Lab 5 - Respuestas Otoño 2014

1) $f : [a,b] \to \mathbb{R}$ continua

a) $\exists c \in [a,b]$ t. $f(c) = \frac{1}{3} f(a) + \frac{2}{3} f(b)$

SPG (sin perdida de generalidad)

Sup. $f(a) \leq f(b)$ ... ①

$\Leftrightarrow \frac{1}{3} f(a) \leq \frac{1}{3} f(b) \Leftrightarrow \frac{1}{3} f(a) + \frac{2}{3} f(b) \leq \frac{1}{3} f(b) + \frac{2}{3} f(b)$

$\Leftrightarrow \boxed{\frac{1}{3} f(a) + \frac{2}{3} f(b) \leq f(b)}$ ... ②

Además

① $\Rightarrow \frac{2}{3} f(a) \leq \frac{2}{3} f(b) \Leftrightarrow \frac{2}{3} f(a) + \frac{1}{3} f(a) \leq \frac{2}{3} f(b) + \frac{1}{3} f(a)$

$\Leftrightarrow \boxed{f(a) \leq \frac{2}{3} f(b) + \frac{1}{3} f(a)}$ ... ③

De ② y ③

$f(a) \leq \frac{2}{3} f(b) + \frac{1}{3} f(a) \leq f(b)$

Por TVI

$\exists c \in [a,b]$ t. $f(c) = \frac{2}{3} f(b) + \frac{1}{3} f(a)$

b) Para cada $\alpha \in [0,1]$ $\exists c_\alpha \in [a,b]$ t. $f(c_\alpha) = \alpha f(a) + (1-\alpha) f(b)$

Sea $\alpha \in [0,1]$

Sup $f(a) \leq f(b)$

$\Rightarrow \alpha f(a) \leq \alpha f(b)$ y $(1-\alpha) f(a) \leq (1-\alpha) f(b)$

$\Rightarrow \underbrace{\alpha f(a) + (1-\alpha) f(b) \leq f(b)}_{①}$ y $\underbrace{f(a) \leq (1-\alpha) f(b) + \alpha f(a)}_{②}$

De ① y ② $\Rightarrow f(a) \leq \alpha f(a) + (1-\alpha) f(b) \leq f(b)$

Por TVI $\Rightarrow \exists c_\alpha \in [a,b]$ t. $f(c_\alpha) = \alpha f(a) + (1-\alpha) f(b)$

2) $T : [0, 2\pi] \to \mathbb{R}$ cont. con $T(0) = T(2\pi)$

p.d. $\exists c \in [0,\pi]$ t. $T(c) = T(c+\pi)$

Sea $f(\theta) = T(\theta) - T(\theta + \pi)$

p.d $\exists c \in [0,\pi]$ t. $f(c) = 0$

$f(0) = T(0) - T(\pi)$

$f(\pi) = T(\pi) - T(2\pi) = T(\pi) - T(0) = -(T(0) - T(\pi))$

$f(0) \cdot f(\pi) = -(T(0) - T(\pi))^2 \leq 0$

Por Bolzano $\Rightarrow \exists c \in [0,\pi]$ t. $f(c) = 0 \Rightarrow T(c) = T(c+\pi)$

3) Sea $p(x) = a_0 x^{2n+1} + a_1 x^{2n} + \dots + a_{2n+1}$ con $a_0 \neq 0$

Caso 1: $a_0 > 0$

$\left.\begin{array}{l} \lim_{x \to \infty} p(x) = \infty \\ \lim_{x \to -\infty} p(x) = -\infty \end{array}\right\}$ $\exists a > 0$ y $b < 0$ t. $p(a) > 0$ y $p(b) < 0$

$\Rightarrow p(a) \cdot p(b) < 0$

Por Bolzano $\exists c \in [a,b]$ t. $p(c) = 0$

Caso 2: $a_0 < 0$

es análogo.

