POLITECNICO
MILANO 1863

DIPARTIMENTO DI ELETTRONICA
INFORMAZIONE E BIOINGEGNERIA

1D Digital filters

Filter definition in MATLAB

1. Given H(z) = B(z) / A(z):
   - [H(omega), omega] = freqz(B(z), A(z), N, ‘whole’): for both FIR and IIR.
   - h(n) = filter(B(z), A(z), delta(n)): precise with FIR, only an approximation for IIR.
2. Given h(n):
   - H(k)= fft(h)

Filter definition in MATLAB

- H(k) is defined over N samples.
- The DFT is PERIODIC:
  - In frequency domain, period = Fs, f = [0, Fs) or f = [-Fs/2, Fs/2) [Hz]
  - In angular frequency domain, period = 2πFs, ω= [0, 2πFs) or ω =[-πFs, πFs) [rad/s]
  - In normalized frequency, period = 1, $\tilde{f}$= [0, 1) or $\tilde{f}$= [-0.5, 0.5)
  - In normalized angular frequency, period = 2π, $\tilde{\omega}$= [0, 2π) or $\tilde{\omega}$= [-π,π)

How to relate the MATLAB result with the actual Fourier spectrum?
→ How to express MATLAB samples as real frequencies in Hz or normalized frequencies?

MATLAB metrics conversion

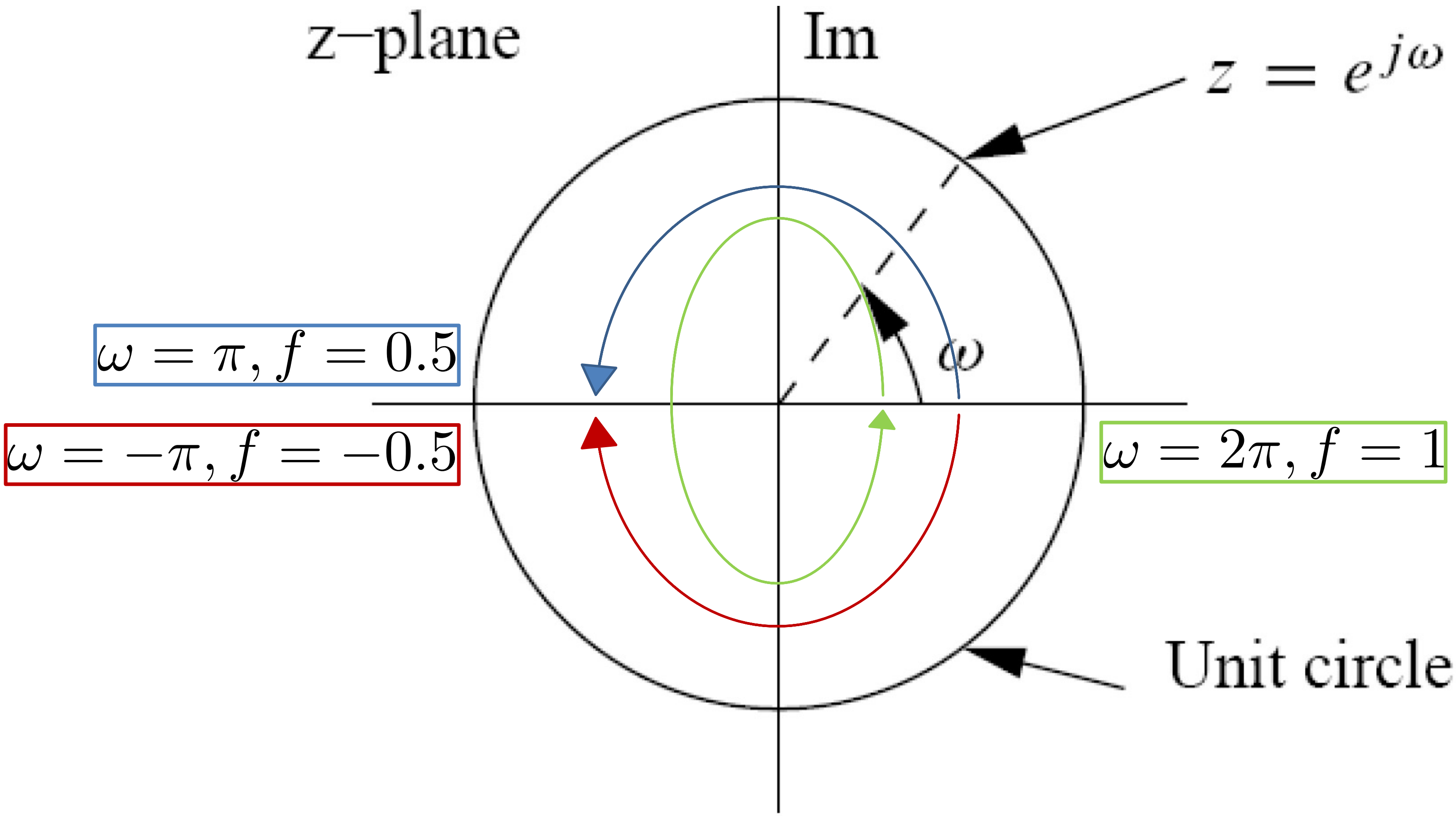
The N-th sample in MATLAB corresponds to the maximum frequency over one period of the periodic spectrum.

Given the array of MATLAB samples: $\mathbf{n} = [0, 1, 2, ..., N-1]$

- The frequency axis [Hz] = [0, Fs) is obtained as $\mathbf{n} \cdot \frac{\mathrm{F}_s}{N}$
- The angular frequency axis [rad/s] = $[0, 2\pi\mathrm{F}_s)$ is obtained as $\mathbf{n} \cdot \frac{2\pi\mathrm{F}_s}{N}$
- The normalized frequency axis = [0, 1) is obtained as $\mathbf{n} \cdot \frac{1}{N}$
- The normalized angular frequency axis = $[0, 2\pi)$ is obtained as $\mathbf{n} \cdot \frac{2\pi}{N}$

LEARNING BY HEART IS NOT NEEDED! *Just think at the units of measure*

From Z domain to normalized frequency domain



To pass from normalized domain to real frequency domain, multiply by Fs

FIR vs IRR filters in MATLAB

Given a FIR filter and an input signal x(n),

the output y(n) is obtained by:

- Function ‘conv’ if filter is expressed in time domain
- Function ‘filter’ if you have H(z) or h(n)
- The product of ‘fft’s in frequency domain
- The product of signal ‘fft’ and filter ‘freqz’ in f domain

Given an IIR filter and an input signal x(n),

the output y(n) is obtained by:

- You cannot use ‘conv’! The result will be just an approximation because the filter has infinite duration
- Function ‘filter’ if you have H(z)

Zeros and poles recall

$$H(z) = \frac{\sum_{k=0}^{N} b_k z^{-k}}{\sum_{k=0}^{D} a_k z^{-k}} = z^{D-N} \frac{b_0}{a_0} \frac{\prod_{i=1}^{N} (z - z_i)}{\prod_{i=1}^{D} (z - p_i)}$$

$$= \frac{b_0}{a_0} \frac{\prod_{i=1}^{N} (1 - z_i z^{-1})}{\prod_{i=1}^{D} (1 - p_i z^{-1})}$$

- z_i = roots of numerator, called ‘zeros’
- p_i = roots of denominator, called ‘poles’

Zeros and poles recall: *the poles*

- The poles are associated with the autoregressive part of the filter → they generate IIR filters.
- The filter amplitude response enhances frequencies which are near the poles.
- If poles are outside the unit circle and the filter is causal, the system is unstable.

