Determining the Number of Priced State Variables in the ICAPM

Eugene F. Fama*

Abstract

Suppose the ICAPM governs asset prices and there is a total of S state variables that might be of hedging concern to investors. Can we determine which state variables are, in fact, of hedging concern? What does it mean to say that these state variables are priced, that is, that they give rise to special risk premiums in expected returns? The goal of this paper is to formulate this problem clearly and show when it can and cannot be solved. Ignoring estimation problems, it is possible to find the set of priced state variables when the state variables are identified (named). When we know the number of state variables, but not their names, confident conclusions about even the number of them that produce special risk premiums are probably impossible, unless the number is zero, so the ICAPM collapses to the CAPM.

I. Introduction

Suppose asset pricing conforms to the discrete-time version of Merton's (1973) Intertemporal Capital Asset Pricing Model (ICAPM) in Fama (1996). Suppose we know there is a total of S state variables that are potentially of special hedging concern to investors because they capture uncertainty about the evolution of consumption-investment opportunities. The questions addressed here are:

i) If the S state variables are identified (named), how can we determine the subset of them that are in fact of hedging concern?

ii) Are the state variables of hedging concern priced, that is, do they produce special risk premiums in expected returns? What exactly is meant by a special risk premium?

iii) If the state variables are not named, we obviously cannot identify which of them are priced. Can we at least determine how many of them generate special premiums?

Note that I am not looking for the minimum number of risk premiums that suffice to describe expected returns. That number is always one. As always, any

*Graduate School of Business, University of Chicago, 1101 East 58th Street Chicago, IL 60637.
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minimum-variance portfolio, that is, any portfolio that minimizes variance, given its expected return, can be used to describe the expected returns on all assets (Fama (1976), Roll (1977)). For example, with risk-free borrowing and lending, the expected return on any asset i can be described as

$$(1) \qquad E(r_i) - r_f \quad = \quad [E(r_T) - r_f]\beta_{iT},$$

where r_f is the risk-free rate, r_T is the return on the mean-variance-efficient (MVE) tangency portfolio (the portfolio that, along with f, generates all of Markowitz' (1959) mean-variance-efficient portfolios), and β_{iT} is the slope in the regression of $r_i - r_f$ on $r_T - r_f$. But equation (1) is just algebra; it is the algebraic condition on the weights in a minimum-variance portfolio, applied to the MVE portfolio T. Equation (1) holds whatever the model generating asset prices.

In contrast, the questions I seek to answer go to the heart of the economics of the ICAPM. Specifically, given ICAPM asset pricing, and given that there is a total of S state variables potentially of hedging concern to investors, i) how can we determine which of these state variables are in fact of hedging concern, and ii) in what sense do these state variables produce special risk premiums in expected returns?

The answers to these questions are implied by i) the nature of optimal portfolios in the ICAPM, and ii) the special role of the value-weight market portfolio in an ICAPM equilibrium. To preview, suppose the state variables of hedging concern are the subset $N(L)$, where N is the number ($\leq S$) of state variables and L is the list of their names. The portfolios of ICAPM investors are then multifactor-minimum-variance with respect to the state variables in L. Specifically, investor portfolios are MMV$(N(L),\, E(r))$: they have the smallest return variances, given their expected returns, $E(r)$, and given their loadings (regression slopes) on the $N(L)$ state variables. An ICAPM market equilibrium then says assets must be priced so that the market portfolio, M, is MMV$(N(L),\, E(r))$. And M almost surely is not in any MMV$(N^*(L^*),\, E(r))$ set that does not control loadings on all the state variables in L. Thus:

(C1) We can determine the state variables of hedging concern to investors if we can find the least constrained set of MMV$(N(L), E(r))$ portfolios, that is, the MMV$(N(L),\, E(r))$ set that controls the smallest $N(L)$ set of state variable loadings, while also including the market portfolio M.

Given that we know the names of the S state variables that might be of hedging concern and assuming away the thorny problems of estimation, we can find the least constrained MMV$(N(L),\, E(r))$ set that includes M. Any MMV$(N(L),\, E(r))$ set can be constructed by combining mimicking portfolios for the $N(L)$ state variables with the risk-free security and any other MMV$(N(L),\, E(r))$ portfolio that is not itself a portfolio of the risk-free security and the $N(L)$ state variable mimicking portfolios. If investors are concerned with hedging uncertainty about the $N(L)$ state variables, an ICAPM equilibrium requires that assets are priced so that the market portfolio is MMV$(N(L),\, E(r))$. Thus, M can be the additional portfolio used, along with f and the mimicking portfolios for the $N(L)$ state variables, to generate the MMV$(N(L),\, E(r))$ portfolios relevant for choice by investors.