4.a) $\lim_{x\to 0} \frac{\tan(x)-\operatorname{sen}(x)}{x^3} = \lim_{x\to 0} \frac{\frac{\operatorname{sen}(x)}{\cos(x)} - \operatorname{sen}(x)}{x^3}$

$= \lim_{x\to 0} \frac{\frac{\operatorname{sen}(x)-\operatorname{sen}(x)\cos(x)}{\cos(x)}}{x^3} = \lim_{x\to 0} \frac{\operatorname{sen}(x)-\operatorname{sen}(x)\cos(x)}{x^3\cos(x)}$

$= \lim_{x\to 0} \frac{\operatorname{sen}(x)(1-\cos(x))}{x^3\cos(x)} = \lim_{x\to 0} \frac{\operatorname{sen}(x)}{x} \cdot \frac{(1-\cos(x))}{x^2\cos(x)} \cdot \frac{1+\cos(x)}{1+\cos(x)}$

$= \lim_{x\to 0} \frac{\operatorname{sen}(x)}{x} \cdot \frac{1-\cos^2(x)}{x^2} \cdot \frac{1}{\cos(x)(1+\cos(x))}$

$= \lim_{x\to 0} \frac{\operatorname{sen}(x)}{x^2} \cdot \lim_{x\to 0} \frac{\operatorname{sen}^2(x)}{x^2} \cdot \frac{1}{\cos(x)(1+\cos(x))} = \frac{1}{2}$

4.b) $\lim_{x\to \frac{\pi}{2}^-} \frac{|x-\frac{\pi}{2}|}{\cos(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{-(x-\frac{\pi}{2})}{\cos(x)} = \lim_{x\to\frac{\pi}{2}^-} \frac{\frac{\pi}{2}-x}{\sin\left(\frac{\pi}{2}-x\right)} = \lim_{\theta\to 0} \frac{\theta}{\sin(\theta)} = 1$

4.c) $\lim_{x\to 0} \frac{\tan(2x)}{2x^2+3x} = \lim_{x\to 0} \frac{\frac{\sin(2x)}{\cos(2x)}}{x(2x+3)} = \lim_{x\to 0} \frac{\sin(2x)}{x(2x+3)\cos(2x)}$

$= \lim_{x\to 0} \frac{\sin(2x)}{2x} \lim_{x\to 0} \frac{2}{(2x+3)\cos(2x)} = 1 \cdot \frac{2}{3} = \frac{2}{3}$

4.d) $\lim_{x\to\infty} 2x\operatorname{sen}\left(\frac{3}{x}\right) = \lim_{x\to\infty} \frac{2\operatorname{sen}\left(\frac{3}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\frac{2}{3}\operatorname{sen}\left(\frac{3}{x}\right)}{\frac{3}{x}} = \lim_{\theta\to 0} \frac{\frac{2}{3}\operatorname{sen}(\theta)}{\theta}$

$= \frac{2}{3}$

4.e) $\lim_{x\to 0} \frac{\operatorname{sen}(x+2x^2)}{3x} = \lim_{x\to 0} \frac{\operatorname{sen}(x+2x^2)}{(x+2x^2)} \frac{x+2x^2}{3x} = \frac{1+2x}{3} = \frac{1}{3}$

$\theta = \frac{3}{x}$ $\lim_{\theta\to 0} = 0$

4.f) $\lim_{x\to 0} \frac{\sqrt{4+\operatorname{sen}(x)}-2}{x} = \lim_{x\to 0} \frac{4+\operatorname{sen}(x)-4}{x(\sqrt{4+\operatorname{sen}(x)}-2)} = \lim_{x\to 0} \frac{\operatorname{sen}(x)}{x(\sqrt{4+\operatorname{sen}(x)}+2)} = \frac{1}{2+2} = \frac{1}{4}$

5) $$\lim_{x\to 0} \frac{\operatorname{sen}(\pi + ax)}{bx} = 2$$

$$\lim \operatorname{sen}(\pi + ax) = -\operatorname{sen}(ax)$$

$$\Rightarrow \lim_{x\to 0} \frac{\operatorname{sen}(\pi+ax)}{bx} = \lim_{x\to 0} \frac{-\operatorname{sen}(ax)}{bx} = \lim_{x\to 0} \frac{-\operatorname{sen}(ax)}{b(ax)} \cdot a = -\frac{a}{b} = 2$$

$$\therefore \lim_{x\to 0} \frac{\operatorname{sen}(\pi+ax)}{bx} = 2 \iff \frac{a}{b} = -2$$