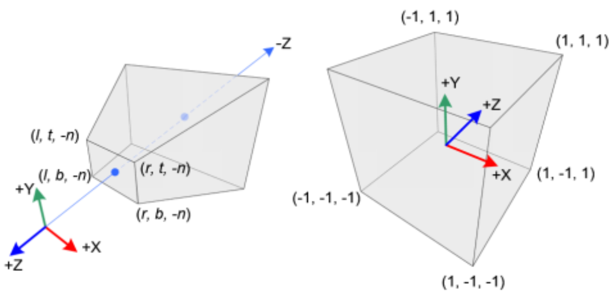
Projection math: reminder

16 June 2025 07:51

Note the viewing frustum.
Next, note that -z is in front of you, not behind you.



1. Projecting onto the near plane

This works by taking a point (x, y, z) and drawing a line from there to the origin.
Then, you find at what point that line intersects the near plane.

line: $\lambda \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} ? \\ ? \\ -n \end{pmatrix}$

$\lambda z = -n \quad \Rightarrow \quad \lambda = -\frac{n}{z}$

point is $\begin{pmatrix} -\frac{xn}{z} \\ -\frac{yn}{z} \\ -n \end{pmatrix}$

Now the x and y coordinates are basically just the projected coordinates.
Although not normalized yet.
Can we represent this as a matrix?
No, since we cannot do the division by z. However, we introduce a new coordinate w, and say that the coordinates of something are (x/w, y/w, z/w). Example:

$$\begin{pmatrix} n & 0 & 0 & 0 \\ 0 & n & 0 & 0 \\ 0 & 0 & n & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} nx \\ ny \\ -n \\ -z \end{pmatrix}$$

As you can see, w is initially set to 1. After the transformation it takes the value of -z.
If you divide your x, y and z components by -z, you get back to what you wanted.

Now we want to get things normalized. For example, an x value of r should map to +1, and a y value of -b should map to -1. Also, a z value of -n should map to -1, and a z value of -f should map to 1.
This is the matrix we want to get to.

$$\begin{pmatrix} & & & \\ & ? & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -z x_n \\ -z y_n \\ -z z_n \\ -z \end{pmatrix}$$

Where x_n is normalized x coordinate, y_n is normalized y coordinate, etc.
Next, let x_p be the un-normalized coordinate. So:

$$x_p = -\frac{nx}{z}, \quad y_p = -\frac{yn}{z}, \quad z_p = -n$$

Now, let's solve for x. If the first row of the matrix is lambda, mu, k, c, then we can write:

$$\lambda x + \mu y + kz + c = -z x_n$$

① when $x_p = r$, $x_n = 1$

② when $x_p = l$, $x_n = -1$

It's clear we can solve this using only lambda and k. So let's write:

$$\lambda x + \mu z = -z x_n$$

① : $x_p = r = -\frac{nx}{z} \Rightarrow x = -\frac{rz}{n}$, $x_n = 1$

$$\lambda\left(-\frac{rz}{n}\right) + \mu z = -z(1) = -z$$

$$\Rightarrow \mu - \frac{\lambda r}{n} = -1 \quad \Rightarrow \underline{\mu = \frac{\lambda r}{n} - 1}$$

② : $x_p = l = -\frac{nx}{z} \Rightarrow x = \frac{-lz}{n}$, $x_n = -1$

$$\Rightarrow \lambda\left(-\frac{lz}{n}\right) + \mu z = -z(-1) = z$$

$$\Rightarrow -\lambda\left(\frac{l\cancel{z}}{n}\right) + \cancel{z}\left(\frac{\lambda r}{n} - 1\right) = \cancel{z}$$

$$\Rightarrow -\frac{\lambda l}{n} + \frac{\lambda r}{n} - 1 = 1$$

$$\Rightarrow \underline{\lambda = \frac{2n}{r-l}}$$

$$\mu = \frac{\lambda r}{n} - 1 = \frac{r}{\cancel{n}}\left(\frac{2\cancel{n}}{r-l}\right) - 1$$

$$= \frac{2r}{r-l} - 1 = \frac{2r-(r-l)}{r-l}$$

$$= \underline{\underline{\frac{r+l}{r-l}}}$$

You can repeat this process for y in a very similar way, and you get:

$$\begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \lambda & \mu \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -z x_n \\ -z y_n \\ -z z_n \\ -z \end{pmatrix}$$

