VV285 RC Part VI

Differential Calculus

Curve

Xingjian Zhang

Univerity of Michigan-Shanghai Jiao Tong University Joint Institute

June 25, 2020

Exercise 5.1

The determinant is a map $\det\colon \operatorname{Mat}(n \times n, \mathbb{R}) \to \mathbb{R}$.

i) Use a suitable norm and the Leibniz representation to show that $\det\colon \operatorname{Mat}(n \times n, \mathbb{R}) \to \mathbb{R}$ is continuous. (This was shown in class; you are asked to fill in all the details.)
(2 Marks)

ii) Use a suitable norm and the Leibniz formula for the determinant to show that

$$\det(\mathbb{1} + \varepsilon A) = 1 + \varepsilon \cdot \operatorname{tr} A + o(\varepsilon), \qquad \varepsilon \searrow 0,$$

where $\operatorname{tr} A = \sum_{i=1}^{n} a_{ii}$ is the trace of the matrix $A \in \operatorname{Mat}(n \times n, \mathbb{R})$.
(2 Marks)

iii) For $A \in \operatorname{GL}(n, \mathbb{R})$, prove that

$$(D \det)|_A H = \det A \cdot \operatorname{tr}(A^{-1}H), \qquad H \in \operatorname{Mat}(n \times n, \mathbb{R}). \qquad (*)$$

(You should use the properties of the determinant and consider $\det(A + H)$. Use that A is invertible.)
(2 Marks)

iv) Extend the derivative of the determinant to any $A \in \operatorname{Mat}(n \times n, \mathbb{R})$.
Instructions: Express A^{-1} using the adjugated matrix. Show that every non-invertible matrix is the limit of a sequence of invertible matrices and extend $(*)$ to all matrices by continuity.
(2 Marks)

Assignment 5 Recap II

i)

ii)

iii)

iv)

Assignment 5 Recap VI

Exercise 5.2

i) Let $f, g\colon \mathbb{R} \to \mathbb{R}$, $F\colon \mathbb{R}^2 \to \mathbb{R}$. Show that

$$\frac{d}{dt}F(f(t), g(t)) = f'(t)\partial_1 F(f(t), g(t)) + g'(t)\partial_2 F(f(t), g(t))$$

where $\partial_1 F(x, y) = \frac{\partial}{\partial x}F(x, y)$ and $\partial_2 F(x, y) = \frac{\partial}{\partial y}F(x, y)$. *Hint:* Consider $F(f(t), g(t))$ as a composition of the map $G\colon \mathbb{R} \to \mathbb{R}^2$, $G(t) = (f(t), g(t))$ with $F\colon \mathbb{R}^2 \to \mathbb{R}$ and apply the chain rule.
(2 Marks)

ii) Let $f(x, y)$ be a differentiable function over $(a, b) \times (c, d)$ and let $\alpha, \beta\colon \mathbb{R} \ni [c, d] \to [a, b] \in \mathbb{R}$ be differentiable. Then the integral

$$I(y) = \int_{\alpha(y)}^{\beta(y)} f(x, y)\, dx$$

is a differentiable function of y in $[c, d]$. Assuming that $\frac{\partial f}{\partial y}(x, y)$ is continuous and $\alpha'(y), \beta'(y)$ exist in $[a, b] \times [c, d]$, give a formula for $I'(y)$.
(3 Marks)

Assignment 5 Recap VII

i)

ii)

Outline

- Definition of Curves
- Orientation of Curves
- Tangent Vector of Curves
- Curve Length
- Line Integral of a Scalar Function
- Normal Vector of Curves, Curvature
- *Radius of Curvature
- After Class Exercise

Something you need to pay attention to...

Think More and Be Interactive!

- Do think more about the question in "()".
  e.g. "(How to prove?)"
- You are welcome to ask questions in a adequate manner.
- Please open your camera so that I can receive more feedbacks from you. (Makes our life easier!)
- The class is designed to be interactive. However, if you really do not want to be asked at all, please type an "_" before your zoom name.

Definition of Curves

Let V be a finite-dimensional vector space and $I \subset \mathbb{R}$ an interval.

- A set $\mathcal{C} \subset V$ for which there exists a **continuous, surjective and locally injective** map $\gamma : I \to \mathcal{C}$ is called a *curve*.
- The map γ is called a *parametrization* of $\mathcal{C}$.
- A curve $\mathcal{C}$ together with a parametrization is called a *parametrized* curve.

Remark: Curves have several non-equivalent definitions. For example, topological curves, algebraic curves, and what we have learned, parametric curves. Additionally, in majority of the practical cases, we expect our parametrization to be differentiable. We then call these curves differentiable curves.

