Homework 6

Math 3607, Autumn 2021

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Problem 1.

1. (Understanding matrix multiplication) Do **LM** 12.5–3.

a) i^{th} row of rw^T where $r, w \in \mathbb{R}$?
What is size of matrix?

$$rw^T = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix} \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$$

$$= \begin{bmatrix} r_1 w_1 & r_1 w_2 & \cdots & r_1 w_j & \cdots & r_1 w_n \\ r_2 w_1 & r_2 w_2 & \cdots & r_2 w_j & \cdots & r_2 w_n \\ \vdots & \vdots & & \vdots & & \vdots \\ r_i w_1 & r_i w_2 & \cdots & r_i w_j & \cdots & r_i w_n \\ \vdots & \vdots & & \vdots & & \vdots \\ r_n w_1 & r_n w_2 & \cdots & r_n w_j & \cdots & r_n w_n \end{bmatrix} \quad n \times n$$

$$\boxed{\begin{array}{l} i^{th} \text{ row is } \begin{bmatrix} r_i w_1 & r_i w_2 & \cdots & r_i w_j & \cdots & r_i w_n \end{bmatrix} \\ \text{Size is } n \times n \text{ matrix} \end{array}}$$

b.) j^{th} element of row vector $b^T A$? What is size of the vector?

$A \in \mathbb{R}^{m \times n}$ $b \in \mathbb{R}^m$

$$b^T A = \underbrace{\begin{bmatrix} b_1 & b_2 & \cdots & b_m \end{bmatrix}}_{1 \times m} \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}}_{m \times n}$$

size $1 \times n$ row vector

$b \in \mathbb{R}^{m\times 1}$ $\quad b^T \in \mathbb{R}^{1\times m}$ $\quad A \in \mathbb{R}^{m\times n}$

$\Rightarrow b^T A \in {}^{1\times n}$, a row vector with n elements. The j^{th} element of $b^T A$ is

$b_1 \cdot a_{1j} + b_2 a_{2j} + \ldots\ldots + b_m \cdot a_{mj}$

$$\boxed{\sum_{k=1}^{m} b_k a_{kj}}$$

C.) What is c_{ij} if $C = AB$ and $B \in \mathbb{R}^{n\times p}$? What is the size of the new matrix?

$A \in \mathbb{R}^{m\times n}$

$AB =$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1j} & \ldots & b_{1p} \\ b_{21} & b_{22} & \ldots & b_{2j} & \ldots & b_{2p} \\ \vdots & \vdots & & \vdots & & \vdots \\ b_{i1} & b_{i2} & \ldots & b_{ij} & \ldots & b_{ip} \\ b_{n1} & b_{n2} & \ldots & b_{nj} & \ldots & b_{np} \end{bmatrix}}_{n\times p}$$

$c_{ij} = a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{ij} \cdot b_{ij} + a_{in} \cdot b_{nj}$

$$\boxed{\sum_{k=1}^{n} a_{ik} \cdot b_{kj} \quad \text{with an } n\times p \text{ matrix}}$$

d.) What is d_{ij} if $D = B^T A^T$. What is the size of resulting matrix?

$B \in \mathbb{R}^{n \times p}$ $\quad A \in \mathbb{R}^{m \times n}$

$B^T \in \mathbb{R}^{p \times n}$ $\quad A^T \in \mathbb{R}^{n \times m}$

$$\underset{p \times n}{B^T} \begin{bmatrix} b_{11} & b_{12} & \cdots b_{1j} \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots b_{2j} \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{i1} & b_{i2} & \cdots b_{ij} \cdots & b_{in} \\ \vdots & \vdots & \vdots & \vdots \\ b_{p1} & b_{p2} & \cdots b_{pj} \cdots & b_{pn} \end{bmatrix} \quad \underset{n \times m}{A^T} \begin{bmatrix} a_{11} & a_{12} & \cdots a_{1j} \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots a_{2j} \cdots & a_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & \cdots a_{ij} \cdots & a_{im} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots a_{nj} \cdots & a_{nm} \end{bmatrix}$$