Zeros and poles recall: *the zeros*

- The zeros are associated with the moving average part of the filter → they generate FIR filters
- The filter amplitude response attenuates frequencies which are near the zeros
- Zeros influence also the phase of the filter:
  - Minimum phase zeros if $z < 1$
  - Maximum phase zeros if $z \geq 1$

Filter design using zeros&poles

- Place poles close to the unit circle in frequencies that must be emphasized
- Place zeros according to the desired phase response
    - The closer they are to the unit circle, the higher the frequency attenuation

Filter design using zeros&poles

Open ‘zpgui.m’

Remarkable LTI filters

Magnitude square function

- The magnitude response of a LTI system is:

$$M(f) = |H(f)|^2 = H(f) \cdot H^*(f) = H(z) \cdot H^*(z^{-1})\Big|_{|z|=1}$$

- Given a generic rational transfer function

$$H(z) = \frac{b_0}{a_0} \frac{\prod_{i=1}^{N}(1 - z_i z^{-1})}{\prod_{i=1}^{D}(1 - p_i z^{-1})}$$

$$\Downarrow$$

$$M(z) = H(z)H^*(z^{-1}) = \frac{|b_0|^2}{|a_0|^2} \frac{\prod_{i=1}^{N}(1 - z_i z^{-1})(1 - z_i^* z)}{\prod_{i=1}^{D}(1 - p_i z^{-1})(1 - p_i^* z)}$$

Magnitude square function

$$M(z) = H(z)H^*(z^{-1}) = \frac{|b_0|^2}{|a_0|^2} \frac{\prod_{i=1}^{N}(1 - z_i z^{-1})(1 - z_i^* z)}{\prod_{i=1}^{D}(1 - p_i z^{-1})(1 - p_i^* z)}$$

- For each zero z_i of H(z), there is another zero at $\dfrac{1}{z_i^*}$
- For each pole p_i of H(z), there is another pole at $\dfrac{1}{p_i^*}$
- M(z) presents poles and zeros in conjugate reciprocal pairs

Magnitude square function

- Given a magnitude response requirement M(z) for H(z)
- Given stability and causality requirements for H(z)

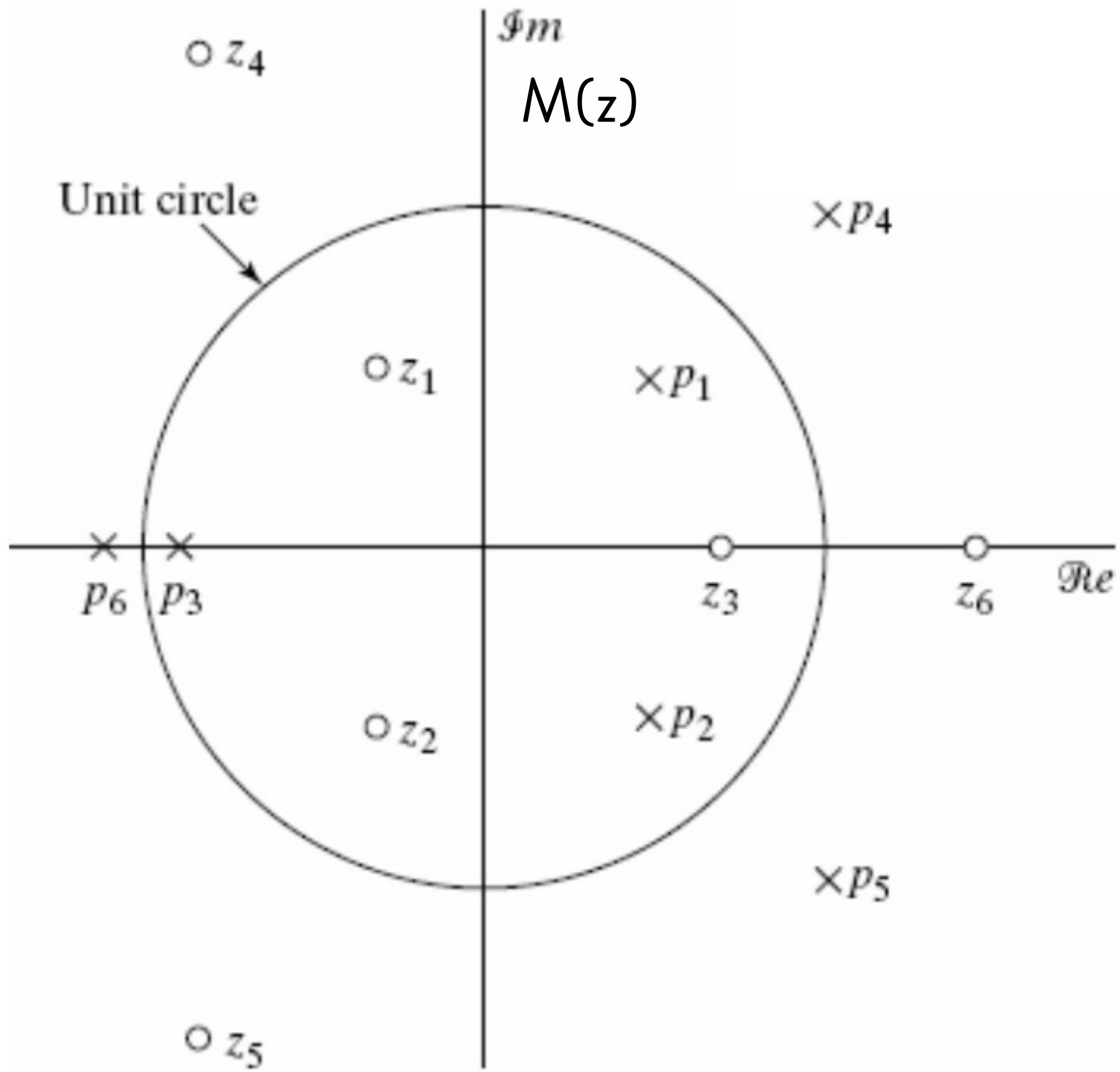
$\Downarrow$

- The poles of H(z) are those of M(z) inside the unit circle and are uniquely identified
- The zeros of H(z) are **not** uniquely identified
- Given a causal FIR filter H(z) of order N, it has the same magnitude response M(z) of the causal FIR filter:

$$G(z) = z^{-N} H^*(z^{-1})$$

From M(z) to H(z)

How to get a ***causal stable*** *system with* ***real*** *coefficients?*



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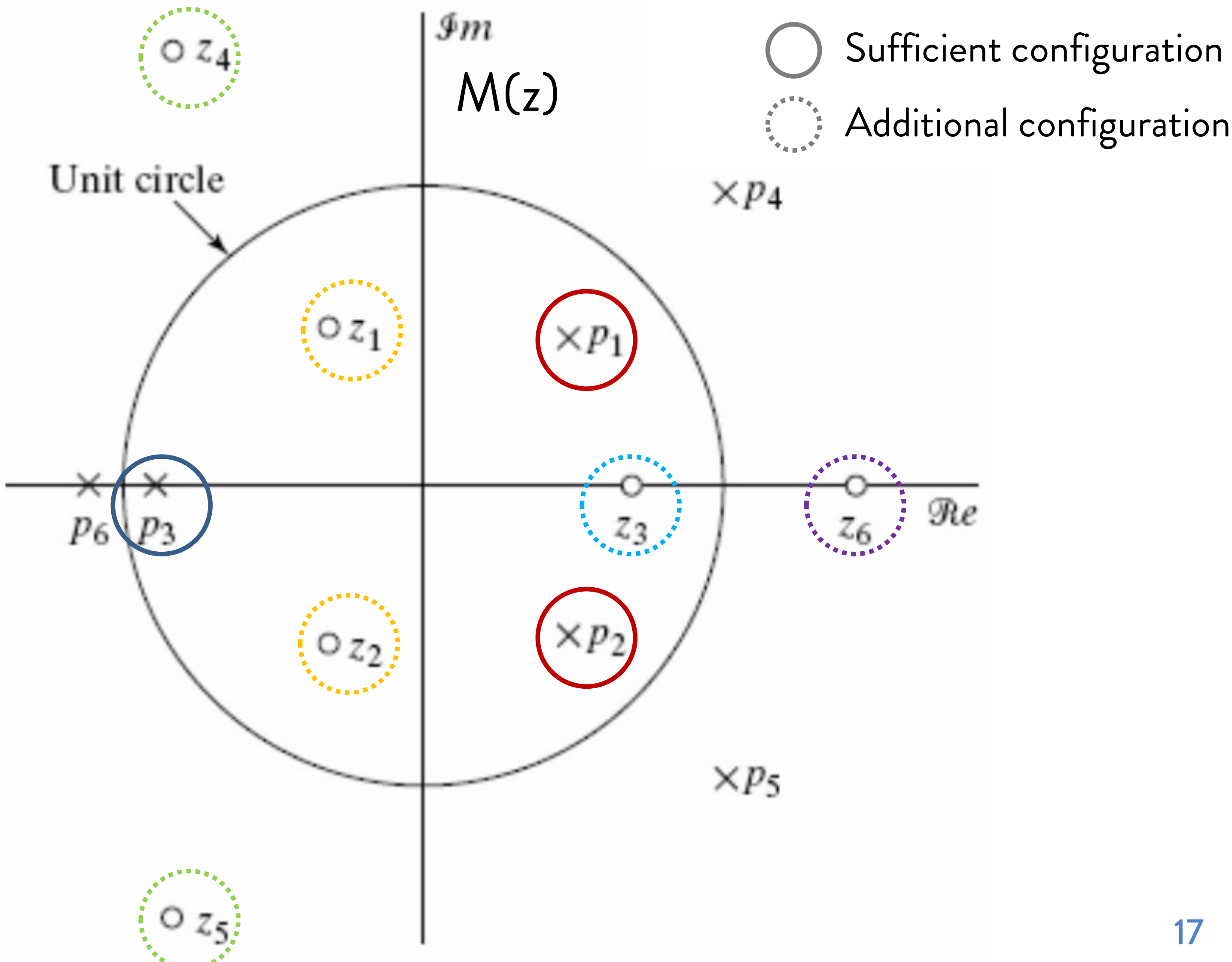