Properties of Curves I

Let $\mathcal{C} \subset V$ be a curve possessing a parametrization $\gamma : I \to \mathcal{C}$ with $\operatorname{int} I = (a, b)$ for $-\infty \leq a < b \leq \infty$.

(i) If γ is (globally) injective parametrization we say that $\mathcal{C}$ is a *simple curve*. (No intersections)

(ii) If **there exists some** γ, such that

$$\lim_{t\to a} \gamma(t) = \lim_{t\to b} \gamma(t),$$

the curve $\mathcal{C}$ is said to be *closed*.

(iii) If a curve is not closed, it is said to be *open*. The points

$$x := \lim_{t\to a} \gamma(t) \qquad \text{and} \qquad y := \lim_{t\to b} \gamma(t)$$

are called the *initial point* and the *final point* of the parametrized curve $(\mathcal{C}, \gamma)$. The open curve is said t join x and y.

Properties of Curves II

Remark: Notice the closeness and simpleness of a curve is based on the **existence** of some parametrization γ that meets $(i), (ii)$. If a curve is found to be closed by some parametrization γ, it is still closed no matter what we choose as a new parametrization.

Let $\mathcal{C} \subset V$ be a curve with parametrization $\gamma : I \to \mathcal{C}$.

(i) Let $J \subset \mathbb{R}$ be an interval. A continuous, bijective map $r : J \to I$ is called a *reparametrization* of the parametrized curve $(\mathcal{C}, \gamma)$.

(ii) If r is *increasing* the reparametrization is said to be *orientation-preserving*.

(iii) If r is *decreasing* the reparametrization is said to be *orientation-reversing*.

Let $(\mathcal{C}, \gamma)$ be a parametrized curve and r a reparametrization of $(\mathcal{C}, \gamma)$. The curve $(\mathcal{C}, \widetilde{\gamma})$ with $\widetilde{\gamma} = \gamma \circ r$ is said to have the *same orientation* as $(\mathcal{C}, \gamma)$ if r is orientation-preserving. Otherwise it is said to have *reverse orientation*.

Orientation of Curves II

Only in below case, we can define positive orientation explicitly.

Let $(\mathcal{C}, \gamma)$ be a parametrized, simple, closed curve in $\mathbb{R}^2$. Then $\mathcal{C}$ is said to have *positive orientation* if γ traverses $\mathcal{C}$ in a *counter-clockwise* direction.

Tangent Vector of Curves I

A curve $\mathcal{C} \subset V$ is said to be *smooth* if there exists a parametrization $\gamma : I \to \mathcal{C}$ such that

(i) γ is continuously differentiable on $\operatorname{int} I$ and

(ii) $D\gamma|_t \neq 0$ for all $t \in \operatorname{int} I$.

A *smooth reparametrization* is a reparametrization that is continuously differentiable with non-vanishing derivative in the interior of its domain.

If $V = \mathbb{R}^n$, the Jacobian $D\gamma = \gamma'$ of a smooth curve $\gamma : I \to \mathbb{R}^n$ is given by

$$\gamma'(t) = \begin{pmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{pmatrix}, \qquad\qquad t \in \operatorname{int} I.$$

Tangent Vector of Curves II

For smooth curves, we can further define the tangent vectors.

Let $\mathcal{C}^* \subset \mathbb{R}^n$ be an oriented smooth curve and $p \in \mathcal{C}^*$. Let $\gamma : I \to \mathbb{R}^n$ be a parametrization of $\mathcal{C}^*$. Then we define the *unit tangent vector* to $\mathcal{C}^*$ at $p = \gamma(t)$ by

$$T \circ \gamma(t) := \frac{\gamma'(t)}{\|\gamma'(t)\|}, \qquad t \in \operatorname{int} I. \tag{1}$$

This defines the *tangent vector field* $T : \mathcal{C}^* \to \mathbb{R}^n$ on $\mathcal{C}$. Notice that the unit tangent vector does not depend on the choice of parametrization.

Question: Consider the difference between

$$T \circ \gamma(t), T \circ \gamma, T, \gamma', \gamma'(t)$$

.

