VV285 RC Part I

Elements of Linear Algebra

"Matrices are just linear maps!"

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Overview of Linear Algebra

1. Systems of Linear Equations
2. Finite-Dimensional Vector Spaces
3. Inner Product Spaces
4. Linear Maps
5. Matrices
6. Theory of Systems of Linear Equations
7. Determinants

Overview

1. Linear System
   Homogeneous vs. Inhomogeneous
   Underdetermined vs. Overdetermined
2. Equivalency of Linear System
3. The Gauß – Jordan Algorithm
4. Diagonalizable (Existence and Uniqueness of Linear System)
5. **Fundamental Lemma for Homogeneous Equations**

Linear System

A *linear system* of m (algebraic) equations in n unknowns $x_1, x_2, \dots, x_n \in V$ is a set of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{1}$$

where $b_1, b_2, \dots, b_m \in V$ and $a_{ij} \in \mathbb{F}, i = 1, \dots, m,\ j = 1, \dots, n$.
If $b_1 = b_2 = \cdots = b_m = 0$, then (1) is called a *homogeneous system*. Otherwise, it is called an *inhomogeneous system.*
If $m < n$ we say that the system in *underdetermined*, if $m > n$ the system is called *overdetermined*. A solution of a linear system of equations (1) is a tuple of elements $(y_1, y_2, \dots, y_n) \in V^n$ such that the predicate (1) becomes a true statement.

We say that two systems of linear equations are *equivalent* if any solution of the first system is also a solution of the second system and vice-versa. Thus the systems

$$\begin{aligned} x_1 + 3x_2 - x_3 &= 1 \\ -5x_2 + x_3 &= 1 \\ 10x_2 + x_3 &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1 &= 2 \\ x_2 &= 0 \\ x_3 &= 1 \end{aligned}$$

are *equivalent*.

Gauß-Jordan Algorithm

The goal of the *Gauß-Jordan algorithm* (also called Gaußian elimination) is to transform a system

$$\begin{array}{ccc|c} * & * & * & \diamond \\ * & * & * & \diamond \\ * & * & * & \diamond \end{array} \qquad * \in \mathbb{R} \text{ or } \mathbb{C}, \quad \diamond \in V$$

first into the form

$$\begin{array}{ccc|c} 1 & * & * & \diamond \\ 0 & 1 & * & \diamond \\ 0 & 0 & 1 & \diamond \end{array} \tag{2}$$

and subsequently into

$$\begin{array}{ccc|c} 1 & 0 & 0 & \diamond \\ 0 & 1 & 0 & \diamond \\ 0 & 0 & 1 & \diamond \end{array} \tag{3}$$

Include:

1. Swapping (interchanging) two rows,
2. Multiplying each element in a row with a number,
3. Adding a multiple of one row to another row.

Result: Transform a system into a equivalent system. Since each row represents an equation, we are essentially **manipulating equations**.

Extension: The application of Gauß-Jordan Algorithm

Diagonalization

A system of m equations with n unknowns will have a unique solution if and only if it is *diagonalizable*. i.e. It can be transformed into diagonal form.

Remark: *Diagonalization* turns out to be an important topic in VV286, especially in terms of *ordinary differential equation systems*.

The homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned}$$

of m equations in n real or complex unknowns $x_1, x_2, \ldots, x_n$ has a **non-trivial** solution if $n > m$.

Remark: This fundamental lemma contributes to prove that any basis of a vector space has the same length.

Overview

Let V be a real or complex vector space and $v_1, v_2, \ldots, v_n \in V$. Then the vectors $v_1, v_2, \ldots, v_n$ are said to be *independent* if for all $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$

$$\sum_{k=1}^{n} \lambda_k v_k = 0 \qquad \Rightarrow \qquad \lambda_1 = \lambda_2 = \cdots = \lambda_n = 0.$$

A finite set $M \subset V$ is called an *independent set* if the elements of M are independent.

