Logics of Programs

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Logics of Programs

- Are modal logics that allow describing properties of transition systems

- Examples:
  - Hennessy-Milner Logic
  - Propositional Dynamic Logics
  - Mu-calculus

- Perfectly suited for describing transition systems: they can tell apart transition systems modulo **bisimulation**

Hennessy-Milner Logic

HM Logic aka (multi) modal logic Ki

- Syntax:

Φ := p *(atomic propositions)*

[a]Φ | <a>Φ *(modal operators)*

$\neg\Phi$ | $\Phi_1 \wedge \Phi_2$ | $\Phi_1 \vee \Phi_2$ | true | false *(closed under booleans)*

- Propositions are used to denote atomic properties of states
- <a>Φ means there exists an a-transition that leads to a state where Φ holds; i.e., expresses the capability of executing action a bringing about Φ
- [a]Φ means that all a-transitions lead to states where Φ holds; i.e., express that executing action a brings about Φ

Hennessy-Milner Logic

- Semantics: assigns meaning to the formulas.

- Given a TS $T = \langle P, A, S, s_0, \delta, \Pi \rangle$, a state $s \in S$, and a formula Φ, we define (by structural induction) the "truth relation"

$$T,s \models \Phi$$

  - $T,s \models p$ if $p \in \Pi(s)$;
  - $T,s \models [a]\Phi$ if **for all** s' such that $s \rightarrow_a s'$ we have $T,s' \models \Phi$;
  - $T,s \models \langle a \rangle \Phi$ if **exists** s' such that $s \rightarrow_a s'$ and $T,s' \models \Phi$;
  - $T,s \models \neg\Phi$ if it is not the case that $T,s \models \Phi$;
  - $T,s \models \Phi_1 \vee \Phi_2$ if $T,s \models \Phi_1$ or $T,s \models \Phi_2$;
  - $T,s \models \Phi_1 \wedge \Phi_2$ if $T,s \models \Phi_1$ and $T,s \models \Phi_2$;
  - $T,s \models$ true always;
  - $T,s \models$ false never.

Hennessy-Milner Logic

- Another way to give the same semantics to formulas: formulas extension in a transition system assigns meaning to the formulas.

- Given a TS $T = < P, A, S, S^0, \delta, \Pi >$ "the extension of a formula Φ in T", denote by $(\Phi)^T$, is defined as follows:

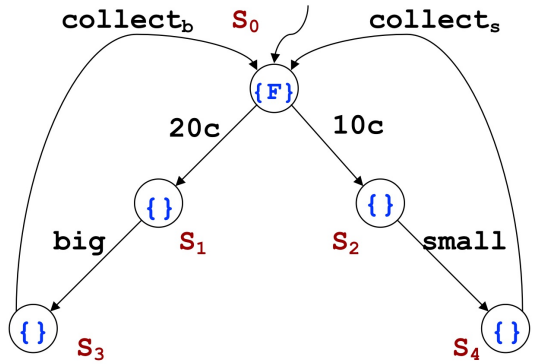
  - $(p)^T = p^T = \{s \mid p \in \Pi(s)\}$;
  - $([a]\Phi)^T = \{s \mid \forall s'.\ s \rightarrow_a s' \text{ implies } s' \in (\Phi)^T\}$;
  - $(\langle a \rangle \Phi)^T = \{s \mid \exists s'.\ s \rightarrow_a s' \text{ and } s' \in (\Phi)^T\}$;
  - $(\neg\Phi)^T = S - (\Phi)^T$;
  - $(\Phi_1 \vee \Phi_2)^T = (\Phi_1)^T \cup (\Phi_2)^T$;
  - $(\Phi_1 \wedge \Phi_2)^T = (\Phi_1)^T \cap (\Phi_2)^T$;
  - $(\text{true})^T = S$;
  - $(\text{false})^T = \emptyset$.

- Note: $T,s \models \Phi$ now written as $s \in (\Phi)^T$

Model Checking

- Given a TS T, one of its states s, and a formula Φ verify whether the formula holds in s. Formally:

$$T,s \models \Phi \quad \text{or} \quad s \in (\Phi)^T$$



- Examples (TS is our vending machine):
  - $S_0 \models$ Final
  - $S_0 \models$ <10c>true — *capability of performing action 10c*
  - $S_2 \models$ [big]false — *inability of performing action big*
  - $S_0 \models$ [10c][big]false — *after 10c cannot execute big*

- Model checking variant (aka "query answering"):
  - Given a TS T … — *– the database*
  - … compute the extension of Φ — *– the query*

Formally: compute the set $(\Phi)^T$ which is equal to $\{s \mid T,s \models \Phi\}$

