

# Applied Complex Analysis (2021)

## 1 Lecture 25: Laplace transforms and half-Fourier transforms

A key tool in the Wiener–Hopf method will be the half-Fourier transforms

$$\int_{-\infty}^0 u(t)e^{-ist}dt \quad \text{and} \quad \int_0^{\infty} u(t)e^{-ist}dt$$

and the Laplace transform

$$\int_0^{\infty} u(t)e^{-zt}dt$$

in particular we are interested in the analyticity properties with respect to  $s/z$ .

Outline:

2. Analyticity properties of Fourier transforms
  - Inverse Fourier transform on shifted contours
3. Half-Fourier transforms
  - Inverting the Half-Fourier transform
  - Relationship to Laplace transform
4. Application: solving differential equations on the half-line

## 1.1 Analyticity properties of Fourier transforms

Consider the Fourier transform of

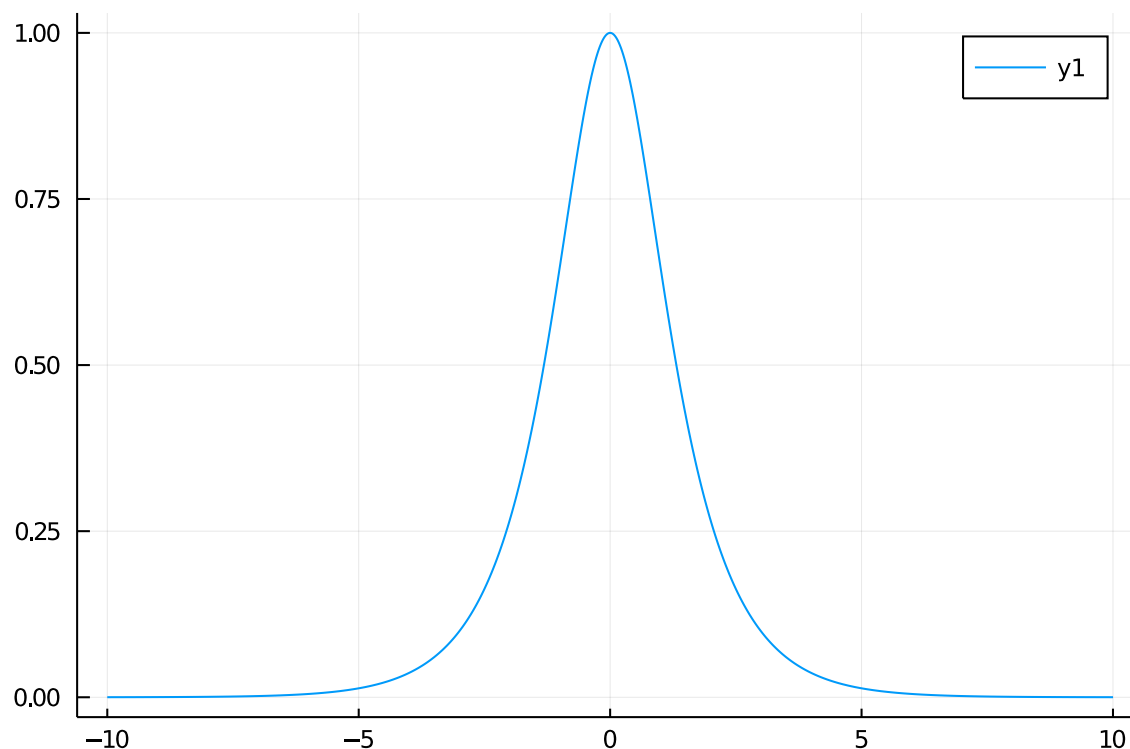
$$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$$

This function has exponential decay in both directions:

```
using Plots, ApproxFun, OscillatoryIntegrals, ComplexPhasePortrait,  
LinearAlgebra
```

```
xx = -10:0.01:10
```

```
plot(xx,sech.(xx))
```



