Least and Greatest Fixpoints

Transfinite Approximates

Giuseppe De Giacomo

Fixpoints

We briefly recall few notions on fixpoints.

- Consider the equation:

$$X = f(X)$$

  where f is an operator from $2^{\mathcal{S}}$ to $2^{\mathcal{S}}$ ($2^{\mathcal{S}}$ denotes the set of all subsets of a set $\mathcal{S}$).
- Every solution $\mathcal{E}$ of this equation is called a **fixpoint** of the operator f
- every set $\mathcal{E}$ such that $f(\mathcal{E}) \subseteq \mathcal{E}$ is called **pre-fixpoint**, and
- every set $\mathcal{E}$ such that $\mathcal{E} \subseteq f(\mathcal{E})$ is called **post-fixpoint**.
- In general, an equation as the one above may have either no solution, a finite number of solutions, or an infinite number of them. Among the various solutions, the smallest and the greatest solutions (with respect to set-inclusion) have a prominent position, if they exist.
- The the smallest and the greatest solutions are called **least fixpoint** and **greatest fixpoint**, respectively.

Tarski-Knaster fixpoint theorem

We say that f is **monotonic** wrt $\subseteq$ (set-inclusion) whenever $\mathcal{E}_1 \subseteq \mathcal{E}_2$ implies $f(\mathcal{E}_1) \subseteq f(\mathcal{E}_2)$.

Theorem (Tarski'55)

Let $\mathcal{S}$ be a set, and f an operator from $2^{\mathcal{S}}$ to $2^{\mathcal{S}}$ that is monotonic wrt $\subseteq$. Then:

- *There exists a unique least fixpoint of f, which is given by $\bigcap\{\mathcal{E} \subseteq \mathcal{S} \mid f(\mathcal{E}) \subseteq \mathcal{E}\}$.*
- *There exists a unique greatest fixpoint of f, which is given by $\bigcup\{\mathcal{E} \subseteq \mathcal{S} \mid \mathcal{E} \subseteq f(\mathcal{E})\}$.*

Proof of Tarski-Knaster theorem: least fixpoint

We start by showing the proof for the **least fixpoint** part. (The proof for the greatest fixpoint is analogous, see later).

Let us define $\mathcal{L} = \bigcap\{\mathcal{E} \subseteq \mathcal{S} \mid \ f(\mathcal{E}) \subseteq \mathcal{E}\}$.

Lemma

$f(\mathcal{L}) \subseteq \mathcal{L}$

Proof.

- For every $\mathcal{E}$ such that $f(\mathcal{E}) \subseteq \mathcal{E}$, we have $\mathcal{L} \subseteq \mathcal{E}$, by definition of $\mathcal{L}$.
- By monotonicity of f, we have $f(\mathcal{L}) \subseteq f(\mathcal{E})$.
- Hence $f(\mathcal{L}) \subseteq \mathcal{E}$ (for every $\mathcal{E}$ such that $f(\mathcal{E}) \subseteq \mathcal{E}$).
- But then $f(\mathcal{L})$ is contained in the intersection of all such $\mathcal{E}$, so we have $f(\mathcal{L}) \subseteq \mathcal{L}$.

$\square$

Proof of Tarski-Knaster theorem: least fixpoint

Lemma

$\mathcal{L} \subseteq f(\mathcal{L})$

Proof.

- By the previous lemma, we have $f(\mathcal{L}) \subseteq \mathcal{L}$.
- But then $f(f(\mathcal{L})) \subseteq f(\mathcal{L})$, by monotonicity.
- Hence, $\bar{\mathcal{E}} = f(\mathcal{L})$ is such that $f(\bar{\mathcal{E}}) \subseteq \bar{\mathcal{E}}$.
- Thus, $\mathcal{L} \subseteq f(\mathcal{L})$, by definition of $\mathcal{L}$.

□

Proof of Tarski-Knaster theorem: least fixpoint

The previous two lemmas together show that $\mathcal{L}$ is indeed a fixpoint: $\mathcal{L} = f(\mathcal{L})$. We still need to show that is the **least** fixpoint.

