



POLITECNICO
MILANO 1863

DIPARTIMENTO DI ELETTRONICA
INFORMAZIONE E BIOINGEGNERIA

Z transform

Z transform definition

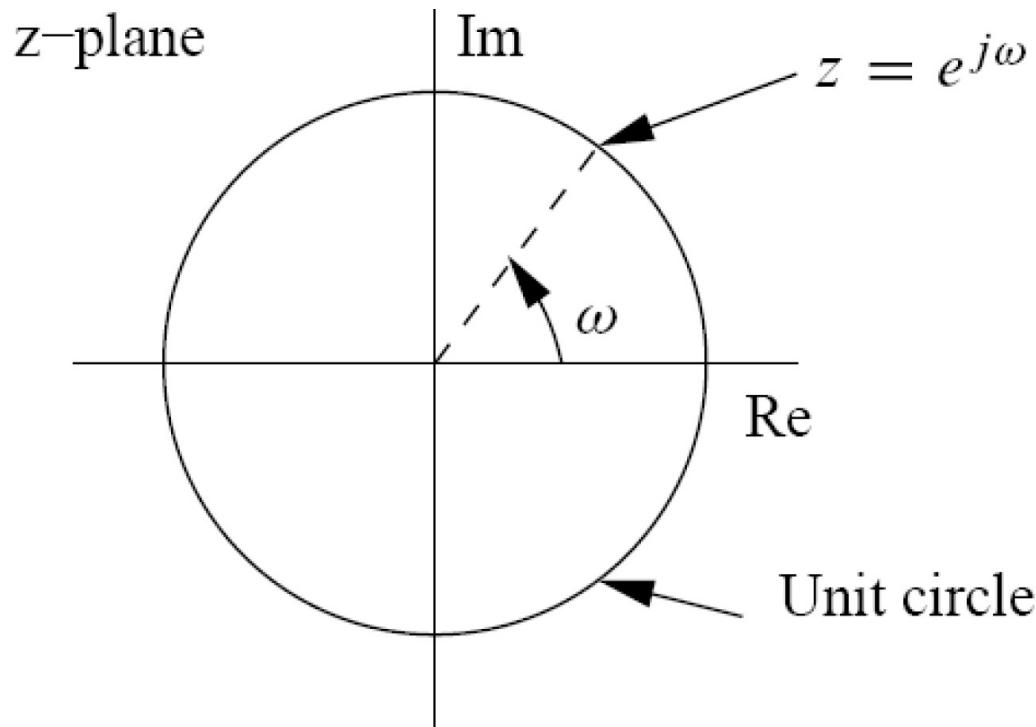
- The Z transform of a sequence $x(n)$ is defined as

$$X(z) = \sum_{-\infty}^{+\infty} x(n) \cdot z^{-n}$$

- z is a complex variable, $z = \rho e^{j2\pi f}$

Z transform definition

- Since z is a complex number, we can represent it in the complex plane.



Z transform example

- Given the sequence

$$x(n) = \delta(n) + \delta(n + 1) + 2\delta(n - 2)$$

$$X(z) = \sum_{-\infty}^{\infty} x(n)z^{-n}$$



$$X(z) = \sum_{-\infty}^{\infty} \delta(n)z^{-n} + \sum_{-\infty}^{\infty} \delta(n + 1)z^{-n} + \sum_{-\infty}^{\infty} 2\delta(n - 2)z^{-n}$$

$$X(z) = 1 + z + 2z^{-2}$$

Z transform example

- Given the sequence $x(n) = a^n u(n)$, $|a| < 1$

$$X(z) = \sum_{-\infty}^{\infty} a^n u(n) z^{-n}$$

$$X(z) = \sum_0^{\infty} a^n z^{-n} = \sum_0^{\infty} (az^{-1})^n$$

- Recalling geometric series:

$$\sum_0^K a^n = \frac{1 - a^{K+1}}{1 - a}$$

$$\sum_0^{\infty} a^n = \frac{1}{1 - a}$$

$$x(n) = a^n u(n) \implies X(z) = \frac{1}{1 - az^{-1}}$$

Z transform properties

- $Z\{ax(n) + by(n)\} = aX(z) + bY(z)$
- $Z\{x(n - k)\} = X(z)z^{-k}$
- $Z\{x(n)a^n\} = X\left(\frac{z}{a}\right)$
- $Z\{x(-n)\} = X(z^{-1})$
- $Z\{nx(n)\} = -z \frac{dX(z)}{dz}$
- $Z\{x(n) * y(n)\} = X(z) \cdot Y(z)$

