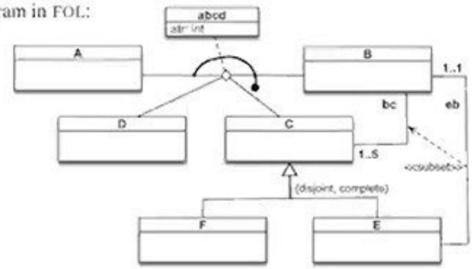
SAPIENZA Università di Roma – MSc. in Engineering in Computer Science

Formal Methods – July 16, 2025

(Time to complete the test online: 2:00 hours)

Exercise 1 Use FOL for the following exercises.

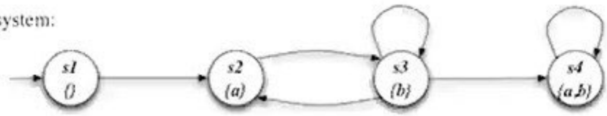
- **Exercise 1.1:** Express the following UML class diagram in FOL:



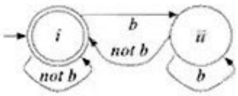
- **Exercise 1.2:** Consider the following predicates: $Teacher(x, y)$, teacher x works at school y; $Subject(x, y)$ subject x is of type y (e.g., "arts", "classics", "science"); $Teaches(x, y)$, teacher x teaches subject y. Express in First-Order Logic (FOL) the following queries, and state which ones are Conjunctive Queries (CQs).

  1. *There exists a teacher working at Marconi who teaches a arts subject.*
  2. *There exists a teacher working at Marconi who teaches at least two arts subjects.*
  3. *There exists a teacher working at Marconi who teaches all arts subjects.*
  4. *There exists a teacher working at Marconi who teaches only arts subjects.*
  5. *Return the pairs of teachers such that the first teacher teaches all subjects that the second one teaches.*
  6. *Return the pairs of teachers such that the first teacher teaches all subjects that the second one teaches and at least one that the second teacher does not teach.*

Exercise 2 Consider the following transition system:



- **Exercise 2.1:** Model check the Mu-Calculus formula: $\mu X.\nu Y.((a \wedge \langle next\rangle X) \vee (b \wedge \langle next\rangle Y))$
- **Exercise 2.2:** Model check the CTL formula $AF(a \wedge AX(EGb))$, by translating it in Mu-Calculus.
- **Exercise 2.3** Model check the LTL formula $\Diamond\Box(b)$, by considering that the Büchi automaton for $\neg\Diamond\Box(b)$ (i.e., $\Box\Diamond(\neg b)$) is:



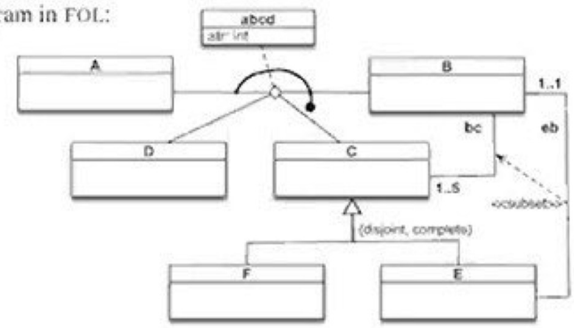
Exercise 3 Given the following ~~boolean~~ conjunctive queries:

```
q1(x) :- r(x,x),b(x,y),b(y,x)
q2(x) :- r(x,y),b(y,z),b(z,x)
```

check whether $q1$ is contained into $q2$, explaining the technique used and, in case of containment, showing the homomorphism between the canonical databases.

Exercise 1 Use FOL for the following exercises.