Now the Fourier transform of  $\operatorname{sech} x$  is

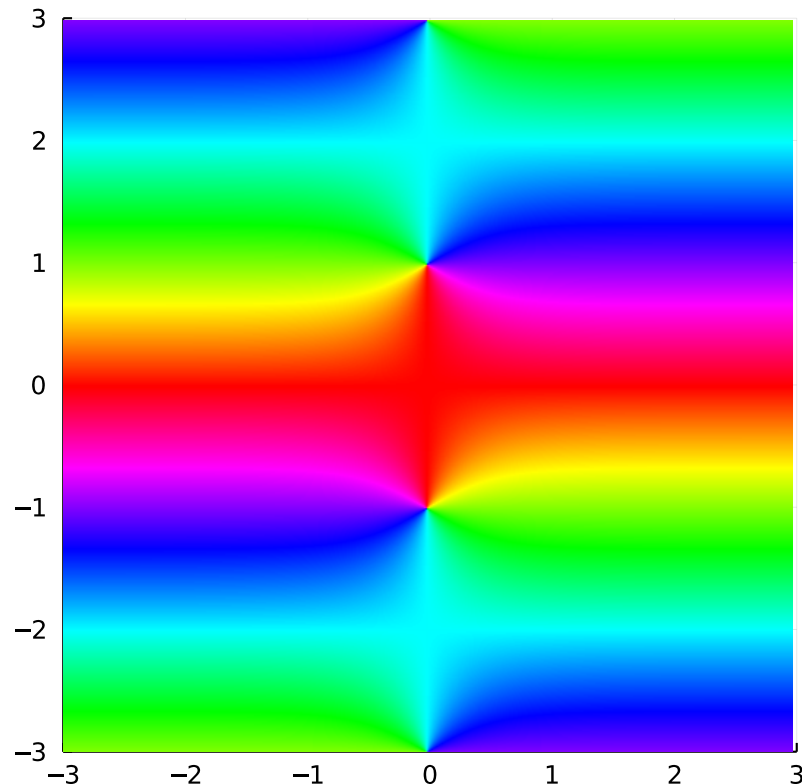
$$\mathcal{F}\operatorname{sech}(s) = \int_{-\infty}^{\infty} \operatorname{sech} t e^{-ist} dt = \pi \operatorname{sech} \frac{\pi s}{2}$$

This is calculated via Residue theorem with a bit of work. Note that

$$\pi \operatorname{sech} \frac{\pi z}{2} = \frac{2\pi}{e^{\frac{\pi z}{2}} + e^{-\frac{\pi z}{2}}}$$

is analytic for  $-1 < \Im z < 1$ .

```
phaseplot(-3..3, -3..3, z -> pi*sech(pi*z/2))
```



This is because of the exponential decay.

**Theorem (Analyticity of Fourier transforms)** Suppose  $|f(x)e^{\gamma x}| < |M(x)|$  where  $M$  is absolutely integrable for all  $a < \gamma < b$ . Then

$$\hat{f}(z) = \int_{-\infty}^{\infty} f(t)e^{-izt} dt$$

is analytic for  $a < \Im z < b$ .

**Proof** Let  $z = s + i\gamma$  and note that  $|f(t)e^{-izt}| = |f(t)e^{\gamma t}|$ . Thus for  $a < \gamma < b$ , we can exchange differentiation and integration to get

$$\frac{d\hat{f}}{dz} = -iz\hat{f}(z)$$

■

**Remark** We don't need  $f$  to be analytic at all! Decay in  $f$  gives analyticity.

In the case of  $\operatorname{sech} x$ , we get exponential decay in both directions: that is  $\operatorname{sech} xe^{\gamma x}$  is absolutely integrable for  $|\gamma| < 1$ .

Another example is  $e^{-x^2/2}$ , which is absolutely integrable for any  $\gamma$ . Therefore, it's Fourier transform is in fact entire:

$$\mathcal{F}[e^{-x^2/2}](z) = \sqrt{2\pi}e^{-\frac{z^2}{2}}$$

### 1.1.1 Inverse Fourier transform on shifted contours

A neglected fact of the Fourier transform is that we can think of  $\hat{f}(z)$  living on any line  $(-\infty + i\gamma, \infty + i\gamma)$ , and in fact we can recover  $f$  from the Fourier transform only on this line. This works even if  $\hat{f}(s)$  is not defined on the real-axis, the real-axis is NOT special!