Curve Length

We define the *curve length* by

$$\ell(\mathcal{C}) := \sup_{\text{partitions } \mathcal{P}} \ell_{\mathcal{P}}(\mathcal{C}).$$

However, in practice, we calculate the curve length by

$$\ell(\mathcal{C}) = \int_a^b \|\gamma'(t)\| dt$$

We immediately find that ℓ^{-1} is a *natural parametrization* of the curve. i.e., we can parametrize $\mathcal{C}$ using

$$\gamma = \ell^{-1} : I \to \mathcal{C}, \qquad \text{int } I = (0, \ell(\mathcal{C})).$$

Exercise

In coordinates, we have some function $r = r(\varphi), 0 < \varphi < 2\pi$. Prove its curve length is

$$\ell = \int_0^{2\pi} \sqrt{r^2(\varphi) + r'^2(\varphi)} d\varphi$$

Line Integral of a Scalar Function

The basic idea of the line integral of a scalar function is to calculate "the total mass of a non-uniform string".

Let $\Omega \subset \mathbb{R}^n$, $f : \Omega \to \mathbb{R}$ be a continuous potential function and $\mathcal{C}^* \subset \Omega$ an oriented smooth curve with parametrization $\gamma : I \to \mathcal{C}$.

We then define the *line integral of the potential f along $\mathcal{C}^*$* by

$$\int_{\mathcal{C}^*} f\, d\ell := \int_I (f \circ \gamma)(t) \cdot \|\gamma'(t)\|\, dt$$

Remark: The line integral of scalar function (scalar field) does not have many applications in engineering or physics. However, we will soon learn the line integral of vector function, which has many uses in physics.

Normal Vector of Curves, Curvature

The *unit normal vector* $N : \mathcal{C} \to \mathbb{R}$ is defined by

$$N \circ \gamma(t) := \frac{(T \circ \gamma)'(t)}{\|(T \circ \gamma)'(t)\|}, \qquad t \in \operatorname{int} I. \tag{2}$$

The *curvature* of a smooth C^2-curve $\mathcal{C} \subset V$ is

$$\kappa : \mathcal{C} \to \mathbb{R}, \qquad \kappa \circ \ell^{-1}(s) := \left\| \frac{d}{ds}\left(T \circ \ell^{-1}(s)\right) \right\|$$

where T is the unit tangent vector and $\ell^{-1} : I \to \mathcal{C}$ is the curve length parametrization of $\mathcal{C}$. In fact, we also have

$$\kappa \circ \gamma(t) = \kappa \circ \ell^{-1}(s)\big|_{s = \ell \circ \gamma(t)} = \frac{\|(T \circ \gamma)'(t)\|}{\|\gamma'(t)\|}$$

Note that, both N, κ also does not depend on the orientation of $\mathcal{C}$.

*Curvatrue for Function $y = y(x)$

A (twice continuously differentiable) function is a smooth curve, so we can find its curvature.

$$\kappa = \left|\frac{y''}{(1 + y'^2)^{\frac{3}{2}}}\right|$$

(You can verify this using the formula of curvature.)

Exercise: Projectile I

Let's do some simple physics! Given gravity acceleration g. An object with mass m is thrown with an initial velocity (v_x, v_y) from the origin. (Both $v_x, v_y > 0$) Find

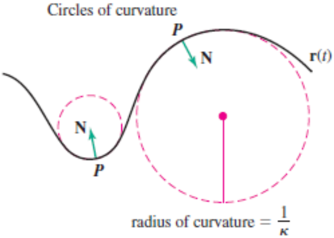
1. a suitable parametrization.
2. the trajectory using parametrization.
3. the length of trajectory before landing.
4. the curvature of the trajectory at any point.

Exercise: Projectile II

Answer:

We can then define the *radius of curvature* by κ:

$$\rho := \frac{1}{\kappa}$$



Projectile Again I

Find

1. the radius of curvature of the trajectory at a given point.
2. the velocity at the same point.
3. the normal component of the gravitational acceleration at the same point.
4. what's their relations?

Projectile Again II

Answer:

After Class Exercise

Define the parametrization of the cycloid $\gamma(t), t \in (0, 2\pi)$

$$\gamma(t) = \begin{pmatrix} t - \sin t \\ 1 - \cos t \end{pmatrix}$$

Find the tangent vector, normal vector, curve length function and curvature of this curve.

$$T \circ \gamma(t) = \begin{pmatrix} \sin \frac{t}{2} \\ \cos \frac{t}{2} \end{pmatrix}$$

$$N \circ \gamma(t) = \begin{pmatrix} \cos \frac{t}{2} \\ -\sin \frac{t}{2} \end{pmatrix}$$

$$l \circ \gamma(t) = 4\left(1 - \cos \frac{t}{2}\right)$$

$$\kappa \circ \gamma(t) = \frac{1}{4 \sin \frac{t}{2}}$$

About Assignment 6

6.1 *binormal vector* and *torsion*

6.3 engineering application

6.6 **physics application**, relation (similarity and difference) between maths and physics in line integral. (Very important!)

Have Fun

And

Learn Well!