- **Exercise 1.1:** Express the following UML class diagram in FOL:



$A(x),\ B(x),\ C(x),\ D(x),\ E(x),\ F(x)$
$ABCD(x, y, z, w)$
$ATR(x, y, z, w, \tau)$
$BC(x, y)$
$EB(x, y)$

$\forall x, y, z, w.\ ABCD(x, y, z, w) \supset A(x) \wedge B(y) \wedge C(z) \wedge D(w)$
$\forall x, y, z, z', w, w'.\ ABCD(x, y, z, w) \wedge AB(x, y, z', w') \supset z = z' \wedge w = w'$
$\forall x, y, z, w, \tau.\ ATR(x, y, z, w, \tau) \supset ABCD(x, y, z, w) \wedge INT(\tau)$

$\forall x, y.\ BC(x, y) \supset B(x) \wedge C(y)$
$\forall x.\ B(x) \supset 1 \leq \#\{y \mid DA(x, y)\} \leq 5$

$\forall x, y.\ EB(x, y) \supset E(x) \wedge B(y)$
$\forall x.\ E(x) \supset 1 \leq \#\{y \mid DA(x, y)\} \leq 1$

$\forall x, y.\ EB(x, y) \supset BC(x, y)$

$\forall x.\ F(x) \supset C(x) \wedge \neg E(x)$
$\forall x.\ E(x) \supset C(x)$
$\forall x.\ C(x) \supset F(x) \vee E(x)$

- **Exercise 1.2:** Consider the following predicates: $Teacher(x, y)$, teacher x works at school y; $Subject(x, y)$ subject x is of type y (e.g., "arts", "classics", "science"); $Teaches(x, y)$, teacher x teaches subject y. Express in First-Order Logic (FOL) the following queries, and state which ones are Conjunctive Queries (CQs).

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1) $\exists x,y.\ TEACHER(x, MARCONI) \wedge S(y, ART) \wedge T(x,y)$ ✓

2) $\exists x,y,y'.\ TEACHER(x, MARC) \wedge S(y, ART) \wedge S(y', ART) \wedge T(x,y) \wedge T(x,y') \wedge y \neq y'$ ✗

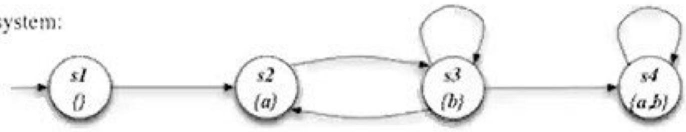
3) $\exists x.\ TEACHER(x, MARC) \wedge \forall y.\ (S(y, ART) \supset T(x,y))$ ✗

4) $\exists x.\ TEACHER(x, MARC) \wedge \forall y.\ (T(x,y) \supset S(y, ART))$ ✗

5) $\forall y.\ (T(s',y) \supset T(s,y))$ ✗

6) $\forall y.\ (T(s',y) \supset T(s,y)) \wedge \exists z.\ T(s,z) \wedge \neg T(s',z))$ ✗

Exercise 2 Consider the following transition system:



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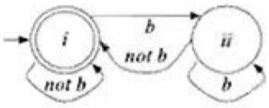

1) $\varphi = \mu X.\nu Y.((a \wedge \langle NEXT \rangle X) \vee (b \wedge \langle NEXT \rangle Y))$

$[X_0] = \phi$

$[X_1] = [\nu Y.((a \wedge \langle NEXT \rangle X_0) \vee (b \wedge \langle NEXT \rangle Y))] =$

$[Y_0] = \{1,2,3,4\}$

$[Y_1] = ([a] \cap PREE(NEXT, X_0)) \cup ([b] \cap PREE(NEXT, Y_0)) =$

$= (\{2,4\} \cap \phi) \cup (\{3,4\} \cap \{1,2,3,4\}) = \{3,4\}$

$[Y_2] = ([a] \cap PREE(NEXT, X_0)) \cup ([b] \cap PREE(NEXT, Y_1)) =$

$= (\{2,4\} \cap \phi) \cup (\{3,4\} \cap \{2,3,4\}) = \{3,4\}$

$[Y_1] = [Y_2] = [X_1] = \{3,4\}$

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$[Y_1] = [Y_2] = [X_2] = \{2,3,4\}$

