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Econophysics meets ABMs: a step towards realism

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Extended abstract

During the 1960s Mandelbrot observed that the exchanges of money that occur in economic interactions are analogous to exchanges of energy between molecules in a gas: econophysics was born.

Since then, econophysics has become a well established branch of physics, where the elegance of mathematic formulae has often been put before the understanding of real world implications of the research, favouring simplicity and beauty to realism.

Meanwhile, economics tried to forge the tools to inform economic policies since the Great Depression of 1929. It went through a series of models, each one addressing the problem of the previous generation of models, and each one with its own fallacies. The state of the art in economic research are the Agent-Based Models (ABMs), which are criticized for relying heavily on simulations and lacking a simple and intuitive mathematical description.

This thesis aims at bringing together the complexity of ABMs with the simple mathematical description of econophysics in order to address the issue of economic inequality and wealth distribution in the long run, in a similar fashion to what the economist Thomas Piketty did in his book "Capital in the 21st century".

This is achieved in two ways:

- Developing new mathematical tools and frameworks to apply the econophysics model of Drăgulescu-Yakovenko (DY model) to real world phenomena like the study of social mobility and the development of optimal fiscal and monetary policy.

- Creating a new model of capital exchange based on the DY model in which the contribution of financial investments, consume, income and taxes are implemented.

The final objective of this thesis will be of practical nature: to address fiscal and monetary policies. But in doing so, a wide range of physical ideas will come to life: trade will be explained with thermodynamics, social mobility will be studied with Brownian motion and path integrals, fiscal policy will be discussed with the Euler-Lagrange equations, and monetary policy will reveal a phenomenology for the DY model quite similar to bands in crystals.

The thesis is structured as follows:

- In chapter 1 the development of economics and econophysics are analysed, trying to stress the difficulties that both disciplines encounter in studying the issue of inequality. The work of Piketty is presented as the starting point of our work.

- In chapter 2 the DY model of capital exchange is presented: we review the thermodynamics of it, write the Boltzmann equation, find the equilibrium wealth distribution, and discuss a procedure of our own to measure social mobility. Finally, the comparison between the DY model predictions and real world data on wealth distribution is explored.

- In chapter 3 it is demonstrated with a proof of our own design, that perturbations to the equilibrium in the DY model decay exponentially in time. In the proof, a structure similar to band in crystal is discovered in the DY model.

- In chapter 4 our extensions of the DY model are presented. The contribution of financial investments, consume, income and taxes are implemented. The time evolution and final shape of the wealth distribution are analysed for each new economic variable, and taxes are found to be the only way polarization between riches and poors can be avoided.

- In chapter 5 we solve the problem of designing optimal fiscal policy in the DY model with the help of production theory. We propose tho paths leading to the optimal taxes: one computational, using stochastic gradient descent, and the other theoretical, using the analogy with the Euler-Lagrange equations for our system.

- In chapter 6 our study of the perturbations around the equilibrium of the DY model is used in the design of monetary policy. The concept of equilibrium time for an economy is presented and calculated in the U.S. economy from the 1950s until today.

- In chapter 7 the conclusions of our work are drawn and our main findings are summarized. Promising leads for future research are identified both in the extension of our model, and in possible new uses of the mathematical tools we developed throughout the work.

Chapter 1

Introduction

1.1 Brief history of Macroeconomic thought

Since the beginning of civilization, economics was an important piece of our social structure. Despite that, it was not until the end of the 18^{th} century that humanity could shed light on economic phenomena and count on more than gut feelings to design economic policy.

The first milestone in this path was "The wealth of nations" by Adam Smith in 1776. In the book, the foundations of division of labour, productivity and free markets were sketched. But even though other works in this field emerged in the subsequent 150 years, see for example Thomas Malthus [13], macroeconomics as a discipline was the product of the Great Depression of 1929, and started with Keynes "The general theory of Employment, Interest and Money" in 1936.
Key to this development was the need to address the problems that led to the Great Depression in order to build a more stable society. Thus it is safe to say that macroeconomics was born as a **tool to inform economic policy**.

In the following 30 years, new models emerged. The most complicated of them probably was the Klein and Goldberg model, which claimed that a set of dozens of equations inspired by the Keynes model governed the economy. Given those equations, the effects of shocks on the economy, like an increase in unemployment, could be forecasted on the variables of interest, like the production, and policy makers could counteract the effect of those shocks by the means of fiscal and monetary policy.

The problem with all these models, is that they lacked a microeconomic foundation. In fact, rather than deriving decision rules for investment, consumption, labour and supply based on an optimization problem for the utility of the agents, those rules were simply assumed.

What further restricted the room for transformative monetary and fiscal policies was the rational expectation hypothesis, first introduced by Lucas in his book "Econometric Policy Evaluation: A Critique" in 1976. In it, Lucas argued that only unexpected changes in fiscal and monetary policy could have any impact on the economy, because the expected ones are already taken into account by the households when they make decisions. So for example to solve the problem of high unemployment in an economy, the general understanding before Lucas was that the government should look at the Phillips curve, that implies a negative correlation between inflation and unemployment, and thus increase inflation (by printing money), causing unemployment to go down.

The full awareness of the failure of Keynesian theory to address economic policy came during the 1970s, when a mix of rising oil prices and decreasing productivity led to having simultaneously high inflation and high unemployment in the USA. Since even the most established relationships between economic quantities, like the Phillips curve, seemed to be unstable over time, then no economic policy can be engineered if it relied solely upon them. And this was Lucas' point: any policy designed to exploit a historical relationship between economic variables without understanding the microeconomic foundation that generated the relationship is misguided.

As a consequence to the Lucas critique, during the 1980s a new class of macroeconomic models was developed with the use of rational expectations econometrics. In 1982, Kydland and Prescott developed a real business cycle model to predict the effects of policies on the macroeconomic variables [11]. In it, the economy was seen as a system at equilibrium, and fluctuations in the macroeconomic variables were explained by exogenous stochastic random fluctuations in the productivity level. Due to this type of description, real business cycle models are seen as the progenitors of a new family of models: DSGE models.

DSGE stands for Dynamic Stochastic General Equilibrium. DSGE models are dynamic because current choices are influenced by the probabilities of future outcomes. They are stochastic because they account for random shocks in some exogenous factors, and the shocks are propagated to all the macro variables. They are general, because the aim is to model the entire economy. And finally they are at equilibrium, because they comply with the general competitive equilibrium theory.

DSGE models are still used today as a forecasting tool by the most prominent institutions. The European Central Bank has developed a DSGE model called the Smets-Wouters model, that uses to analyze the entire economy of the eurozone.

But DSGE fail on an impressive number of metrics: they are unable to account for the occurrence of rare economic crises, mainly due to the assumed distribution of the exogenous shocks. They assume rational agents, that have infinte information and act with infinite intelligence. They can not address distributional issues, since they rely on the representative agent hypothesis, hence all agents are equal. And as a matter of fact, DSGE present an internal contradiction. In fact, despite the rigorous hypothesis we have enumerated so far, some hypothesis, like sticky prices, have to be made ex-post, without a solid justification, just to fit the data. Moreover, the shocks in DSGE models are exogenous, but according to Stiglitz [24], the Great Recession shows how endogenous shocks are the most important to account for economic fluctuations.

As a response to the failure of DSGE models to become less hypothesis driven and more based on reality, a new class of models emerged as an alternative: Agent Based Models (ABMs). In it, agents are modelled with empirical and experimental microeconomic evidence, and their behaviour is derived with the help of cognitive psychology. The all knowing perfectly rational agents are replaced by ones with bounded rationality and adaptive behaviour.

Agent based models do not assume that economy can achieve equilibrium, nor the representative agent paradigm. The top down approach of DSGE models is replaced by this less elegant, less hypothesis driven model, which has been shown to be able to predict even the most complex, nonlinear, out of equilibrium phenomena, like herd behaviour in financial crashes.

The price to pay for this increase in predicting power though, lies in the strong reliance on computer simulations, and the almost complete impossibility to predict the outcome of those simulation before they end. ABMs trade intuition with realism.

1.2 Piketty and inequality

One of the works that most influenced the debate on economic inequality is the Book “Capital in the 21st century” by Piketty [19]. In this work, the cross country properties of wealth distribution are studied from an empirical perspective, by looking at the available time series of wealth and trying to understand the mechanism causing economic inequality, that in the literature became known as polarization.

The relevance of Piketty’s work lies in the overcoming of the representative agent paradigm, that guided DSGE models until then. In essence the representative agent hypothesis states that all agents in the economy are equal thus ruling out the possibility to address distributional issues among them.

Two are the main findings of the author that inspired our work: The second fundamental law of capitalism and the discussion on polarization.

The second fundamental law of capitalism: Take an economy, and define I as the average income of the agents. Then suppose each agent consume a fraction of the income every month, and saves another fraction that we will call s, the saving rate. Therefore calling $w(t)$ the average wealth of the agents at time t we have:

$$w(t+1) = w(t) + sI(t) \tag{1.1}$$

Furthermore, suppose that the income of the agents is growing at a certain rate g, that is

$$I(t+1) = (1+g)I(t) \tag{1.2}$$

Let us also define the parameter $\beta(t) = \frac{w(t)}{I(t)}$.
Given these premises, the second fundamental law of capitalism states that

$$\lim_{t\to+\infty} \beta(t) = \lim_{t\to+\infty} \frac{w(t)}{I(t)} = s/g. \tag{1.3}$$

Since β measures how important accumulated wealth is with respect to current income, this law alone implies that the long term importance of accumulated wealth is increasing with the saving rate and decreases with the growth of the economy g, hence binding macroeconomic aggregate variables, like β and g, to parameters that govern the preferences of the agents, like s.

Piketty derived this law from empirical data on wealth distribution across countries, but actually his conclusions can be drawn by purely analytical reasoning given his premises.

The discussion on polarization: consider an economy with a well defined rate of return on capital r, meaning that eq:

$$K(t+1) = (1+r)K(t) \tag{1.4}$$

must hold for the capital K. Define g as the growth rate of the whole economy as before. Then, according to Piketty, polarization happens if and only if $r > g$. In this scenario rich become richer at a rate greater than the poors, and the distribution of wealth will become more unequal over time.

1.3 A review of the relevant econophysics literature

Since the beginning of the 20th century, it was a widely accepted fact that the income distribution was well fitted for high incomes with a power law tail, meaning the probability of having an income m is $P(m) = Am^{-\alpha}$ with A being the positive constant that normalizes the distribution. This result was first published by the economist Vilfredo Pareto in 1897 with "Cours d'économie politique", and holds its importance still today. The first to lay the foundations of econophysics was probably Mandelbrot with [14], when he said

"There is a great temptation to consider the exchanges of money which occur in economic interactions as analogous to the exchanges of energy which occur in physical shocks between molecules. In the loosest possible terms, both kinds of interactions should lead to similar states of equilibrium. That is, one should be able to explain the law of income distribution by a model similar to that used in statistical thermodynamics..."

He was also quick to address the elephant in the room and said *"Unfortunately the Pareto distribution decreases much more slowly than any of the usual laws of physics..."*, alluding to the fact that the Boltzmann-Gibbs distribution has an exponential tail, in violation of the Pareto power-law prediction.

Since then many progresses have been made in addressing the good points of both sides. It is now understood that the bulk of the population at intermediate value of wealth does not follow a power law behaviour, but rather an exponential one, and the mechanism leading to this distribution can in fact be inquired by pure thermodynamic arguments, leading to a Boltzmann-Gibbs distribution. Such mechanisms have been shown to be so general that they hold not only for the distribution of wealth, but also for energy consumptions and other commodities [21], paving the way for an entropic explanation of the phenomenon [27].

To be totally truthful the consensus on the exponential behaviour of the bulk is not unanimous, and other distributions have been proposed to fit it, the most prominent of which are the Gompertz curve [16] and the Gamma distribution (see for example [2, 12, 15]). Despite the feud being still alive, the other distributions might present better fitting properties for specific realities, but given the simplicity, generality and most importantly the entropic justification of the Boltzmann-Gibbs distribution, the consensus is growing on the relevance of the last.

Guided in part by the ability of the Boltzmann-Gibbs distribution to reproduce the wealth distribution, many agent-based models have been developed inspired by the kinetic theory of gases [18, 7, 17, 5, 6, 23]. These models are referred in the econophysics literature as kinetic exchange models of markets.

The analogy goes as follows:

- The molecules in the gas are replaced by agents in the economic model.

- As the molecules have a positive kinetic energy, the agents have a positive wealth.

- during the interactions, molecules exchange energy as agents exchange money.

Despite the analogy being extremely elegant, the consensus among the economic community is that econophysics is not trying to model real world phenomena, but rather to recognize the physics in it, see Gallegati et al.[9].

1.4 The motivation for this thesis

With this thesis, our aim is to replicate and extend the results found by Piketty on wealth inequality in order to be able to address fiscal and monetary policy.

Our approach will be half way between econophysics and economics: we will try our best to bring together the complexity and real world predictions of ABMs, and the elegance and intuitiveness of econophysics.

Our main focus will be towards real world applications, but in the process a wide range of physical ideas will emerge: thermodynamics, Brownian motion, path integral and bands in crystals are some of them.

Not all the ideas of this thesis can be found in the literature. The original contribution have two origins:

- We complement the mathematical tools and ideas surrounding the Drăgulescu - Yakovenko (DY) model with a focus on real world application

- Recognizing the value of the entropic foundation of the DY model, we propose some extensions of it that can better account for real world economic phenomena, like financial investments, consume, income and taxes.

In the next section we will explore in great detail the DY model, and we will start developing the tools and ideas needed throughout this work.

Chapter 2

The Drăgulescu-Yakovenko model

The Drăgulescu -Yakovenko model, or DY model for short, is a wealth exchange model which takes its name by the authors who developed it in the year 2000 in [7]. In this chapter we review every aspect of it, from the thermodynamics to the Fokker-Planck equations, with an eye on possible applications.

Since the DY model is the simplest model of random exchange and yet presents a rich phenomenology, we are going to spend some time in this chapter developing the mathematical tools and ideas we will use throughout our work. This will not only build our intuition around the DY model, but will introduce us to those ideas in their simplest possible applications.

Not all the idea presented here can be found in the literature: the study of social mobility with Brownian motion and path integral is, to the best of my knowledge, original, and some of our remarks on the temperature of the distribution are not well understood in the literature. But the focus here will not be to expand as much as possible the results given in the literature, but rather to familiarize ourselves with the ideas we will need throughout the work.

This chapter is structured as follows:

- In section 2.1 the specifics of the DY model are presented in detail, the difference between physical and monetary layer is made, and money conservation is motivated.

- In section 2.2 the equilibrium wealth distribution for the DY model is found with a purely thermodynamic argument, showing the powerful analogy with the kinetic theory of gases.

- In section 2.3 this analogy is brought even further: trade deficits and immigration are explained with the help of the second law of thermodynamics.

- In section 2.4 the Fokker-Planck equation for our system is derived from first principles.
- In section 2.5 the simulations we made on the system are shown, in order to familiarize ourselves with its dynamics.
- In section 2.6 the real value of the temperature of the distribution is discussed, addressing some confusion in the literature.
- In section 2.7 the possibility of contracting debt is explored.
- In section 2.8 the motion of a single agent is studied with the use of Brownian motion and path integrals. A way of measuring social mobility is proposed and the results obtained in the DY model are discussed.
- Eventually in section 2.9, the DY model is put to the test and confronted with real world data on income distribution. The agreement of the data with the prediction is almost perfect, and little disagreement in the richer part of the population will reveal a new source of income for the riches: financial investments. This will be crucial for the extensions of the DY model we will explore in chapter 4.

2.1 The model

Let us start by explaining in detail how the DY model works. We have a system of $N \gg 1$ agents. Each agent has certain amount of money m_i. On a very distinct layer, there are physical goods, like cars and phones, things that money can buy, that have a value, but are not money per se. Physical goods can be created or destroyed, but not money: money can be used to buy a physical good by transferring an integer amount of them from one agent to another, but can't be created or destroyed by the agents. This is analogous to the money we are used to, where only central banks have the ability to print it, and in most states they are the only entity for which is legal to destroy it. Since total money will be conserved, we will study the evolution of the monetary layer, and will not be interested in the evolution of the physical one. We stress that money is not only conserved at the global scale, but also in the single transactions among the agents, implying a local conservation.

Now that we understand the rules of the game, let us explain how the evolution of the system works: At the beginning of the process, all agents are endowed with an equal amount of money m_0. Then, for each step of the procedure, an ordered pair of agents A and B is selected. If agent A has a money balance greater than zero, he gives a unity of money (a coin) to agent B, in exchange for an equal valued physical good, that we are not keeping track of. If instead agent A has zero money, no transaction is carried out in this step of the algorithm. The process repeats, and after a sufficient number of steps, the system reaches an equilibrium distribution of money between the agents. We will show in the next section that this distribution is the Boltzmann-Gibbs distribution.

2.2 The Boltzmann-Gibbs distribution

In this section we will review the derivation of the Boltzmann-Gibbs distribution of energy for molecules in a gas, paying particular attention to the hypothesis we will need. By doing so, we notice how the same derivation will apply to an agent based model, in which molecules are replaced by agents, and energy is replaced by money.

The Boltzmann-Gibbs equation: Let us consider a gas in a closed container. Assume there are N distinguishable particles to be placed in a set of discrete energy levels ϵ_i. The total energy of the system will thus be $E = \sum_i N_i \epsilon_i$.

The distribution that will be obtained at thermodynamical equilibrium will be the one that maximizes the number of microstates associated to it in the phase space. The number of different microstates that realizes a single distribution is just the number of permutation with repetition of the particles that leave the populations N_i unchanged:

$$W(N_i) = \frac{N!}{\Pi_i N_i} \tag{2.1}$$

Therefore, for the equilibrium distribution, W will be maximized with the constraints of particle numbers and energy being fixed.

It is useful to take the logarithm of (2.1), to get: $\ln(W) = \ln(N!) - \sum_i \ln(N_i!)$.

We now recall the Stirling approximation, holding for large N: $\ln(N!) \simeq N\ln(N) - N$. Using it on equation (2.1) we can find the system of equations holding at equilibrium:

$$\begin{cases} \sum_i \delta N_i = 0 \\ \sum_i \epsilon_i \delta N_i = 0 \\ \delta(\ln(W)) = -\sum_i \delta(N_i \ln(N_i)) = 0 \end{cases} \tag{2.2}$$

Rewriting the third equation:

$$(\ln(N_i) - 1)\delta N_i = 0, \tag{2.3}$$

the solution can be found by means of Lagrange multipliers:

$$\sum_i (\alpha + \beta\epsilon_i + \ln N_i)\delta N_i = 0 \tag{2.4}$$

Since equation 2.4 must hold for any change in the occupation N_i, it implies $\alpha + \beta\epsilon_i + \ln(N_i) = 0$. Solving for the occupation, we get: $N_i = e^{\alpha-\beta\epsilon_i}$ that we rewrite in a more familiar form introducing the chemical potential μ as $N_i = e^{-\beta(\epsilon_i-\mu)}$. To find the two constants β and μ we write the equations of normalization of probability and the constraint on the average energy. If we write those equation in the continuous form, rather than in the discrete form (more on this can be found in section 2.6), we obtain:

$$\beta = \frac{1}{\langle\epsilon\rangle} = \frac{1}{T} \qquad \text{and} \qquad \mu = -T\ln(T) \tag{2.5}$$

where we decided for convenience to rename the average energy with the letter T, that will be the temperature of our system.

The economic analogue: Let us now reformulate the problem in the model with agents: instead of having molecules that collide exchanging energy, we have agents that make transactions exchanging money. Money is now the conserved quantity in transactions, as energy was conserved in collisions. Moreover, energy was a positive quantity for each molecule, as now is the account of each agent, since no agent is allowed to contract debt. By the very same argument used to demonstrate the Boltzmann-Gibbs distribution for energy in a gas, we can prove that the equilibrium wealth distribution among the agents will be:

$$P(m) = \frac{1}{T}e^{-\frac{m}{T}}, \tag{2.6}$$

where m is the amount of money and the temperature T is the average money in the system.

2.3 Thermodynamics and Trade

In the previous section we explored the thermodynamic origin of the wealth distribution. We defined the temperature as $T = \langle m\rangle$ and the chemical potential $\mu = -T\ln T$. We now bring the analogy with thermodynamics even further, describing trade and immigration as a result of the system going towards thermodynamic equilibrium. More on this analogy can be found in [3].

Let us consider two systems with different money temperatures $T_1 > T_2$. We can interpret those systems as two countries: in country 1, the agents have more average money per capita than in country 2, making country 1 the rich country and country 2 the poor one. Suppose a flow of money and agents is allowed between the two countries. Suppose the flow to be slow enough such that the equilibrium time-scale of the system is faster than the characteristic time scale of the flow. Therefore the two systems can be considered at equilibrium where the variables N and T change with time. We can thus write the first law of thermodynamics for each of the two systems:

$$\delta S = \beta \delta M - \alpha \delta N \quad \leftrightarrow \quad \delta M = T \delta S + \mu \delta N \tag{2.7}$$

Now we call $\delta M_{1\to 2}$ the amount of money leaving system one and entering system 2, and $\delta N_{2\to 1}$ the amount of agents leaving system 2 for system 1. With this notation, we can write the change in total entropy:

$$\delta S_1 + \delta S_2 = \left(\frac{1}{T_2} - \frac{1}{T_2} \right) \delta M_{1\to 2} + \ln \left(\frac{T_1}{T_2} \right) \delta N_{2\to 1} \tag{2.8}$$

Since the second law of thermodynamics ensures that entropy must increase, and both the coefficients of $\delta M_{1\to 2}$ and $\delta N_{2\to 1}$ are positive, we see that if the number of agents is kept fixed, money flows from the rich country to the poor one. And if we consider that an exchange of money in the monetary layer corresponds to an equal and opposite exchange of goods in the physical layer, we just found what in economics is called a trade deficit. The second term in equation 2.8, implies that if no exchange of money is allowed, the agents will flow from the poorer to the richer country, giving rise to immigration. Both trade and immigration have the net effect of bringing the temperatures of the two systems closer together.

