VV285 RC Part IV

Differential Calculus

Sets, Norms, Continuity & Convergence

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Outline

- Insight
- Open Balls & Sets
- Equivalent Norms
- Open, Closed Set
- Closure
- Continuity
- Compact Sets

Insight I

From now on, we will have a study into vector(-valued) function, i.e. functions

$$f : \mathbb{R}^n \to \mathbb{R}^m.$$

In the second part of VV285, we will

1. establish the theory of continuity and convergence based on the definition of open set and equivalence of norm (in finite vector space).

2. investigate the derivative of a vector(-valued) function (a matrix, why?).

3. investigate the curve (a special vector function from $\mathbb{R} \to V$).

4. investigate the potential function (another special vector function from $V \to \mathbb{R}$).

Insight II

5. investigate the second derivative of a potential function (still a matrix, why?).
6. learn some techniques to find the extrema of a potential function.
7. study the extrema of a real potential function under constraints (plays an important role in engineering).

Open Balls

Let $(V, \|\cdot\|)$ be a normed vector space. Then

$$B_\varepsilon(a) := \{x \in V : \|x - a\| < \varepsilon\}, \qquad a \in V, \varepsilon > 0, \tag{1}$$

is called an *open ball* of radius ε about a.

Remark: 1. Open ball in $\mathbb{R}$ is open interval. (It is therefore the generalized open interval in $\mathbb{R}^n$.) 2. The "shape" of open balls depends on the vector space V and the norm $\|\cdot\|$.

Question: Draw the unit open balls in $\mathbb{R}^2$ with norms

$$\begin{aligned} \|x\|_1 &= |x_1| + |x_2|, \\ \|x\|_2 &= \sqrt{|x_1|^2 + |x_2|^2}, \\ \|x\|_\infty &= \max\{|x_1|, |x_2|\} \end{aligned} \tag{2}$$

Open Sets

Let $(V, \|\cdot\|)$ be a normed vector space. A set $U \subset V$ is called *open* if for every $a \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$.

Definition. Let V be a vector space on which we may define two norms $\|\cdot\|_1$ and $\|\cdot\|_2$. Then the two norms are called *equivalent* if there exists two constants $C_1, C_2 > 0$ such that

$$C_1\|x\|_1 \leq \|x\|_2 \leq C_2\|x\|_1 \qquad \text{for all } x \in V. \tag{3}$$

Theorem. In a finite-dimensional vector space, all norms are equivalent.

Remark: In the proof, we have a basic norm inequality: Let $(V, \|\cdot\|)$ be a finite- or infinite-dimensional normed vector space and $\{v_1, \ldots, v_n\}$ an **independent** set in V. Then there exists a $C > 0$ such that for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$

$$\|\lambda_1 v_1 + \cdots + \lambda_n v_n\| \geq C\left(|\lambda_1| + \cdots + |\lambda_n|\right). \tag{4}$$

Theorem of *Bolzano-Weierstraß* in $\mathbb{R}^n$

In VV186, we proved: Every bounded sequence of real numbers has a convergent subsequence.

Now we extend it to $\mathbb{R}^n$

Let $(x^{(m)})_{m\in\mathbb{N}}$ be a sequence of vectors in $\mathbb{R}^n$, i.e., $x^{(m)} = \left(x_1^{(m)}, \ldots, x_n^{(m)}\right)$. Suppose that there exists a constant $C > 0$ such that $\left|x_k^{(m)}\right| < C$ for all $m \in \mathbb{N}$ and each $k = 1, \ldots, n$. Then there exists a subsequence $\left(x^{(m_j)}\right)_{j\in\mathbb{N}}$ that converges to a vector $y \in \mathbb{R}^n$. (What is the key of proof?)

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Remark: The key is to construct sub-sub-...-subsequence one by one.

Theorem of *Bolzano-Weierstraß* in $\mathbb{R}^n$

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No.
Counterexample: $\ell^1 := \{(a_n) : \sum_{n=0}^{\infty} |a_n| < \infty\}$ denotes summable sequences. Consider $e^{(n)} = (0, 0, 0, \ldots, 1, 0, 0, \ldots)$, where $\left\|e^{(n)}\right\|_1 = 1$. But $\left(e^{(n)}\right)$ does not converge. (Why?)