Lemma

$\mathcal{L}$ is the **least** *fixpoint: for every $f(\mathcal{E}) = \mathcal{E}$ we have $\mathcal{L} \subseteq \mathcal{E}$.*

Proof.

By contradiction.

- Suppose not. Then there exists an $\hat{\mathcal{E}}$ such that $f(\hat{\mathcal{E}}) = \hat{\mathcal{E}}$ and $\hat{\mathcal{E}} \subset \mathcal{L}$.
- Being $\hat{\mathcal{E}}$ a fixpoint (i.e., $f(\hat{\mathcal{E}}) = \hat{\mathcal{E}}$), we have in particular $f(\hat{\mathcal{E}}) \subseteq \hat{\mathcal{E}}$.
- Hence by definition of $\mathcal{L}$, we get $\mathcal{L} \subseteq \hat{\mathcal{E}}$. Contradiction.

$\square$

Proof of Tarski-Knaster theorem: greatest fixpoint

Now we prove the **greatest fixpoint** part.

Let us define $\mathcal{G} = \bigcup\{\mathcal{E} \subseteq \mathcal{S} \mid \mathcal{E} \subseteq f(\mathcal{E})\}$.

Lemma

$\mathcal{G} \subseteq f(\mathcal{G})$

Proof.

- For every $\mathcal{E}$ such that $\mathcal{E} \subseteq f(\mathcal{E})$, we have $\mathcal{E} \subseteq \mathcal{G}$, by definition of $\mathcal{G}$.
- Consider now $e \in \mathcal{G}$. Then there exists an $\hat{\mathcal{E}}$ such that $\hat{\mathcal{E}} \subseteq f(\hat{\mathcal{E}})$, $e \in \hat{\mathcal{E}}$, by definition of $\mathcal{G}$.
- But $\hat{\mathcal{E}} \subseteq \mathcal{G}$, and by monotonicity $f(\hat{\mathcal{E}}) \subseteq f(\mathcal{G})$, hence $e \in f(\mathcal{G})$.

$\square$

Proof of Tarski-Knaster theorem: greatest fixpoint

Lemma

$f(\mathcal{G}) \subseteq \mathcal{G}$

Proof.

- By the previous lemma we have $\mathcal{G} \subseteq f(G)$
- But then, we have that $f(\mathcal{G}) \subseteq f(f(\mathcal{G}))$, by monotonicity.
- Hence, $\bar{\mathcal{E}} = f(\mathcal{G})$ is such that $\bar{\mathcal{E}} \subseteq f(\bar{\mathcal{E}})$.
- Thus, $f(\mathcal{G}) \subseteq \mathcal{G}$, by definition of $\mathcal{G}$.

$\square$

Proof of Tarski-Knaster theorem: greatest fixpoint

The previous two lemmas together show that $\mathcal{L}$ is indeed a fixpoint: $\mathcal{G} = f(\mathcal{G})$. We still need to show that is the **greatest** fixpoint.

Lemma

$\mathcal{G}$ is the **greatest** *fixpoint: for every $\mathcal{E} = f(\mathcal{E})$ we have $\mathcal{E} \subseteq \mathcal{G}$.*

Proof.

By contradiction.

- Suppose not. Then there exists an $\hat{\mathcal{E}}$ such that $\hat{\mathcal{E}} = f(\hat{\mathcal{E}})$ and $\mathcal{G} \subset \hat{\mathcal{E}}$.
- Being $\hat{\mathcal{E}}$ a fixpoint, we have $\hat{\mathcal{E}} \subseteq f(\hat{\mathcal{E}})$.
- Hence by definition of $\mathcal{G}$, we get $\hat{\mathcal{E}} \subseteq \mathcal{G}$. Contradiction.

$\square$

Ordinals

We will need to do more iterations than the natural numbers, so we will use ordinals.