Let $v_1, v_2, \ldots, v_n \in V$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{F}$. Then the expression

$$\sum_{k=1}^{n} \lambda_k v_k = \lambda_1 v_1 + \cdots + \lambda_n v_n$$

is called a *linear combination* of the vectors $v_1, v_2, \ldots, v_n$.
The set

$$\operatorname{span}\{v_1, \ldots, v_n\} = \left\{ y \in V : y = \sum_{k=1}^{n} \lambda_k v_k, \lambda_1, \ldots, \lambda_n \in \mathbb{F} \right\}$$

is called the *(linear) span* or the *linear hull* of the vectors $v_1, v_2, \ldots, v_n$.

The vectors $v_1, v_2, \ldots, v_n \in V$ are independent if and only if **none of them is contained in the span of all the others**.

(How to prove?)

Basis

Let V be a real or complex vector space. An n-tuple $\mathcal{B} = (b_1, \dots, b_n) \in V^n$ is called an *(ordered and finite) basis* of V if every vector v has a **unique** representation

$$v = \sum_{i=1}^{n} \lambda_i b_i, \qquad \lambda_i \in \mathbb{F}. \tag{4}$$

The numbers λ_i are called the *coordinates* of v with respect to $\mathcal{B}$.

The tuple of vectors $(e_1, e_2, \dots, e_n)$, $e_i \in \mathbb{R}^n$,

$$e_i = (0, \dots 0, \underset{\substack{\uparrow \\ i\text{th} \\ \text{entry}}}{1}, 0, \dots, 0), \qquad i = 1, \dots, n,$$

is called the *standard basis* or *canonical basis* of $\mathbb{R}^n$.

Basis = Independence + Span

Let V be a real or complex vector space.
An n-tuple $\mathcal{B} = (b_1, \ldots, b_n) \in V^n$ is a basis of V if and only if

1. the vectors $b_1, b_2, \ldots, b_n$ are linearly independent, i.e., $\mathcal{B}$ is an independent set,
2. $V = \operatorname{span} \mathcal{B}$.

(How to prove?)

Remark: This theorem is more practical than the definition of basis when proving some set is a basis of some vector space. It helps one decompose the proof into two parts: 1. prove linear independence (uniqueness of (4)) 2. prove the span is large enough (existence of (4)).

Let V be a real or complex finite-dimensional vector space, $V \neq \{0\}$. **Then any basis of V has the same length** (number of elements).

Remark: This theorem can be proved by contradiction (Use the definition of basis and the fundamental lemma for homogeneous equations). With such a premise, we can then define the *dimension* of vector space.

Let V be a real or complex vector space. Then V is called *finite-dimensional* if either

- $V = 0$ or
- V possesses a finite basis.

If V is not finite-dimensional, we say that it is *infinite-dimensional*.

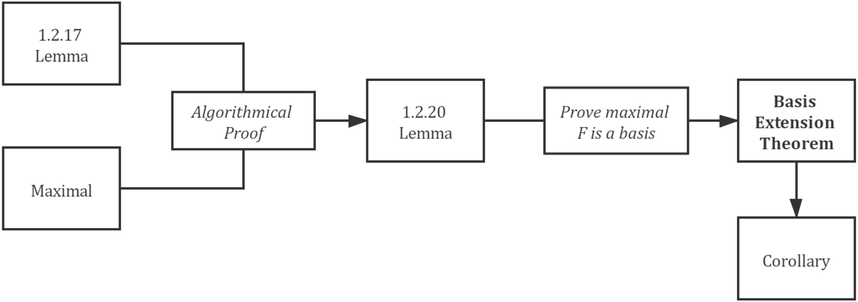

Figure: Logic Flow of Basic Extension Theorem

An interpretation of "**maximal**": the max (in size) independent subset of some set.