Moreover, if M, f, and the $N(L)$ state variable mimicking portfolios span (can generate) the set of MMV$(N(L), E(r))$ portfolios, the expected return on any asset i, is

$$(2) \qquad E(r_i) - r_f = \beta_{iM}[E(r_M) - r_f] + \sum_{s=1}^{N(L)} \beta_{is}[E(r_s) - r_f],$$

where $r_s, s = 1, \ldots, N(L)$ are the returns on the mimicking portfolios for the $N(L)$ state variables, and β_{iM} and the β_{is} are the slopes from the multiple regression of $r_i - r_f$ on $r_M - r_f$ and $r_s - r_f$, $s = 1, \ldots, N(L)$. Most important, if $N(L)$ is the set of state variables of hedging concern to investors, no smaller set of state variable mimicking portfolios can be used, along with M and f, to explain expected returns in the manner of (2).

The prescription for determining the relevant $N(L)$ subset of the S state variables is now clear:

(C2) An ICAPM equilibrium implies that we can identify the state variables of hedging concern to investors by finding the smallest set $N(L)$ of state variable mimicking portfolios that, along with M and f, describe expected returns on all assets in the manner of (2).

The argument also leads to a precise statement of the sense in which the $N(L)$ state variables of hedging concern are priced:

(C3) *Priced State Variables* (Definition 1)—The $N(L)$ state variables of hedging concern to investors carry special risk premiums (they are priced) in the sense that they provide the smallest set of premiums, $E(r_s) - r_f$, that can be used, along with the market premium, $E(r_M) - r_f$, to describe expected returns on all assets in the specific manner of (2).

Weeding out irrelevant state variables does not, however, require that we use competing versions of the ICAPM expected return equation (2) to explain the expected returns on all assets. We shall see that we can identify priced state variables and, thus, the relevant version of (2), by using (2) to explain the expected returns on mimicking portfolios for irrelevant state variables. Specifically:

(C4) *Priced State Variables* (Definition 2)—We have identified the $N(L)$ priced state variables when, using (2), a) the market premium and the premiums on the mimicking portfolios for the $N(L)$ state variables can explain the premiums on the mimicking portfolios for all state variables not in $N(L)$, and b) the premium on the mimicking portfolio for any state variable in $N(L)$ cannot be explained by the market premium and the premiums on the other state variable mimicking portfolios in $N(L)$.

The story proceeds as follows. Section II briefly reviews results from Fama (1996) on discrete-time ICAPM portfolio theory and asset pricing. Section III uses Section II as the basis of the approach, outlined above, to identify the state variables that produce special risk premiums in the ideal case where the names (economic identities) of the S state variables that might be of hedging concern are known.

Section IV considers the case where asset pricing conforms to the ICAPM, the total number of state variables, S, is known, but the state variables are not identified. If we do not know the identities of the state variables, we clearly cannot determine the specific list of them of hedging concern. The more surprising bad news is that conclusions about even the number of priced state variables are unlikely, unless the number is zero, so the ICAPM collapses to the CAPM of Sharpe (1964), Lintner (1965), and Black (1972). The problem is that, to determine the number of priced state variables, we must be able to determine that the portfolios we use to explain expected returns are MMV$(N(L), E(r))$ with respect to specific $N(L)$ subsets of the state variables. This is probably impossible if the state variables are not identified.

Finally, the paper addresses only the logical problems in determining the number of state variables that are priced because they are of special hedging concern to investors. Estimation problems, which clearly complicate the task (e.g., Brown (1989)), are ignored. Thus, when we talk about parameters, true values are implied, and they are assumed to be known.

II. ICAPM Portfolio Theory and Asset Pricing

Suppose asset pricing conforms to the discrete-time version of Merton's (1973) ICAPM in Fama (1996), and suppose there are S state variables potentially of hedging concern to investors. Fama (1996) assumes that the joint distribution of asset returns and the S state variables is multivariate normal. Thus, the relation between the return on any asset i and the state variables is described by the linear regression,

$$\text{(3)} \qquad \begin{aligned} r_i &= E(r_i) + \sum_{s=1}^{S} b_{is} k_s + \epsilon_i, \quad E(\epsilon_i) = 0, \\ \operatorname{cov}(\epsilon_i, k_s) &= 0, \quad s = 1, \ldots, S, \end{aligned}$$

where b_{is} is the multiple regression slope or loading of r_i on state variable s, and where, without loss of generality, the state variables k_s, $s = 1, \ldots, S$ are scaled to have zero means.