Examples

- Useful abbreviation (let actions A = $\{a_{1,\ldots,} a_n\}$):
  - $\langle any \rangle \Phi$ stands for $\langle a_1 \rangle \Phi \vee \cdots \vee \langle a_n \rangle \Phi$
  - $[any] \Phi$ stands for $[a_1] \Phi \wedge \cdots \wedge [a_n] \Phi$
  - $\langle any - a_1 \rangle \Phi$ stands for $\langle a_2 \rangle \Phi \vee \cdots \vee \langle a_v \rangle \Phi$
  - $[any - a_1] \Phi$ stands for $[a_2] \Phi \wedge \cdots \wedge [a_v] \Phi$

- Examples:
  - $\langle a \rangle true$ — *capability of performing action a*
  - $[a]false$ — *inability of performing action a*
  - $\neg Final \wedge \langle any \rangle true \wedge [any\text{-}a]false$ — *necessity/inevitability of performing action a (i.e., action a is the only action possible)*
  - $\neg Final \wedge [any]false$ — *deadlock!*

Satisfiability

- Observe that a formula Φ may be used to select among all TS T those such that for a given state s we have that $T,s \models \Phi$

- SATISFIABILITY: Given a formula Φ verify whether there exists a TS T and a state s such that. Formally:

  check whether exists T, s such that $T,s \models \Phi$

Satisfiability

- Satisfiability: given a formula Φ verify whether there exists a (finite/infinite) TS T and a state of T such that the formula holds in s.

  SAT: check the existence of T,s such that $T,s \models \Phi$

- Validity: given a formula Φ verify whether in every (finite/infinite) TS T and in every state of T the formula holds in s.

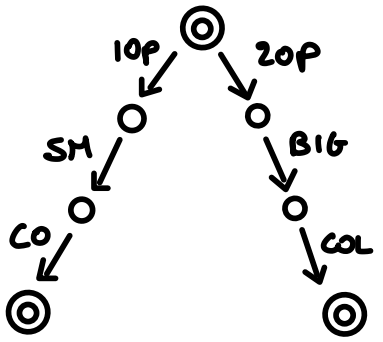
  VAL: check the nonexistence of T,s such that $T,s \models \neg\Phi$

  Note: VAL = UNSAT

Examples: check the satisfiability/validity of the following formulas:

- $\langle 10p\rangle\langle small\rangle\langle collect_s\rangle Final$ ✓

- $Final \rightarrow ((\langle 10p\rangle\langle small\rangle\langle collect_s\rangle Final) \wedge (\langle 20p\rangle\langle big\rangle\langle collect_b\rangle Final))$ ✓



- $\langle 10p\rangle\langle small\rangle\langle collect_s\rangle Final \wedge [10p]false$ ✗ UNSAT

$\neg(x) = [10p][SMALL][COLLECTS]\neg FINAL \vee \langle 10p\rangle TRUE$

VALID

Logic of Programs and Bisimulation

- Consider two TS, $T = (P,A,S,s_0,\delta, F)$ and $T' = (P,A,S',t_0,\delta', F')$.
- Let L be the language formed by all Hennessy-Milner Logic formulas.

- We define:
  - $\sim_L = \{(s,t) \mid \text{for all } \Phi \text{ of L we have } T,s \models \Phi \text{ iff } T',t \models \Phi\}$

  - $\sim = \{(s,t) \mid \text{exists a bisimulation R s.t., } R(s,t)\}$

- **Theorem**: $s \sim_L t$ iff $s \sim t$

- Proof: we show that
  - $s \sim t$ implies $s \sim_L t$ by structural induction on formulas of L.
  - $s \sim_L t$ implies $s \sim t$ by co-induction showing that $s \sim_L t$ is a bisimulation.

This theorem says that Hennessy-Milner Logic has exactly the same distinguishing power of bisimulation. So, it is the right logic to predicate on transition systems.

The same result holds also for PDL and Mu-Calculus introduced below.

Logic of Programs and Bisimulation

Show: s ~ t implies s $\sim_L$ t by structural induction on formulas of L.

Proofs by induction

Show that **property** (s~t) ***is closed wrt the rules of the inductively defined set*** (formation rules for formulas in L)

That is:

- Show **Base Cases** (atomic formulas)
- Show **Recursive Cases** by assuming property holds for smaller cases (**inductive hypothesis**)

Logic of Programs and Bisimulation

Show: $s \sim_L t$ implies $s \sim t$ by coinduction showing that $s \sim_L t$ is a bisimulation.

Proofs by coinduction

Show that **property** ($s \sim_L t$) ***is closed wrt the rules of the coinductively defined set*** (bisimulation $s \sim t$)

That is:

- Assume **property** holds, show that applying the **recursive rules** it continues to hold.