Remark: The proof of the Theorem of *Bolzano-Weierstraß* in $\mathbb{R}^n$ gives us a primary concept of "a stopping point" when obtaining the subsequences. However, in infinite-dimensional vector space, the theorem does not hold.

Non Equivalent Norms in ∞-dim VS

Consider the space of continuous functions on $[0,1]$, $C([0,1])$. We can define the two norms

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|, \qquad \|f\|_1 = \int_0^1 |f(x)|dx.$$

Consider function f:

Interior, Exterior and Boundary Points

Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$.

(i) A point $x \in M$ is called an *interior point of M* if there exists an $\varepsilon > 0$ such that $B_\varepsilon(x) \subset M$.

(ii) The set of interior points of M is denoted by $\operatorname{int} M$.

(iii) A point $x \in V$ is called a *boundary point of M* if for every $\varepsilon > 0$, $B_\varepsilon(x) \cap M \neq \emptyset$ and $B_\varepsilon(x) \cap (V \setminus M) \neq \emptyset$.

(iv) The set of boundary points of M is denoted by ∂M.

(v) A point that is neither a boundary nor an interior point of M is called an *exterior point of M*.

(vi) An exterior point of M is an interior point of $V \setminus M$.

Open & Closed Set

Open Set

1. Let $(V, \|\cdot\|)$ be a normed vector space. A set $U \subset V$ is called *open* if for every $a \in U$ there exists an $\varepsilon > 0$ such that $B_\varepsilon(a) \subset U$.
2. The set M is open if and only if $M = \text{int } M$.
3. The set M is open if and only if $M^c = V \backslash M$ is closed.

Closed Set

1. Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$. Then M is said to be *closed* if its complement $V \setminus M$ is open. (The set M is closed if and only if $M^c = V \backslash M$ is open.)
2. The set M is closed if and only if it contains all of its boundary points ($\partial M \subset M$).
3. The set M is closed if and only if it coincides with its closure ($M = \overline{M}$).

Exercise

Consider the following subsets of $\mathbb{R}^2$

$$A = \{(x, y) : 0 < x < 1, \ln x < y < 0\}$$

For this set, state

1. whether it is open, closed or neither,
2. $\overline{A}$
3. int $A, \partial A$, and set of exterior points of A.
4. $\partial A \cap A$ (the boundary points that are part of the set).

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1. It's open.
2. $\overline{A} = \{(x, y) : 0 < x \leq 1, \ln x \leq y \leq 0\} \cup \{(0, y) : y \leq 0\}$
3. int $A = A$, $\partial A = \{(0, y) : y \leq 0\} \cup \{(x, 0) : 0 < x < 1\} \cup \{(x, \ln x) : 0 < x \leq 1\}$, ext $A = \mathbb{R}^2/(\text{ int } A \cup \partial A)$
4. $\emptyset$

Closure

Let $(V, \|\cdot\|)$ be a normed vector space and $M \subset V$. Then

$$\overline{M} := M \cup \partial M$$

is called the *closure* of M. Closure can also be defined using sequences equivalently:

$$\overline{M} = \left\{ x \in V : \underset{(x_n)_{n\in\mathbb{N}}}{\exists} x_n \in M \text{ and } x_n \to x \right\} \tag{5}$$

Exercise

Now let's do some conceptually interesting exercise. Is the following set open, closed, both or neither? Find their closure.

1. $\emptyset$,
2. $\mathbb{R}^n$, the whole vector space,
3. $\{a\}$, set of a single point,
4. the set of all symmetric matrices: $\left\{A \in \mathrm{Mat}(n \times n; \mathbb{R}) : A = A^T\right\}$
5. the set of all invertible matrices (so-called *General Linear Group*): $\{A \in \mathrm{Mat}(n \times n; \mathbb{R}) : \det A \neq 0\}$
6. the set $\Omega = \{A \in \mathrm{Mat}(2 \times 2) : \det A = 1\}$
7. * the set of all polynomials in $C([-1, 1])$ (Are all the subspaces closed in an infinite-dimensional vector space?)
8. find a neither open nor closed set.