You can see we still have two constants to find, lambda and mu. This is to normalize z. We set the first 2 elements of the row to zero (you can try to solve them not equal to zero, but there's no point.)

$$\lambda z + \mu = -z z_n$$

① when $z = -n$, $z_n = -1$

② when $z = -f$, $z_n = 1$

$$\dots + \mu = \dots z_n$$

① when $z = -n$, $z_n = -1$

② when $z = -f$, $z_n = 1$

$$① : -n\lambda + \mu = -(-n)(-1) = -n$$

$$\Rightarrow \mu = n\lambda - n$$

$$② : -f\lambda + \mu = -(-f)(1) = f$$

$$\Rightarrow \mu = f\lambda + f = n\lambda - n$$

$$\Rightarrow \lambda = \frac{n+f}{n-f}$$

$$\mu = n\left(\frac{n+f}{n-f}\right) - n = \frac{n^2 + fn - n^2 + fn}{n-f}$$

$$= \frac{2fn}{n-f}$$

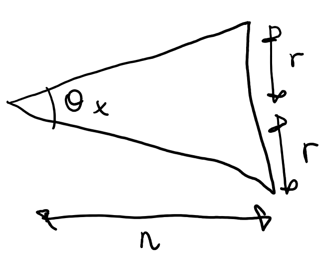
Finally, we're done! And we have the following:

$$\begin{pmatrix} \frac{2n}{r-l} & 0 & \frac{r+l}{r-l} & 0 \\ 0 & \frac{2n}{t-b} & \frac{t+b}{t-b} & 0 \\ 0 & 0 & \frac{n+f}{n-f} & \frac{2fn}{n-f} \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -z x_n \\ -z y_n \\ -z z_n \\ -z \end{pmatrix}$$

However, this can be simplified, by a lot. First of all, the frustum is usually symmetrical, so r = -l and t = -b. This means we get:

$$\begin{pmatrix} \frac{n}{r} & 0 & 0 & 0 \\ 0 & \frac{n}{t} & 0 & 0 \\ 0 & 0 & \frac{n+f}{n-f} & \frac{2fn}{n-f} \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -z x_n \\ -z y_n \\ -z z_n \\ -z \end{pmatrix}$$

However, usually in a game you wouldn't set r and t, but you would rather set your field of view (much more intuitive). Looking at the frustum from above:



θ_x is x FOV

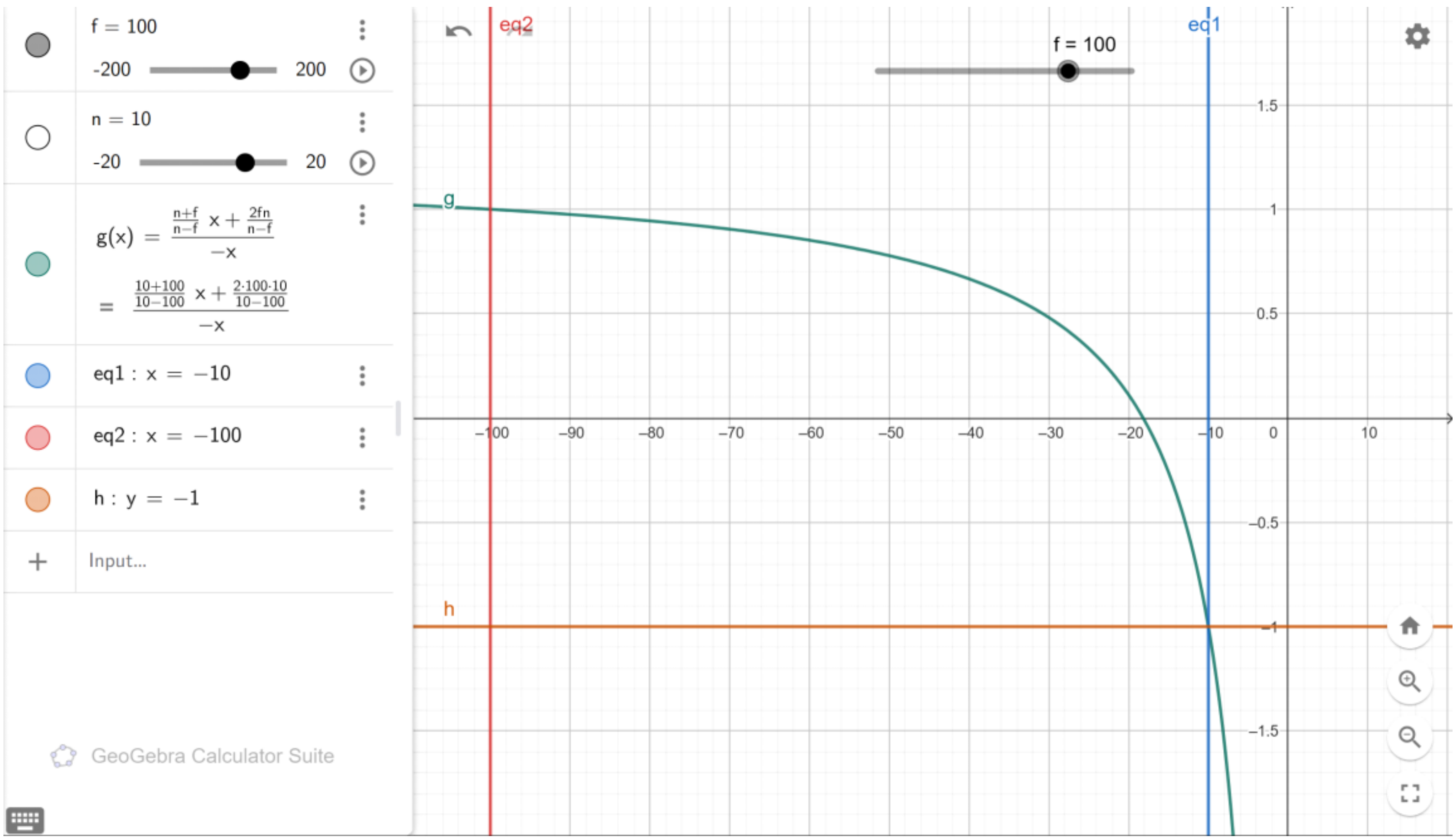
$$\tan\left(\frac{\theta_x}{2}\right) = \frac{r}{n}$$

Similarly, for y, you get $\tan\left(\frac{\theta_y}{2}\right) = \frac{t}{n}$

So we can rewrite the projection as:

$$\begin{pmatrix} \cot\left(\frac{\theta_x}{2}\right) & 0 & 0 & 0 \\ 0 & \cot\left(\frac{\theta_y}{2}\right) & 0 & 0 \\ 0 & 0 & \frac{n+f}{n-f} & \frac{2fn}{n-f} \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} = \begin{pmatrix} -z x_n \\ -z y_n \\ -z z_n \\ -z \end{pmatrix}$$

As a note, let's graph z_n against z. You can see the values of f and n on the left.



You can see how $z = -n$, $z_n = -1$, and for $z = -f$, $z_n = 1$.
However, note how the graph gets flat towards $z = -f$. This can cause errors where two z values are two close to each other, that two objects start glitching in and out of each other. This is called z fighting.

For the final element, think about something: when's the last time you adjusted your y fov? Only x fov is really used, and that's because your y fov can be decided using your aspect ratio. Specifically, aspect ratio = $a = r/t$, so $t = r/a$, so

$$\cot\frac{\theta_y}{2} = \frac{n}{t} = \frac{n}{\frac{r}{a}} = a\left(\frac{n}{r}\right) = a\cot\frac{\theta_x}{2}$$

So, the very final projection matrix is:

$$\begin{pmatrix} \cot\left(\frac{\theta_x}{2}\right) & 0 & 0 & 0 \\ 0 & a\cot\left(\frac{\theta_x}{2}\right) & 0 & 0 \\ 0 & 0 & \frac{n+f}{n-f} & \frac{2fn}{n-f} \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

Remember that there's still other matrices like the lookat matrix, which is in another pdf.