**Theorem** Suppose  $f(x)e^{\gamma x}$  is square integrable. Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty+i\gamma}^{\infty+i\gamma} \hat{f}(\zeta) e^{ix\zeta} d\zeta$$

**Proof** Note for  $g(x) = f(x)e^{\gamma x}$

$$\hat{g}(s) = \int_{-\infty}^{\infty} f(t) e^{\gamma t - ist} dt = \hat{f}(s + i\gamma).$$

Therefore we have

$$\begin{aligned} e^{\gamma x} f(x) &= g(x) = \mathcal{F}^{-1} \hat{g}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{g}(s) e^{ixs} ds = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(s + i\gamma) e^{ixs} ds \\ &= \frac{1}{2\pi} \int_{-\infty+i\gamma}^{\infty+i\gamma} \hat{f}(\zeta) e^{ix(\zeta - i\gamma)} d\zeta \\ &= \frac{1}{2\pi} e^{\gamma x} \int_{-\infty+i\gamma}^{\infty+i\gamma} \hat{f}(\zeta) e^{ix\zeta} d\zeta \end{aligned}$$

Which shows the result by cancelling out  $e^{\gamma x}$ .

## 1.2 Half-Fourier transforms

Consider now

$$\int_0^{\infty} f(t)e^{-ist}dt$$

This is in fact the Fourier transform of  $f$  extended to the negative real axis by zero:

$$\int_{-\infty}^{\infty} \begin{cases} f(t) & t \geq 0 \\ 0 & \text{otherwise} \end{cases} e^{-ist}dt$$

To make sure we remember the domain of definition, we introduce the notation:

$$f_R(x) = \begin{cases} f(t) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_L(x) = \begin{cases} f(t) & t < 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\widehat{f_R}(s) = \int_0^{\infty} f(t)e^{-ist}dt \quad \text{and} \quad \widehat{f_L}(s) = \int_{-\infty}^0 f(t)e^{-ist}dt$$

Because it is identically zero on the negative real axis, we immediately get the following:

**Corollary (analyticity of Half-Fourier transform)** Suppose  $f(x)$  is bounded for  $x \geq 0$ . Then  $\widehat{f_R}(z)$  is analytic in the lower half-plane

$$\mathbb{H}_- = \{z : \Im z < 0\}.$$

More generally,  $f$  can even have exponential decay: if  $f(x)e^{\gamma x}$  is bounded then  $\widehat{f_R}(z)$  is analytic in  $\{z : \Im z < \gamma\}$ . As before, the same inversion formula follows:

**Corollary (inverting Half-Fourier transform)** Suppose  $f(x)e^{\gamma x}$  is square integrable for  $x \geq 0$ . Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \widehat{f_R}(\zeta) e^{ix\zeta} d\zeta$$

for any choice of  $-\infty < M \leq \gamma$ .

**Example** Consider  $f(x) = xe^{-x}$  for  $0 \leq x < \infty$ . Note that  $f(x)e^{\gamma x}$  is square integrable for any  $\gamma < 1$ , and we have

$$\widehat{f_R}(z) = \int_0^\infty te^{-t-izt}dt = \frac{1}{(1+iz)^2} = -\frac{1}{(z-i)^2}$$

is analytic for  $\Im z < 1$ . Thus for any  $M < 1$  we have

$$\begin{aligned} f(x) &= \frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \widehat{f_R}(\zeta) e^{ix\zeta} d\zeta \\ &= \frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \frac{1}{(1+i\zeta)^2} e^{ix\zeta} d\zeta \end{aligned}$$

Since  $x > 0$ , we can use Residue calculus in the upper-half plane, which confirms the result:

$$\operatorname{Res}_{z=i} \frac{e^{ixz}}{(1+iz)^2} = \operatorname{Res}_{z=i} \frac{e^{-x} + ixe^{-x}(z-i) + O(z-i)^2}{-(z-i)^2} = -ixe^{-x}.$$



**Example** Consider  $f(x) = x$ . This function is not square-integrable, but we have  $f(x)e^{\gamma x}$  is square integrable for any  $\gamma < 0$ , and we find for  $\Im z < 0$

$$\widehat{f_R}(z) = -\frac{1}{z^2}$$

Thus we can still use the result to say, for any  $M < 0$ ,

$$f(x) = -\frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \frac{1}{\zeta^2} e^{ix\zeta} d\zeta$$

Note that the results have corresponding analogues for  $f_L$ :

**Corollary (analyticity of left Half-Fourier transform)** Suppose  $f(x)e^{\gamma x}$  is bounded for  $x \leq 0$ . Then  $\widehat{f}_L(z)$  is analytic for  $\{z : \Im z > \gamma\}$ .