A. MMV Portfolios

Suppose $N(L)$ is the subset of the state variables that are, in fact, of hedging concern. The portfolios of ICAPM investors are then multifactor-minimum-variance with respect to the $N(L)$ state variables. With risk-free borrowing and lending, each such MMV$(N(L), E(r))$ portfolio is a solution to the problem,

$$\text{(4a)} \qquad \min \sigma^2(r_e) = \sum_{i=1}^{N(L)} \sum_{j=1}^{N(L)} x_{ie} x_{je} \sigma_{ij}, \quad \text{subject to,}$$

$$\text{(4b)} \qquad \sum_{i=1}^{N(L)} x_{ie} b_{is} = b_{es}, \quad s = 1, \ldots, N(L),$$

$$\text{(4c)} \qquad \sum_{i=1}^{N(L)} x_{ie}[E(r_i) - r_f] \quad = \quad E(r_e) - r_f,$$

where $\sigma^2(r_e)$ is the variance of the portfolio's return, and σ_{ij} is the covariance between the returns on assets i and j. The set of MMV($N(L)$, $E(r)$) portfolios is defined by the solutions to (4) for all combinations ($-\infty$ to ∞) of the target expected return, $E(r_e)$, and the target loadings on the state variables, b_{es}, $s = 1, \ldots, N(L)$. Since ICAPM investors are risk averse, they only consider MMV($N(L)$, $E(r)$) portfolios that are also multifactor efficient, ME($N(L)$, $E(r)$), that is, MMV($N(L)$, $E(r)$) portfolios that have maximum expected returns, given their return variances and state variable loadings. For our purposes, it is the MMV($N(L)$, $E(r)$) properties of ME($N(L)$, $E(r)$) portfolios that are important.

A result, used often below, is that any MMV($N(L)$, $E(r)$) set includes all MMV($N^*(L^*)$, $E(r)$) sets that constrain the loadings on only a subset, $N^*(L^*)$, of the state variables in $N(L)$. Such MMV($N^*(L^*)$, $E(r)$) portfolios have the smallest return variances, given their expected returns and given whatever happen to be their loadings on state variables in $N(L)$ that are not in $N^*(L^*)$. MMV($N^*(L^*)$, $E(r)$) portfolios are, thus, MMV($N(L)$, $E(r)$) for any $N(L)$ set that includes the $N^*(L^*)$ set. In economic terms, any MMV($N(L)$, $E(r)$) set includes all the MMV($N^*(L^*)$, $E(r)$) subsets relevant for investors concerned with hedging only subsets of the $N(L)$ state variables. And every MMV($N(L)$, $E(r)$) set contains Markowitz' (1959) mean-variance-efficient set, that is, the MVE = ME($0, E(r)$) portfolios relevant for investors who do not use their portfolio decisions to hedge uncertainty about any state variables.

We can now be precise about what is meant by the $N(L)$ set of state variables of hedging concern to investors. The $N(L)$ set is the union of the $N^*(L^*)$ sets of hedging concern to different investors.

Fama (1996) shows that the solution to (4) points to a simple way to build (span) any MMV($N(L)$, $E(r)$) set. One starts with Markowitz' (1959) mean-variance-efficient (MVE) portfolios, which are in every MMV($N(L)$, $E(r)$) set. With a risk-free security f, MVE portfolios combine f and the MVE tangency portfolio, T. To generate the entire MMV($N(L)$, $E(r)$) set, one then combines f and T with mimicking portfolios for the $N(L)$ state variables.

The mimicking portfolios are defined by the solution to (4) for $N(L) = S$. They have an attractive quality. The mimicking portfolio for state variable s is MMV($1(s)$,); it has the smallest return variance, given its loading on state variable s, with no constraints on expected return or loadings on other state variables. This means the MMV($1(s)$,) mimicking portfolio for state variable s minimizes variance, given whatever happen to be its expected return and loadings on other state variables. Thus, an MMV($1(s)$,) mimicking portfolio is in every MMV($N(L)$, $E(r)$) set, where $N(L)$ includes s. These mimicking portfolios are then natural building blocks, along with f and T, for spanning different sets of MMV portfolios.[1]

[1]The MMV($1(s)$,) mimicking portfolios are also among those that Huberman, Kandel, and Stambaugh (1987) shows can be used in tests of multifactor asset pricing models.

The economic logic behind the spanning result is important. In the ICAPM, the number of securities is finite and MMV portfolios have positive residual variances in (3). The residual variances of MMV portfolios are undiversifiable risks that, with risk-averse investors, are compensated in expected returns. This is why the MMV($N(L)$, $E(r)$) portfolios defined by (4) allow variation in expected returns unrelated to state variable loadings. Adding the MVE portfolio T to the spanning set that includes f and the $N(L)$ state variable mimicking portfolios allows investors to trade expected return for return variance that is not due to the $N(L)$ state variables, as well as control exposures to the state variables.

Two additional implications of the spanning results are used below. First, variation in expected returns independent of loadings on the state variables implies that the MVE tangency portfolio, T, cannot be a portfolio of f and any $N(L)$ mimicking portfolios. Second, since every MMV($N(L)$, $E(r)$) set includes the MVE set, if one cannot generate T as a portfolio of f and state variable mimicking portfolios, one cannot generate any MMV($N(L)$, $E(r)$) set using only f and any set of the S mimicking portfolios.