2.4 The Boltzmann Equation

In this section we write the general Boltzmann equation for a wealth exchange model, we show how under relatively loose hypothesis the implied stationary wealth distribution is an exponential, and then we proceed in writing the specific Boltzmann equation for the DY model in a pedagogical way. This will allow us to familiarize ourselves with the construction of this equation, a skill we will need in chapter 4 and 5.

Let us start by writing down the general Boltzmann Equation for a process.

$$\begin{aligned} \frac{dP(m)}{dt} = \int \int [& -w_{[m,m']\to[m-\Delta,m'+\Delta]} P(m) P(m') \\ & + w_{[m-\Delta,m'+\Delta]\to[m,m']} P(m-\Delta) P(m'+\Delta)] dm' d\Delta \end{aligned} \tag{2.9}$$

where $w_{[m,m']\to[m-\Delta,m'+\Delta]}$ is the rate of transfer of an amount of money Δ from an agent with money m to one with money m'.

Now a very interesting point: if the model in question has time reversal symmetry, then the two transfer coefficients w are equal, hence the stationary solution is easily shown to be of the exponential type. Normalization of probabilities force us to choose the decreasing exponential and leave only one degree of freedom, namely the temperature.

Notice that the only assumption we made to arrive to the exponential solution at equilibrium is the time reversal symmetry, which means that if we look at an exchange between two agents in time reverse, that should also be a valid exchange under our exchange rule. For this reason, unitary exchanges between agents described by the DY model lead to the exponential distribution of money, whereas other exchange rules, like the proportional money transfer, lead to other distributions. In fact, if an agent gives a fraction of what he owns to another agent, looking it in reverse would imply receiving a fraction of our money from the giver, which is a totally different exchange rule.

We now proceed in finding the Boltzmann equation for the DY model for $m > 0$. Let us start by observing that, since only exchanges of one coin are allowed in the DY model, we can write the right hand side of equation of the Boltzmann equation in four contributions. We could be tempted to think that our system is governed by a purely diffusive equation. So every agent that is extracted leaves his wealth column, and goes half of the times in the right one and the other half in the left one. On the other hand, every column receives intake from the two adjacent columns, proportionally to their populations. This would lead us to:

$$\frac{1}{r}\frac{\partial P(m)}{\partial t} = P(m+1) - 2P(m) + P(m-1), \tag{2.10}$$

where the value of r is half the fraction of people extracted per unit time. But eq 2.10 is just the heat equation: if we impose reflective boundary conditions at $m = 0$, meaning that no agent is allowed to have a negative wealth and that agents can not enter or leave the system at $m = 0$, we know that the solution would just diffuse towards infinity. This would not conserve the total wealth of the agents, which we showed was protected by a local conservation law. What did we do wrong?

As soon as at least one agent has zero wealth, he can not be selected for an exchange as a loser, causing a different probability of being a loser or a winner also for all the other agents. Hence calling α_l and α_w the two probabilities of being selected as a loser and as a winner respectively, we get the new Fokker-Plank equations:

$$\frac{1}{r}\frac{\partial P(m)}{\partial t} = \alpha_l P(m+1) - (\alpha_l + \alpha_w)P(m) + \alpha_w P(m-1). \tag{2.11}$$

We now recognize that $\alpha_w = (1 - P(0))\alpha_l$, because when an agent is selected as a winner, he can actually win if and only if the other agent selected has a positive money balance, whereas when the agent is selected as a loser he can always lose, since the we are writing the Boltzmann equation for $m > 0$. We can then rewrite the equation 2.11 as:

$$\frac{1}{r}\frac{\partial P(m)}{\partial t} = P(m+1) - 2P(m) + P(m-1) + P(0)(P(m) - P(m-1)) \quad (2.12)$$

which could cast in the continuous form by using the Laplace operator and the left derivative:

$$\frac{1}{r}\frac{\partial P(m)}{\partial t} = \nabla^2 P(m) + P(0)\partial_m P(m) \quad (2.13)$$

Since the general form of the Fokker-Plank equation is written as

$$\frac{\partial P(m)}{\partial t} = \partial_m(A(m)P(m) + \partial_m(B(m)P(m))) \quad (2.14)$$

with coefficients A and B potentially depending on m, and being equal to:

$$A(m) = -\left\langle \frac{\Delta m}{\Delta t} \right\rangle, \quad B(m) = \left\langle \frac{(\Delta m)^2}{2\Delta t} \right\rangle \quad (2.15)$$

Therefore we can interpret the motion of a single agent as a **Brownian motion**, with drift velocity $\langle \frac{\Delta m}{\Delta t} \rangle = -rP(0)$, and diffusion coefficient $\left\langle \frac{(\Delta m)^2}{2\Delta t} \right\rangle = r$. Interestingly the fact that the drift velocity is negative means that over long enough timeframes the probability for a single agent of not visiting zero wealth goes to zero. This is true because on average the distance traveled for an agent due to the drift velocity is linear in time, whereas the one due to diffusion is proportional to $\sqrt{t}$. It has to be kept in mind though that equation 2.13 only holds for $m > 0$. For zero wealth, with analogous reasoning and keeping in mind that we imposed reflective boundary conditions, we instead have

$$\frac{1}{r}\frac{\partial P(0)}{\partial t} = P(1) - P(0) + P(0)^2 \quad (2.16)$$

The drift velocity for $m = 0$ can be found recalling that it is the difference between the rate to go right and the one to go left. Since agents can only go right and with probability $1 - P(0)$ for each extraction that selects them, we have $-A = \langle \frac{\Delta m}{\Delta t} \rangle = r(1 - P(0))$, making the velocity at $m = 0$ positive and with absolute magnitude grater or equal to the one of the drift velocity for $m \neq 0$. This was expected since, due to conservation of money, the following equation must hold at all times

$$\sum_{m=0}^{+\infty} P(m) \left\langle \frac{\Delta m}{\Delta t} \right\rangle = 0. \quad (2.17)$$

There is a growing trend in the literature, to call the Boltzamann-Gibbs equation we just obtained as the Fokker-Planck equation of the system. This is rigorously wrong since eq 2.12 is non-linear and non-local in the probability $P(m)$, due to the first term containing the product of the probability at zero and the probability at m, thus not be eligible to become a Kolmogorov-forward equation. The official excuse that is often given is that when solving for $P(m \neq 0)$, $P(0)$ is considered as a constant. Even though we are less than persuaded from this justification, we choose to adhere to this terminology, in order to facilitate readers that might approach this work while being familiar with the literature.

2.5 Simulations

In this section we explore the behaviour of our model with the help of computer simulations. In each simulation, the agents are initially given an amount of money m_0, and the system is allowed to evolved for a certain number of interactions.

In figure 2.1, we can observe the Probability distribution function (Pdf) of the wealth of the agents as time elapses. As you can see, at first the system undergoes pure diffusion. This is a consequence of the fact that drift velocity of the agents calculated in the previous section is $\left\langle \frac{\Delta m}{\Delta t} \right\rangle = -rP(0)$, and $P_0 = 0$ at the beginning. After a while, the bin P_0 starts to popolate, and this causes the distribution to be skewed, and all the agents gain a negative drift velocity. After a sufficiently large amount of time, the distribution thermalizes and the exponential is reached.

In figure 2.2 some relevant quantities of the system are graphed. Notice how, as the system approaches equilibrium, the Shannon entropy reaches a plateau. Also notice how inequality develops between initially perfectly equal agents due to the stochastic nature of the exchanges. For the reader that might be unfamiliar with Lorenz curves and the Gini coefficient, a detailed explanation of this construction is given in Appendix A.

2.6 Remarks on the temperature of the distribution

We saw that the equilibrium wealth distribution in our system follows an exponential distribution given by

$$P(m) = \alpha e^{-\frac{m}{T}}.$$

To calculate the value of α we impose the normalization condition on the Probabilities $P(m)$. What [27], [3] and many others do in the literature is to just take the distribution in the continuous form and integrate:

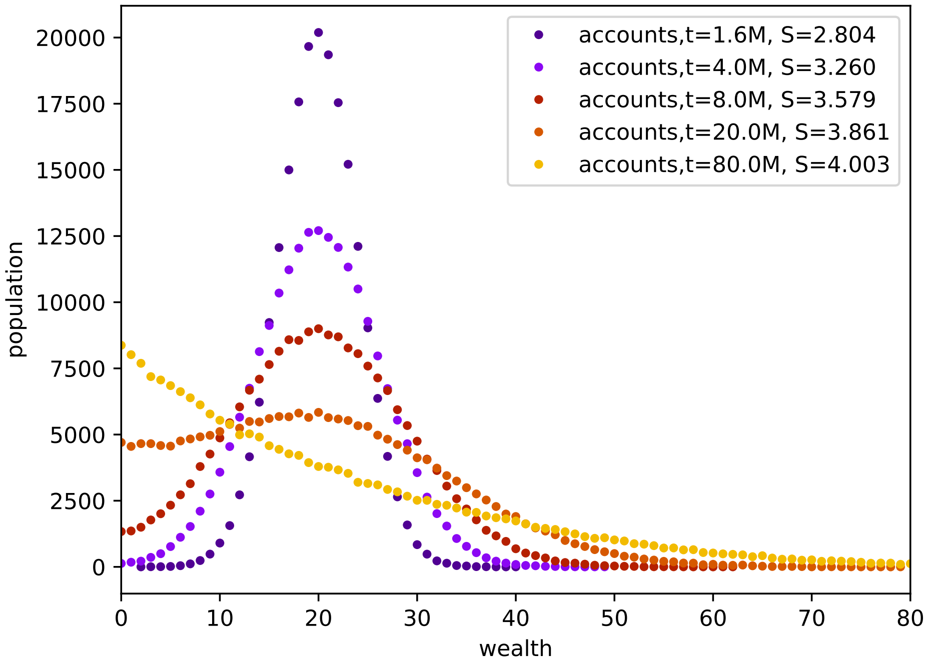


Figure 2.1: A population of $2 \cdot 10^5$ people with $m_0 = 20$ was evolved for 80 million exchanges. Snapshots of the evolution of the Pdf are taken in the times mentioned in the legend. S is the Von Neumann entropy of the distribution.

$$1 = \int_0^{+\infty} P(m)dm = \int_0^{+\infty} \alpha e^{-\frac{m}{T}} dm = \alpha T \Gamma(1) = \alpha T \tag{2.18}$$

where we switched to the a-dimensional variable $x = m/T$, and then recognized the Euler Γ function. This implies $\alpha = 1/T$.

However, if we take that to be the normalization constant, we can easily calculate the temperature in the system with respect to the average amount of money of the agents m_0:

$$m_0 = \int_0^{+\infty} mP(m)dm = \int_0^{+\infty} \frac{m}{T} e^{-\frac{m}{T}} dm = \Gamma(2)T = T. \tag{2.19}$$

We just found that the temperature of the system is equal to m_0. However if we run a simulation for a system with any number of agents, we see that the temperature obtained from the fit is always higher than m_0 and the two quantities are not compatible within the standard error, as shown in figures 2.3 and 2.4.

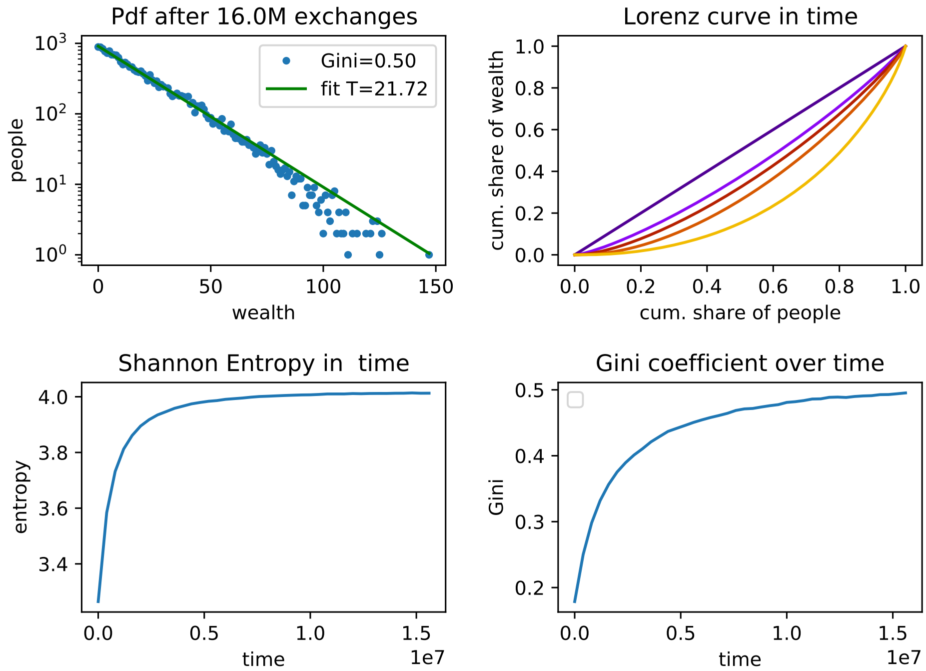


Figure 2.2: A population of 20 thousand agents with $m_0 = 20$ is evolved for 16 million exchanges. On the top left panel, the final Pdf is plotted in semilogarithmic scale, and we can see it is almost thermalized to the exponential distribution. on the top right panel we see the evolution of the Lorenz curve, plotted after 0, 0.4, 0.8, 1.6 and 16 million of exchanges. In the bottom panels we have the time evolution of the Shannon entropy and the Gini coefficient.

This discrepancy arises from the fact that we gave all the agents an integer amount of money, and they exchange one coin at a time, so we know the money variable will be integer valued. Therefore, the condition 2.18 should be written as:

$$1 = \sum_{m=0}^{+\infty} \alpha e^{-\frac{m}{T}} = \alpha \frac{1}{1 - e^{-\frac{1}{T}}} \tag{2.20}$$

In the second equal sign we summed the geometric series. We now have $\alpha = 1 - e^{-\frac{1}{T}}$, which is equal to the one found before only at first order in the parameter $\frac{1}{T}$. Using this result we can again calculate the relation between the temperature T and m_0:

$$m_0 = \sum_{m=0}^{+\infty} mP(m) = \sum_{m=0}^{+\infty} (1 - e^{-\frac{1}{T}}) m e^{-\frac{m}{T}} = \frac{1}{e^{\frac{1}{T}} - 1}. \tag{2.21}$$

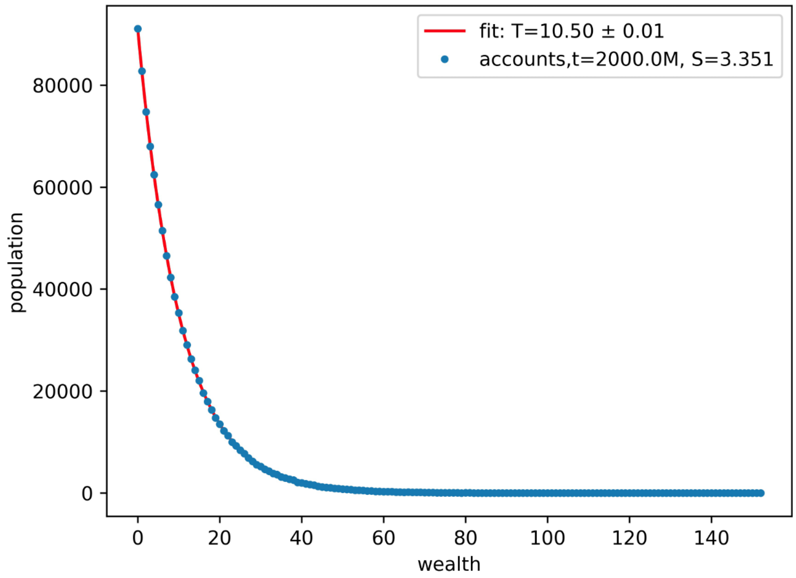


Figure 2.3: Simulated wealth distribution for a population with 10^6 agents with $m_0 = 10$ evolved for $2 \cdot 10^9$ exchanges

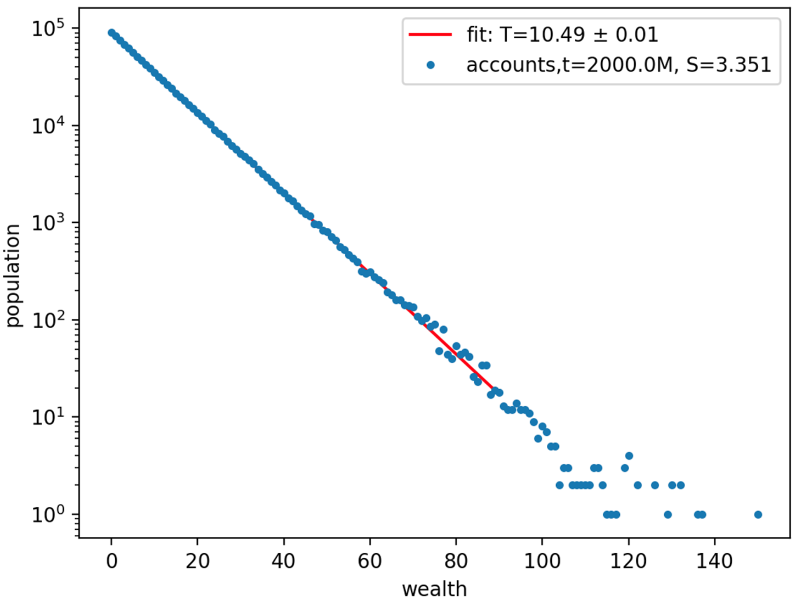


Figure 2.4: Simulated wealth distribution for a population with 10^6 agents with $m_0 = 10$ evolved for $2 \cdot 10^9$ exchanges and plotted in semilog scale.

Now solving for T yields:

$$T = \frac{1}{\ln(1 + \frac{1}{m_0})} \tag{2.22}$$

And taking the Taylor expansion in the small parameter $1/m_0$ we can explore the difference between the continuous case, where $T = m_0$ and the discrete case:

$$T \simeq \frac{1}{\frac{1}{m_0} - \frac{1}{2m_0^2} + \frac{1}{3m_0^3} - \ldots} = m_0 + \frac{1}{2} - \frac{1}{12m_0} + \ldots \tag{2.23}$$

Therefore for high m_0 the difference between T and m_0 approaches $1/2$, in good agreement with our simulations.

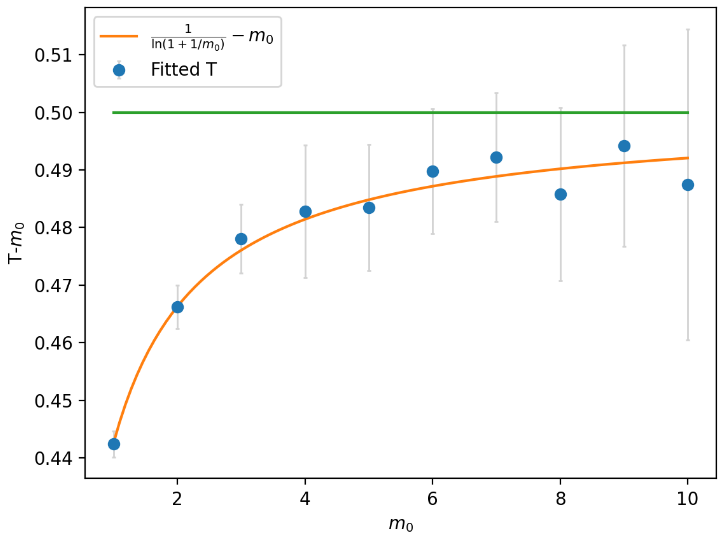


Figure 2.5: Fitted T for every m_0 are obtained by evolving a system of $3 \cdot 10^5$ people for 10^{10} interactions. Standard errors are estimated by evolving 20 copies of such system and taking the standard deviations of the set of fitted temperatures. Notice how when temperature increases, the difference between T and m_0 approaches $1/2$ (the green line in the figure).

To remove every doubt we also show in figure 2.5 the theoretical prediction alongside the fitted temperatures for every m_0. Notice how each measure of the temperature is in good agreement with the theoretical curve and not compatible with zero.

2.7 Economy in presence of debt

What happens if debt is permitted among the agents? If an infinite amount of debt is acceptable then the system undergoes unbounded diffusion. We already wrote the Fokker Planck governing the system in section 2.4, and we rewrite it here for convenience:

$$\frac{1}{r}\frac{\partial P(m)}{\partial t} = \nabla^2 P(m) \tag{2.24}$$

Given this equation, the time evolution of a Dirac delta placed in m_0 at $t = 0$ is:

$$P(m) = \frac{1}{\sqrt{2\pi\sigma(t)}} e^{-\frac{(m-m_0)^2}{2\sigma(t)^2}} \tag{2.25}$$

where $\sigma(t) = \sqrt{rt}$.

Snapshots of the evolution are shown in figure 2.6.

Notice how agents spread symmetrically on both sides of the initial wealth. The distributions widens over time, and some agents are going towards infinitively negative wealth.

The behaviour $\sigma(t) \propto t$ can be observed in figure 2.7, where the fitted sigma is compared with the square root prediction.

If instead only a finite amount of debt m_d is allowed for each agent, then the distribution stabilizes in an exponentially decreasing one, with starting wealth at $-m_d$ and temperature $T = m_0 + m_d$, as shown in figure 2.8. By this observation alone it can be shown that allowing debt increases inequality in the system, because the temperature increases. This fact has been proven true by the introduction of credit cards, that is widely credited with having increased inequality (see for example [20]).

An interesting consequence of the mathematical solution of equation 2.24 is that we could think not to impose a constraint on maximal debt, but on maximal wealth. In this case, if any debt is permitted, the solution is an increasing exponential. This scenario is resemblant of a Soviet-era economic policy, where wealth is confiscated from people richer than a certain threshold.

Xi, Ding, and Wang (2005) [26] considered another boundary condition for debt, which is credited to be more realistic. They constrained not the debt of each agents, but rather the total debt of all the agents in the system.

