

1 Lecture 25: Laplace transforms and half-Fourier transforms

A key tool in the WienerHopf method will be the half-Fourier transforms

$$\int_{-\infty}^0 u(t)e^{-ist}dt \quad \text{and} \quad \int_0^{\infty} u(t)e^{-ist}dt$$

and the Laplace transform

$$\int_0^{\infty} u(t)e^{-zt}dt$$

in particular we are interested in the analyticity properties with respect to s/z .

Outline:

2. Analyticity properties of Fourier transforms
 - Inverse Fourier transform on shifted contours
3. Half-Fourier transforms
 - Inverting the Half-Fourier transform
 - Relationship to Laplace transform
4. Application: solving differential equations on the half-line

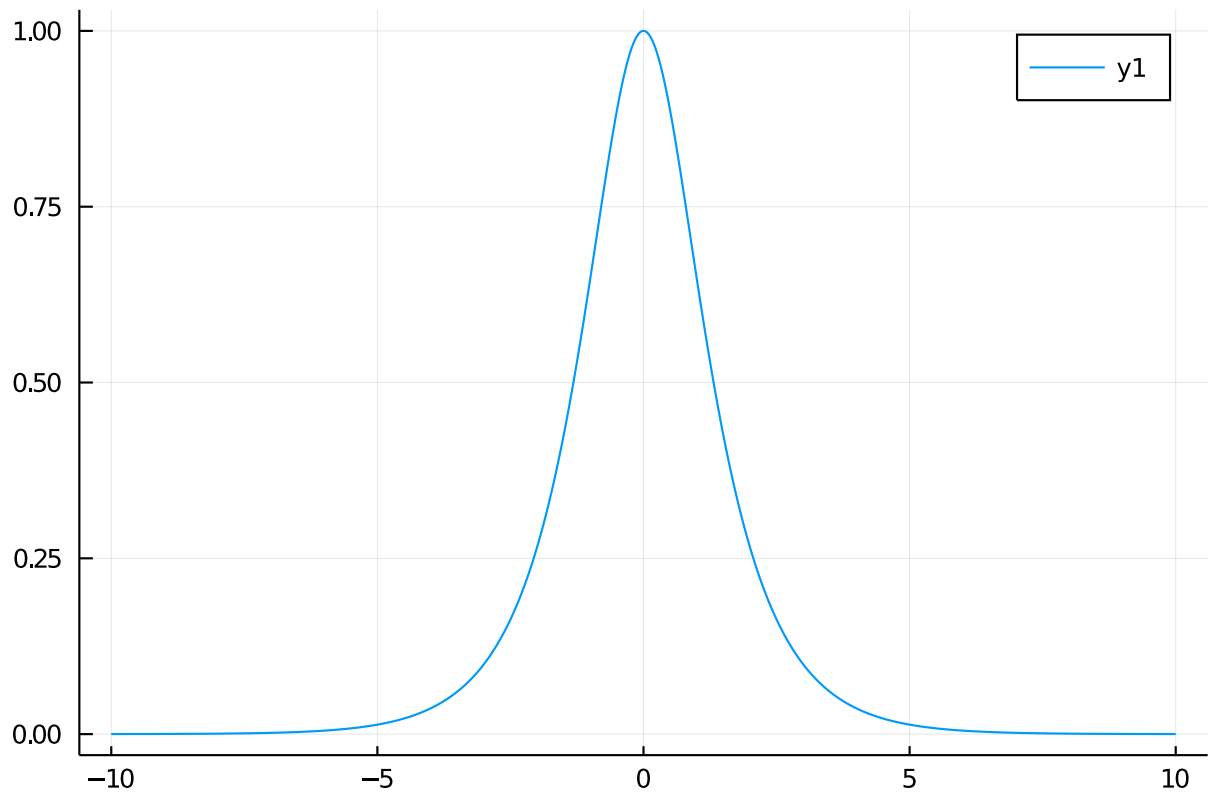
1.1 Analyticity properties of Fourier transforms

Consider the Fourier transform of

$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

This function has exponential decay in both directions:

```
using Plots, ApproxFun, OscillatoryIntegrals, ComplexPhasePortrait, LinearAlgebra
xx = -10:0.01:10
plot(xx, sech.(xx))
```



