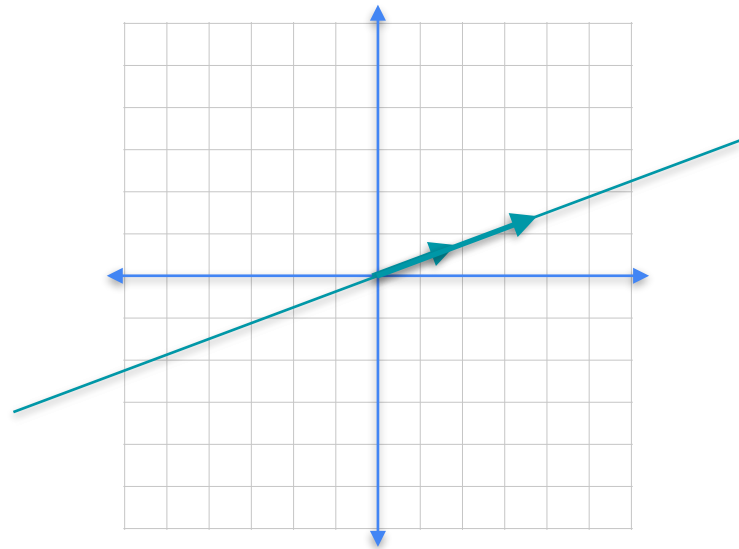
Is this a basis for something?



A basis is a minimal spanning set

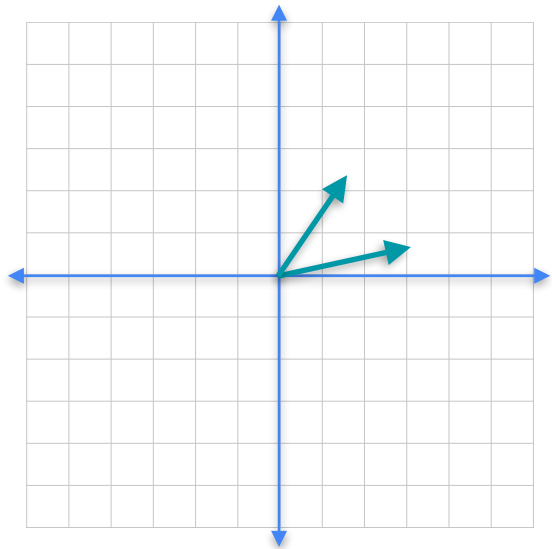


Basis

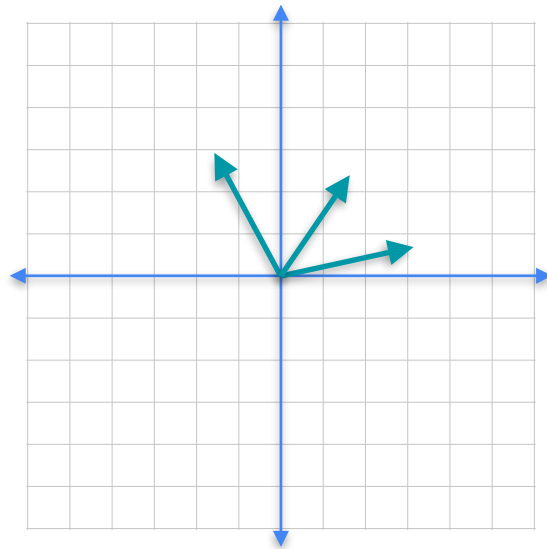


Not a basis

A basis is a minimal spanning set

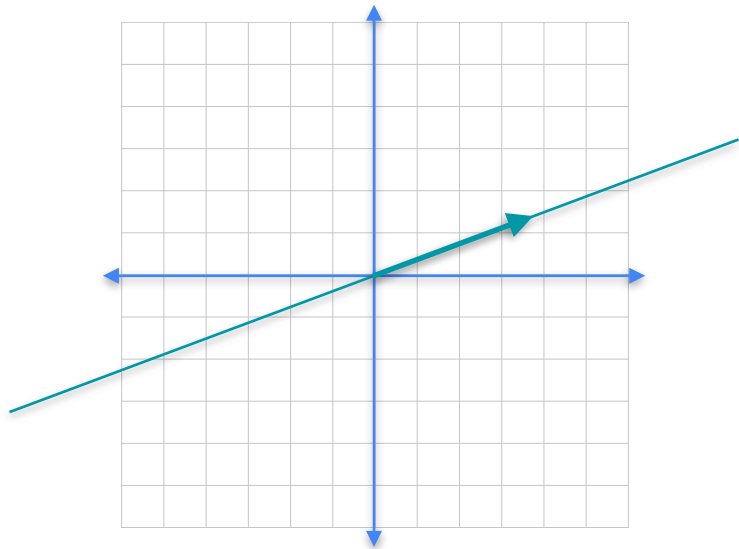


Basis

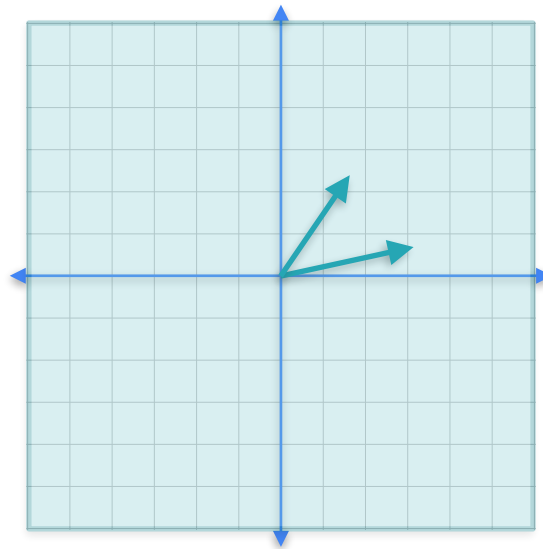


Not a basis

Number of elements in the basis is the dimension

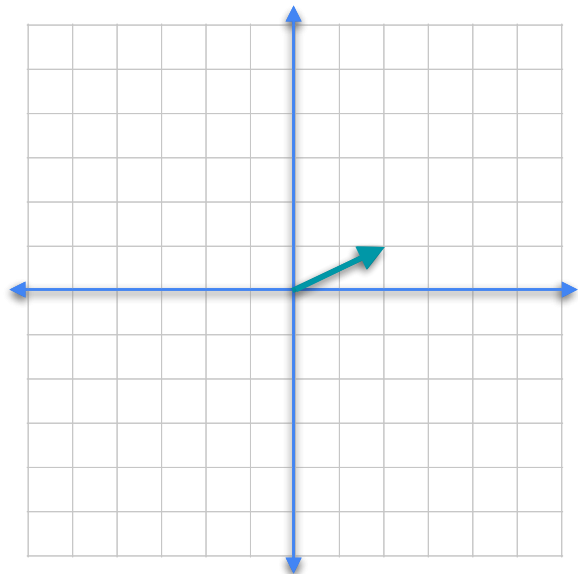


Dimensions: 1
1 element in the basis



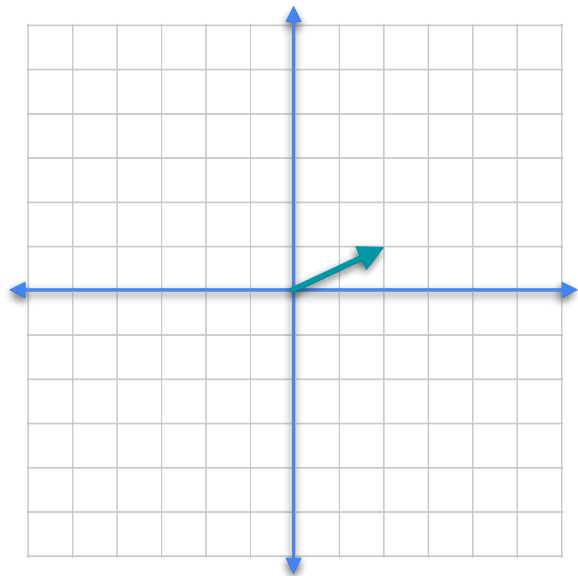
Dimensions: 2
2 elements in the basis

Linearly independent and linearly dependent vectors

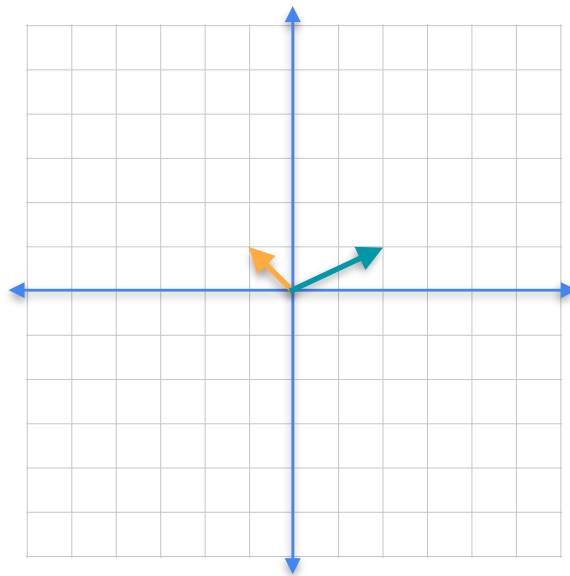


Linearly independent

Linearly independent and linearly dependent vectors

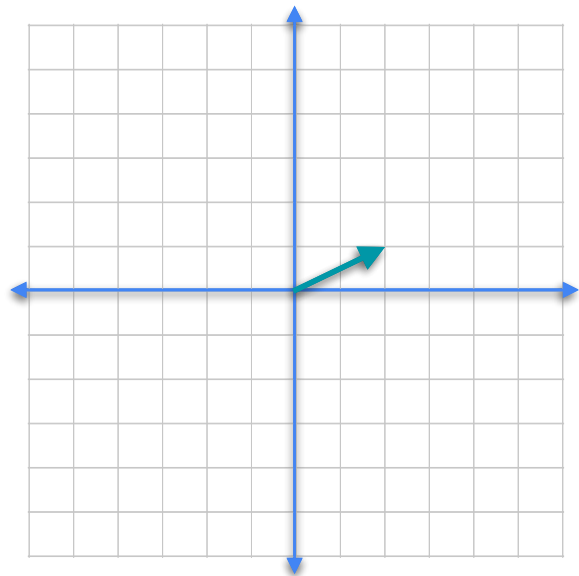


Linearly independent

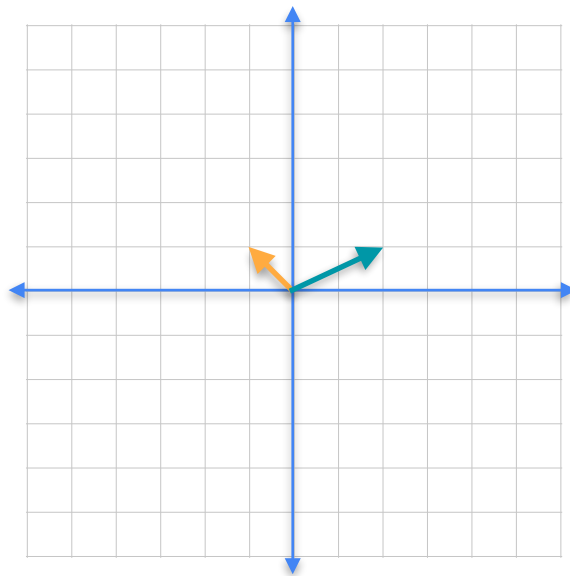


Linearly independent

Linearly independent and linearly dependent vectors

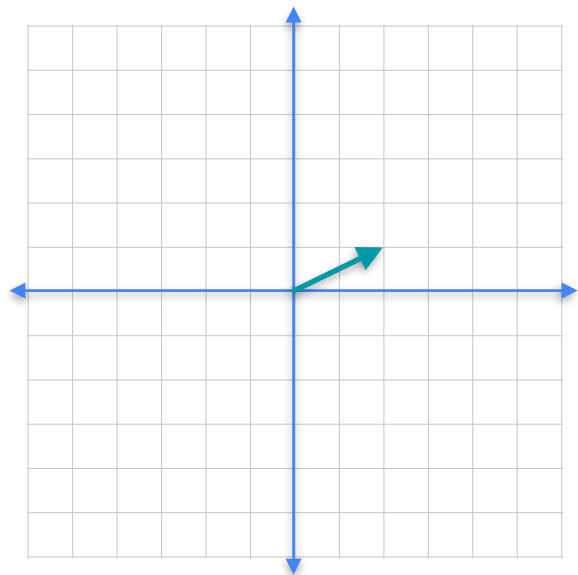


Linearly independent

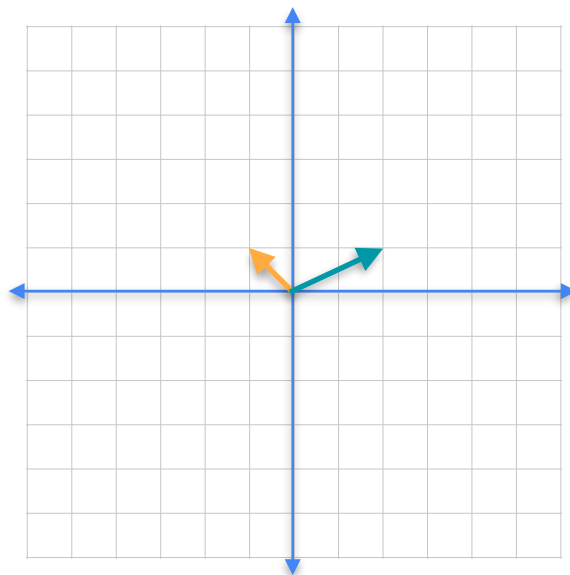


Linearly independent

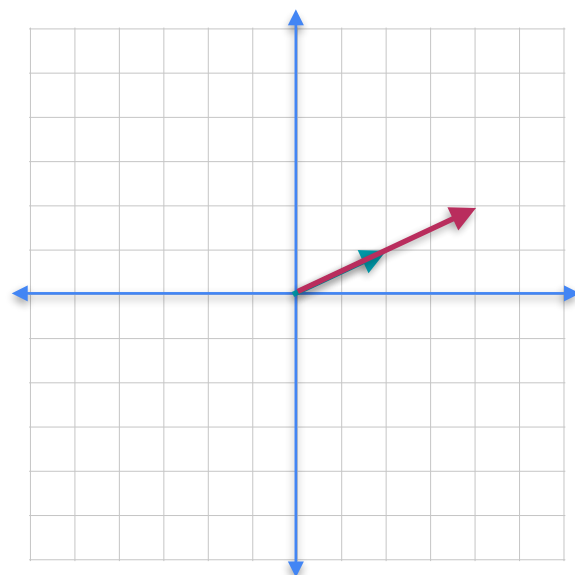
Linearly independent and linearly dependent vectors



Linearly independent

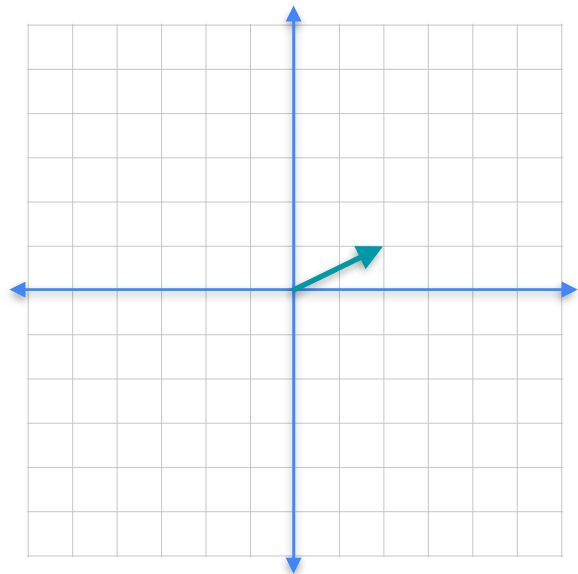


Linearly independent

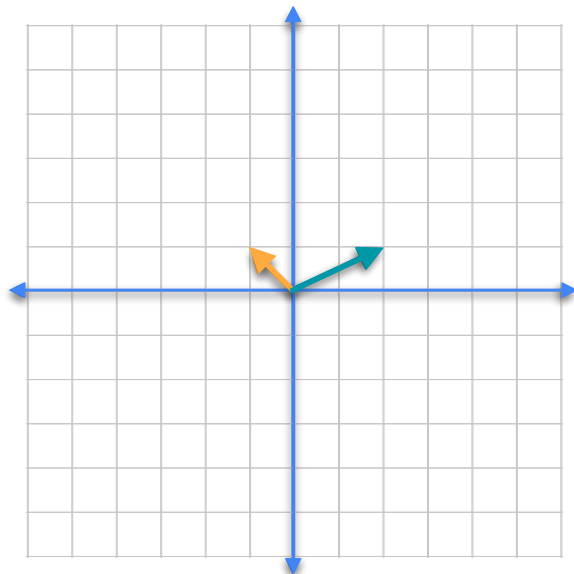


Linearly dependent

Linearly independent and linearly dependent vectors

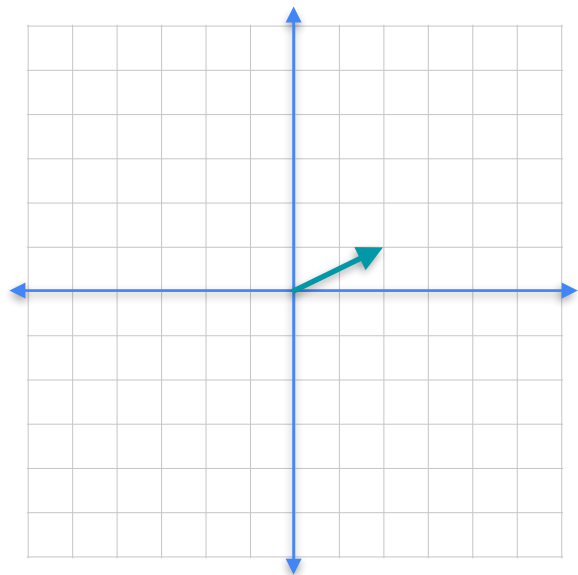


Linearly independent

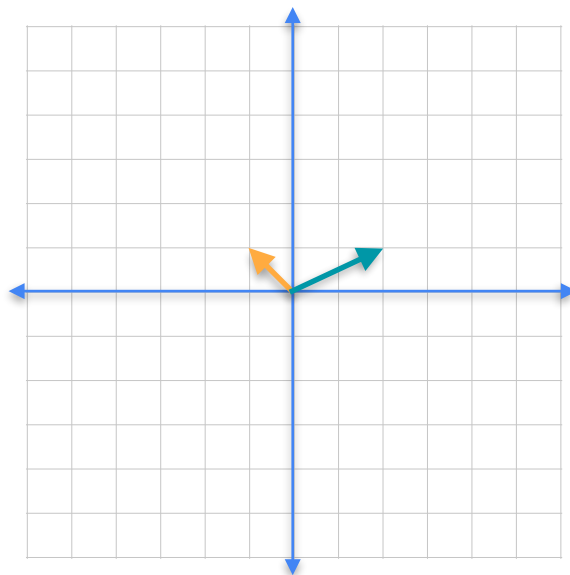


Linearly independent

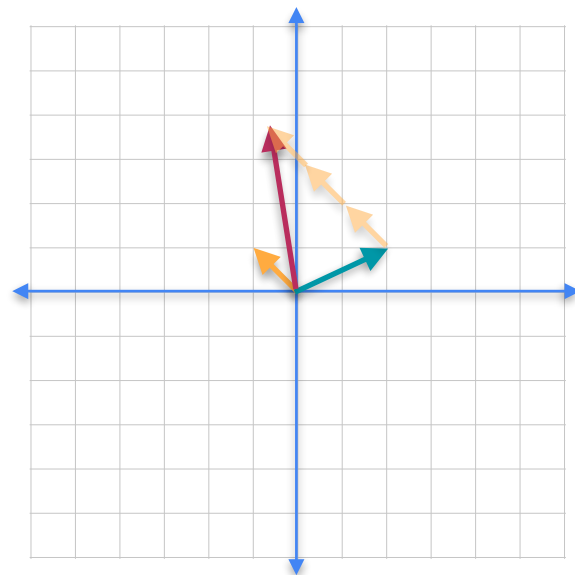
Linearly independent and linearly dependent vectors



Linearly independent

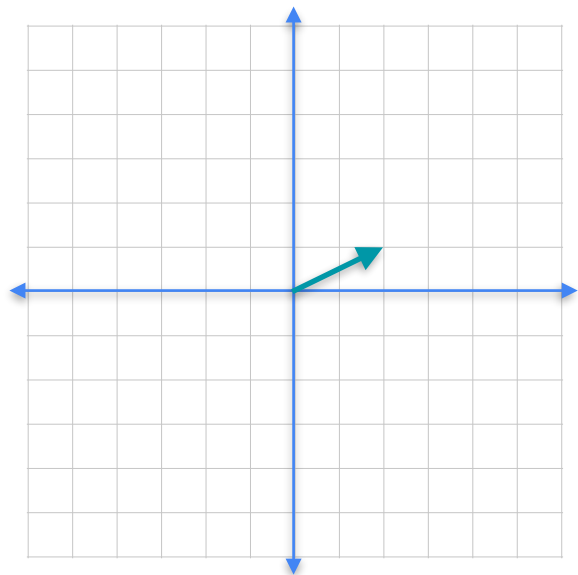


Linearly independent

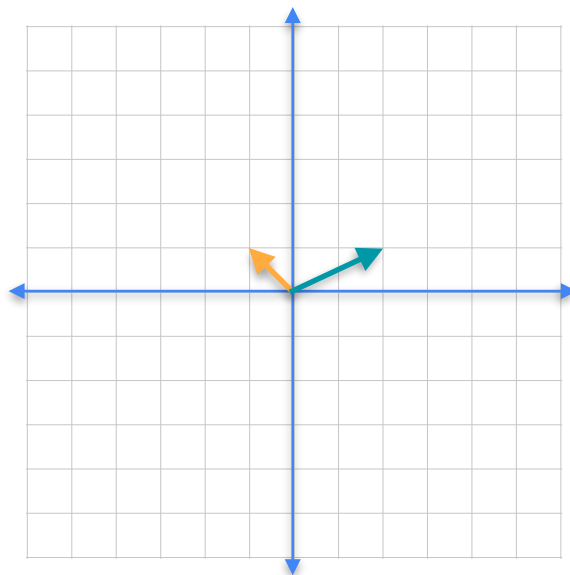


Linearly dependent

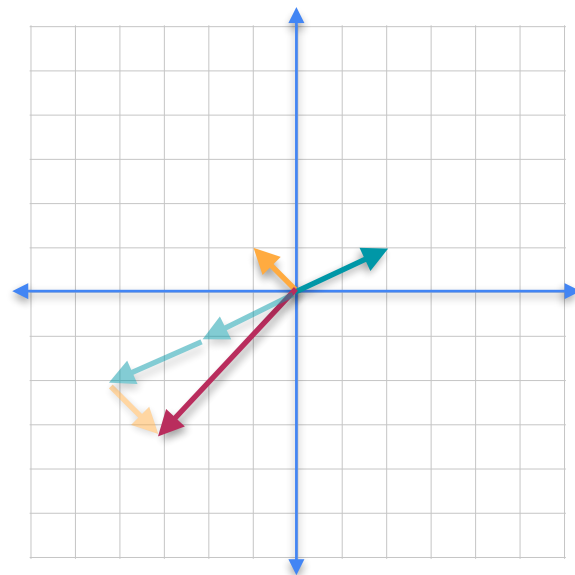
Linearly independent and linearly dependent vectors



Linearly independent

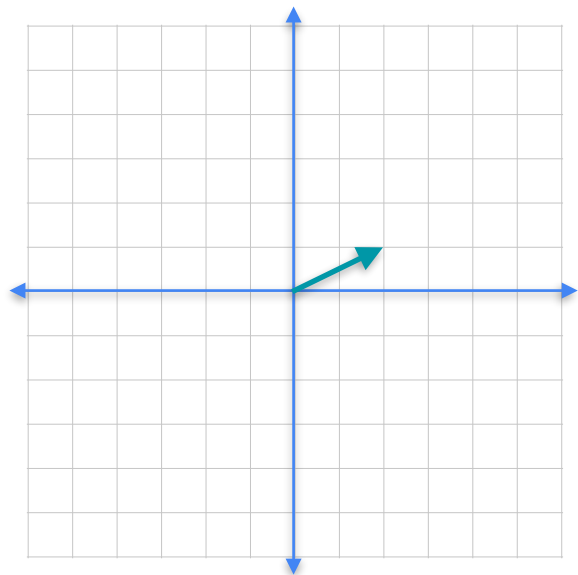


Linearly independent

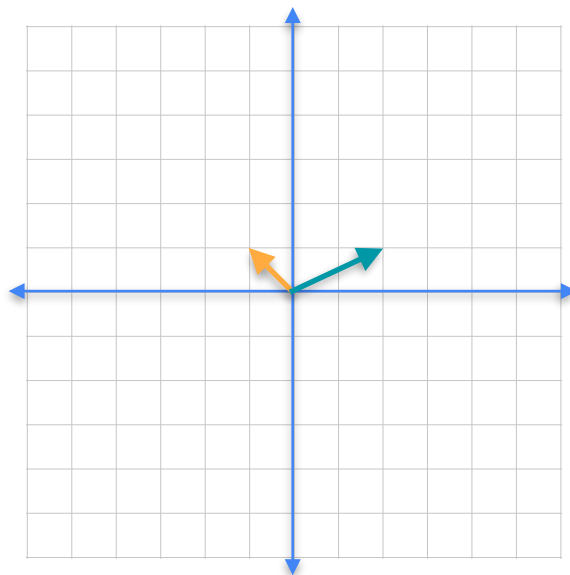


Linearly dependent

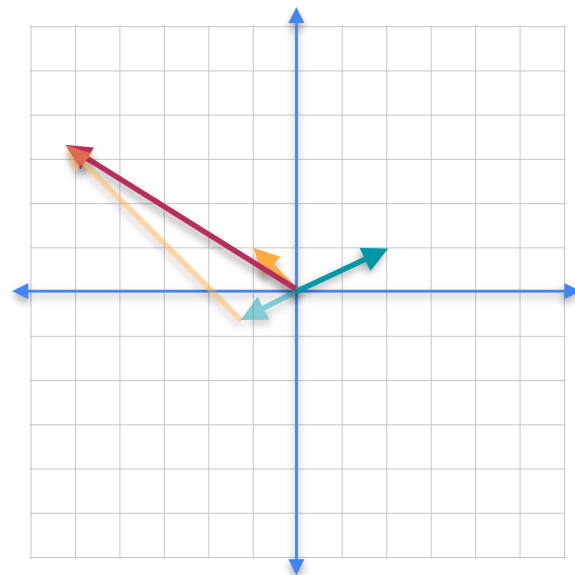
Linearly independent and linearly dependent vectors



Linearly independent

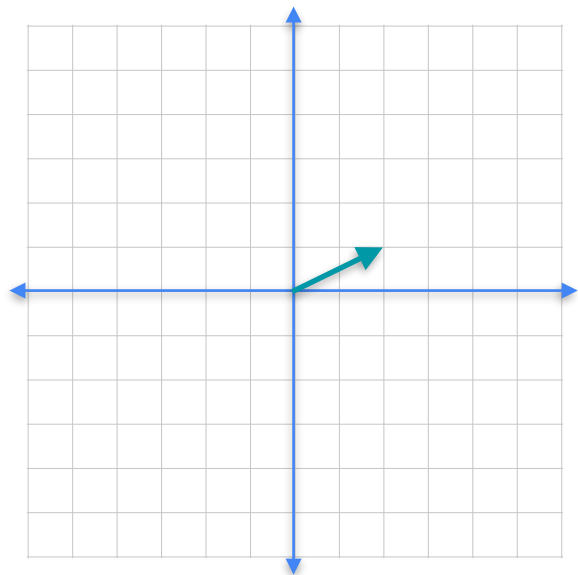


Linearly independent

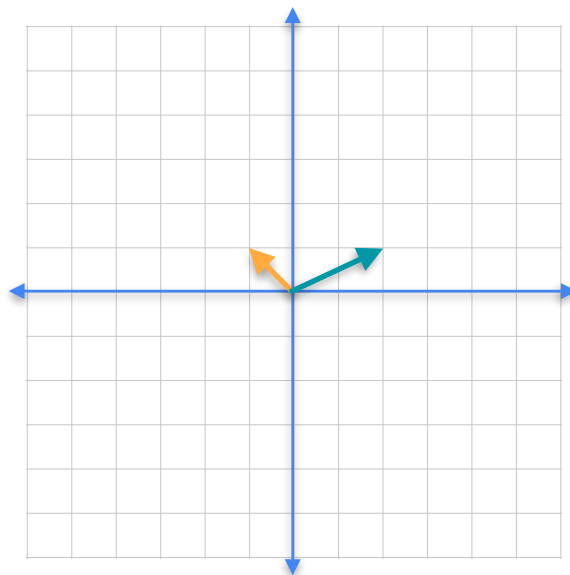


Linearly dependent

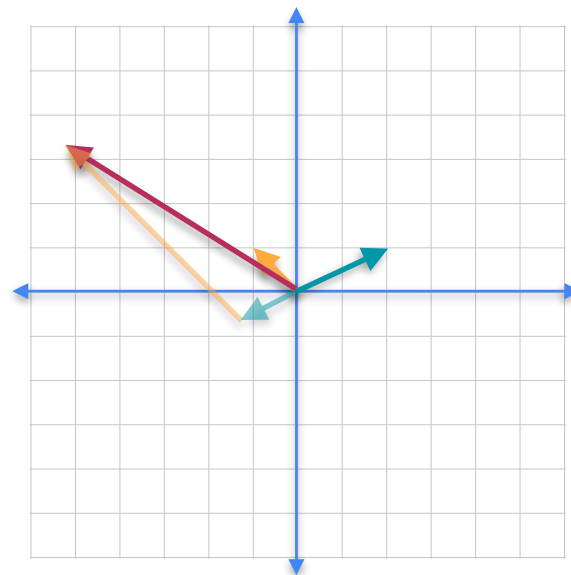
Linearly independent and linearly dependent vectors



Linearly independent

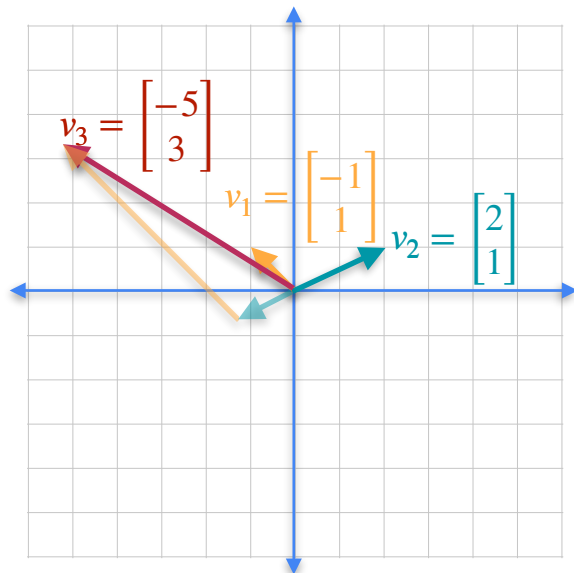


Linearly independent



Linearly dependent

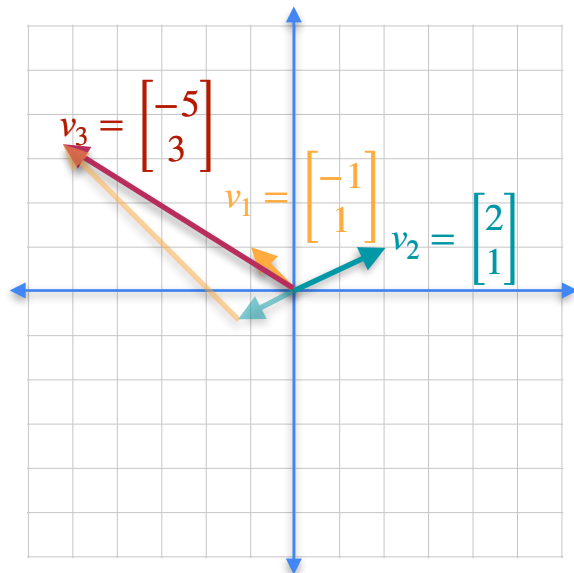
Let's see how to check for linear dependence



$$\alpha + \beta =$$

Linearly dependent

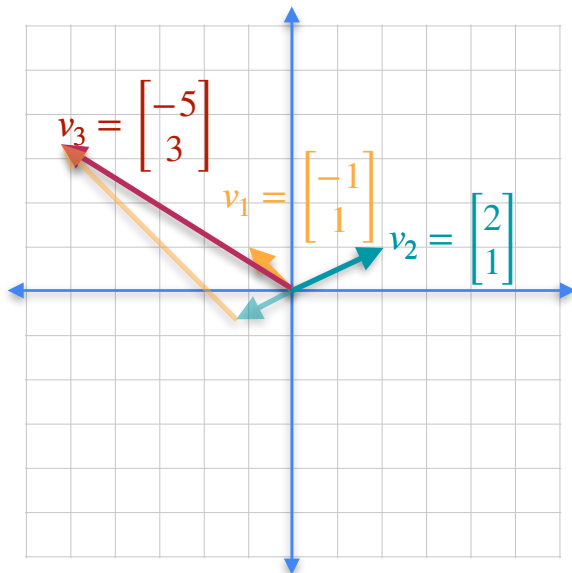
Let's see how to check for linear dependence



$$\alpha v_1 + \beta v_2 = v_3$$

Linearly dependent

Let's see how to check for linear dependence



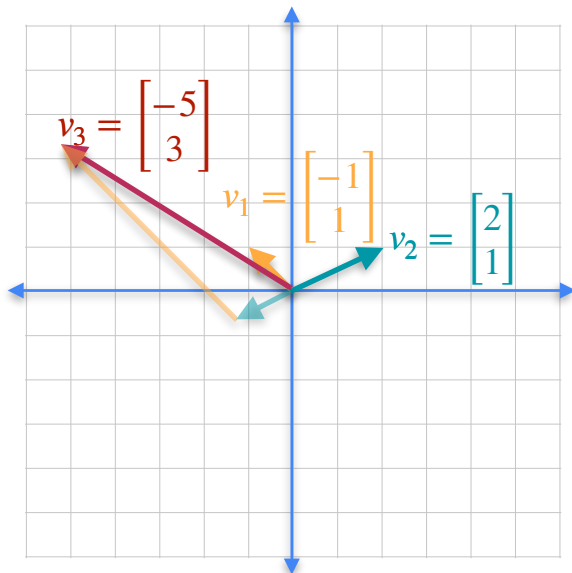
Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$

$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

| | |
|---|-------------------------|
| 1 | $-\alpha + 2\beta = -5$ |
| 2 | $\alpha + \beta = 3$ |

Let's see how to check for linear dependence



Linearly dependent

$$\alpha v_1 + \beta v_2 = v_3$$
$$\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

v_3 is a linear combination
of v_1 and v_2

1 $-\alpha + 2\beta = -5$

2 $\alpha + \beta = 3$

1 + 2

$$3\beta = -2 \rightarrow \beta = -\frac{2}{3}$$

2

$$\alpha - \frac{2}{3} = 3 \rightarrow \alpha = \frac{11}{3}$$

Quiz

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \quad -1 \quad =$$

Linearly dependent

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly dependent

Solution

Are these vectors linearly independent?