$p \times m$ matrix

$$d_{ij} = b_{i1} \cdot a_{1j} + b_{i2} \cdot a_{2j} + b_{ij} \cdot a_{ij} + b_{in} \cdot a_{nj}$$

$$= \boxed{\sum_{k=1}^{n} b_{ik} \cdot a_{kj}}$$

e.) $C = AB = \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$ with an $m \times p$ matrix

$$D = B^T A^T = \sum_{k=1}^{n} b_{ik} \cdot a_{kj}$$

$$(AB)^T = \sum_{k=1}^{n} [C^T]_{ji} = \sum_{k=1}^{n} [C]_{ij} = \sum_{k=1}^{n} C_{ij}$$

$$= \sum_{k=1}^{n} b_{ik} \cdot a_{kj}$$ therefore $d_{ij} = C_{ji}$

Since the column swaps with the row, this means that the a and b swap positions f.) What is the $(i, j)^{th}$ element of $A^T A$? What is the size of the resulting matrix?

$$\underset{n \times m}{\overset{A^T = B}{\begin{bmatrix} a_{11} & a_{12} \dots a_{1j} \dots & a_{1m} \\ a_{21} & a_{22} \dots a_{2j} \dots & a_{2m} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{i1} & a_{i2} \dots a_{ij} \dots & a_{im} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{n1} & a_{n2} \dots a_{nj} \dots & a_{nm} \end{bmatrix}}} \quad \underset{m \times n}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} \dots a_{1j} \dots & a_{1n} \\ a_{21} & a_{22} \dots a_{2j} \dots & a_{2n} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{i1} & a_{i2} \dots a_{ij} \dots & a_{in} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{m1} & a_{m2} \dots a_{mj} \dots & a_{mn} \end{bmatrix}}}$$

$n \times n$ size matrix

$$b_{i1} \cdot a_{1j} + b_{i2} \cdot a_{2j} + \dots + b_{ij} \cdot a_{ij} + \dots + b_{im} \cdot a_{mj}$$

$$(i,j)^{th} = \sum_{k=1}^{m} b_{ik} a_{kj}$$

g.) What is the $(i,j)^{th}$ element of AA^T? What is the size of the resulting matrix?

$$\underset{m \times n}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} \dots a_{1j} \dots & a_{1n} \\ a_{21} & a_{22} \dots a_{2j} \dots & a_{2n} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{i1} & a_{i2} \dots a_{ij} \dots & a_{in} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{m1} & a_{m2} \dots a_{mj} \dots & a_{mn} \end{bmatrix}}} \quad \underset{n \times m}{\overset{A^T = B}{\begin{bmatrix} a_{11} & a_{12} \dots a_{1j} \dots & a_{1m} \\ a_{21} & a_{22} \dots a_{2j} \dots & a_{2m} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{i1} & a_{i2} \dots a_{ij} \dots & a_{im} \\ \vdots & \vdots \quad\quad \vdots & \vdots \\ a_{n1} & a_{n2} \dots a_{nj} \dots & a_{nm} \end{bmatrix}}}$$

$$a_{i1} \cdot b_{1j} + a_{i2} \cdot b_{2j} + a_{ij} \cdot b_{ij} + a_{in} b_{nj}$$

$$= \sum_{k=1}^{n} a_{ik} \cdot b_{kj}$$ with a matrix of size $m \times m$

h.) What is the i^{th} element of $A \cdot A^T b$? What is the size of the resulting matrix? $b \in \mathbb{R}^m$

$$\underset{m \times n}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} \dots & a_{1j} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2j} \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} \dots & a_{ij} \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \dots & a_{mj} \dots & a_{mn} \end{bmatrix}}} \quad \underset{n \times m}{\overset{A^T = C}{\begin{bmatrix} c_{11} & c_{12} \dots & c_{1j} \dots & c_{1m} \\ c_{21} & c_{22} \dots & c_{2j} \dots & c_{2m} \\ \vdots & \vdots & \vdots & \vdots \\ c_{i1} & c_{i2} \dots & c_{ij} \dots & c_{im} \\ \vdots & \vdots & \vdots & \vdots \\ c_{n1} & c_{n2} \dots & c_{nj} \dots & c_{nm} \end{bmatrix}}} \quad \underset{m \times 1}{\overset{b}{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_i \\ \vdots \\ b_m \end{bmatrix}}}$$