Notice: no base cases, only recursive cases!!!

Propositional Dynamic Logic

- Φ := p | *(atomic propositions)*
  $\neg \Phi \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid$ *(closed under boolean operators)*
  $[r]\Phi \mid \langle r \rangle \Phi$ *(modal operators)*

  r := a | $r_1 + r_2 \mid r_1;r_2 \mid r^* \mid \Phi?$ *(complex actions as regular expressions)*

- Essentially add the capability of expressing partial correctness assertions via formulas of the form
  - $\Phi_1 \rightarrow [r]\Phi_2$ *under the conditions Φ_1 all possible executions of r that terminate reach a state of the TS where Φ_2 holds*
- Also add the ability of asserting that a property holds in all nodes of the transition system
  - $[(a_1 + \cdots + a_\nu)^*]\Phi$ *in every reachable state of the TS Φ holds*
- Useful abbereviations:
  - any stands for $(a_1 + \cdots + a_\nu)$ *Note that + can be expressed also in Hennessy-Milner Logic*
  - u stands for any* *any* is often refferred as the «master/universal modality»

Mu-Calculus

- Φ := P | *(atomic propositions)*
  $\neg \Phi \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid$ *(closed under boolean operators)*
  $[r]\Phi \mid \langle r \rangle \Phi$ *(modal operators)*
  $\mu X.\Phi(X) \mid \nu X.\Phi(X)$ *(fixpoint operators)*

- It is the most expressive logic of the family of logics of programs.

- It subsumes
  - PDL (modalities involving complex actions are translated into formulas involving fixpoints)
  - LTL (linear time temporal logic),
  - CTS, CTS* (branching time temporal logics)

Examples:

- $[any^*]\Phi$ can be expressed as $\nu X.\ \Phi \wedge [any]X$
- $\mu X.\ \Phi \vee [any]X$ *along all runs eventually Φ*
- $\mu X.\ \Phi \vee \langle any \rangle X$ *along some run eventually Φ*
- $\nu X.\ [a](\mu Y.\ \langle any \rangle true \wedge [any\text{-}b]Y) \wedge X$ *every run that contains a contains later b*

Mu-Calculus

- To understand fixpoint operators one has to consider them as fixpoint of equations:
- Namely given $\mu X.\Phi(X)$ and $\nu X.\Phi(X)$ consider the equation

$$X = \Phi(X)$$

Then:
- $\mu X.\Phi(X)$ stands for the smallest predicate X such that $X = \Phi(X)$ – or $\Phi(X) \rightarrow X$
- $\nu X.\Phi(X)$ stands for the largest predicate X such that $X = \Phi(X)$ – or $X \rightarrow \Phi(X)$

Notice:
- $\mu X.\Phi(X)$ is defined by induction and computed by the least fixpoint algorithm over the TS
- $\nu X.\Phi(X)$ is defined by coinduction and computed by the greatest fixpoint algorithm over the TS

- Examples:
  - ν X. $\Phi \wedge$ [any]X - gfp of X = $\Phi \wedge$ [any]X
  - μ X. $\Phi \vee$ [any]X - lfp of X = $\Phi \vee$ [any]X
  - μ X. $\Phi \vee$ <any>X - lfp of X = $\Phi \vee$ <any>X
  - ν X. [a](μ Y. <any>true $\wedge$ [any-b]Y) $\wedge$ X
    - lfp of y = <any>true $\wedge$ [any-b]Y
    - gfp of X=[a](*lfp above*) $\wedge$ X

Examples of Mu-Calculus

Examples (TS is our vending machine):

- $S_0 \models$ Final
- $S_0 \models$ <10c>true — *capability of performing action 10c*
- $S_2 \models$ [big]false — *inability of performing action big*
- $S_0 \models$ [10c][big]false — *after 10c cannot execute big*
- $S_i \models \mu$ X. Final $\vee$ [any] X — *eventually a final state is reached*
- $S_0 \models \nu$ Z. (μ X. Final $\vee$ [any] X) $\wedge$ [any] Z — *or equivalently*

  $S_0 \models$ [any*](μ X. Final $\vee$ [any] X) — *from everywhere eventually final*

