# Basics of Theory of Dynamic systems

**What is a system:** An object, device or Phenomenon which is synthetized as an aggregation or interConnection of Parts.

**Automation Engineering Goal**

How does a system evolve?

. Study how a sequence of events happens in an autonomous way (without human intervention)

↳ Evolve : a certain number of internal attributes of the system varies over time (time and space)

**Automation Engineering is Composed of**

. Analysis : Predict output of system knowing its inputs (System's theory)

$$u \longrightarrow \boxed{S} \longrightarrow y$$

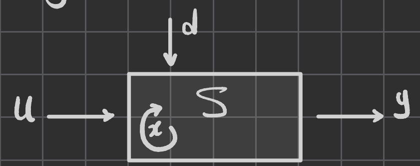
. Synthesis : Obtaining a desired outputs from a system by feeding in the right inputs (automatic Control)

$$u \longrightarrow \boxed{S} \longrightarrow y$$

. Identification : Mathematically define a system knowing only inputs and outputs.

$$u \longrightarrow \boxed{S} \longrightarrow y$$

**Dynamical System** An oriented system which evolves over time.




. oriented means Producing an "effect" given a "Cause"

. the State 'x' internal variables

. $u$ : input 'independent variables'

. $y$ : output 'dependent variables'

. $d$ : disturbance 'external variables' Not a standard input for the system

**Dynamic model** State space representation :

$\dot{x} = f(x, u) \qquad y = g(x, u)$

$\dot{x} = Ax + Bu \qquad y = Cx + Du$ : linear time-invariant

. Free evolution (zero-input response) : $\dot{x} = Ax$ & $y = Cx$

. Forced evolution (zero-state response) : $\dot{x} = Bu$ & $y = Du$

**Problem Setting** $\dot{x} = Ax + Bu \quad , \quad y = Cx + Du$

. linear

. Time-invariant — Dynamic matrices do not depend on time!

. Continous time


$F = m\dot{v}$, if no force $\rightarrow$ $0 = m\dot{v}$ $\therefore$ $\dot{v} = 0 \Rightarrow v_t = v_0$ initial velocity
↳ Newton's 1st law

$$\dot{v} = a = \frac{F}{m} \xrightarrow{\int} v(t) = v_0 + \frac{1}{m}\int_0^t F(\tau)\,d\tau$$

double the force: $v(t) = \frac{2}{m}\int_0^t F(\tau)\,d\tau$

linear combination of causes implies linear combination of effects.

## Stability of a linear System

Study the behavior of the system under perturbations of the initial state.

Stable $\rightarrow$ Small perturbations cause small variations.
Unstable $\rightarrow$ Small perturbations cause big variations.

Equilibria: a state is an equilibrium if it causes a degenerate trajectory.

$$x_e \in \mathbb{R}^N : x_0 = x_e \Rightarrow x_t = x_e, \forall t > 0$$
$$x_e : Ax_e = 0$$

A system in equilibrium doesn't evolve $\rightarrow$ but it doesn't mean it's stable!

An equilibrium point is stable if: $|x_0 - x_e| < \delta \Rightarrow |x(t) - x_e| < \epsilon, \forall t > 0$

If the system starts close enough to the equilibrium
↳ the evolution stays close enough to it

Asymptotic stability: An equilibrium is asymptotically stable if it is stable and: $|x_0 - x_e| < \delta_a \Rightarrow \lim_{t \to +\infty} |x(t) - x_e| = 0$

if the system starts close enough to the equilibrium
it converges back to it eventually

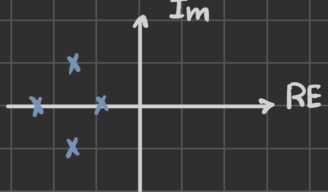
- Pendulum is example of Asymptotic stability
- local concept defined by the basin of attraction '$\delta_a$'

## Stability and eigenvalues

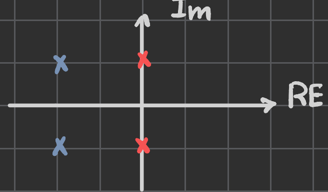
- A system is asymptotically stable if and only if all eigenvalues have strictly negative real part. $Re\{\lambda_i\} < 0, \forall \lambda_i : P_A(\lambda_i) = 0$

- A system is (marginally) stable if and only if all eigenvalues have non positive real part. $Re\{\lambda_i\} \leq 0, \forall \lambda_i : P_A(\lambda_i) = 0$

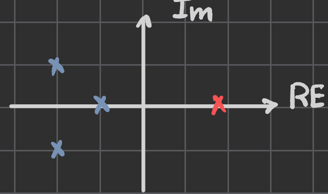
- A system is unstable if and only if there is at least one eigenvalue with positive real part. $\exists \lambda_i : P_A(\lambda_i) = 0 \wedge Re\{\lambda\} > 0$




asymptotic stable




marginal stable




Instability

# Solution of linear System

$\Phi(t)$ $H(t-\tau)$

- $x(t) = e^{At} x_0 + \int_0^t e^{A(t-\tau)} B u(\tau)\, d\tau$
- $y(t) = C e^{At} x_0 + {}_0\int^t [C e^{A(t-\tau)} B + D\delta(t-\tau)]\, u(\tau)\, d\tau$

$\Psi(t)$ $W(t-\tau)$

- $x(t) = \Phi(t)\, x_0 + {}_0\int^t H(t-\tau)\, u(\tau)\, d\tau$
- $y(t) = \Psi(t)\, x_0 + {}_0\int^t W(t-\tau)\, u(\tau)\, d\tau$

$\Rightarrow$ Integrals are hard to solve ↙
Laplace Transform
$F(s) = {}_0\int^\infty f(t)\, e^{-st}\, dt$

## Laplace Transform Properties

- linearity: $\mathcal{L}[\alpha f(t) + \beta g(t)] = \alpha \mathcal{L}[f(t)] + \beta \mathcal{L}[g(t)]$
- Derivative: $\mathcal{L}[\frac{df(t)}{dt}] = s\,\mathcal{L}[f(t)] - f(0)$
- Convolution: $\mathcal{L}[{}_0\int^t f(t-\tau)\, g(\tau)\, d\tau] = F(s)\, G(s)$

## Frequency Domain

- $\dot{x} = Ax + Bu \longrightarrow s\,X(s) - x_0 = A\,X(s) + B\,U(s)$

$$X(s) = (sI - A)^{-1} x_0 + (sI - A)^{-1} B\, U(s)$$

- $Y(s) = C(sI - A)^{-1} x_0 + [C(sI - A)^{-1} B + D]\, U(s)$
- $W(s) = C(sI - A)^{-1} B + D$ : describes the input-output behavior. "Transfer function"

## Transfer Function

- $W(s) = \frac{N(s)}{D(s)}$ Poles are eigenvalues.
  Not all eigenvalues are Poles.
- An eigenvalue appears as a root (Pole) of the transfer function if :

  - $\text{rank}(A - \lambda_i I \mid B) = n$ Controllable
  - $\text{rank}\begin{pmatrix} A - \lambda_i I \\ C \end{pmatrix} = n$ Observable

  and