Ex 20: magnitude response

- Given the filters:

$$H_1(z) = \frac{2(1 - z^{-1})(1 + 0.5z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}$$

$$H_2(z) = \frac{(1 - z^{-1})(1 + 2z^{-1})}{(1 - 0.8e^{j\pi/4}z^{-1})(1 - 0.8e^{-j\pi/4}z^{-1})}$$

- Derive A(z) and B(z) for $H_1(z)$ and $H_2(z)$
- Plot the zeros and the poles in the Z-plane using 'zplane'
- Plot in the same figure the magnitude responses as a function of normalized omega in [0, 2pi), using N = 1024 samples
- How are the magnitudes related? Why?

Allpass filters

Allpass filters are designed to have constant gain and any phase response:

$$|H_{ap}(f)| = |H_{ap}(z)|\Big|_{|z|=1} = 1$$

- Given the previous considerations, a generic causal allpass filter is:

$$H_{ap}(z) = z^{-K} e^{j\phi} \frac{A(z)}{\tilde{A}(z)}, \quad K \geq 0$$

where

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2} + ... + a_N z^{-N}$$

$$\tilde{A}(z) = z^{-N} A^*(z^{-1}) = a_N^* + a_{N-1}^* z^{-1} + ... + a_2^* z^{2-N} + a_1^* z^{1-N} + z^{-N}$$

Allpass filters

- Given an allpass filter:

$$H_{ap}(z) = \frac{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}}{a_N^* + a_{N-1}^* z^{-1} + \dots + a_2^* z^{2-N} + a_1^* z^{1-N} + z^{-N}}$$

a general form to represent an **allpass real** valued impulse response is:

$$H_{ap}(z) = c_0 \prod_{k=1}^{M_r} \frac{z^{-1} - d_k}{1 - d_k z^{-1}} \prod_{k=1}^{M_c} \frac{(z^{-1} - e_k)(z^{-1} - e_k^*)}{(1 - e_k^* z^{-1})(1 - e_k z^{-1})}$$

Zeros and poles occur in conjugate reciprocal pairs

Allpass filter properties

- The cascade of two allpass filters is again an allpass filter
- Each pole of an allpass system is associated with a conjugate reciprocal zero
- The magnitude of many cascaded allpass filters is always the same

Ex 21: allpass systems

- Write a MATLAB function 'allpass.m' like this:

  '[z_out, p_out, b_out, a_out] = allpass(b,a)'
- Inputs: b, a = numerator and denominator of H(z)
- Outputs: z_out, p_out, b_out, a_out = zeros, poles, numerator, denominator of the allpass transfer function related to H(z)
- Use the function 'allpass' to compute the allpass transfer function $H_{ap}(z)$ related to the causal filter $H(z) = \dfrac{1+3z^{-1}}{1-0.5z^{-1}}$
- Plot the amplitude of $H_{ap}(f)$ vs normalized frequencies in [0, 1), using N = 512 samples
- Is the filter stable? How do you expect the phase to behave?

Minimum phase filters

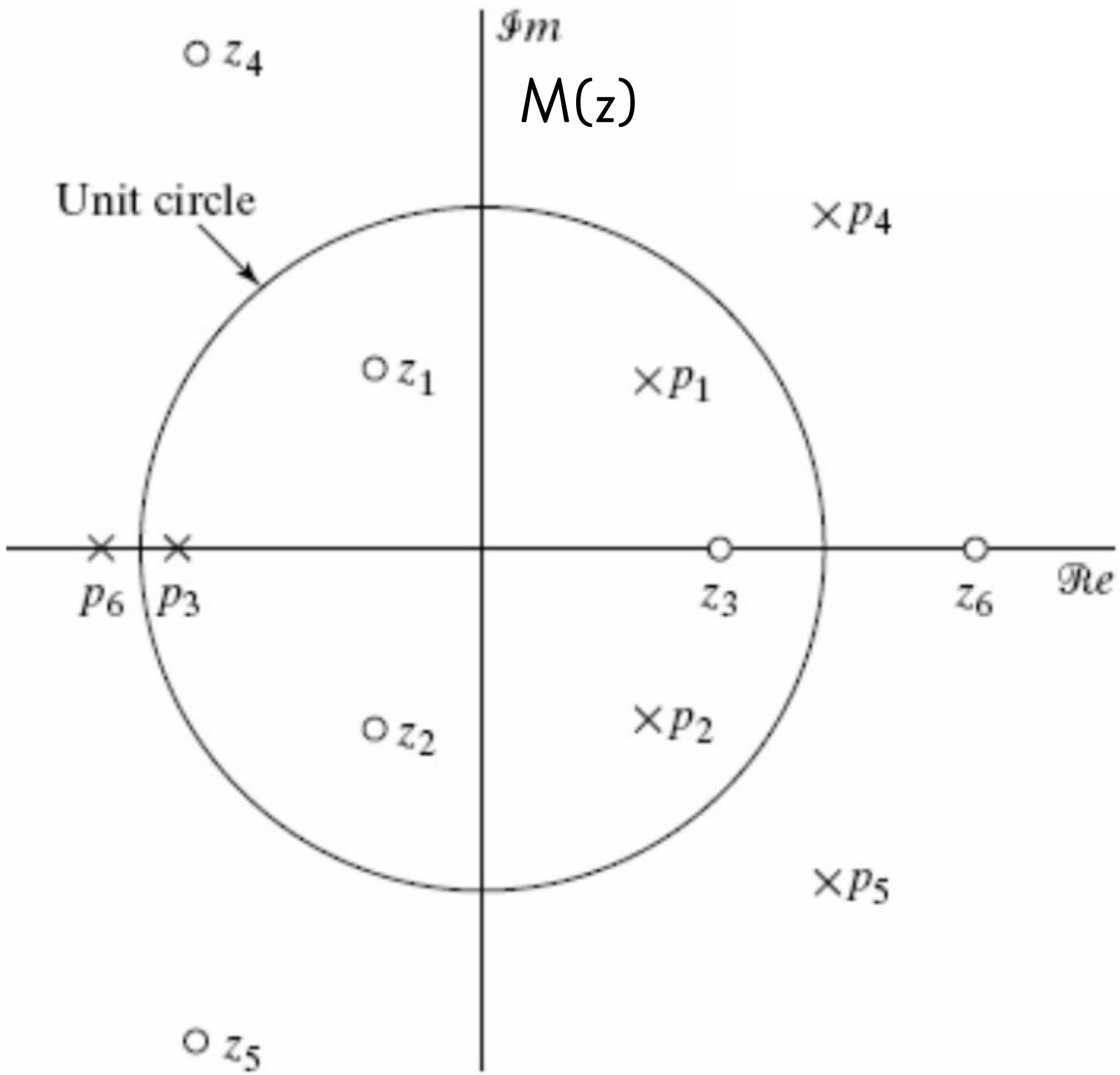
- Minimum phase filters are such that both $H(z)$ and $1/H(z)$ are stable and causal