Ordinals

$$
\begin{array}{l}
0 \doteq \emptyset \\
1 \doteq \{0\} = \{\emptyset\} \\
2 \doteq \{0,1\} = \{\emptyset,\{\emptyset\}\} \\
3 \doteq \{0,1,2\} = \{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\} \\
\ldots \\
\omega \doteq \{0,1,2,3,\ldots\} \quad \text{-these are the natural numbers, sometimes denoted by } \omega_0 \\
\omega+1 \doteq \{\omega,\{\omega\}\} \\
\omega+2 \doteq \{\omega,\{\omega\},\{\omega,\{\omega\}\}\} \\
\omega+3 = \{\omega,\{\omega\},\{\omega,\{\omega\}\},\{\omega,\{\omega\},\{\omega,\{\omega\}\}\}\} \\
\ldots \\
\omega_1 \\
\omega_1+1 \doteq \{\omega_1,\{\omega_1\}\} \\
\omega_1+2 \doteq \{\omega_1,\{\omega_1\},\{\omega_1,\{\omega_1\}\}\} \\
\omega_1+3 \doteq \{\omega_1,\{\omega_1\},\{\omega_1,\{\omega_1\}\},\{\omega_1,\{\omega_1\},\{\omega_1,\{\omega_1\}\}\}\} \\
\ldots
\end{array}
$$

An ordinal λ is a limit ordinal iff $\lambda = \cup_{i<\lambda} i$, otherwise is a successor ordinal.
Eg.: $1,2,3,\ldots$ are successor ordinals, ω is the first limit ordinal, $\omega+1,\omega+2,\omega+3,\ldots$ are successor ordinals, ω_1 is the second limit ordinal, etc.

Approximates of least fixpoints

Approximates for a least fixpoint

The approximates for a least fixpoint $\mathcal{L} = \bigcap\{\mathcal{E} \subseteq \mathcal{S} \mid \; f(\mathcal{E}) \subseteq \mathcal{E}\}$ are as follows:

$$
\begin{array}{l}
Z_0 \doteq \emptyset \\
Z_1 \doteq f(Z_0) \\
Z_2 \doteq f(Z_1) \\
\ldots \\
Z_\omega \doteq \bigcup_{i<\omega} Z_i \\
Z_{\omega+1} \doteq f(Z_\omega) \\
Z_{\omega+2} \doteq f(Z_{\omega+1}) \\
\ldots \\
Z_{\omega_1} \doteq \bigcup_{i<\omega_1} Z_i \\
Z_{\omega_1+1} \doteq f(Z_{\omega_1}) \\
Z_{\omega_1+2} \doteq f(Z_{\omega_1+1}) \\
\ldots
\end{array}
$$

Approximates of least fixpoints

Lemma

For all successor ordinals i, $Z_i \subseteq Z_{i+1}$ and for all limit ordinals λ, $Z_j \subseteq Z_\lambda$ for all $j < \lambda$.

Proof.

By transfinite induction on i.

- Zero: $i = 0$. By definition $Z_0 = \emptyset$, and trivially $\emptyset \subseteq Z_1$.
- Successor ordinals: $i = k + 1$. By tranfinite-inductive hypothesis we assume $Z_{k-1} \subseteq Z_k$, and we show that $Z_k \subseteq Z_{k+1}$.
  - $f(Z_{k-1}) \subseteq f(Z_k)$, by monotonicity.
  - But $f(Z_{k-1}) = Z_k$ and $f(Z_k) = Z_{k+1}$, hence we have $Z_k \subseteq Z_{k+1}$.
- Limit ordinals: $i = \lambda$, $Z_\lambda = \bigcup Z_{j<\lambda}$ hence $Z_j \subseteq Z_\lambda$ for all $j < \lambda$.

□

Approximates of least fixpoints

Lemma

For all ordinals i, $Z_i \subseteq \mathcal{L}$.

Proof.

By transfinite induction on i.