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Not a basis!

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

Linearly independent

$$\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

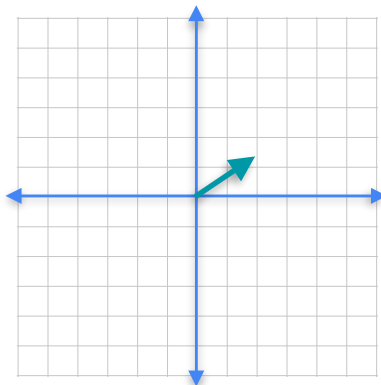
Basis: a formal definition

A basis is a set of vectors that:

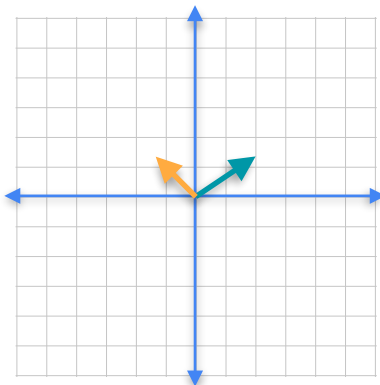
- Spans a vector space
- Is linearly independent



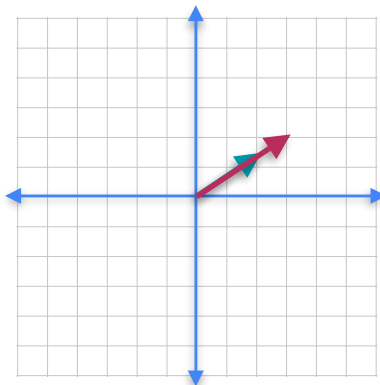
Not all sets of N vectors are a basis
for N -dimensional space



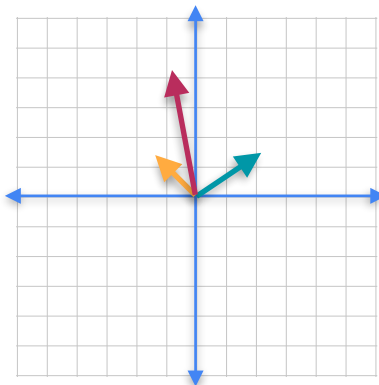
Spans a line
Linearly independent
Is a basis



Spans the plane
Linearly independent
Is a basis



Spans a line
Linearly dependent
Not a basis



Spans the plane
Linearly dependent
Not a basis

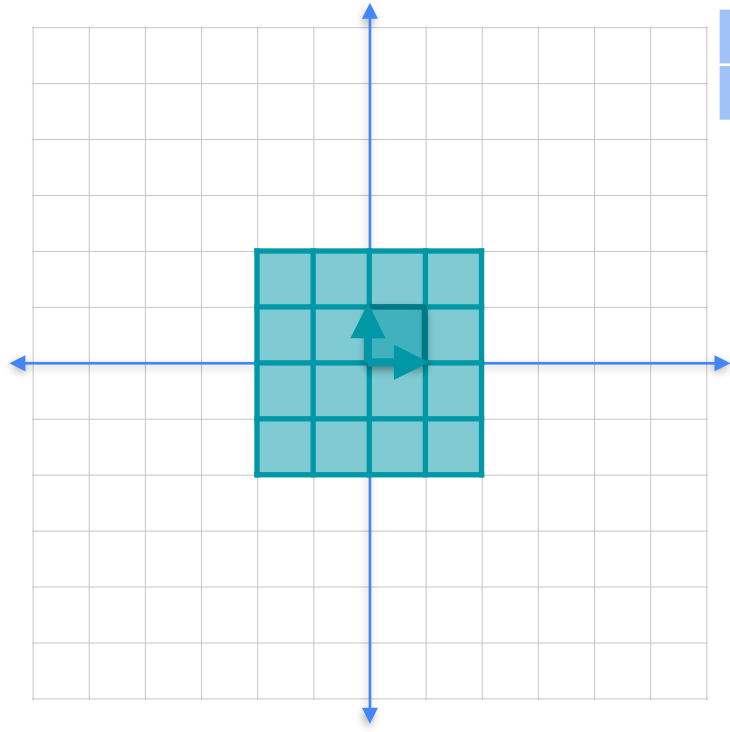


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Determinants and Eigenvectors

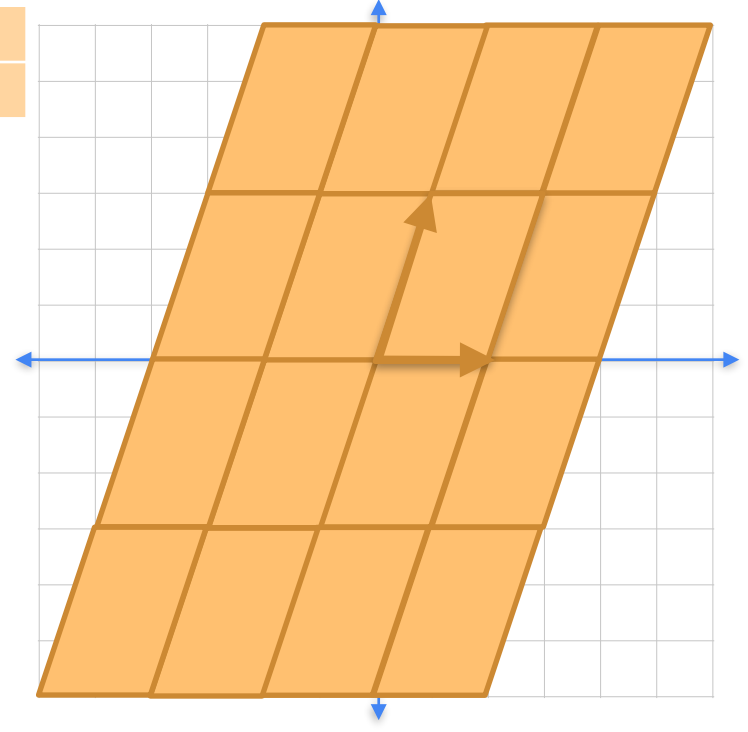
Eigenbasis

Basis

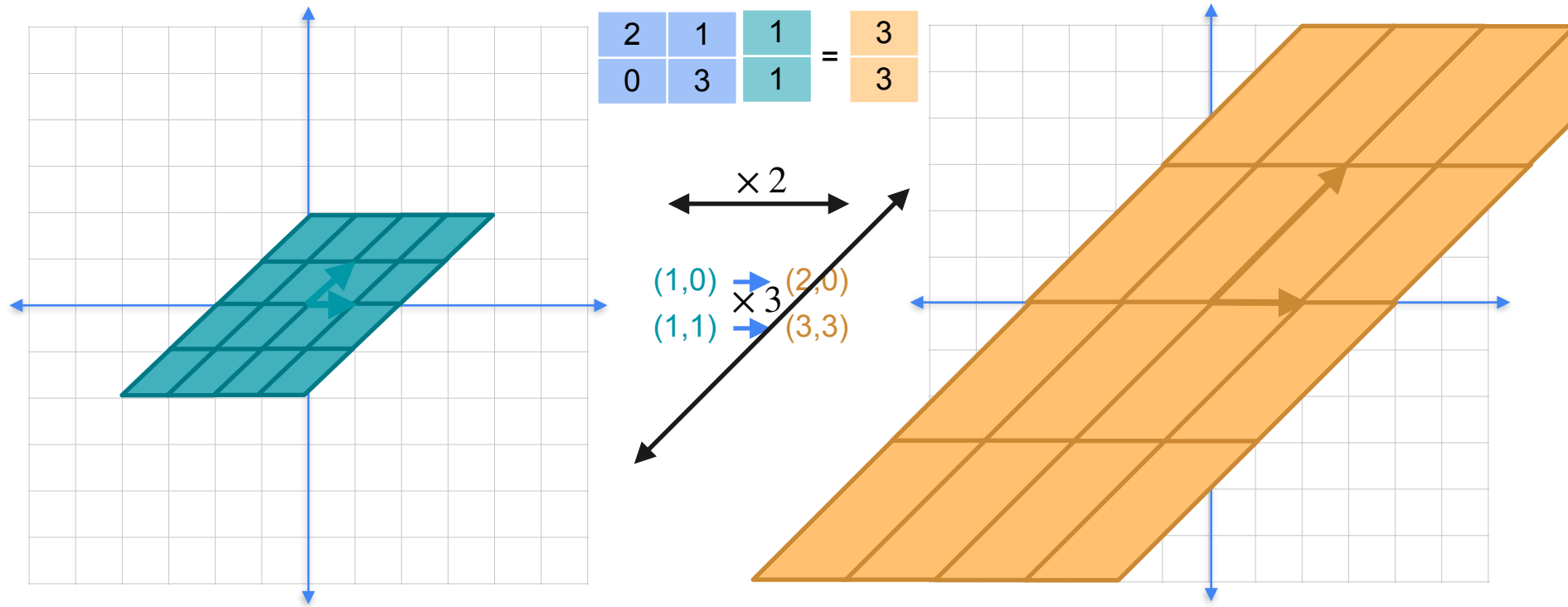


| | | | | |
|---|---|---|---|---|
| 2 | 1 | 0 | = | 1 |
| 0 | 3 | 1 | | 3 |

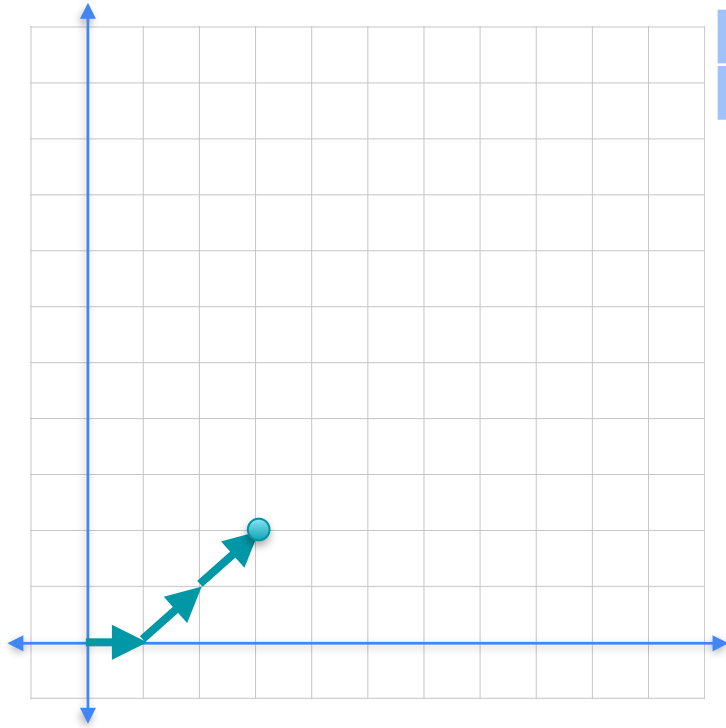
$(1,0) \rightarrow (2,0)$
 $(0,1) \rightarrow (1,3)$



Eigenbasis

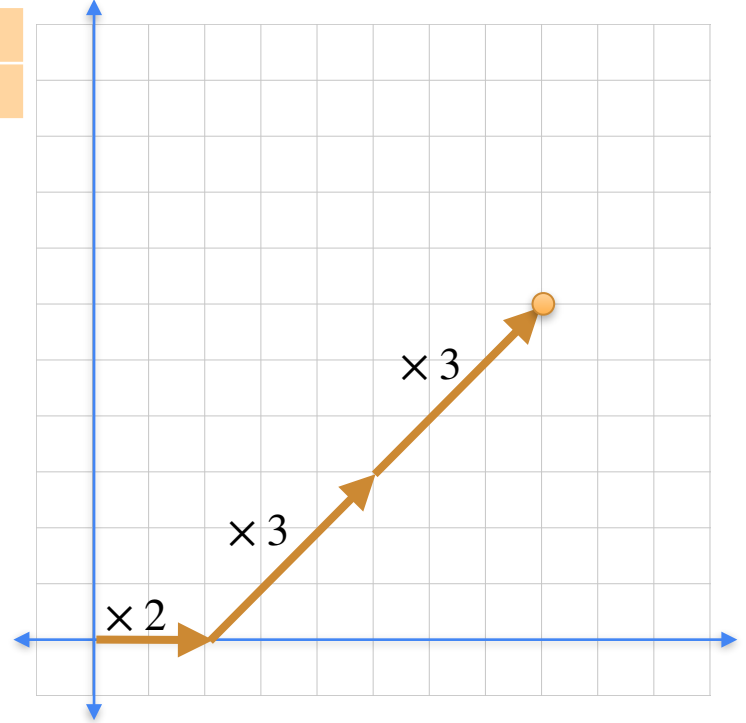


Eigenbasis



| | | | | |
|---|---|---|---|---|
| 2 | 1 | 3 | = | 8 |
| 0 | 3 | 2 | | 6 |

$$(3,2) \rightarrow (8,6)$$



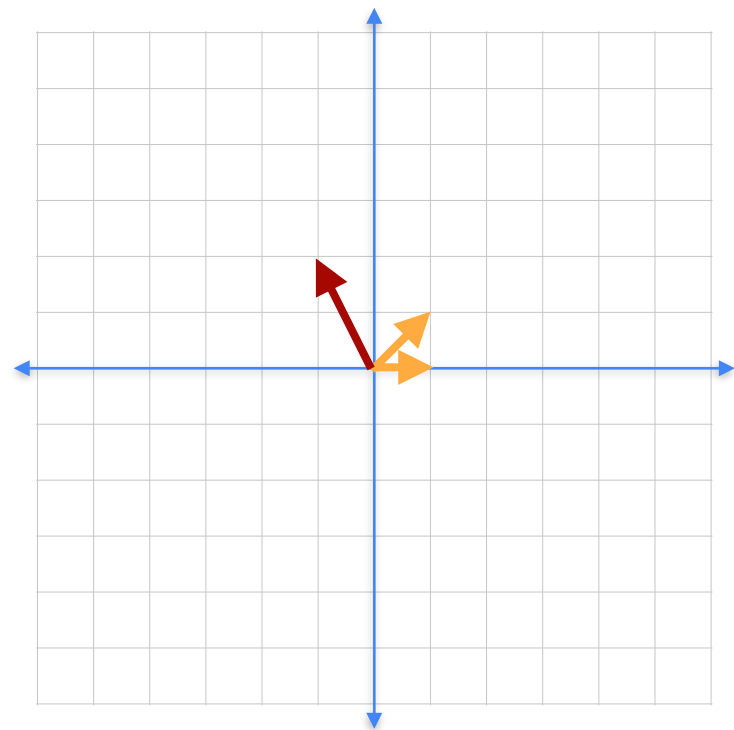


DeepLearning.AI

Determinants and Eigenvectors

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors

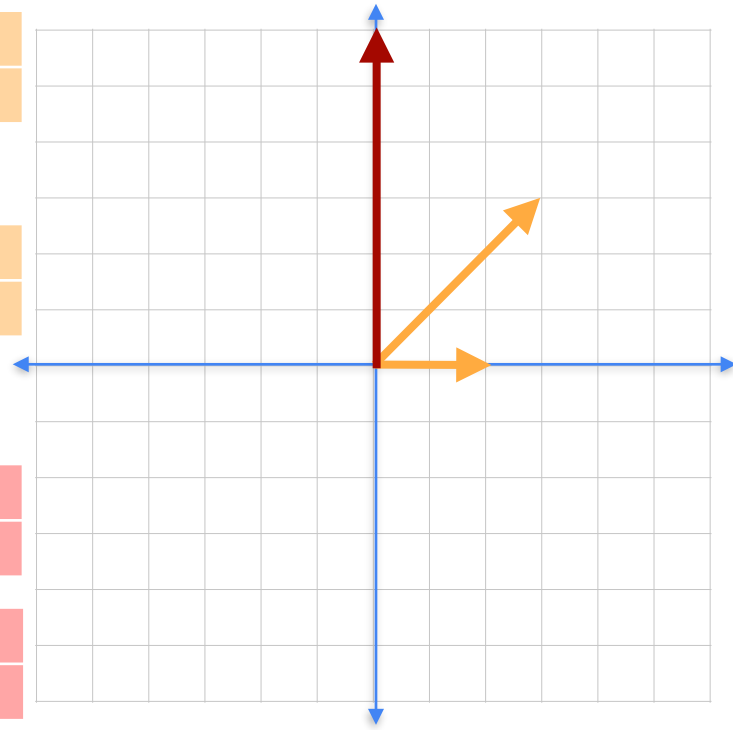


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

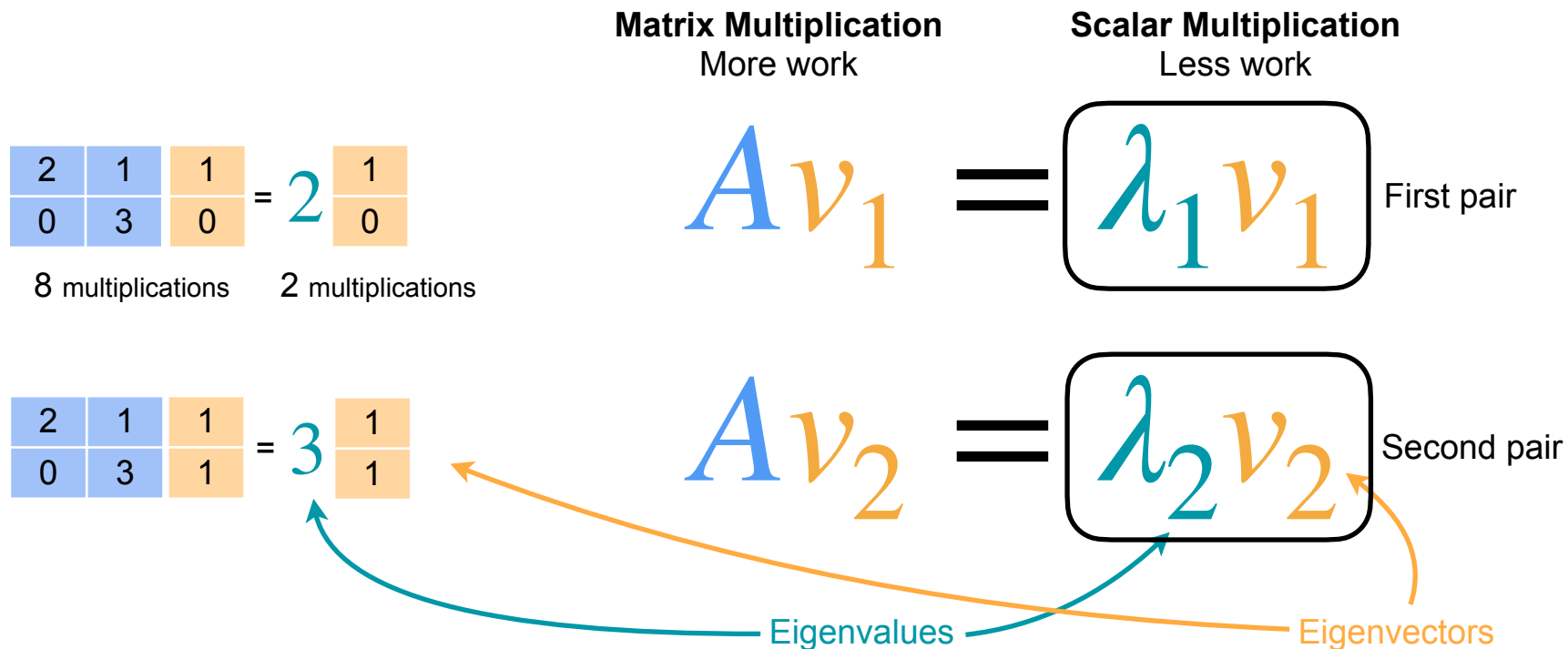
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$

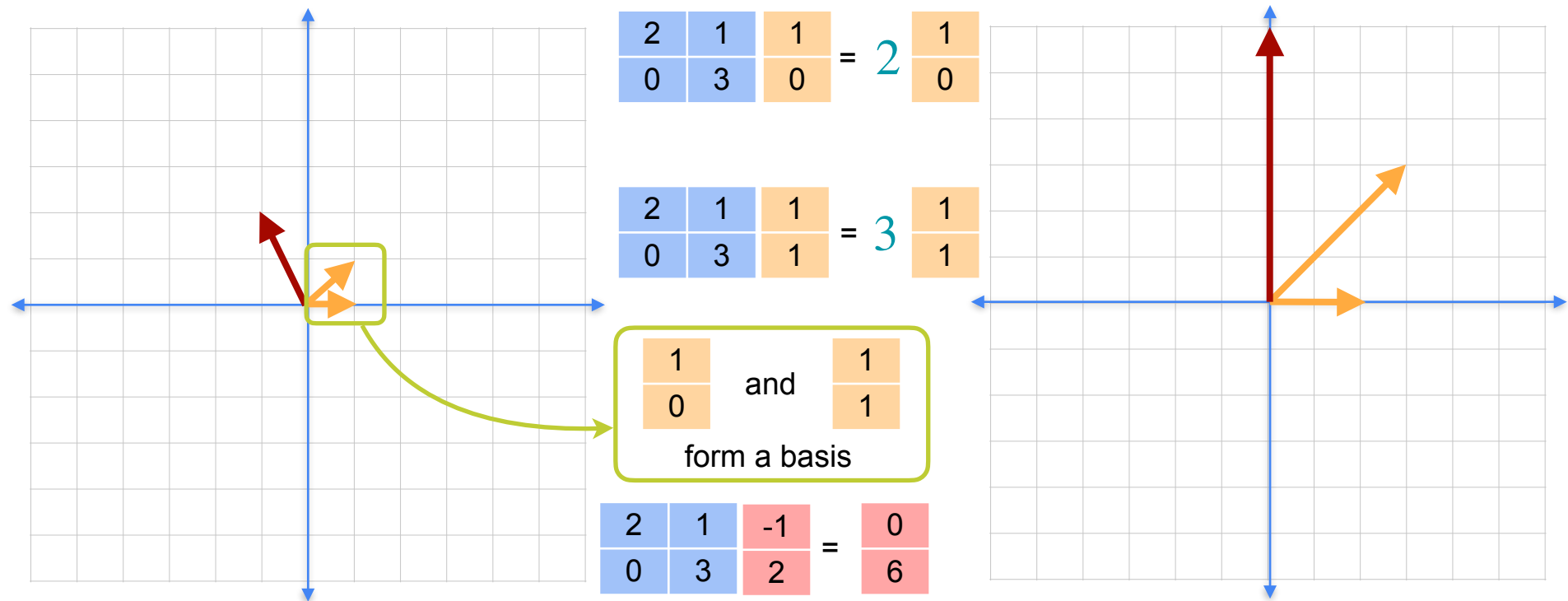
$$\neq \lambda \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$



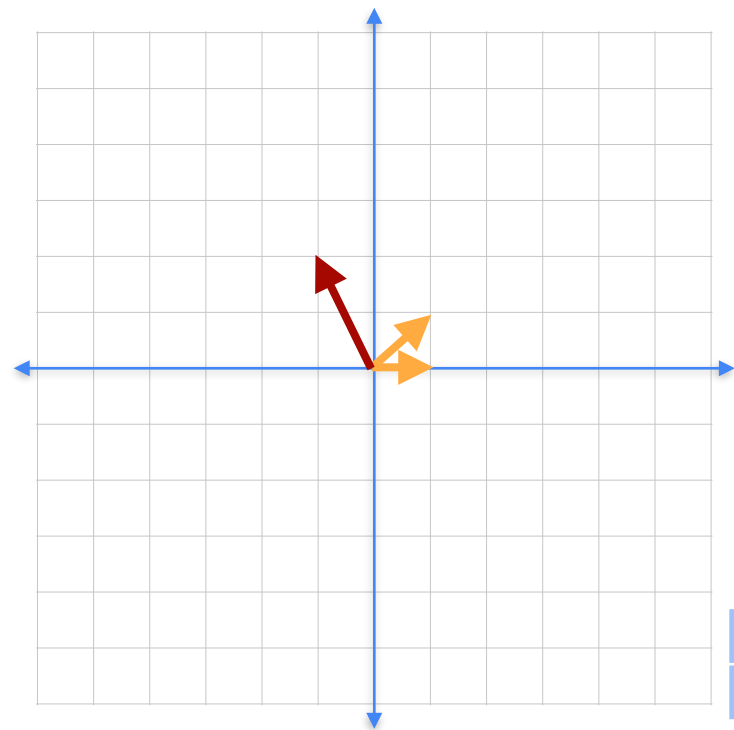
Eigenvalues and eigenvectors



Eigenvalues and eigenvectors



Eigenvalues and eigenvectors

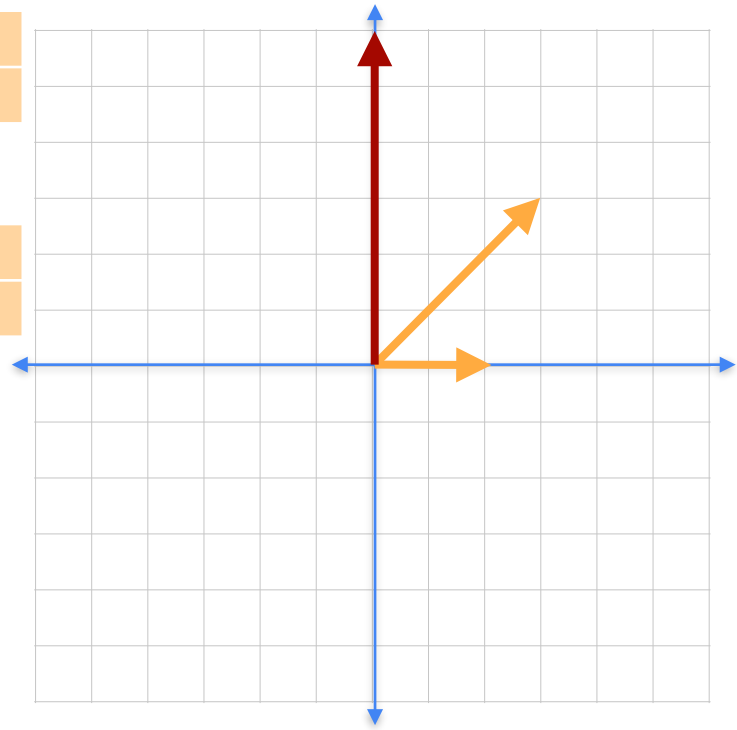


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

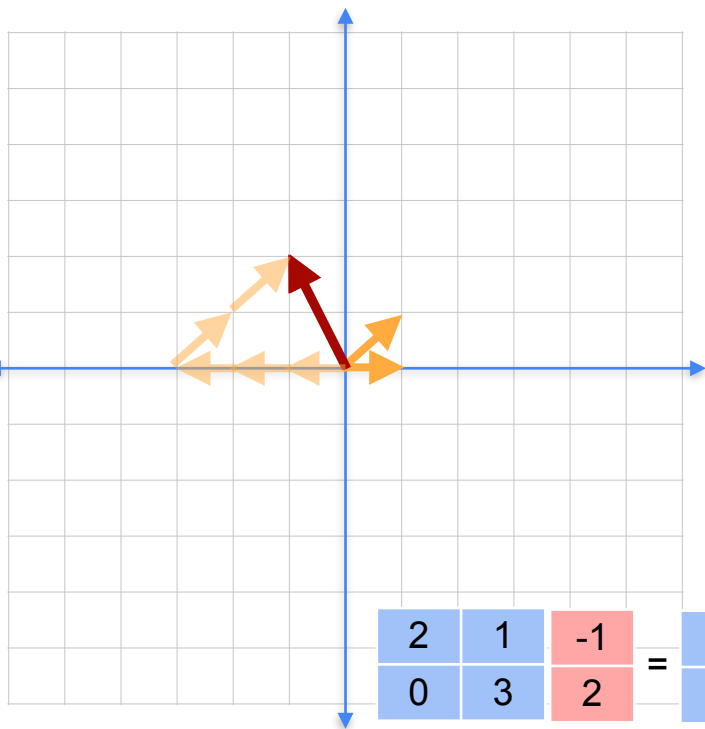
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
form a basis

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \end{bmatrix}$$



Eigenvalues and eigenvectors

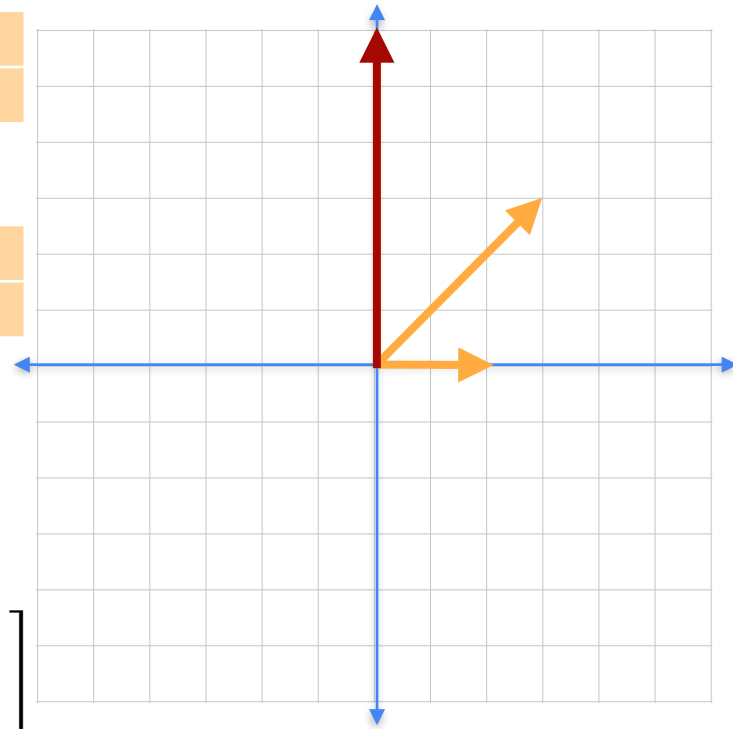


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

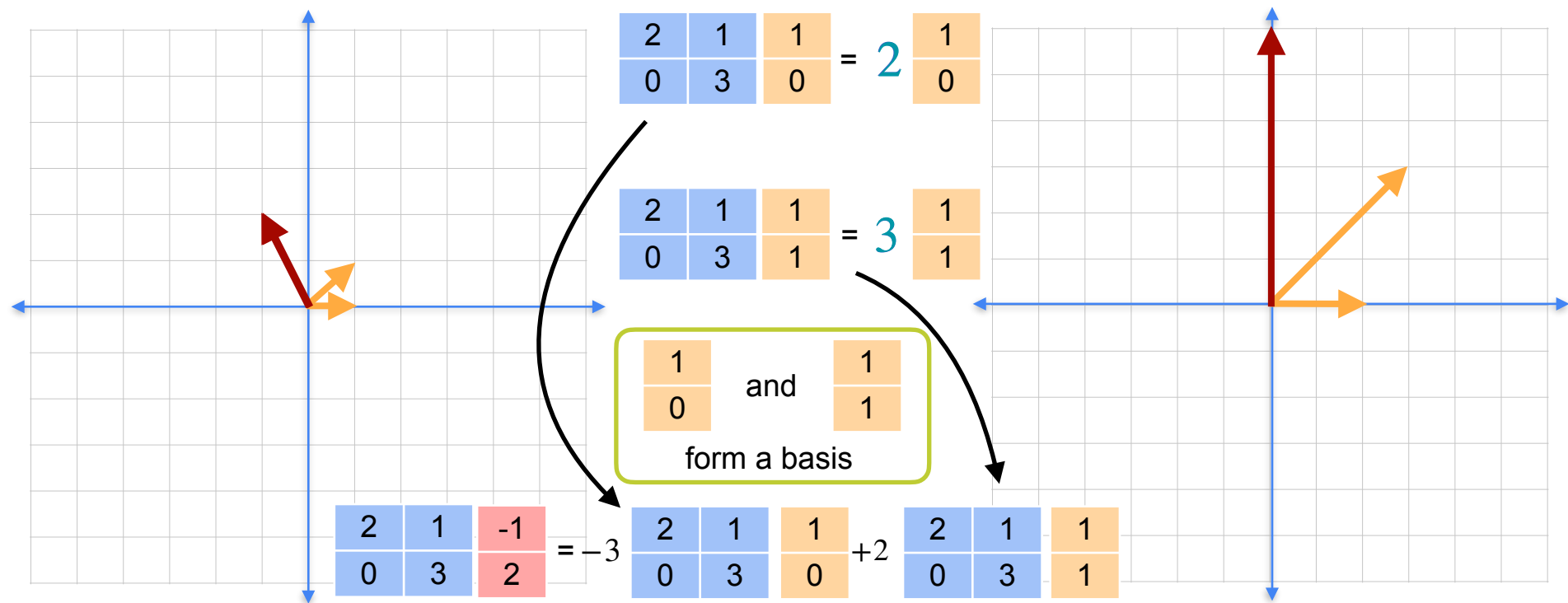
$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
 form a basis