⇓

- The poles must be inside the unit circle
- The zeros must be inside the unit circle
- Given a square magnitude response $M(z)$, there is a **unique** system whose zeros and poles are inside the unit circle and it is called minimum phase system

From M(z) to H(z)

*How to get a **minimum phase** system with real coefficients?*



From M(z) to H(z)

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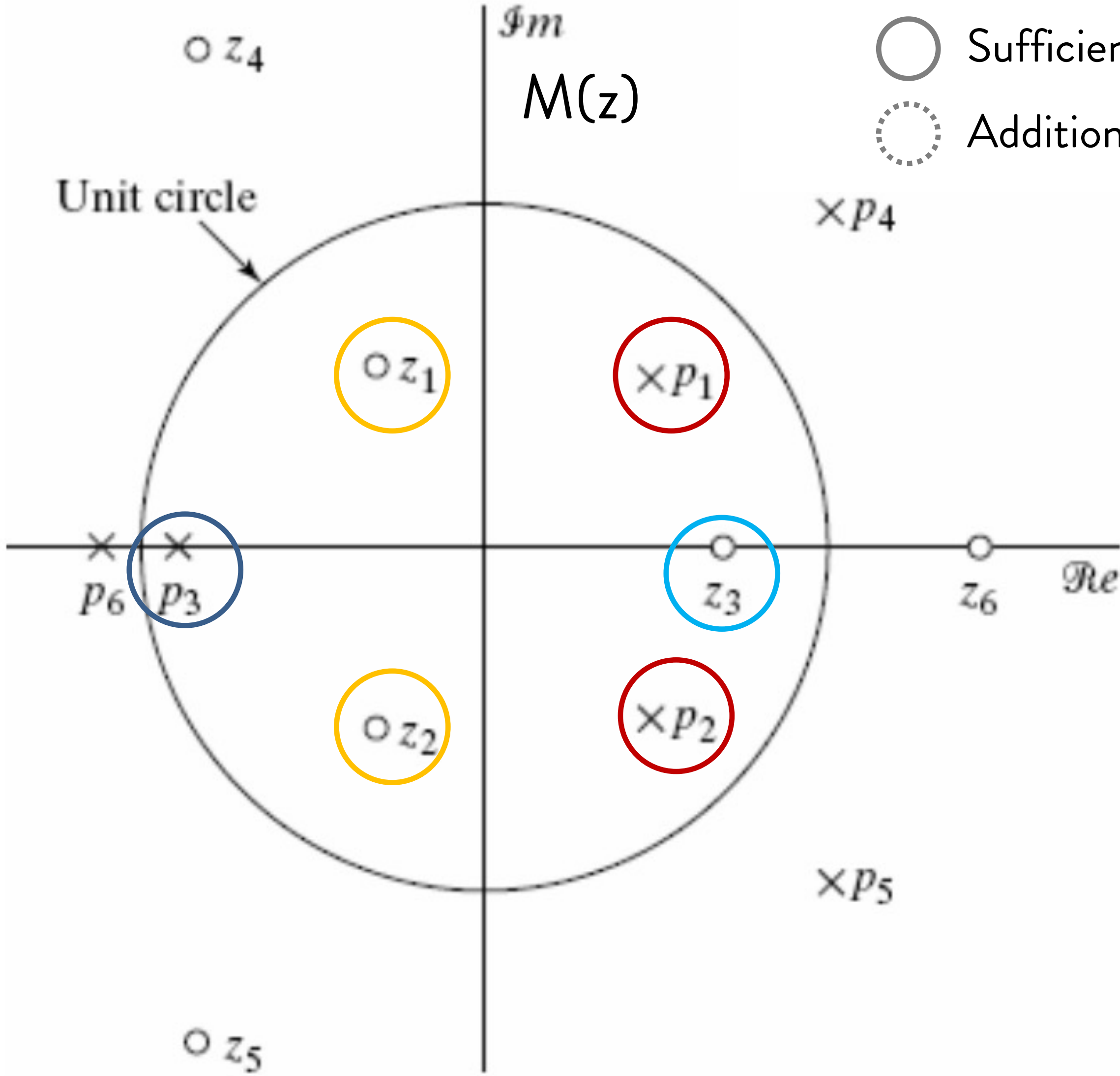
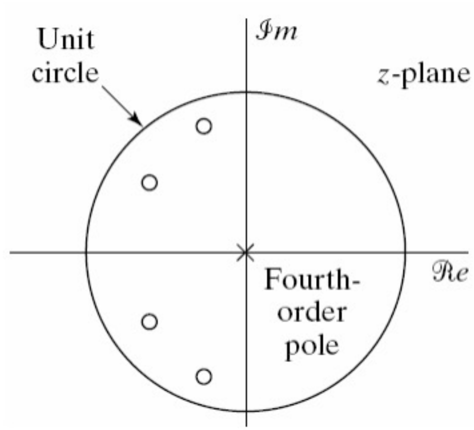


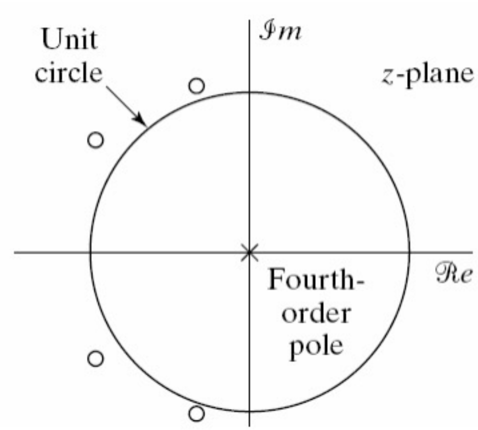
Sufficient configuration

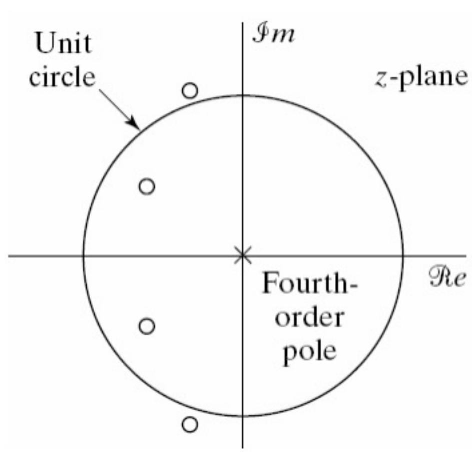
Additional configuration

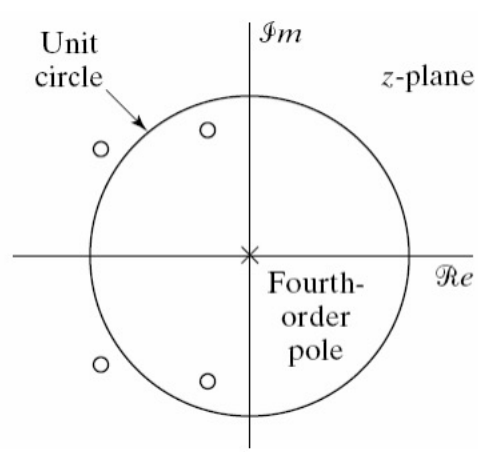
From M(z) to H(z)