2.8 Brownian motion and social mobility

The majority of our work will study the system from the perspective of the wealth distribution, focusing on variables that depend on all the agents. In this section instead, we will try to focus on the wealth of a single agent, to see how

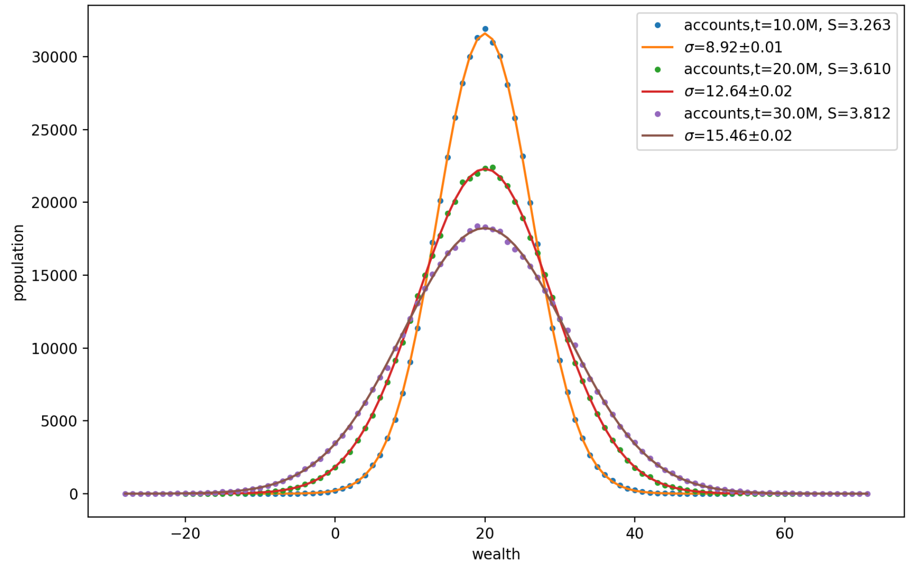


Figure 2.6: A system of $5 \cdot 10^5$ agents with initial wealth of 20 units is evolved for $3 \cdot 10^7$ interactions when debt is permitted. It can be seen that both the Shannon entropy and the fitted sigma of the gaussian increase over time.

it evolves over time. We saw in section 2.4 that our system obeys drift-diffusion equations. We now use that information to try to assess social mobility for the agents.

What is social mobility? It is usually defined as the capability of individuals to move along the social ladder. Since in our model the only information we have for the agents is their wealth, we will need to define it based on the probability of going from a given wealth to another one in a given time frame.

We can use the drift and diffusion coefficients derived in section 2.4 to write down the probability distribution of wealth at time t for an agent that at time t_0 had a wealth of w_0. We will assume for simplicity that the system is already thermalized, in order to be able to consider $P(0)$ as time independent. We will also assume that the time interval $t - t_0$ is short enough that we can neglect the possibility of the agent of having visited zero wealth, in order to be able to use the purely diffusive equation for the whole interval $t - t_0$.

$$P_0(w, t|w_0, t_0) = \frac{1}{\sqrt{2\pi\sigma(t)}} e^{-\frac{(w-w_c(t))^2}{2\sigma^2(t)}} \tag{2.26}$$

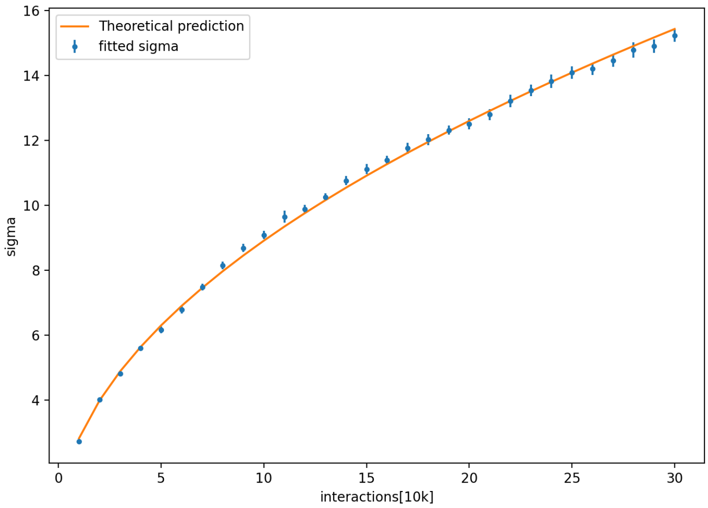


Figure 2.7: A system of 5 thousand agents is evolved for $3 \cdot 10^5$ interactions, and the sigma of the distribution is plotted over time. The $\sqrt{t}$ behaviour is observed in the data.

Where we used:

$$\begin{cases} \sigma^2(t) = B(t-t_0) = r(t-t_0) \\ w_c(t) = w_0 + A(t-t_0) = w_0 - rP(0)(t-t_0) \end{cases} \tag{2.27}$$

We now have the explicit form of the probability of transiting from a given wealth $w_0 \neq 0$ to any other wealth for short timeframes. But if we are interested in long timeframes, we can not neglect the probability of hitting the wall at $m = 0$. In fact, the space traveled due to a negative and constant drift velocity is linear in time whereas the σ of the gaussian is increasing with the square root of time.

But here comes in our rescue a powerful instrument in theoretical physics: the path integral formulation.

The idea is that we can write the real probability $P(w,t|w_0,t_0)$ as the probability of being in (w,t) without ever passing through $w = 0$, and then sum the probability of being there having hit the wall once, end so on. In formulae:

$$P(w,t|w_0,0) = P_0(w,t|w_0,0) + P_1(w,t|w_0,0) + P_2(w,t|w_0,0) + ... \tag{2.28}$$

Where we exploited the time translation invariance of the system at equilibrium and assumed $t_0 = 0$, and the P_0 term is just the free gaussian propagator calculated in 2.26, that we will now rename $G(w,t|w_0,t_0)$.

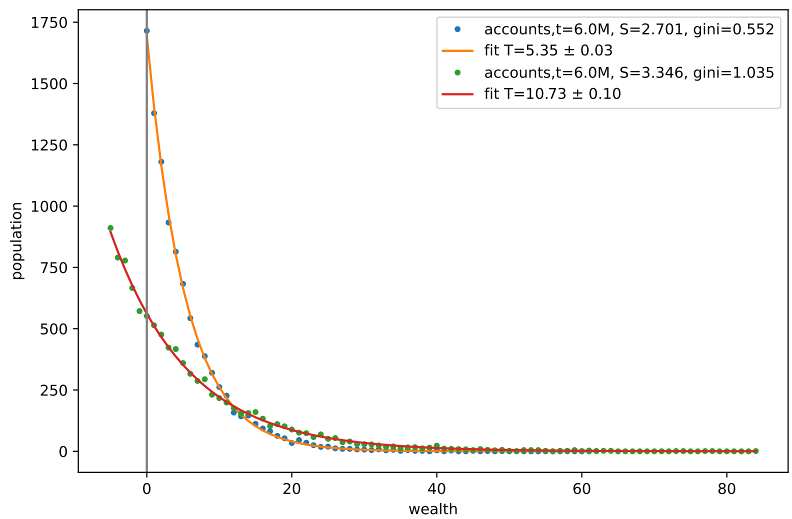


Figure 2.8: A population of 10 thousand agents is evolved until thermalized. Notice how inequality calculated based on the Gini index increases when debt is allowed. A Gini index grater than one is made possible by the fact that the poorest agents have negative wealth, thus the Lorenz curve has negative slope near the origin.

We already wrote the propagator for $w \neq 0$. Let us write now the propagator for $w = 0$ and we will combine them to write the path integral:

$$D(1, t|0, t') = 1 - (1 - P(0))^{t-t'}. \tag{2.29}$$

Having found both propagators, we can now write the probability of going from $(w_0, 0)$ to (w, t) having passed through zero wealth exactly once:

$$P_1(w, t|w_0, 0) = \int_0^t dt' \int_{t'}^t dt'' G(w, t|1, t'')D(1, t''|0, t')G(0, t'|w_0, 0). \tag{2.30}$$

Recognizing the pattern, we can write the complete probability of eq 2.28 as the time-ordered exponential:

$$P(w, t|w_0, t_0) = G\overrightarrow{\exp}(DG) = \overrightarrow{\exp}(GD)G \tag{2.31}$$

Where the factorial in the n-th term of the Taylor expansion of the exponential are given by the fact that we are allowed to integrate only in the volume of the n-simplex where $t' < t'' < ... < t$.

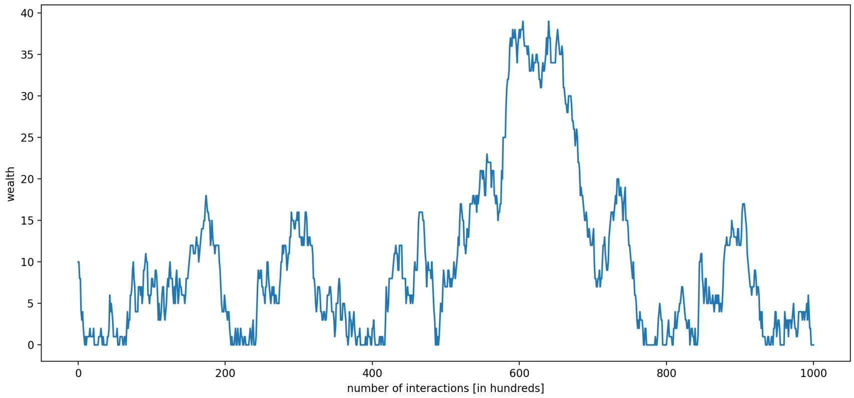


Figure 2.9: The random walk of an agent in a system with $m_0 = 10$. The system was allowed to thermalize before recording his path, in order to let the drift coefficient $A = rP(0)$ be constant in time and equal to $A = \frac{r}{m_0+1}$.

Since even writing the explicit solution as time ordered exponential would give a very cumbersome result and we would need to evaluate the integrals numerically, we decided to adopt a computational approach and calculate the matrix element with a computer simulation.

A population of $5 \cdot 10^5$ agents with initial wealth $m_0 = 2$ was evolved until thermalized. Now 10^4 agents are selected for every wealth between 0 and 5. In figure 2.10 can be seen where those agents are located after time evolution. The element of column n and row m represents the probability of the agent with initial wealth $n-1$ of having wealth $m-1$ at time t.

We can see how, after a long enough time frame, the system loses memory of the initial position of the agents, and the probability of having a given wealth is independent of the initial wealth and distributed along the exponential curve.

To better visualize the independence on the initial condition, we also produced figure 2.11, where each row is divided by the probability of the agents to have that wealth at equilibrium. After a long enough timeframe, the matrix has lost all the information about the initial positions, and is uniform in colour.

The time scale in which this matrix becomes uniform in colour can be taken as an indicator of social mobility: thus models with shorter timescales will be judged more mobile than model with longer timescales. Notice how the fact that uniformity in the matrix is achieved is a particularity of the DY model, and in principle we can think of models where having a certain wealth at time t_0 changes forever the probabilities of reaching other wealths. This will be the case for the

DY model with investment that we will explore in chapter 4, and in general is possible whenever the distribution does not reach a stationary state.

We stress that the method we developed to visualize social mobility can not only be used for the DY model, but also for any model for which we know the exchange rules between the agents.

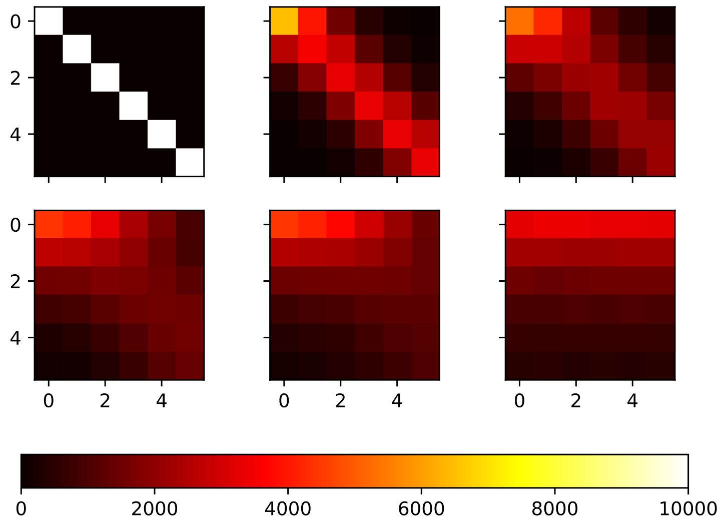


Figure 2.10: measure of position of the agents while time elapses. The evolution routine makes $\Delta t = 5 \cdot 10^5$ exchanges, and has been called (from top left to bottom right): (0,1,2,4,8,100) times

2.9 Confronting the data

In this section we are going to compare the theoretical prediction on the wealth distribution by the DY model with real data from all over the world. For the DY model to be confirmed by empirical evidences we would need to see exponential distribution of wealth across all geographies and for each historic periods. Unfortunately wealth distribution is hard to find, because in most nations is not compulsory to declare wealth, and thus can only be estimated by using proxies, like inheritance taxes. But what is widely accessible to the public is the income distribution, because people have to declare income to pay income taxes. So in this section we will explore the income distribution across geographies and across time.

Let us start by figure 2.12, in which income distribution in the U.S and U.K. is

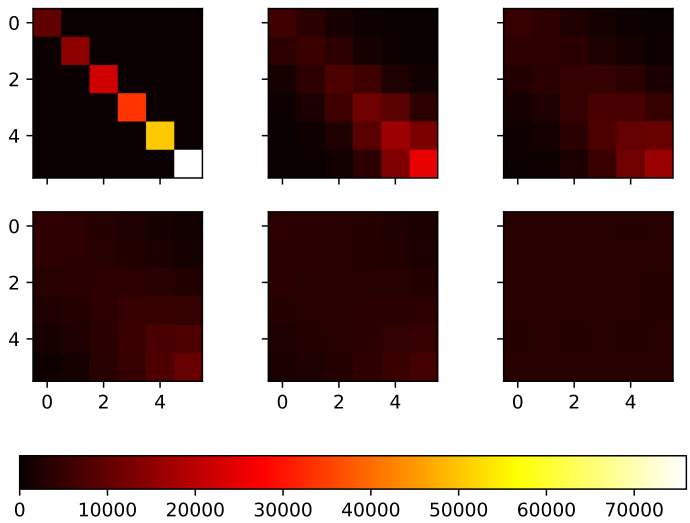
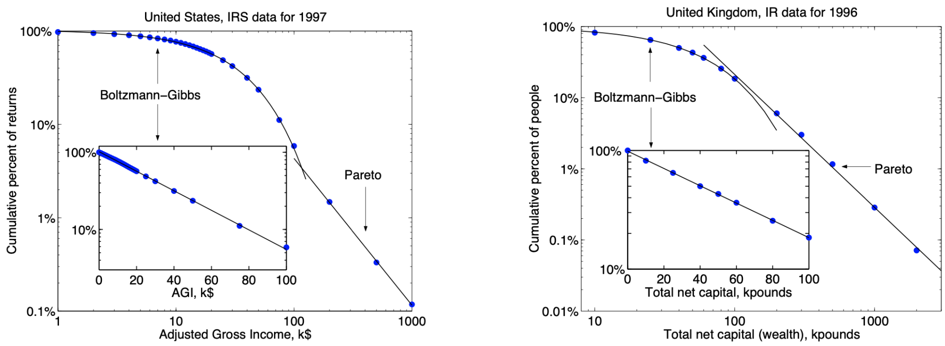


Figure 2.11: measure of position of the agents while time elapses. The evolution routine makes $\Delta t = 5 \cdot 10^5$ exchanges, and has been called(from top left to bottom right) : (0,1,2,4,8,100) times. The rows are normalized with the Equilibrium probability factor for that wealth. Notice how all the information about initial wealth is lost.

shown. In both countries the lower 95% of the income distribution is well fitted with an exponential. The remaining tail is well fitted by a power law.

We claim, and we will prove it later in the chapter, that the two behaviours of the distribution are due to different sources of income: the poorest 95% of the people gains mainly with labour income, whereas the top 5% gains mainly with financial investments.

In figure 2.13 the income distribution in 67 countries is plotted. The exponential behaviour is observed across the board. We can point out two interesting features: First, in the countries where welfare measure are important, like in Scandinavia, the exponential behaviour interrupts for wealths of the order of the average wealth. This is the result of subsidies from the government, that cause the poor population to be drastically reduced with respect to the exponential prediction. Another interesting feature can be seen to the other end of the spectrum: the location of rightest point is an indication of when capital gains play a major role in the income with respect to labour. For business friendly country, like the U.S. and Singapore, the exponential interrupts at 3 times the average wealth, whereas for European countries it extends to 5-7 times the average wealth.



(a) (b)

Figure 2.12: (a) Income distribution in the U.S. in 1997. (b) Income distribution in the U.K in 1996. Both the images are taken from Dragulesku & Yakovenko [8].

In figure 2.14(a) we can see how the income distribution varies over time. If we rescale the incomes of a given year for the average income in that year, we see that the exponential part of the distribution is extremely stable year after year.

The Pareto tail on the other hand, always keeps the power law behaviour, but the parameters have big swings in the years. This phenomenon can be explained by looking at figure 2.14(b). The percentage of total income that goes to the tails f, is very sensitive to the trends of the financial market, whereas the temperature of the exponential bulk is almost indifferent to it. In the next subsection, we are going to motivate this difference with the help of the Fokker Planck equations

2.9.1 Fokker-Planck equations

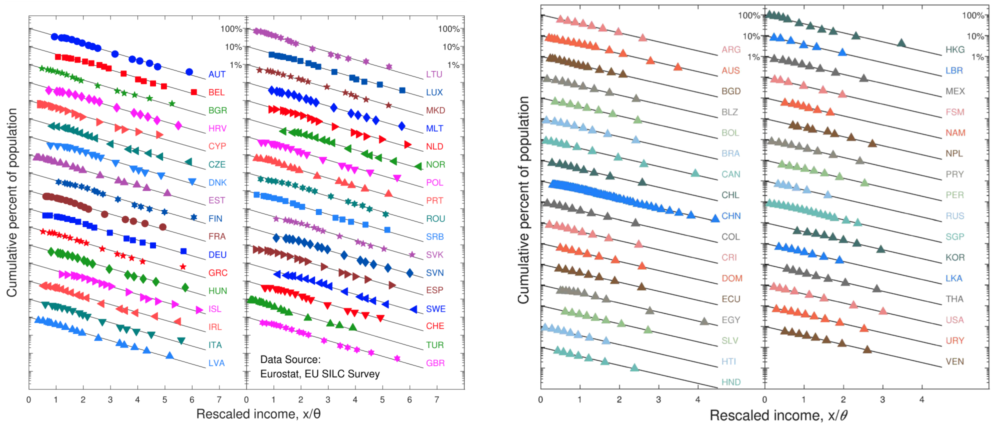
We already encountered the general form of the Fokker-Plank equation in equation 2.14, we rewrite it here for convenience:

$$\frac{\partial P(m)}{\partial t} = \partial_m(A(m)P(m) + \partial_m(B(m)P(m))) \tag{2.32}$$

with coefficients A and B potentially depending on m, and being equal to

$$A(m) = -\left\langle \frac{\Delta m}{\Delta t} \right\rangle, \quad B(m) = \left\langle \frac{(\Delta m)^2}{2\Delta t} \right\rangle \tag{2.33}$$

We also observed that in the particular case of the DY model the coefficient A and B were independent of m. The process described by the Fokker-Planck



(a) (b)

Figure 2.13: Income distribution in (a) Europe and (b) The rest of the world. Images are obtained by rescaling the incomes with respect to the average income of the country θ and plotting in semilogarithmic scale. Both the images are taken from Yong Tao et al. [25].

equation with these coefficient is known as additive diffusion. We can explicitly find the stationary solution to 2.32. First, let us observe that we can get rid of a derivative, and get:

$$\frac{\partial}{\partial m}(BP(m)) = -AP(m), \tag{2.34}$$

This is true because we integrate and write equation 2.34 with the addition of a current indipendent on m. But since in our case we have reflective boundary conditions at $m = 0$, meaning no agent can enter or leave the system from $m = 0$, the current is zero there, so it is zero everywhere. We can now solve equation 2.34 and get $P(m) = \alpha e^{-m/T}$, with α being fixed by the normalization condition, and $T = \frac{B}{A}$. This re-demonstrates the Boltzmann-Gibbs distribution.
Interestingly enough, if we assume that the length of the random walk steps of the agents to be linear with their wealth, and scale the new A and B coefficient accordingly:

$$A(m) = -\left\langle \frac{\Delta m}{\Delta t} \right\rangle = am, \quad B(m) = \left\langle \frac{(\Delta m)^2}{2\Delta t} \right\rangle = bm^2 \tag{2.35}$$

we can still solve Equation 2.34 and find the stationary solution:

$$P(m) = \alpha \frac{1}{m^{1+\beta}} \qquad \text{with} \qquad \beta = 1 + \frac{a}{b} \tag{2.36}$$

Which is the distribution found by Pareto. Retrospectively assumption 2.35 does not seem too much out of the blue if we consider that the majority of the income for the richest is due to capital gains, and that they are naturally proportional to the initial wealth. Therefore this solution should describe the richest tail of the population.

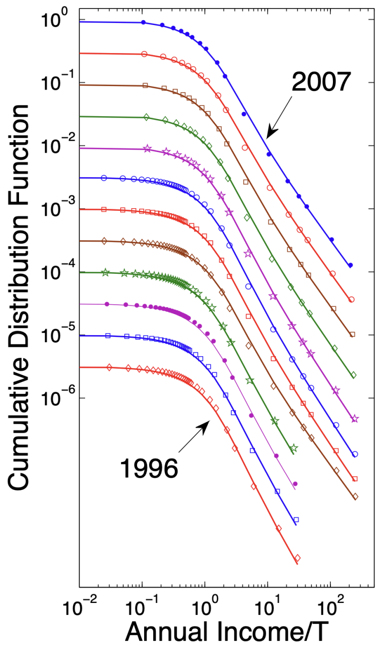


Figure 2.15: Distribution of income in the U.S. form 1996 to 2007, with a and b as fit parameters. Taken from Banerjee [3]. The curves are shifted vertically to better visualize the data, that would otherwise have an almost complete superimposition.

