Internship Notes

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1 Wave-function fractality

After seeing the fractality of the energy spectrum we now tried to study the fractality of the wave-function of the ground-state close to criticality (in the scenario $V = V_1 = V_2$). We already know that for $V \ll V_c$ the wave-function of the ground-state will be delocalized while for $V \gg V_c$ the wave function is exponentially localized. Assuming the transition between the two is smooth we expect to observe a wave-function that is neither localized nor delocalized, which we imagined to be fractal, close to criticality i.e. for $V \approx V_c$. Our first way of studying this fractality was to compute $\tau_q/(q-1)$ since it is an indicator of the fractal dimension of the wave-function. We know from theory that $IPR_0^{(q)} \sim L^{-\tau_q}$, so in order to compute τ_q we fixed V and then varied L from 50 to 10 000. We then performed a fit on 5 different intervals. The first interval we took is the whole data-set, the second is the last 80% the third is the last 60%, and so on until the fifth which is the last 20% of the data-set.

2 Envelope removal

We know from theoretical results in simpler models (i.e. AA limit) that the wave-function will be contained within an envelope $\sin(\pi x/L)$. We then write:

$$\psi(x) = \sin(\pi x/L)\phi(x)$$

Where ψ is the wavefunction obtained numerically and ϕ is the carrier wave. Then we get that:

$$\mathrm{IPR}_\psi^{(q)} = \frac{\int_0^L \left(\sin(\pi x/L)\phi(x)\right)^{2q} dx}{\left(\int_0^L \left(\sin(\pi x/L)\phi(x)\right)^2\right)^q}$$

Now using the fact that the frequency of the sinus is negligible with respect to the frequency of the carrier wave we get that:

$$\mathrm{IPR}_\psi^{(q)} = \frac{\frac{1}{L}\int_0^L \sin(\pi x/L)^{2q} dx \int_0^L \phi(x)^{2q} dx}{\left(\frac{1}{L}\int_0^L \sin^2(\pi x/L) dx \int_0^L \phi(x)^2 dx\right)^q}$$

Now assuming that $q > -1/2$ we also have that:

$$\int_0^L \sin(\pi x/L)^{2q} dx = L\frac{\Gamma(1/2+q)}{\sqrt{\pi}\Gamma(1+q)}$$

And it is trivial to compute that:

$$\left(\int_0^L \sin(\pi x/L)^2 dx\right)^q = 2^{-q}L^q$$

Therefore we get that:

$$\mathrm{IPR}_\psi^{(q)} = \frac{2^q\Gamma(\frac{1}{2}+q)}{\sqrt{\pi}\Gamma(1+q)} \cdot \frac{\int_0^L \phi(x)^{2q} dx}{\left(\int_0^L \phi(x)^2 dx\right)^q}$$

In conclusion, we notice from the equation above that:

$$\mathrm{IPR}_\psi^{(q)} \underset{L}{\propto} \mathrm{IPR}_\phi^{(q)}$$

In consequence we see that the finite size effect is not pathological when we look at the $\mathrm{IPR}^{(q)}$. On the other hand the envelope might conceal some features of the wave-function. Therefore, in the following whenever we consider or plot a wave-function we will first divide by $\sin\left(\frac{\pi x}{L}\right)$ to get clearer results.

3 τ_q study

The result from the numerical computations are shown in figure 2. From these wee see that for increasing values of L the transition of τ_q seems to converge to a discontinuous step function, nonetheless we observe a fixed point at $V = V_c$ invariant by any change of range of L (see fig. 3), so we can hypothesize that even though the transition will be discontinuous there will still exist a fractal wave-function at exactly $V = V_c$. The finite size of our system allows us to observe this fractality even at values close to V_c since V_c is not exactly determined. So in figure 4 we observe for example a wave-function that appears to be perfectly fractal at $L = 50\,000$ but when we try to compute the same wave-function taking $L = 10^6$ (see fig. 5) we see immediately that this actually corresponds to a localized state at higher precision. To be truly convinced that figure 4 actually correspond to a fractal wave-function we also zoomed the graph on different peaks and we see that the graph is perfectly self-similar (See fig. 6, 7 and 8).