Now the Fourier transform of $\text{sech } x$ is

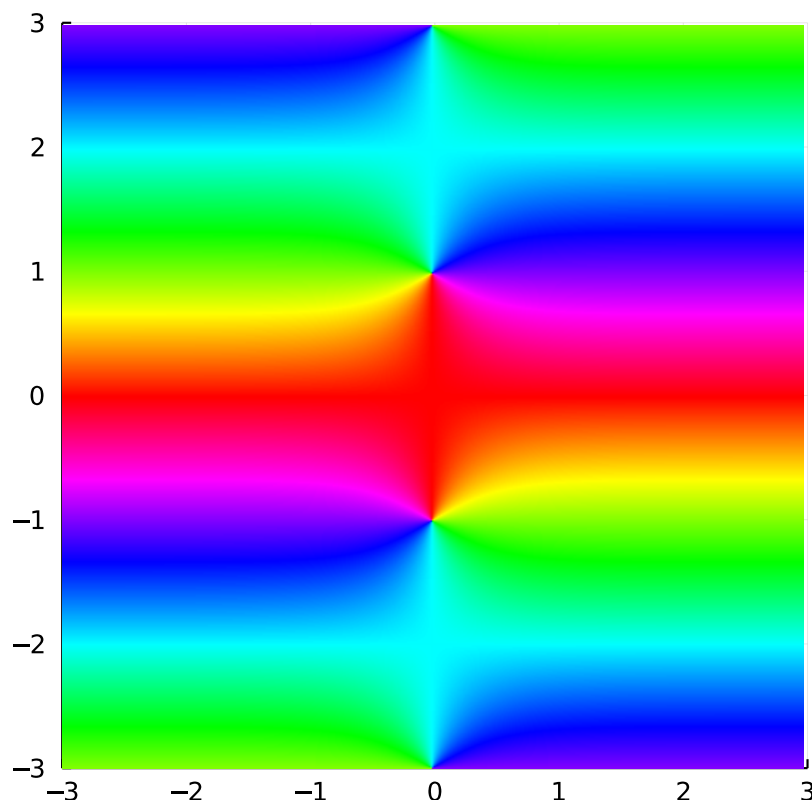
$$\mathcal{F}\text{sech}(z) = \int_{-\infty}^{\infty} \text{sech } t e^{-izt} dt = \pi \text{sech} \frac{\pi z}{2}$$

This is calculated via Residue theorem with a bit of work. Note that

$$\pi \text{sech} \frac{\pi z}{2} = \frac{2\pi}{e^{\frac{\pi z}{2}} + e^{-\frac{\pi z}{2}}}$$

is analytic for $-1 < \Im z < 1$.

`phaseplot(-3..3, -3..3, z -> pi*sech(pi*z/2))`



This is because of the exponential decay.

Theorem (Analyticity of Fourier transforms) Suppose $|f(x)e^{\gamma x}| < |M(x)|$ where M is absolutely integrable for all $a < \gamma < b$. Then

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(t)e^{-izt} dt$$

is analytic for $a < \Im z < b$.

Proof Let $z = s + i\gamma$ and note that $|f(t)e^{-izt}| = |f(t)|e^{\gamma t}$. Thus for $a < \gamma < b$, we can exchange differentiation and integration to get

$$\frac{d\hat{f}}{dz} = -iz\hat{f}(z)$$

■

Remark We don't need f to be analytic at all! Decay in f gives analyticity.

In the case of $\text{sech } x$, we get exponential decay in both directions: that is $\text{sech } xe^{\gamma x}$ is absolutely integrable for $|\gamma| < 1$.

Another example is $e^{-x^2/2}$, which is absolutely integrable for any γ . Therefore, its Fourier transform is in fact entire:

$$\mathcal{F}[e^{-x^2/2}](z) = \sqrt{2\pi}e^{-\frac{z^2}{2}}$$

1.1.1 Inverse Fourier transform on shifted contours

A neglected fact of the Fourier transform is that we can think of $\hat{f}(z)$ living on any line $(-\infty + i\gamma, \infty + i\gamma)$, and in fact we can recover f from the Fourier transform only on this line. This works even if $\hat{f}(s)$ is not defined on the real-axis, the real-axis is NOT special!