**Corollary (inverting left Half-Fourier transform)** Suppose  $f(x)e^{\gamma x}$  is square integrable for  $x < 0$ . Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty+iM}^{\infty+iM} \widehat{f}_L(\zeta) e^{ix\zeta} d\zeta$$

for any choice of  $\gamma \leq M < \infty$ .

## 1.2.1 Laplace transforms

Now consider the Laplace transform

$$\check{f}(z) = \int_0^\infty f(t)e^{-zt}dt$$

but this is just the half Fourier transform rotated by  $+\pi/2$ :

$$\check{f}(\mathrm{i}z) = \widehat{f_{\mathrm{R}}}(z)$$

Thus if  $f(x)e^{\gamma x}$  is square integrable, then  $\check{f}(z)$  is well-defined for  $\Re z \geq \gamma$ .

*NEVER* think of the Laplace transform as a real-valued object: it only makes sense as a complex object. This is seen from the inverse Laplace transform

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty-M}^{\mathrm{i}\infty-M} \check{f}(\zeta)e^{\zeta x}d\zeta$$

which is of course just the inverse Fourier transform in disguise.

### 1.3 Application: solving differential equations on the half-line

Consider the following ODE for  $x \geq 0$ :

$$u''(x) + 2u'(x) + u(x) = f(x)$$

with initial conditions  $u(0) = u'(0) = 0$ . Note that we have by integration-by-parts

$$\begin{aligned}\check{u}'(z) &= \int_0^\infty u'(t)e^{-zt}dt = u(0) + z \int_0^\infty u(t)e^{-zt}dt = u(0) + z\check{u}(z) \\ \check{u}''(z) &= u'(0) + z\check{u}'(z) = u'(0) + zu(0) + z^2\check{u}(0)\end{aligned}$$

Thus taking into account the initial conditions, the equation in Laplace space becomes

$$(z^2 + 2z + 1)\check{u}(z) = \check{f}(z)$$

Hence we have

$$\check{u}(z) = \frac{1}{2\pi i} \int_{-i\infty-M}^{i\infty-M} \frac{\check{f}(\zeta)}{\zeta^2 + 2\zeta + 1} e^{x\zeta} d\zeta$$

Consider the case  $f(x) = x$ , so that

$$\check{f}(z) = \frac{1}{z^2}$$

Here we need  $M < 0$  hence we are integrating on a contour in the right-half plane. Using Residue calculus, we have

$$\begin{aligned} u(x) &= \left( \operatorname{Res}_{z=-1} + \operatorname{Res}_{z=0} \right) \frac{e^{zx}}{z^2(z+1)^2} \\ &= \operatorname{Res}_{z=-1} \frac{e^{-x} + e^{-x}(x+2)(z+1) + O(z+1)^2}{(z+1)^2} + \operatorname{Res}_{z=0} \frac{1 + (x-2)z + O(z)^2}{z^2} \\ &= (x+2)e^{-x} + x - 2 \end{aligned}$$

## 1.4 Laplace transform of rational functions

We now consider the question of calculating Laplace transforms (or equivalently, half-Fourier transforms)

$$\check{f}(s) = \int_0^\infty f(t)e^{-st}dt$$

where  $f$  is rational. We're going to do something seemingly crazy: we'll first calculate the Cauchy transform

$$\mathcal{C}[fe^{-s\Diamond}](z) = \frac{1}{2\pi i} \int_0^\infty \frac{f(t)e^{-st}}{t-z}dt$$

so that

$$\check{f}(s) = -2\pi i \lim_{z \rightarrow \infty} z\mathcal{C}[fe^{-s\Diamond}](z)$$

Note that the exponential decay in the integrand allows us to use Plemelj's lemma: if we find a function  $\phi(z)$  such that

1.