Convolution theorem

Ex 8a: Z-transform convolution property

- Given a signal $x(n) = [3, 2, 1, 0, 1]$, n in $[-2, 2]$
- Given a LTI system with $h(n) = [1, 3, 2.5, 4, 2]$, n in $[0, 4]$
- Compute the output of the system using ‘conv’. Which is the support of $y(n)$?
- Write the expression of $H(z)$.
- Exploiting the convolution theorem, compute $Y(z) = X(z) H(z)$
- Which is the order of polynomial $H(z)$?



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Z-transform expressions

Z transform expressions

There are several ways to represent Z-transform

$$1. \ X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_N z^{-N}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_D z^{-D}} ,$$

- $N(z)$ and $D(z)$ are polynomial expressed in z^{-1}
- Useful to compute the Inverse Z transform

$$2. \ X(z) = z^{D-N} \frac{b_0}{a_0} \frac{\prod_{i=1}^N (z - z_i)}{\prod_{i=1}^D (z - p_i)}$$

- Useful for filter characterization
- z_i are called ‘zeros’, p_i are called ‘poles’

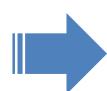
Z transform relationship with LTI systems

Given $x(n)$ and $h(n)$ (impulse response of LTI system):

$$y(n) = x(n) * h(n)$$

The same system can also be described by a linear difference equation with constant coefficients

$$\sum_{k=0}^D a_k y(n - k) = \sum_{k=0}^N b_k x(n - k)$$



$$y(n) = \boxed{\sum_{k=0}^N b_k x(n - k)} - \boxed{\sum_{k=1}^D a_k y(n - k)}$$

Moving Average
(FIR)

Autoregressive
(IIR)

Z transform relationship with LTI systems

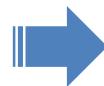
$$\sum_{k=0}^D a_k y(n-k) = \sum_{k=0}^N b_k x(n-k)$$

Converting in Z domain

$$Y(z) \sum_{k=0}^D a_k z^{-k} = X(z) \sum_{k=0}^N b_k z^{-k}$$

Since $y(n) = x(n) * h(n)$,

Thanks to the convolution theorem: $Y(z) = X(z)H(z)$


$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^N b_k z^{-k}}{\sum_{k=0}^D a_k z^{-k}}$$

Inverse Z transform

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^N b_k z^{-k}}{\sum_{k=0}^D a_k z^{-k}}$$

⇒ $h(n) = Z^{-1}\{H(z)\}$

$h(n)$ can be computed in different ways:

1. Long division
2. Partial fract expansion
3. Viewing $H(z)$ as cascade of filters $h_1(n) * h_2(n) * h_3(n) \dots$

Inversion of a polynomial Z transform

- Given the Z-transform of $h(n)$, $H(z) = \sum_{n=0}^k h(n)z^{-n}$
- We can compute its root decomposition:

$$H(z) = h_0 \prod_{n=1}^k (1 - z_n z^{-1}), h_0 = H(n=0)$$

- z_n are called roots of the polynomial $H(z)$, $H(z = z_n) = 0$
- Thanks to the convolution theorem,