- Zero: $i = 0$. By definition $Z_0 = \emptyset$, and trivially $\emptyset \subseteq \mathcal{L}$.
- Successor ordinals: $i = k + 1$. By transfinite-inductive hypothesis we assume $Z_k \subseteq \mathcal{L}$, and we show that $Z_{k+1} \subseteq \mathcal{L}$.
    - $f(Z_k) \subseteq f(\mathcal{L})$, by monotonicity.
    - But then $f(Z_k) \subseteq \mathcal{L}$, since $\mathcal{L} = f(\mathcal{L})$.
    - Hence, considering that $f(Z_k) = Z_{k+1}$, we have $Z_{k+1} \subseteq \mathcal{L}$.
- Limit ordinals: $i = \lambda$, $Z_\lambda = \bigcup Z_{j<\lambda}$, since $Z_j \subseteq \mathcal{L}$ for all $j < \lambda$, by transfinite-induction we have that $Z_\lambda \subseteq \mathcal{L}$.

$\square$

Approximates of least fixpoints

Theorem (Tarski-Knaster on approximates of least fixpoints)

If for an ordinal α, $Z_{\alpha+1} = Z_\alpha$, then $Z_\alpha = \mathcal{L}$. Moreover such ordinal α always exists!

Proof.

We show only the first part of the theorem (the second part is highly nontrivial).

- $Z_\alpha \subseteq \mathcal{L}$ by the above lemma.
- On the other hand, since $Z_{\alpha+1} = f(Z_\alpha) = Z_\alpha$, we trivially get $f(Z_\alpha) \subseteq Z_\alpha$, and hence $\mathcal{L} \subseteq Z_\alpha$ by definition of $\mathcal{L}$.

$\square$

Observe also that once for some α, $Z_{\alpha+1} = Z_\alpha$, then for all $\beta \geq \alpha$ we have $Z_{\beta+1} = Z_\beta$, by definition of approximates.

Approximates of least fixpoints

The above theorem gives us a simple sound procedure to compute the least fixpoint:

Least fixpoint algorithm

$$
\begin{array}{l}
Z_{old} := \emptyset; \\
Z := f(Z_{old}); \\
\textbf{while } (Z \neq Z_{old})\{ \\
\qquad Z_{old} := Z; \\
\qquad Z := f(Z); \\
\}
\end{array}
$$

If in $\mathcal{L} = \bigcap\{\mathcal{E} \subseteq \mathcal{S} \mid \ f(\mathcal{E}) \subseteq \mathcal{E}\}$ the set $\mathcal{S}$ **is finite** then the above procedure **terminates** in $|\mathcal{S}|$ steps and becomes **sound and complete**.

Notice the above procedure is **polynomial** in the size of $\mathcal{S}$.

Approximates of greatest fixpoints

Approximates for a greatest fixpoint

The approximates for the greatest fixpoint $\mathcal{G} = \bigcup\{\mathcal{E} \subseteq \mathcal{S} \mid \ \mathcal{E} \subseteq f(\mathcal{E})\}$ are:

$$
\begin{array}{l}
Z_0 \doteq \mathcal{S} \\
Z_1 \doteq f(Z_0) \\
Z_2 \doteq f(Z_1) \\
\ldots \\
Z_\omega \doteq \bigcap_{i<\omega} Z_i \\
Z_{\omega+1} \doteq f(Z_\omega) \\
Z_{\omega+2} \doteq f(Z_{\omega+1}) \\
\ldots \\
Z_{\omega_1} \doteq \bigcap_{i<\omega_1} Z_i \\
Z_{\omega_1+1} \doteq f(Z_{\omega_1}) \\
Z_{\omega_1+2} \doteq f(Z_{\omega_1+1}) \\
\ldots
\end{array}
$$

Approximates of greatest fixpoints

Lemma

For all successor ordinals i, $Z_{i+1} \subseteq Z_i$ and for all limit ordinals λ, $Z_\lambda \subseteq Z_j$ for all $j < \lambda$.

Proof.

By transfinite induction on i.