$[X_3] = [\nu Y.((a \wedge \langle NEXT \rangle X_2) \vee (b \wedge \langle NEXT \rangle Y))] =$

$[Y_0] = \{1,2,3,4\}$

$[Y_1] = ([a] \cap PREE(NEXT, X_2)) \cup ([b] \cap PREE(NEXT, Y_0)) =$

$= (\{2,4\} \cap \{1,2,3,4\}) \cup (\{3,4\} \cap \{1,2,3,4\}) = \{2,3,4\}$

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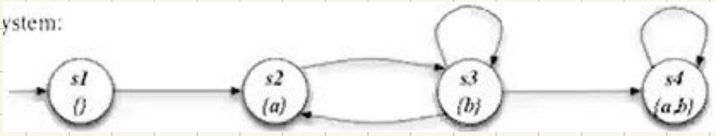
$= (\{2,4\} \cap \{1,2,3,4\}) \cup (\{3,4\} \cap \{1,2,3,4\}) = \{2,3,4\}$

$[Y_1] = [Y_2] = [X_3] = \{2,3,4\}$

$[X_2] = [X_3] = \{2,3,4\}$

$s_1 \in [\varphi] = \{2,3,4\}$? NO!

2) $\underbrace{AF(\underbrace{a \wedge \underbrace{AX(\underbrace{EG\ b}_{\alpha})}_{\beta}}_{\gamma})}_{\delta}$



$[\alpha] = [EG\ b] = [\nu Z.\ b \wedge \langle NEXT \rangle Z] =$

$[Z_0] = \{1,2,3,4\}$

$[Z_1] = [b] \cap PREE(NEXT, Z_0) =$

$= \{3,4\} \cap \{1,2,3,4\} = \{3,4\}$

$[Z_2] = [b] \cap PREE(NEXT, Z_1) =$

$= \{3,4\} \cap \{2,3,4\} = \{3,4\}$

$[Z_1] = [Z_2] = [\alpha] = \{3,4\}$

$[\beta] = [AX\ \alpha] = [[NEXT]\alpha] = PREA(NEXT, \alpha) = \{2,4\} = [\beta]$

$[\gamma] = [a \wedge \beta] = [a] \cap [\beta] = \{2,4\} \cap \{2,4\} = \{2,4\} = [\gamma]$

$[\delta] = [AF\ \gamma] = [\mu Z.\ \gamma \vee [NEXT] Z] =$

$[Z_0] = \emptyset$

$[Z_1] = [\gamma] \cup PREA(NEXT, Z_0) =$

$= \{2,4\} \cup \emptyset = \{2,4\}$

$[Z_2] = [\gamma] \cup PREA(NEXT, Z_1) =$

$= \{2,4\} \cup \{1,4\} = \{1,2,4\}$

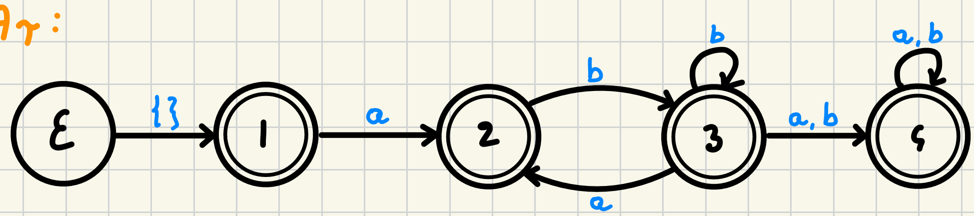
$[Z_3] = [\gamma] \cup PREA(NEXT, Z_2) =$

$= \{2,4\} \cup \{1,4\} = \{1,2,4\}$

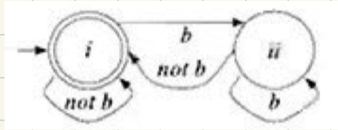
$s_1 \in \delta = ?$ YES!