*Which is the **minimum phase** system?*



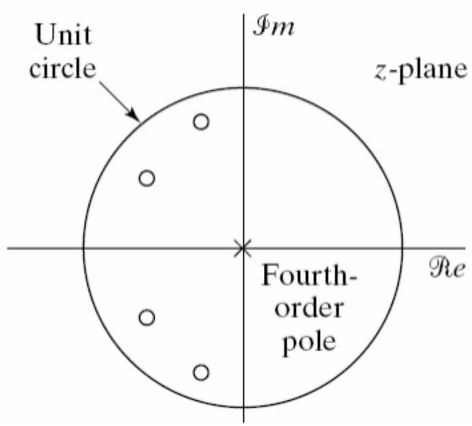


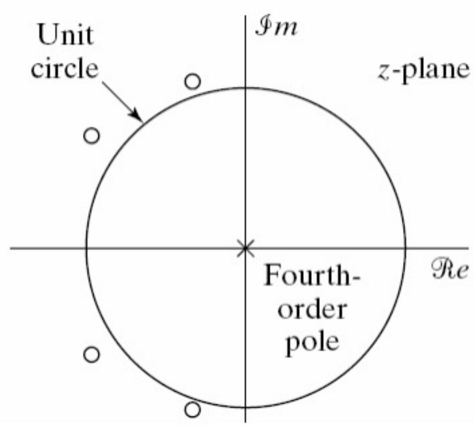


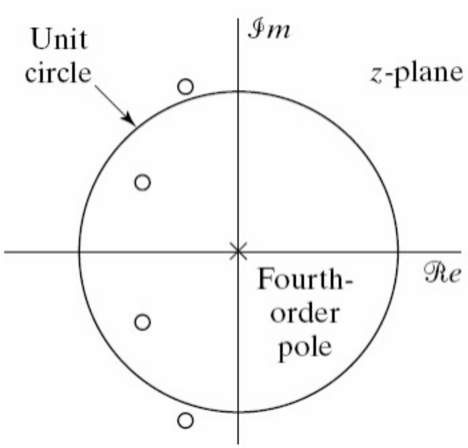


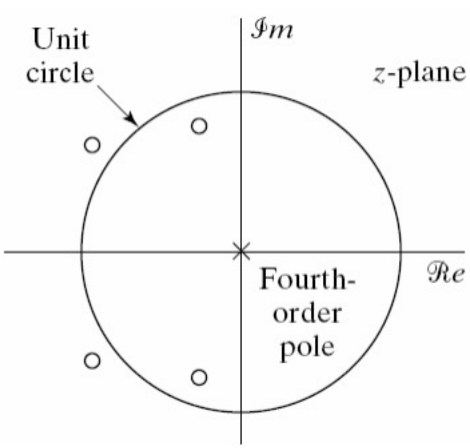
From M(z) to H(z)

Which is the ***minimum phase*** *system?*









Ex 22: minimum phase systems

- Write a MATLAB function 'typeOfFilter(b, a)' that receives as input the numerator and denominator coefficients of a causal filter H(z)=B(z)/A(z) and it returns:
  - -1 if the filter is not stable
  - 1 if the filter is stable and it is minimum phase
  - 0 if the filter is stable but it is not minimum phase
- If you test this function on a FIR filter, which can be the possible outputs?
- Test the function on $H(z) = \dfrac{1 - 2z^{-1} - 0.5z^{-3} + 0.2z^{-4}}{1 + 0.08z^{-1} + 2z^{-3}}$

Properties of Allpass – Minimum phase filters

Any rational causal stable system can be decomposed into the multiplication of a minimum phase system and an allpass system

$$H(z) = H_{min}(z)H_{ap}(z)$$

- $H_{min}(z)$ contains:
  - the poles and zeros of H(z) that lie inside the unit circle
  - zeros that are conjugate reciprocals of the zeros of H(z) lying outside the unit circle.
- $H_{ap}(z)$ contains:
  - all the zeros of H(z) that lie outside the unit circle
  - poles which are conjugate reciprocals of the zeros of H(z) lying outside the unit circle

Ex 23: Allpass-minimum phase conversion

- Given the filter with B(z)=[1, -1.98, 1.77, -0.17, 0.21, 0.34], A(z)=[1, 0.08, 0.40 ,0.27]
- Compute the allpass-minimum phase decomposition of H(z)
- Check the results using 'zplane'
- Plot the amplitude of $H_{ap}(f)$ (DTFT) vs normalized angular frequencies in [0, 2pi), using N = 1024 samples
- Plot the first 50 samples of $h_{min}(n)$

Digital filters' design

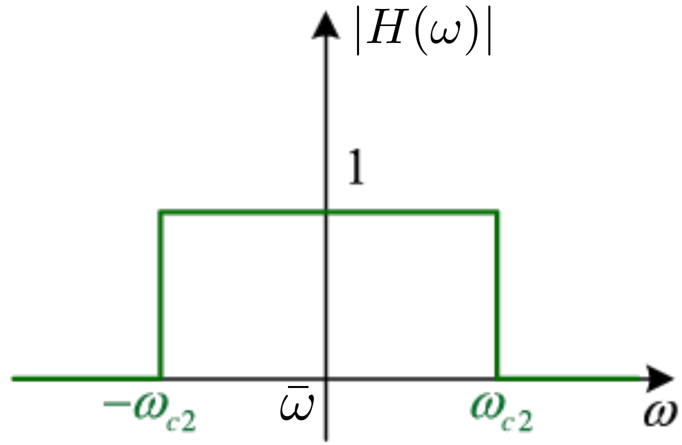
How to design filters

1. Specify always the characteristics of the filter in frequency domain, not in time domain (e.g., lowpass, highpass, bandpass..)
2. Approximate these properties using a discrete-time system → find the filter coefficients
3. Realize the system using finite precision arithmetic

Ideal filter

Ideal filter:

- low-pass if $\bar{\omega} = 0$
- band-pass if $0 < \bar{\omega} < \pi$
- high-pass if $\bar{\omega} = \pi$