$$H(z) = h_0 H_1(z) H_2(z) H_3(z) \dots H_k(z)$$

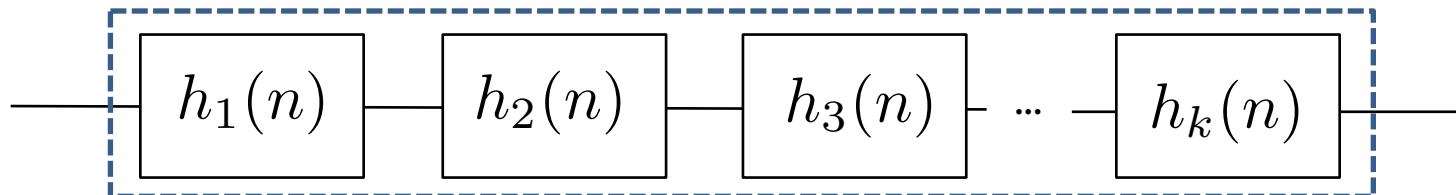


$$h(n) = h_0 \cdot h_1(n) * h_2(n) * h_3(n) * \dots * h_k(n)$$

Inversion of a polynomial Z transform

- $H(z) = h_0 \prod_{n=1}^k (1 - z_n z^{-1})$
- Given the roots z_n , $H(z = z_n) = 0$
- Thanks to the convolution theorem we can derive the impulse response as the cascade of multiple filters written as elementary sequences . For example,

$$h_1(n) = Z^{-1}\{1 - z_1 z^{-1}\} = \delta(n) - z_1 \delta(n - 1)$$



$$h(n) = h_0 \cdot h_1(n) * h_2(n) * h_3(n) * \dots * h_k(n)$$

Ex 8b: Z-transform convolution property

- Given a signal $x(n) = [3, 2, 1, 0, 1]$, n in $[-2, 2]$
- Given a LTI system with $h(n) = [1, 3, 2.5, 4, 2]$, n in $[0, 4]$
- Compute the output of the system using ‘conv’. Which is the support of $y(n)$?
- Write the expression of $H(z)$.
- Exploiting the convolution theorem, compute $Y(z) = X(z) H(z)$
- Which is the order of polynomial $H(z)$?
- Compute the roots of $H(z)$.
- Write $y_1(n)$ as the convolution of $x(n)$ with the filter cascade:
$$h(n) = h_0 \cdot h_1(n) * h_2(n) * h_3(n) * \dots * h_k(n)$$
- Plot $y(n)$ and $y_1(n)$ in the same figure and check if $y_1(n) = y(n)$

Partial fract expansion for computing Z^{-1}

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^N b_k z^{-k}}{\sum_{k=0}^D a_k z^{-k}}$$

⇒ $h(n) = Z^{-1}\{H(z)\}$

$$H(z) = \sum_{k=1}^D \sum_{m=1}^M \frac{r_{k_m}}{(1 - p_k z^{-1})^m} + \boxed{\sum_{k=0}^{N-D} c_k z^{-k}}$$

$N \geq D$

M is the multiplicity of the root (or ‘pole’) p_k .

The Z transform inversion is the sum of simple inversions.

Partial fract expansion for computing Z^{-1}

$$H(z) = \sum_{k=1}^D \sum_{m=1}^M \frac{r_{k_m}}{(1 - p_k z^{-1})^m} + \boxed{\sum_{k=0}^{N-D} c_k z^{-k}}$$

$N \geq D$

The Z transform inversion is the sum of simple inversions (causal):

- $Z^{-1} \left\{ \frac{r_{k_1}}{(1 - p_k z^{-1})} \right\} = r_{k_1} \cdot (p_k)^n u(n)$
- $Z^{-1} \left\{ \frac{r_{k_2}}{(1 - p_k z^{-1})^2} \right\} = r_{k_2} \cdot (n+1)(p_k)^n u(n)$
- $Z^{-1} \left\{ c_k z^{-k} \right\} = c_k \cdot \delta(n - k)$

Partial fract expansion for computing Z^{-1}

$$H(z) = \frac{\sum_{k=0}^N b_k z^{-k}}{\sum_{k=0}^D a_k z^{-k}} = \sum_{k=1}^D \sum_{m=1}^M \frac{r_{k_m}}{(1 - p_k z^{-1})^m} + \sum_{k=0}^{N-D} c_k z^{-k}$$

- Residues r_{k_m} , poles p_k and c_k can be found using the MATLAB function ‘[residues, poles, c_k] = residuez(b, a)’
- ‘b’ is the vector of the numerator coefficients (ordered from b_0 to b_N)
- ‘a’ is the vector of the denominator coefficients (ordered from a_0 to a_D).