Continuous Function

Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces and $f : U \to V$ a function. Then f is *continuous at $a \in U$* if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in U}{\forall}\ \|x-a\|_1 < \delta \qquad \Rightarrow \qquad \|f(x)-f(a)\|_2 < \varepsilon. \tag{6}$$

We can rewrite the definition in the form of open ball. (How?)

Of course, we can prove as usual the following:

Theorem. Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces and $f : U \to V$ a function. Then f is *continuous at $a \in U$* if and only if

$$\underset{\substack{(x_n)_{n\in\mathbb{N}} \\ x_n\in U}}{\forall}\ x_n \to a \qquad \Rightarrow \qquad f(x_n) \to f(a). \tag{7}$$

Exercise

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by $f(x, 0) = 0$ and

$$f(x, y) = \left(1 - \cos\frac{x^2}{y}\right)\sqrt{x^2 + y^2}$$

for $y \neq 0$. Show that f is continuous at (0,0) by definition.

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We show that $\lim_{\sqrt{x^2+y^2}\to 0} f(x,y) = 0$. We have for all $(x,y) \in \mathbb{R}^2$

$$|f(x,y)| \leq \sqrt{x^2+y^2}\left|1 - \cos\frac{x^2}{y}\right| \leq 2\sqrt{x^2+y^2} \to 0$$

Exercise

Find the limit if it exists

$$\lim_{(x,y)\to(0,0)} \frac{x^2 y}{x^4 + y^2}$$

Images and Pre-Images of Sets

Suppose that $f : M \to N$, where M, N are any sets. Let $A \subset M$. Then we define the *image of A* by

$$f(A) := \{y \in N : y = f(x) \text{ for some } x \in A\}.$$

In particular, we can write

$$\operatorname{ran} f = f(M).$$

Similarly, for $B \subset N$ we define the *pre-image of B* by

$$f^{-1}(B) := \{x \in M : f(x) = y \text{ for some } y \in B\}. \tag{8}$$

(T/F?)

$$f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$$
$$f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$$

Determinant is continuous function. (How do we prove this?)

Det

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We basically do nothing but claim that determinant takes the form of polynomials. And a polynomial is continuous clearly. Formally, we need to first define a norm, and use sequence to prove its continuity.

Tricky question*: Is a determinant a uniformly continuous function?

Compact Sets

Let $(V, \|\cdot\|)$ be a normed vector space and $K \subset V$. Then K is said to be *compact* if every sequence in K has a convergent subsequence with limit contained in K.

Remark: In infinite-dimensional vector space,

$$\text{compact} \Rightarrow \text{closed and bounded.}$$

In finite-dimensional vector space,

$$\text{compact} \Leftrightarrow \text{closed and bounded.}$$

Functions on Compact Sets

Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ be normed vector spaces and $K \subset U$ is compact. The function $f : K \to V$ is continuous. Then we know

1. $f(K)$ is compact in V
2. f has a least upper bound on K
3. f is uniformly continuous on K

Remark: Compact set is the $\mathbb{R}^n$ generalization of closed interval in $\mathbb{R}$.

Uniform Continuity

Continuity. Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces and $f : U \to V$ a function. Then f is *continuous at $a \in U$* if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in U}{\forall}\ \|x-a\|_1 < \delta \qquad \Rightarrow \qquad \|f(x)-f(a)\|_2 < \varepsilon. \tag{9}$$

Uniform Continuity. Let $(U, \|\cdot\|_1)$ and $(V, \|\cdot\|_2)$ be normed vector spaces, $\Omega \subset U$ and $f : \Omega \to V$ a function. Then f is *uniformly continuous in Ω* if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in\Omega}{\forall}\ \|x-y\|_1 < \delta \qquad \Rightarrow \qquad \|f(x)-f(y)\|_2 < \varepsilon. \tag{10}$$

Compare the difference between the two definition.

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Compare the difference between the two definition. * Prove the determinant is **not** uniformly continuous.

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* Prove the determinant is **not** uniformly continuous.

About Assignment

Some points you'd better understand in the assignment 4:

1. *distance* of a point x to M,
2. *distance* between a compact set K and a closed set M,
3. *non-equivalence* of norm in infinite-dimensional vector space,
4. closed, bounded, but **not** compact set in infinite-dimensional vector space.

Have Fun
And
Learn Well!