Basis Extension Theorem

Let V be a finite-dimensional vector space and $A' \subset V$ an independent set. Then there exists a basis of V containing A'.

Remark:
The basis extension theorem is fundamental. It tells us that for any independent subset A' of a finite-dimensional vector space V, we can always find and add $\dim V - |A'|$ elements to A' to extend it into a basis of V. And two useful corollaries follow immediately:

Let V be an n-dimensional vector space, $n \in \mathbb{N}$. Then

1. any independent set A with n elements is a basis of V.
2. an independent set A may have at most n elements.

(How to prove?)

Let V be a real or complex vector space and U, W be sets in V.

(i) We define the *sum of U and W* by

$$U + W := \left\{ v \in V : \underset{u \in U}{\exists} \; \underset{w \in W}{\exists} : v = u + w \right\}.$$

(ii) If U and W are subspaces of V with $U \cap W = \{0\}$, the sum $U + W$ is called *direct*, and we denote it by $U \oplus W$.

Two properties about sum of vector space:

1. The sum $U + W$ of vector spaces U, W is direct if and only if all $x \in U + W$, $x \neq 0$, have a **unique** representation $x = u + w, \; u \in U, w \in W$.
2. Let V be a vector space and $U, W \subset V$ be finite-dimensional subspaces of V. Then

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W.$$

Proof I

Suppose

$$\{v_1, \ldots, v_p\}$$

is a basis for $U \cap W$. By *Basis Extension Theorem*, we can find a basis

$$\{v_1, \ldots, v_p, u_1, \ldots, u_q\}$$

for U and a basis

$$\{v_1, \ldots, v_p, w_1, \ldots, w_r\}$$

for W.
Then we just need to show that

$$B = \{v_1, \ldots, v_p, u_1, \ldots, u_q, w_1, \ldots, w_r\}$$

is a basis for $U + W$

Suppose

$$\alpha_1 v_1 + \cdots + \alpha_p v_p + \beta_1 u_1 + \cdots + \beta_q u_q + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0$$

Then

$$x = \underbrace{\alpha_1 v_1 + \cdots + \alpha_p v_p + \beta_1 u_1 + \cdots + \beta_q u_q}_{\epsilon U} = -(\underbrace{\gamma_1 w_1 + \cdots + \gamma_r w_r}_{\in W})$$

belongs to $U \cap W$. Thus

$$x = \delta_1 v_1 + \cdots + \delta_p v_p$$

and therefore

$$\delta_1 v_1 + \cdots + \delta_p v_p = -(\gamma_1 w_1 + \cdots + \gamma_r w_r)$$

Proof III

so that

$$\delta_1 v_1 + \cdots + \delta_p v_p + \gamma_1 w_1 + \cdots + \gamma_r w_r = 0$$

Since the set $\{v_1, \ldots, v_p, w_1, \ldots, w_r\}$ is linearly independent, we conclude

$$\delta_1 = 0, \quad \ldots, \quad \delta_p = 0, \quad \gamma_1 = 0, \quad \ldots, \quad \gamma_r = 0$$

and also that

$$\alpha_1 v_1 + \cdots + \alpha_p v_p + \beta_1 u_1 + \cdots + \beta_q u_q = 0$$

So, from linear independence of $\{v_1, \ldots, v_p, u_1, \ldots, u_q\}$ we get

$$\alpha_1 = 0, \quad \ldots, \quad \alpha_p = 0, \quad \beta_1 = 0, \quad \ldots, \quad \beta_q = 0$$

Therefore, the set B is independent. It is clear that $\operatorname{span} B = U + W$. So we conclude B is a basis for $U + W$, and furthermore,

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W.$$

Corollary

Let V be a vector space and $U, W \subset V$ be finite-dimensional subspaces of V. Then

$$\dim(U + W) \leq \dim U + \dim W.$$

The condition for "$=$": the sum is direct. i.e.

$$\dim(U \oplus W) = \dim U + \dim W.$$