We point out that the assumption of random walk steps proportional to wealth, can not be justified by a pairwise money exchange model with time reversal symmetry, like the DY model, as can be understood from the discussion we had in section 2.4, and so can be thought as coming from a single particle approach, where money is conserved only on average. This resembles the difference between the microcanonical ensemble, where conservation of energy is imposed as a constraint, and the canonical one, where it is enforced only on average due to energy exchanges with the heat bath.

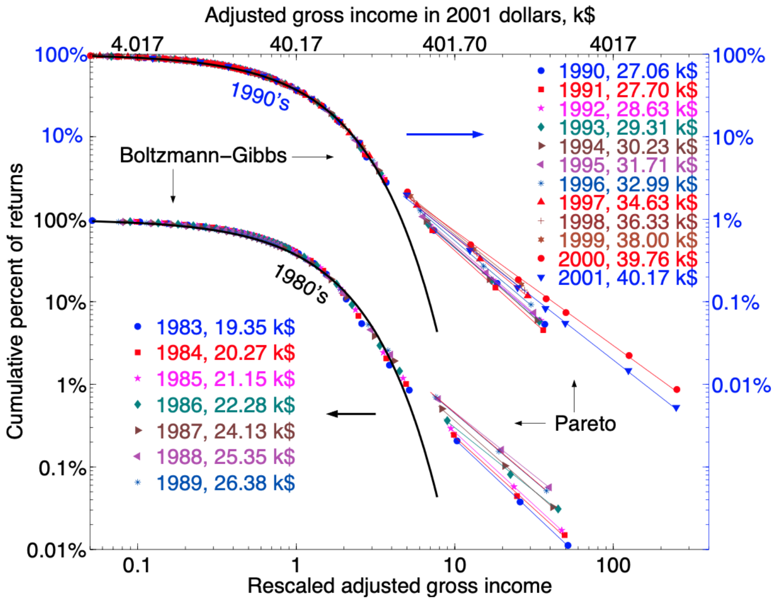
If we consider both the sources of income, being labour income, which is independent of wealth, and capital gains, which is proportional to wealth, then we can rewrite the coefficients A and B as

$$A(m) = A_0 + am, \quad B(m) = B_0 + bm^2 \tag{2.37}$$

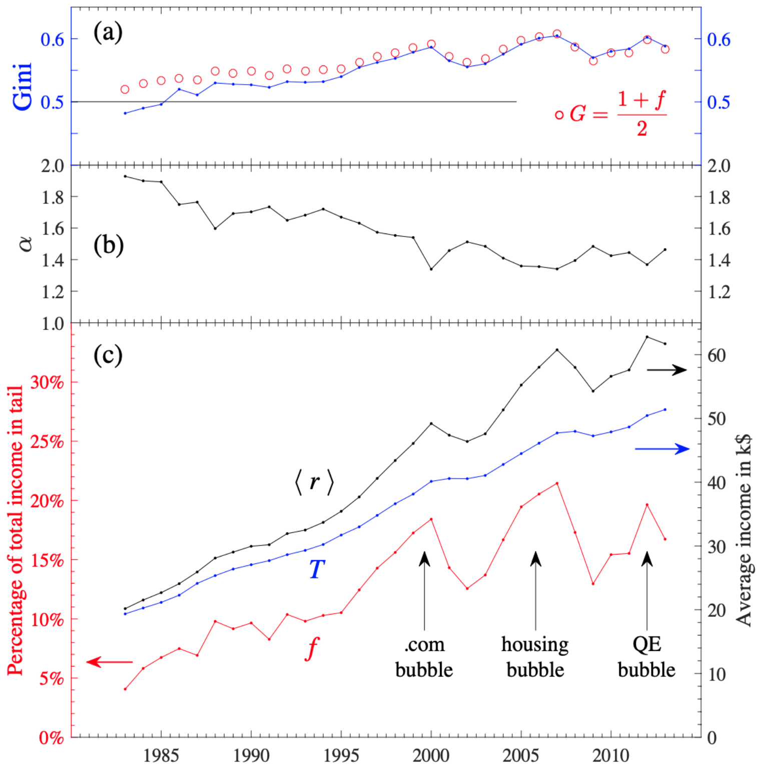
and solving equation 2.34 with these coefficient yields:

$$P(m) = \frac{Ce^{-\frac{m_0}{T}\arctan\frac{m}{m_0}}}{\left[1 + (\frac{m}{m_0})^2\right]^{1+\frac{a}{2b}}}. \tag{2.38}$$

In figure 2.15 we can see how the real income distribution in the U.S. economy from 1996 to 2007 seems to match the predicted distribution.



(a)



(b)

Figure 2.14: (a) Income distribution in the U.S. from 1983 to 2001 taken from [22]. (b) plot over time of the Gini coefficient G, the percentage of income in the tail f, the average income in the whole system $\langle r \rangle$, the temperature of the exponential bulk T, the Pareto exponent α, taken from [28]

Chapter 3

Thermalization time in the DY model

In this chapter we will investigate the behaviour of the perturbation of the equilibrium distribution, and demonstrate that they fade away in an exponential fashion. We will discover that our system presents bands similar to the one we encountered in solid state cristals, where only perturbations with a specific relation between energy and momentum are allowed. This study will be critical in chapter 6, in order to study the effects of one-time redistribution policies.

To the best of my knowledge, none of the contributions given in this chapter can be found in the literature, and should therefore be considered as original.

3.1 The problem

It might be hard to assess how much time will it take to relax to the equilibrium from a general configuration, but we can definitely try to study the functional form of the distance between our distributions and the equilibrium one over time. If we get lucky, and we are able to find a kind of distance that decreases exponentially with time, then we can take the characteristic time scale of the exponential to be the relaxation time. Notice how this reasoning would not work for a distance that decreases in a power law way for example, since power laws are scale invariant.

Let us begin by recalling the Fokker-Plank equation for our model. We will use the operators ∂_m and ∇^2 intending them in the discrete form, in order to make the notation more compact. The partial derivative operator ∂_m is intended as the left derivative, such that $\partial_m A(m) = A(m) - A(m-1)$, whereas the Laplacian is as usual $\nabla^2 A(m) = A(m+1) + A(m-1) - 2A(m)$.

$$\begin{cases} \frac{1}{r}\frac{\partial P(m)}{\partial t} = P(0)\partial_m P(m) + \nabla^2 P(m) & \text{if } m > 0 \\ \frac{1}{r}\frac{\partial P(0)}{\partial t} = P(1) - P(0) + P(0)^2 & \text{if } m = 0 \end{cases} \tag{3.1}$$

The equations are not linear. Since we are far better at solving linear equations, and we are interested in knowing the relaxation time scale for distributions near the equilibrium, the idea is to take a distribution that is close to the equilibrium one, and thus can be written as:

$$P(m) = P_{eq}(m) + \epsilon\delta(m), \tag{3.2}$$

where ϵ is assumed to be small, and the normalization condition requires

$$\sum_{m=0}^{\infty} \delta(m) = 0.$$

Using that P_{eq} is defined by requiring that equations 3.1 hold with the left hand-side equal to zero, we can find the equations for the distribution $\delta(m)$. We will only take the first order in ϵ:

$$\begin{cases} \frac{1}{r}\frac{\partial \delta(m)}{\partial t} = \delta(0)\partial_m P_{eq}(m) + P_{eq}(0)\partial_m \delta(m) + \nabla^2\delta(m) + O(\epsilon) & \text{for } m > 0 \\ \frac{1}{r}\frac{\partial \delta(0)}{\partial t} = \delta(1) - \delta(0) + 2P(0)\delta(0) & \text{for } m = 0 \end{cases} \tag{3.3}$$

Having found these equations, we now give a rigorous proof of the fact that all the $P(m)$ will tend to the equilibrium ones exponentially in time, in other words the $\delta(m)$ will go to zero exponentially in time for each m.

3.2 Proof

Lets consider the system of equation for our model for the variations $\delta(m)$: If we remember that ∂_m and ∇^2 are finite difference operators, we can rewrite the whole system as:

$$\frac{1}{r}\frac{\partial \vec{\delta}}{\partial t} = M \cdot \vec{\delta}, \tag{3.4}$$

where we defined $\vec{\delta}$ as the infinite vector having components $(\delta(0), \delta(1), ...)$, and M as the matrix that enforces the system of equations 3.3. From this perspective, we can write the solution as:

$$\vec{\delta}(t) = e^{Mrt}\vec{\delta}(0), \tag{3.5}$$

making it clear that the perturbations decay exponentially in time if the matrix M is negative definite. Unfortunately we were not able to show such a statement in a simple way, for example by localizing the eigenvalues with the Gershgorin theorems. But the proof can be carried out nonetheless. Consider the matrix M:

$$\begin{pmatrix} 2P_E(0)-1 & 1 & 0 & 0 & 0 & ... \\ \partial_m P_E(1)+1-P_E(0) & P_E(0)-2 & 1 & 0 & 0 & ... \\ \vdots & 0 & 1-P_E(0) & P_E(0)-2 & 1 & ... \\ \partial_m P_E(m) & 0 & 0 & 1-P_E(0) & P_E(0)-2 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.6}$$

It can be decomposed as the sum of two matrices: the matrix A that has just the first column, and matrix B that is tridiagonal:

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & ... \\ \partial_m P_E(1) & 0 & 0 & 0 & 0 & ... \\ \partial_m P_E(2) & 0 & 0 & 0 & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & ... \\ \partial_m P_E(m) & 0 & 0 & 0 & 0 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix} \tag{3.7}$$

and

$$B = \begin{pmatrix} 2P_E(0)-1 & 1 & 0 & 0 & 0 & ... \\ 1-P_E(0) & P_E(0)-2 & 1 & 0 & 0 & ... \\ 0 & 1-P_E(0) & P_E(0)-2 & 1 & 0 & ... \\ 0 & 0 & 1-P_E(0) & P_E(0)-2 & 1 & ... \\ 0 & 0 & 0 & 1-P_E(0) & P_E(0)-2 & ... \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}. \tag{3.8}$$

Notice how matrix B contains terms of order 1, whereas matrix A contains terms of order $1/T$. This is true because $P_E(m) = \alpha e^{-\frac{m}{T}}$, and A contains the derivatives of it.

For this reason, we will focus for now on finding the eigenvalues of the matrix B, knowing the perturbation of A will be negligible for high temperatures, and find the exact eigenvalues of M in subsection 3.2.2

3.2.1 Finding the eigenvalues of B

The general eigenvector of B will be in the form:

$$\begin{pmatrix} 1 \\ \alpha_1 \\ \vdots \\ \alpha_n \\ \vdots \end{pmatrix} \tag{3.9}$$

where the first component can not be 0 otherwise for any eigenvalue also the second component can be shown to be zero, and, thanks to the tridiagonal form, this can extend to all the components.

The eigenvalue λ is determined by α_1 and is obtained by the first eigenvalue equation:

$$\lambda = \alpha_1 - 1 + 2P_E(0) \tag{3.10}$$

From now on we will replace $P_E(0)$ with P_0 in order to get the notation lighter. By the tridiagonal nature of the matrix, it is easy to build a recursive relation binding the coefficients α_i:

$$\alpha_{n+1} = (\lambda - P_0 + 2)\alpha_n + (P_0 - 1)\alpha_{n-1} \tag{3.11}$$

This recursive equation can be solved by assuming the functional form of the coefficients to be $\alpha_n = k_\lambda^n$, where the subscript λ is placed to keep in mind that k is a function of λ. With this assumption we can solve the second order equation for k_λ:

$$k_\lambda = \frac{1}{2}\left(\lambda - P_0 + 2 \pm \sqrt{(\lambda - P_0 + 2)^2 - 4(1 - P_0)}\right). \tag{3.12}$$

We just found that our system presents a band behaviour very similar in nature to the ones we could find in solid state physics, where the eigenvalue λ serves the purpose of energy, and the eigenvector label k_λ plays the role of momentum. Now we proceed in our findings and further constrict the allowed relaxation times for the perturbations.

In figure 3.1 and 3.2 the bands are plotted for respectively infinite temperature, and the lowest possible temperature $T = 1$. Now in order not to make the coefficient α_n diverge we need $|k_\lambda| < 1$, which constrains the possible eigenvalues.

Furthermore, we are now going to show that we must discard as non physical all the eigenvalues related to positive k_λ. In fact, assume there is at least a perturbation on our system that has a non-null component in the eigenvector $0 < k_\lambda < 1$ (if there is more than one consider the one with maximum λ).

After a while, since the eigenvalue λ of this perturbation is greater than all the perturbation for $k_\lambda < 1$, as can be seen from figures 3.1 and 3.2, this perturbation

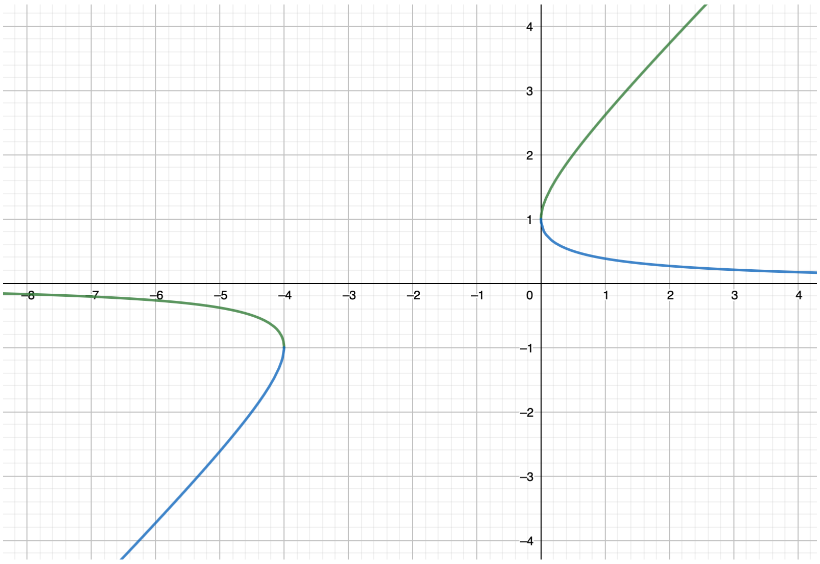


Figure 3.1: The bands for infinite temperature. On the x-axis we have the eigenvalue λ, and on the y-axis we have k_λ. As you can see, only eigenvalues $\lambda < 4$ are admissible.

will dominate the system. If the coefficients of such eigenvector were all positive, this would be in stark violation of the condition

$$\sum_{m=0}^{\infty} \delta(m) = 0, \tag{3.13}$$

that is equivalent to the normalization of probabilities and must hold at all times.

We now show that this is the case, and no perturbation can be generated in the system for $k_\lambda > 0$ and $\lambda > -2 + P_0$. Furthermore, due to the bounds on $P_0 = \frac{1}{m_0+1} < 1/2$ and $P_0 > 0$, perturbations should be searched for in the left branch of the figures 3.1 and 3.2.

Let us start with the proof: consider the eigenvalue equation for the coefficient α_n:

$$\lambda \alpha_n = (1 - P_0)\alpha_{n-1} + \alpha_n(P_0 - 2) + \alpha_{n+1} \tag{3.14}$$

Now solving for α_n yields:

$$\alpha_n = \frac{(P_0 - 1)\alpha_{n-1} - \alpha_{n+1}}{P_0 - 2 - \lambda}. \tag{3.15}$$

And rearranging eq 3.15 to have one coefficient as a function of the following two we get:

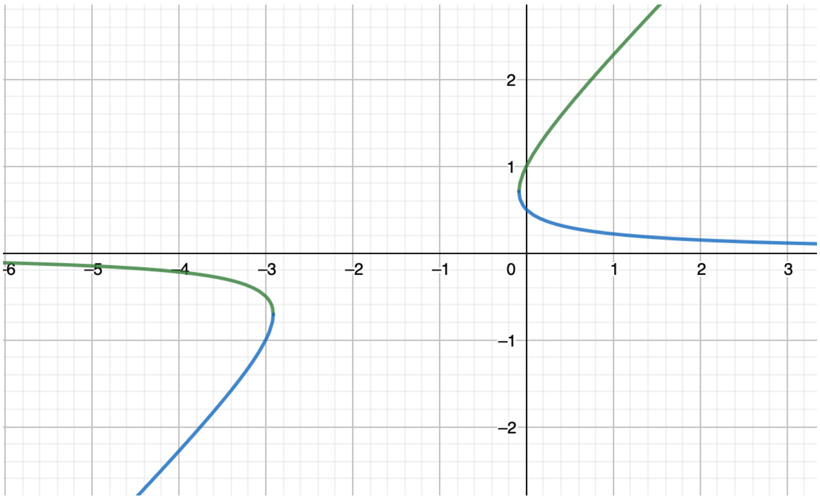


Figure 3.2: The bands for the lowest temperature $m_0 = 1$. In this case, only eigenvalues $\lambda < -\frac{3}{2} - \sqrt{2}$ are admissible.

$$\alpha_n = \frac{(P_0 - 2 - \lambda)\alpha_{n+1} + \alpha_{n+2}}{P_0 - 1} \tag{3.16}$$

We can now prove that all the alphas will be positive using induction and going backwards from infinity. If we fix a small number ϵ, for n sufficiently high, the relation $|\alpha_{n+2} - k_\lambda \alpha_{n+1}| < \epsilon$ holds, implying for such n that α_n will satisfy:

$$\left| \alpha_n - \frac{P_0 - 2 - \lambda + k_\lambda}{P_0 - 1} \right| < 2\epsilon. \tag{3.17}$$

Since for these hypothetical eigenvalues $\lambda > -1$ and $k_\lambda > 0$ (from figures 3.1 and 3.2 and the dispersion relation 3.12), then α_n will have the same sign as α_{n+1} and α_{n+2}. But this is absurd since a perturbation with all the coefficients of equal sign can not satisfy the normalization of probabilities 3.13.

With the aforementioned considerations, we observe that, since the highest λ will give the longer thermalization time, we can explicitly find the relation linking the thermalization time scale to the temperature of the system by looking at the left branch in figures 3.1 and 3.2

$$\lambda = -2 + \frac{1}{m_0 + 1} - 2\sqrt{\frac{m_0}{m_0 + 1}} \tag{3.18}$$

We have to keep in mind though, that not all the perturbations in the left branch are compatible with the normalization of probabilities, so in principle the longer lasting perturbation will be the one of the left branch to also satisfy condition 3.13. Since an analytical solution to 3.13 is not possible, we evaluated it numerically as a function of λ with a Python script you can find in appendix B and obtained for infinite temperature figure 3.3. As you can see, for infinite temperature, only the **perturbations with $\lambda = -4$ are allowed**.

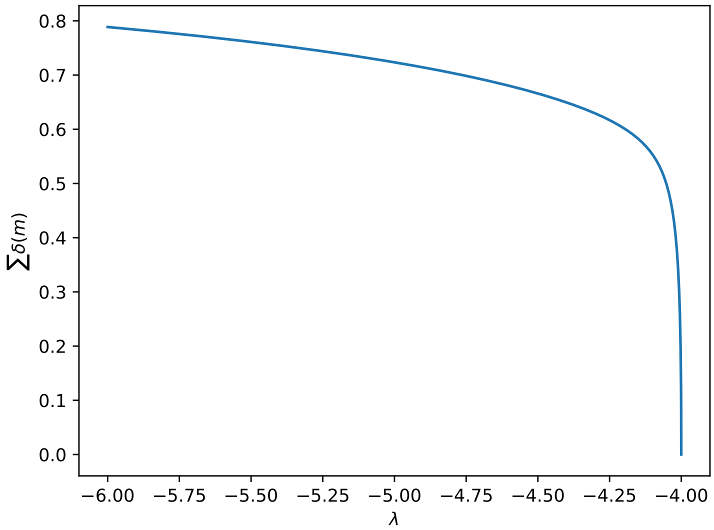


Figure 3.3: The sum of coefficient of the perturbation $\delta(m)$ as a function of λ for infinite temperature. Only the perturbations for $\lambda = -4$ are allowed.

3.2.2 Finding the eigenvalues of M

We now end the proof by demonstrating that the eigenvalue $\lambda = -4$ is also the maximum allowed eigenvalue for the complete matrix $M = A + B$. The idea will be to find the spectrum of the new matrix M in the same manner in which we found the spectrum of B. As we saw before, the allowed bands for the eigenvalues of B were found imposing the convergence of the components of the eigenvectors.

We now do the same for M. Consider the new eigenvalue equations:

$$
\begin{pmatrix}
2P_E(0)-1 & 1 & 0 & 0 & \dots \\
1-P_E(0)+\partial_m P_E(1) & P_E(0)-2 & 1 & 0 & \dots \\
\partial_m P_E(2) & 1-P_E(0) & P_E(0)-2 & 1 & \dots \\
\vdots & 0 & 1-P_E(0) & P_E(0)-2 & \dots \\
\partial_m P_E(m) & 0 & 0 & 1-P_E(0) & \dots \\
\vdots & \vdots & \vdots & \vdots &
\end{pmatrix}
\begin{pmatrix}
1 \\ \alpha+\delta\alpha \\ \beta+\delta\beta \\ \gamma+\delta\gamma \\ \epsilon+\delta\epsilon \\ \vdots
\end{pmatrix}
= \lambda
\begin{pmatrix}
1 \\ \alpha+\delta\alpha \\ \beta+\delta\beta \\ \gamma+\delta\gamma \\ \epsilon+\delta\epsilon \\ \vdots
\end{pmatrix}
\tag{3.19}
$$

Where we did not write $\lambda+\delta\lambda$ as the eigenvalue because we used our reparametrization degree of freedom to set $\delta\lambda = 0$. In other words the spectrum of the matrix M has divergencies for some λ, and we want to find those λ. We can find the new coefficients of the eigenvector using that α, β etc. were defined to hold with matrix B, thus the equations for the $\delta\alpha_n$ are:

$$
\begin{cases}
\delta\alpha = 0 \\
\delta\beta = \partial_m P_E(1) \\
\delta\gamma = (\lambda+2-P_0)\delta\beta - \partial P_E(2) \\
\vdots
\end{cases}
\tag{3.20}
$$

And from the fourth coefficient onwards we have the general formula:

$$
\delta\alpha_{n+1} = (\lambda+2-P_0)\delta\alpha_n + (P_0-1)\delta\alpha_{n-1} - \partial_m P_E(n) \tag{3.21}
$$

where we renamed the n-th coefficient α_n.