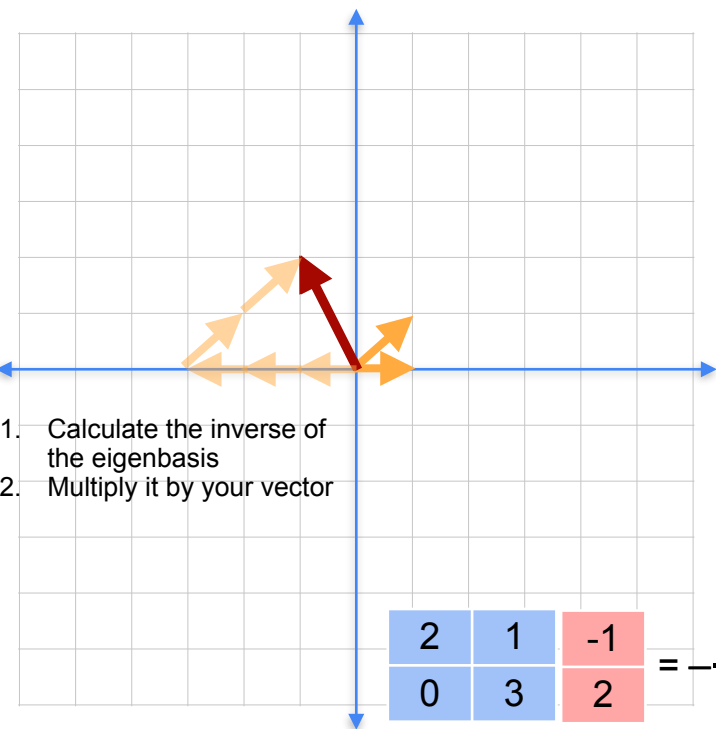
$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} -3 & +2 \end{bmatrix}$$



Eigenvalues and eigenvectors



Eigenvalues and eigenvectors

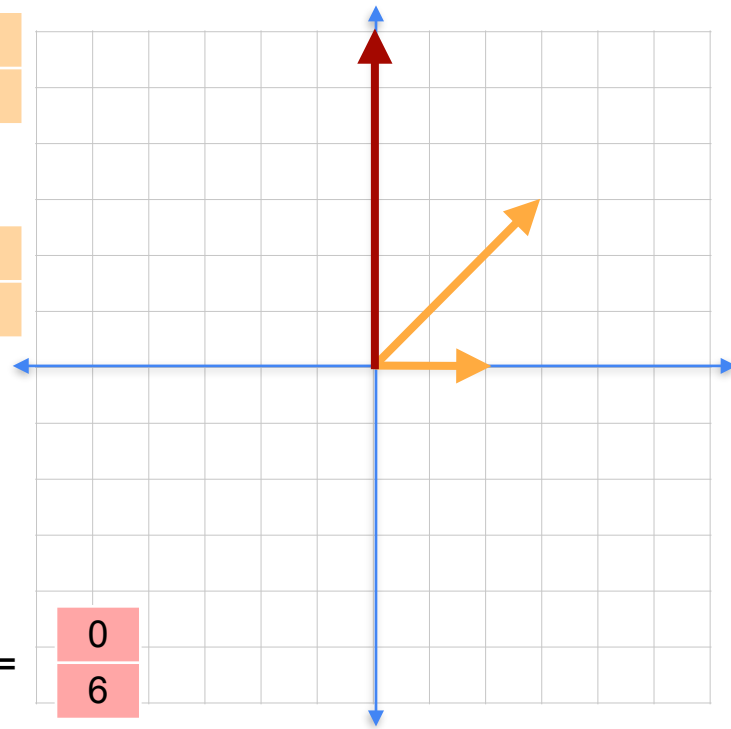


$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ form a basis}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 6 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Eigenvalues and eigenvectors

- $Av = \lambda v$ for each eigenvector / eigenvalue
- Eigenvectors: direction of stretch
- Eigenvalues: how much stretch
- Eigenbasis: the set of a matrix's eigenvectors, can be arranged as a matrix with one eigenvector in each column
- Save work and characterize a transformation

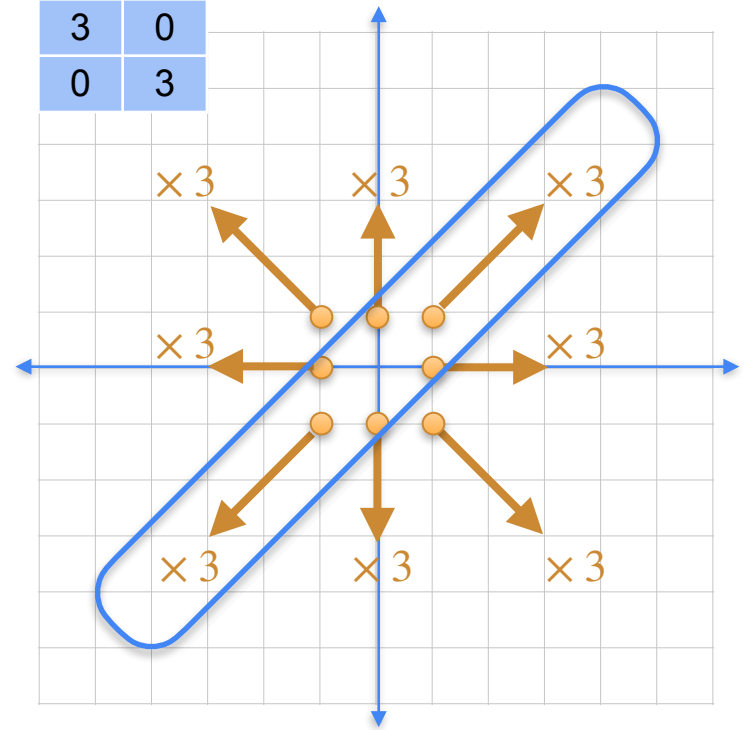
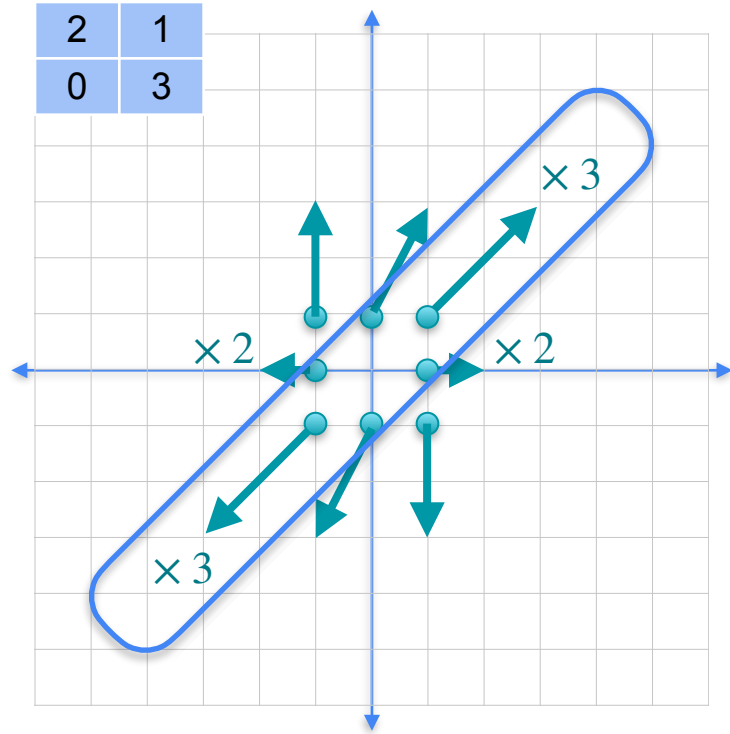


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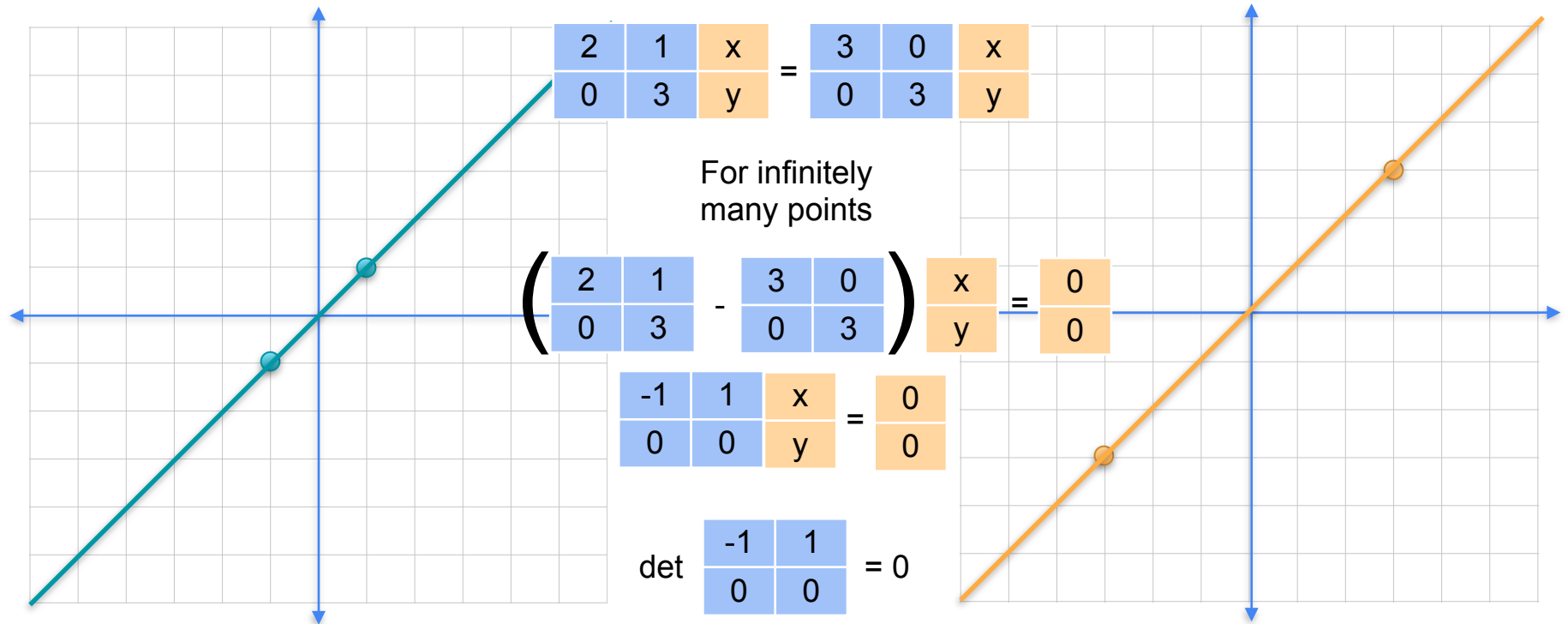
Determinants and Eigenvectors

Calculating eigenvalues and eigenvectors

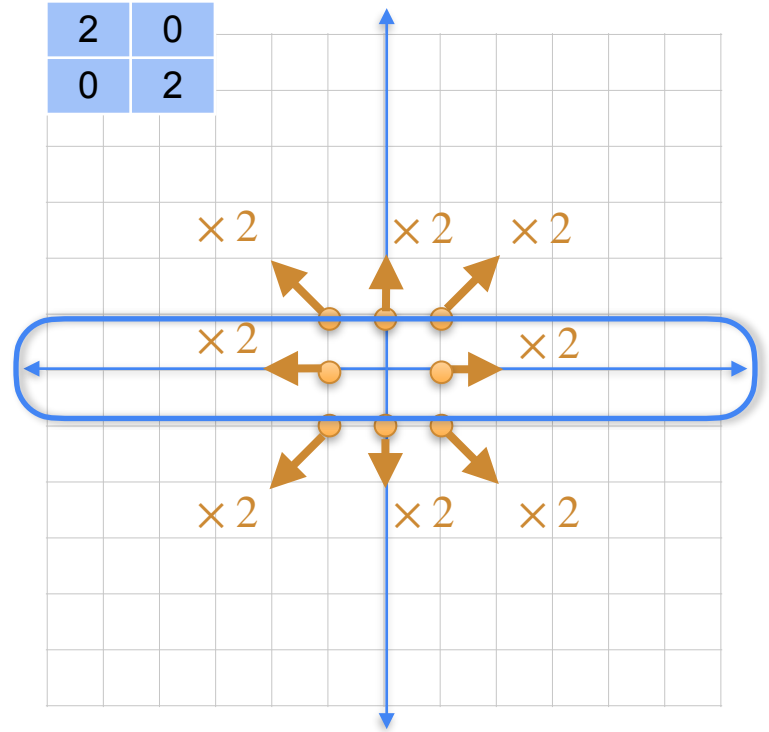
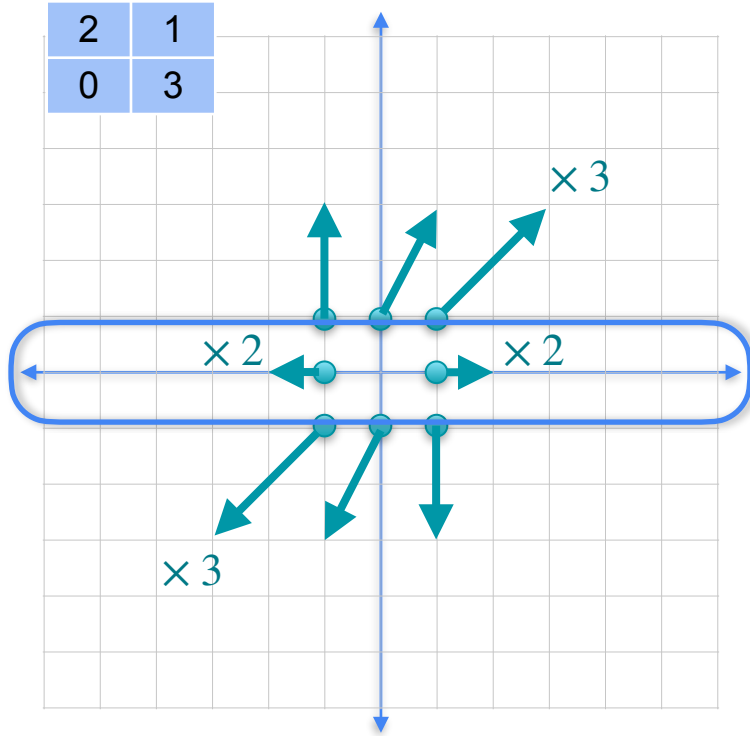
Finding eigenvalues



Finding eigenvalues



Finding eigenvalues



Finding eigenvalues

$$\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

For infinitely many points

$$\left(\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \right) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\det \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} = 0$$

Finding eigenvalues

If λ is an eigenvalue:

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

For infinitely many (x,y)

$$\begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Has infinitely many solutions

$$\det \begin{bmatrix} 2-\lambda & 1 \\ 0 & 3-\lambda \end{bmatrix} = 0$$

Characteristic polynomial

$$(2 - \lambda)(3 - \lambda) - 1 \cdot 0 = 0$$

$$\begin{aligned} \lambda &= 2 \\ \lambda &= 3 \end{aligned}$$

Finding eigenvectors

Eigenvalues: $\lambda = 2$
 $\lambda = 3$

Solve the equations

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 2 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 2x$$

$$0x + 3y = 2y$$

$$x = 1$$

$$y = 0$$

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 3 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$2x + y = 3x$$

$$0x + 3y = 3y$$

$$x = 1$$

$$y = 1$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Quiz

- Find the eigenvalues and eigenvectors of this matrix:

| | |
|---|---|
| 9 | 4 |
| 4 | 3 |

Solution

- Eigenvalues: 11, 1
- Eigenvectors: (2,1), (-1,2)

| | |
|---|---|
| 9 | 4 |
| 4 | 3 |

- The characteristic polynomial is

$$\det \begin{bmatrix} 9-\lambda & 4 \\ 4 & 3-\lambda \end{bmatrix} = (9-\lambda)(3-\lambda) - 4 \cdot 4 = 0$$

- Which factors as $\lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$

- The solutions are $\lambda = 11$
 $\lambda = 1$

Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

$$\lambda I = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

Characteristic polynomial: $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 2 - \lambda & 1 & -1 \\ 1 & -\lambda & -3 \\ -1 & -3 & -\lambda \end{bmatrix} = 0$$

$$(2 - \lambda)\lambda^2 + 3 - 3 - 9(2 - \lambda) + \lambda + \lambda = -\lambda^3 + 2\lambda^2 + 11\lambda - 12 = 0$$

$$-(\lambda + 3)(\lambda - 1)(\lambda - 4) = 0$$

Eigenvalues: $-3, 1, 4$

Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \quad \text{Eigenvalues: } -3, 1, 4$$

$$Av = \lambda v$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

Eigenvalues: $-3, 1, 4$

$$Av = \lambda v$$

$$\begin{bmatrix} 2 & 1 & -1 & x_1 \\ 1 & 0 & -3 & x_2 \\ -1 & -3 & 0 & x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 + x_2 - x_3 \\ x_1 - 3x_3 \\ -x_1 - 3x_2 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4x_1 \\ x_1 - 3x_3 &= 4x_2 \\ -x_1 - 3x_2 &= 4x_3 \end{aligned}$$

$$\begin{aligned} R_1 \quad -2x_1 + x_2 - x_3 &= 0 \\ R_2 \quad x_1 - 4x_2 - 3x_3 &= 0 \\ R_3 \quad -x_1 - 3x_2 - 4x_3 &= 0 \end{aligned}$$

$$\begin{aligned} R_2 + R_3 \quad -7x_2 - 7x_3 &= 0 & 3R_1 + R_3 \quad -7x_1 - 7x_3 &= 0 \\ x_2 = -x_3 & & x_1 = -x_3 & \end{aligned}$$

$$\begin{aligned} x_1 &= k \\ x_2 &= k \\ x_3 &= -k \end{aligned}$$

infinite solutions
of this form

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= -1 \end{aligned}$$

this works!

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 2 \\ x_3 &= -2 \end{aligned}$$

so does this!

Eigenvector:

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Finding eigenvalues

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

Eigenvalues

$\lambda_1 = 4$

$\lambda_2 = 1$

$\lambda_3 = -3$

Eigenvectors

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Note on dimensions

Eigenvalues \longrightarrow Determinant \longrightarrow Square Matrix

| | |
|---|---|
| 9 | 4 |
| 4 | 3 |



| | | |
|---|---|----|
| 9 | 4 | 5 |
| 4 | 3 | -2 |





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Determinants and Eigenvectors

**On the number of
eigenvectors**

Number of eigenvectors

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -3 \\ -1 & -3 & 0 \end{bmatrix}$$

3 by 3
matrix

?

3 distinct
eigenvalues

?

3 distinct
eigenvectors

Eigenvalues

$$\lambda_1 = 4 \quad \lambda_2 = 1 \quad \lambda_3 = -3$$

Eigenvectors

| | | |
|----|---|----|
| 1 | 0 | 2 |
| 1 | 1 | -1 |
| -1 | 1 | 1 |



Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Characteristic polynomial = $\det(A - \lambda I) = \det$

$$\begin{bmatrix} 2 - \lambda & 0 & 0 \\ 1 & 4 - \lambda & 0.5 \\ 0 & 0 & 2 - \lambda \end{bmatrix}$$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0 = 0$$

Eigenvalues: 4, 2, 2 Repeated eigenvalue

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$
$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 2x_3 &= 4x_3 \end{aligned}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ -2x_3 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= \text{any number} \\ x_3 &= 0 \end{aligned}$$

Eigenvector

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$
$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 & x_1 \\ -1 & 4 & -0.5 & x_2 \\ 0 & 0 & 2 & x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 2x_3 \end{bmatrix}$$

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 2x_3 &= 2x_3 \end{aligned}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 0 &= 0 \end{aligned}$$

$$x_1 = 2x_2 - 0.5x_3$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 2 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= 1 \\ x_3 &= 2 \end{aligned}$$

Point in different directions

Different eigenvectors

Repeated eigenvalues - Example 1

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 0 & 0 & 2 \end{bmatrix}$$

Eigenvalues

$\lambda_1 = 4$

$\lambda_2 = 2$

$\lambda_3 = 2$

Eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Characteristic polynomial = $\det(A - \lambda I) = \det$

$$\begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 4-\lambda & 0.5 \\ -4 & 0 & 2-\lambda \end{bmatrix}$$

$$(2 - \lambda)^2(4 - \lambda) + 0 + 0 - 0 - 0 - 0$$

Eigenvalues: 4, 2, 2 Repeated eigenvalue

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 4$$

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 4

$$\begin{aligned} 2x_1 &= 4x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 4x_2 \\ 4x_1 + 2x_3 &= 4x_3 \end{aligned}$$

$$Av = 4v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 4 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4x_1 \\ 4x_2 \\ 4x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

$$\begin{aligned} -2x_1 &= 0 \\ -x_1 - 0.5x_3 &= 0 \\ 4x_1 - 2x_3 &= 0 \end{aligned}$$

$$x_1 = 0 \quad x_3 = 0 \quad x_2 = \text{any number}$$

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 0 \end{aligned}$$

Same as before!

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \quad \text{Eigenvalue: } 2$$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$\begin{bmatrix} 0 \\ k \\ 4k \end{bmatrix}$$

$$x_1 = 0$$

$$x_3 = 4x_2$$

$$\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 1 \\ x_3 &= 4 \end{aligned}$$

$$\begin{bmatrix} 0 \\ 0.5 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 0 \\ x_2 &= 0.5 \\ x_3 &= 2 \end{aligned}$$

On the same line
Same eigenvector

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalue: 2

$$\begin{aligned} 2x_1 &= 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 &= 2x_2 \\ 4x_1 + 2x_3 &= 2x_3 \end{aligned}$$

$$Av = 2v$$

$$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 2 \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix}$$

$$\begin{bmatrix} 2x_1 \\ -x_1 + 4x_2 - 0.5x_3 \\ 4x_1 + 2x_3 \end{bmatrix}$$

$$\begin{aligned} 0 &= 0 \\ -x_1 + 2x_2 - 0.5x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

$$x_1 = 0 \quad x_3 = 4x_2$$

$$\begin{bmatrix} 0 \\ k \\ 4k \end{bmatrix}$$

Repeated eigenvalues - Example 2

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 4 & -0.5 \\ 4 & 0 & 2 \end{bmatrix}$$

Eigenvalues

$\lambda_1 = 4$

$\lambda_2 = 2$

$\lambda_3 = 2$

Eigenvectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$


Can't create an eigenbasis
from this matrix



Summary

| | |
|---|---|
| a | b |
| c | d |

Eigenvalues

λ_1, λ_2



If $\lambda_1 \neq \lambda_2$  2 eigenvectors
(2 different directions)




If $\lambda_1 = \lambda_2$  1 eigenvector
(1 direction)
 2 eigenvectors
(2 different directions)

| | | |
|---|---|---|
| a | b | c |
| d | e | f |
| g | h | i |

$\lambda_1, \lambda_2, \lambda_3$

If $\lambda_1 \neq \lambda_2 \neq \lambda_3$  3 eigenvectors
(3 different directions)

If $\lambda_1 = \lambda_2 \neq \lambda_3$  2 eigenvectors
(2 different directions)
 3 eigenvectors
(3 different directions)

If $\lambda_1 = \lambda_2 = \lambda_3$  1 eigenvector
(1 direction)
 2 eigenvectors
(2 different directions)
 3 eigenvectors
(3 different directions)



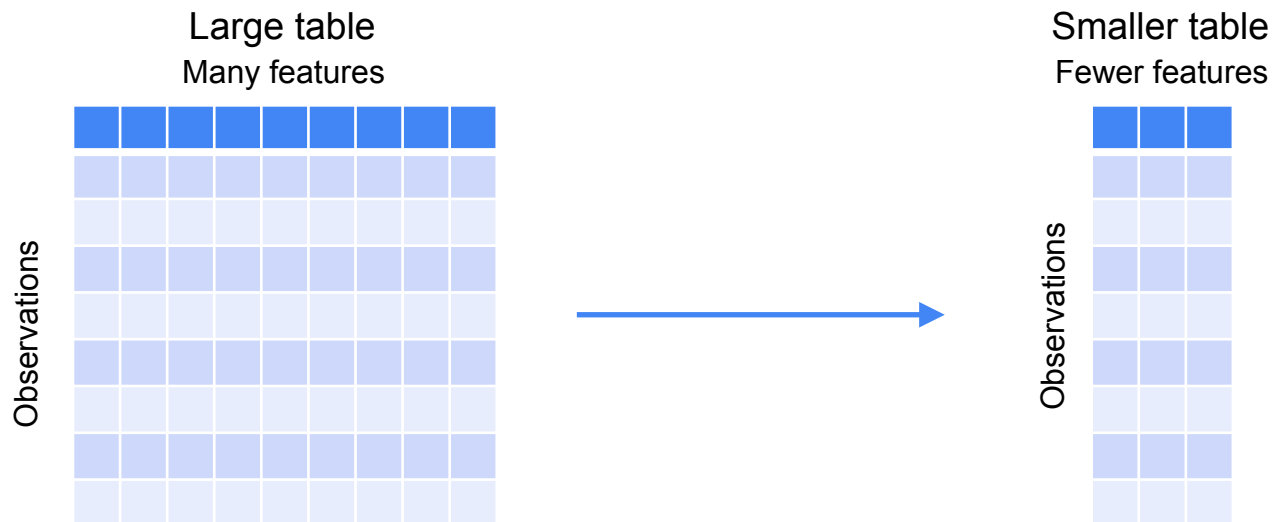
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Determinants and Eigenvectors

**Dimensionality reduction
and projection**

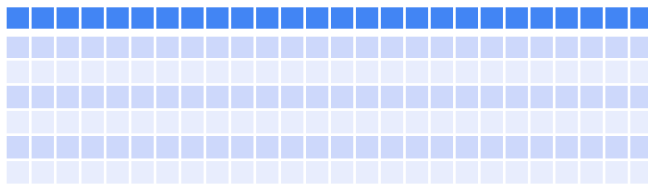
Dimensionality Reduction

- Reduce dimensions (# of columns) of dataset
- Preserve as much information as possible



Dimensionality Reduction

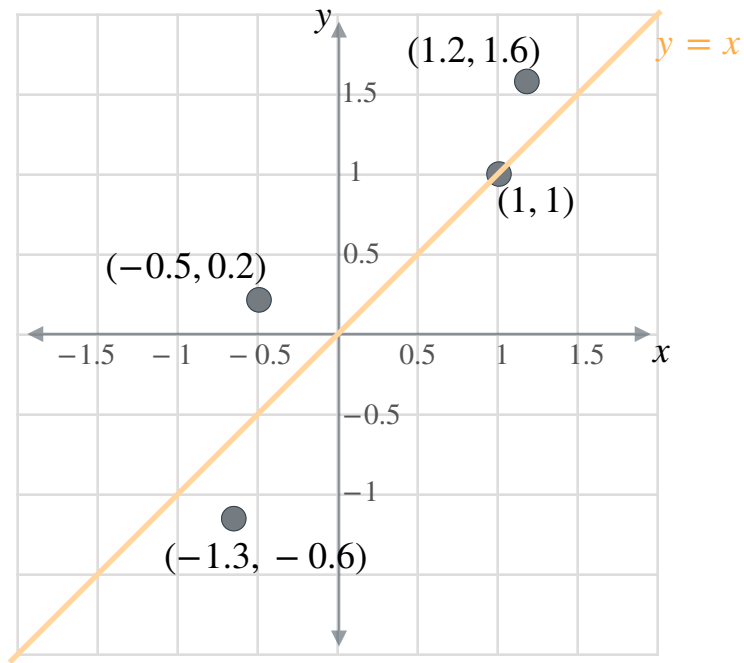
- Leads to smaller datasets
- Easier to visualize



| Customer Age | Account Age | Days Since Login | Total Purchases | Total \$ Spent |
|--------------|-------------|------------------|--|----------------|
| 23 | 1 month | 10 days | 1 | \$100 |
| 71 | 45 months | 2 days | Easy approach - just delete columns Loses valuable information | |
| 54 | 30 months | 15 days | | |
| 36 | 22 months | 12 days | | |
| | | | 2 | \$70 |
| | | | 4 | \$210 |

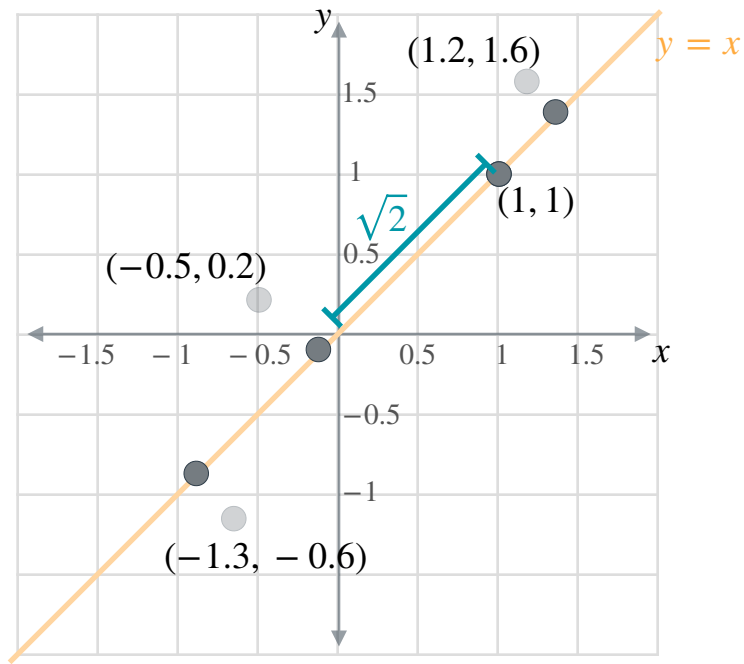
Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |



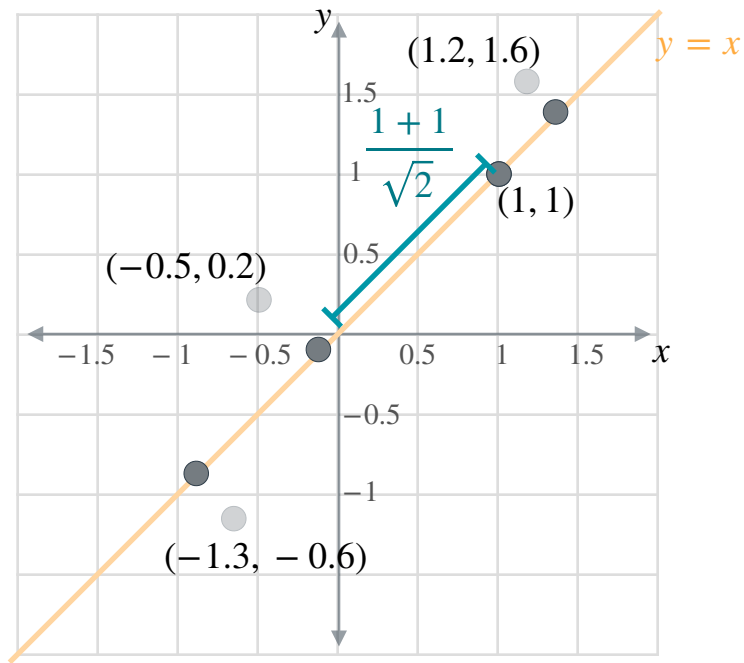
Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |



Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |



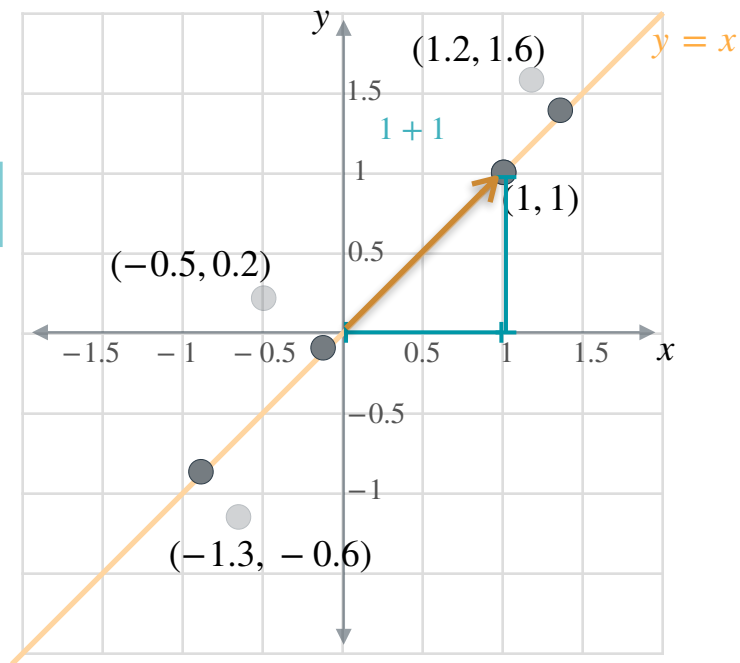
Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

1
1

=

(1 + 1)