Mu-Calculus extends PDL

We can easily translate in Mu-Calculus all PDL formulas.
Here is the translation function T: PDL → Mu Calculus:

$$
\begin{aligned}
T(\langle a\rangle\Phi) &= \langle a\rangle T(\Phi)\\
T(\langle r_1 + r_2\rangle\Phi) &= T(\langle r_1\rangle\Phi) \vee T(\langle r_2\rangle\Phi)\\
T(\langle r_1;r_2\rangle\Phi) &= T(\langle r_1\rangle \langle r_2\rangle\Phi)\\
T(\langle r^*\rangle \Phi) &= \mu X.\ T(\Phi) \vee T(\langle r\rangle X)\\
T(\langle\Phi_1?\rangle\Phi_2) &= T(\Phi_1) \wedge T(\Phi_2)
\end{aligned}
$$

$$
\begin{aligned}
T([a]\Phi) &= [a]T(\Phi)\\
T([r_1 + r_2]\Phi) &= T([r_1]\Phi) \wedge T([r_2]\Phi)\\
T([r_1;r_2]\ \Phi) &= T([r_1][r_2]\Phi)\\
T([r^*]\ \Phi) &= \nu X.\ T(\Phi) \wedge T([r]X)\\
T([\Phi_1?]\Phi_2) &= T(\Phi_1) \rightarrow T(\Phi_2)
\end{aligned}
$$

$$
\begin{aligned}
T(p) &= p\\
T(\neg\Phi) &= \neg T(\Phi)\\
T(\Phi_1 \wedge \Phi_2) &= T(\Phi_1) \wedge T(\Phi_2)\\
T(\Phi_1 \wedge \Phi_2) &= T(\Phi_1) \vee T(\Phi_2)
\end{aligned}
$$

$$T(X) = X$$

(although X is not a PDL formula we need this auxiliary definition in the translation)

Notice: no alternation of least and greatest fixpoints!!!

Mu-Calculus extends CTL

We can easily translate in Mu-Calculus all CTL formulas.
Here is the translation function T: CTL → Mu Calculus:

CTL formulas:

$\Phi := p \mid \neg\Phi \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid$ *(atomic propositions) (boolean operators)*

$EX\Phi \mid EF\Phi \mid EG\Phi \mid \Phi_1 \, EU \, \Phi_2 \mid$ *(temporal (modal) operators on a path)*

$AX\Phi \mid AF\Phi \mid AG\Phi \mid \Phi_1 \, AU \, \Phi_2$ *(temporal (modal) operators on all paths)*

$T(EX\,\Phi) = \langle\text{-}\rangle T(\Phi)$
$T(EF\Phi) = \mu Z.\ T(\Phi) \vee \langle\text{-}\rangle Z$
$T(EG\Phi) = \nu Z.\ T(\Phi) \wedge \langle\text{-}\rangle Z$
$T(\Phi_1 EU \Phi_2) = \mu Z.\ T(\Phi_2) \vee \Phi_1 \wedge \langle\text{-}\rangle Z$

$T(AX\,\Phi) = [\text{-}]T(\Phi)$
$T(AF\Phi) = \mu Z.\ T(\Phi) \vee [\text{-}]Z$
$T(AG\Phi) = \nu Z.\ T(\Phi) \wedge [\text{-}]Z$
$T(\Phi_1 AU \Phi_2) = \mu Z.\ T(\Phi_2) \vee \Phi_1 \wedge [\text{-}]Z$

$T(p) = p$
$T(\neg\Phi) = \neg T(\Phi)$
$T(\Phi_1 \wedge \Phi_2) = T(\Phi_1) \wedge T(\Phi_2)$
$T(\Phi_1 \wedge \Phi_2) = T(\Phi_1) \vee T(\Phi_2)$

Notice: no alternation of least and greatest fixpoints!!!

*Mu-Calculus extends CTL**

We can translate in Mu-Calculus all CTL* formulas.
Here is the translation function T: CTL → Mu-Calculus:

CTL formulas:

$\Phi := P \mid$	*(atomic propositions)*
$\neg \Phi \mid \Phi_1 \wedge \Phi_2 \mid \Phi_1 \vee \Phi_2 \mid$	*(boolean operators)*
$E\Theta \mid A\Theta$	*(Exist a path/Forall paths)*
where Θ any LTL formula	*(LTL temporal formula on a path)*

The translation function is not trivial (the translation may generate an exponential formula).

Important: the resulting formula has at most one alternation of the least and greatest fixpoint.

Model Checking/Satisfiability

- Model checking is polynomial in the size of the TS for
  - Hennessy-Milner Logic
  - PDL
  - Mu-Calculus

- Also, model checking wrt the formula is
  - Polynomial for Hennessy-Miner Logic
  - Polynomial for PDL
  - Polynomial for Mu-Calculus with bounded alternation of nested fixpoints, and NP∩coNP in general

- Satisfiability is decidable for the three logics, and the complexity (in the size of the formula) is as follows:
  - Hennessy-Milner Logic: PSPACE-complete
  - PDL: EXPTIME-complete
  - Mu-Calculus: EXPTIME-complete

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