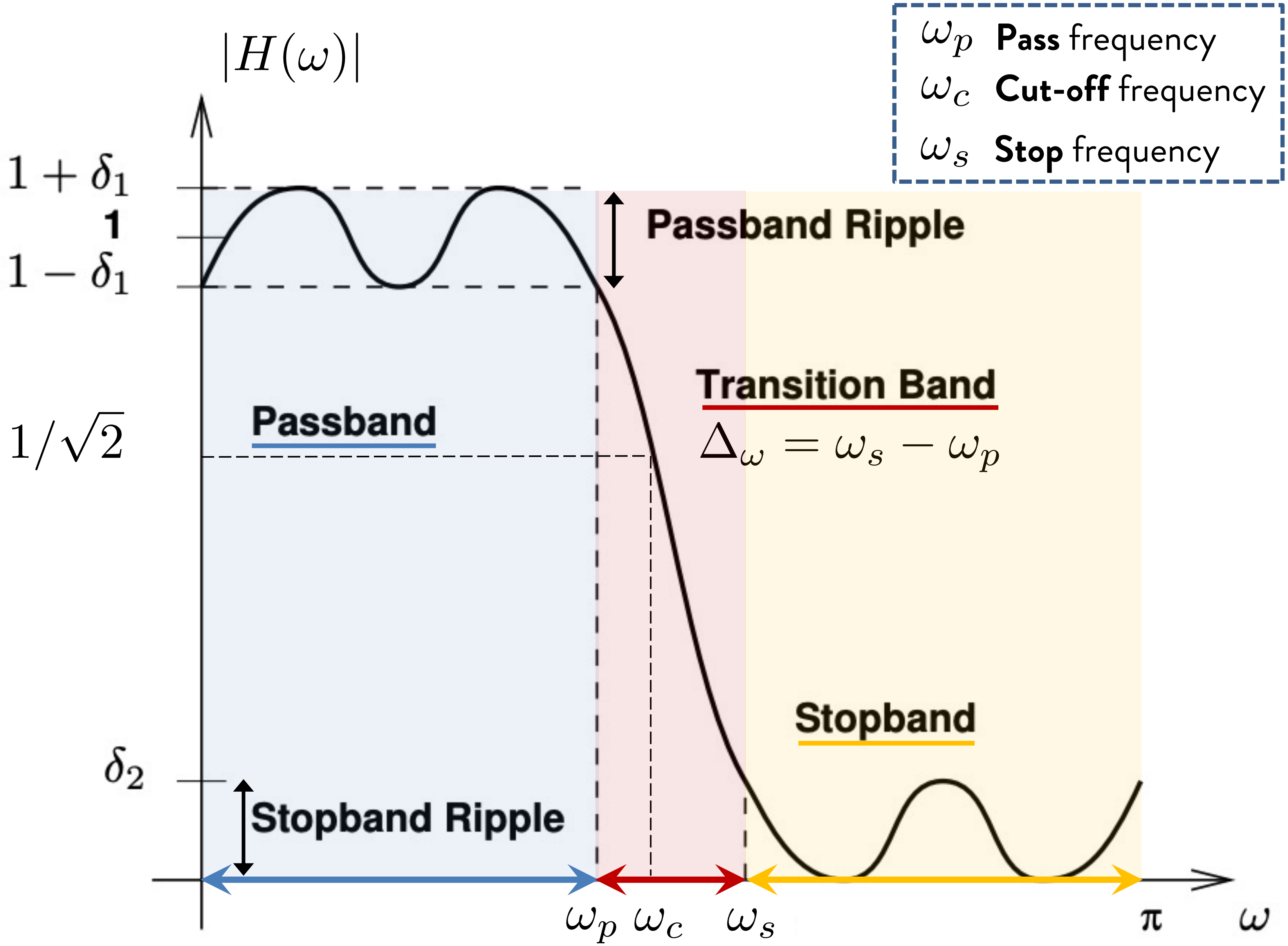


The impulse response of this filter is ≈ the sinc function.

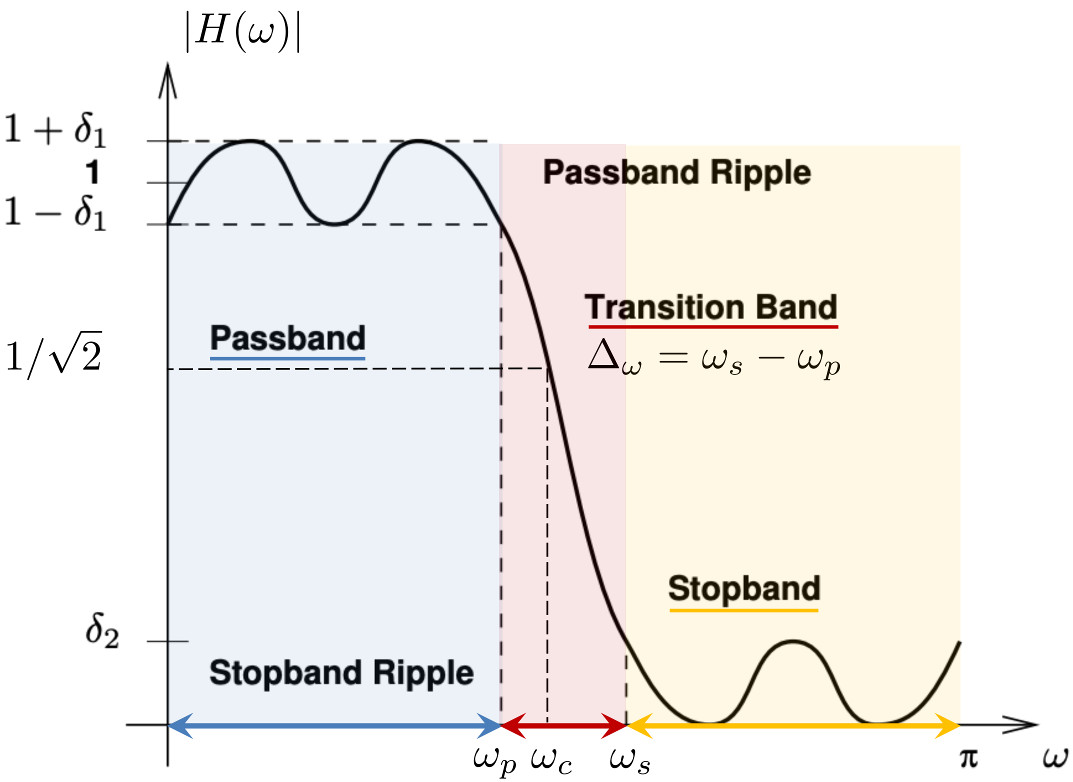
It is non-causal with an infinite delay →

Real systems can only approximate it

Real filters



Real filters



δ_1 Peak passband ripple

$1-\delta_1$ Minimum passband gain

δ_2 Peak stobband ripple

ω_c 3dB or cut-off frequency

Towards ideal filters:

- Peak ripple → 0
- Transition band → 0

IIR vs FIR filters

FIR:

- Only zeros
- Always stable
- Can be linear phase
- It should be high order for best performances

IIR:

- Poles and zeros
- May be unstable
- Difficult to control phase
- Lower order (1/10-th of FIR) for high performances

IIR vs FIR

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We saw IIR design with poles&zeros

How to design FIR filters?

FIR filter design: windowing method

The ideal filter has an infinite time duration and infinite delay.

Idea: obtain a FIR filter by truncating an infinite duration impulse response

- Given the ideal $h_i(n)$, build $h(n) = h_i(n)w(n)$
- $w(n)$ is a finite duration window

  → in frequency domain, product becomes convolution
- $H(f)$ is a blurred version of the ideal filter $H_i(f)$

FIR filter design: windowing method

How to choose the window?

- As short as possible (in time) to minimize the cost of the FIR filter
- As narrow as possible in frequency to approach the ideal filter

FIR filter design: windowing method

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FIR filter design: windowing method

How to choose the window?

- As short as possible (in time) to minimize the cost of the FIR filter
- As narrow as possible in frequency to approach the ideal filter