$[Z_2] = [Z_3] = [\delta] = \{1,2,4\}$

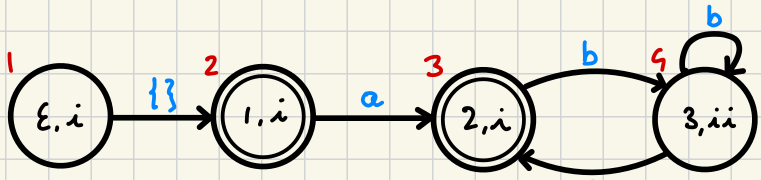
3) A_γ:



$A_{\neg\varphi}$:



$A_\gamma \cap A_{\neg\varphi}$:



$$\varphi = \nu X . \mu Y \left(F \wedge \langle NEXT \rangle X \vee \langle NEXT \rangle Y \right)$$

$$[X_0] = \{1,2,3,4\}$$

$$[X_1] = [\mu Y . (F \wedge \langle NEXT \rangle X_0 \vee \langle NEXT \rangle Y)]$$

$$[Y_0] = \emptyset$$

$$[Y_1] = [F] \cap PREE(NEXT, X_0) \cup PREE(NEXT, Y_0) =$$
$$= \{2,3\} \cap \{1,2,3,4\} \cup \emptyset = \{2,3\}$$

$$[Y_2] = [F] \cap PREE(NEXT, X_0) \cup PREE(NEXT, Y_1) =$$
$$= \{2,3\} \cap \{1,2,3,4\} \cup \{1,2,4\} = \{1,2,3,4\}$$

$$[Y_3] = [F] \cap PREE(NEXT, X_0) \cup PREE(NEXT, Y_2) =$$
$$= \{2,3\} \cap \{1,2,3,4\} \cup \{1,2,3,4\} = \{1,2,3,4\}$$

$$[Y_2] = [Y_2] = [X_1] = \{1,2,3,4\}$$

$$[X_0] = [X_1] = \{1,2,3,4\}$$

$s_1 \in [\varphi] = ?$ YES!

Exercise 3 Given the following ~~boolean~~ conjunctive queries:

```
q1(x) :- r(x,x),b(x,y),b(y,x)
q2(x) :- r(x,y),b(y,z),b(z,x)
```

check whether $q1$ is contained into $q2$, explaining the technique used and, in case of containment, showing the homomorphism between the canonical databases.

FREEZE

$$q_1(c) \subseteq q_2(c) \qquad \begin{cases} q_1(c): \ r(c,c),\ b(c,y),\ b(y,c) \\ q_2(c): \ r(c,y),\ b(y,z),\ b(z,c) \end{cases}$$

BUILD CANONICAL INTERPRETATION

$I_{q_1(c)}$: $\Delta^{q_1(c)}: \{c, y\}$

$r^{q_1(c)}: \{<c,c>\}$

$b^{q_1(c)}: \{<c,y>, <y,c>\}$

$I_{q_2(c)}$: $\Delta^{q_2(c)}: \{c, y, z\}$

$r^{q_2(c)}: \{<c,y>\}$

$b^{q_2(c)}: \{<y,z>, <z,c>\}$

QUERY ANSWERING

$$I_{q_1(c)} \models q_2(c)\ ? \longrightarrow \begin{matrix} \alpha(y)=c \\ \alpha(z)=y \end{matrix} \longrightarrow I_{q_1(c)}, \alpha \models q_2(c)$$

HOMOMORPHISM

$h(c)=c$
$h(y)=\alpha(y)=c$
$h(z)=\alpha(z)=y$

$(c,y) \in r^{q_2(c)} \Rightarrow (h(c), h(y)) \in r^{q_1(c)}$

$(y,z) \in b^{q_2(c)} \Rightarrow (h(y), h(z)) \in b^{q_1(c)}$

$(z,c) \in b^{q_2(c)} \Rightarrow (h(z), h(c)) \in b^{q_1(c)}$