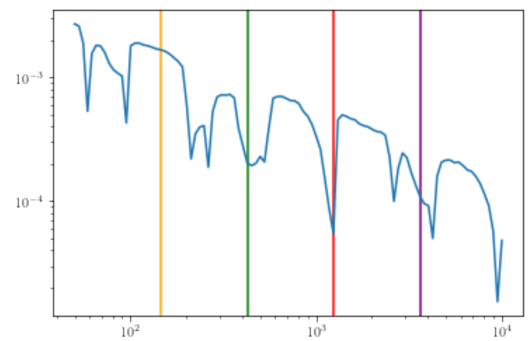

Figure 1: $IPR_0^{(3)}$ as a function of L

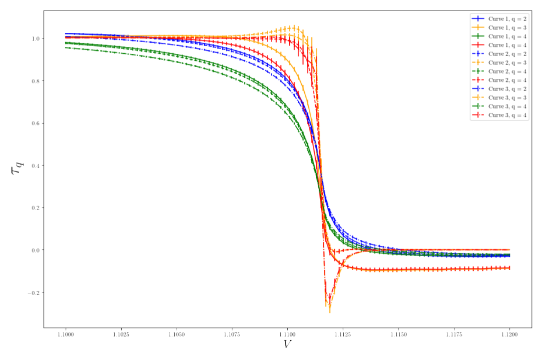

Figure 2: τ_q as a function of V

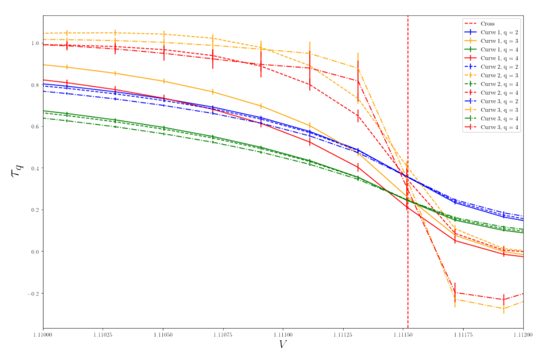

Figure 3: τ_q as a function of V

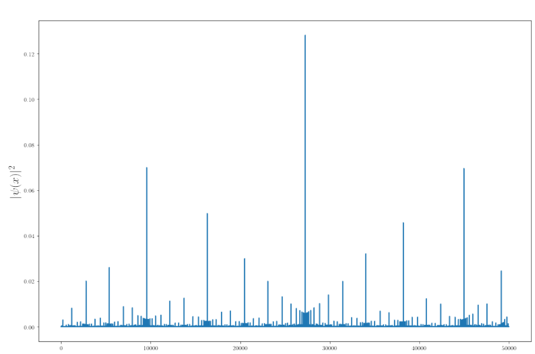

Figure 4: $|\psi(x)|^2$ for the ground state at $V = 1.11152$ with $L = 5 \cdot 10^4$

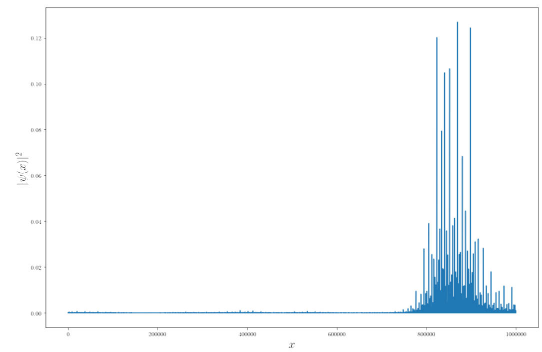

Figure 5: $|\psi(x)|^2$ for the ground state at $V = 1.11152$ with $L = 10^6$.

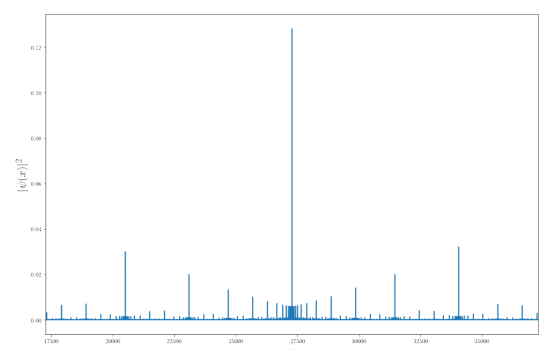

Figure 6: $|\psi(x)|^2$ for the ground state at $V = 1.11152$ with $L = 5 \cdot 10^4$ for $x \in [17275, 37275]$.

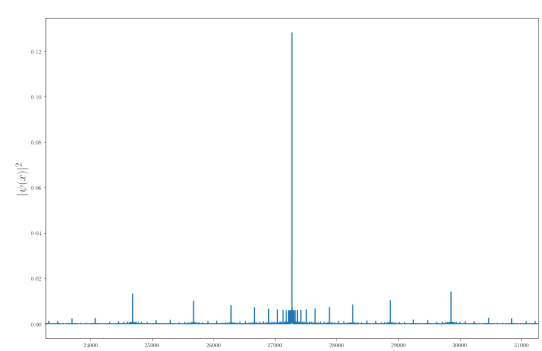

Figure 7: $|\psi(x)|^2$ for the ground state at $V = 1.11152$ with $L = 5 \cdot 10^4$ for $x \in [23275, 31275]$.

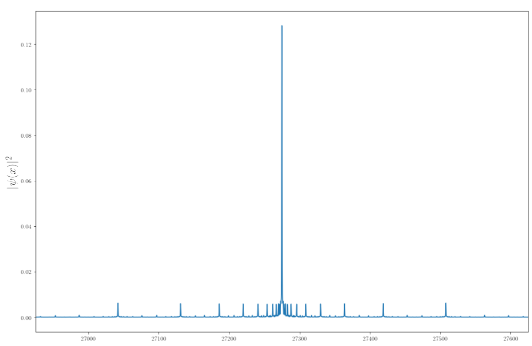

Figure 8: $|\psi(x)|^2$ for the ground state at $V = 1.11152$ with $L = 5 \cdot 10^4$ for $x \in [26925, 27625]$.