- Zero: $i = 0$. By definition $Z_0 = \mathcal{S}$, and trivially $Z_1 \subseteq \mathcal{S}$.
- Successor ordinals: $i = k + 1$: by inductive hypothesis we assume $Z_k \subseteq Z_{k-1}$, and we show that $Z_{k+1} \subseteq Z_k$.
  - $f(Z_k) \subseteq f(Z_{k-1})$, by monotonicity.
  - But $f(Z_k = Z_{k+1}$ and $f(Z_{k-1}) = Z_k$ hence $Z_{k+1} \subseteq Z_k$.
- Limit ordinals: $i = \lambda$, $Z_\lambda = \bigcap Z_{j<\lambda}$ hence $Z_\lambda \subseteq Z_j$ for all $j < \lambda$.

□

Approximates of greatest fixpoints

Lemma

For all ordinals i, $\mathcal{G} \subseteq Z_i$.

Proof.

By transfinite induction on i.

- Zero: $i = 0$. By definition $Z_0 = \mathcal{S}$, and trivially $\mathcal{G} \subseteq \mathcal{S}$.
- Successor ordinals: $i = k + 1$: by inductive hypothesis we assume $\mathcal{G} \subseteq Z_k$, and we show that $G \subseteq Z_{k+1}$.
  - $f(\mathcal{G}) \subseteq f(Z_k)$, by monotonicity.
  - But then $\mathcal{G} \subseteq f(Z_k)$, since $\mathcal{G} = f(\mathcal{G})$.
  - Hence, considering that $f(Z_k) = Z_{k+1}$, we get $\mathcal{G} \subseteq Z_{k+1}$.
- Limit ordinals: $i = \lambda$, $Z_\lambda = \bigcap Z_{j<\lambda}$, since $\mathcal{G} \subseteq Z_j$ for all $j < \lambda$, by transfinite-induction we have that $\mathcal{G} \subseteq Z_\lambda$.

$\square$

Approximates of greatest fixpoints

Theorem (Tarski-Knaster on approximates of greatest fixpoint)

If for some ordinal α, $Z_{\alpha+1} = Z_\alpha$, then $Z_\alpha = \mathcal{G}$. Moreover such ordinal α always exists!

Proof.

We show only the first part of the theorem (the second part is highly nontrivial).

- $\mathcal{G} \subseteq Z_\alpha$ by the above lemma.
- On the other hand, since $Z_{\alpha+1} = f(Z_\alpha) = Z_\alpha$, we trivially get $Z_\alpha \subseteq f(Z_\alpha)$, and hence $Z_\alpha \subseteq \mathcal{G}$ by definition of $\mathcal{G}$.

$\square$

Observe also that once for some α, $Z_{\alpha+1} = Z_\alpha$, then for all $\beta \geq alpha$ we have $Z_{\beta+1} = Z_\beta$, by definition of approximates.

Approximates of greatest fixpoints

The above theorem gives us a simple sound procedure to compute the greatest fixpoint:

Greatest fixpoint algorithm

$$
\begin{array}{l}
Z_{old} := \mathcal{S}; \\
Z := f(Z_{old}); \\
\textbf{while } (Z \neq Z_{old})\{ \\
\qquad Z_{old} := Z; \\
\qquad Z := f(Z); \\
\}
\end{array}
$$

If in $\mathcal{G} = \bigcup\{\mathcal{E} \subseteq \mathcal{S} \mid \ \mathcal{E} \subseteq f(\mathcal{E})\}$ the set $\mathcal{S}$ **is finite** then the above procedure **terminates** in $|\mathcal{S}|$ steps and becomes **sound and complete**.

Notice the above procedure is **polynomial** in the size of $\mathcal{S}$.

Discussion

For simplicity we have considered fixpoint wrt set-inclusion. In fact, the only property of set inclusion that we have used is the **lattice** implicitly defined by it.

We recall that a lattice is a the partial order (defined by set inclusion in our case), with the minimal element ($\emptyset$ in our case) and maximal element ($\mathcal{S}$ in our case).

We can immediately extend all the results presented here to arbitrary lattices substituting to the relation $\subseteq$ the relation $\leq$ of the lattice.