$$\phi(z)$$

is analytic off  $[0, \infty)$

2.

$$\lim_{z \rightarrow \infty} \phi(z) = 0$$

for any angle of approach

3.

$$\phi$$

has weaker than pole singularities at 0

4.

$$\phi_+(x) - \phi_-(x) = f(x)e^{-sx}$$

Then we have calculated the Cauchy transform:

$$\phi(z) = \mathcal{C}[fe^{-s\Diamond}](z).$$

Let's start with  $f(x) = 1$  and  $s = 1$ , that is, what is the Cauchy transform of  $e^{-x}$ ? Consider the exponential integral:

$$\text{Ei}(z) = \int_{-\infty}^z \frac{e^{\zeta}}{\zeta} d\zeta$$

Without loss of generality, the contour of integration is

$$(-\infty, -1) \cup [-1, z)$$

that is, a straight line to  $-1$  and a straightline from  $-1$  to  $z$ . Thus we have a branch cut on  $[0, \infty)$  which has the jump

$$\text{Ei}^+(x) - \text{Ei}^-(x) = - \oint \frac{e^{\zeta}}{\zeta} d\zeta = -2\pi i$$

where

$$\text{Ei}^{\pm}(x) = \lim_{\epsilon \rightarrow 0} \text{Ei}(x \pm i\epsilon)$$

Consider

$$\phi(z) = -\frac{e^{-z}\text{Ei}(z)}{2\pi i}$$

1. This is analytic off  $[0, \infty)$
2. Integrating by parts we have decay at  $\infty$  in all directions:

$$e^{-z}\text{Ei}(z) = \frac{1}{z} - \int_{-\infty}^z \frac{e^{\zeta-z}}{\zeta^2} d\zeta = O(z^{-1})$$

3.

$$\phi$$

has a logarithmic singularity at 0

4.

$$\phi_+(x) - \phi_-(x) = e^{-x}.$$

Thus

$$\mathcal{C}[e^{-s\Diamond}](z) = \phi(sz) = -\frac{e^{-sz}\text{Ei}(sz)}{2\pi i}$$



Let's make sure we didn't make a mistake. Here we first define Ei:

```
const ei_-_1 = let ζ = Fun(-50 .. -1)
  sum(exp(ζ)/ζ)
end
function ei(z)
  ζ = Fun(Segment(-1 , z))
  ei_-_1 + sum(exp(ζ)/ζ)
end
```

```
φ = (z) -> -exp(-z)*ei(z)/(2π*im)
```

#3 (generic function with 1 method)

The expression matches the Cauchy transform:

```
t = Fun(0 .. 50)
s = 2.0
z = 2.0+2.0im
sum(exp(-s*t)/(t-z))/(2π*im), φ(s*z)

(0.02534853710699083 + 0.017828329563678146im, 0.02534853710699053 +
0.0178
28329563677806im)
```

We then recover the Laplace transform by taking the limit:

$$\check{1}(s) = -2\pi i \lim_{z \rightarrow \infty} z \phi(sz) = \lim_{z \rightarrow \infty} \frac{z}{sz} = \frac{1}{s}$$

What about rational  $f$ ? Do the same trick of subtracting off the singularities. For example, consider  $f(z) = 1/(z+1)$ . Then

$$\frac{\phi(sz) - \phi(-s)}{z+1}$$

satisfies all the necessary properties.

```
t = Fun(0 .. 50)
```

```
s = 2.0
```

```
z = 2.0+2.0im
```

```
f = 1/(t+1)
```

```
sum(exp(-s*t)*f/(t-z))/(2*pi*im), (phi(s*z)-phi(-s))/(z+1)
```

```
(0.017439749335766613 + 0.013485355450154743im, 0.01743974933576627 +  
0.013
```

```
48535545015414im)
```

Therefore,

$$\check{f}(s) = -2\pi i \lim_{z \rightarrow \infty} z \frac{\phi(sz) - \phi(-s)}{z+1} = 2\pi i \phi(-s) = -e^s \text{Ei}(-s)$$

```
sum(exp(-s*t)*f), 2*pi*im*phi(-s), -exp(s)*ei(-s)
```

```
(0.36132861688822454, 0.3613286168882152 + 0.0im, 0.3613286168882152)
```