Theorem Suppose $f(x)e^{\gamma x}$ is square integrable. Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty+i\gamma}^{\infty+i\gamma} \hat{f}(\zeta) e^{ix\zeta} d\zeta$$

Proof Note for $g(x) = f(x)e^{\gamma x}$

$$\hat{g}(s) = \int_{-\infty}^{\infty} f(t) e^{\gamma t - ist} dt = \hat{f}(s + i\gamma).$$

Therefore we have

$$\begin{aligned} e^{\gamma x} f(x) &= g(x) = \mathcal{F}^{-1} \hat{g}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(s) e^{ixs} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s + i\gamma) e^{ixs} ds \\ &= \frac{1}{2\pi} \int_{-\infty+i\gamma}^{\infty+i\gamma} \hat{f}(\zeta) e^{ix(\zeta - i\gamma)} d\zeta \\ &= \frac{1}{2\pi} e^{\gamma x} \int_{-\infty+i\gamma}^{\infty+i\gamma} \hat{f}(\zeta) e^{ix\zeta} d\zeta \end{aligned}$$

Which shows the result by cancelling out $e^{\gamma x}$.

1.2 Half-Fourier transforms

Consider now

$$\int_0^{\infty} f(t) e^{-ist} dt$$

This is in fact the Fourier transform of f extended to the negative real axis by zero:

$$\int_{-\infty}^{\infty} \begin{cases} f(t) & t \geq 0 \\ 0 & \text{otherwise} \end{cases} e^{-ist} dt$$

To make sure we remember the domain of definition, we introduce the notation:

$$f_R(x) = \begin{cases} f(t) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_L(x) = \begin{cases} f(t) & t < 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\widehat{f}_R(s) = \int_0^\infty f(t)e^{-ist}dt \quad \text{and} \quad \widehat{f}_L(s) = \int_{-\infty}^0 f(t)e^{-ist}dt$$

Because it is identically zero on the negative real axis, we immediately get the following:

Corollary (analyticity of Half-Fourier transform) Suppose $f(x)$ is bounded for $x \geq 0$. Then $\widehat{f}_R(z)$ is analytic in the lower half plane

$$\mathbb{H}_- = \{z : \Im z < 0\}.$$

More generally, f can even have exponential decay: if $f(x)e^{\gamma x}$ is bounded then $\widehat{f}_R(z)$ is analytic in $\{z : \Im z < \gamma\}$. As before, the same inversion formula follows:

Corollary (inverting Half-Fourier transform) Suppose $f(x)e^{\gamma x}$ is square integrable for $x \geq 0$. Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \widehat{f}_R(\zeta) e^{ix\zeta} d\zeta$$

for any choice of $-\infty < M \leq \gamma$.

Example Consider $f(x) = xe^{-x}$ for $0 \leq x < \infty$. Note that $f(x)e^{\gamma x}$ is square integrable for any $\gamma < 1$, and we have

$$\widehat{f}_R(z) = \int_0^\infty te^{-t-izt}dt = \frac{1}{(1+iz)^2} = -\frac{1}{(z-i)^2}$$

is analytic for $\Im z < 1$. Thus for any $M < 1$ we have

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \widehat{f}_R(\zeta) e^{ix\zeta} d\zeta \\ &= \frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \frac{1}{(1+i\zeta)^2} e^{ix\zeta} d\zeta \end{aligned}$$

Since $x > 0$, we can use Residue calculus in the upper-half plane, which confirms the result:

$$\operatorname{Res}_{z=i} \frac{e^{ixz}}{(1+iz)^2} = \operatorname{Res}_{z=i} \frac{e^{-x} + ixe^{-x}(z-i) + O(z-i)^2}{-(z-i)^2} = -ixe^{-x}.$$

Example Consider $f(x) = x$. This function is not square-integrable, but we have $f(x)e^{\gamma x}$ is square integrable for any $\gamma < 0$, and we find for $\Im z < 0$

$$\widehat{f}_R(z) = -\frac{1}{z^2}$$

Thus we can still use the result to say, for any $M < 0$,

$$f(x) = -\frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \frac{1}{\zeta^2} e^{ix\zeta} d\zeta$$

Note that the results have corresponding analogues for f_L :

Corollary (analyticity of left Half-Fourier transform) Suppose $f(x)e^{\gamma x}$ is bounded for $x \leq 0$. Then $\widehat{f}_L(z)$ is analytic for $\{z : \Im z > \gamma\}$.