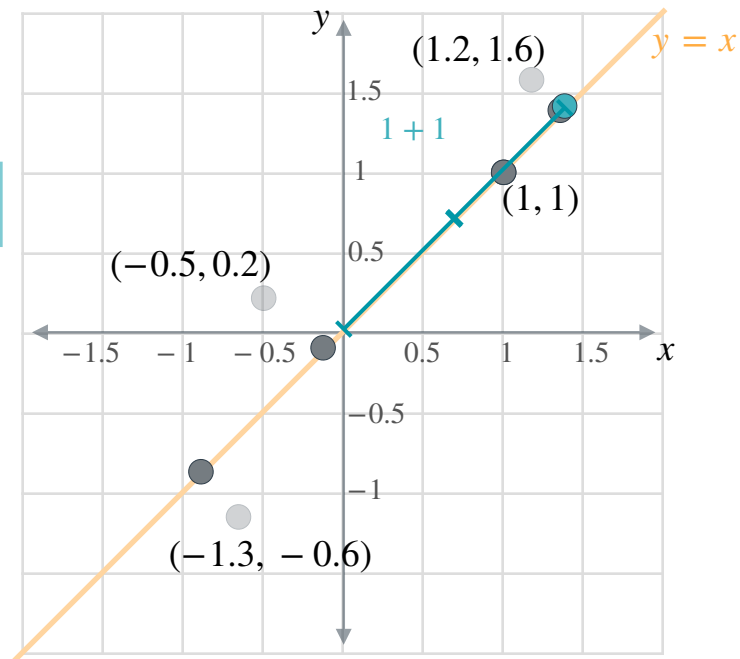
Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

1
1

=

(1 + 1)

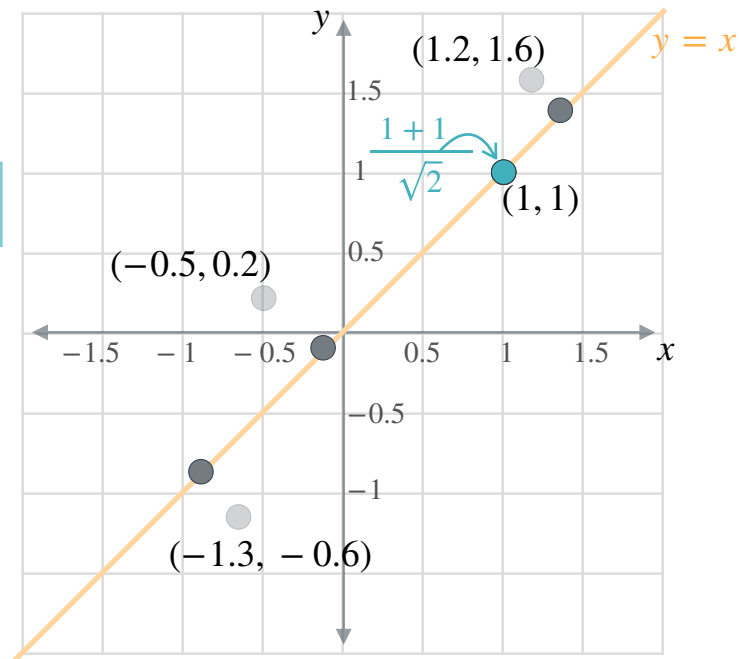


Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$



Projections

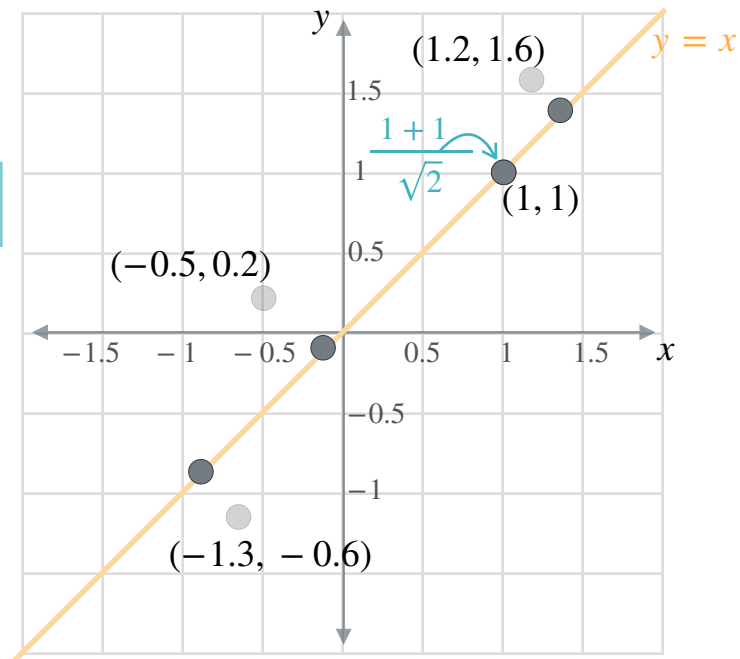
| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

Norm of 1

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$\frac{1}{\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\|_2}$$



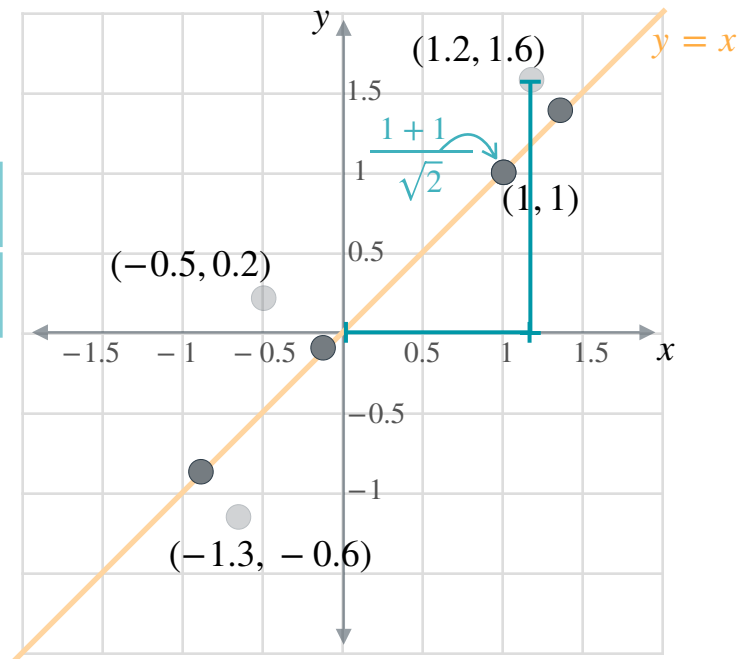
Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

1
1

$$\frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

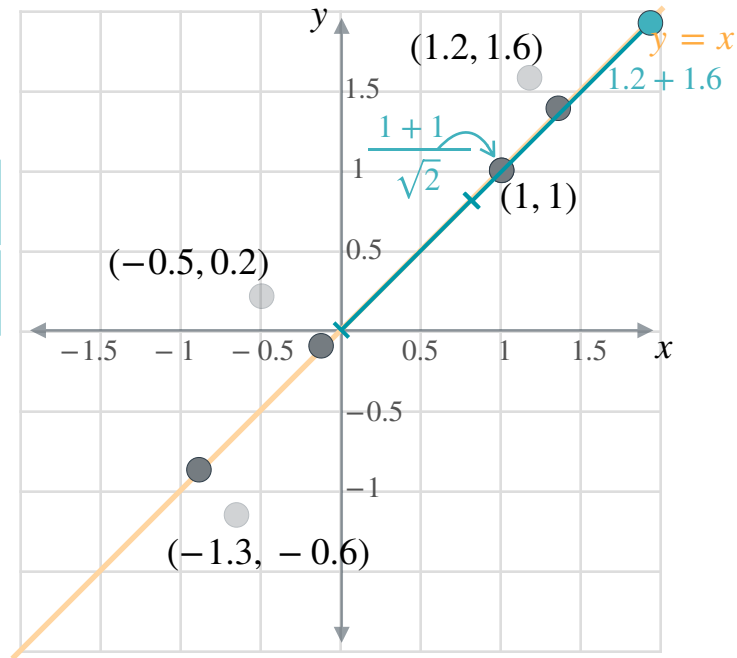


Projections

$$\frac{1}{1} \cdot \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

(1.2 + 1.6)



Projections

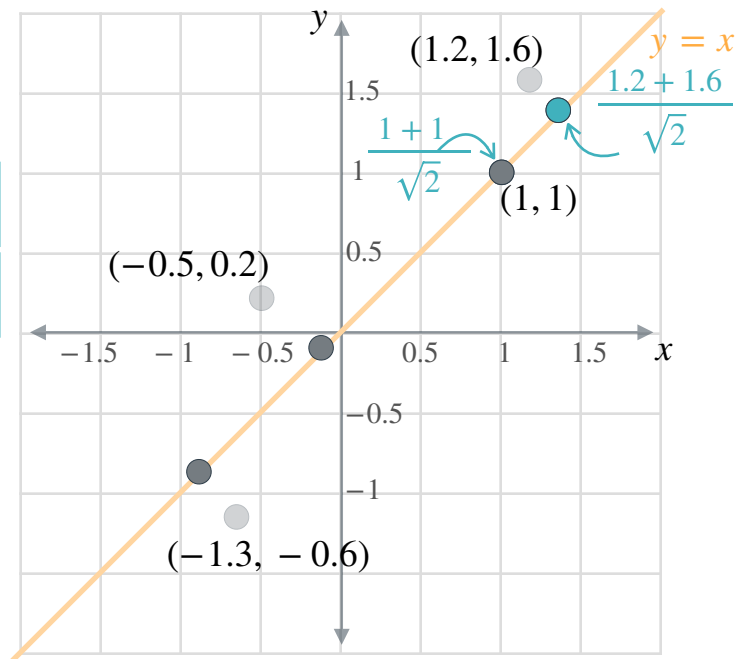
| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}}$$

=

$$\begin{bmatrix} (1 + 1) / \sqrt{2} \\ (1.2 + 1.6) / \sqrt{2} \end{bmatrix}$$



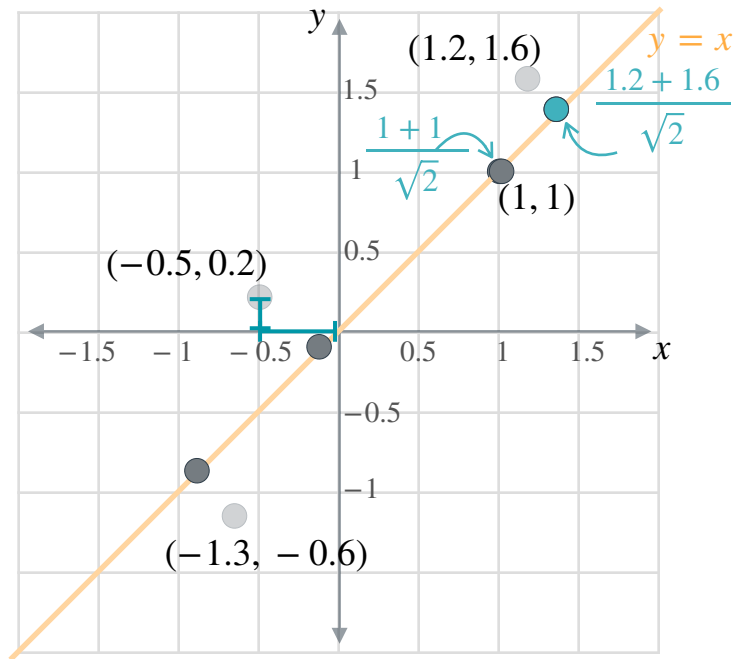
Projections



$$\frac{1}{1} \cdot \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$(1.2 + 1.6)/\sqrt{2}$$



Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

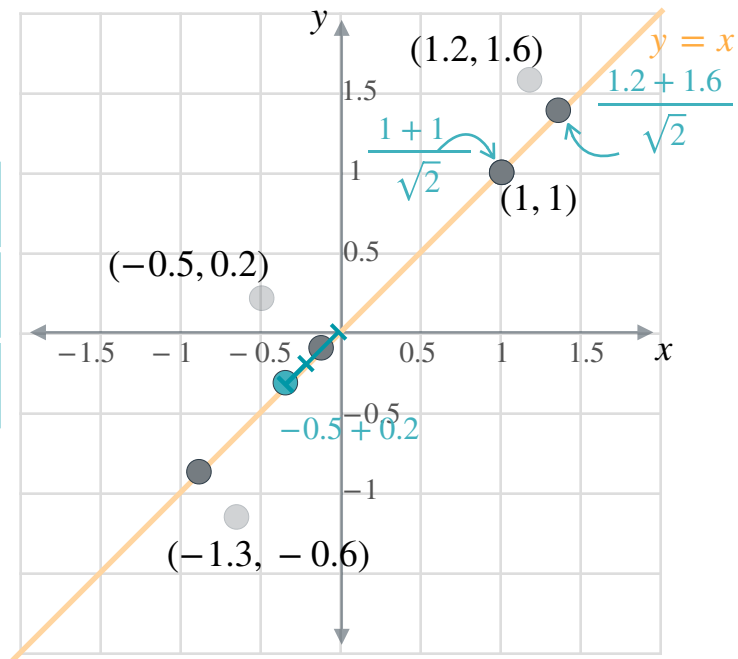
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$(1.2 + 1.6) / \sqrt{2}$$

$$(-0.5 + 0.2)$$



Projections

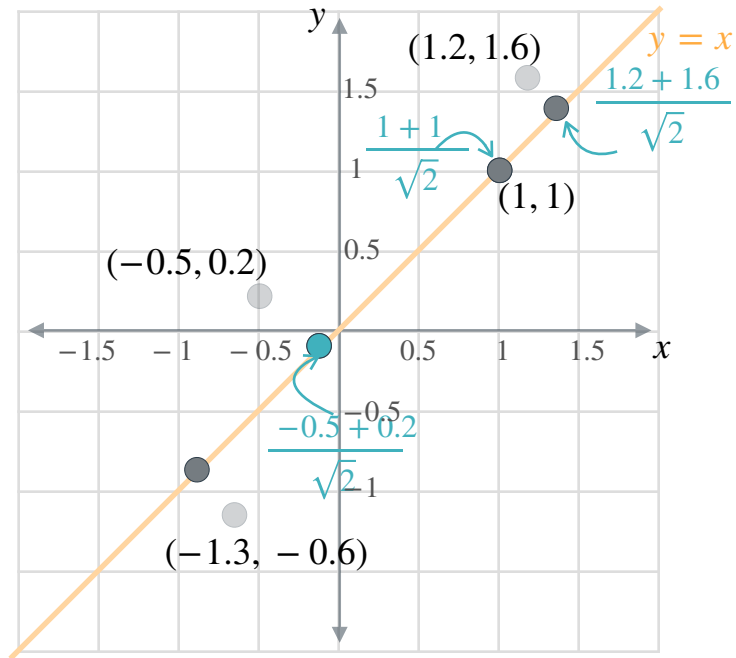


$$\frac{1}{1} \cdot \frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$(1.2 + 1.6)/\sqrt{2}$$

$$(-0.5 + 0.2)/\sqrt{2}$$



Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

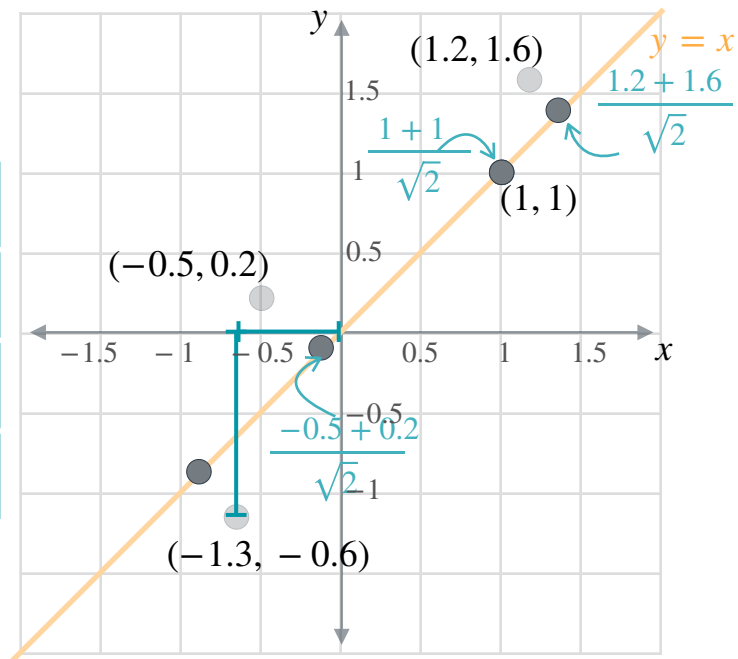
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\frac{1}{\sqrt{2}} =$$

$$(1 + 1) / \sqrt{2}$$

$$(1.2 + 1.6) / \sqrt{2}$$

$$(-0.5 + 0.2) / \sqrt{2}$$

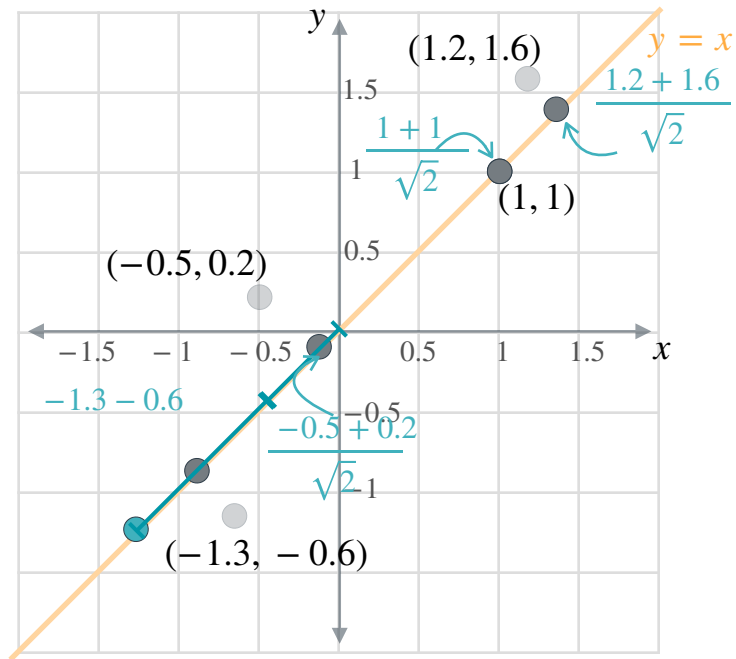


Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

$$\begin{aligned} & (1 + 1) / \sqrt{2} \\ & (1.2 + 1.6) / \sqrt{2} \\ & (-0.5 + 0.2) / \sqrt{2} \\ & (-1.3, -0.6) \end{aligned}$$



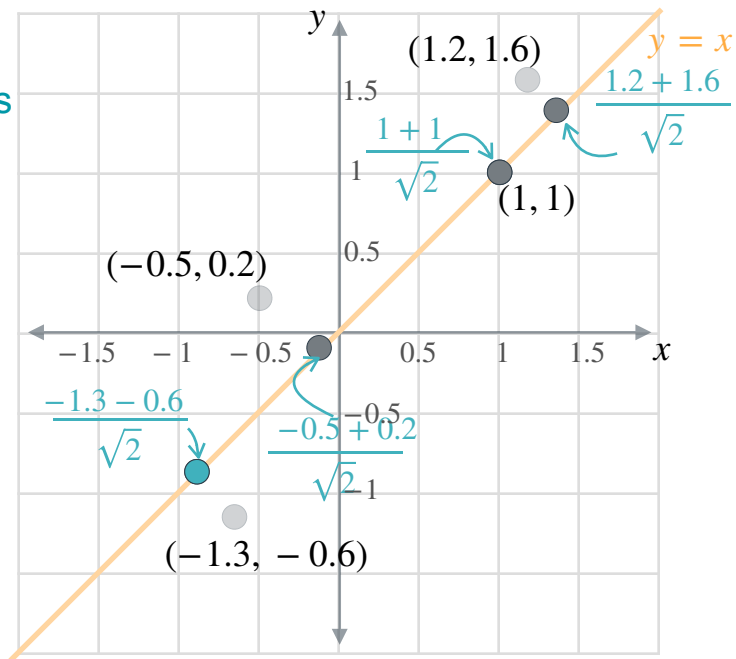
Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

$$\begin{aligned} & (1 + 1) / \sqrt{2} \\ & (1.2 + 1.6) / \sqrt{2} \\ & (-0.5 + 0.2) / \sqrt{2} \\ & (-1.3 - 0.6) / \sqrt{2} \end{aligned}$$



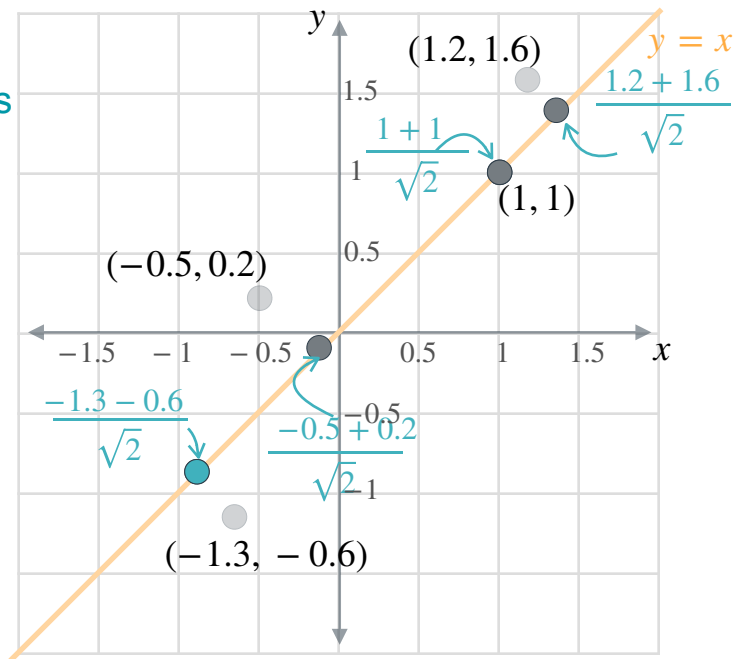
Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

| |
|---------|
| 1.4142 |
| 1.9799 |
| -0.2121 |
| -1.344 |



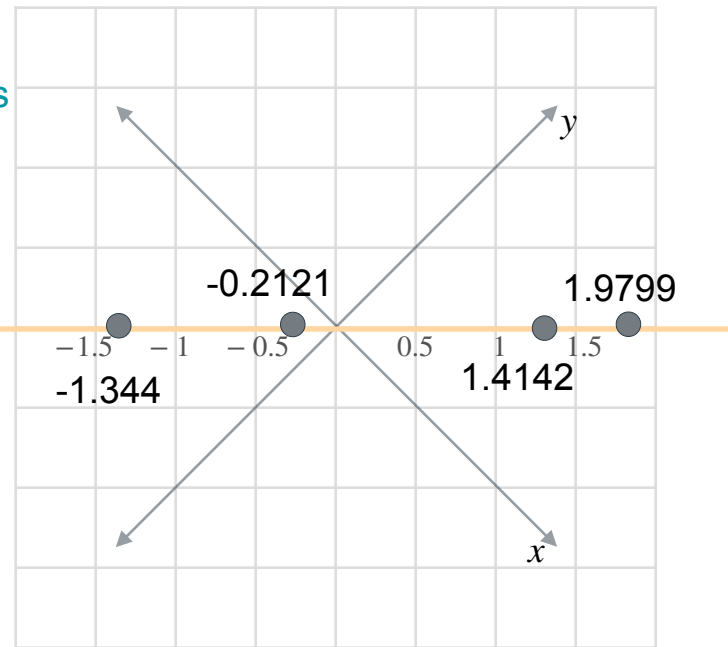
Projections

| x | y |
|------|------|
| 1.0 | 1.0 |
| 1.2 | 1.6 |
| -0.5 | 0.2 |
| -1.3 | -0.6 |

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{\sqrt{2}} =$$

Final coordinates

| |
|---------|
| 1.4142 |
| 1.9799 |
| -0.2121 |
| -1.344 |



Projections

To project a matrix A onto a vector v

$$A_P = A \frac{v}{\|v\|_2}$$

$r \times 1 \qquad r \times c \qquad c \times 1$

Projections

To project a matrix A onto vectors v_1 and v_2

$$A_P = A \overbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}^V$$

$r \times 2$ $r \times c$ $c \times 2$

Projections

To project a matrix A onto vectors v_1 and v_2

$$A_P = A \overbrace{\begin{bmatrix} \frac{v_1}{\|v_1\|_2} & \frac{v_2}{\|v_2\|_2} \end{bmatrix}}^V$$

$r \times 2$ $\boxed{r \times c}$ $c \times \boxed{2}$

Projections

To project a matrix A onto vectors v_1 and v_2

$$A_P = AV$$

$$r \times 2 \quad r \times c \quad c \times 2$$

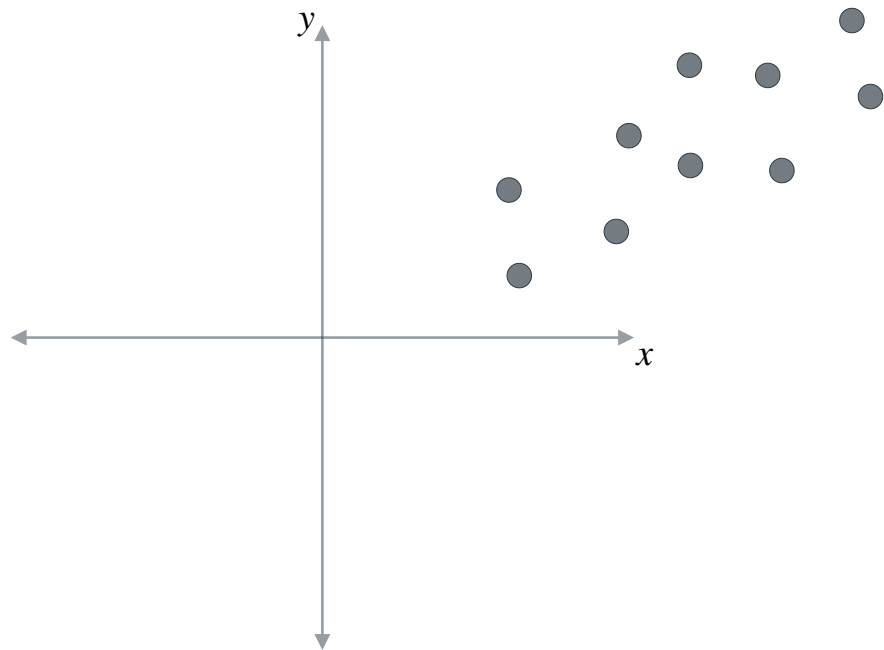


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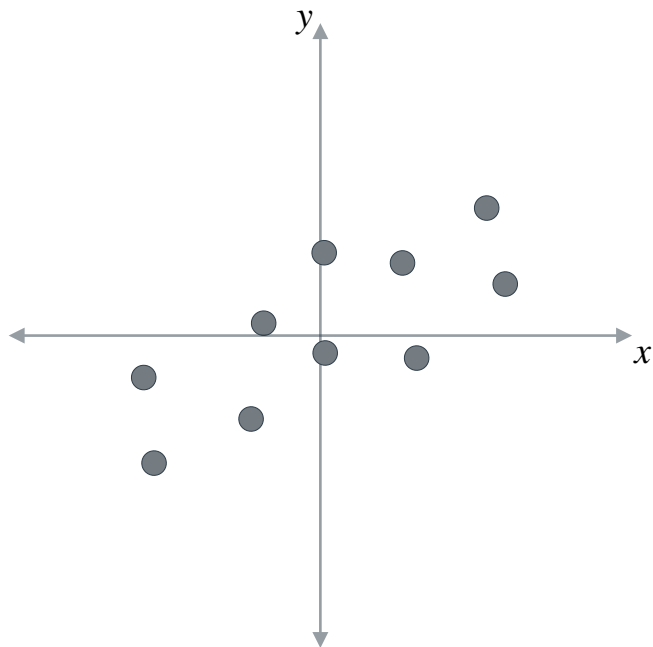
Determinants and Eigenvectors

Motivating PCA

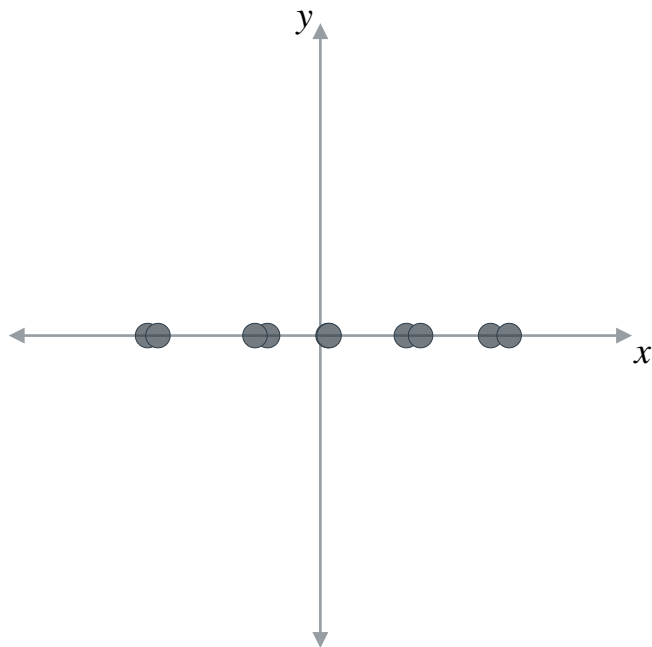
Dimensionality Reduction



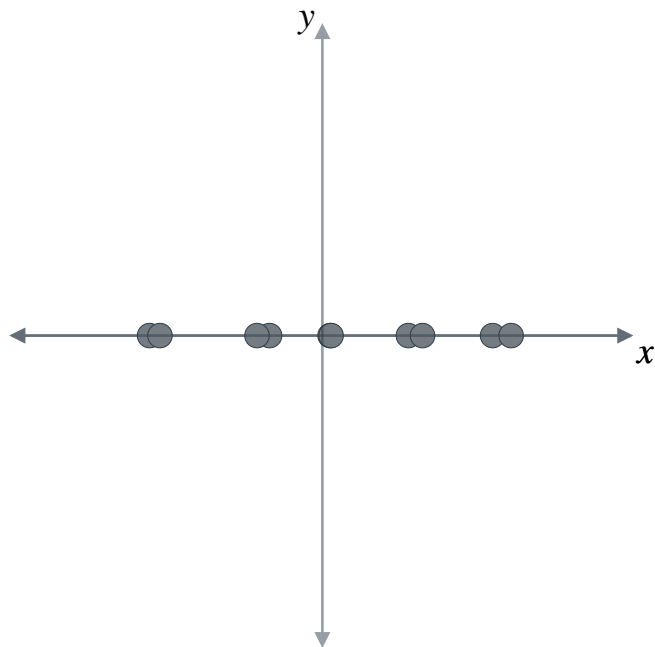
Principal Component Analysis (PCA)



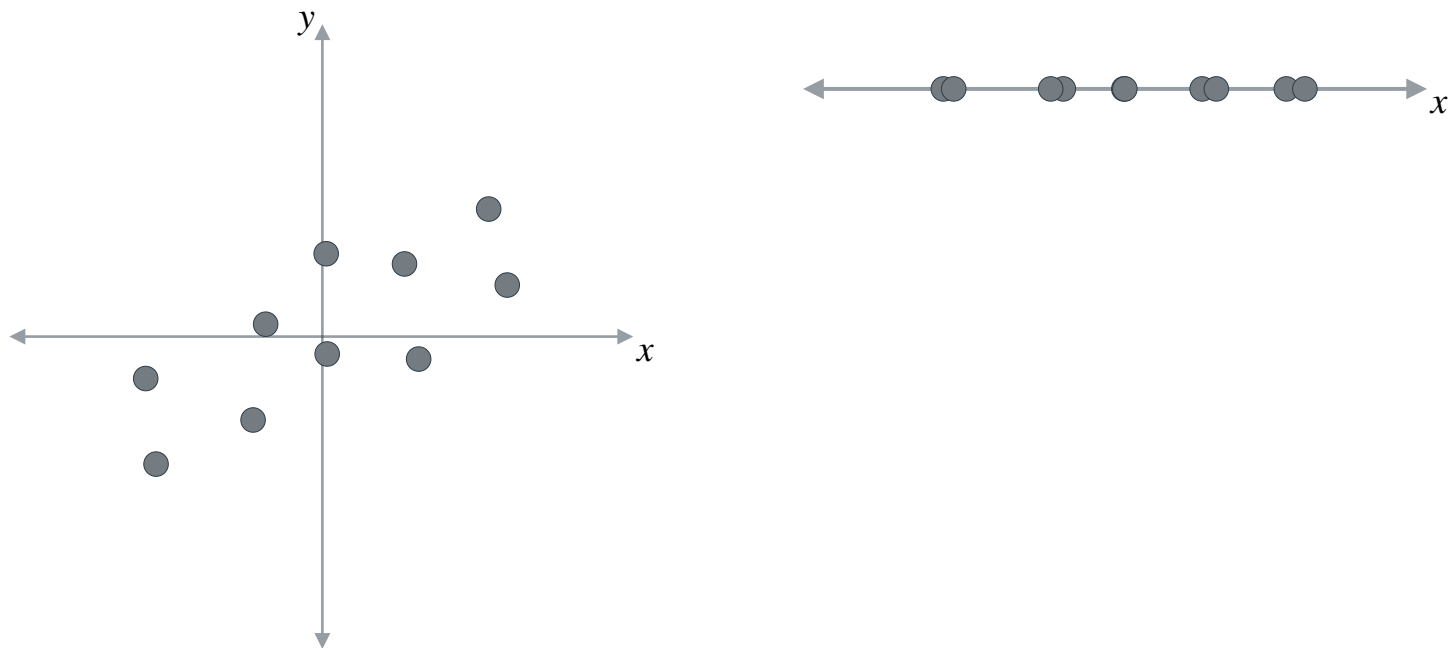
Principal Component Analysis (PCA)