$$\underset{m \times n}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} \dots & a_{1j} \dots & a_{1n} \\ a_{21} & a_{22} \dots & a_{2j} \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} \dots & a_{ij} \dots & a_{in} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \dots & a_{mj} \dots & a_{mn} \end{bmatrix}}} \quad \underset{n \times 1}{\overset{Cb}{\begin{bmatrix} c_{11} \cdot b_1 + c_{12} \cdot b_2 + \dots + c_{1j} \cdot b_i + \dots + c_{1m} \cdot b_m \\ c_{21} \cdot b_1 + c_{22} \cdot b_2 + \dots + c_{2j} \cdot b_i + \dots + c_{2m} \cdot b_m \\ c_{i1} \cdot b_1 + c_{i2} \cdot b_2 + \dots + c_{ij} \cdot b_i + \dots + c_{im} \cdot b_m \\ c_{n1} \cdot b_1 + c_{n2} \cdot b_2 + \dots + c_{nj} \cdot b_i + \dots + c_{nm} \cdot b_m \end{bmatrix}}}$$

i^{th} element

$$\sum_{K=1}^{m} a_{iK} \cdot c_{Kj} \cdot b_K$$

with a column vector of $m \times 1$

Problem 2.

2. (Gram-Schmidt in MATLAB) Do **LM** 12.6–2.

```
%A = [1 1 1 1; 1 0 0 1; 4 2 -2 1]';
m = 500;
n = 20;
A = randn(m,n);
[Q, R] = qr(A, 0);
[Q, R] = GramSchmidt(A);
%Check Q^T*Q - I = 0
%check A - Q*R = 0
norm(Q'*Q - eye(n,n))
```

```
ans = 4.6463e-16
```

```
norm(A - Q*R)
```

```
ans = 5.1706e-15
```

Problem 3.

Part a.)

3. (Periodic fit; **FNC** 3.1.3) In this problem you are trying to find an approximation to the periodic function $f(t) = e^{\sin(t-1)}$ over one period, $0 \leq t \leq 2\pi$. In MATLAB, let `t=linspace(0,2*pi,200)'` and let `b` be a column vector of evaluations of f at those points.

   (a) Find the coefficients of the least square fit

$$f(t) \approx c_1 + c_2 t + \cdots + c_7 t^6.$$

```
t=linspace(0,2*pi,20)';
y = exp(sin(t - 1));
n = 7;
V = t .^ (0:n-1);
c = V \ y;
```

Part b.)

(b) Find the coefficients of the least squares fit

$$f(t) \approx d_1 + d_2 \cos(t) + d_3 \sin(t) + d_4 \cos(2t) + d_5 \sin(2t).$$

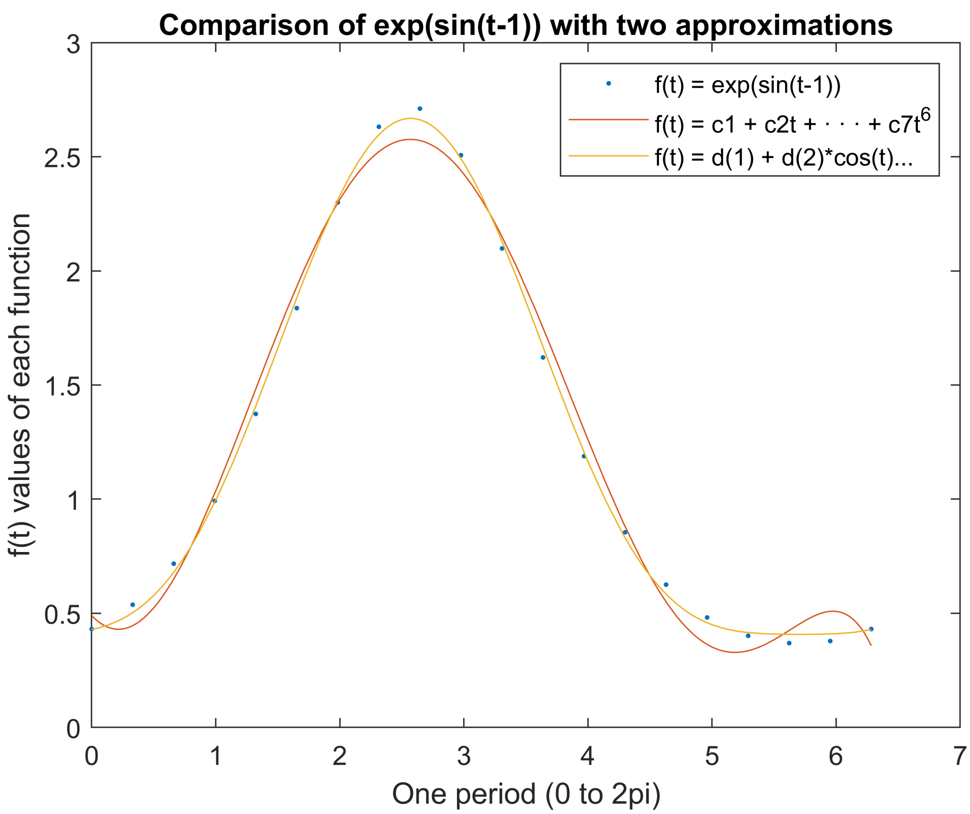
```
X = [ones(size(t)) cos(t) sin(t) cos(2*t) sin(2*t)];
% d2 = cos(t);
% d3 = sin(t);
% d4 = cos(2.*t);
% d5 = sin(2.*t);
% X = [d1 d2 d3 d4 d5]
d = X \ y;
```

