# Basics of Theory of Dynamic systems

**What is a system** An object, device, or Phenomenon which is synthetized as an aggregation or interconnection of Parts.

**Automation Engineering - Goal** How does a system evolve? ⇒ Study how Sequence of events happens in autonomous way (without human intervention)

evolve: A certain number of internal attributes of the system varies over time (and space)

Automation Engineering is Composed of: analysis, Synthesis and identification.

analysis: Predict the output of a system knowing it's inputs. (system's theory) $u \to \boxed{S} \to y$

Synthesis: obtaining desired outputs from a system by feeding it the right inputs. (automatic control) $u \to \boxed{S} \to y$

Identification: Mathematically define a system knowing only inputs and outputs. (system identification) $u \to \boxed{S} \to y$

Control Engineering is analysis + synthesis

**Dynamical Systems** An oriented system which evolves over time.

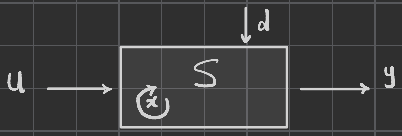
Oriented means Producing an "effect" given a "Cause"

$x$ : the state "internal variables"

$u$ : the input (independent variable) the "Causes"

$y$ : the output (dependent variable) the "effects"

$d$ : the disturbance (external variable)




**Dynamic model**

- $\dot{x} = f(x,u)$ & $y = g(x,u)$ : State Space representation
- $\dot{x} = Ax + Bu$ & $y = Cx + Du$ : linear time-invariant state-space representation
- $\dot{x} = Ax + \cancel{Bu}$ & $y = Cx + \cancel{Du}$ : Zero-input response (free evolution)
- $\dot{x} = \cancel{Ax} + Bu$ & $y = \cancel{Cx} + Du$ : Zero-state response (forced evolution)

**Problem Setting** $\dot{x} = Ax + Bu$ & $y = Cx + Du$

- linear
- Time-invariant: Dynamic matrices do not depend on time!
- Continuous time
- linear Combination of causes implies linear Combination of effects.

**Stability of a linear System**

- $\dot{x} = Ax$ , $x(0) = 0$
- Consider a system in free evolution with an initial state value.
- what is the Concept of "stability" ⇒ Study the behavior of the system under the Perturbations of the initial state.
- Stable : Small Perturbations Cause Small variations
- Unstable: Small Perturbations Cause big variations

**Stability - Equilibria (linear)**

- $\dot{x} = f(x)$ , A state is an equilibrium if it causes a degenerate trajectory.

$x_e \in \mathbb{R}^N$ : $x_0 = x_e \to x(t) = x_e$ , $\forall t > 0 \Rightarrow x_e : Ax_e = 0$

A system in equilibrium does NOT evolve ! but it doesn't mean it's stable !

**Stability**

- An equilibrium Point is stable if: $\forall \epsilon, \exists \delta(\epsilon) : |x_0 - x_e| < \delta \to |x(t) - x_e| < \epsilon, \forall t > 0$

if the system starts "close enough" to the equilibrium , the evolution stays "close enough" to it !

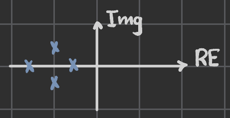
- An equilibrium Point is unstable if it is not stable. (Condition above doesn't hold)
- An equilibrium is asymptotically if it is stable and : $\exists \delta_a : |x_0 - x_e| < \delta_a \to \lim_{t \to +\infty} |x(t) - x_e| = 0$,

if the system starts "close enough" to the equilibrium it converges back to it (eventually) !

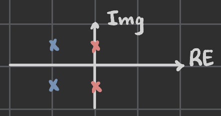
- Asymptotic stability is a local Concept defined by the basin of attraction $\delta_a$.