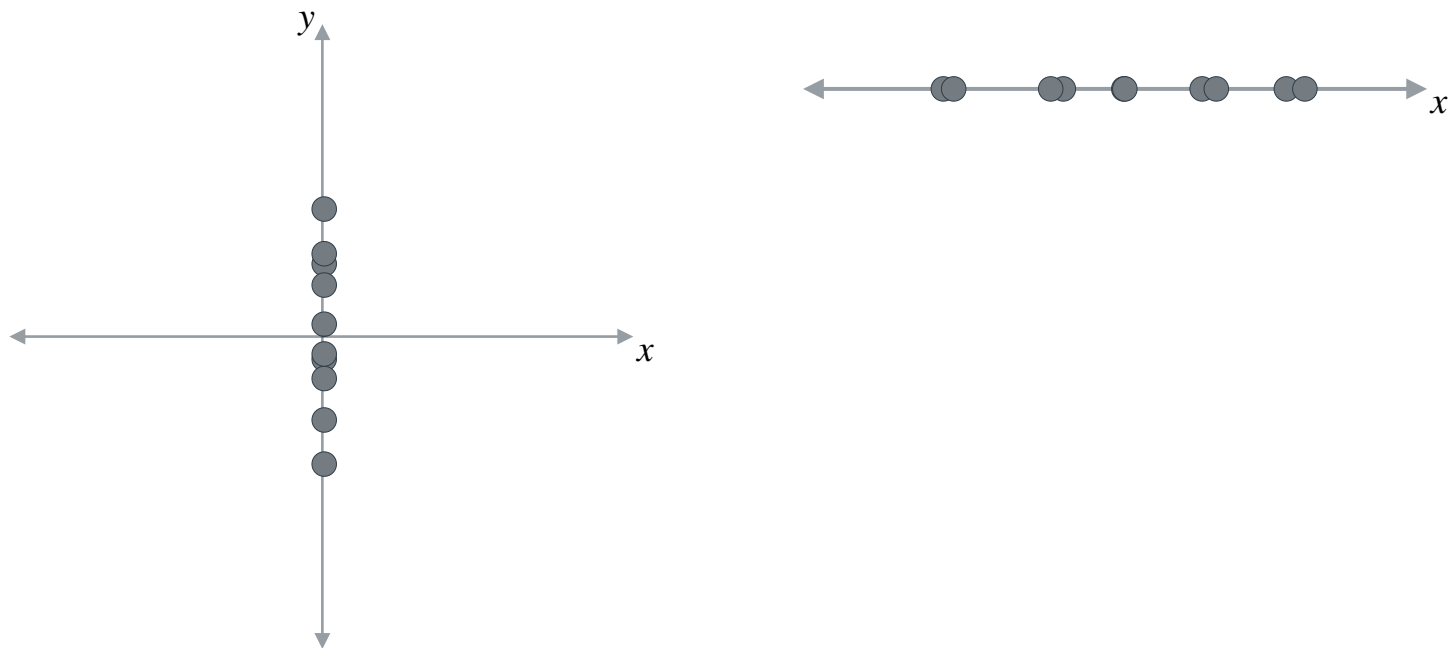
Principal Component Analysis (PCA)



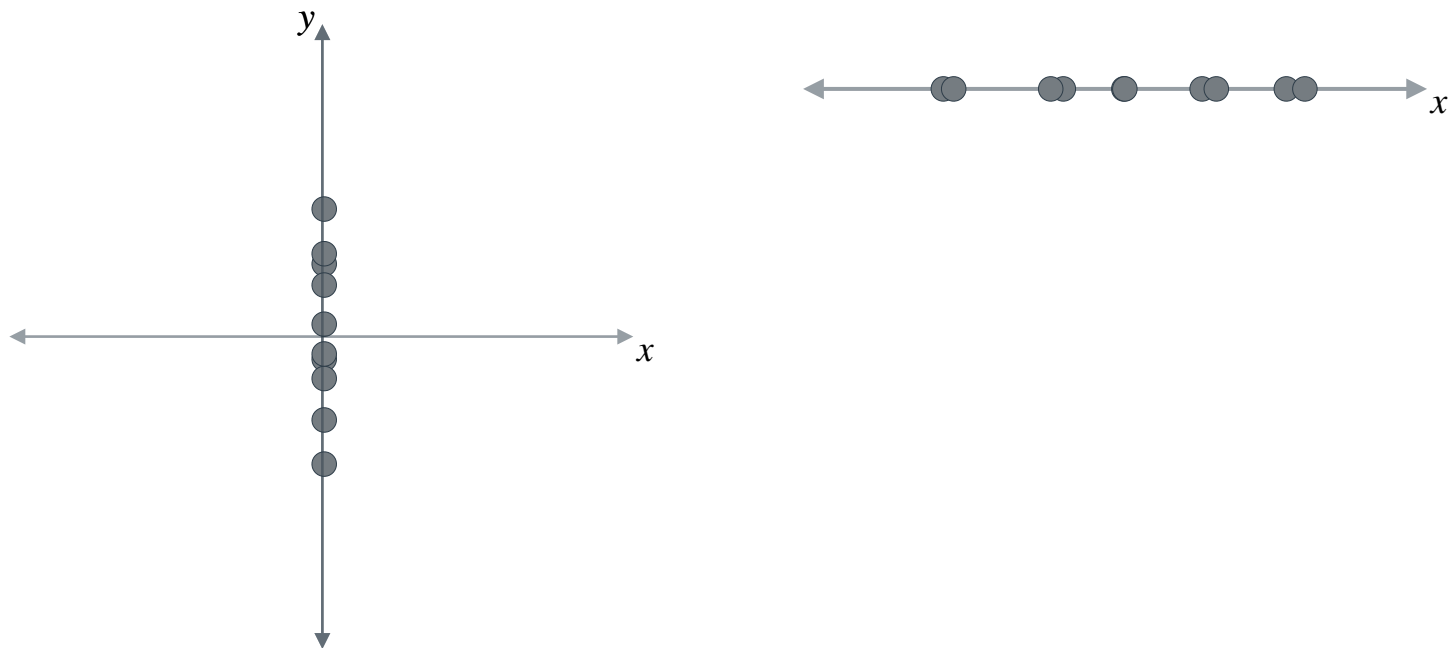
Principal Component Analysis (PCA)



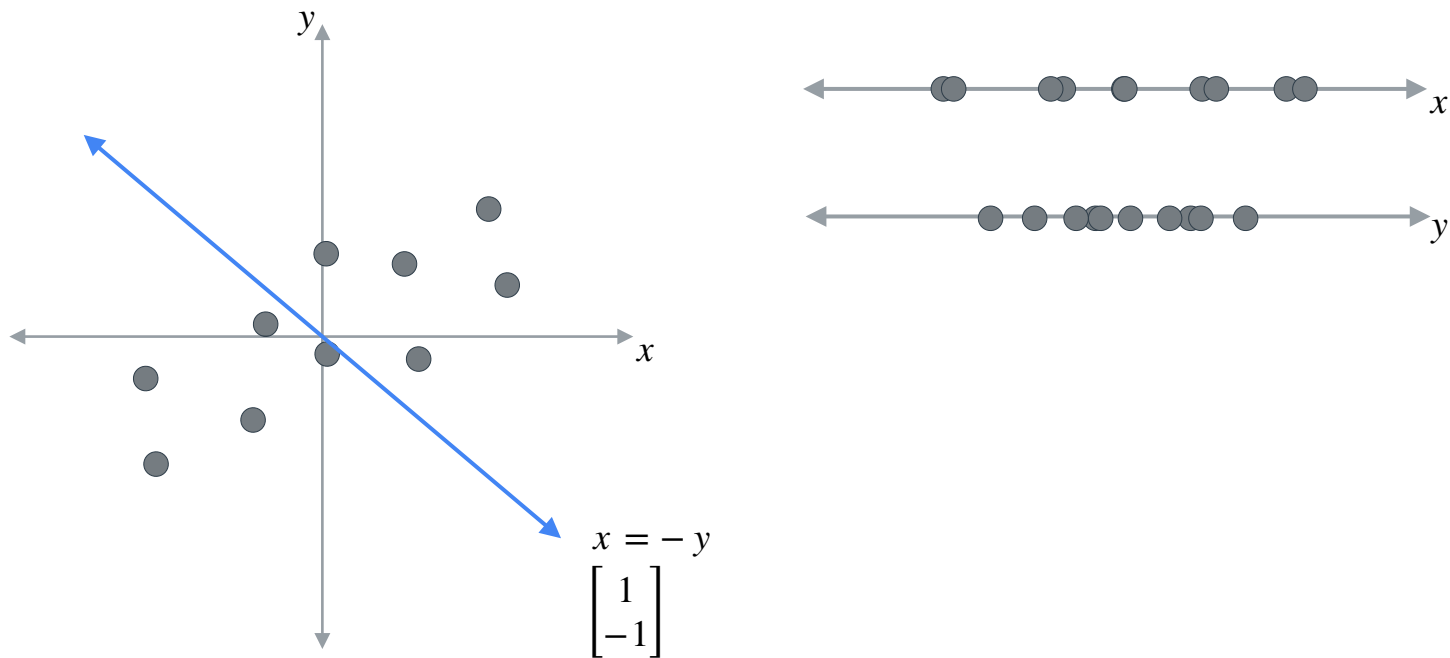
Principal Component Analysis (PCA)



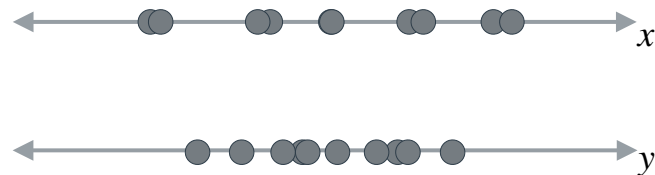
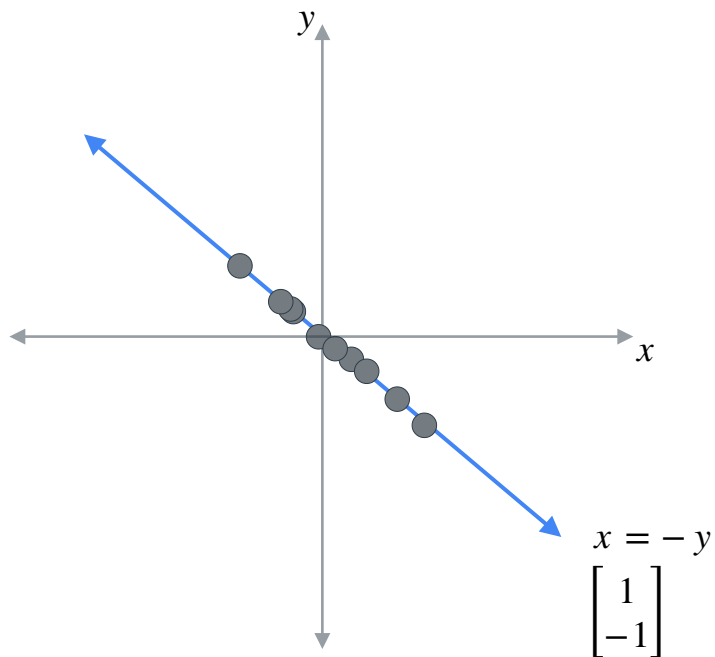
Principal Component Analysis (PCA)



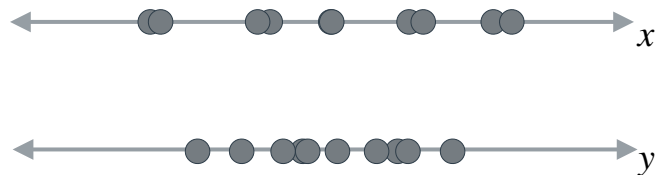
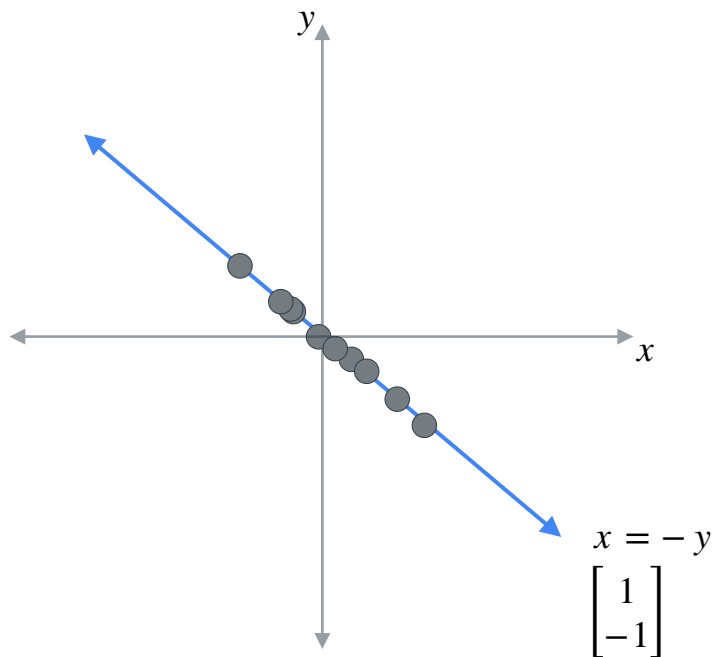
Principal Component Analysis (PCA)



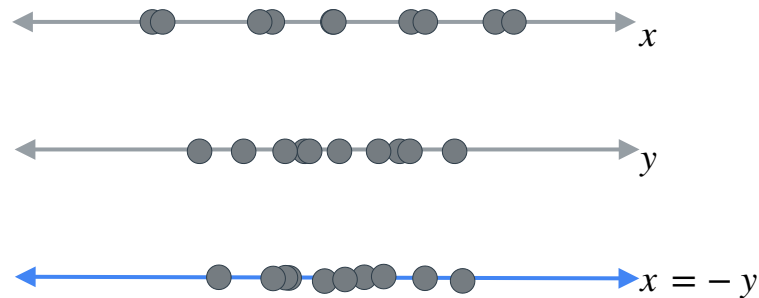
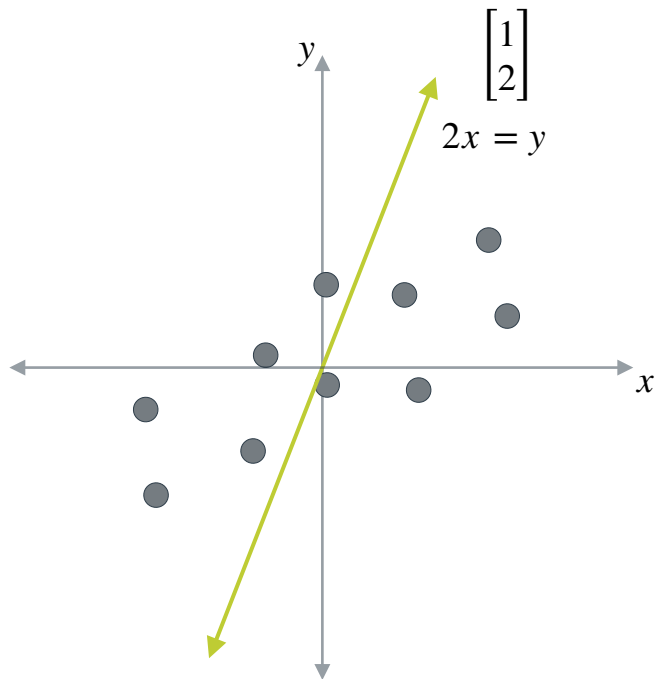
Principal Component Analysis (PCA)



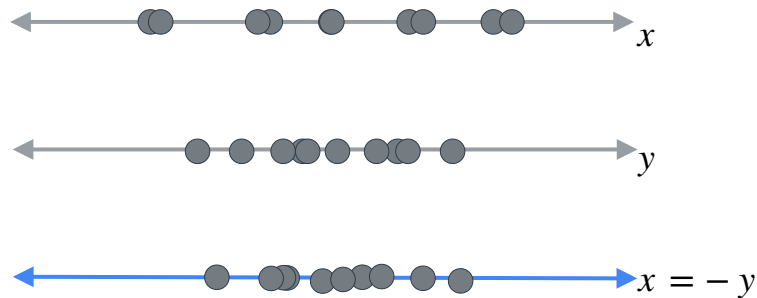
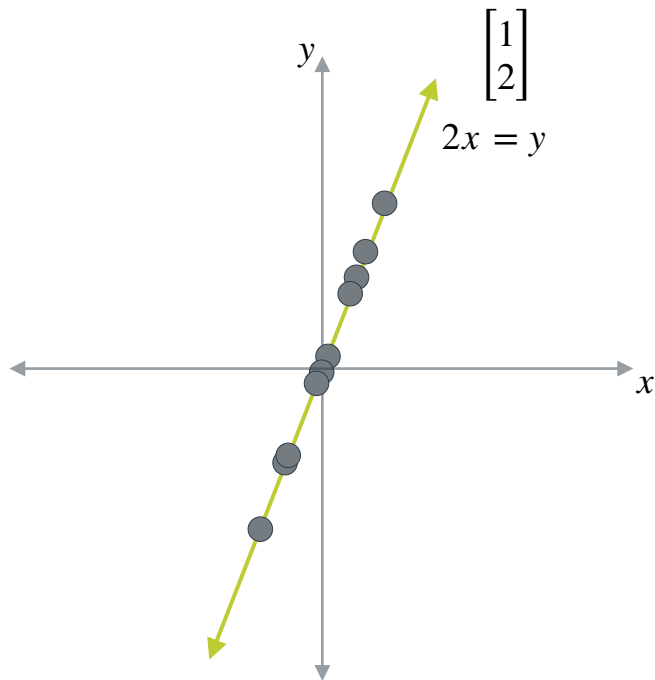
Principal Component Analysis (PCA)



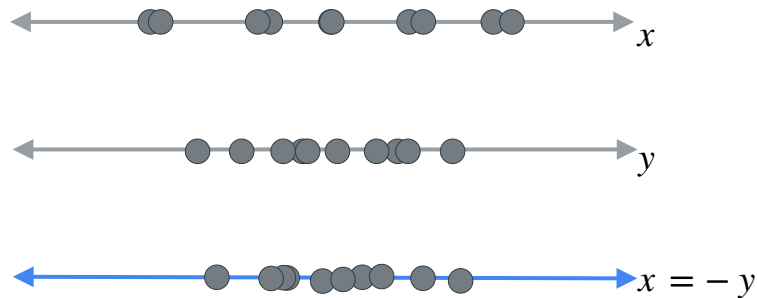
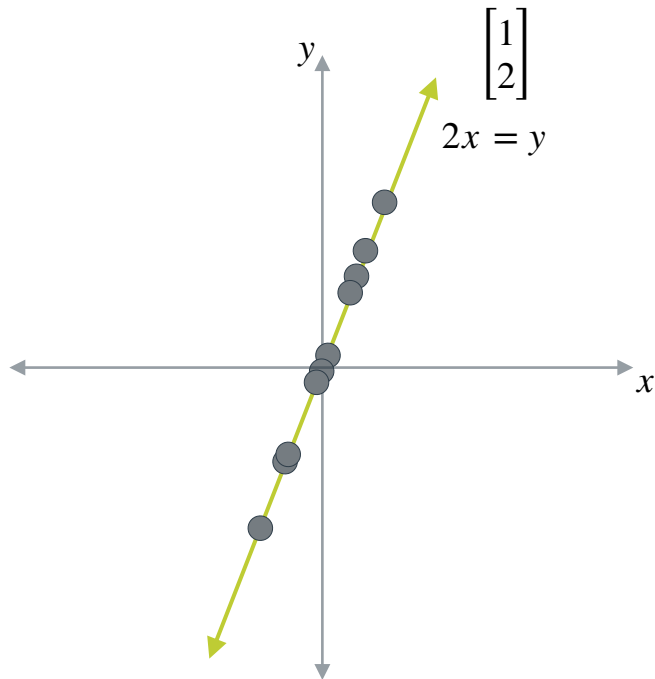
Principal Component Analysis (PCA)



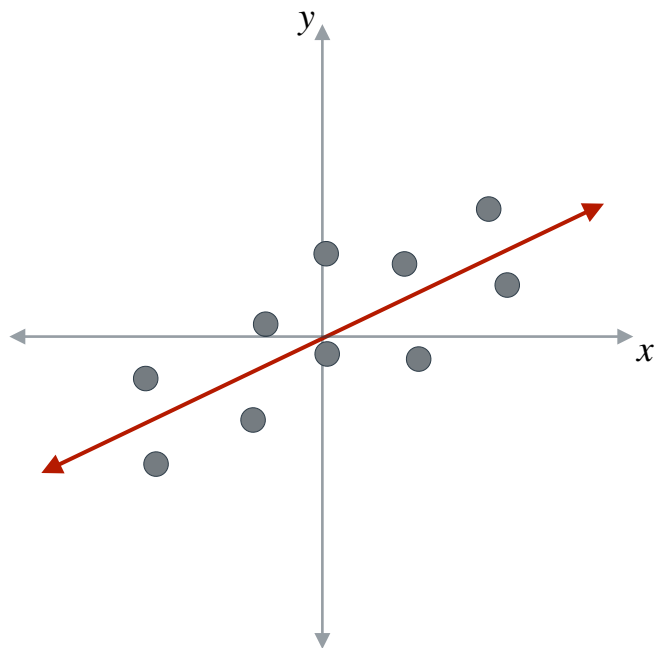
Principal Component Analysis (PCA)



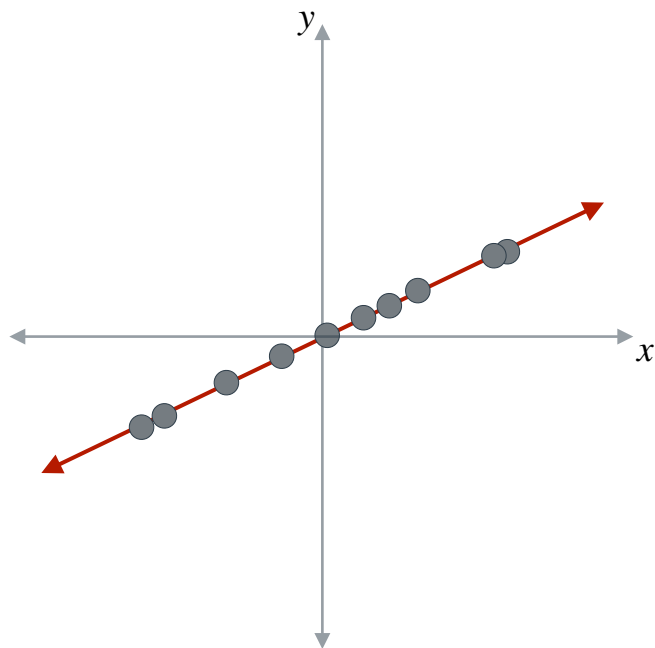
Principal Component Analysis (PCA)



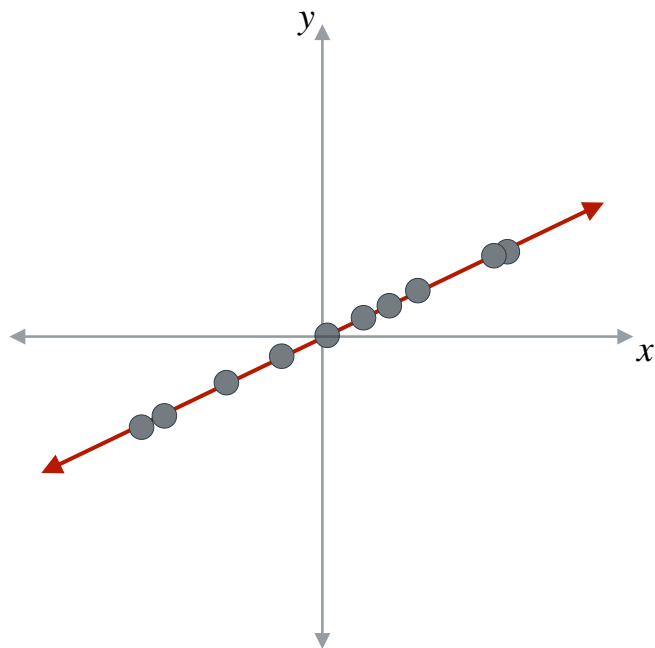
Principal Component Analysis (PCA)



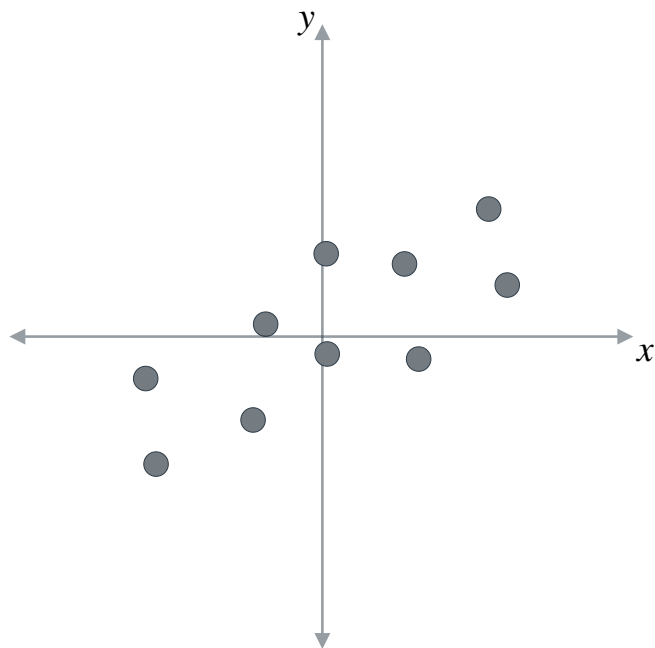
Principal Component Analysis (PCA)



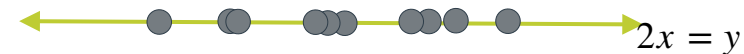
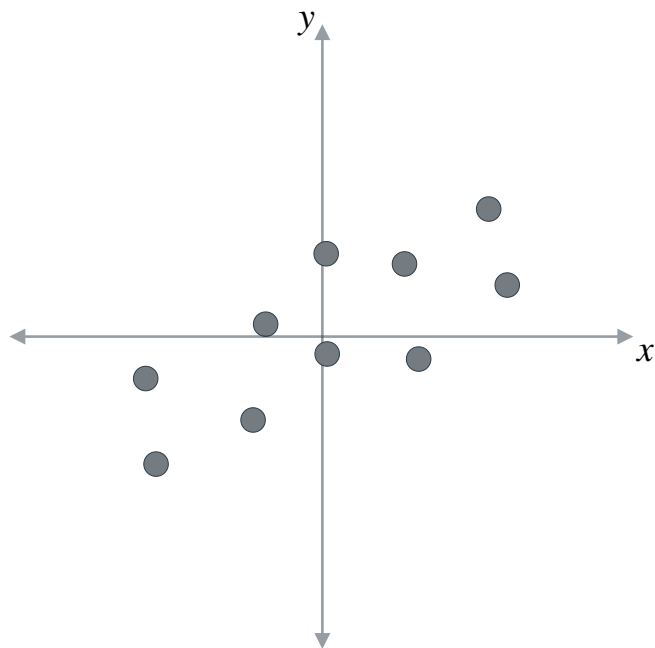
Principal Component Analysis (PCA)



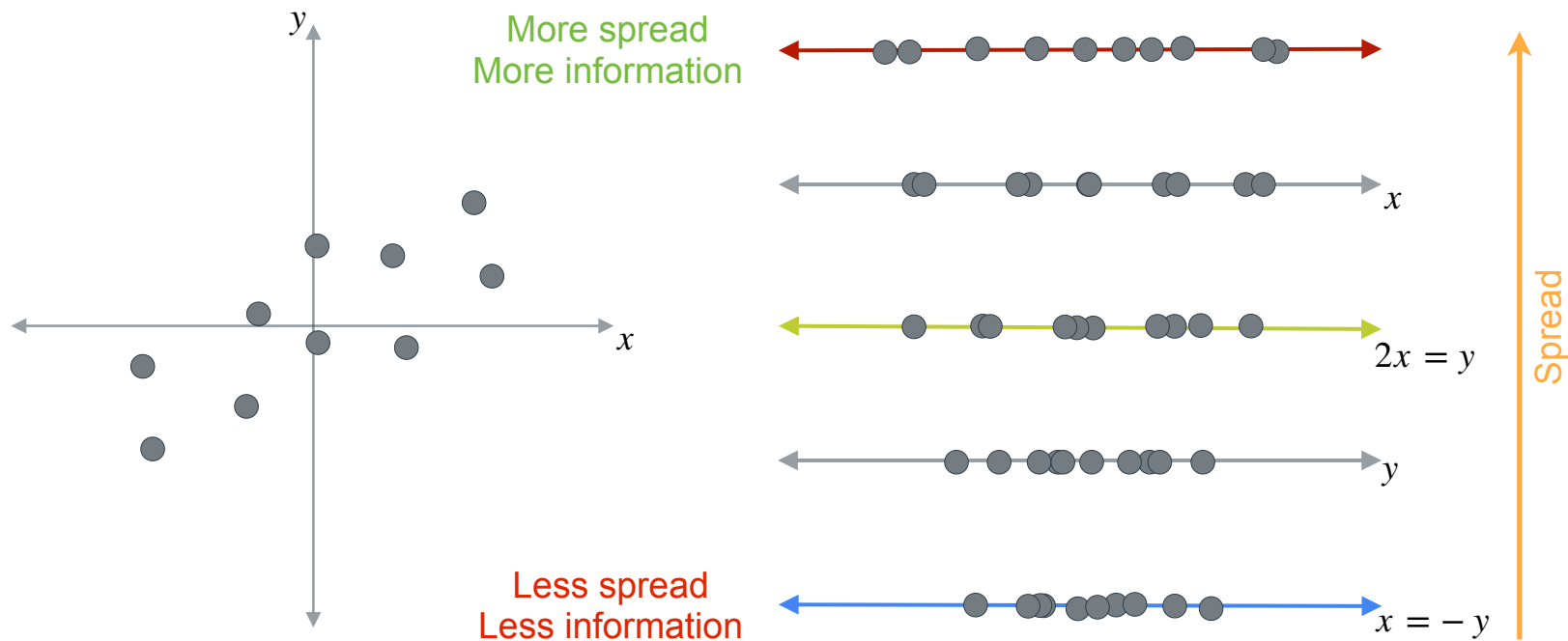
Principal Component Analysis (PCA)



Principal Component Analysis (PCA)

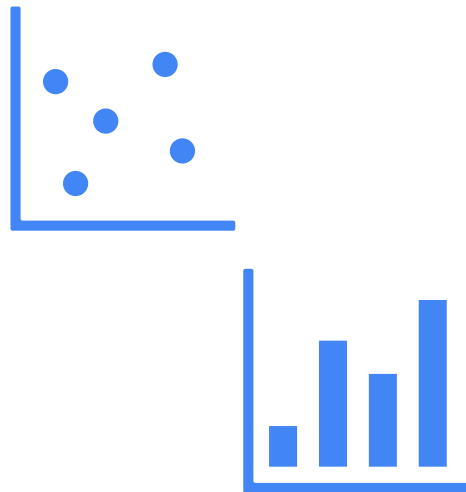
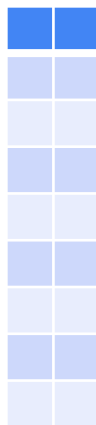
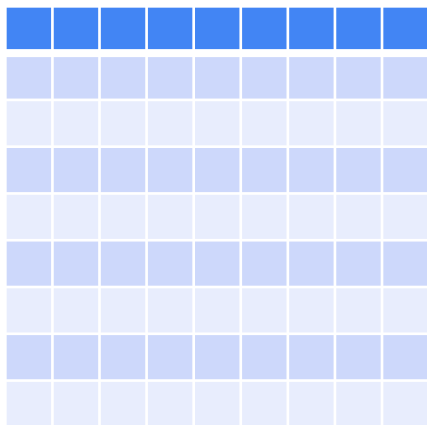


Principal Component Analysis (PCA)



Benefits of Dimensionality Reduction

- Easier dataset to manage
- PCA reduces dimensions while minimizing information loss
- Simpler visualization





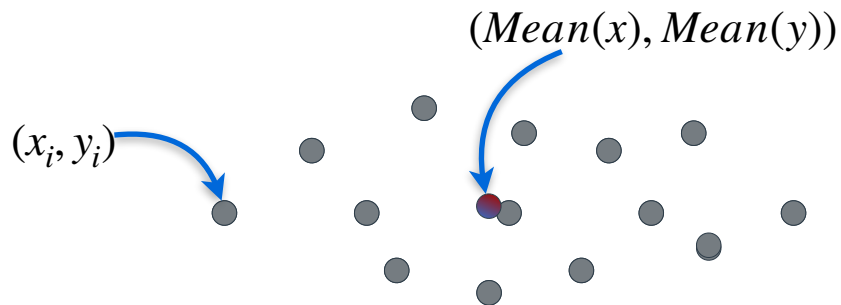
DeepLearning.AI

Determinants and Eigenvectors

Variance and covariance

Mean

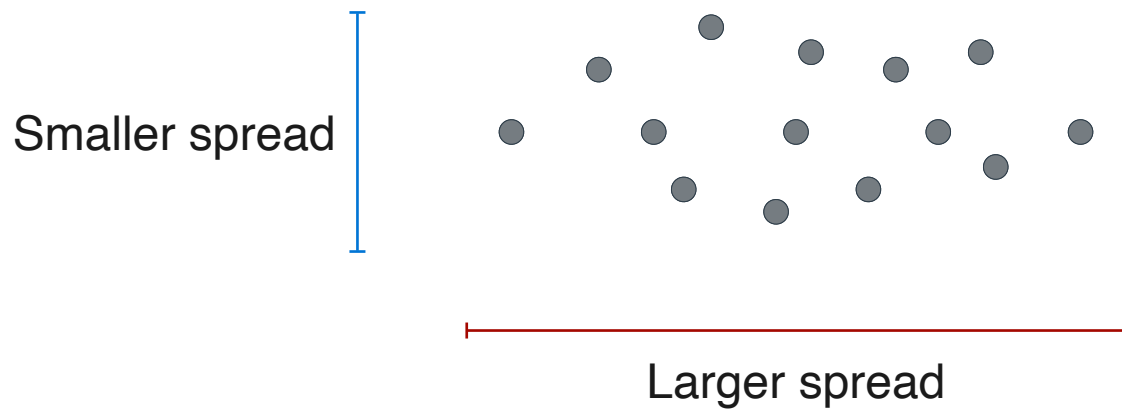
“The average of the data”