Let us now assume the relation for the coefficients: $\alpha_n = ce^{-n/T}$ which leads to this expression for the constant c:

$$
c = -\frac{1}{T} \cdot \frac{1-e^{-1/T}}{e^{-1/T} - (\lambda+2-P_0) + (1-P_0)e^{-1/T}} \tag{3.22}
$$

For $\lambda < 0$ this expression is always negative and never diverging, thus the coefficients $\delta\alpha_n$ never diverge for the eigenvalues we are interested in. Therefore the eigenvalues λ that cause the coefficients to diverge are the ones already found when diagonalizing B.

$$
\lambda = -2 + \frac{1}{m_0+1} - 2\sqrt{\frac{m_0}{m_0+1}} \tag{3.23}
$$

Chapter 4

DY model extensions

In chapter 1 we explored agent-based macroeconomics, focusing on his multiple advantages over more traditional and established economic models, like the DSGE models.

An important downside to agent based simulations though, is that is often hard to guess which will be the outcome of a simulation with a given set of parameters. This feature, not only makes this models less intuitive and harder to study, but it also makes it more difficult to trust in a real world setting. If one is not able to measure precisely all the parameters needed to run a simulation, it would still be possible to tell with a certain accuracy which will be the outcome. This can be done most of the times with Monte-Carlo methods, evolving different copies of the system with different parameters and looking at the outcome, but some of the intuition around it is lost, and for large sets of parameters, the computational power to perform this kind of simulations is not always at reach.
The bottom line is that we would like to be able to address economic policy with a more predictable and intuitive instrument.

On the other side of the spectrum lies econophysics, for which the elegance of the formulation seems to be more important than the ability to explain real world phenomena.

In this chapter we will try to fuse together the complexity of agent-based models and the predictability of econophysics to give birth to a new model.

As we will see in chapter 5 and 6, conserving both the mathematical formalism of econophysics and the ABM approach will give rise to applications that would not be possible doing only one of the two.

To the best of my knowledge none of the additions to the DY model we will discuss in this chapter is already in the literature, and should therefore be considered as original.

4.1 Road map

We will first debate inequality for the pure DY model and for the multiplicative random exchange model, two well known models in the literature. Then, our addition to the DY model are proposed, and the effects of each one are analyzed with computer simulations. The first extension will be the financial investments, as a way in which money can be generated. Observing the polarization this brings between rich and poors, we will try to add consume to even up growing economic inequality. By observing the failure of consume to stabilize the wealth gap, we will add an endowed income, that acts like a welfare measure. We will show that this will not stop polarization, but will substantially decrease the number of poor people. We will find that the only way polarization can be avoided is by means of a wealth tax that will absorb the profits made from the financial investments of the rich. In the last section, the Fokker-Planck terms associated to every contribution are discussed, bringing the intuitiveness of econophysics in this set of real world economic phenomena.

We stress here two points that are central to the development of our model: The first is that mechanism of money creation and destruction we introduced are crucial to recover the dynamical aspect of the wealth distribution, hence enabling us to compare our work directly with Piketty's one. The second point concerns our desire to start from the DY model and then add the mentioned contributions, and not from any other model of exchange. This choice is not accidental: in fact, as we already saw in chapter 2, the exponential distribution is the one that maximizes the statistical weight, and hence the one with more volume associated to it in the phase space. We can see the DY exchanges in the phase space as a sort of attracting force that takes any configuration and brings it to the exponential. With this picture the DY model can be seen as a way of accounting for unknown mechanisms in economic interactions, that assuming they are unbiased, will naturally lead to the most probable distribution.

In our work, the ratio of number of DY interactions to the other interactions like investments etc., was a parameter that we had to set arbitrarily. We argue that with the addition of further economic phenomena to the model, the contribution of the DY model can be reduced, until almost completely disappearing.

4.2 Empirical evidences

In this section we explain in detail how we are going to implement consume, financial investments, income, and taxes, and motivate our choices basing our assumptions on empirical evidences.

We bring forward two main points: the first is the empirical evidence on buffer stock by Carroll [4], that indicates that in a given country people save a relatively fixed number of salaries, and consume the remaining wealth.

The marginal propensity to consume *mpc* of the people is defined as how many cents people consume for every dollar that they gain on top of what they already have. It is usually a function of wealth, so if an agent that has 100 consumes 30 cents for an additional unit of wealth, his *mpc* is 0.30, but the *mpc* might change for an agent with wealth of 1000. In order to estimate the *mpc* of our model, we use the second empirical evidence by Violante [10], that strongly suggests that the marginal propensity to consume of the people is a decreasing function of their wealth.

Another important choice we have to make concerns the amount of capital that each agent chooses to invest given his wealth. We divide financial investments into two types: savings, which are more typical for agents with low wealth and are characterised by low risk and low percentage returns, and what we simply call investments, that are more typical of high wealth individuals, with higher percentage returns, and are usually riskier. In the model we will develop in the next section, this distinction will not be crucial, and savings and investment can be though as being equal. Further extension of this model though, might benefit from this distinction.

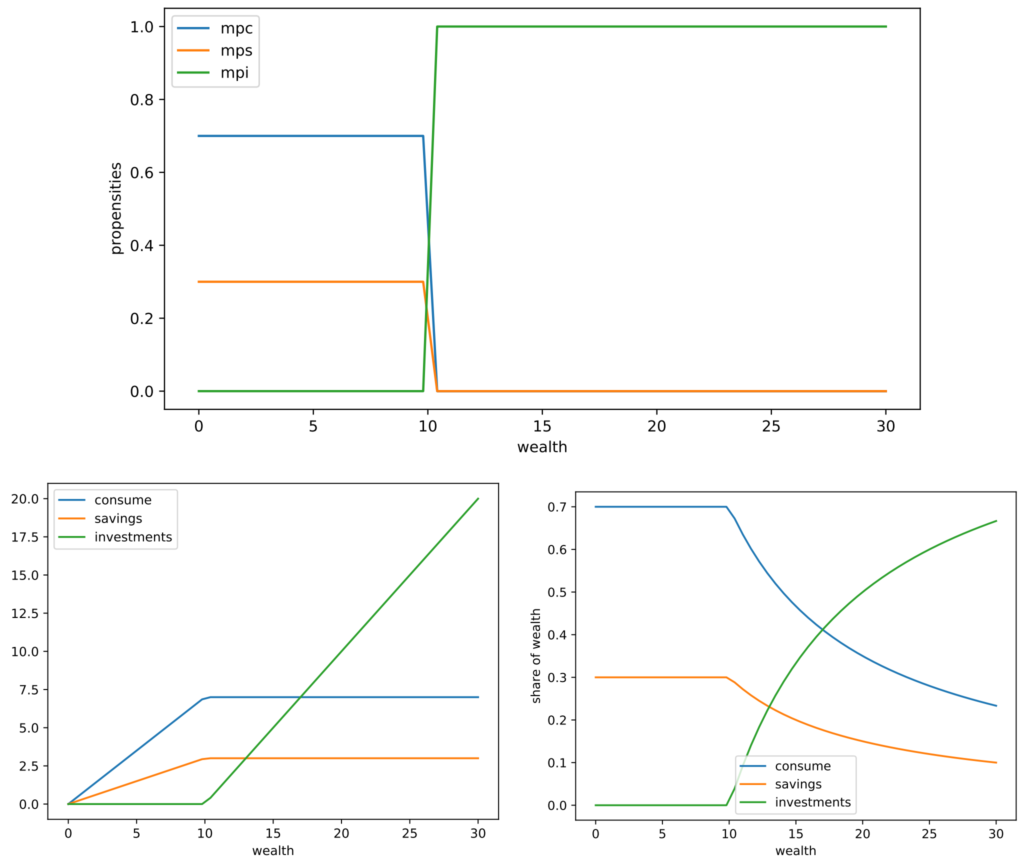
Based on all these facts, we will assume the consume, investment and savings functions to be of the type sketched in figure 4.1.

The growth rate of capital is exogenous for now, but we can think of future generalizations of this model where to a certain Pdf is assigned a production function, for example the Cobb-Douglas one, and so the available capital and working hours of the agents produce real new wealth by means of real investments, thus making the interest rate endogenous. More details on this process can be found in chapter 5, when addressing fiscal policy.

We try to bring some uniformity in the following discussion by showing the same quantities plotted for all the models, in order to facilitate comparisons.

4.3 Drăgulescu-Yakovenko model

We show the wealth distribution (pdf), the Lorenz curve, and other relevant quantities for the pure DY model, in order to see how these change with the proposed contributions. As can be seen from figures 4.2, the system thermalizes to the exponential distribution, and inequality develops. In figure 4.3, money is showed to be a constant in the system, and the time evolution of entropy and the Gini coefficient are presented.



(a) (b)

Figure 4.1: (Top) The chosen marginal propensity to consume, save, and invest as a function of wealth. (bottom left) The implied cumulative consume, saving and investment functions. (bottom right) The fraction of wealth that is consumed, saved and invested. For all the three figures we chose the threshold wealth to be 10.

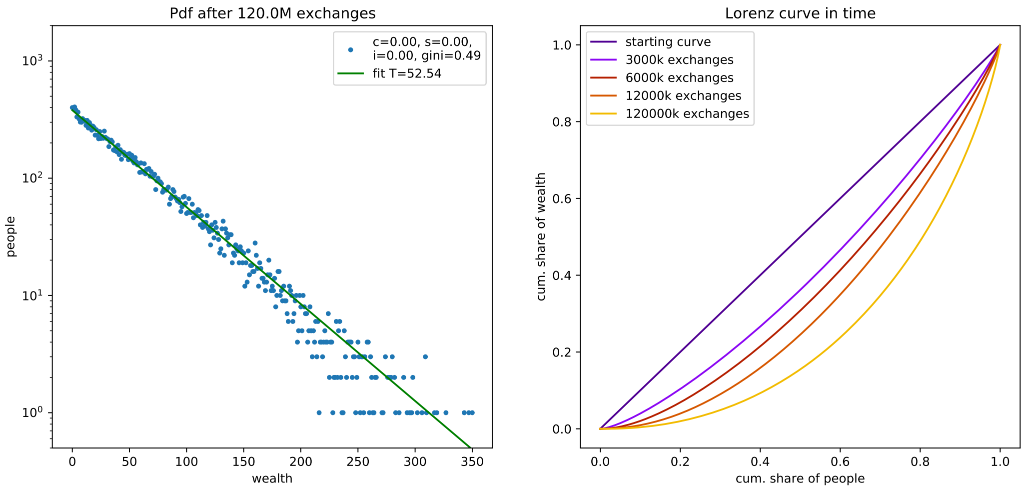


Figure 4.2: Pdf and Lorenz Curve for a DY model with 10^4 people and $m_0 = 50$. On the left the pdf is shown after 120 million of exchanges. On the right, the Lorenz curve of the system is plotted after a number of exchanges written in the legend. As the system thermalizes, inequality develops.

Figure 4.3: The macroeconomic variables of interest are plotted for the same system of figure 4.2. On the x-axis in all the plots we find time measured in units of 3 million exchanges. All the variables, exception made for the entropy and the Gini coefficient, are plotted on a per agent basis, so for example investments is the total money invested divided by the number of agents. Notice how, when financial investments, savings, and consumption are absent, the average amount of money per agent is a constant. Also notice how the Gini coefficient and the entropy increase over time. Entropy in this model reaches the maximum value allowed by constraints on total money and number of people, as discussed in chapter 2. The two graphs representing *investments* and *saving* + *investments* are empty since both quantities are zero and are plotted in semilogarithmic scale.

4.4 Random multiplicative exchange

In this section we quickly explore a non DY model, where the exchange of money between the agents in a transaction is an random number evenly distributed between 0 and a constant fraction of their wealth γ. Randomization is essential: otherwise exchanging a constant fraction of the wealth would result in a pile-up of the agents due to rounding up errors. Those errors would occur since agents can exchange only integer amounts of money, whereas the fraction of agents wealth will not in general be an integer. The theoretical shape of the wealth distribution for this model is discussed for example in [2][12][15], and is shown to be a gamma distribution.

In figure 4.4 the equilibrium wealth distribution for this model is showed, alongside the evolution in time of the Lorenz curve. In figure 4.5 the Gini coefficient and the Shannon entropy are plotted: notice how both these quantities are lower than the ones calculated for the DY model with the same number of agents,

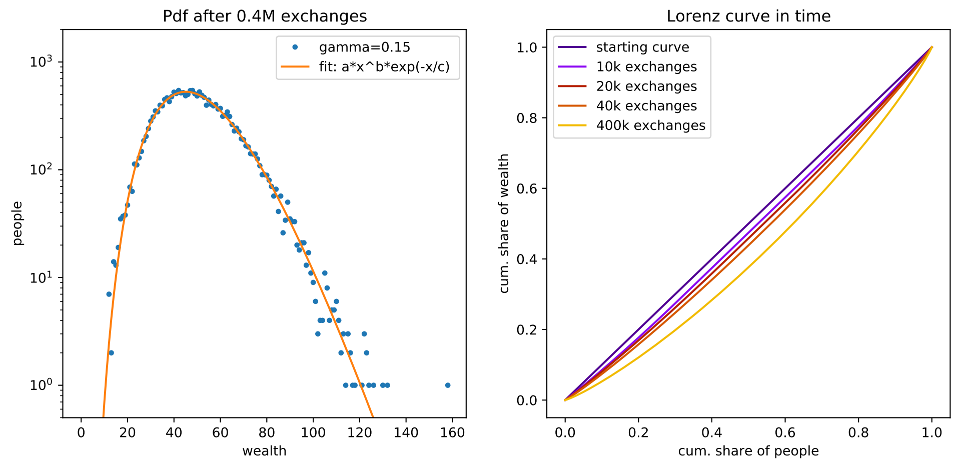


Figure 4.4: A population of $2 \cdot 10^4$ agents with average wealth $m_0 = 50$ is evolved for 400k exchanges. The pdf and the Lorenz curves are shown. We see how the poors are drastically reduced with respect to the DY model.

4.5 DY model with financial investment

We now show our first generalization of the DY model. We assume that agents can not only make transactions between them, but, with a certain rate, they can also make a financial investment.

When an agent makes a financial investment, he has a return on it, that is the capital he invested times an interest rate r, that is exogenous, meaning we fix it as a parameter, and is not determined by what happens within the simulation.

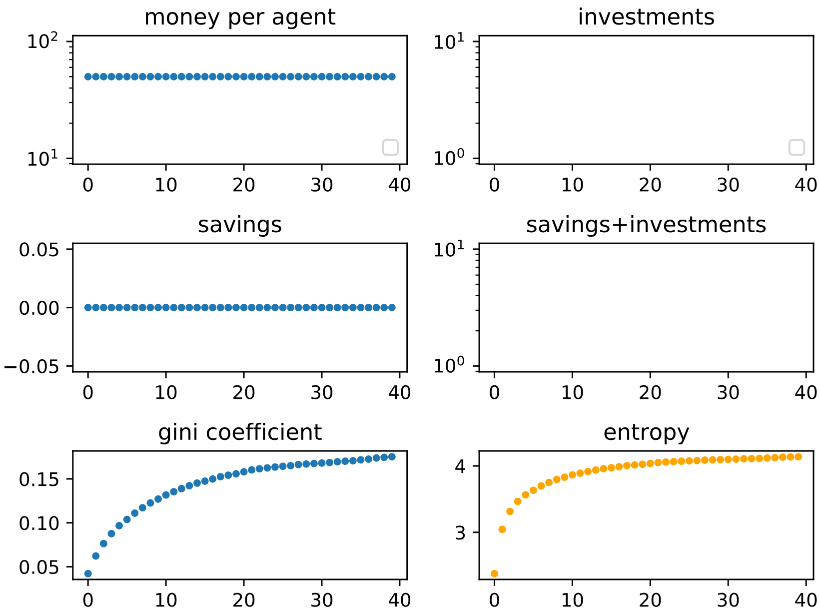
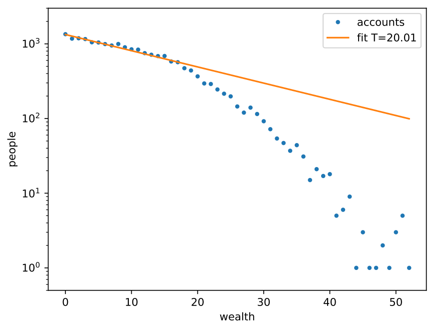


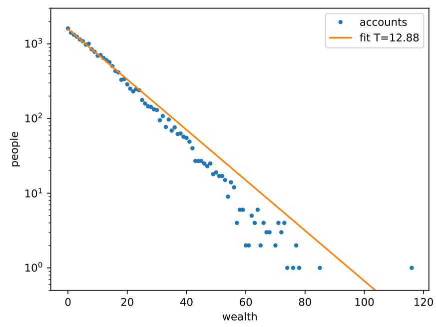
Figure 4.5: macroeconomic variables of interest for the same system of figure 4.4. On the x-axis in all the plot we find time measured in units of 10^5 exchanges. All the variables, exception made for the entropy and the Gini coefficient, are plotted on a per agent basis, so for example investments is the total money invested divided by the number of agents.We can see how the Gini coefficient stabilizes on a lower value than the DY model, implying a greater equality among the agents. Entropy is also lower than the DY model, as expected.

Therefore we assume the profits of a financial investment are known with certainty, and there is no volatility. An agent makes an investment on average every ten times that he is extracted to make the exchange. Then, with the standard randomization procedure we already talked about in the previous section, the agent is assigned a random integer profit uniformly distributed between 0 and the maximum profit $p = r \cdot K$, where K is the invested capital.

Let us now look at the results: in figure 4.6 we can see how the distribution of wealth first gets towards an exponential, but then the investments generate profits for the riches, that leave the exponential curve. In figure 4.7 on the left we see that the growth is entirely due to the riches, and fitting only the temperature of the poor people, we find almost the same temperature as we did without any investments, meaning the poorer tail of the population was not influenced in any way by the increase in money in the system. This in in stark contradiction with a theory called “trickle down” [1], that claims rich people are the cause of economic growth, but the poorer will indirectly benefit from it.



(a)



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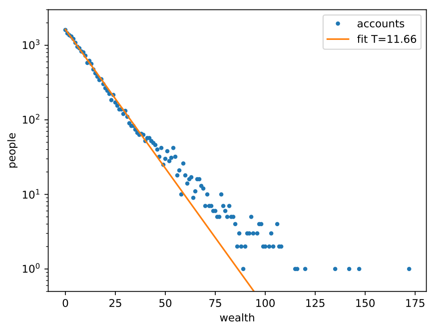

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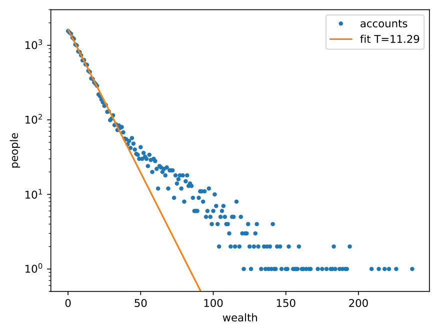


(d)

Figure 4.6: The evolution of the wealth distribution of 10^4 people and $m_0 = 20$ in the DY model with investments. Snapshots of the evolution are shown every 1 million exchanges. The rich people become richer over time.

In figure 4.8 we see money in the system growing at an exponential rate, after an initial sub-exponential phase. We are also remembered that this economic growth is happening to the expenses of equality, that is reducing, as the increase in the Gini coefficient shows. The system does not allow for a stationary state, due to the continue increase of money.

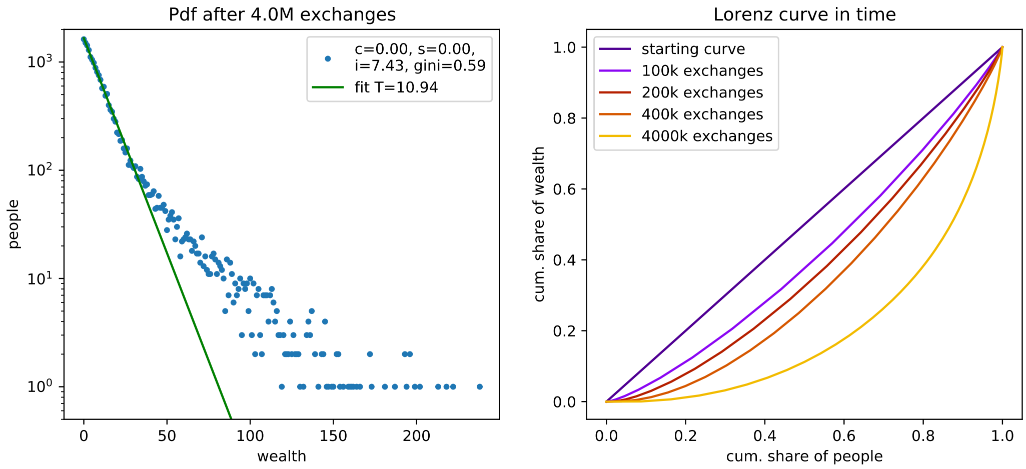


Figure 4.7: Pdf and Lorenz curves for the same system of figure 4.6. After 4 million exchanges, the poors still thermalize to a temperature of less than 11, meaning all the growth of wealth is due to the riches.