## Stability and Eigenvalues

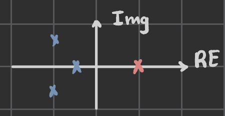
- A system is asymptotically stable if and only if all eigenvalues have strictly negative real Part.
- A system is marginally stable if and only if all eigenvalues have non Positive real Part.
- A system is unstable if and only if there is at least one eigenvalue with Positive real Part.




asymptotic stable




marginal stable




instability

## Solution of linear dynamical System

- $x(t) = \underbrace{e^{At}}_{\Phi(t)} x_0 + \int_0^t \underbrace{e^{A(t-\tau)} B}_{H(t-\tau)} u(\tau)\, d\tau$
- $y(t) = \underbrace{C e^{At}}_{\Psi(t)} x_0 + \int_0^t \left[ \underbrace{C e^{A(t-\tau)} B + D\, \delta(t-\tau)}_{W(t-\tau)} \right] u(\tau)\, d\tau$

## Laplace Transform

- From a real-valued function to a Complex-valued one.

  $f(t), t \in \mathbb{R} \xrightarrow{\mathcal{L}} F(s), \; s \in \mathbb{C}$

  where : $F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

## Transfer Function

- $W(s) = C(sI - A)^{-1} B + D$ : it describes the input-output behavior !
- $W(s) = \frac{N(s)}{D(s)}$ : $D(s) \longrightarrow$ Poles, Poles are eigenvalues, not all eigenvalues are poles!

# Feedback Control Systems

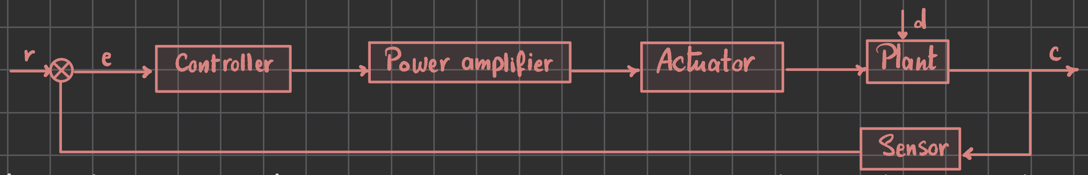
## Chapter1: Introduction and linearized models Notes

One classification of Control Systems is the following:

- Process Control or regulator systems: The Controlled variable, output, must be held as close as Possible to a usually Constant desired value, or input, despite any disturbances.
- Servomechanisms: The input varies and the output must be made to follow it as closely as Possible.

Among many other possible classifications of Control systems is the distinction between Continous and discrete systems.
The classification into linear and nonlinear Control system is also important. Yet most systems become nonlinear if the variables move over wide enough ranges. The importance in practice of linear techniques relies on linearization based on the assumption that the Variables stay close enough to a given operating Point.

Closed loop Control or feedback Control: . The output c is measured Continuously and fed back to be Compared with the input r
. The error $e = r - c$ is used to adjust the Control valve by means of an actuator.




if The Control System is any good the error e will usually be Small, ideally Zero. Therefore, it is quite inadequate to operate an actuator. A task of Controller is to amplify the error Signal. The Controller output, however, will still be at a low Power level. The Power amplifier raises Power to the levels needed for the actuator.

Control system analysis and design: Can be summarized in terms of the following two questions:

1. analysis: What is the Performance of a given system in response to changes of inputs or disturbances?
2. design: if the Performance is unsatisfactory, how it Can be improved without changing the Process, actuator, and Power amplifier blocks?

Motivations for feedback Can be listed as follows:

- reducing the effects of Parameter variations.
- reducing the effects of disturbance inputs.
- improving transient response characteristics.
- reducing steady-state errors.

Stability is always the Primary Concern in feedback Control design. But to be useful a system must also possess adequate relative stability; that is the overshoot of a step response must be acceptably Small, and this response must not be unduly oscillatory during the transient period. Relative stability Considerations usually impose an upper limit on gain, and hence on accuracy and speed of response. Much of Control system design Can be Summarized as being Concerned with achieving a satisfactory Compromise between these features.