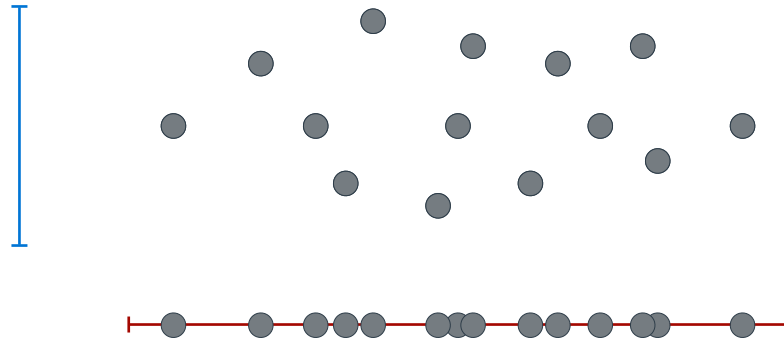
$$Mean(x) = \frac{1}{n} \sum_{i=1}^n x_i$$

$$Mean(y) = \frac{1}{n} \sum_{i=1}^n y_i$$

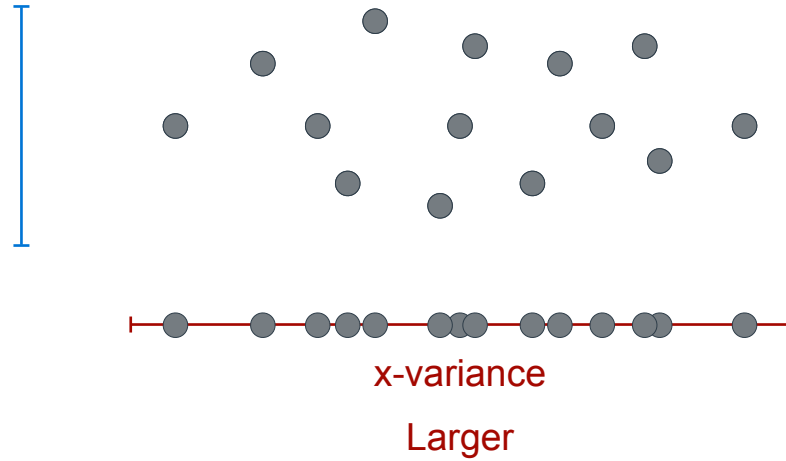
Variance



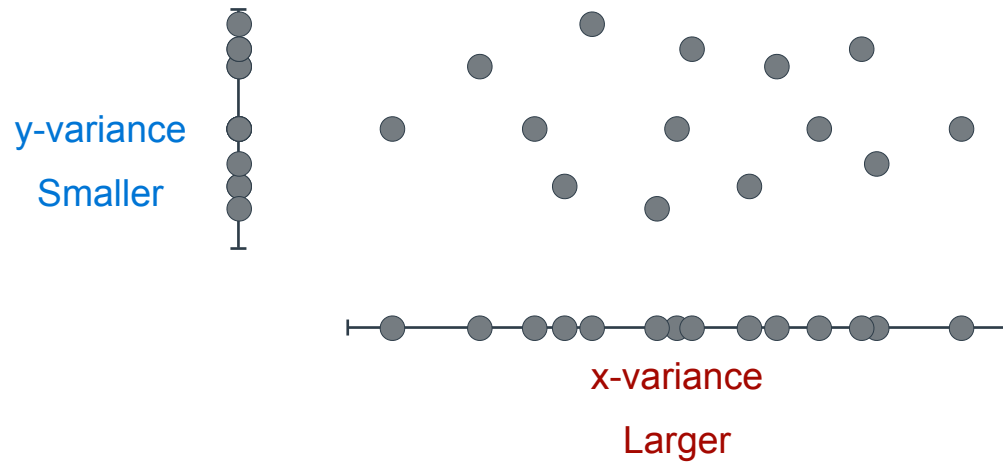
Variance



Variance



Variance



Variance

$$\text{Variance}(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \text{Mean}(x))^2 = 16$$

| | x_i | $x_i - \text{Mean}(x)$ | $(x_i - \text{Mean}(x))^2$ |
|---|-------|------------------------|----------------------------|
| 1 | 10 | 1 | 1 |
| 2 | 4 | -5 | 25 |
| 3 | 11 | 2 | 4 |
| 4 | 14 | 5 | 25 |
| 5 | 6 | -3 | 9 |

→ 64

$$\text{Mean}(x) = 9$$

Variance

$$\text{Variance}(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \text{Mean}(x))^2$$

$$\text{Var}(x)$$

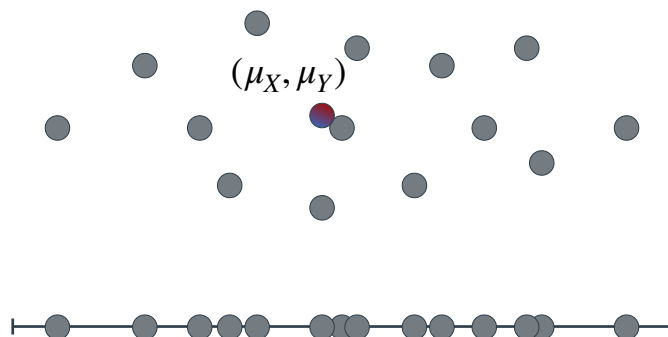
$$\mu_x$$

“The average squared distance from the mean”

Variance

$$Var(y) = \frac{1}{n-1} \sum_{i=1}^n (y_i - \mu_Y)^2$$

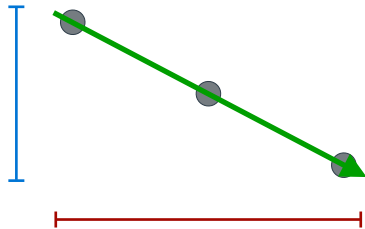
y-variance
Smaller



x-variance
Larger

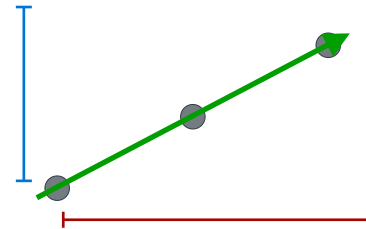
$$Var(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2$$

Problem



Negative covariance

Solution: Covariance



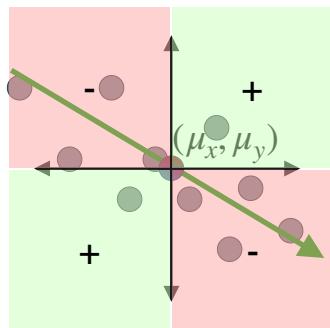
Positive covariance

Covariance

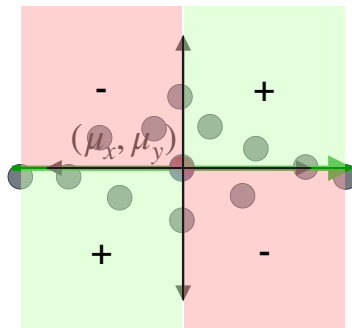
“Take the average”

$$Cov(x, y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

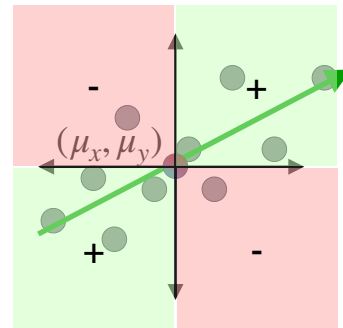
$$Var(x) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2$$



negative
covariance



covariance zero
(or very small)

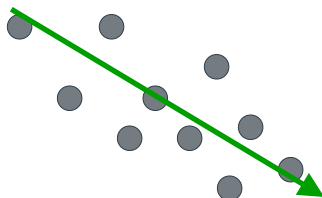


positive
covariance

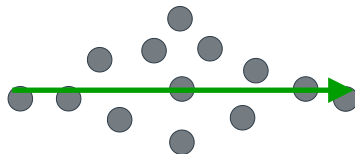
Covariance

$$\text{Cov}(x, y) = \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y)$$

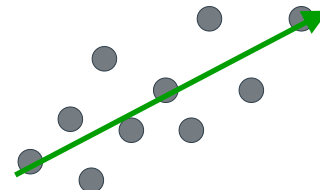
“The direction of the relationship between two variables”



negative
covariance



covariance zero
(or very small)



positive
covariance

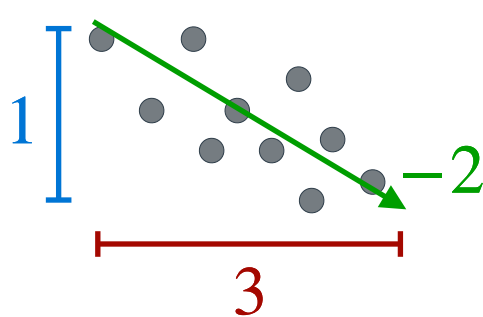


DeepLearning.AI

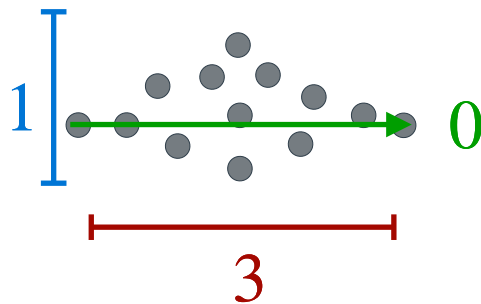
Determinants and Eigenvectors

The covariance matrix

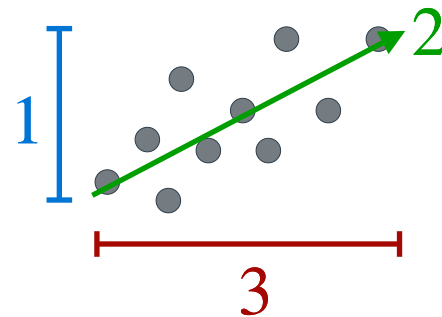
Covariance matrix



$$\begin{bmatrix} & \\ & \end{bmatrix}$$

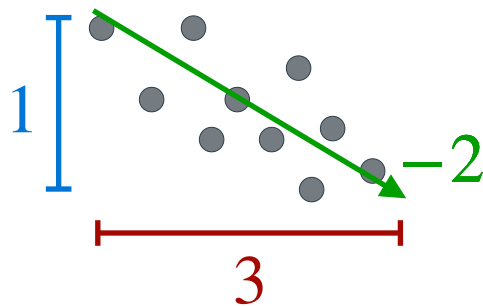


$$\begin{bmatrix} & \\ & \end{bmatrix}$$

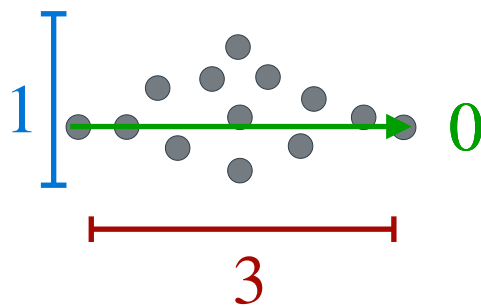


$$\begin{bmatrix} & \\ & \end{bmatrix}$$

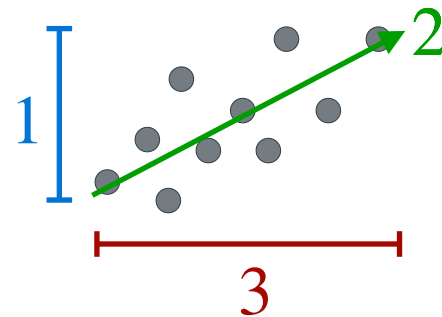
Covariance matrix



$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

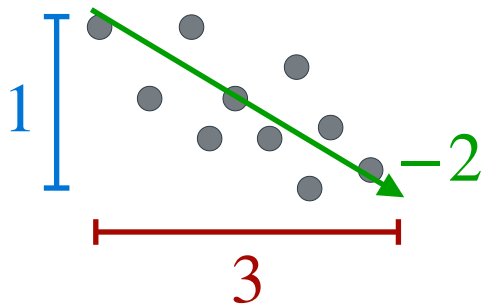


$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

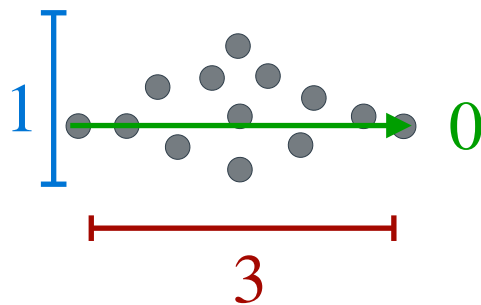


$$\begin{bmatrix} 3 & \\ & 1 \end{bmatrix}$$

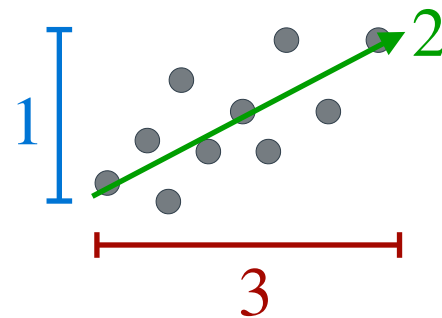
Covariance matrix



$$\begin{bmatrix} 3 & -2 \\ -2 & 1 \end{bmatrix}$$

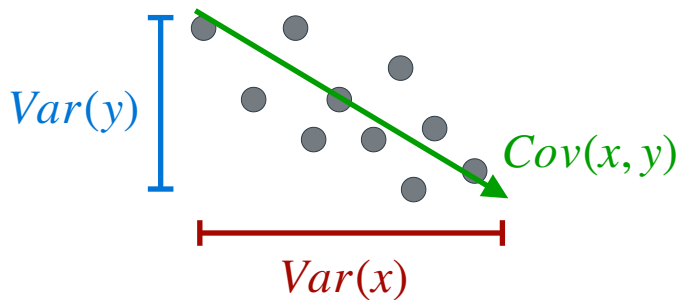


$$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$



$$\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

Covariance matrix



$$C = \begin{matrix} & \begin{matrix} x & y \end{matrix} \\ \begin{matrix} x \\ y \end{matrix} & \begin{bmatrix} \text{Var}(x) & Cov(x, y) \\ Cov(y, x) & \text{Var}(y) \end{bmatrix} \end{matrix}$$

$$Cov(x, x) = Var(x)$$

Covariance matrix

$$\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1} \left(\begin{array}{c} \\ \\ \\ \end{array} - \begin{array}{c} \\ \\ \\ \end{array} \right)^T \left(\begin{array}{c} \\ \\ \\ \end{array} - \begin{array}{c} \\ \\ \\ \end{array} \right)$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1}(A - \mu)^T(A - \mu)$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$C = \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{array}{cc} & \\ & \\ & \\ & \end{array} \right)^T \left(\begin{array}{cc} & \\ & \\ & \\ & \end{array} \right)$$
$$= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}$$

Covariance matrix

$$A = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} \quad \mu = \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix}$$

$$\begin{aligned} C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\ &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \end{aligned}$$

Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &\quad \quad \quad \boxed{2} \times n \quad \quad \quad n \times \boxed{2}
 \end{aligned}$$

Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}
 \end{aligned}$$

$$(x_1 - \mu_x)(x_1 - \mu_x) + (x_2 - \mu_x)(x_2 - \mu_x) + \dots + (x_n - \mu_x)(x_n - \mu_x)$$

Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}
 \end{aligned}$$

$\sum_{i=1}^n (x_i - \mu_x)^2 = \text{Var}(x)$

Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix} \\
 &\quad \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)^2 = \text{Var}(x)
 \end{aligned}$$

Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix} \\
 &\quad (x_1 - \mu_x)(y_1 - \mu_y) + (x_2 - \mu_x)(y_2 - \mu_y) + \dots + (x_n - \mu_x)(y_n - \mu_y)
 \end{aligned}$$

Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix}
 \end{aligned}$$

$\sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = \text{Cov}(x, y)$

Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \end{bmatrix} \\
 &= \frac{1}{n-1} \sum_{i=1}^n (x_i - \mu_x)(y_i - \mu_y) = \text{Cov}(x, y)
 \end{aligned}$$

Covariance matrix

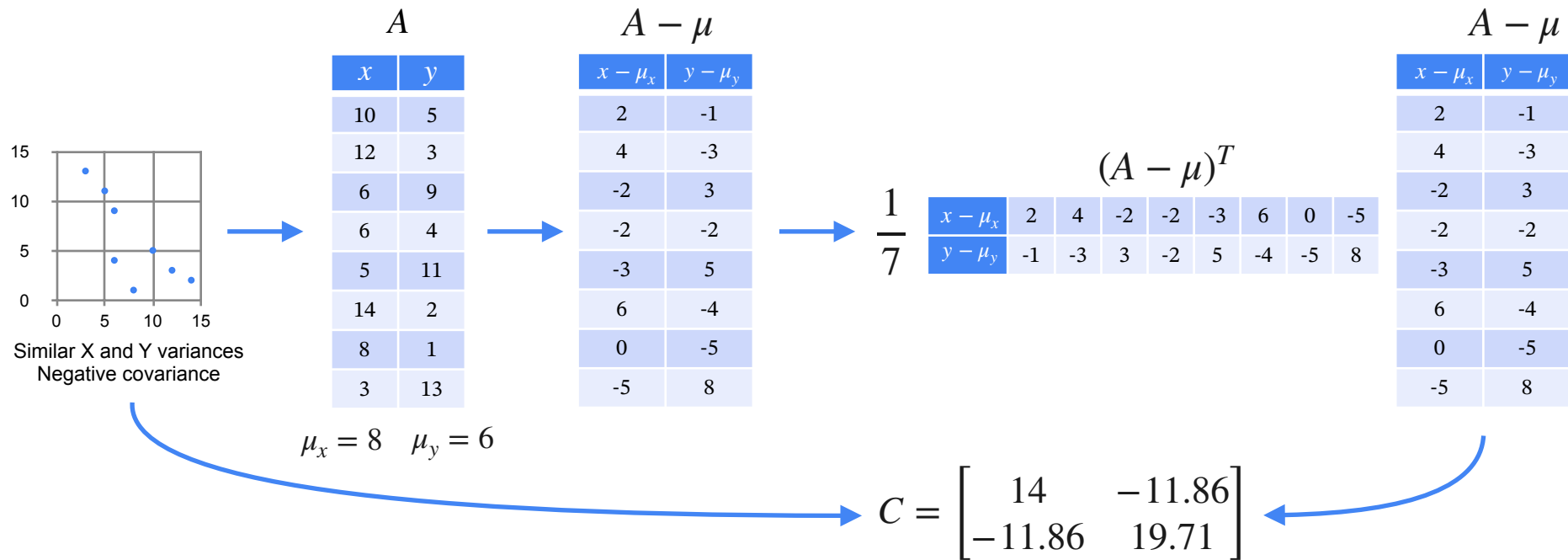
$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix}
 \end{aligned}$$

Covariance matrix

$$\begin{aligned}
 C &= \frac{1}{n-1} (A - \mu)^T (A - \mu) = \frac{1}{n-1} \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right)^T \left(\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{bmatrix} - \begin{bmatrix} \mu_x & \mu_y \\ \mu_x & \mu_y \\ \vdots & \vdots \\ \mu_x & \mu_y \end{bmatrix} \right) \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix}^T \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \\
 &= \frac{1}{n-1} \begin{bmatrix} x_1 - \mu_x & x_2 - \mu_x & \dots & x_n - \mu_x \\ y_1 - \mu_y & y_2 - \mu_y & \dots & y_n - \mu_y \end{bmatrix} \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ x_2 - \mu_x & y_2 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} = \begin{bmatrix} \text{Var}(x) & \text{Cov}(x, y) \\ \text{Cov}(y, x) & \text{Var}(y) \end{bmatrix}
 \end{aligned}$$

Matrix formula

$$A - \mu = \begin{bmatrix} x_1 - \mu_x & y_1 - \mu_y \\ \vdots & \vdots \\ x_n - \mu_x & y_n - \mu_y \end{bmatrix} \quad C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$



Matrix formula

$$A = \begin{bmatrix} x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & z_n \end{bmatrix} \quad C = \frac{1}{n-1} (A - \mu)^T (A - \mu)$$

1. Arrange data with a different feature in each column
2. Calculate column averages
3. Subtract each average from their respective column to generate $A - \mu$
4. $\frac{1}{n-1} (A - \mu)^T (A - \mu)$ gives the covariance matrix C

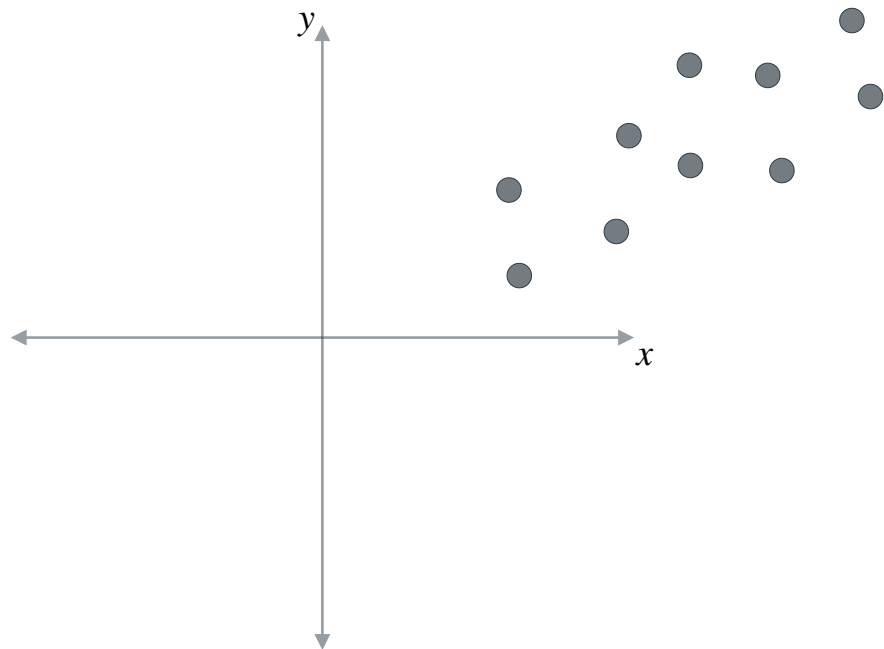


DeepLearning.AI

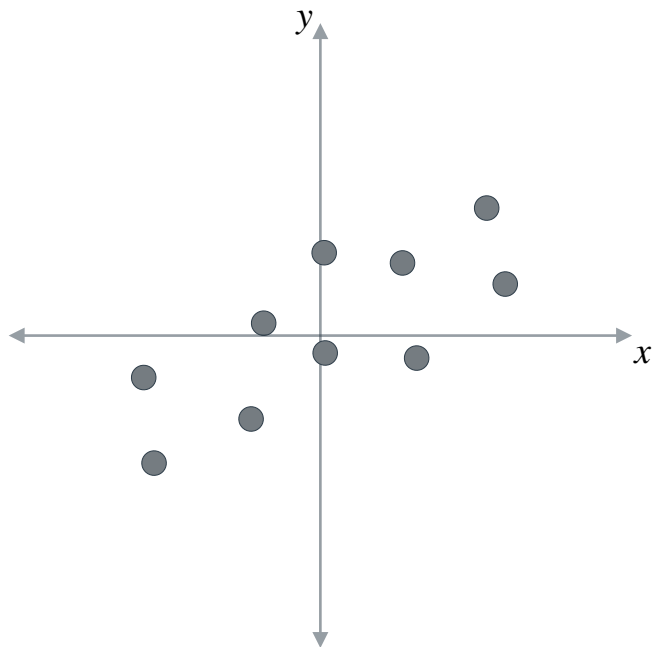
Determinants and Eigenvectors

PCA - Overview

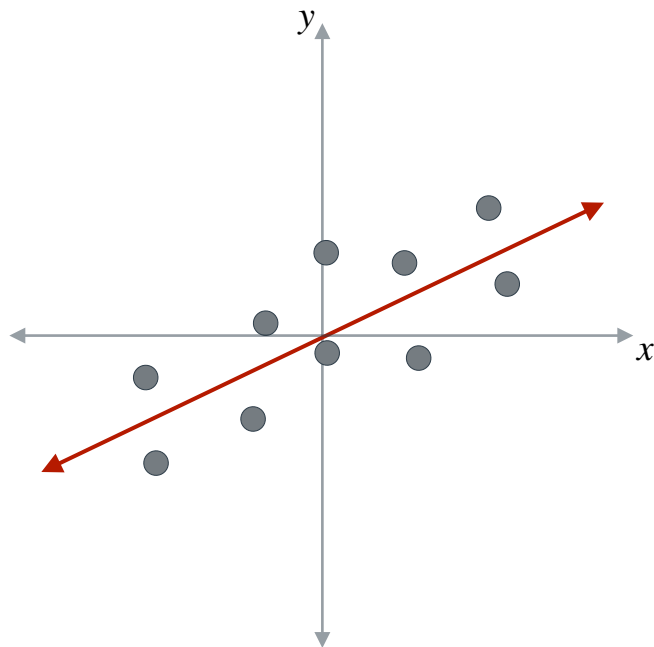
Principal Component Analysis (PCA)



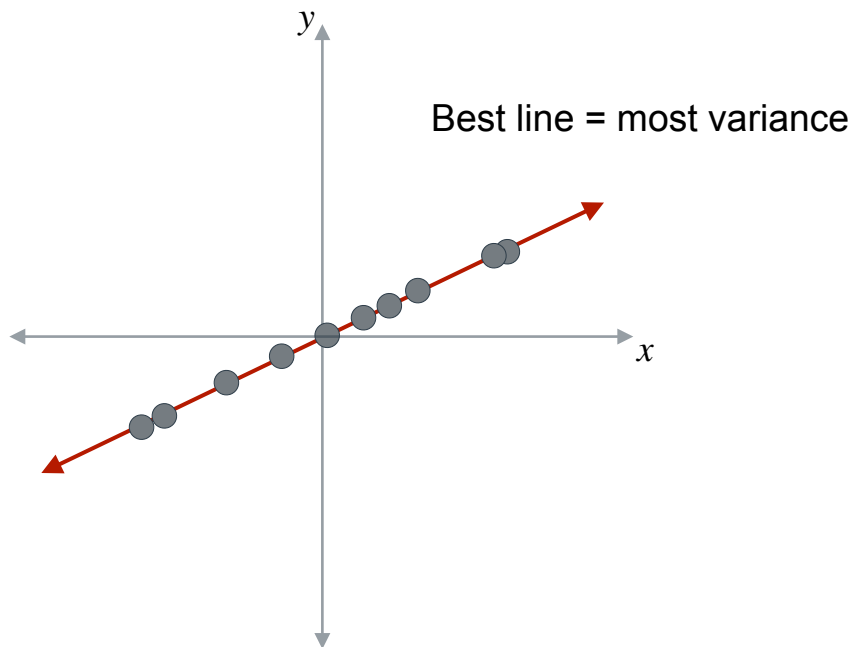
Principal Component Analysis (PCA)



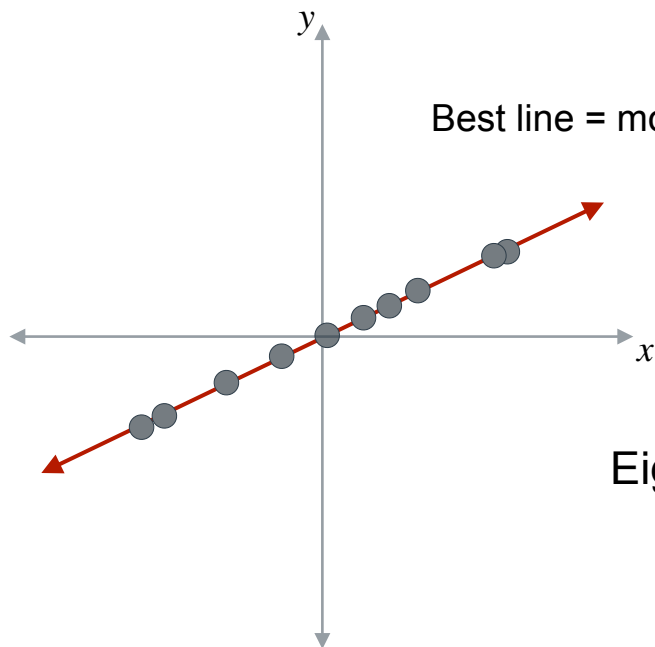
Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



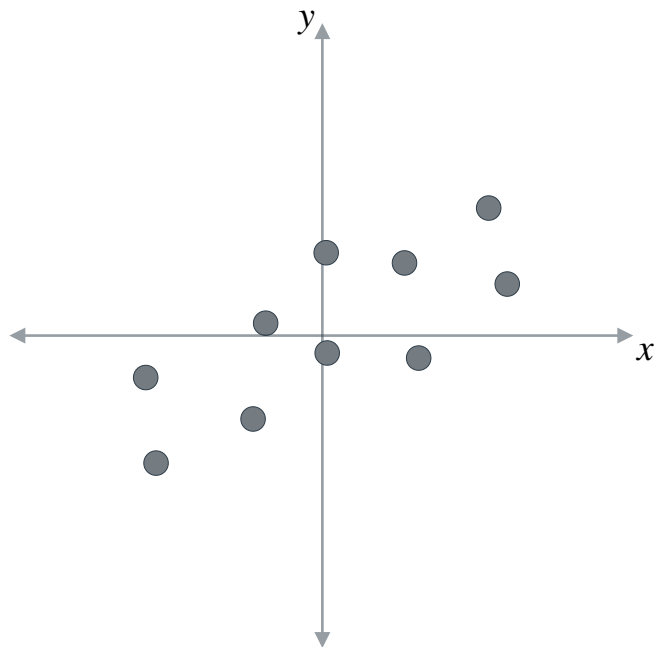
Projections

Eigenvalues / Eigenvectors

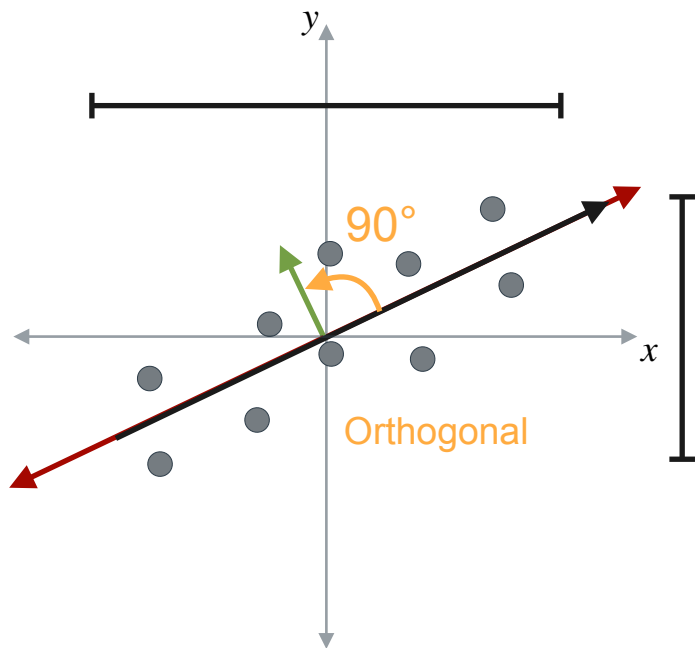
Covariance Matrix

PCA

Principal Component Analysis (PCA)



Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

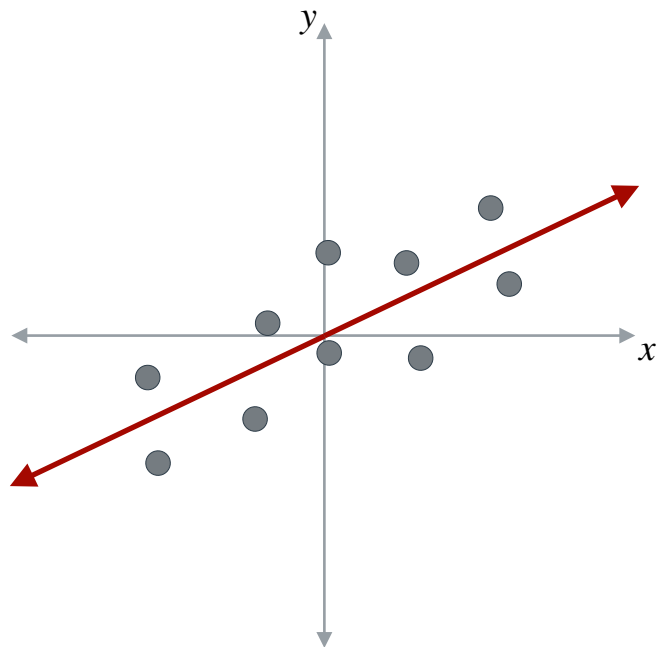
Eigenvectors
(direction)

$$11$$


$$1$$

Eigenvalues
(magnitude)