Part c.)

(c) Plot the original function $f(t)$ and the two approximations from (a) and (b) together on a well-labeled graph.

```
clf
plot(t, y, '.'), hold on
p = @(t) polyval(flip(c), t);
f = @(t) d(1) + d(2)*cos(t) + d(3)*sin(t) + d(4)*cos(2*t) + d(5)*sin(2*t);
fplot(p, [0 2*pi])
fplot(f, [0 2*pi])
xlabel('One period (0 to 2pi)')
```

```
ylabel('f(t) values of each function')
title('Comparison of exp(sin(t-1)) with two approximations')
legend('f(t) = exp(sin(t-1))', 'f(t) = c1 + c2t + · · · + c7t^6', ...
    'f(t) = d(1) + d(2)*cos(t)...')
```



Problem 4.

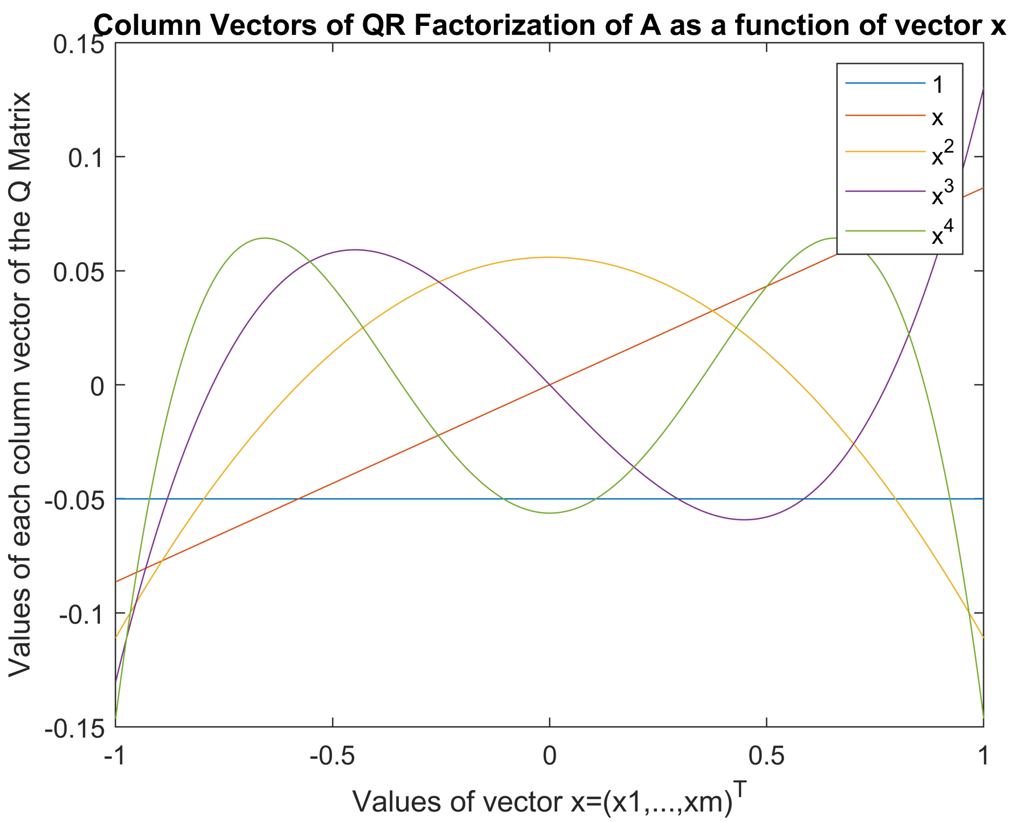
4. (Adapted from **FNC** 3.3.3.) Let $x_1, x_2, \ldots, x_m$ be m equally spaced points in $[-1, 1]$ and V be the Vandermonde-type matrix