B. Expected Returns

The MMV($N(L)$, $E(r)$) set obtained by combining f, T, and the mimicking portfolios for the $N(L)$ state variables includes Markowitz's (1959) MVE set. The expected returns on assets can, thus, be expressed as (Huberman and Kandel (1987)),

(5) $$E(r_i) - r_f \quad = \quad \beta_{iT}[E(r_T) - r_f] + \sum_{s=1}^{N(L)} \beta_{is}[E(r_s) - r_f],$$

where the βs are the slopes from the multiple regression of the excess return on i on the excess returns on T and the mimicking portfolios for the $N(L)$ state variables. Equation (5), however, is no help in identifying the state variables of hedging concern. The problem is that there is a version of (5) for any $N(L)$, including $N(L) = 0$. The problem arises because the MVE tangency portfolio T is in every MMV($N(L)$,$E(r)$) set, and any MMV($N(L)$, $E(r)$) set is spanned by T, f, and the mimicking portfolios for the $N(L)$ state variables.

Suppose we replace T in the spanning set with a portfolio of f, T, and the $N(L)$ mimicking portfolios. Suppose this portfolio, call it e, has non-zero weights on T and all the $N(L)$ mimicking portfolios. Portfolio e is MMV($N(L)$, $E(r)$). Since e cannot be constructed as a portfolio of f, T, and a proper subset $N^*(L^*)$ of the $N(L)$ mimicking portfolios, e is not MMV($N^*(L^*)$, $E(r)$) for any subset of state variables strictly smaller than $N(L)$. Since e has a non-zero weight on T, however, portfolios of f, e, and the mimicking portfolios for the $N(L)$ state variables span the MVE set. Thus, e can replace T in (5),

(6) $$E(r_i) - r_f \quad = \quad \beta_{ie}[E(r_e) - r_f] + \sum_{s=1}^{N(L)} \beta_{is}[E(r_s) - r_f],$$

where the βs are the slopes from the multiple regression of the excess return on i on the excess returns on e and the mimicking portfolios for the $N(L)$ state variables.

Because e has non-zero weights on T and all the $N(L)$ mimicking portfolios, expressing T as a portfolio of f, e, and the $N(L)$ mimicking portfolios requires non-zero weights on e and all the $N(L)$ mimicking portfolios. This implies that spanning the MVE set with portfolios of f, e, and the state variable mimicking portfolios requires, f, e, and all the $N(L)$ mimicking portfolios. This, in turn, implies that $N(L)$ is the smallest set of state variables that can be used, along with e, to describe expected returns in the specific manner of (6). In other words, in contrast to (5), no subset of the premiums on the right-hand side of (6) can describe expected returns on all assets. This result is due to the fact that e is MMV$(N(L), E(r))$, but e is not MMV$(N^*(L^*), E(r))$ for any set of state variables that does not include all those in $N(L)$.

I argue next that market equilibrium in the ICAPM says that if $N(L)$ is the set of state variables of hedging concern, the market portfolio, M, must have the properties of e in (6). Substituting M for e in (6) then leads to an approach to identifying the state variables of hedging concern to investors.

C. Market Equilibrium

Suppose $N(L)$ is the set of state variables of hedging concern to ICAPM investors. Investor portfolios are MMV$(N(L), E(r)$, and they can be constructed as portfolios of the risk-free security, f, the MVE tangency portfolio T, and the mimicking portfolios for the $N(L)$ state variables. In a market equilibrium, the market portfolio is a portfolio of the portfolios chosen by investors. Thus, M is a portfolio of f, T, and the $N(L)$ mimicking portfolios. Since $N(L)$ is the union of the sets of state variables of hedging concern to different investors, the portfolios of individual ICAPM investors do not necessarily have non-zero weights on T and all the $N(L)$ mimicking portfolios. But the market portfolio almost surely does. This means M has the properties of e in (6); that is, in an ICAPM equilibrium, M is MMV$(N(L), E(r))$, but M is not MMV$(N^*(L^*), E(r))$ for any $N^*(L^*)$ set that does not include $N(L)$. We can, thus, substitute M for e in (6),

$$E(r_i) - r_f \quad = \quad \beta_{iM}[E(r_M) - r_f] + \sum_{s=1}^{N(L)} \beta_{is}[E(r_s) - r_f]. \tag{7}$$

Because M is not MMV$(N^*(L^*), E(r))$ for any $N^*(L^*)$ set that does not include the $N(L)$ state variables of hedging concern, one cannot recover the tangency portfolio T as a portfolio of f, M, and a proper subset of the $N(L)$ state variable mimicking portfolios. This implies that the $N(L)$ state variables of hedging concern are all priced in the sense that $E(r_s) - r_f$, $s = 1, \ldots, N(L)$, is the smallest set of mimicking portfolio premiums that can be used, along with the market premium, $E(r_M) - r_f$, to explain expected returns on all assets in the specific manner of (7). This is the definition (C3) of priced state variables in the Introduction.