Principal Component Analysis (PCA)



$$C = \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

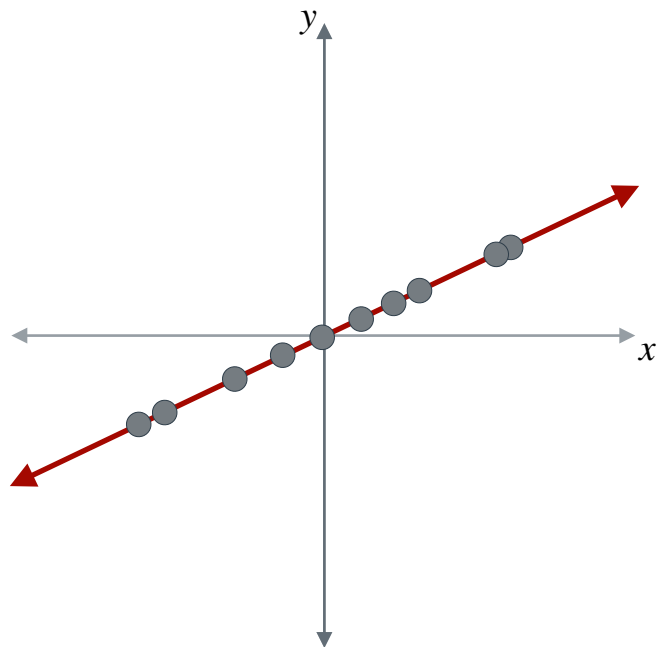
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors
(direction)

$$11$$

Eigenvalues
(magnitude)

Principal Component Analysis (PCA)



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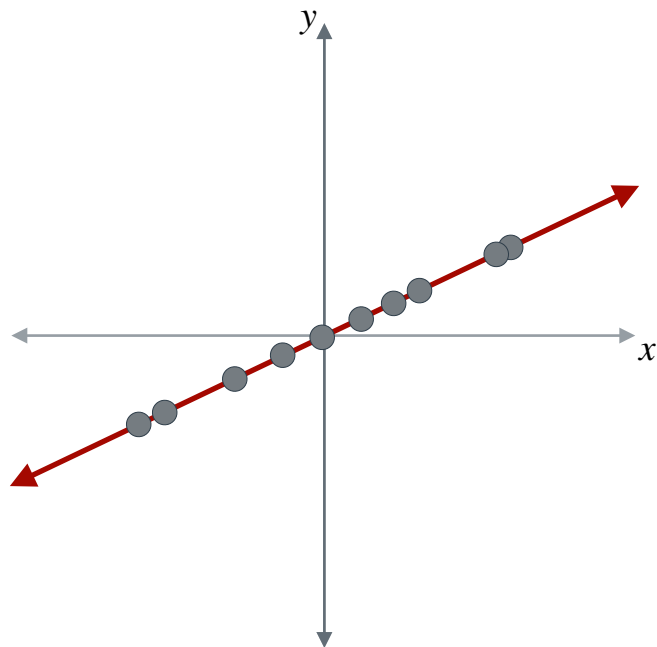
$$\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Eigenvectors
(direction)

$$11$$

Eigenvalues
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Principal Component Analysis (PCA)



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Eigenvectors
(direction)

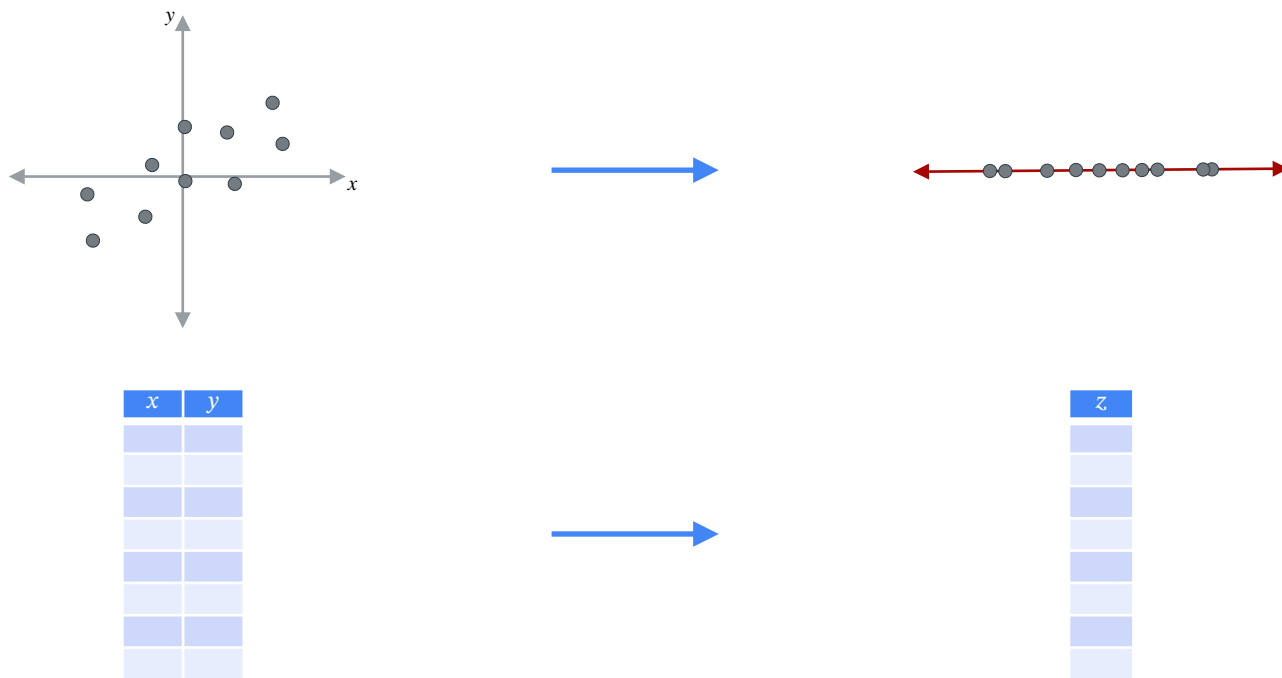
$$11$$

Eigenvalues
(magnitude)

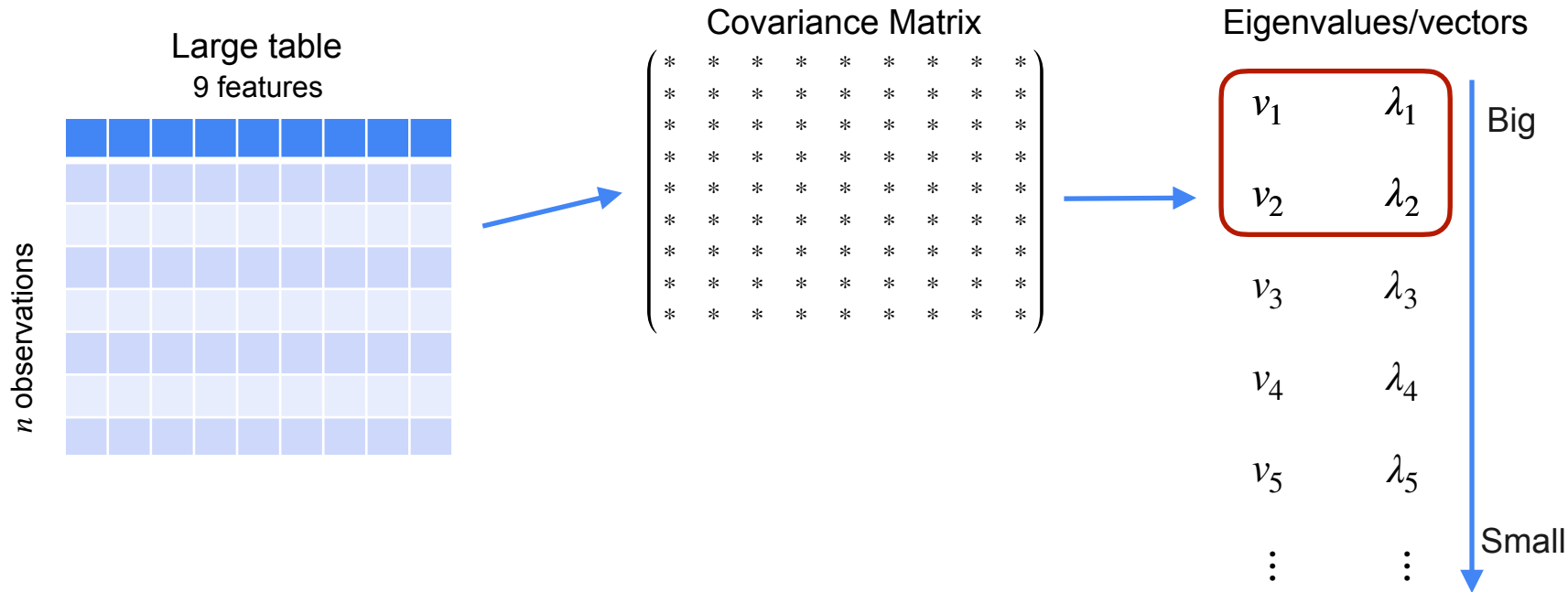
Principal Component Analysis (PCA)



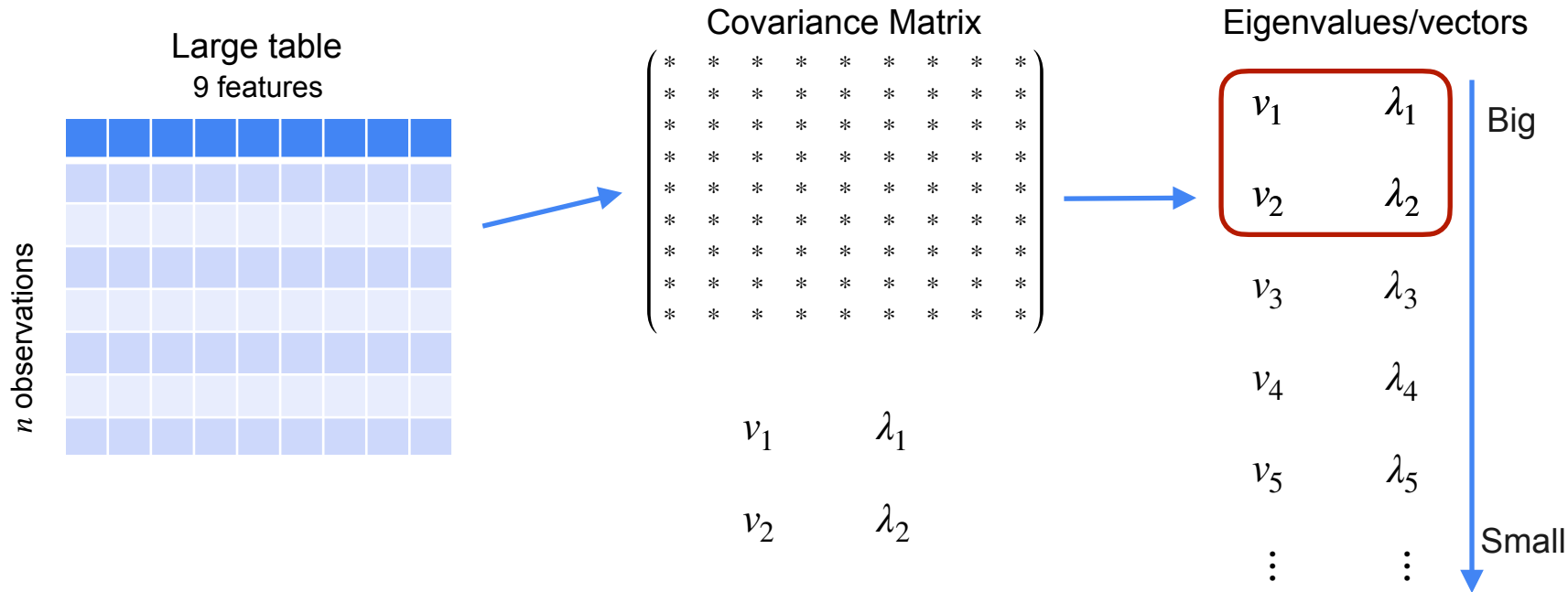
Principal Component Analysis (PCA)



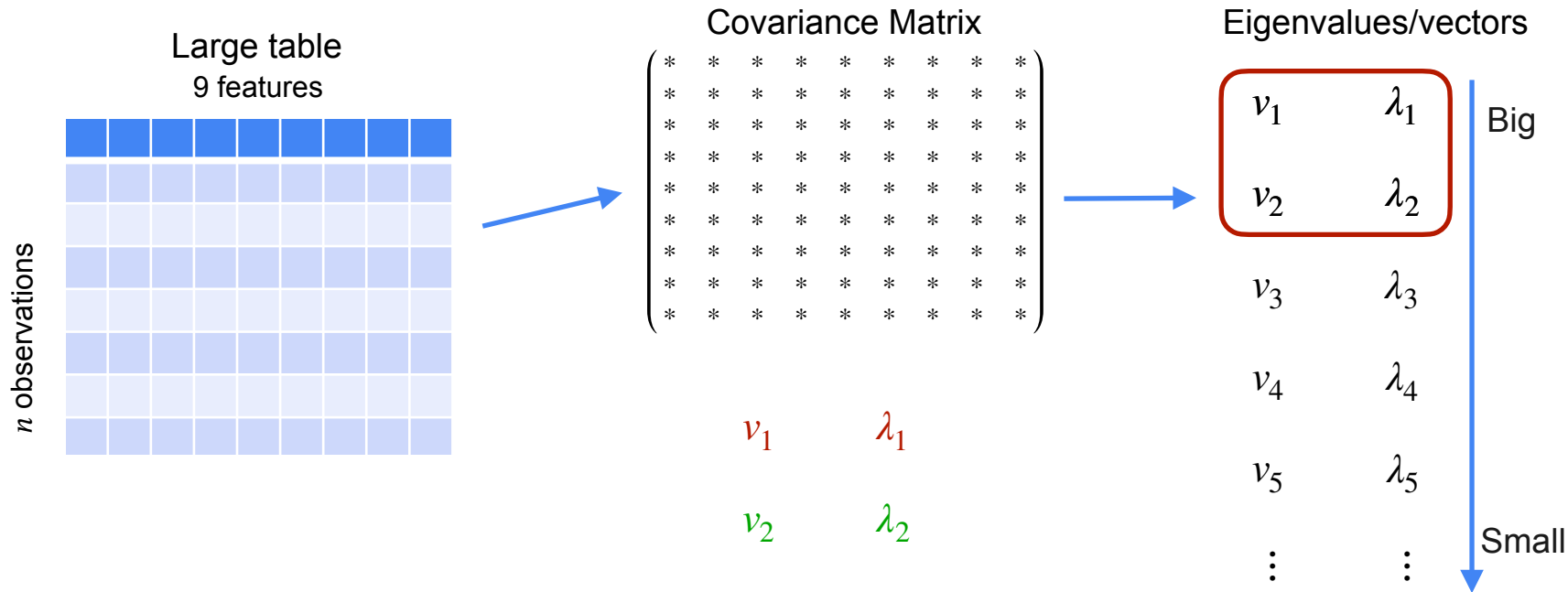
PCA: Principal Component Analysis



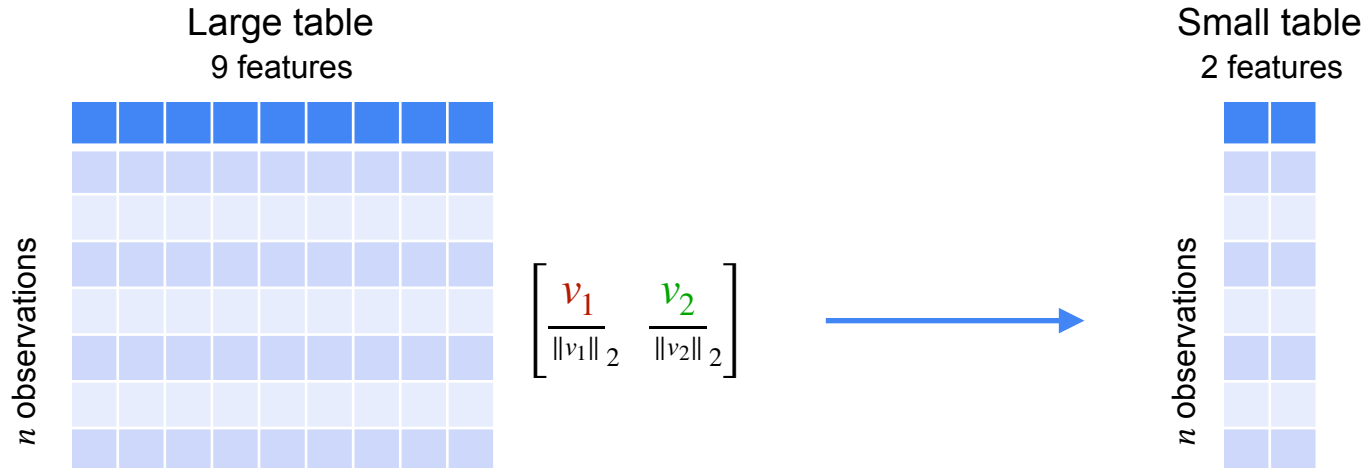
PCA: Principal Component Analysis



PCA: Principal Component Analysis



PCA: Principal Component Analysis



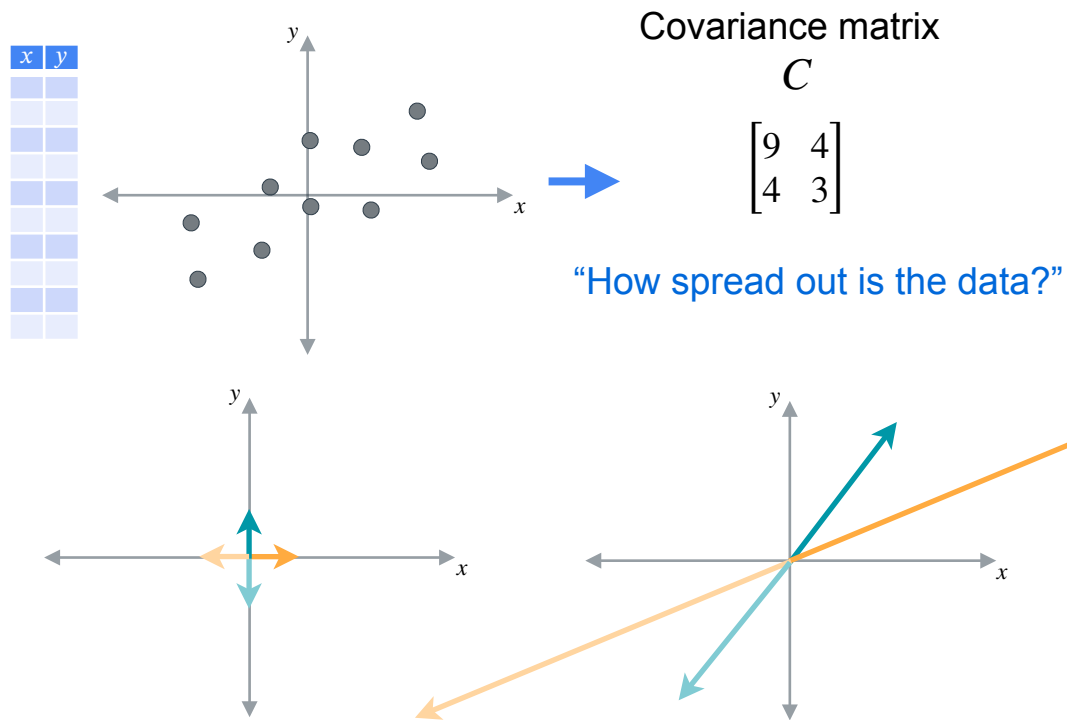


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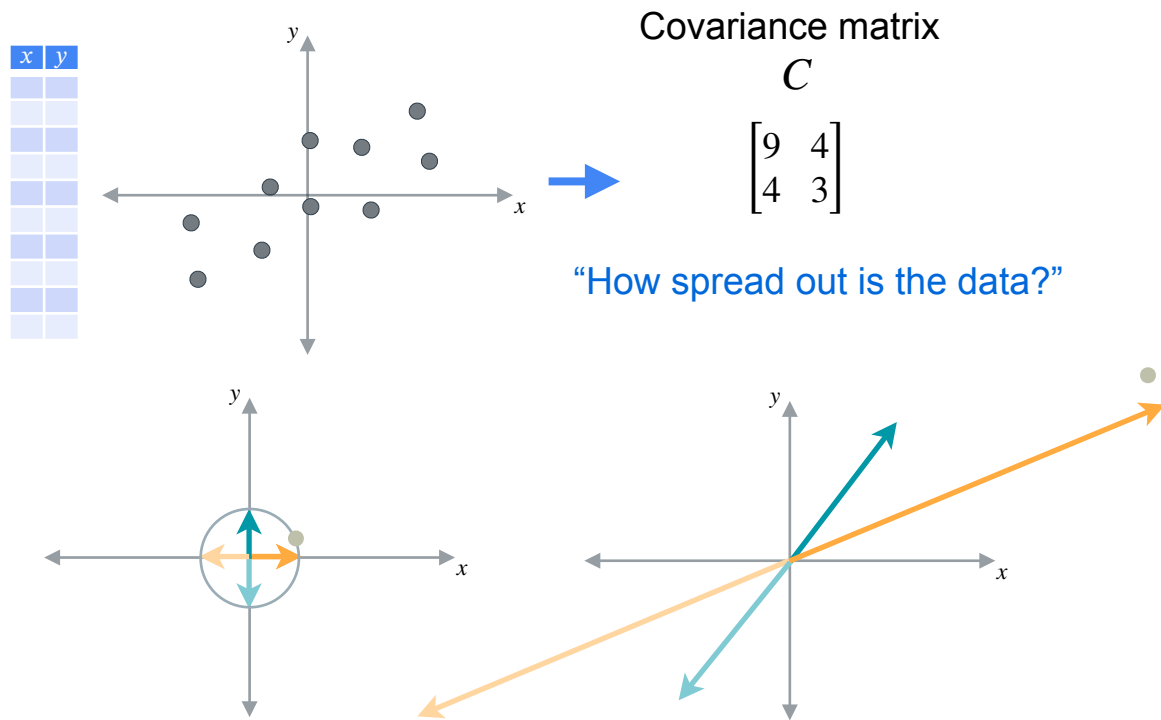
Determinants and Eigenvectors