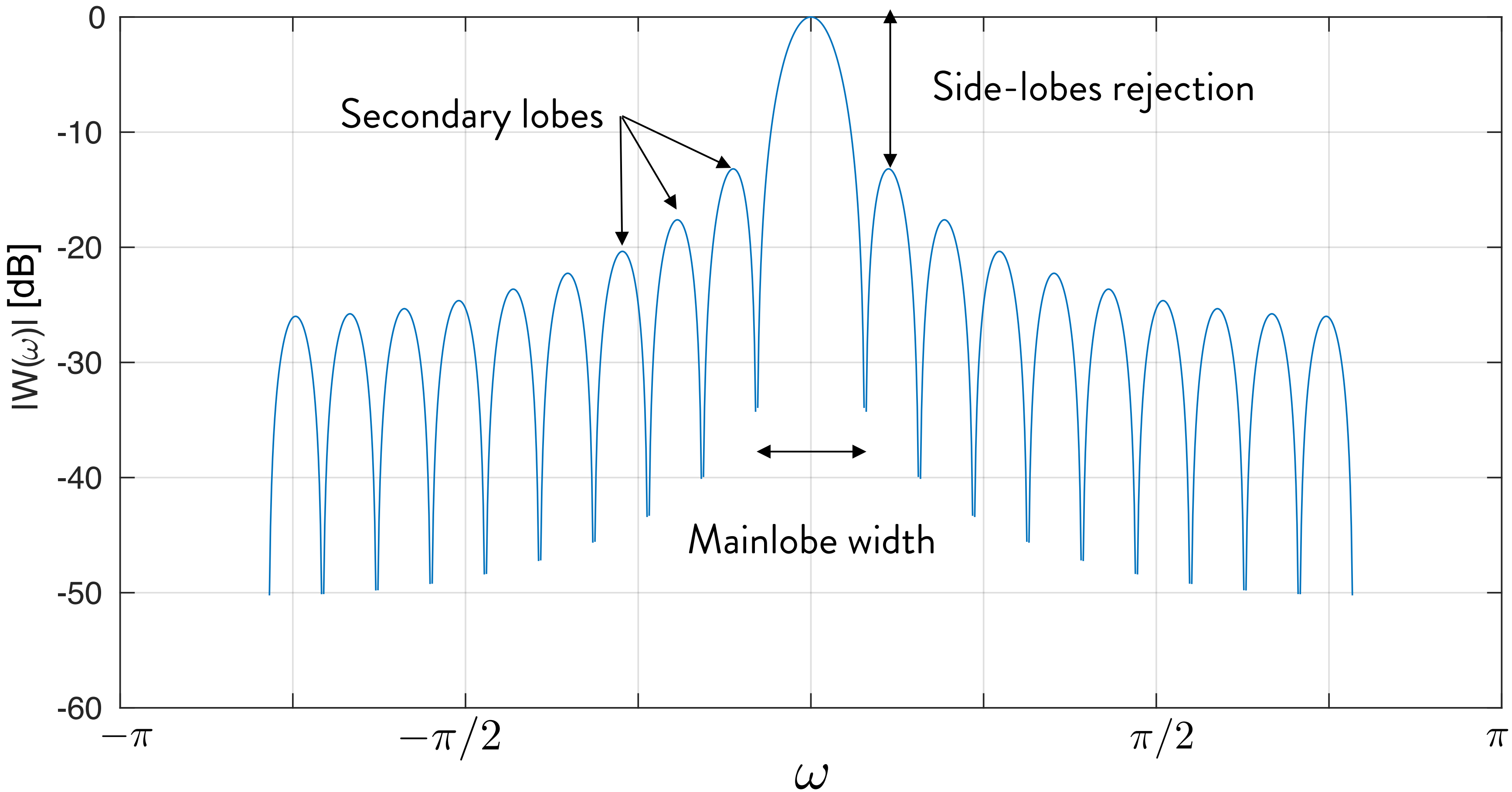
↓

Even though these requirements conflict each other, a good window is defined as the one introducing the minimum distortion.

→ Without considering the filter cost, $W(f)$ should look like a $\delta(f)$:

- its energy must be concentrated around $f = 0$
- $W(f)$ should decay fast as frequency increases

FIR filter design: windowing method



FIR filter design: windowing method

Every window is characterized by:

- Main-lobe width: it decreases as the window length increases
- Side-lobes rejection: ratio between the main-lobe peak and 1° secondary lobe peak [dB]
- Side-lobes roll off: asymptotic decay of the side-lobe peaks vs frequency octave [dB/octave] or frequency decade [dB/decade]

Examples of windows:

- Rectangular → 'rectwin' in MATLAB
- Hanning → 'hann' in MATLAB
- Hamming → 'hamming' in MATLAB
- Blackman → 'blackman' in MATLAB and many others…

Window design in MATLAB

- You can use the function ‘window’ to design windows:

  ‘w = window(@window_name, Nsamples)’

- Otherwise, you can call specific functions named as the window, for instance:
  - ‘rect_w = rectwin(Nsamples)’
  - ‘hamming_w = hamming(Nsamples)’
  - ‘hann_w = hann(Nsamples)’
  - …

FIR design in MATLAB

'fir1' is used to implement window-based FIR filter design

h = fir1(filter_order,cut-off,filter_type,window_type(filter_order+1))

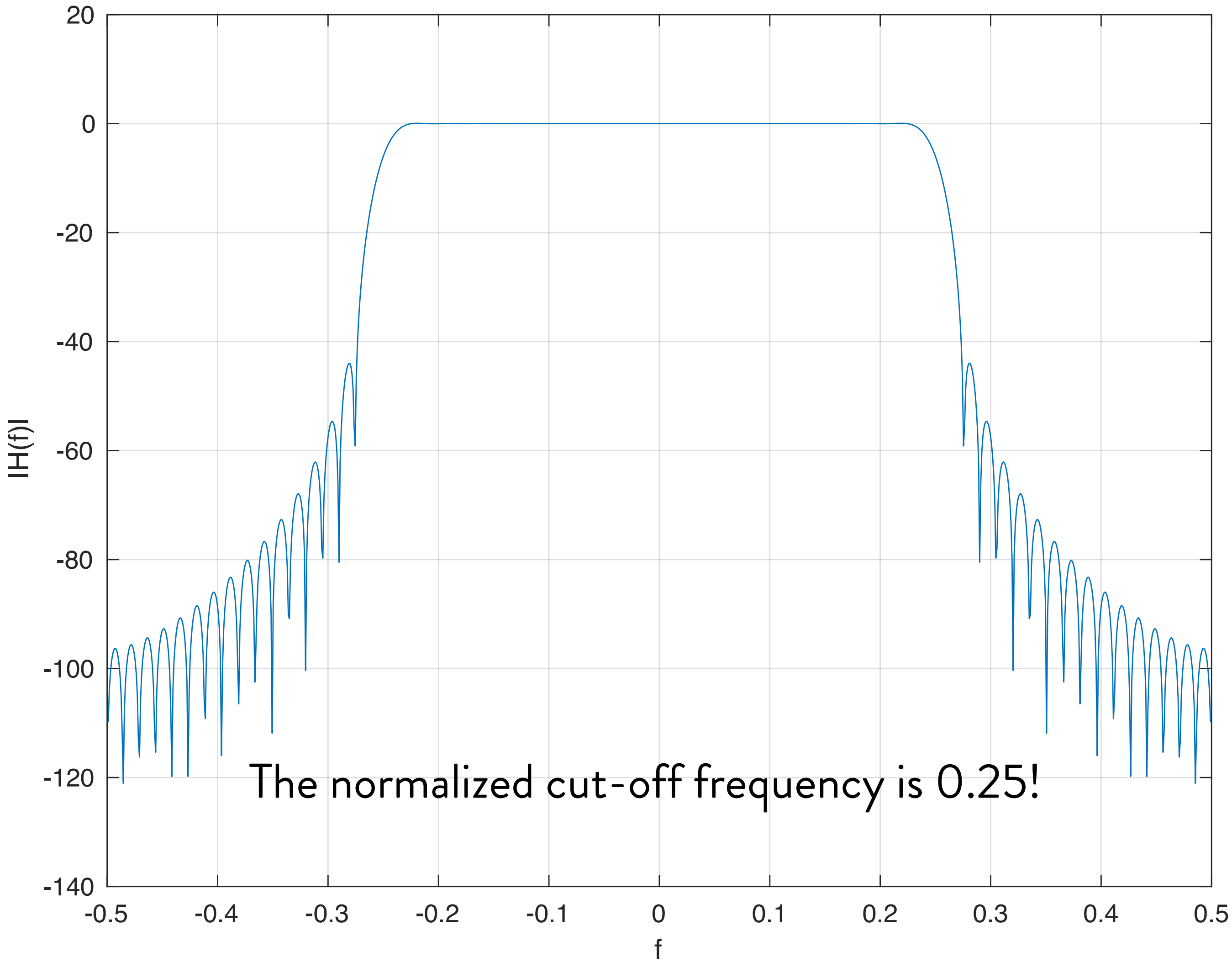
NB:

- 'cut-off' sets the 6dB point (when $|H(f)|$ is 6dB lower than the maximum peak)
- 'cut-off' for MATLAB is between 0 and 1, but 1 corresponds to half the sampling frequency!