Ex 9.a: Partial fract expansion of Z transform

- Given a LTI system with this transfer function:

$$H(z) = \frac{z^{-5} + z^{-4} - 3z^{-3} - 8z^{-2} + 7z^{-1} + 9}{z^{-3} - 2z^{-2} - z^{-1} + 2}$$

- Find its partial fract expansion:
 - Save in a vector r the residues
 - Save in a vector p the poles
 - Save in a vector c the coefficients of the polynomial $\sum_{k=0}^{N-D} c_k z^{-k}$
- Find $h(n)$ as the sum of elementary filters found with the partial fract expansion, $n = 0:100$.

Another MATLAB solution for Z^{-1}

- Given $H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^N b_k z^{-k}}{\sum_{k=0}^D a_k z^{-k}}$
- The inverse Z transform can be found using the MATLAB function ‘`h= filter(b, a, x)`’
- ‘`b`’ is the vector of the numerator coefficients (ordered from `b_0` to `b_N`)
- ‘`a`’ is the vector of the denominator coefficients (ordered from `a_0` to `a_D`)
- ‘`x`’ is the input signal to the system. To find $h(n)$, `x` must be...?

Ex 9.b: Partial fract expansion of Z transform

- Given a LTI system with this transfer function:

$$H(z) = \frac{z^{-5} + z^{-4} - 3z^{-3} - 8z^{-2} + 7z^{-1} + 9}{z^{-3} - 2z^{-2} - z^{-1} + 2}$$

- Find its partial fract expansion:
 - Save in a vector r the residues
 - Save in a vector p the poles
 - Save in a vector c the coefficients of the polynomial $\sum_{k=0}^{N-D} c_k z^{-k}$
- Find $h(n)$ as the sum of elementary filters found with the partial fract expansion, $n = 0:100$.
- Find $h(n)$ using ‘filter’, $n = 0:100$.



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Zeros-Poles factorization

Zeros-Poles factorization

$$H(z) = \frac{\sum_{k=0}^N b_k z^{-k}}{\sum_{k=0}^D a_k z^{-k}} = z^{D-N} \frac{b_0}{a_0} \frac{\prod_{i=1}^N (z - z_i)}{\prod_{i=1}^D (z - p_i)}$$

- Roots of numerator are called ‘zeros’
- Roots of denominator are called ‘poles’
- In MATLAB we can plot poles and zeros with the function
- ‘zplane(z, p)’ (zeros, poles, in column vectors)
- ‘zplane(b, a)’ (numerator, denominator, in row vectors)

System stability

- For a system to be stable,

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

- If all the poles of $H(z)$ are inside the unitary circle ($|p_i| < 1, \forall i$), the system is stable.
- If one positive zero is inside the unitary circle, it is called ‘minimum phase’
- If one positive zero is outside the unitary circle, it is called ‘maximum phase’

Ex 10: zeros-poles factorization

- Given $y(n) = x(n) - bx(n - 1) + ay(n - 1)$
- Which is the expression of $H(z)$?
- Which is the value of $h(0)$? Derive it without computing $h(n)$.

What about $y(0)$?

- Compute and plot (in the Z-plane) the zeros and poles.
- Plot $h(n)$ for n in $[0, 50]$ for $b = 0.5$ and $a = 0.2$.
- Plot $h(n)$ for n in $[0, 50]$ for $b = 1.2$ and $a = 0.2$
- Plot $h(n)$ for n in $[0, 50]$ for $b = 1.2$ and $a = 1.1$
- In which situations is the system stable?
- When are the zeros minimum phase?

Ex 11: zeros-poles factorization

- Given

$$y(n) = 0.5x(n) - 2x(n-1) + x(n-2) - 2\rho \cos(\theta)y(n-1) - \rho^2 y(n-2)$$

- $\rho = 0.9, \theta = \pi/8$
- Which is the expression of $H(z)$?
- Which are the values of $h(0)$ and $y(0)$?
- For n in $[0, 200]$, which is the expression of $h(n)$? Use the function ‘filter’
- Compute its zeros and poles.
- Plot its zeros and poles.