Applied Complex Analysis (2021)

1 Mastery solutions

1. It follows from the orthogonality of the Chebyshev polynomials that

$$c_k = \frac{\langle f, T_k \rangle}{\langle T_k, T_k \rangle} = \begin{cases} \frac{1}{\pi}\langle f, T_0 \rangle & \text{if } k = 0, \\ \frac{2}{\pi}\langle f, T_k \rangle & \text{if } k > 0, \end{cases}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product with respect to the Chebyshev weight:

$$\langle f, g \rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} \mathrm{d}x.$$

2. For $k = 1, \ldots, n$ and $j \geq 0$,

$$T_{(2j+1)n}(x_k) = \cos\left((2j+1)n\frac{2k-1}{2n}\pi\right) = 0,$$

and for $0 \leq m \leq n-1$,

$$T_{2jn\pm m}(x_k) = \cos\left((2jn \pm m)\frac{2k-1}{2n}\pi\right) = (-1)^j \cos\left(m\frac{2k-1}{2n}\pi\right) = (-1)^j T_m(x_k).$$

3. We can write

$$\begin{aligned} f(x) = &\sum_{m=0}^{n-1} c_m T_m(x) + \sum_{j=0}^{\infty} c_{(2j+1)n} T_{(2j+1)n}(x) + \sum_{j=1}^{\infty} c_{2jn} T_{2jn}(x) \\ &+ \sum_{m=1}^{n-1}\sum_{j=1}^{\infty} \left[c_{2jn+m} T_{2jn+m}(x) + c_{2jn-m} T_{2jn-m}(x)\right] \end{aligned}$$

then

$$\begin{aligned} f(x_k) &= \sum_{m=0}^{n-1} c_m T_m(x_k) + \sum_{j=1}^{\infty} c_{2jn}(-1)^j T_0(x) + \sum_{m=1}^{n-1} T_m(x_k) \sum_{j=1}^{\infty} (-1)^j \left[c_{2jn+m} + c_{2jn-m}\right] \\ &= p_n(x_k) \\ &= \sum_{m=0}^{n-1} a_m T_m(x_k), \end{aligned}$$

and the results follow by comparing coefficients.

4.

$$
\begin{aligned}
p_n(x) - f(x) &= \sum_{k=0}^{n-1} a_k T_k(x) - \sum_{k=0}^{\infty} c_k T_k(x) \\
&= \sum_{k=0}^{n-1} (a_k - c_k)\, T_k(x) - \sum_{k=n}^{\infty} c_k T_k(x) \\
&= \sum_{k=0}^{n-1} (a_k - c_k)\, T_k(x) - (f(x) - f_n(x))
\end{aligned}
$$

5. Recall from Lecture 19 that

$$
|f(x) - f_n(x)| \leq \frac{2M\rho^{-n}}{1-\rho^{-1}}.
$$

From the formulae for the a_k, it follows that

$$
\left|\sum_{k=0}^{n-1} (a_k - c_k)\, T_k(x)\right| \leq \sum_{k=0}^{n-1} |a_k - c_k| \leq \sum_{k=n}^{\infty} |c_k| \leq \frac{2M\rho^{-n}}{1-\rho^{-1}}.
$$

The latter inequality follows from the bound

$$
|c_k| \leq 2M\rho^{-k}
$$

see Lecture 19. We conclude that

$$
|f(x) - p_n(x)| \leq \frac{4M\rho^{-n}}{1-\rho^{-1}}.
$$