III. Testing for Priced State Variables when the State Variables are Identified

Equation (7) is the basic tool for identifying priced state variables. We begin with general principles and then move on to the details of the approach.

A. General Principles

The task is to find the smallest set of state variables that, along with M and the risk-free security, describe expected returns on all assets in the manner of (7). We begin by arguing that any version of (7) that describes expected returns on all assets will not understate the number of priced state variables. But it is easy to overstate the number of priced state variables.

The ICAPM equilibrium condition says that if only $N(L)$ state variables are of hedging concern, the market portfolio is MMV$(N(L), E(r))$, but M is not MMV with respect to any set of state variables that does not include $N(L)$. This means expected returns on all assets cannot be described by any version of (7) that does not include the mimicking portfolios for all the $N(L)$ state variables. Thus, if we find a version of (7) that describes the expected returns on all assets, it will not understate the number of priced state variables.

It is, however, easy to overstate the number of priced state variables. If $N(L) < S$ state variables are of hedging concern, the market portfolio is MMV $(N(L), E(r))$. But M is also MMV with respect to any broader set of state variables that includes $N(L)$. Thus, any version of (7) that includes the premiums for a broader set of mimicking portfolios also describes expected returns. For example, the S state variable expression,

$$E(r_i) - r_f = \beta_{iM}[E(r_M) - r_f] + \sum_{s=1}^{S} \beta_{is}[E(r_s) - r_f], \tag{8}$$

holds irrespective of the number of priced state variables.

In short, to identify priced state variables, we must work from smaller to larger sets of state variable mimicking portfolios until we find the smallest set that, along with the market portfolio and the risk-free security, describes expected returns on all assets in the manner of (7).

B. Narrowing the Search

It is not necessary to test alternative ICAPMs on all assets. I argue next that, to identify priced state variables, it suffices to find the smallest $N(L)$ set of state variable mimicking portfolios that, along with M and f, explain the expected returns on other state variable mimicking portfolios in the manner of (7). (Since M contains all assets, however, the tests clearly require all assets.)

Suppose $N(L)$ is the set of state variables of hedging concern, so expected returns are given by (7). Since (7) holds for all assets, it explains the expected returns on state variable mimicking portfolios not in $N(L)$. Thus, as for any asset, the intercept in the regression,

$$(9) \qquad r_k - r_f = \alpha_k + \beta_{kM}(r_M - r_f) + \sum_{s=1}^{N(L)} \beta_{ks}(r_s - r_f) + \epsilon_k,$$

is zero for any mimicking portfolio k not in $N(L)$. Intuitively, the intercept α_k in (9) is the information about expected returns in the premium for state variable k, $E(r_k) - r_f$, that is not captured by the market premium, $E(r_M) - r_f$, and the premiums for the state variables in $N(L)$, $E(r_s) - r_f$, $s = 1, \ldots, N(L)$. If $N(L)$ is the set of state variables of hedging concern, state variables that are not in $N(L)$ have no such additional information about expected returns, and their intercepts in (9) are zero.

Conversely, suppose q is a state variable in the $N(L)$ set of hedging concern. Consider the regression of the excess return on q on the excess market return and the excess returns on the other state variables in $N(L)$,

$$(10) \qquad r_q - r_f = \alpha_q + \beta_{qM}(r_M - r_f) + \sum_{s=1}^{N(L)-1} \beta_{qs}(r_s - r_f) + \epsilon_q.$$

Given that the $N(L)$ state variables are of hedging concern, α_q in (10) cannot be zero; that is, the premium (expected excess return) on the mimicking portfolio for state variable q cannot be explained by the market premium and the premiums for the other state variables in $N(L)$. To establish this result, let us replace $r_q - r_f$ on the right-hand side of (9) with $\alpha_q + \epsilon_q$ from (10),

$$(11) \quad r_k - r_f = \alpha_k + \beta_{kM}(r_M - r_f) + \sum_{s=1}^{N(L)-1} \beta_{ks}(r_s - r_f) + \beta_{kq}(\alpha_q + \epsilon_q) + \epsilon_k.$$

The intercepts in (11) are the same as those in (9). If $N(L)$ is the set of state variables of hedging concern, then, as in (9), the intercepts α_k in (11) are zero for all assets. Moreover, since $\alpha_q + \epsilon_q$ is uncorrelated with the other explanatory variables in (11), the regression slopes in

$$(12) \qquad r_k - r_f = \alpha_k + \beta_{kM}(r_M - r_f) + \sum_{s=1}^{N(L)-1} \beta_{ks}(r_s - r_f) + \epsilon_k$$

are the same as those for the corresponding explanatory returns in (11). If α_q in (10) is zero for a state variable q in $N(L)$, however, the expected value of $\alpha_q + \epsilon_q$ in (11) is zero. The intercepts in (12) are then the same as those in (11), which are the same as those in (9). If the intercepts in (9) are zero for all assets, the intercepts in (12) are zero for all assets, and expected returns are

$$(13) \qquad E(r_k) - r_f = \beta_{kM}[E(r_M) - r_f] + \sum_{s=1}^{N(L)-1} \beta_{ks}[E(r_s) - r_f],$$

where β_{kM} and the β_{ks} are the regression slopes from (12). But (13) contradicts the proposition that $N(L)$ is the set of state variables of hedging concern to investors, which implies that the premiums for M and all the $N(L)$ mimicking portfolios are necessary to explain expected returns. Thus, if $N(L)$ is the set of state variables of hedging concern, the intercept α_q in (10) must be non-zero for any state variable q in $N(L)$.