$$V = \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_m & x_m^2 & \cdots & x_m^{n-1} \end{bmatrix},$$

where $m = 400$ and $n = 5$. Find the thin QR factorization of $V = \hat{Q}\hat{R}$, and, on a single graph, plot every column of $\hat{Q}$ as a function of the vector $\mathbf{x} = (x_1, x_2, \ldots, x_m)^{\mathrm{T}}$.

```
xpre = linspace(-1,1,400)';
n = 5;
V = [ones(400,1) xpre .^ (1:n-1)];
[Q, R] = qr(V, 0);
% one = Q(:,1)
% x1 = Q(:,2)
% x2 = Q(:,3)
% x3 = Q(:,4)
```

```
% x4 = Q(:,5)
clf
%plot(xpre, one, xpre, x1, xpre, x2, xpre, x3, xpre, x4)
plot(xpre, Q)
xlabel('Values of vector x=(x1,...,xm)^T')
ylabel('Values of each column vector of the Q Matrix')
title('Column Vectors of QR Factorization of A as a function of vector x')
legend('1','x','x^2','x^3','x^4')
```



Problem 5.

Part a.)

(a) Modify and develop the script into a MATLAB function `visMatrixNorm` which takes two inputs

- A, a 2×2 matrix and
- p, a number which can be either 1, 2, or ∞,

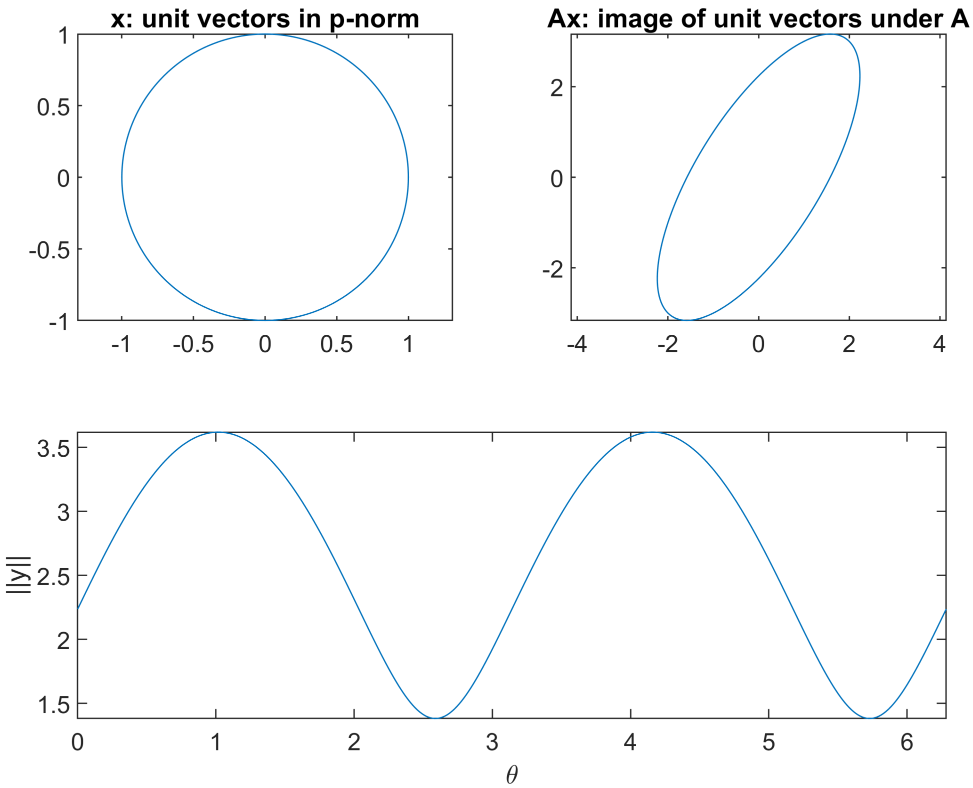
and carries out the same tasks as above, namely,

- approximating $\|A\|_p$ using (1) and
- producing a figure such as Figure 1.