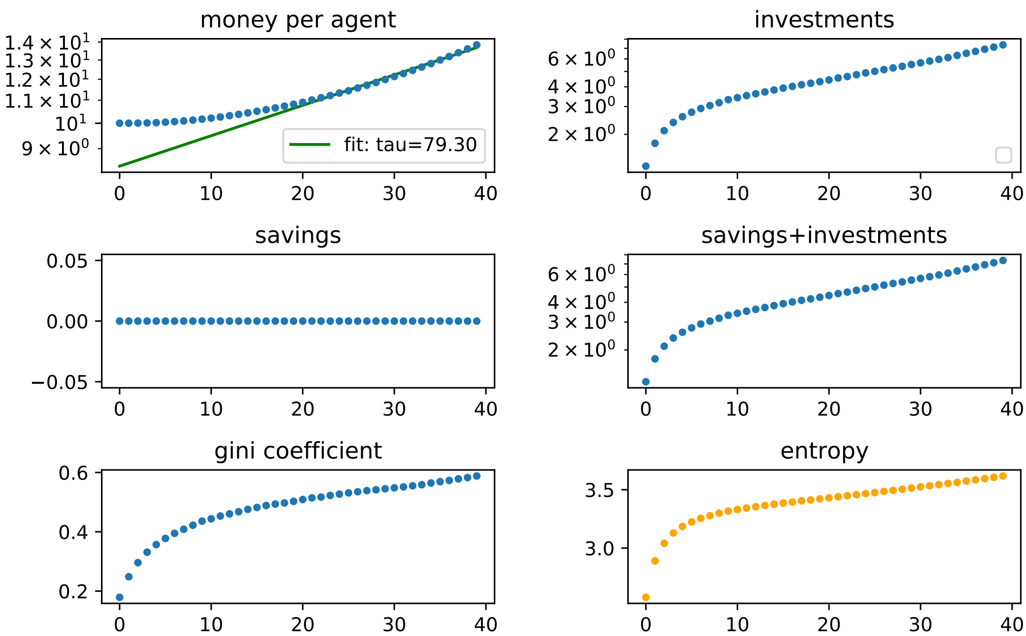


Figure 4.8: Relevant macroeconomic quantities of the same system of figure 4.6 plotted every 10^5 exchanges. We see the money in the system growing at an exponential rate, after an initial sub-exponential phase.

4.6 DY+investments+consume

In the last section we introduced investment as a way in which money can be created. By doing so, we were trying to reflect the fact that real wealth can be created in the physical layer with investments. We now introduce a way in which money can be destroyed, to reflect that wealth can be lost: the consumption. If agents can consume a fraction of their wealth given by figure 4.1, then two behaviours are possible for the agents. In fact, since consumption is bounded from above, whereas the profits of investments are not, there will be a threshold wealth above which agents will gain on average more in investment than what they lose in consumption, whereas agents below this threshold will do the opposite on average. This wealth depends on the parameter of our model, like the return on investment, the *mpc*, the *mpi*, and the value of wealth for which investments start to take place, that we will call β. In formulae, calling the threshold wealth w_t

$$r(w_t - \beta) = mpc \cdot \beta \quad \leftrightarrow \quad w_t = (1 + \frac{mpc}{r})\beta \tag{4.1}$$

So for long time frames, agents above this threshold wealth will see their wealth increase exponentially in time, whereas agents below will get even poorer with respect to the model with investment and without consumption.

If the threshold wealth is too high compared to the initial wealth of the agents, then no agent will reach it, and all agents will in the end lose all the money.
If instead one or a group of agents succeed in exceeding the threshold wealth in the long run, then the economy will grow at an exponential rate determined by his/their wealth.

From this observations, we can say that consumption can not stop polarization.

4.7 DY+investments+consume+income

We saw in the last section how consume was insufficient to stop the growing divide between rich and poors due to investments. In this section we will investigate whether a welfare measure like an endowed income from the government can be sufficient to stop polarization.
Consider an endowed income that is equal for all agents. Let us now explore the role income plays in the wealth distribution.
In figure 4.9, the equilibrium wealth distribution is plotted for a DY system with investments, consume and income. As you can see, the number of poors is drastically reduced by this welfare measure, but, as we will see in section 4.9 by looking at the Fokker Planck equation, a constant income is just a shift to the right of all the agents, and this alone can not prohibit the riches from getting

richer more than the poors do. Hence polarization can not be stopped with this measure, but in the next section we will finally find a mechanism that can stabilize the wealth distribution and stop polarization: taxes.

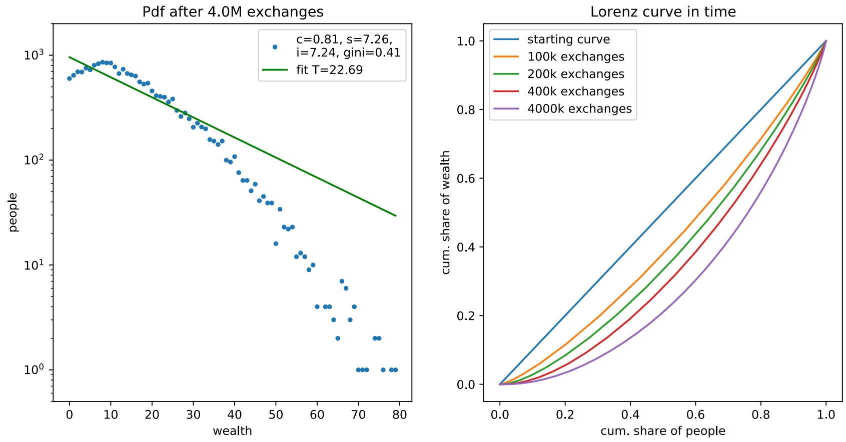


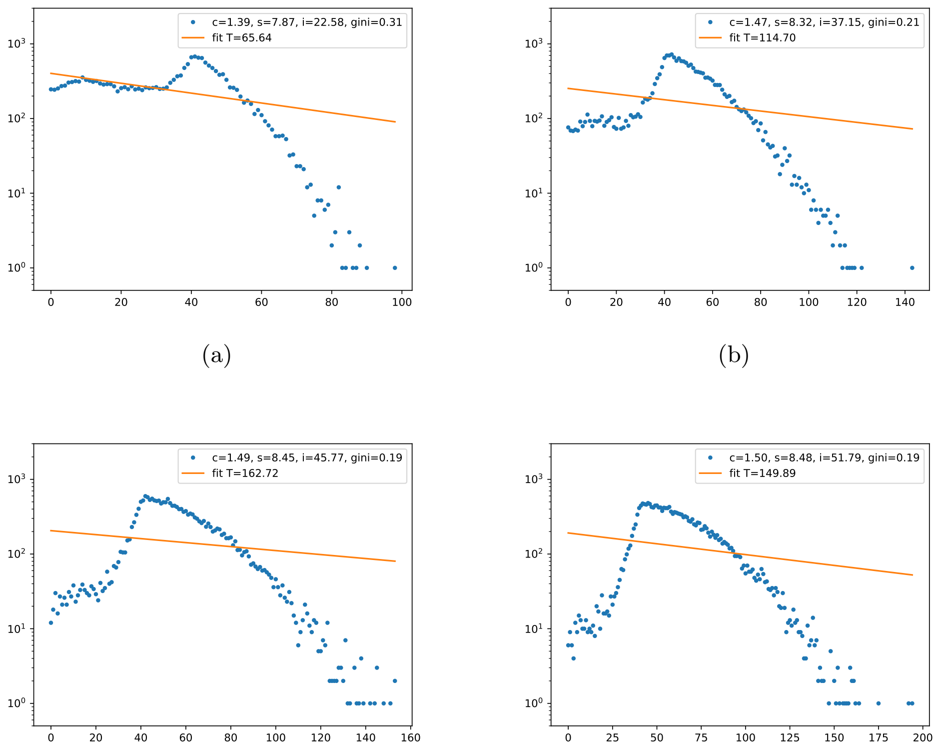
Figure 4.9: A population of 20 thousand with $m_0 = 20$ people is evolved considering the DY exchanges, consume and income, until the equilibrium distribution is reached. The pdf and the Lorenz curves are plotted. As you can see, the poorer population is substantially reduced with respect to the case without income.

4.8 DY+investments+consume+income+taxes

In this section we add a wealth tax to the system, as a way to counterbalance the profits made from the investments, that destabilize the wealth distribution. We will assume the tax to be:

$$\begin{cases} wtax = \beta m & \text{for} \quad m \geq \alpha \\ wtax = 0 & \text{for} \quad m < \alpha \end{cases} \tag{4.2}$$

where β is the fraction of your wealth that is paid in taxes on average every time an agent is extracted. If the fraction β of the wealth that has to be paid in taxes is grater than the return on investment, then the system is always stable, meaning that for long timeframes wealth distribution has a definite shape and economic growth stops. With the addition of this tax, a rich phenomenology can be observed. In figure 4.10, we see four snapshots of the wealth distribution. At first, the riches are reduced by the taxes and the poors are slightly reduced by the subsidies. After a while though, the discontinuity in the wealth tax causes a point of non-derivability for $m = \alpha = 40$, and the distribution converges to the shark-fin one. Notice in figure 4.11 on the right the inversion in the Lorenz curves,



(a) (b) (c) (d)

Figure 4.10: The time evolution of the wealth distribution of a system of 20 thousands agents, where investment, consume, income and taxes are taken into account. Snapshots of the evolution are taken every 4 million exchanges.

where the one for long timeframes is over al the others, meaning inequality has been reduced over time. The same conclusion can be drawn looking at figure 4.12, where the Gini coefficient first increases and then decreases until a plateau is reached. Notice in figure 4.12 how for number of exchanges from 2 million to 8 million, which corresponds to times from 5 to 20 in the figure, we have simultaneously economic growth and reduction of inequality. The entropy is plotted here to remember ourselves that despite inequality is reducing over time, the system is still evolving towards the equilibrium.

With this section we exhausted the phenomenology of our extensions to the DY model. In the next section, we will find the explicit mathematical equation that can explain what we observed so far.

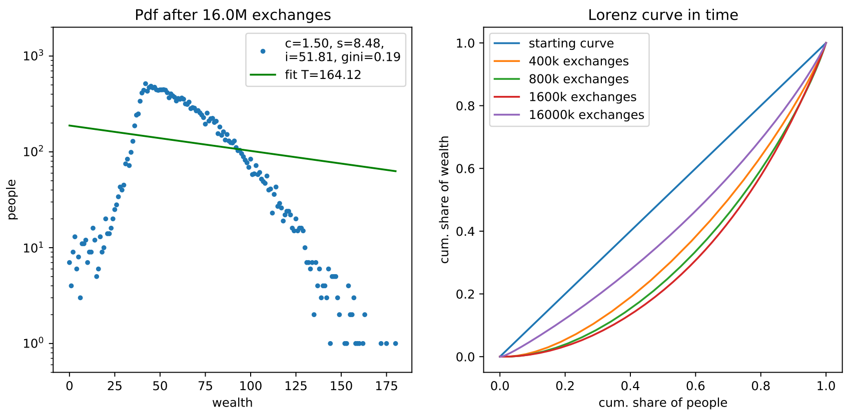


Figure 4.11: On the left, final wealth distribution for a system of 20 thousand agents after 16 million exchanges where investment, consume, income and taxes are taken into account. On the right the various Lorenz curves are plotted for the number of exchanges written in the legend. Notice how the purple curve, for 16 million exchanges, is above all the others.

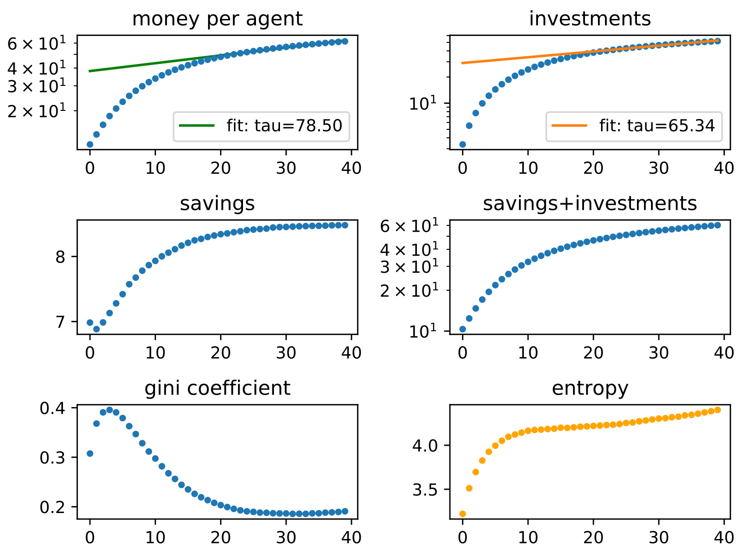


Figure 4.12: A plot of the macroeconomic variables of a system of 20 thousand agents evolved for 16 million of exchanges where investment, consume, income and taxes are taken into account. On the x-axis in all the plot we find time measured in units of 400 thousand exchanges. All the variables, exception made for the entropy and the Gini coefficient, are plotted on a per agent bases, so for example investments is the total money invested divided by the number of agents.

4.9 Fokker-Planck equation

Finally, in this section, the Fokker Planck terms for all the contributions proposed above are found and put together in the final equation governing the wealth distribution of the system.

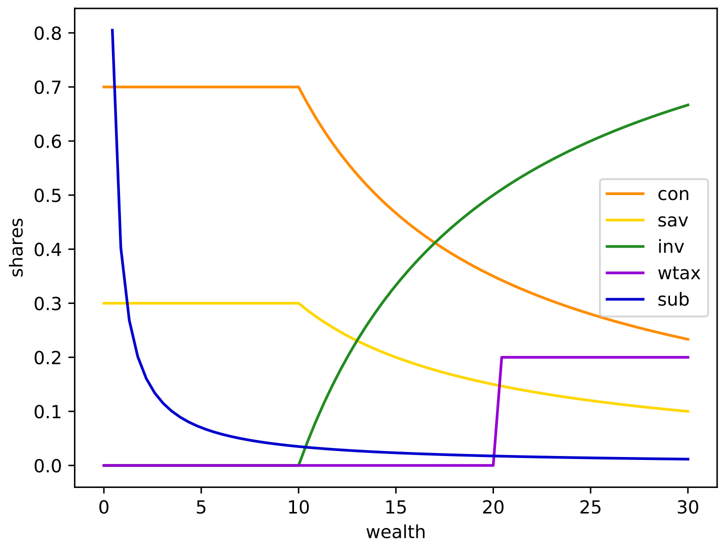


Figure 4.13: The assumed consume, savings, investments, wealth tax, and subsidies as a function of wealth expressed as a fraction of the wealth.

In chapter 2 we explained in great detail how to write the Fokker Plank term for a process. Essentially the idea was to count the inflow of people in a given wealth bin and subtract the outflow.

Without further ado let us write the complete equation for all the contributions for $m > 0$:

$$\begin{aligned}\frac{\partial P(m)}{\partial t} &= r\alpha\partial_m\Big[\big(wtax(m) - inv(m)\cdot r_{inv} - sav(m)\cdot r_{sav} + con(m) \\ &\qquad - sub(m)\big)\cdot mP(m)\Big] + \partial_m\big[A(m)P(m)\big] + \partial_m^2\big[B(m)P(m)\big] \\ &= \partial_m\big[\tilde{A}(m)P(m)\big] + \partial_m\big[A(m)P(m)\big] + \partial_m^2\big[B(m)P(m)\big]\end{aligned} \tag{4.3}$$

where the drift and diffusion terms A and B respectively were shown in the pure DY model to be: $A = rP(0)$ and $B = 1$, r_{inv} and r_{sav} are the return on investments and savings, that can potentially be different, and α is the reciprocal of the rate of frequencies in which a DY exchange is made and all the other actions are performed. For our simulations we chose $\alpha = 0.1$
We did a small abuse of the notation in the formula above, since ∂_m has to be considered as the right derivative for all the terms that cause the agents to loose

money, like consume and taxes, and has to be considered as the left derivative otherwise. The last equal sign has to be considered as the definition of the coefficient $\tilde{A}(m)$, which can be thought as the drift term taking into accounts all the addition we did to the DY model. So having chosen the DY model and the functions displayed in figure 4.13, the complete Fokker-Plank equation has been written for $m > 0$.

Analogously it can be shown that for $m = 0$ the following equation holds for the DY model:

$$\frac{1}{r}\frac{\partial P(m)}{\partial t} = \alpha[wtax(1) + con(1) - sub(0)] + P(1) - P(0) + P(0)^2 \tag{4.4}$$

where the last three terms were already present in the DY model, and the terms regarding savings and investment are not present since agents of zero wealth have nothing to invest or save.

Using this set of equations, the time evolution of the wealth distribution can be explicitly found.

The analytical solution for the wealth distribution for $m > 0$ can be found by direct integration analogously to what we did in section 2.9.1, just replacing the old drift term A with the new $A + \tilde{A}$:

$$P(m) = c \cdot e^{-\int_0^m \frac{A(m') + \tilde{A}(m') + \partial_m B(m')}{B(m')} dm'} \tag{4.5}$$

If an analytical integration of eq. 4.5 is not possible due to the specific form of the chosen function in figure 4.13, numerical integration can be performed.

For the reader that might want to see this derivation in grater detail we advise to check out section 5.1, where this derivation is carried out explicitly in the case where only taxes are added, thus $\tilde{A}(m) = T(m)$

We do not calculate the explicit wealth distribution for the set of functions in figure 4.13. Our aim in this chapter was showing that it is possible to preserve the formalism of the Fokker-Planck equations even when various economic phenomena are taken into account.

Chapter 5

Applications: optimal fiscal policy

In this chapter we will try to use the results we obtained so far with our model to address fiscal and monetary policy.

First we will see how we can achieve any target wealth distribution by the means of redistributive policies like taxes and subsidies, provided we are able to write down the Fokker-Planck equation for the model, like we did for the DY model. This result is particularly handy since in chapter 4 we wrote the Fokker-Planck equations governing all the economic processes, like consume, financial investments, taxes and subsidies. We then propose a way to link the shape of the wealth distribution to the production of the economy, using the Cobb-Douglas production function. Since we linked taxes with $P(m)$, and $P(m)$ with production, it is now possible to find optimal fiscal policy, where optimality is defined as the maximization of any function of choice depending only from the $P(m)$ and the production $Y(P(m))$.

To the best of my knowledge, none of the ideas discussed in this chapter are already in the literature, apart from the link between taxes and the wealth distribution discussed in section 5.1 that can be found, in a slightly less general form in [29]. These contributions should therefore be considered as original.

5.1 Taxes and wealth distribution

In chapter 2 it was demonstrated that the distribution of wealth was exponential from the Fokker-Planck equation given by the DY model. We are now interested in the opposite problem: given a distribution of interest P(m), are we able to write down a Fokker-Planck equation that has $P(m)$ as a solution?

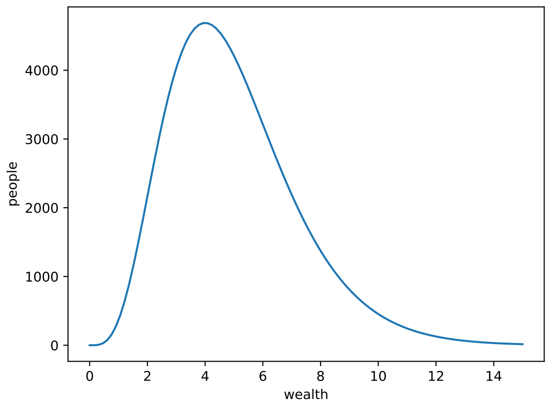


Figure 5.1: An hypothetic target distribution to achieve with fiscal policy.

This is not only of academic interest. Suppose you are in charge of the economy of a country, and you are required to achieve with the means of economic policy the distribution of figure 5.1, either because this would increase production, or to tackle economic inequality, or for some other reason. What can you do to achieve that distribution? The answer is that you can always put taxes in place such that the target distribution is achieved, provided it is at least continuous and its first moment (which is the average wealth of the target system) is equal to the average wealth of your system. To see how, it is first handy to find how taxes change the Fokker-Planck equation of your system. We will consider only taxes that depend only from the wealth of the agents, and not any other factor like income.

$$\frac{1}{r}\frac{\partial P_T(m)}{\partial t} = -T(m)P(m) + T(m+1)P(m+1) = \partial_m(T(m)P(m)) \tag{5.1}$$

where the first term represents the outflow of people from the m-th bin due to paid taxes and the second term the inflow from the $(m+1)$-th bin. If taxes could also add wealth instead of only diminishing it, in equation 5.1 we would need to sum to the left derivative also the right derivative, reflecting the intake of the m-th bin from the $(m-1)$-th bin. Equation 5.1 clearly shows that taxes only add a drift term to the Fokker-Planck equation. We can thus write the complete Fokker-Planck equation fo a generic model using the drift and diffusion coefficient A and B.

$$\frac{1}{r}\frac{\partial P_T(m)}{\partial t} = \partial_m[(A(m) + T(m))P(m) + \partial_m(B(m)P(m))] = -\partial_m J(m) \tag{5.2}$$

Equation 5.2 is a continuity equation for the probability $P(m)$, whose associated current is $J(m)$. When stationarity is achieved, eq. 5.2 implies $J(m)$

constant for every m. But since no agent is allow to exit or enter our system from the boundary $m = 0$, this implies that $J(m)$ is identically null for every m. The new Fokker-Planck equation for the stationary probability can thus be written as:

$$(A(m) + T(m))P(m) = -\partial_m(B(m)P(m)). \tag{5.3}$$

And solving for the taxes gets

$$T(m) = -A(m) - \frac{B(m)}{P(m)}\partial_m P(m) - \partial_m B(m). \tag{5.4}$$

For the DY model $A = rP(0)$, implying that taxes increase for everyone when $P(0) \neq 0$. Also notice that if we wanted to achieve the exponential distribution in the DY model with continuous variable m we would need to impose $T(m) \equiv 0$.

We would like to stress two points that are not present in the literature: the first is that fixed the amount of taxes $T(m)$ a family of distributions is still possible. For example in the DY model, with $T(m) \equiv 0$, we already found the equilibrium distribution in the continuous form to be:

$$P(m) = \frac{1}{\tau}e^{-m/\tau} \tag{5.5}$$

where the temperature τ is a parameter that is determined by the constraint on total money into the system, not by the condition of zero taxes.

We can rewrite equation 5.4, recalling $A = \frac{r}{\tau}$ and $B = r$

$$0 = -\frac{r}{\tau} + \frac{r}{\tau} - 0 \tag{5.6}$$

Which holds for any temperature τ. As you can see taxes are zero for any exponential distribution regardless of the temperature, and the right distribution is found with the constraint that the first moment of the P(m) can not change with taxes, since average money in the system must be conserved. The reason for the conservation of money, is that we put in T(m) the difference between the taxes collected and the subsidies, since the government can not take money for itself, and always redistributes to the citizens.