Ignoring estimation issues, the prescription for finding the state variables of hedging concern is now clear. We must find the smallest $N(L)$ set of state variables that produce zero intercepts in (9) for all state variables that are not in $N(L)$. The state variables that are not in $N(L)$ are unpriced in the sense that the premiums (expected excess returns) for their mimicking portfolios are explained by the market premium and the premiums on the mimicking portfolios for the state variables in $N(L)$. Every state variable in $N(L)$ is priced in the sense that the premium for its mimicking portfolio cannot be explained by the market premium and the premiums on the mimicking portfolios for the other state variables in $N(L)$. These are the conditions on priced state variables stated in (C4) in the Introduction.

Note that the approach to identifying priced state variables centers on the intercepts in (9) and (10), not on the expected excess returns on the explanatory portfolios. It is common knowledge that the expected excess returns on the explanatory portfolios are not informative about whether a state variable is priced. The problem is that "rotating" (taking linear combinations of) the explanatory portfolios produces new explanatory portfolios with different expected excess returns but with the same power to explain the expected returns on assets (Roll and Ross (1980), Chen (1983)). The intercepts in (9) and (10), are, however, unaffected by rotations of the explanatory returns.

The path to finding the set of priced state variables is quite narrow. We must find the smallest set of state variables that, along with the market portfolio, explain expected returns in the specific manner of (7). The path is, however, strewn with ICAPM intuition. To identify priced state variables, we use the ICAPM equilibrium condition; that is, if $N(L)$ is the set of state variables of hedging concern, then assets must be priced so that the market portfolio is MMV($N(L)$, $E(r)$), and M is not MMV with respect to any set of state variables that does not include all those in $N(L)$. This implies that M and the $N(L)$ state variable mimicking portfolios are all necessary to explain expected returns in the manner of (7). And (7) clearly captures the economics of ICAPM pricing in a world where $N(L)$ state variables are of hedging concern to investors.

Finally, we can now appreciate the power of defining the mimicking portfolio for state variable s without constraints on its loadings on other state variables. Each of these state variable mimicking portfolios is MMV($1(s)$,) for a different $s = 1, \ldots, S$. This means each is in any MMV($N(L)$, $E(r)$) set where $N(L)$ includes state variable s. And this is why these mimicking portfolios can be used, along with M and f, to determine $N(L)$ the number, and names of priced state variables. Moreover, the procedure works because no set of mimicking portfolios can be used, along with f, to span any MMV($N(L)$, $E(r)$) set, so f and mimicking portfolios cannot be used to recover MVE portfolios, including the MVE tangency portfolio T.

In contrast, another definition of a mimicking portfolio for state variable s is that it minimizes variance given its loading on s, but subject to being uncorrelated with (orthogonal to) other state variables. These mimicking portfolios are $\text{MMV}(S,)$, but because the orthogonality condition constrains the loadings on all state variables, they are not $\text{MMV}(N(L),)$ for any $N(L) < S$. This means that no proper subset of these orthogonalized mimicking portfolios can be used, along with M and f, to generate $\text{MMV}(N(L), E(r))$ portfolios and describe expected returns, and the orthogonalized mimicking portfolios cannot be used, along with M and f, to determine the set of priced state variables.

IV. S is Known, but the State Variables are Not Identified

Tests for ICAPM pricing often do not identify the S state variables potentially of hedging concern to investors. For example, Fama and French (FF hereafter (1993), (1996)) use a market proxy and two diversified portfolios formed on the basis of firm size (market capitalization) and book-to-market equity (BE/ME) to explain returns. There is, however, no presumption that size and BE/ME are state variables. Rather the idea (based on work like that in Fama and French (1992)) is that there seem to be two dimensions of expected return that are missed by the market βs of the CAPM and are exposed by sorting firms on size and BE/ME. One can then interpret the three-factor model in FF (1993), (1996) as proposing that i) expected returns are governed by a two-state-variable ICAPM, so the risk-free security, the market portfolio, and two $\text{MMV}(2(L), E(r))$ portfolios span $\text{MMV}(2(L), E(r))$ portfolios and describe expected returns, and ii) the two FF portfolios formed on size and BE/ME are $\text{MMV}(2(L), E(r))$.