  MATLAB Cut-off = 1 ←→ normalized frequency = 0.5
- 'filter_order' corresponds to the number of samples - 1

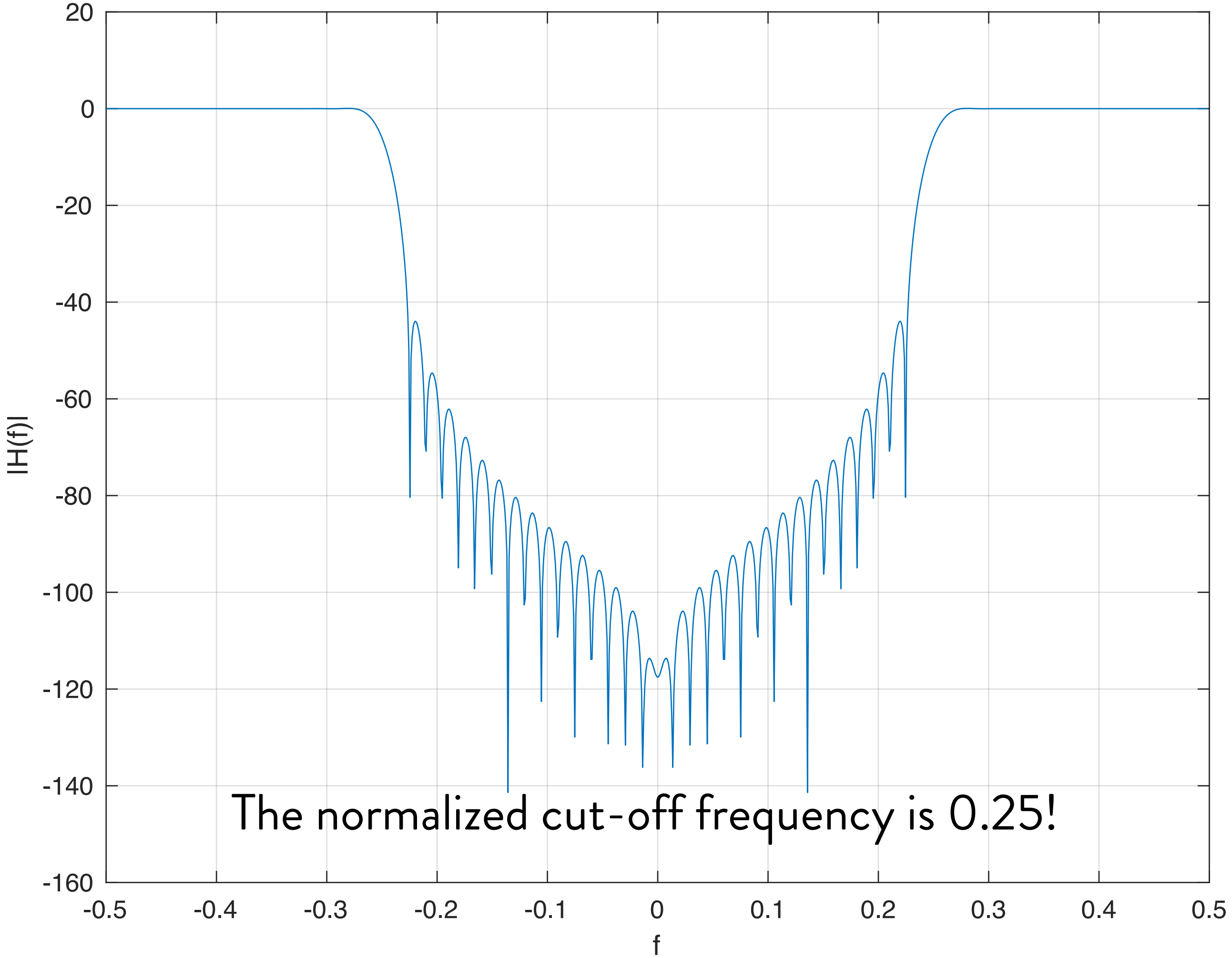
FIR design in MATLAB:fir1

Low-pass: ‘h = fir1(66, 0.5, hann(67))’



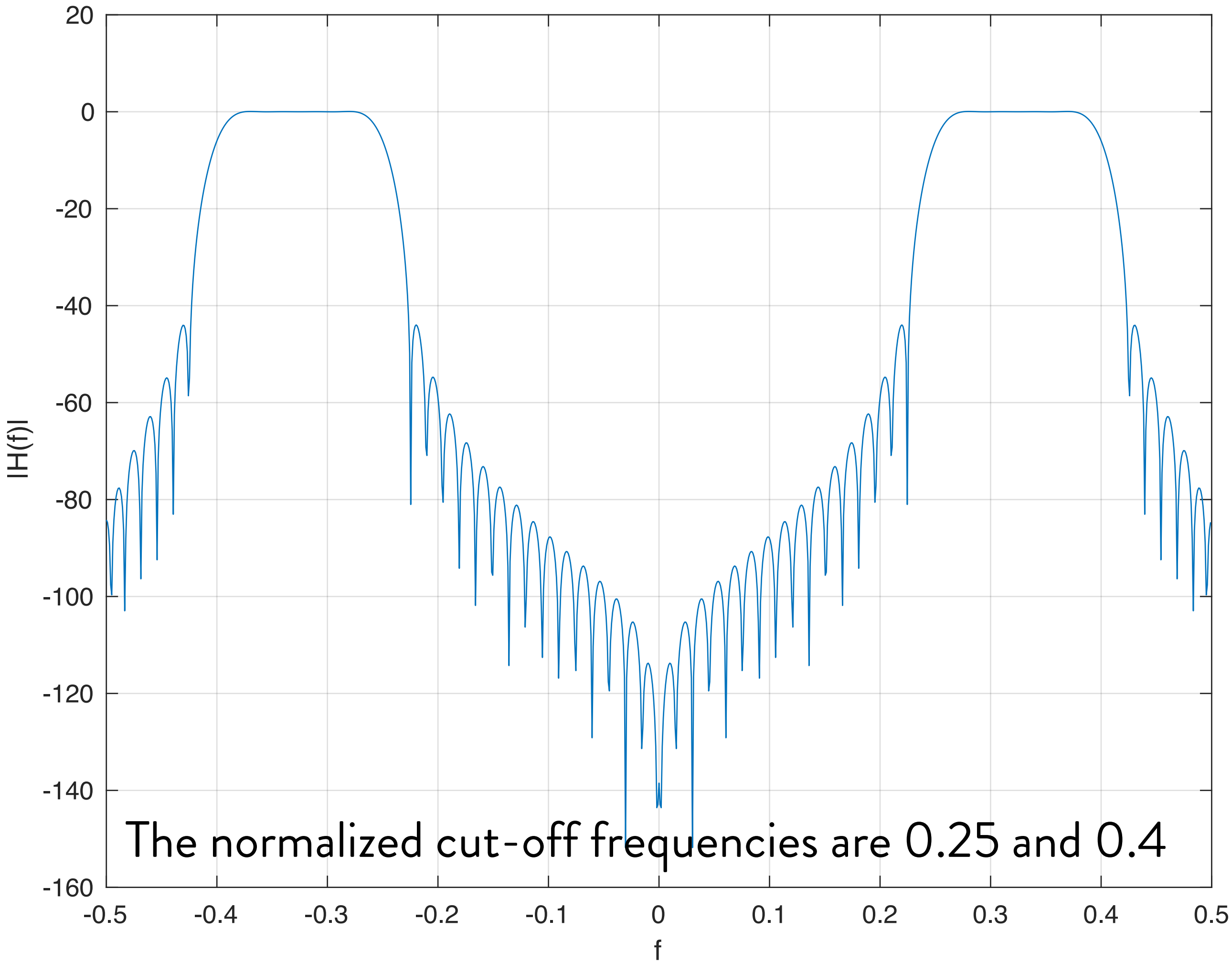
FIR design in MATLAB:fir1

High-pass: ‘h = fir1(66, 0.5, ‘high’, hann(67))



FIR design in MATLAB:fir1

Band-pass: ‘h = fir1(66, [0.5, 0.8], hann(67))



Ex 24: windowing

- Given x as a cosine wave sampled at Fs = 8KHz, defined from 0 to 1 second, amplitude 1.5, frequency 1.1KHz, phase 45 deg
- Compute y as x filtered with a low-pass filter with normalized cut-off frequency of 0.4 and 64 samples
- Plot the amplitude of H(f) vs normalized frequencies in [-0.5, 0.5)
- Apply a Hanning window to select the first 512 samples of y
- Plot the amplitude of the DFT of the windowed y vs frequency in Hz, defined in [-Fs/2, Fs/2).
- If you change the cut-off frequency to 0.05, what do you expect to see in the spectrum of y?