The second point we highlight is that in general taxes have one regularity class less than the target distribution they want to achieve, as can be seen from the term containing $\partial_m P(m)$ in eq. 5.4. This implies that it is possible to achieve a non differentiable wealth distribution using a discontinuous tax, and one could think, completely abandoning the economic sense of this work, that even a discontinuous wealth distribution could be achieved by having a Dirac delta in the taxes.

We just found the distribution of taxes that achieves a target distribution. What if we wanted to do the opposite, and find the wealth distribution implied by a certain tax function?

We could then solve for P(m) eq.5.3 and get:

$$P(m) = c \cdot e^{-\int_0^m \frac{A(m')+T(m')+\partial_m B(m')}{B(m')} dm'} \tag{5.7}$$

where the constant c is found requiring probability to be normalized.

A point worth stressing is that if we are not interested in finding the best tax function, but just ranking different tax functions with respect to a given figure of merit, then the figure of merit can be easily calculated given the taxes, and taxes can be ordered based on it. The bottom line as we said, is that comparing a finite set of taxes is simple, while finding the optimal one is hard.

5.2 The production function

In economics, a production function $Y(x_1, x_2, ..., x_n)$ is a relation between quantities that are the input of a certain productive process, and the maximal obtainable output of that process. The inputs are called productive factors, and can be economic objects like labour, capital, available land, entrepenuership etc. For the purpose of this thesis, we will be interested in linking the wealth distribution to the production of the economy, and so our production factors should only depend on the $P(m)$. We choose labour and capital to be the production factors. Now two other steps are needed. The first is to figure out how to determine available labour and capital based on the $P(m)$, and the second is which is the output given the available labour and capital, namely to choose the appropriate production function. Let us start by the choice of the production function by reviewing the two main alternatives.

5.2.1 The Leontief production function

The Leontief production function or fixed proportions function, is a production function that implies that all the factors of productions will be used in fixed proportions as determined by technology, therefore not allowing for substitutability between factors. In our case, being the factors labour L and capital K the production y would be:

$$Y = \min\left(\frac{L}{a}, \frac{K}{b}\right) \tag{5.8}$$

where a and b are in the required ratios of labour and capital needed for production, and are determined by technology.

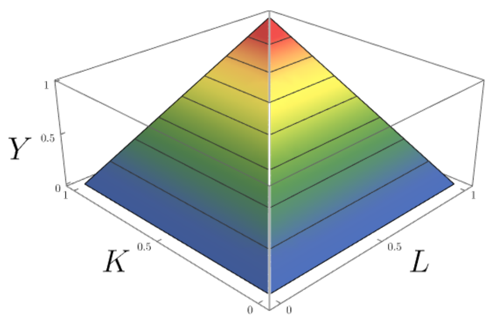


Figure 5.2: The Leontief Production function with isoquants, by Luca Verginer - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=40187600

The Leontief production function is a sensible assumption when considering the production of a good, with production factors being the "ingredients" needed to make it. For example, you would produce less laptops if instead of having two screens and two keyboards you had 1 screen and any other number of keyboards (we assumed for simplicity keyboards and screens to be the only factors of production in laptop manufacturing).

But we are interested in the output of the whole economy, and history teaches us that human labour is often replaceable with costly robots (the capital in our analysis), so our production function must allow for some sort of substitution between the two. This leads us to the Cobb-Douglass production function.

5.2.2 The Cobb-Douglass production function

The Cobb-Douglass production function has the form:

$$Y = A(t)L^{\beta}K^{\alpha} \tag{5.9}$$

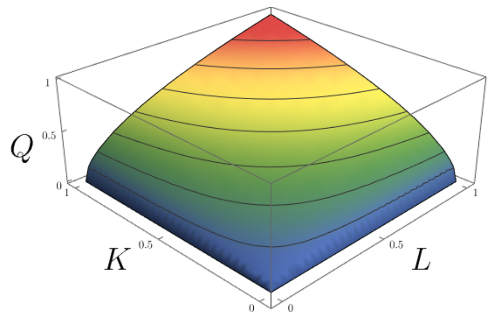


Figure 5.3: The Cobb–Douglas production function with isoquants By Luca Verginer Own work, CC BY-SA 4.0, https:// commons.wikimedia.org/w/index.php?curid=40182147

where A is the total factor productivity, while α and β are the output elasticities of capital and labour, and are assumed indipendent of capital and labour themselves, in what is called the "constant elasticity of substitution" assumption. Sometimes for Cobb-Douglass production function is meant the function with $\alpha + \beta = 1$, thus assuming constant return to scale. This is a reasonable assumption in our case since $\alpha + \beta \geq 1$ is a sensible assumption when land is not the limiting factor of production and $\alpha + \beta \leq 1$ holds when total factor productivity varies slowly with time, thus implying progresses in an industry do not significantly affect the productivity in the other industries. The new Cobb-Douglass production function is then:

$$Y = A(t)L^{1-\alpha}K^{\alpha} \tag{5.10}$$

The only parameter we still need to estimate is the exponent α. It can be shown that α and $1 - \alpha$ are respectively capital and labour share of GDP. The share of labour is plotted in figure 5.4, and can be observed to be roughly 0.6 with small fluctuations over time and a slowly decreasing trend due to technological advances. Taking $\alpha = 0.4$, we now have a framework to estimate the production of the economy.

5.2.3 The available factors of productivity

We already discussed that labour and capital will be the only two factors of production. Now we need to assess available labour and capital based on the wealth distribution $P(m)$. An important point here is that we are not trying to be accurate from an economic perspective: being meticulous on the values would bring us too far from the econophysics framework we are working on. Remember the focus of our work will be to show the path to optimal fiscal policy.

That being said, we will assume the capital in the system to be the already calculated financial investment of chapter 4, and the available labour to be a constant, the reason being that the working schedule of the riches is not that different from that of the poors.

We now have all the ingredients to move forward and find the optimal fiscal policy.

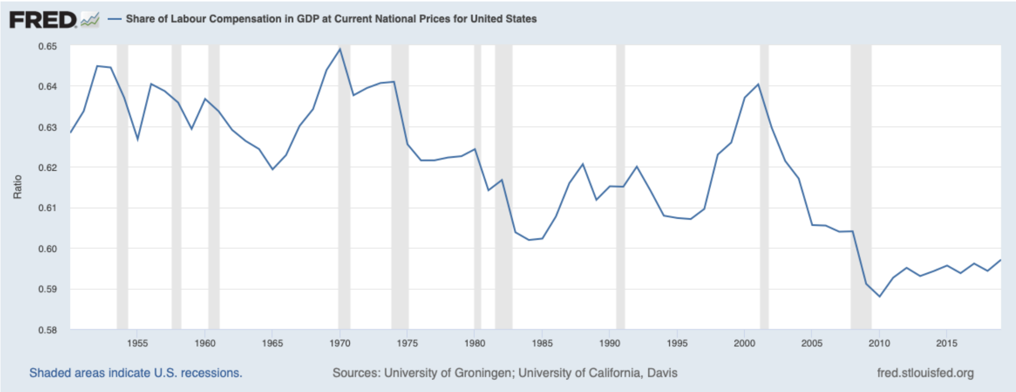


Figure 5.4: The labour share of GDP in the U.S. from the 1950s until today.

5.3 Optimal fiscal policy

In equation 5.7 the wealth distribution $P(m)$ generated by a tax distribution $T(m)$ was found explicitly for any model of exchange. Then in section 5.2 we found the output of the economy given the $P(m)$, hence linking the taxes to the economic output. It is now possible, by direct calculation, to rank a set of taxes based on a figure of merit O provided it depends solely from $P(m)$ and $Y(P(m))$. The procedure is simple: we would need to calculate the $P(m)$ with the taxes using formula 5.7, then calculate the figure of merit $O(P, Y(P))$, and finally rank the taxes based on their associated value of the figure of merit.

Given the ease of performing this operation, we now focus on the hard part of the problem: finding the optimal taxation $T(m)$ among all the $T : \mathbb{N} \to \mathbb{R}$ with the constraint

$$\int_0^\infty T(x)dx = 0 \tag{5.11}$$

Our set of function has the cardinality $\aleph_1$, so clearly we will not be able to enumerate them. If we could make a continuous bijection between the $T(x)$ and real numbers, which have the same cardinality, then we could plot taxes as a function of a real variable. Unfortunately the bijection exists, but is not continuous in general. The bottom line is that is very simple to compare two tax functions for any given criteria, but is much harder to find the better tax function of all the $\aleph_1$ possibilities.

To solve this problem, we will first propose a computational method to find the optimal taxes for any given objective function. Then the same problem will be approached by the elegant formalism of Euler-Lagrange equations.

The rule for the exchange between agents will be the one of the DY model for convenience, but eq. 5.7 and hence our procedure, holds for any model provided we are able to write down the Fokker-Plank equation for it.

5.3.1 Computational approach

The computational method we would choose is called **stochastic gradient descent** method, and is widely used in machine learning to find the optimal weights for a neural network. In general it can be used any time one is interested in finding the local maximum of a function, which in our case is the objective function. Very smart implementation of this algorithm, like the Adam method, can be found in the literature, but we propose one of our own that is the simplest for the purpose of our work.

Our proposed rudimental implementation goes as follows: suppose we are interested in finding the best taxes only among the tax functions that are null for $m > M$. Then start from a tax function at random in this set. Select two bins among the ones with $m < M$ with a random process. Alter by the fixed parameter ϵ the value of taxes of the two bins respecting the constraint 5.11, so if one is increased of ϵ the other one is decreased by the same quantity. Then calculate the objective function for the new taxes: this means find the $P(m)$ for this taxes, then calculate the objective function resulting from it. If it is higher than before, keep the change in the bins and repeat the algorithm, else reverse the change and repeat the algorithm. After a certain number of interaction, the objective function reaches a plateau, where statistical fluctuations dominate the growth trend. At that moment reduce the quantity ϵ and increment the value of M. Keep reducing ϵ and incrementing M until you are satisfied with the precision reached in determining the tax function.

This method has has the disadvantage of finding only local maximum, but presents fast convergence and good precision in the tax function.

5.3.2 Theoretical approach

Let us state the problem in a way that will look much more familiar to theoretical physicists: for every model of exchange, we have a Fokker-Planck equation. Given the model and the tax function we decide to impose $T(m)$, we have a determined equilibrium distribution $P(m)$. In turn, the equilibrium distribution univocally determines the value of the objective function O, so it is safe to say that if we fix the model, there exist a functional link between $T(m)$ and O, that we will explicit for this derivation writing $O = O(P(T(m)))$.

Given the hypotheses on the objective function, it can be written as:

$$O(P(T(m))) = \int_0^\infty dm f(P(T(m)), m) \tag{5.12}$$

But from here the parallelism with the least action principle is clear: in physics we have a functional S that is expressed as the time integral of the Lagrangian:

$$S = \int_0^t dt' L(\vec{x}(t'), \dot{\vec{x}}(t'), t') \tag{5.13}$$

S depends on the functional form of the trajectory $S(\vec{x}(t), \dot{\vec{x}}(t))$, as much as O depends on the tax function $O(P(T(m)))$. In physics the least action principle helps us in finding the trajectory $x(t)$, imposing the stationarity of the action functional S: this leads to the Euler-Lagrange equations. Unfortunately the taxes that will maximize our objective function O can't be found imposing:

$$\frac{\delta O(T(m))}{\delta T(m)} = 0 \tag{5.14}$$

and writing the functional derivatives, because unlike the Lagrangian, that can be evaluated at time t only knowing positions and velocity at time t, our function $f(m)$ depends the $P(m)$ everywhere that in turn depends on the taxation $T(m)$ everywhere.

This was the reason we decided to approach the problem from a computational perspective in the first place, but we wanted to present this analogy as a possible lead for future research. The functional derivative method can be applied successfully if the function O seen as a function of the taxes, $\tilde{O}(T(m))$, is linear in the taxes. Unfortunately such objective functions have to be constructed ad-hoc, and none of the most used figure of merit falls in this category.

Chapter 6

Applications: monetary policy

In chapter 3 we studied in great detail the time evolution of the perturbations of the DY model, showing they fade away in an exponential fashion and finding the characteristic time scale of decay for high temperatures:

$$t_{eq} = \frac{1}{4r} \tag{6.1}$$

where r is the rate of extractions in the system divided by 2, or equivalently the number of exchanges per unit time divided by the number of agents. In this section, we are going to develop a strategy to calculate the relaxation time for an economy expressed not as a function of the number of exchanges, but based on three measurable quantities of an economy: the gross domestic product GDP, the average transaction size T_s and the total population p.

We will explicitly find the equilibrium time predicted by our model for the American economy from the 1950s until today, and notice how this instrument can be used as a diagnostic tool for an economy. By doing so, we will be able to evaluate the importance of the COVID-19 crisis, and see how the effectiveness of monetary policy is changing in the U.S. and around the globe.

The title of this chapter should more properly be "Applications: one-time redistribution policies", but since one-time redistribution policies almost always happen during recessions and at the expenses of quantitative easing, we chose to name the most useful application and not the most general one.

As usual, none of the work of this chapter apart from the explanations provided in chapter 6.1 can be found in the literature, and should be therefore considered as original.

6.1 Setting the stage

Central Banks are formally independent entities not under direct control of the government. They have a mandate, which in developed economies is: 1) full employment and 2) price stability. During economic recessions, the central bank's mandate is to bring the economy back on the right track. In the last two decades in USA and the EU, this was achieved with a practice called Quantitative Easing (QE), that has been both praised and disliked by many. How QE works is usually as follows: the central bank enlarges its currency reserves, effectively creating new money in the system, usually digital money and not physical currency bills. Then uses a good portion of that money to buy financial assets in the free market, thus inflating the price of those assets. Usually the biggest share of the money goes towards buying long term domestic treasury bonds, which are an obligation of a sovereign state to pay some coupons to the holder of the bond until it expires. This means the central bank is effectively lending money to the state at a rate that is close to zero and sometimes negative. Since this operation is performed in the open market, it also increases the liquidity of the economy. since many previous treasury bond holders are now favoured to sell their bond to an inflated price and have money in return.

On the short term, everyone seems well off: the government can finance any action to relieve poverty and restart the economy with cheap debt hold by the central bank, and the previous holder of the bonds have now more money and more liquidity than when it all started, and this liquidity will be transferred to all the people in the economy incrementing consumption and employment. But if this is the picture, why would anyone criticise QE?

The answer lies in the long term: after the first wave of new money enters the system, the prices of goods and services will slowly start to rise, since richer agents will have more money thus more purchasing power, in a phenomenon known as inflation. Citing the famous american economist Milton Friedman: "Inflation is always and everywhere a monetary phenomenon", meaning that "printing" new money is never free in the long term, and actually what happens is that money simply looses value as opposed to people gaining the ability to buy more goods with it.

Using the language we developed in our work, we would say that adding money into the system results in an increase in the average temperature, and in the long run the system should stabilise in the equilibrium distribution for the new temperature.

6.2 The equilibrium time in the U.S.

Since we know from eq. 6.1 how to relate the equilibrium time to the rate of extractions r, let us find r for a general economy.

Remember r was defined as half the reciprocal of the time it takes on average to select all the agents for an exchange. If we assumed every exchange to be unitary, then the number of exchanges per unit time is simply the GDP of an economy. This is true for two reasons: 1) in order to count in the GDP, a good has not only to be produced, but also sold, thus causing the monetary exchange we are counting, and 2) in the GDP, only added values are counted, so for example if tyres are used to make a car, only the final value of the car is counted, the tyres are not.

This considerations would imply:

$$r = \frac{GDP[\$/yr]}{2 \cdot p} \tag{6.2}$$

but this can not be the solution, since the unity of measure \$ can not enter our characteristic timescale of equilibrium, that should be a quantity independent of the exchange rate of our currency. For this to work we need to realize that the average size of a transaction in an economy is not one unit of currency. Defining T_s the average transaction size, the right expression for r is:

$$r = \frac{GDP[\$/yr]}{2 \cdot p \cdot T_s[\$]} \tag{6.3}$$

which is measured in 1/[yr], as it should be. Using now eq. 6.1 we can explicitly find the thermalization time of the economy in the limit of infinite temperature to be:

$$t_{eq} = \frac{1}{4r} = \frac{p \cdot T_s}{2 \cdot GDP}[yr] \tag{6.4}$$

We are interested in exploring the evolution of t_{eq} in the United States. To do so, we will make an economic assumption: we assume the average transaction size T_s to be a fraction α of personal disposable income of the agents. Again our focus here is not on economic accuracy, but rather to show the path to the designing of monetary policy, and we leave the duty of empirically finding α to the economists.

This assumption allows us to plot the equilibrium time up to the constant α, that in figure 6.1 we assumed equal to 1 for convenience.

In figure 6.1, we clearly see how, during the unfolding of the COVID-19 pandemic in the U.S., the equilibrium time of the economy increased by about 20% in one quarter, despite presenting fluctuations of less than 10% for the last 70 years.

Let us see how this is relevant: suppose a recession hits, and the central bank has in mind an economic manoeuvre to reshape the wealth distribution in a way

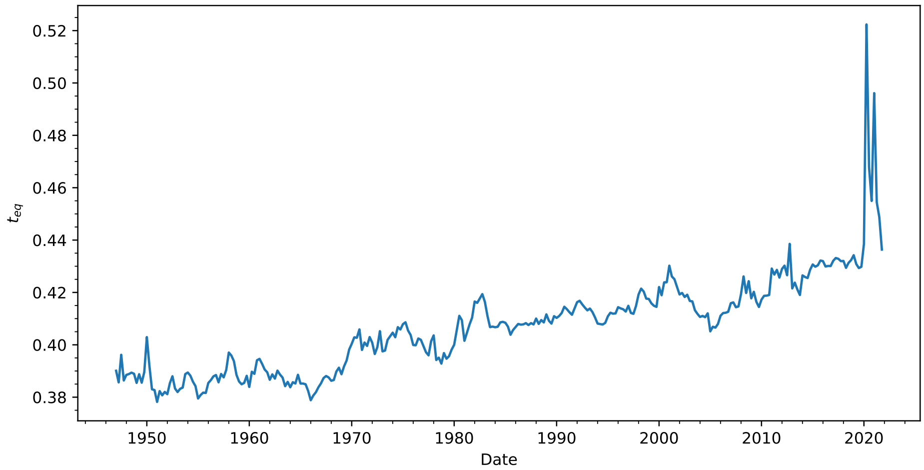


Figure 6.1: The equilibrium time of the U.S. economy from 1948 to 2021.

that would increase consumption and investment. And let us say that to exit the recession, the wealth distribution needs to be in that shape for 3 months, and then the increased productivity will bring the economy back on track. Then it is of great concern to know whether the wealth distribution will remain in the desired shape for the needed three months or not.

One could think that if the stimulus from the central bank falls short in saving the economy, then further injection of money into the system in subsequent waves could be just as good as one big intervention at the beginning of the crisis. But actually only unexpected inflation induces people to consume and invest more, giving a stimulus to the economy. Since after the first intervention people start expecting similar measures, then the required level of inflation needed to have a similar effect on the economy exponentially increases, leading to what the American economist Milton Friedman defined as "inflationary spiral".

Chapter 7

Conclusions and future outlook

In this thesis we have studied the issue of economic inequality from a physics perspective, extending the work of the French economist Thomas Piketty. Our starting point was the DY model of capital exchange, and our contribution was twofold:

1. We expanded significantly the set of mathematical tools and ideas within the DY model
2. We created a new model based on the DY model that can capture the dynamic evolution of the wealth distribution, and for this reason we argue it is better suited to describe the mutable world we live in.

We briefly review our main original contributions, and propose some leads for future research:

- In section 2.8 we developed a method to study social mobility with Brownian motion and path integrals in any model of exchange, and explicitly visualised the social mobility of the DY model with a computer simulation.

  A promising lead for future research would be to use this method to verify the Great Gatsby curve. The Great Gatsby curve is an empirical economic positive correlation between social mobility and equality in a society. Since we know how to calculate both those quantities for any model of exchange (for equality the Gini index can be used), the correlation can be verified by direct calculation. If for any reason the correlation between those two quantities would not be observed, then we could either think that the real world observed correlation is a coincidence, or use the lack of this correlation as an indicator that a model should be discarded as it fails in predicting this evidence of the real world.

- In chapter 3 we developed an analytical argument to demonstrate that the thermalization for the DY model is exponential, and found explicitly the thermalization time as a function of the temperature. In doing so we discovered a band-like structure in the DY model, where only perturbation with a given dispersion relation are admissible.

- In chapter 4 we proposed some extensions to the DY model to account for real world economic phenomena, like financial investment, consume, income and taxes. We found that investments increase polarization, and the only way to stop it is by the means of a wealth tax.

  In addition, one could think of implementing volatility in the investments and also a variable return on investment that can also be negative for some timeframes: this should enable our model to model financial crashes. The next step in this direction would be to find a mechanism that endogenizes the return on investment, meaning we don't set it as a parameter, but depends on quantities of the evolving economy. This endogenizations of return on investment closely resembles the challenge real business cycle theories had to deal with, and we suggest to borrow some of their solutions in our research. Other leads for future research could also enlarge the range of economic phenomena taken into account, and reimplement all the proposed contributions using more empirical evidence from real world data.

- In chapter 5 we solved the problem of designing optimal fiscal policy in the DY model. We used the relation between taxes and wealth distribution that can be found in the literature, and complemented it with notions of production theory, to find the optimal taxes given any figure of merit.

  Given the mathematical complexity of the procedure we described, two methods have been proposed to find the solution of optimal taxes: stochastic gradient descent and the Euler-Lagrange equations. We see potential for future research both on trying to rewrite in a more explicit form the Euler-Lagrange equations for the most common figures of merit, and in ranking real world taxes based on those figures of merit. Moreover, one could think of proceeding with the opposite derivation of what we did, and given any tax function, write a figure of merit for which that taxation is optimal.

  This would shed light on the criteria government use to impose taxes on their citizens.

- In chapter 6 we applied to the real world the study of perturbation to the DY equilibrium done in chapter 3. This led us to the definition of the thermalization time for an economy, a quantity we calculated explicitly for the U.S. economy from the 1950s until today. We also showed how this quantity has great relevance for monetary policy. Here we see two important directions for future research: the first is to find explicitly the average transaction size in an economy using empirical data, in order to better estimate our thermalization time. The second concerns the upward trend that the thermalization time has seen in the last 70 years in the US.

