VV285 RC Mid1

Elements of Linear Algebra

Determinant

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Outline

- Determinants in two- and three-dimensional space
- Definition of Determinants
- Properties of Determinants
- Practical Ways to Calculate Determinants
- Determinants and Systems of Equations

Determinant in $\mathbb{R}^2$ and $\mathbb{R}^3$

- Determinant in $\mathbb{R}^2$

$$\det : \mathrm{Mat}(2\times 2;\mathbb{R}) \to \mathbb{R}, \qquad \det\begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = a_1 b_2 - a_2 b_1. \tag{1}$$

- Determinant in $\mathbb{R}^3$

$$\det : \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \qquad \det(a, b, c) = \langle\, a \times b\,,\, c\,\rangle. \tag{2}$$

- Cross Product

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{e_1} & \mathbf{e_2} & \mathbf{e_3} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \tag{3}$$

Formulas of Determinant

For every $n \in \mathbb{N}, n > 1$, there exists a unique, normed, alternating n-multilinear form $\det : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n \text{ times}} \cong \mathrm{Mat}(n \times n; \mathbb{R}) \to \mathbb{R}$.

Furthermore,
(Using Permutation: *Leibnitz Formula*)

$$\det(a_1, \ldots, a_n) = \det A = \sum_{\pi \in S_n} \operatorname{sgn} \pi \, a_{\pi(1)1} \cdots a_{\pi(n)n}, \tag{4}$$

(Using Recursion: *Laplace Expansion*)

$$\det A = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det A_{ij} \tag{5}$$

Properties of Determinant I

Three basis properties:

- n-multilinear
- normed
- alternating
  (when multilinearity holds,)
  (i) f is alternating
  (ii) $$f(a_1, \dots, a_{j-1}, a_j, a_{j+1}, \dots, a_{k-1}, a_k, a_{k+1}, \dots, a_p) = -f(a_1, \dots, a_{j-1}, a_k, a_{j+1}, \dots, a_{k-1}, a_j, a_{k+1}, \dots, a_p)$$
  (iii) $f(a_1, \dots, a_p) = 0$ if $a_1, a_2, \dots, a_p$ are linearly dependent.

Remark: The property of alternating enables determinant to test whether a matrix's column vectors are linearly independent.

Properties of Determinant II

Properties regarding elementary column operations:

- The determinant of a matrix A changes sign if two columns of A are interchanged, e.g.,

$$\det(a_2, a_1, \ldots, a_n) = -\det(a_1, a_2, \ldots, a_n)$$

- Multiplying all the entries in a column with a number λ leads to the determinant being multiplied by this constant:

$$\det(a_1, \ldots, \lambda a_j, \ldots, a_n) = \lambda \det(a_1, \ldots, a_j, \ldots, \ldots, a_n)$$

- Adding a multiple of a column to another column does not change the value of the determinant:

$$\det(a_1, \ldots, a_j, \ldots, a_k + \lambda a_j, \ldots, a_n) = \det(a_1, \ldots, a_j, \ldots, a_k, \ldots, a_n)$$

Remark: Properties regarding elementary row operations are analogous.

Properties of Determinant III

Properties regarding matrix operations:

- Matrix Transpose

$$\det A = \det A^T$$

- Matrix Product

$$\det(AB) = \det A \det B$$

- Matrix Inverse

$$\det A^{-1} = \frac{1}{\det A}$$

- Calculate Matrix Inverse

$$A^{-1} = \frac{1}{\det A} A^{\sharp}$$

Remark: Another way to find inverse is to use the Gauß-Jordan Algorithm.

Practical Ways to Calculate Determinants

Let $A \in \mathrm{Mat}(n \times n)$ have upper triangular form, i.e.,

$$A = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

for diagonal elements $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ and arbitrary values (denoted by $*$) above the diagonal. Then

$$\det A = \lambda_1 \cdots \lambda_n.$$

Remark: This method can be applied to calculate determinants of matrices $A \in \mathrm{Mat}(n \times n)$ when A is first transformed to upper triangular form using elementary matrix manipulations. This is of practical use for $n \geq 4$.

Determinants and Systems of Equations I

Determinants have a deep relation with the solutions of systems of equations. For a $n \times n$ matrix A,

When $\det A \neq 0$, we have

1. A is invertible,
2. $\ker A = \{0\}$ and $\dim \ker A = 0$,
3. $\text{ran } A = \mathbb{R}^n$ and $\dim \text{ran } A = n$,
4. The column vectors $a_{\cdot k}$ and row vectors $a_{j \cdot}$ of A are linear independent.
5. $\text{rank } A = n$
6. $Ax = b$ has a unique solution $x = A^{-1}b$ for any $b \in \mathbb{R}^n$.

When $\det A = 0$, we have

1. A is not invertible,
2. $\ker A \neq \{0\}$ and $\dim \ker A > 0$,
3. ran $A \subsetneq \mathbb{R}^n$ and $\dim$ ran $A < n$,
4. The column vectors $a_{.k}$ and row vectors $a_{j.}$ of A are linear dependent.
5. rank $A < n$
6. $Ax = 0$ has (infinite) non-trivial solutions $x \in \ker A$.

Remark: These arguments connect everything we learned, from systems of linear equations to determinants. Ask yourself whether you can understand the relationship between each argument.