In the spirit of this line of work, consider the following problem. The ICAPM governs asset prices, and the number of state variables that might be of hedging concern, S, is known. The state variables are not named, however, and we do not have state variable mimicking portfolios. Instead, we are given S $\text{MMV}(S, E(r))$ portfolios that, with M and f, span $\text{MMV}(S, E(r))$ portfolios and describe expected returns as

$$E(r_i) - r_f \quad = \quad \beta_{iM}[E(r_M) - r_f] + \sum_{n=1}^{S} \beta_{in}[E(r_n) - r_f], \tag{14}$$

where r_n, $n = 1, \ldots, S$, are the returns on the S additional $\text{MMV}(S, E(r))$ portfolios, and β_{iM} and $\beta_{in}, n = 1, \ldots, S$, are the slopes from the multiple regression of $r_i - r_f$ on $r_M - r_f$ and $r_n - r_f$, $n = 1, \ldots, S$. The question then is: without naming the state variables, can we determine the number of them that are priced? The answer, unfortunately, is no, except when the number of priced state variables is zero and the ICAPM collapses to the CAPM.

If no state variables are priced, expected returns are given by the CAPM,

$$E(r_i) - r_f \quad = \quad \beta_{iM}[E(r_M) - r_f], \tag{15}$$

where β_{iM} is slope in the regression of $r_i - r_f$ on $r_M - r_f$. And the intercept α_i in the regression,

$$r_i - r_f \quad = \quad \alpha_i + \beta_{iM}(r_M - r_f) + \epsilon_i, \tag{16}$$

is zero for all assets i. If we know that the S state variable ICAPM expected return expression (14) also holds, however, we do not have to test the CAPM expression (15) on all assets. A simple extension of the argument in Section III implies that if the intercepts in (16) are zero for the S MMV$(S, E(r))$ portfolios in (14), then the description of expected returns in (14) collapses to (15). We are in the world of the CAPM.

Suppose the CAPM is rejected. What do we know about an ICAPM market equilibrium that might help identify priced state variables? In an ICAPM equilibrium where $N(L)$ state variables are of hedging concern, the market portfolio M is MMV$(N(L), E(r))$. Thus, the risk-free security, M, and $N(L)$ additional MMV$(N(L), E(r))$ portfolios span the MMV$(N(L), E(r))$ portfolios of investors. And expected returns are

$$E(r_i) - r_f \quad = \quad \beta_{iM}[E(r_M) - r_f] + \sum_{n=1}^{N(L)} \beta_{in}[E(r_n) - r_f], \tag{17}$$

where r_n, $n = 1, \ldots, N(L)$ are the returns on the $N(L)$ additional MMV$(N(L), E(r))$ portfolios, and β_{iM} and β_{in} are the slopes from the multiple regression of $r_i - r_f$ on $r_M - r_f$ and $r_n - r_f$, $n = 1, \ldots, N(L)$. Moreover, if $N(L)$ is the set of state variables of hedging concern, the market portfolio is MMV$(N(L), E(r))$, but M is not MMV for any set of state variables that does not include $N(L)$. Thus, identifying $N(L)$ again involves finding the least constrained set of MMV$(N(L), E(r))$ portfolios that includes M.

Can (17) and the methods of Section III be used to identify $N(L)$? To work with (17), we must find $N(L)$ MMV$(N(L), E(r))$ portfolios that can be used along with the market portfolio to describe expected returns. Moreover, to avoid understating the number of priced state variables, we must be sure that the risk-free security and the additional $N(L)$MMV$(N(L), E(r))$ do not, by chance, span the MVE tangency portfolio T. Finally, we must identify MMV$(N(L), E(r))$ portfolios with these properties for all combinations of N and the list of state variables in L. How would we do all this without identifying the state variables?

It is tempting to proceed as in Section III, but using M and the S additional MMV$(S, E(r))$ portfolios in our possession, rather than M and the S state variable mimicking portfolios (which we no longer have), to search for the number of priced state variables. For example, to test for one state variable ICAPM pricing, we might examine whether the premiums for M and one of the S additional MMV$(S, E(r))$ portfolios describe expected returns in the manner of (17). But this approach works only if the S MMV$(S, E(r))$ portfolios are also MMV$(1(s), E(r))$, and each for a different state variable s. In other words, without knowing the state variables, we have somehow stumbled on mimicking portfolios for them. This, to say the least, is unlikely.

We may luck out and find that, using the methods of Section III applied to (17), the expected premiums for M and $N < S$ of the additional MMV$(S, E(r))$ portfolios in our possession describe expected returns. Such a result, however, would not allow us to conclude that at least N of the state variables carry special premiums. The problem, again, is that if N of the S MMV$(S, E(r))$ portfolios are also MMV$(N(L), E(r))$ for an $N < S$, this does not mean they are MMV$(N^*(L^*)$,

$E(r)$) for any smaller $N^*(L^*)$ set. Thus, they may not allow us to test whether a smaller $N^*(L^*)$ set of state variables is of hedging concern. Conversely, finding that the premiums for M and $N < S$ of the additional MMV$(S, E(r))$ portfolios describe expected returns also does not allow us to conclude that the number of priced state variables is at most N. We may simply have stumbled on a set of N portfolios that, along with the risk-free security, span the MVE tangency portfolio T.