PCA - Why it works

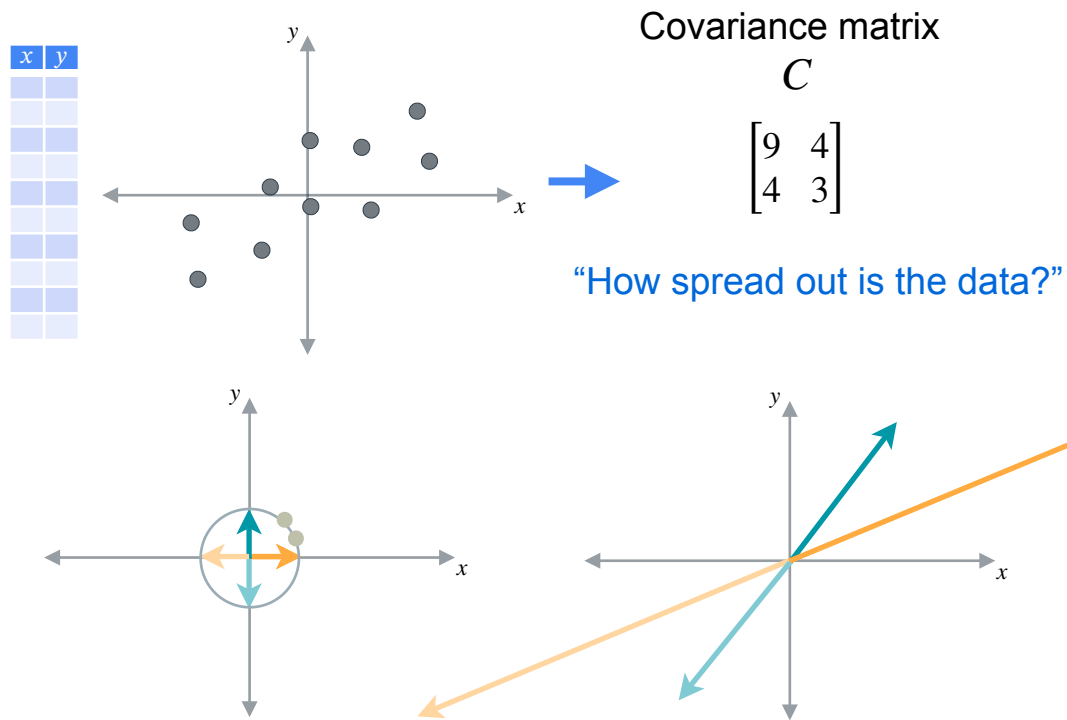
PCA: Why It Works



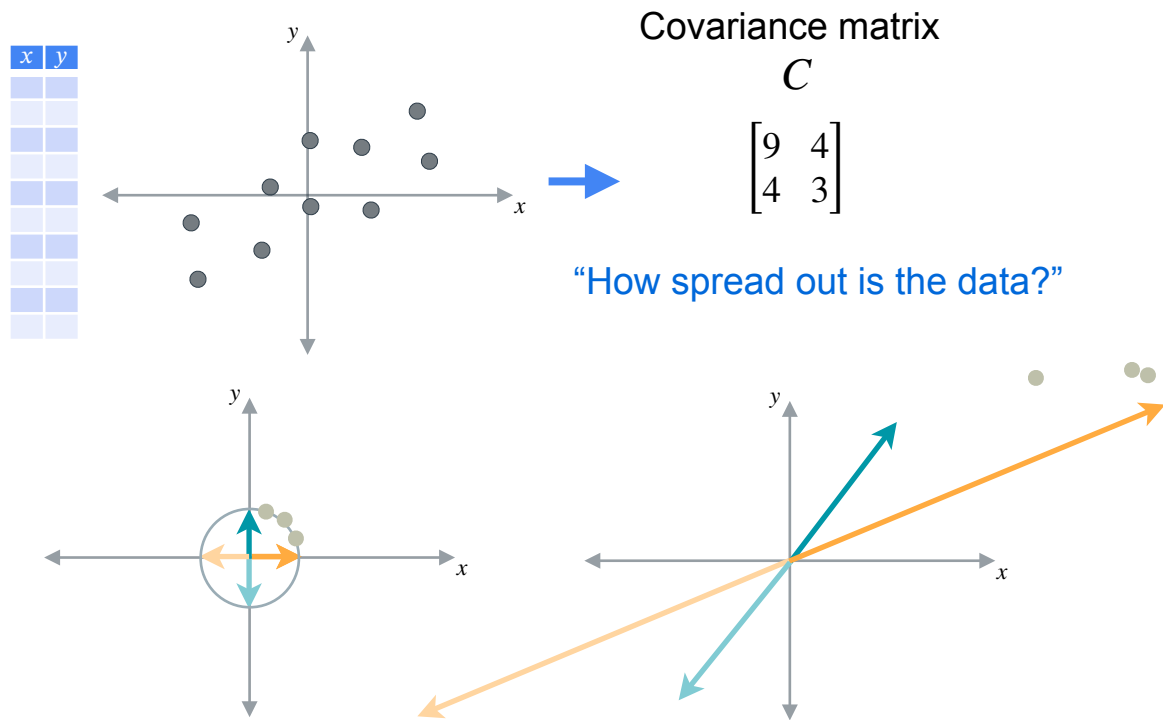
PCA: Why It Works



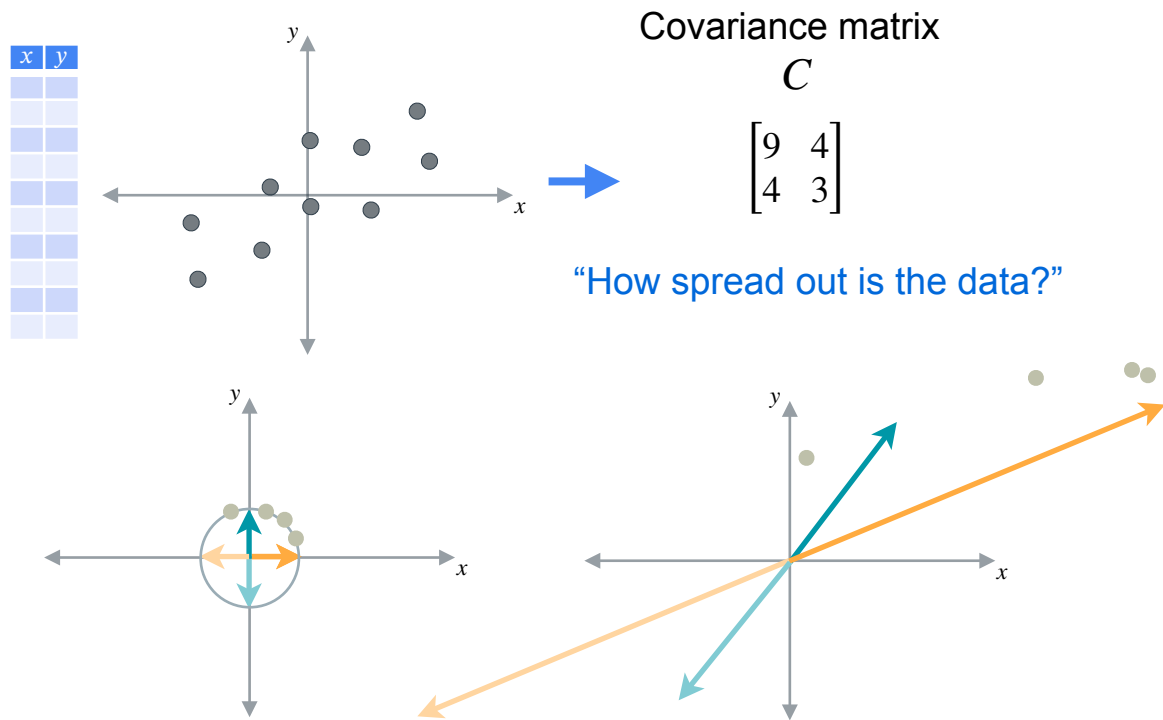
PCA: Why It Works



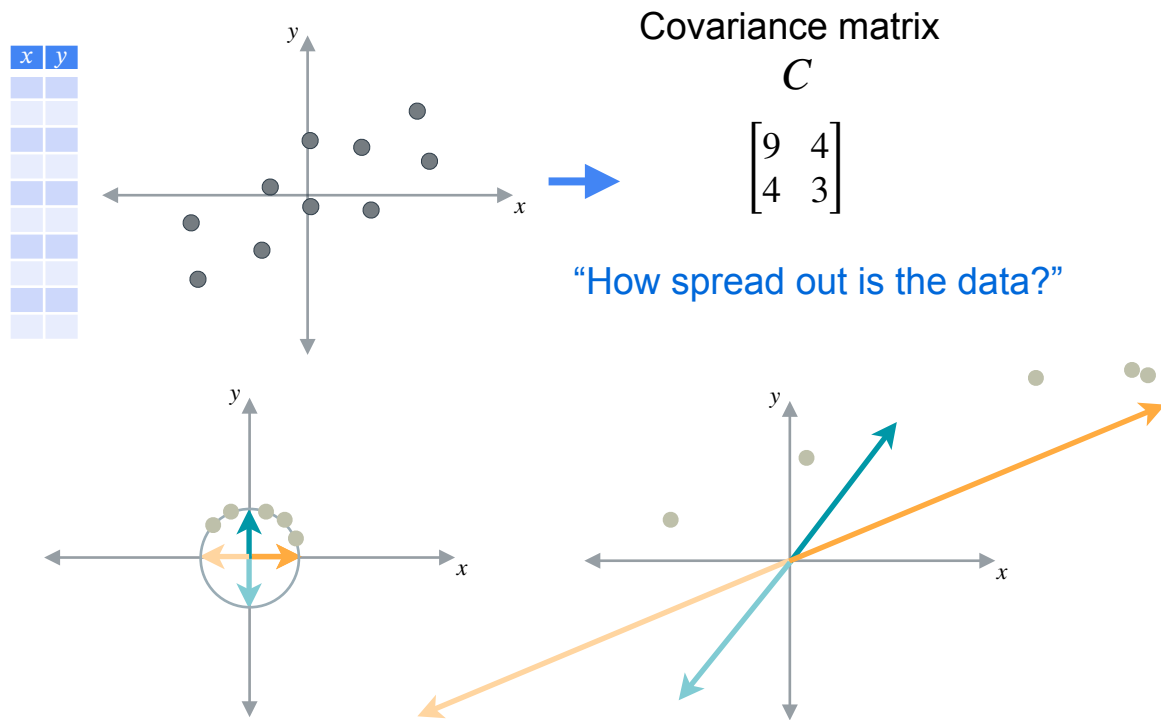
PCA: Why It Works



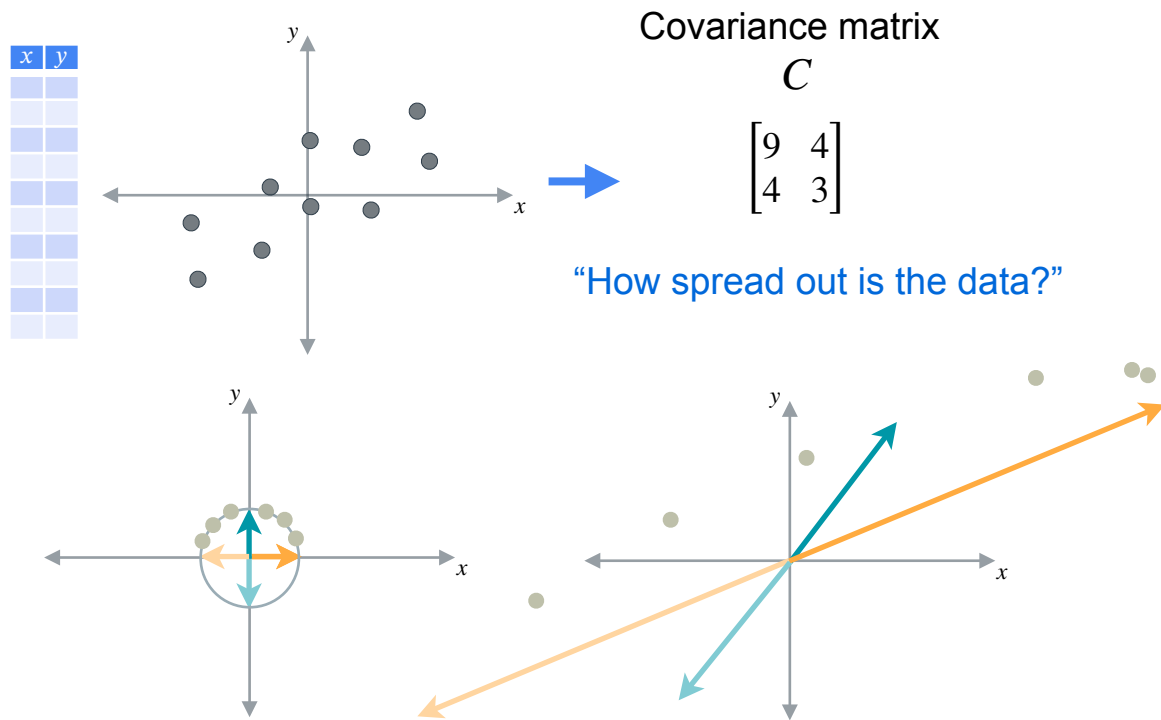
PCA: Why It Works



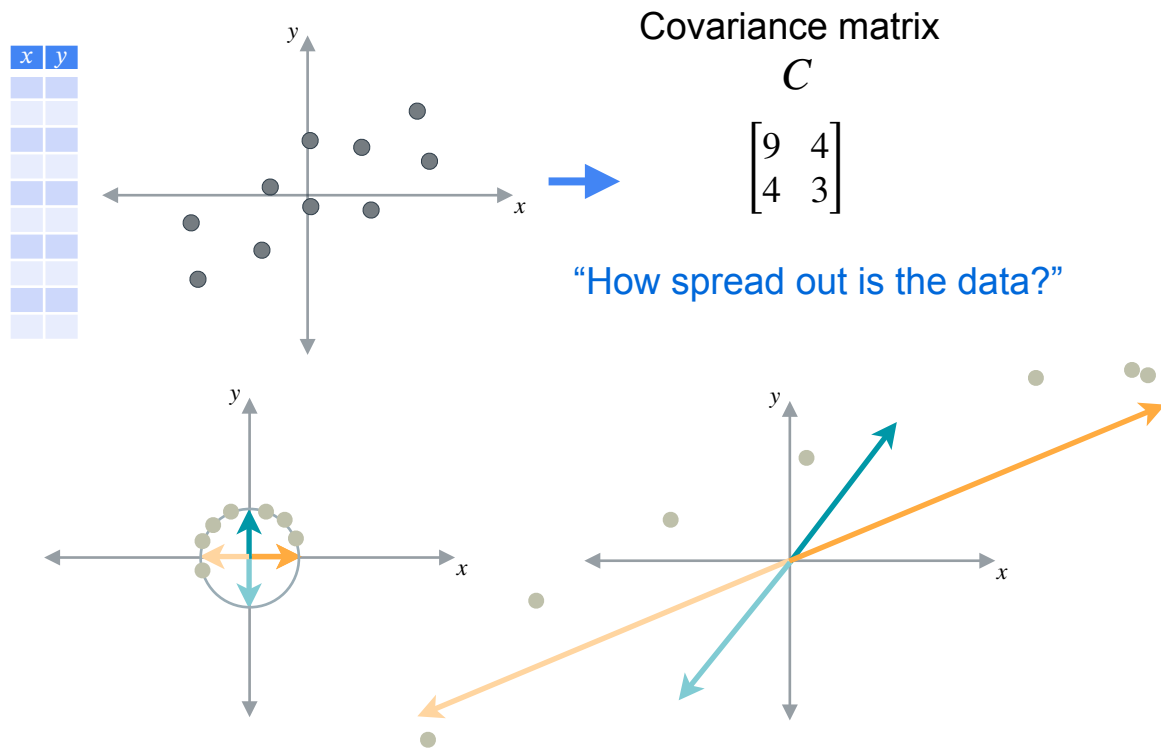
PCA: Why It Works



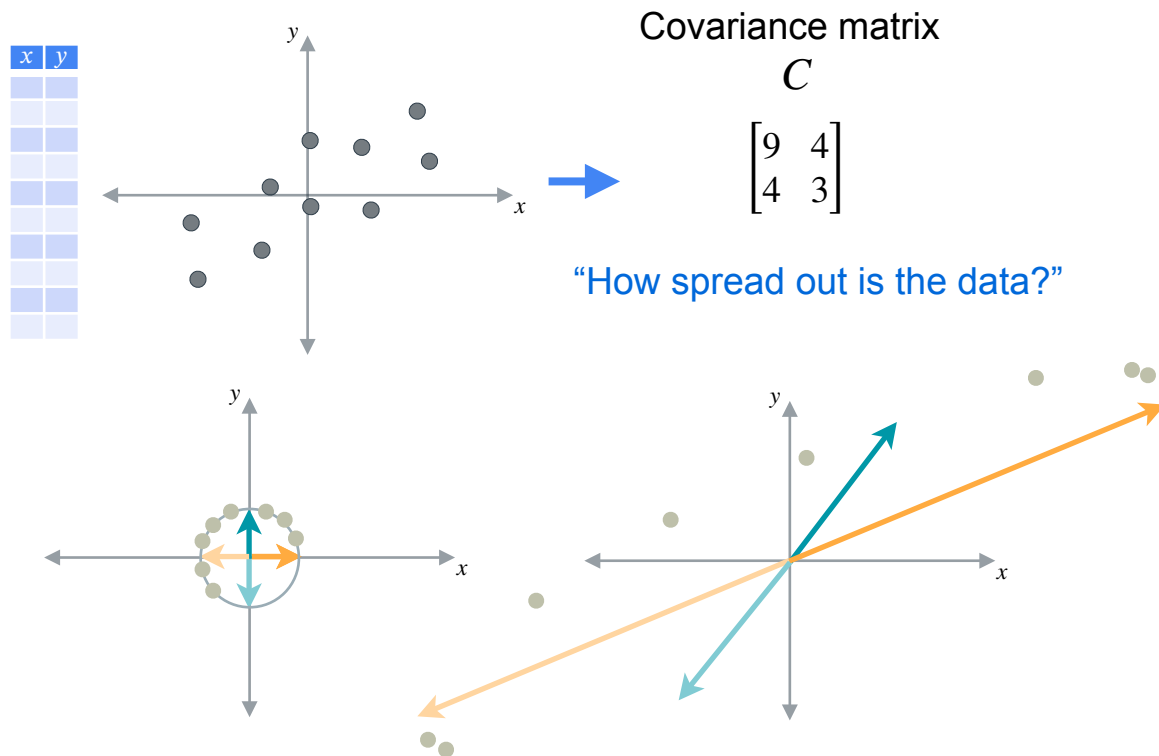
PCA: Why It Works



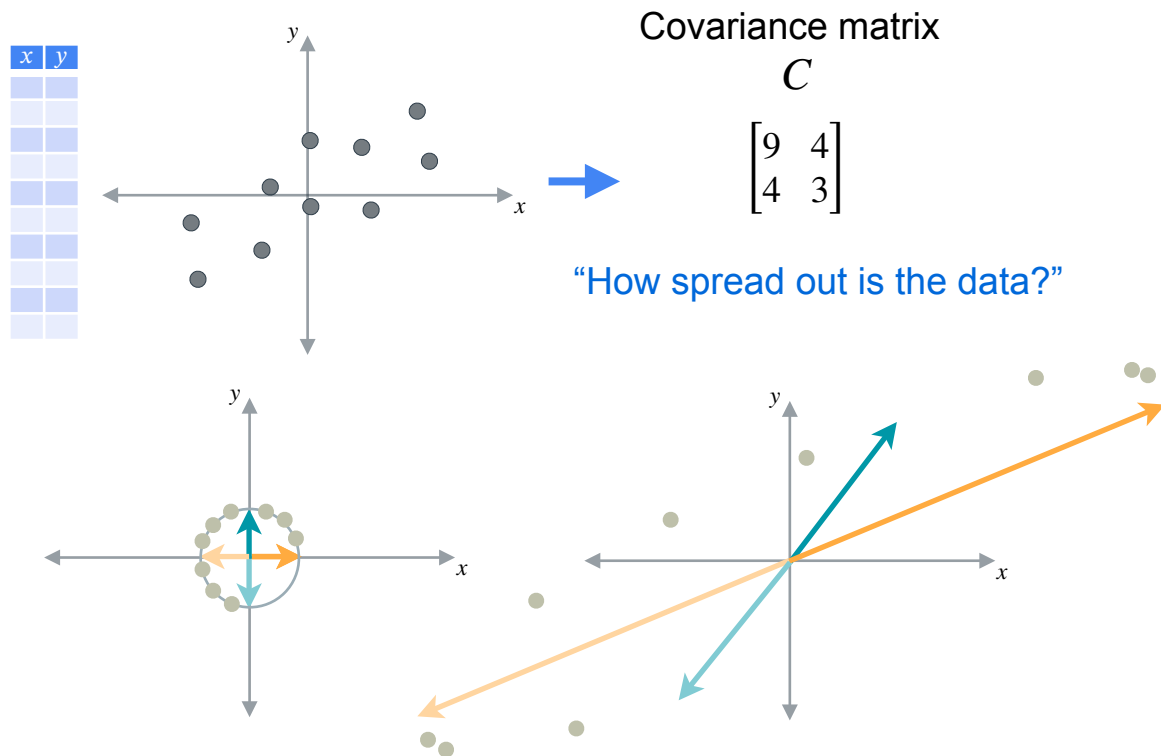
PCA: Why It Works



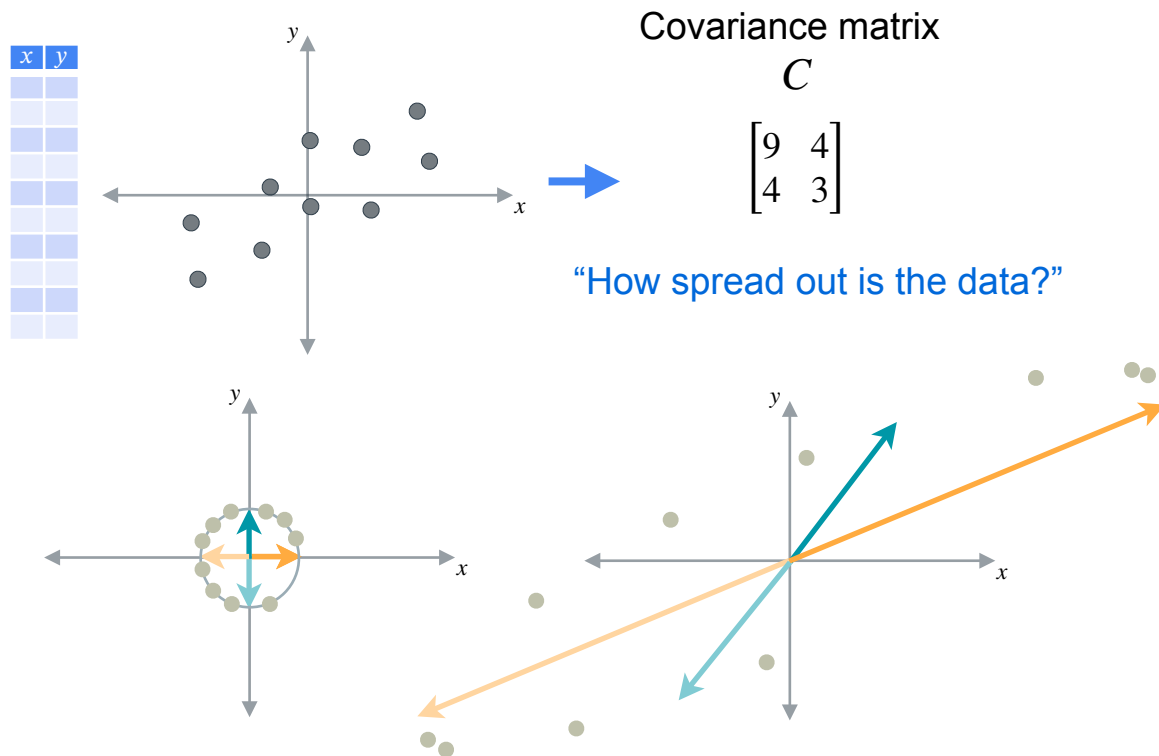
PCA: Why It Works



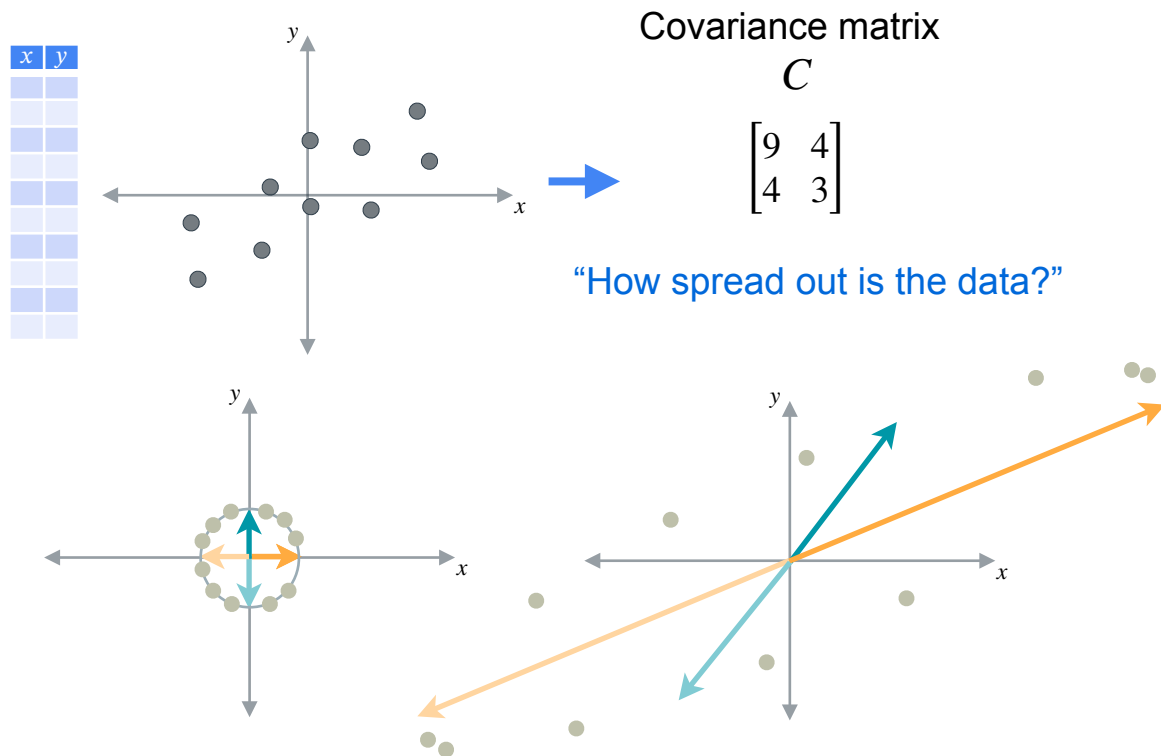
PCA: Why It Works



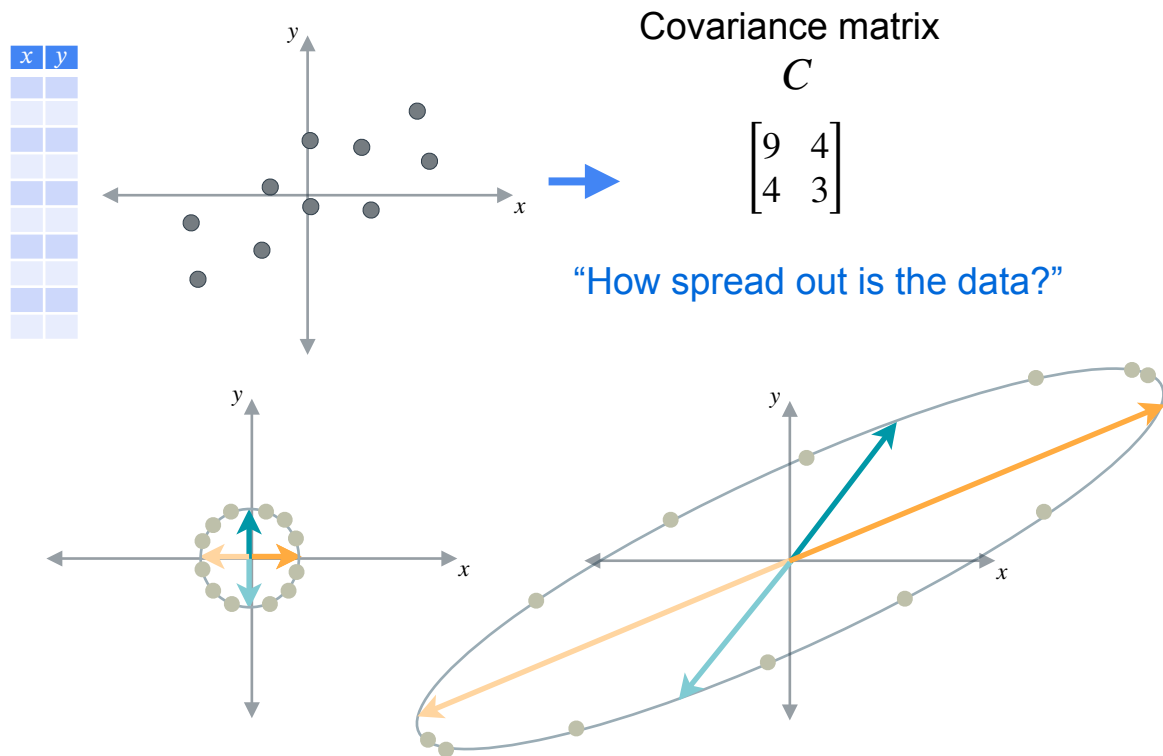
PCA: Why It Works



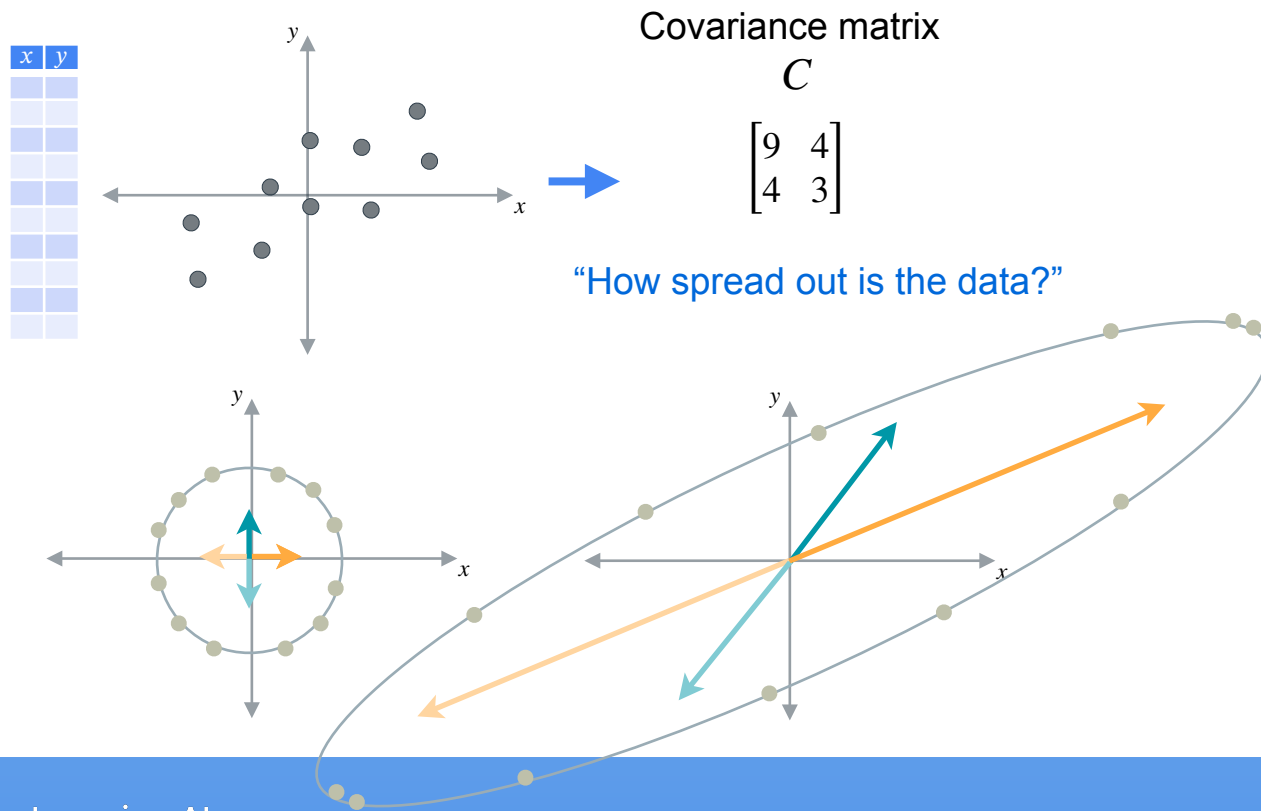
PCA: Why It Works



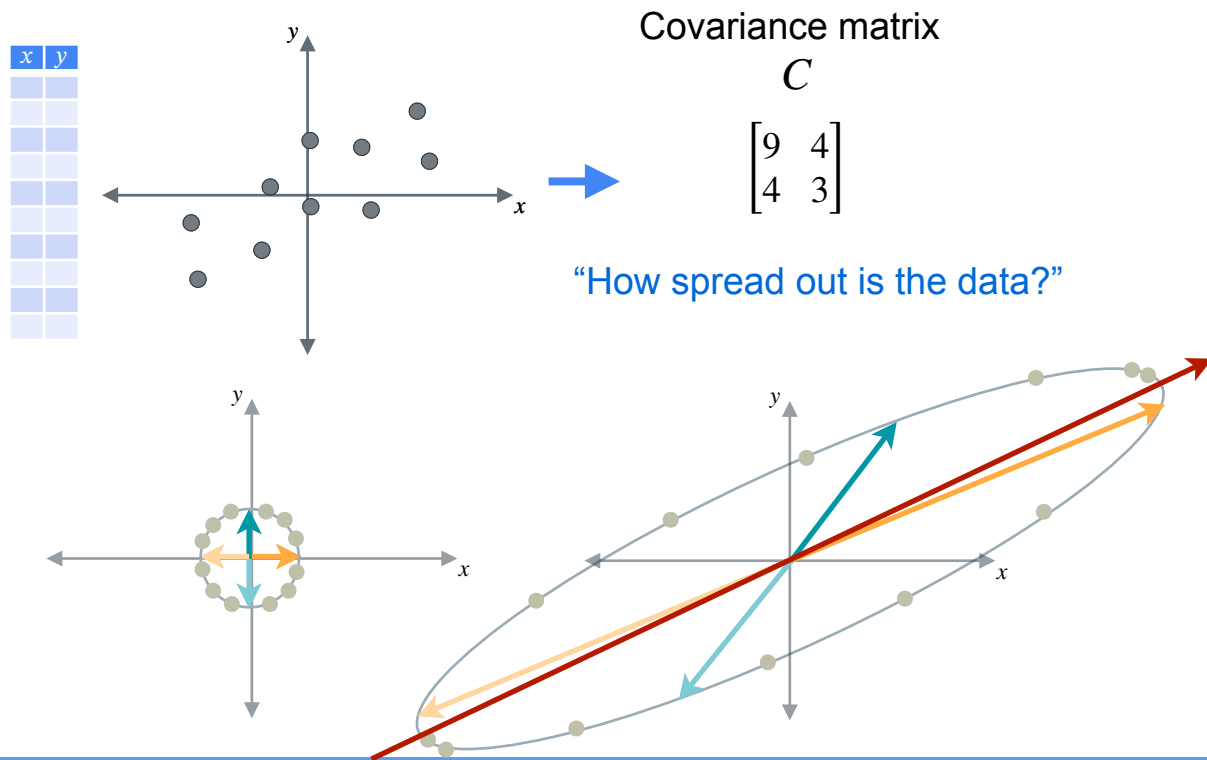
PCA: Why It Works



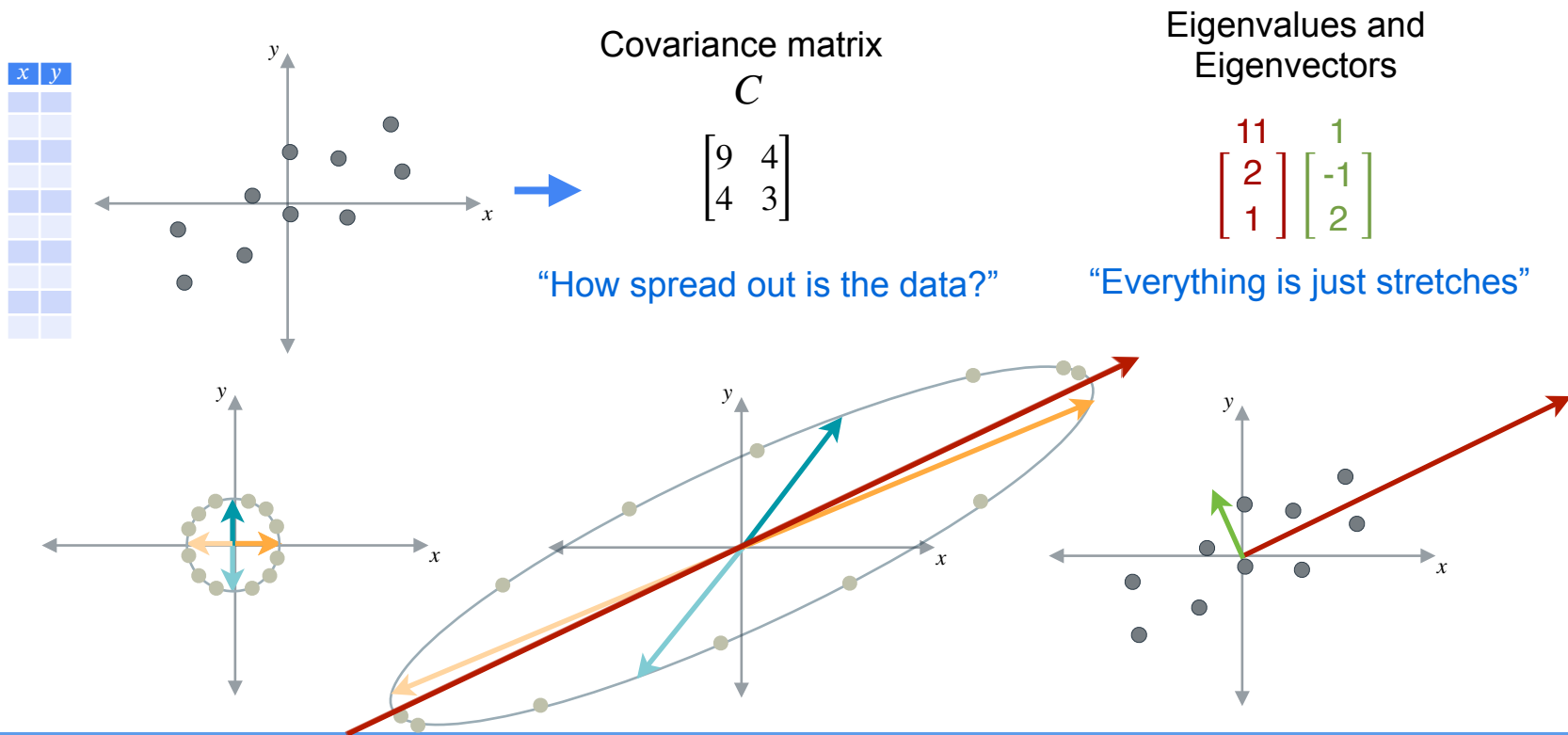
PCA: Why It Works



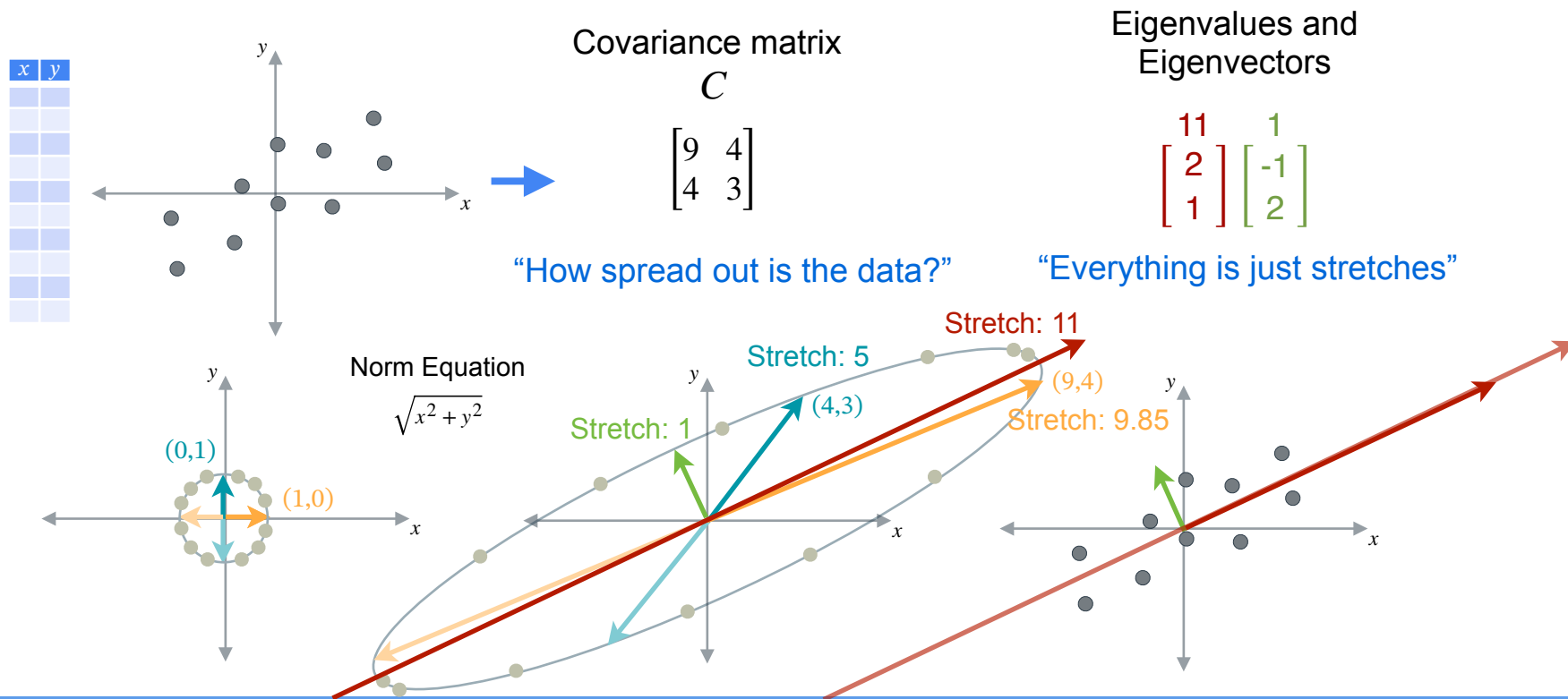
PCA: Why It Works



PCA: Why It Works



PCA: Why It Works





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Determinants and Eigenvectors

PCA - Mathematical formulation

PCA Mathematical formulation

You have n observations of 5 variables (x_1, x_2, x_3, x_4, x_5)

Goal: Reduce to 2 variables

1 Create matrix

$$X = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{15} \\ x_{21} & x_{22} & \dots & x_{25} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{n5} \end{bmatrix}$$

5 variables

n Observations

2 Center the data

$$X - \mu = \begin{bmatrix} x_{11} - \mu_1 & x_{12} - \mu_2 & \dots & x_{15} - \mu_5 \\ x_{21} - \mu_1 & x_{22} - \mu_2 & \dots & x_{25} - \mu_5 \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} - \mu_1 & x_{n2} - \mu_2 & \dots & x_{n5} - \mu_5 \end{bmatrix}$$

PCA Mathematical formulation

You have n observations of 5 variables (x_1, x_2, x_3, x_4, x_5)

Goal: Reduce to 2 variables

3 Calculate Covariance Matrix

$$C = \frac{1}{n-1}(X - \mu)^T(X - \mu) = \begin{bmatrix} \text{Var}(X_1) & \text{Cov}(X_1, X_2) & \text{Cov}(X_1, X_3) & \text{Cov}(X_1, X_4) & \text{Cov}(X_1, X_5) \\ \text{Cov}(X_1, X_2) & \text{Var}(X_2) & \text{Cov}(X_2, X_3) & \text{Cov}(X_2, X_4) & \text{Cov}(X_2, X_5) \\ \text{Cov}(X_1, X_3) & \text{Cov}(X_2, X_3) & \text{Var}(X_3) & \text{Cov}(X_3, X_4) & \text{Cov}(X_3, X_5) \\ \text{Cov}(X_1, X_4) & \text{Cov}(X_2, X_4) & \text{Cov}(X_3, X_4) & \text{Var}(X_4) & \text{Cov}(X_4, X_5) \\ \text{Cov}(X_1, X_5) & \text{Cov}(X_2, X_5) & \text{Cov}(X_3, X_5) & \text{Cov}(X_4, X_5) & \text{Var}(X_5) \end{bmatrix}$$

PCA Mathematical formulation

You have n observations of 5 variables (x_1, x_2, x_3, x_4, x_5)

Goal: Reduce to 2 variables

4 Calculate Eigenvectors and Eigenvalues

Big \uparrow

| | |
|-------------------|-------|
| λ_1 | v_1 |
| λ_2 | v_2 |
| λ_3 | v_3 |
| λ_4 | v_4 |
| Small λ_5 | v_5 |

5 Create Projection Matrix

$$V = \begin{bmatrix} \overline{\|v_1\|_2} & \overline{\|v_2\|_2} \end{bmatrix}$$

6 Project Centered Data

$$X_{PCA} = (X - \mu)V$$



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Determinants and Eigenvectors

Conclusion