Be sure to print out the value of p, the approximate norm, and the norm computed using MATLAB's `norm` function.

```
A = [2 1;
     1 3];
p = 2;
```

```
[~] = visMatrixNorm(A, p)
```

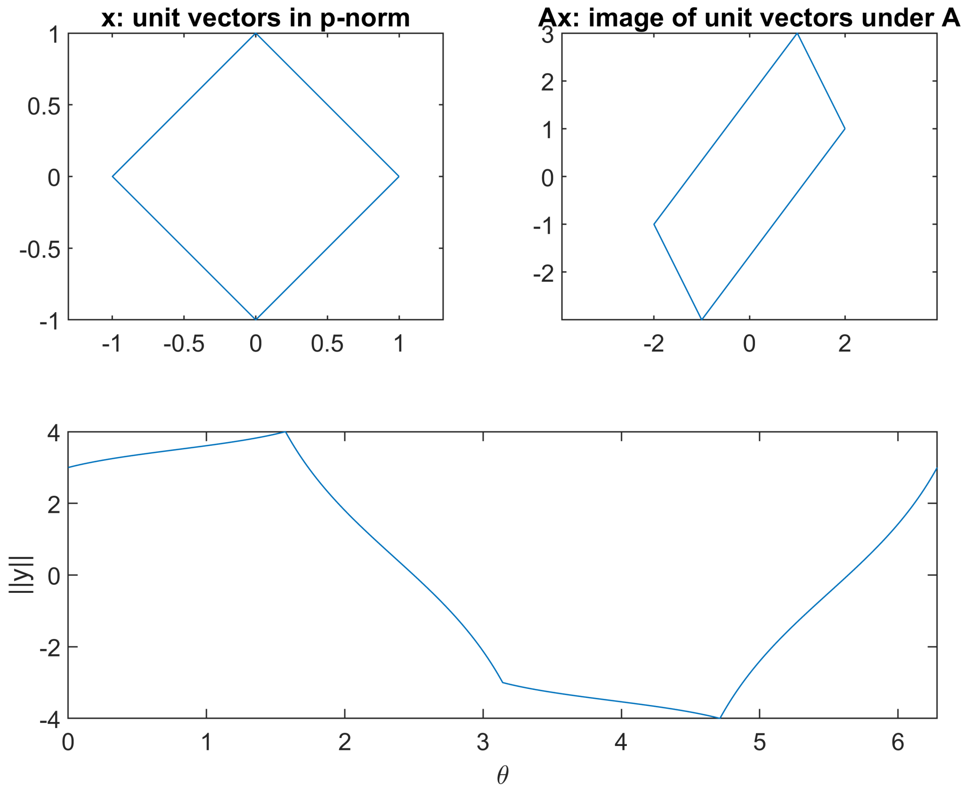



```
p = 2.0000000000000000
approx. norm: 3.6179964204609893
actual norm: 3.6180339887498953
```

Part b.)