Corollary (inverting left Half-Fourier transform) Suppose $f(x)e^{\gamma x}$ is square integrable for $x < 0$. Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \widehat{f_L}(\zeta) e^{ix\zeta} d\zeta$$

for any choice of $\gamma \leq M < \infty$.

1.2.1 Laplace transforms

Now consider the Laplace transform

$$\check{f}(z) = \int_0^\infty f(t) e^{-zt} dt$$

but this is just the half Fourier transform rotated by $+\pi/2$:

$$\check{f}(iz) = \widehat{f_R}(z)$$

Thus if $f(x)e^{\gamma x}$ is square integrable, then $\check{f}(z)$ is well-defined for $\Re z \geq \gamma$.

NEVER think of the Laplace transform as a real-valued object: it only makes sense as a complex object. This is seen from the inverse Laplace transform

$$f(x) = \frac{1}{2\pi i} \int_{-i\infty-M}^{i\infty-M} \check{f}(\zeta) e^{\zeta x} d\zeta$$

which is of course just the inverse Fourier transform in disguise.

1.3 Application: solving differential equations on the half-line

Consider the following ODE for $x \geq 0$:

$$u''(x) + 2u'(x) + u(x) = f(x)$$

with initial conditions $u(0) = u'(0) = 0$. Note that we have by integration-by-parts

$$\begin{aligned} \check{u}'(z) &= \int_0^\infty u'(t) e^{-zt} dt = u(0) + z \int_0^\infty u(t) e^{-zt} dt = u(0) + z\check{u}(z) \\ \check{u}''(z) &= u'(0) + z\check{u}'(z) = u'(0) + zu(0) + z^2\check{u}(z) \end{aligned}$$

Thus taking into account the initial conditions, the equation in Laplace space becomes

$$(z^2 + 2z + 1)\check{u}(z) = \check{f}(z)$$

Hence we have

$$u(x) = \frac{1}{2\pi i} \int_{-i\infty-M}^{i\infty-M} \frac{\check{f}(\zeta)}{\zeta^2 + 2\zeta + 1} e^{x\zeta} d\zeta$$

Consider the case $f(x) = x$, so that

$$\check{f}(z) = \frac{1}{z^2}$$

Here we need $M < 0$ hence we are integrating on a contour in the right-half plane. Using Residue calculus, we have

$$\begin{aligned} u(x) &= \left(\operatorname{Res}_{z=-1} + \operatorname{Res}_{z=0} \right) \frac{e^{zx}}{z^2(z+1)^2} \\ &= \operatorname{Res}_{z=-1} \frac{e^{-x} + e^{-x}(x+2)(z+1) + O(z+1)^2}{(z+1)^2} + \operatorname{Res}_{z=0} \frac{1 + (x-2)z + O(z)^2}{z^2} \\ &= (x+2)e^{-x} + x - 2 \end{aligned}$$

1.4 Laplace transform of rational functions

We now consider the question of calculating Laplace transforms (or equivalently, half-Fourier transforms)

$$\check{f}(s) = \int_0^\infty f(t)e^{-st} dt$$

where f is rational. We're going to do something seemingly crazy: we'll first calculate the Cauchy transform

$$\mathcal{C}[fe^{-s\circ}](z) = \frac{1}{2\pi i} \int_0^\infty \frac{f(t)e^{-st}}{t-z} dt$$

so that

$$\check{f}(s) = -2\pi i \lim_{z \rightarrow \infty} z \mathcal{C}[fe^{-s\circ}](z)$$

Note that the exponential decay in the integrand allows us to use Plemelj's lemma: if we find a function $\phi(z)$ such that