The reason behind this trend should be clarified, as well as its possible link with signs of a weakening efficacy of the monetary policy.

Appendix A

The Gini coefficient

In this Appendix, we review two standard instruments to study economic inequality: the Lorenz curve and the Gini coefficient.

The Lorenz curve: take a set of N agents, each endowed with some wealth w_i. Now order the agents according to increasing wealth. With this ordering let us define the cumulative share of wealth :

$$W_C(n) = \frac{\sum_{i=0}^{n} w_i}{\sum_{i=0}^{N} w_i} \tag{A.1}$$

A Lorenz curve is a plot where the $W_C(n)$ is plotted with respect to the rescaled variable n/N, hence a plot of the cumulative share of wealth with respect to the cumulative share of people.

An example can be found in figure A.1, where to a wealth distribution on the left is associated the Lorenz curve to the right by direct construction. The red line on the left figure is the "perfect equality" curve, meaning the one we would observe if all the agents had the same wealth.

The Gini coefficient: intuitively, the more a Lorenz curve is "distant" from the perfect equality curve, the more inequality is high among the agents. The Gini coefficient tries to measure just that. It does so by considering the area between the red and the yellow curve in figure A.1. The surface of this area is at least zero, and, if we allow only positive wealth, it is at most half the area of the square, which is one, hence $0 < A < 0.5$. The Gini coefficient G is defined as:

$$G = 2A \tag{A.2}$$

and therefore it is a number between 0 and 1. If G is close to zero the Lorenz curve of our system is very close to the perfect equality curve, and hence inequality is low. If $G = 1$ one agent has all the wealth, hence inequality is high.

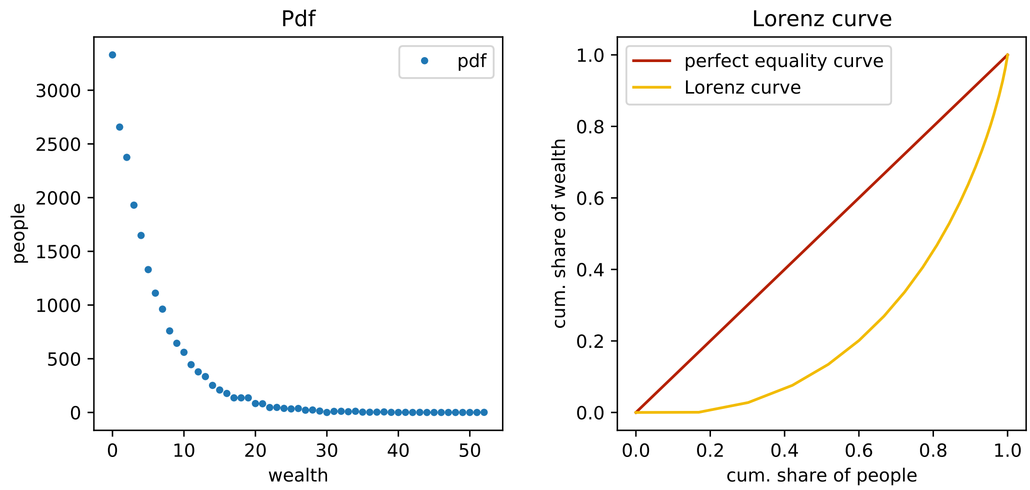


Figure A.1: On the left, a given wealth distribution among the agents is assumed. On the right, the Lorenz curve is plotted for that distribution.

Appendix B

Python code

We show in this appendix the main code used in the original contribution of the project. If you are interested to see the more than 40 codes that generated and analysed all the data in this thesis, go on the github page of the thesis at https://github.com/Marcolino97do/Master-s-Thesis.

The code shown is split in four parts. In the main.py program we can control the parameters of the simulations and also plot results. The engine serves the purpose of evoluting the system from the current state for a specified amount of interactions. We also wrote an engine program in the C language, in order to gain a factor of 100 in execution speed: you can find it on the github page if you are interested. The command.py file is a set of useful functions that I grouped in this library in order to make the code easier to read.

The last code is the one we used to calculate the perturbations that were compatible with normalization of probability: a critical step in our demonstration in chapter 3.

Listing B.1: The code of the main.py program

```
import numpy as np
from decimal import Decimal
import matplotlib.pyplot as plt
import subprocess
from scipy.optimize import curve_fit
import numpy.ma as ma
import engine
import command

import matplotlib
from matplotlib import cm

def retta(x,a,b):
    print(a,b)
    return a*x+b
```

```
def exp(x,a,tau):
    #print(tau,a)
    return a*np.exp(x/tau)

def eugamma(x,a,b,c):
    return a*(x**b)*np.exp(-x/c)

if __name__ == '__main__':

    #INITIALIZING VARIABLES
    time=100000
    people=20000
    initmoney = 5
    runs=40

    wealthshare=0
    exchange=1

    income=0
    alpha=10
    mps=0
    mpc=0
    ros=0.1
    roi=0.1

    beta = 40
    wealthtaxrate = 0
    wlowtaxrate = 0
    whightaxrate =0

    gamma=5
    incometaxrate = 0
    ilowtaxrate=0
    ihightaxrate=0

    nontradefrequency=0
    normalizetrade=0

    fitcutoff=20

    consume,investment,savings,gini=np.zeros(runs),np.zeros(runs),np.zeros(runs),
    np.zeros(runs)
    entropy=np.zeros(runs)
    accounts=np.ones(people)*initmoney

    totmoney=[]
    vecginicurves=np.zeros((runs,people))

    for i in range (1,runs+1):

        #EVOLUTE THE SYSTEM
```

```
    #subprocess.call(["gcc", "-O3", "finalevolution.c"])
    #subprocess.call(["gcc", "-O3", "nonlocalevolution.c"])
    #subprocess.call(["./a.out", str(initmoney), str(people), str(time), str(
alpha),
                      #str(mps),str(mpc),str(ros),str(roi),str(wealthtaxrate),
str(incometaxrate),str(nontradefrequency), str(normalizetrade),str(income),
str(i)])

    accounts=engine.evolution(initmoney,people,time,alpha,mps,mpc,ros,roi,
wealthtaxrate,incometaxrate,
                              nontradefrequency,normalizetrade,income,beta,
wlowtaxrate,whightaxrate,gamma,
                              ilowtaxrate,ihightaxrate,wealthshare,exchange,
accounts)

    #accounts = np.loadtxt("results.txt", unpack=True, dtype=int)

    pdf = command.pdf2(accounts)
    allrichness=np.arange(0, np.max(accounts) + 1)
    totavgmoney=np.sum(accounts)/people
    totmoney.append(totavgmoney)


    #calculating integral quantities based on the pdf
    gini[i - 1],y = command.Gini(accounts,i)
    vecginicurves[i-1][:]=y
    consume[i-1], savings[i-1],investment[i-1]=command.macrovariables(pdf,
alpha,mpc,mps)
    entropy[i-1]=command.entropy(pdf)

    # VISUALIZING DATA
    print(totmoney)
    if(i%10==0):
        #FITTING and maskng
        maskpdf,maskallrichness=ma.masked_array(pdf,allrichness>fitcutoff),ma
.masked_array(allrichness,allrichness>fitcutoff)
        print(maskpdf)
        pdfguess=[1000,-initmoney]
        poptpdf,pcovpdf=curve_fit(exp,maskallrichness[~maskallrichness.mask],
maskpdf[~maskpdf.mask],pdfguess)

        # FITTING THE GAMMA DISTRIBUTION
        #gammaguess = [10e-7, 7, 5]
        #gammapopt, gammapcov = curve_fit(eugamma, allrichness, pdf,
gammaguess)
        #print(gammapopt)

        #rescaledrichnessbin,pdfbin=command.bin_data(allrichness*initmoney/
totavgmoney,pdf,30)
        #plt.semilogy(allrichness,pdf,marker='.',ls='',label='c={:.2f}, s
={:.2f}, i={:.2f}, gini={:.2f}'.format(consume[i-1],savings[i-1],investment[
i-1],gini[i-1]))
```

```
        plt.semilogy(allrichness, pdf, marker='.', ls='',label='accounts')

        #plt.semilogy(allrichness, eugamma(allrichness, *gammapopt), label='
    fit: a*x^b*exp(-x/c)')
        #print(gammapopt)
        #plt.semilogy(rescaledrichnessbin,pdfbin,label='binned distribution')
        plt.semilogy(allrichness,exp(allrichness,*poptpdf),label='fit T={:.2f
    }'.format(Decimal(str(-poptpdf[1])))) #FIT EXP BUONO
        plt.ylim([0.5, 3000])
        plt.xlabel('wealth')
        plt.ylabel('people')

        plt.legend()
        plt.savefig('paolo'+str(i)+'.pdf',format='pdf')
        plt.show()


#FITTING TOTMONEY
points=np.arange(0,runs)
vectotmoney=np.array(totmoney)
masktotmoney=ma.masked_array(vectotmoney,points<runs/2)
maskpoints=ma.masked_array(points,points<runs/2)
#initguess2=[-100,100]
initguess=[initmoney,10]
invinitguess=[1,10]
#popt2,pcov2=curve_fit(retta,maskpoints[~maskpoints.mask], masktotmoney[~
    masktotmoney.mask],initguess2)
popt,pcov=curve_fit(exp,maskpoints[~maskpoints.mask], masktotmoney[~masktotmoney.
    mask],initguess)
#plt.semilogy(points,exp(points,*popt),color='green',label='fit:tau={:.2f}'.
    format(popt[1]))
#plt.semilogy(points,totmoney,marker='.',linestyle='')
#plt.legend()
#plt.show()
maskinv=ma.masked_array(investment,points<runs/2)
invpopt,invpcov=curve_fit(exp,maskpoints[~maskpoints.mask], maskinv[~maskinv.mask
    ],invinitguess)


#PRINT GINI CURVES
consume=np.zeros(runs)

fig, axs = plt.subplots(1, 2)

axs[0].semilogy(allrichness,pdf,marker='.',ls='',label='c={:.2f}, s={:.2f}, \ni
    ={:.2f}, gini={:.2f}'.format(consume[i-1],savings[i-1],investment[i-1],gini[
    i-1]))
#axs[0].semilogy(allrichness,pdf,marker='.',ls='',label='gamma=0.15')
axs[0].semilogy(allrichness,exp(allrichness,*poptpdf),color='green',label='fit T
    ={:.2f}'.format(Decimal(str(-poptpdf[1]))))
#axs[0].semilogy(allrichness,eugamma(allrichness,*gammapopt),label='fit: a*x^b*
    exp(-x/c) ')
```

```
axs[0].set_title("Pdf after {:.1f}M exchanges".format(runs*time/1000000))
axs[0].set_ylim([0.5,2000])
axs[0].legend(loc='upper right')

colors = matplotlib.cm.gnuplot(np.linspace(0.1,0.9,5))
count=0
axs[1].plot(np.arange(0,people)/people,np.arange(0,people)/people,label='starting
     curve',color=colors[count])

for k in range(0,runs):
    if (k==0 or k==1 or k==3 or k==39):
        count += 1
        axs[1].plot(np.arange(0,people)/people,vecginicurves[k][:],color=colors[
    count],label='{:.0f}k exchanges'.format((k+1)*time/1000))
axs[1].set_title("Lorenz curve in time")
axs[0].set_xlabel('wealth')
axs[0].set_ylabel('people')
axs[1].set_xlabel('cum. share of people')
axs[1].set_ylabel('cum. share of wealth')


plt.legend()
#fig.tight_layout()
plt.savefig("PDF+G: mult="+str(wealthshare)+",ros="+str(ros)+",roi="+str(roi)+",
    alpha="+str(alpha)+",m_0="+str(initmoney)+",pop="+str(people)+",timestep="+
    str(time)+".pdf",format='pdf')
plt.show()

#VISUALIZING ALL THE MACRO VARIABLES
print(gini)
fig, axs = plt.subplots(3, 2)
if (popt[1]<300 and popt[1]>-300):
    axs[0, 0].semilogy(points,exp(points,*popt),color='green',label='fit: tau
    ={:.2f}'.format(popt[1]))
axs[0, 0].semilogy(points,totmoney,marker='.',linestyle='')
axs[0, 0].set_title("money per agent")
axs[0, 0].legend(loc='lower right')
axs[1, 0].plot(points,savings,marker='.',linestyle='')
axs[1, 0].set_title("savings")
axs[1, 0].sharex(axs[0, 0])
axs[1, 1].semilogy(points,savings+investment,marker='.',linestyle='')
axs[1, 1].set_title("savings+investments")
axs[0, 1].semilogy(points,investment,marker='.',linestyle='')
#axs[0, 1].semilogy(points,exp(points,*invpopt),marker='',linestyle='-',label='
    fit: tau={:.2f}'.format(invpopt[1]))
axs[0, 1].legend(loc='lower right')
axs[0, 1].set_title("investments")
axs[2, 0].plot(points,gini,marker='.',linestyle='')
axs[2, 0].set_title("gini coefficient")
axs[2, 1].plot(points, entropy,marker='.',linestyle='',color='orange')
axs[2, 1].set_title("entropy")
fig.tight_layout()
```

```
plt.savefig("C:ros="+str(ros)+",roi="+str(roi)+",alpha="+str(alpha)+",m_0="+str(
    initmoney)+",pop="+str(people)+",timestep="+str(time)+".pdf",format='pdf')
plt.show()
```

Listing B.2: The code of the engine.py program

```
import random
import math

def evolution(initmoney,people,time,alpha,mps,mpc,ros,roi,wealthtaxrate,
    incometaxrate,
             nontradefrequency,normalizetrade,income,beta,wlowtaxrate,
    whightaxrate,gamma,
             ilowtaxrate,ihightaxrate,wealthshare, exchange,accounts):

    for t in range(time+1):

        a = random.randint(0, people - 1)
        b = random.randint(0, people - 1)
        multexchange=random.randint(0,math.floor(wealthshare*accounts[a]))
        if (accounts[a] >=exchange+multexchange):
            accounts[a] -= exchange+multexchange
            accounts[b] += exchange+multexchange

        #IMPLEMENTING INVESTMENTS, CONSUMPTION, TAXES FOR AGENT A
        if(random.random()<nontradefrequency):
            if (accounts[a] <= alpha):

                consume = mpc * accounts[a]
                savings = mps * accounts[a]
                investment = 0

            else:

                consume = mpc * alpha
                savings = mps * alpha
                investment = accounts[a] - alpha
            invgain=roi*investment
            savgain=ros*savings

           #CALCULATING TOTAL TRANSFER FOR AGENT A
            tottransfer = savgain + invgain - consume + income
            if(accounts[a]<beta):
                tottransfer-=wlowtaxrate*accounts[a]
            else:
                tottransfer -= whightaxrate * accounts[a]

            if (tottransfer >= 0 and tottransfer<gamma):
                tottransfer -= ilowtaxrate * tottransfer
            elif(tottransfer>=0 and tottransfer>= gamma):
                tottransfer-=ihightaxrate*tottransfer
```

```
            #MAKING THE TRANSFER
            accounts[a]+=math.floor(random.random()*tottransfer)
            if (accounts[a]<0):
                accounts[a]=0

        # IMPLEMENTING INVESTMENTS, CONSUMPTION, TAXES FOR AGENT B
        if (random.random() < nontradefrequency):
            if (accounts[b] <= alpha):

                consume = mpc * accounts[b]
                savings = mps * accounts[b]
                investment = 0

            else:

                consume = mpc * alpha
                savings = mps * alpha
                investment = accounts[b] - alpha
            invgain = roi * investment
            savgain = ros * savings

            # CALCULATING TOTAL TRANSFER FOR AGENT B
            tottransfer = savgain + invgain - consume + income
            if (accounts[b] < beta):
                tottransfer -= wlowtaxrate * accounts[b]
            else:
                tottransfer -= whightaxrate * accounts[b]

            if (tottransfer >= 0 and tottransfer < gamma):
                tottransfer -= ilowtaxrate * tottransfer
            elif (tottransfer >= 0 and tottransfer >= gamma):
                tottransfer -= ihightaxrate * tottransfer

            # MAKING THE TRANSFER
            accounts[b] += math.floor(random.random() * tottransfer)
            if (accounts[b] < 0):
                accounts[b] = 0


    return accounts
```

Listing B.3: The code of the command.py library

```
import matplotlib.pyplot as plt
import numpy as np

#FIND THE PROBABILITY DISTRIBUTION FUNCTION FOR THE ARRAY OF THE ACCOUNTS
def pdf(richaxis):
    allrichness = np.arange(0, np.max(richaxis) + 1)
    temp = np.zeros(len(allrichness))
    h = 0
    for i in allrichness:
```

```
        count = 0
        for j in richaxis:
            # print(i,j,count)
            if (j == i):
                count += 1
        temp[h] = count
        h += 1
    return(temp)


def pdf2(accounts):
    allrichness=np.arange(0,np.max(accounts)+1)
    pdf=np.zeros(len(allrichness))
    for i in range (len(allrichness)):
        count=0
        for j in accounts:
            if(i==j):
                count+=1
        pdf[i]=count
    return pdf

#FIND THE SHANNON ENTROPY OF DISTRIBUTION
def entropy(temp):
    people=np.sum(temp)
    s = 0
    for i in temp:
        if (i != 0):
            s -= (i / people) * np.log(i / people)
    return(s)

#WRITE DATA ON FILE
def write_params(time,people,initmoney,runs):
    f = open('param.txt', 'w')
    f.write('{} {} {} {}'.format(time, people, initmoney,runs))
    f.close()

def calctemp(Tvec):
    avgT=np.mean(Tvec)
    sigmaT=np.sqrt((np.sum((Tvec-avgT)**2))/(len(Tvec)-1))
    return(avgT,sigmaT)

def calc_difference(pdf,temp):
    if(len(pdf)>len(temp)):
        templong= np.append(temp, np.zeros(len(pdf) - len(temp)))
        pdflong=pdf
    elif(len(pdf)<len(temp)):
        pdflong=np.append(pdf,np.zeros(len(temp) - len(pdf)))
        templong=temp
    else:
        pdflong=pdf
        templong=temp
    people=np.sum(pdf)
```

```
    X=np.sum(np.abs(templong-pdflong))/people
    return(X)

def calcD(pdf,m0):
    people=np.sum(pdf)
    allrichness=np.arange(1,len(pdf)+1)
    teopdf=np.zeros(len(pdf))
    for j in allrichness:
        teopdf[j-1] =people* (1 / (m0 + 1)) * (m0 / (m0 + 1)) ** (j)
    diff=np.sum(np.abs(pdf-teopdf))/people
    delta=(m0/(m0+1))**(len(pdf))
    D=diff+delta
    return(D)

def pdf_trunc(richaxis,matrixdimension):
    allrichness = np.arange(0, matrixdimension)
    temp = np.zeros(len(allrichness))
    for i in range(len(allrichness)):
        temp[i] = allrichness[i]
    h = 0
    for i in allrichness:
        count = 0
        for j in richaxis:

            if (j == i):
                count += 1
        temp[h] = count
        h += 1
    return (temp)

def bin_data(x,y,binnumber=1):
    binsyze=int(len(x)/binnumber)
    bins=np.arange(0,binnumber)
    meanx=0
    meany=0
    xbin,ybin=np.zeros(binnumber),np.zeros(binnumber)
    for bin in bins:
        for j in np.arange(0,binsyze):
            meanx+=x[bin*binsyze+j]
            meany+=y[bin*binsyze+j]
        xbin[bin]=meanx/binsyze
        ybin[bin]=meany/binsyze
        meanx=0
        meany=0
    x[binnumber],y[binnumber]=0,0
    return(xbin,ybin)


def macrovariables(pdf,alpha,mpc,mps):
    people=np.sum(pdf)
    consume,investment, savings=[0,0,0]
    for index,value in enumerate(pdf):
```

```
        if(index<=alpha):
            consume+=value*index*mpc
            savings+=value*index*mps
        else:
            consume+=value*alpha*mpc
            savings+=value*alpha*mps
            investment+=(index-alpha)*value

    return consume/people, savings/people, investment/people

def Gini(accounts,j):
    idx = np.argsort(accounts)
    richaxis = accounts[idx]
    #print(richaxis)
    people = len(accounts)
    totmoney = np.sum(richaxis)
    cumrich = np.zeros(people)
    for i in range(0, people):
        cumrich[i] = cumrich[i - 1] + richaxis[i]
    normalcumrich = (cumrich + 0.5) / totmoney
    gini = 1 - 2*np.sum(normalcumrich) / people
    if(j==10):
        plt.plot(np.arange(0,len(normalcumrich))/people,normalcumrich)
        plt.plot(np.arange(0,len(normalcumrich))/people,np.arange(0,len(
    normalcumrich))/people)
        plt.xlabel('share of people')
        plt.ylabel('share of wealth')
        plt.show()

    return gini,normalcumrich
```

Listing B.4: The code used to find the normalized perturbation characteristic timescales

```
import numpy as np
from matplotlib import pyplot as plt


lenvec=500
parameter=0.1
lambdapoints=1000

Pzero=0.5 #depends on the temperature

lambdas=np.linspace(-3,-2.88,lambdapoints)
alpha=np.zeros(lenvec)
alpha[lenvec-1]=parameter


kappas=0.5*(lambdas+2-Pzero-np.sqrt((lambdas+2-Pzero)**2-4*(1-Pzero)))
print(kappas)
```

```
sum=np.zeros(len(lambdas))

numbers=np.arange(0,lenvec-2)

for index,k in enumerate(kappas):
    alpha[lenvec - 1] = parameter
    alpha[lenvec - 2] =alpha[lenvec-1]/k
    for n in numbers[::-1]:
        alpha[n]=((Pzero-2-lambdas[index])*alpha[n+1]+alpha[n+2])/(Pzero-1)
    print(alpha)


    rescalefactor=1/alpha[0]
    sum[index]=rescalefactor*np.sum(alpha)
    alpha = np.zeros(lenvec)


plt.plot(lambdas,sum)
plt.xlabel("$\lambda$")
plt.ylabel("$\sum \delta(m)$")
plt.minorticks_on()
plt.show()
```

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