VV285 RC Part I

Elements of Linear Algebra

"Matrices are just linear maps!"

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Overview of Linear Algebra

1. Systems of Linear Equations
2. Finite-Dimensional Vector Spaces
3. Inner Product Spaces
4. Linear Maps
5. Matrices
6. Theory of Systems of Linear Equations
7. Determinants

1. Linear System
   Homogeneous vs. Inhomogeneous
   Underdetermined vs. Overdetermined
2. Equivalency of Linear System
3. The Gauß – Jordan Algorithm
4. Diagonalizable (Existence and Uniqueness of Linear System)
5. **Fundamental Lemma for Homogeneous Equations**

A *linear system* of m (algebraic) equations in n unknowns $x_1, x_2, \ldots, x_n \in V$ is a set of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where $b_1, b_2, \ldots, b_m \in V$ and $a_{ij} \in \mathbb{F}, i = 1, \ldots, m,\ j = 1, \ldots, n$.
If $b_1 = b_2 = \cdots = b_m = 0$, then (1) is called a *homogeneous system*. Otherwise, it is called an *inhomogeneous system.*
If $m < n$ we say that the system in *underdetermined*, if $m > n$ the system is called *overdetermined*. A solution of a linear system of equations (1) is a tuple of elements $(y_1, y_2, \ldots, y_n) \in V^n$ such that the predicate (1) becomes a true statement.

Linear System

We say that two systems of linear equations are *equivalent* if any solution of the first system is also a solution of the second system and vice-versa. Thus the systems

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 1 \\ -5x_2 + x_3 &= 1 \\ 10x_2 + x_3 &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 &= 2 \\ x_2 &= 0 \\ x_3 &= 1 \end{aligned}$$

are *equivalent*.

Gauß-Jordan Algorithm

The goal of the *Gauß-Jordan algorithm* (also called Gaußian elimination) is to transform a system

$$\begin{array}{ccc|c} * & * & * & \diamond \\ * & * & * & \diamond \\ * & * & * & \diamond \end{array} \qquad\qquad * \in \mathbb{R} \textit{ or } \mathbb{C}, \quad \diamond \in V$$

first into the form

$$\begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \end{array} \tag{2}$$

and subsequently into

$$\begin{array}{ccc|c} 1 & 0 & 0 & \diamond \\ 0 & 1 & 0 & \diamond \\ 0 & 0 & 1 & \diamond \end{array} \tag{3}$$

Elementary Row Manipulations

Include:

1. Swapping (interchanging) two rows,
2. Multiplying each element in a row with a number,
3. Adding a multiple of one row to another row.

Result: Transform a system into a equivalent system. Since each row represents an equation, we are essentially **manipulating equations**.

Diagonalization

A system of m equations with n unknowns will have a unique solution if and only if it is *diagonalizable*. i.e. It can be transformed into diagonal form.

Remark: *Diagonalization* turns out to be an important topic in VV286, especially in terms of *ordinary differential equation systems*.

Fundamental Lemma for Homogeneous Equations

The homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

of m equations in n real or complex unknowns $x_1, x_2, \ldots, x_n$ has a **non-trivial** solution if $n > m$.

Remark: This fundamental lemma contributes to prove that any basis of a vector space has the same length.

1. Linear Independence
2. Span
3. Basis
4. Dimension
5. Basis Extension Theorem
6. Sum of Vector Space

Let V be a real or complex vector space and $v_1, v_2, \dots, v_n \in V$. Then the vectors $v_1, v_2, \dots, v_n$ are said to be *independent* if for all $\lambda_1, \lambda_2, \dots, \lambda_n \in \mathbb{F}$

$$\sum_{k=1}^{n} \lambda_k v_k = 0 \qquad \Rightarrow \qquad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

A finite set $M \subset V$ is called an *independent set* if the elements of M are independent.

Span

Let $v_1, v_2, \ldots, v_n \in V$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$. Then the expression

$$\sum_{k=1}^{n} \lambda_k v_k = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

is called a *linear combination* of the vectors $v_1, v_2, \ldots, v_n$.
The set

$$\operatorname{span}\{v_1, \ldots, v_n\} = \left\{ y \in V : y = \sum_{k=1}^{n} \lambda_k v_k, \lambda_1, \ldots, \lambda_n \in \mathbb{F} \right\}$$

is called the *(linear) span* or the *linear hull* of the vectors $v_1, v_2, \ldots, v_n$.

The vectors $v_1, v_2, \ldots, v_n \in V$ are independent if and only if **none of them is contained in the span of all the others**.

(How to prove?)

Let V be a real or complex vector space. An n-tuple $\mathcal{B} = (b_1, \dots, b_n) \in V^n$ is called an *(ordered and finite) basis* of V if every vector v has a **unique** representation

$$v = \sum_{i=1}^{n} \lambda_i b_i, \qquad \lambda_i \in \mathbb{F}. \tag{4}$$

The numbers λ_i are called the *coordinates* of v with respect to $\mathcal{B}$.

The tuple of vectors $(e_1, e_2, \dots, e_n)$, $e_i \in \mathbb{R}^n$,

$$e_i = (0, \dots 0, \underset{\substack{\uparrow \\ i\text{th} \\ \text{entry}}}{1}, 0, \dots, 0), \qquad i = 1, \dots, n,$$

is called the *standard basis* or *canonical basis* of $\mathbb{R}^n$.

Basis = Independence + Span

Let V be a real or complex vector space.
An n-tuple $\mathcal{B} = (b_1, \dots, b_n) \in V^n$ is a basis of V if and only if

1. the vectors $b_1, b_2, \dots, b_n$ are linearly independent, i.e., $\mathcal{B}$ is an independent set,
2. $V = \operatorname{span} \mathcal{B}$.

(How to prove?)

Remark: This theorem is more practical than the definition of basis when proving some set is a basis of some vector space. It helps one decompose the proof into two parts: 1. prove linear independence (uniqueness of Eq.4) 2. prove the span is large enough (existence of Eq.4).

Dimension

Let V be a real or complex finite-dimensional vector space, $V \neq \{0\}$. **Then any basis of V has the same length** (number of elements).

Remark: This theorem can be proved by contradiction (Use the definition of basis and the fundamental lemma for homogeneous equations). With such a premise, we can then define the *dimension* of vector space.

Let V be a real or complex vector space. Then V is called *finite-dimensional* if either

- $V = 0$ or
- V possesses a finite basis.

If V is not finite-dimensional, we say that it is *infinite-dimensional*.