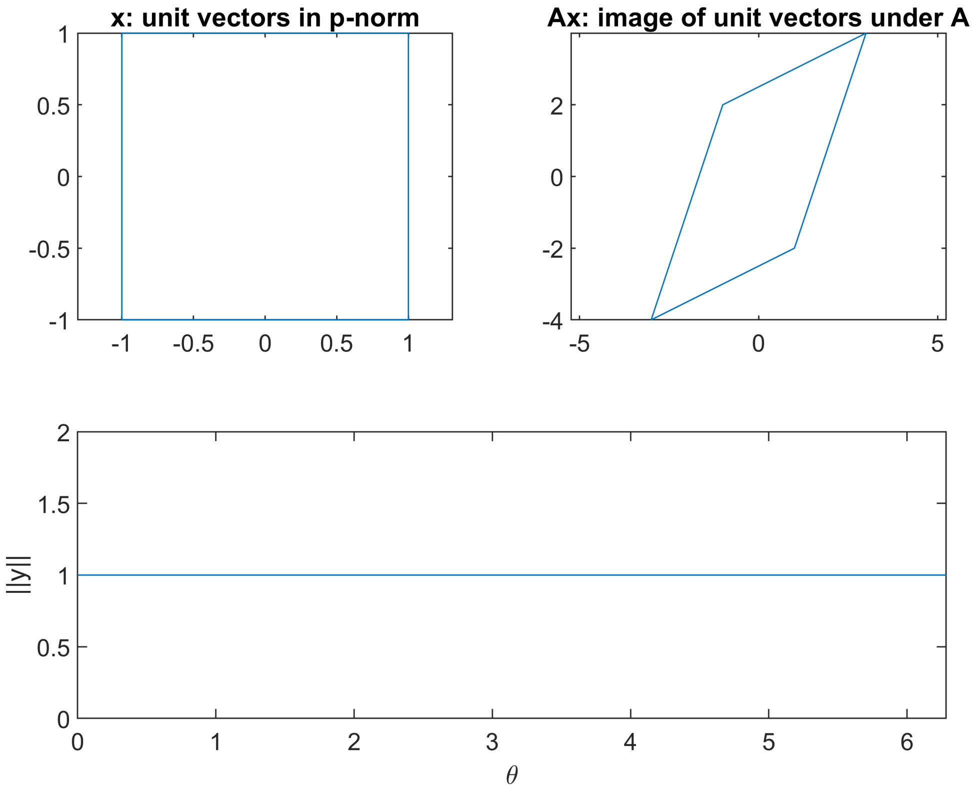
(b) Then run the function with `visMatrixNorm(A, 1)` and `visMatrixNorm(A, Inf)`, where A is as defined in (2).

```
p = 1;
[~] = visMatrixNorm(A, p)
```




```
p = 1.0000000000000000
approx. norm: 4.0000000000000000
actual norm: 4.0000000000000000
```

```
p = inf;
[~] = visMatrixNorm(A, p)
```




```
p =                 Inf
approx. norm: 1.0000000000000000
actual norm: 4.0000000000000000
```

Function Gram-Schmidt

```
function [Q,R] = GramSchmidt(A)
% GramSchmidt computes both Q and R for the procedure of thin QR
% factorization
% input:
% A
% output:
% The orthonormal matrix Q and the upper triangular matrix R
[m,n] = size(A);
Q = A;
R = zeros(n);
for j=1:n
    for i=1:j-1
       R(i,j) = (Q(:,i)' * A(:,j));
       Q(:,j) = Q(:,j) - R(i,j)*Q(:,i);
    end
    R(j,j) = norm(Q(:,j), 2);
    Q(:,j) = Q(:,j) / R(j,j);
end
end
```

Function visMatrixNorm

```
function [Y] = visMatrixNorm(A, p)
% visMatrixNorm computes and prints out the approximate norm versus the actual norm
% given an input of matrix A which is a square matrix and p is the norm value
% The function outputs Y which is the image of the unit vectors under A and also prints out
% the approximate norm versus the actual norm

theta = linspace(0, 2*pi, 361);
X = [cos(theta); sin(theta)];
for j = 1:361
    X(:,j) = X(:,j)/norm(X(:,j),p);  % x: unit vectors in 2-norm
    Y = A*X;                          % y: images of x under A
    norm_Y = (sum(Y.^p, 1)).^(1/p);
end

% visualization
clf
subplot(2,2,1)
plot(X(1,:), X(2,:)), axis equal
title('x: unit vectors in p-norm')
subplot(2,2,2)

plot(Y(1,:), Y(2,:)), axis equal
title('Ax: image of unit vectors under A')
subplot(2,1,2)

plot(theta, norm_Y), axis tight
xlabel('\theta')
ylabel('||y||')

% matrix norm approximation (and comparison)
fprintf(' p = %18.16f\n', p)
fprintf(' approx. norm: %18.16f\n', max(norm_Y))
fprintf(' actual norm: %18.16f\n', norm(A, p))

end
```