Laplace transform: $F(s) = L[f(t)] = \int_0^\infty f(t)e^{-st}dt$, advantage will be found to be that differentiation and integration are changed in algebraic operations.

Transfer Function of a (sub)system is the ratio of the Laplace transforms of its output & input, assuming Zero initial Conditions: $G(s) = \frac{C(s)}{R(s)}$

# Chapter 11: State-Space Analysis

A state-space model is a description in terms of a set of first-order differential equations that are written compactly in a standard matrix form. State models should be directly derived from the original system equations.

The standard form of a state-space model is as follows:

- state equation: $\dot{x} = Ax + Bu$, state variables should be chosen that are measurable and physically meaningful
- output equation: $y = Cx + Du$

## Transfer function matrices and stability:

- $\mathcal{L}[\dot{x} = Ax + Bu] = sX(s) - x_0 = AX(s) + BU(s)$ or $(sI - A)X(s) = BU(s) + x_0$
- $\mathcal{L}[y = Cx] = Y(s) = CX(s)$
- state response to initial conditions alone: $X(s) = (sI - A)^{-1}x_0$
- output response to both initial conditions and input are: $Y(s) = G(s)U(s) + C(sI - A)^{-1}x_0$

where $G(s) = C(sI - A)^{-1}B$ : transfer function matrix because it relates the transform of the input and output vectors for zero initial conditions.

$sX(s) = AX(s) + BU(s) \longrightarrow (sI - A)X(s) = BU(s)$

$X(s) = (sI - A)^{-1}BU(s) \xrightarrow{\text{sub. in } Y} Y(s) = C(sI - A)^{-1}BU(s)$ — zero IC

in General: $Y(s) = [C(sI - A)^{-1}B + D]U(s)$

## Formal Solution: The Transition Matrix

The solution of the scalar equation $\dot{x} = ax + bu$, $x(0) = x_0$ is known to be: $e^{at}x_0 + \int_0^t e^{a(t-\tau)}b\,u(\tau)\,d\tau$

$\dot{x} = Ax$ "Homogenous equation" : $x(t) = e^{At}x_0$

$\dot{x} = Ax + Bu$ : $x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$

matrix exponential:

$$e^{At} = I + At + \frac{1}{2!}A^2t^2 + \frac{1}{3!}A^3t^3 + \dots = \sum_{k=0}^{\infty}\frac{A^kt^k}{k!}$$

$$\frac{d}{dt}e^{At} = \frac{d}{dt}(\quad) = A + \frac{2}{2!}A^2t + \frac{3}{3!}A^3t^2 + \dots = A(I + At + \frac{1}{2!}A^2t^2 + \dots) = Ae^{At}$$

$$\dot{x} = \frac{d}{dt}x(t) = \frac{d}{dt}e^{At}x_0 = Ae^{At}x_0 = Ax \Rightarrow \phi(t) = e^{At}$$ : transition matrix relates the state at $t$ to that at time zero.

### Relate to the lecture Proof

$\dot{x} = Ax + Bu$

$\dot{x} - Ax = Bu$

$e^{-At}\dot{x} - e^{-At}Ax = e^{-At}Bu$ $\qquad \left| \frac{d}{dt}(e^{-At}x) = -Ae^{-At}x + \dot{x}e^{-At} \right.$

$d/dt(e^{-At}x) = e^{-At}Bu$

$e^{-At}x\Big|_0^t = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$

$e^{-At}x(t) - e^{-A0}x_0 = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau \longrightarrow e^{-At}x(t) = x_0 + \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$

$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau$

$y(t) = Cx(t) + Du(t) = C[e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau] + Du(t)$

$= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$ $\qquad \left| Du(t) = \int_0^t D\delta(t-\tau)u(\tau)\,d\tau \right.$

$= Ce^{At}x_0 + \int_0^t [Ce^{A(t-\tau)}B + D\delta(t-\tau)]u(\tau)\,d\tau$