1.

$$\phi(z)$$

is analytic off $[0, \infty)$

2.

$$\lim_{z \rightarrow \infty} \phi(z) = 0$$

for any angle of approach

3.

$$\phi$$

has weaker than pole singularities at 0

4.

$$\phi_+(x) - \phi_-(x) = f(x)e^{-sx}$$

Then we have calculated the Cauchy transform:

$$\phi(z) = \mathcal{C}[fe^{-s\circ}](z).$$

Let's start with $f(x) = 1$ and $s = 1$, that is, what is the Cauchy transform of e^{-x} ? Consider the exponential integral:

$$\text{Ei}(z) = \int_{-\infty}^z \frac{e^{\zeta}}{\zeta} d\zeta$$

Without loss of generality, the contour of integration is

$$(-\infty, -1) \cup [-1, z)$$

that is, a straight line to -1 and a straight line from -1 to z . Thus we have a branch cut on $[0, \infty)$ which has the jump

$$\text{Ei}^+(x) - \text{Ei}^-(x) = -\oint \frac{e^{\zeta}}{\zeta} d\zeta = -2\pi i$$

where

$$\text{Ei}^{\pm}(x) = \lim_{\epsilon \rightarrow 0} \text{Ei}(x \pm i\epsilon)$$

Consider

$$\phi(z) = -\frac{e^{-z}\text{Ei}(z)}{2\pi i}$$

1. This is analytic off $[0, \infty)$
2. Integrating by parts we have decay at ∞ in all directions:

$$e^{-z}\text{Ei}(z) = \frac{1}{z} - \int_{-\infty}^z \frac{e^{\zeta-z}}{\zeta^2} d\zeta = O(z^{-1})$$

- 3.

$$\phi$$

has a logarithmic singularity at 0

- 4.

$$\phi_+(x) - \phi_-(x) = e^{-x}.$$

Thus

$$\mathcal{C}[e^{-s\circ}](z) = \phi(sz) = -\frac{e^{-sz}\text{Ei}(sz)}{2\pi i}$$

Let's make sure we didn't make a mistake. Here we first define Ei:


```

const ei_-_1 = let ζ = Fun(-50 .. -1)
  sum(exp(ζ)/ζ)
end
function ei(z)
  ζ = Fun(Segment(-1 , z))
  ei_-_1 + sum(exp(ζ)/ζ)
end

```

```

φ = (z) -> -exp(-z)*ei(z)/(2π*im)

```

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#3 (generic function with 1 method)

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The expression matches the Cauchy transform:

```

t = Fun(0 .. 50)
s = 2.0
z = 2.0+2.0im
sum(exp(-s*t)/(t-z))/(2π*im), φ(s*z)

```

```

(0.02534853710699083 + 0.017828329563678146im, 0.02534853710699053 + 0.0178
28329563677806im)

```

We then recover the Laplace transform by taking the limit:

$$\check{1}(s) = -2\pi i \lim_{z \rightarrow \infty} z\phi(sz) = \lim_{z \rightarrow \infty} \frac{z}{sz} = \frac{1}{s}$$

What about rational f ? Do the same trick of subtracting off the singularities. For example, consider $f(z) = 1/(z+1)$. Then

$$\frac{\phi(sz) - \phi(-s)}{z+1}$$

satisfies all the necessary properties.

```

t = Fun(0 .. 50)
s = 2.0
z = 2.0+2.0im
f = 1/(t+1)
sum(exp(-s*t)*f/(t-z))/(2π*im), (φ(s*z)-φ(-s))/(z+1)

```

```

(0.017439749335766613 + 0.013485355450154743im, 0.01743974933576627 + 0.013
48535545015414im)

```

Therefore,

$$\check{f}(s) = -2\pi i \lim_{z \rightarrow \infty} z \frac{\phi(sz) - \phi(-s)}{z+1} = 2\pi i \phi(-s) = -e^s \text{Ei}(-s)$$

```

sum(exp(-s*t)*f), 2π*im*φ(-s), -exp(s)*ei(-s)

```

```

(0.36132861688822454, 0.3613286168882152 + 0.0im, 0.3613286168882152)

```