In general, if we know S (the total number of state variables potentially of hedging concern) but the state variables are not identified (named), then ICAPM tests that produce a full description of expected returns leave us in the dark with respect to the number of priced state variables, unless the ICAPM collapses to the CAPM.

Here is some concrete perspective: suppose, for the sake of argument, the FF (1993) three-factor model describes expected returns; that is, the expected premiums on M and the two FF diversified portfolios formed on size and BE/ME describe expected returns on all assets. FF (1996) show that the market premium cannot explain the average premiums for the two additional portfolios, so the ICAPM does not collapse to the CAPM. It is then tempting to conclude that a two-state variable ICAPM explains expected returns, and that M and the two additional FF explanatory portfolios are MMV$(2(L), E(r))$. But it is possible that only one state variable is priced. The two additional explanatory portfolios in FF (1993), (1996) are formed to produce a large spread in average returns, not mimic state variables. Each probably involves a combination of the underlying state variable mimicking portfolios. This means they are MMV$(2(L), E(r))$, but they are not MMV$(1(s), E(r))$. As a result, they cannot produce tests of a one-state variable ICAPM. Conversely, it is also possible that FF have simply found two portfolios that, along with f, span the MVE tangency portfolio T, but the number of priced state variables is, in fact, greater than two.

Finally, the world becomes even murkier if we admit implementation problems. For example, one never has the true market portfolio in tests of the CAPM and the ICAPM. Thus, when we find that a multifactor model describes expected returns, we may simply have found a set of portfolios that spans the MVE tangency portfolio and so produces a parsimonious description of expected returns (FF (1996)). Without naming the state variables that might be priced, we are in the dark about the number of them of hedging concern. And as long as we do not use the true M as an explanatory portfolio, we cannot rule out the possibility that the number of priced state variables is zero.

V. Conclusions

Suppose we know that i) the ICAPM governs asset prices, ii) S is the total number of state variables potentially of hedging concern to investors, and iii) the state variables are named. Then, ignoring implementation problems, we can identify the $N(L)$ state variables that are, in fact, of hedging concern and so produce special premiums in expected returns.

Identifying the $N(L)$ priced state variables involves combining the properties of the MMV$(N(L), E(r))$ portfolios relevant for investors, with the ICAPM pricing condition that the market portfolio must be MMV$(N(L), E(r))$. If investors are concerned with hedging uncertainty about $N(L)$ state variables, their portfolios are MMV$(N(L), E(r))$. These portfolios can be obtained by combining mimicking portfolios for the $N(L)$ state variables with the risk-free security and any other MMV$(N(L), E(r))$ portfolio not itself a combination of f and the $N(L)$ mimicking portfolios. An ICAPM equilibrium requires that the market portfolio is such a portfolio, so M can be used with f and the $N(L)$ mimicking portfolios to span MMV$(N(L), E(r))$ portfolios. This spanning result implies the ICAPM expected return equation (7), which is the tool used to identify priced state variables. The approach works because i) assets are priced so that M is MMV$(N(L), E(r))$, but M is not MMV$(N^*(L^*), E(r))$ for any $N^*(L^*)$ that does not include $N(L)$, and ii) f and the state variable mimicking portfolios do not span the MVE set, so they cannot be used to recover the MVE tangency portfolio T.

If we know the number of state variables but not their names, identifying the number of them that are priced is probably impossible. The problem, in simplest terms, is that, to identify N, the number of priced state variables, we must be able to place the portfolios we use to describe expected returns in specific MMV$(N(L), E(r))$ sets. This is equivalent to requiring that we know the names, L, as well as the number of state variables in different MMV sets. Thus, identifying N basically requires that we identify $N(L)$, which is impossible if the state variables are not named.

Ignoring implementation problems, we can determine when the ICAPM collapses to the CAPM without naming ICAPM state variables. The reason is that, in the CAPM, no state variables are priced so their identities are irrelevant. If, however, we allow for an implementation problem that plagues all CAPM and ICAPM tests, specifically, that the tests never include the true market portfolio M, then unambiguously distinguishing the CAPM from its more complicated ICAPM brothers becomes impossible. The problem is that the implication of the CAPM and the ICAPM that drives tests of the models is that M is in the MMV$(N(L), E(r))$ set that constrains loadings on the state variables of hedging concern. If we do not include the true M as an explanatory portfolio in asset pricing tests, we are not testing the CAPM or the ICAPM. What we are testing is the ad hoc hypothesis that a specific set of explanatory portfolios spans MVE portfolios. An answer to this question says nothing about priced state variables, and $N(L) = 0$ (the CAPM) is not ruled out.

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