Applied Complex Analysis (2021)

1 Solution Sheet 2

1.1 Problem 1

1.

We have

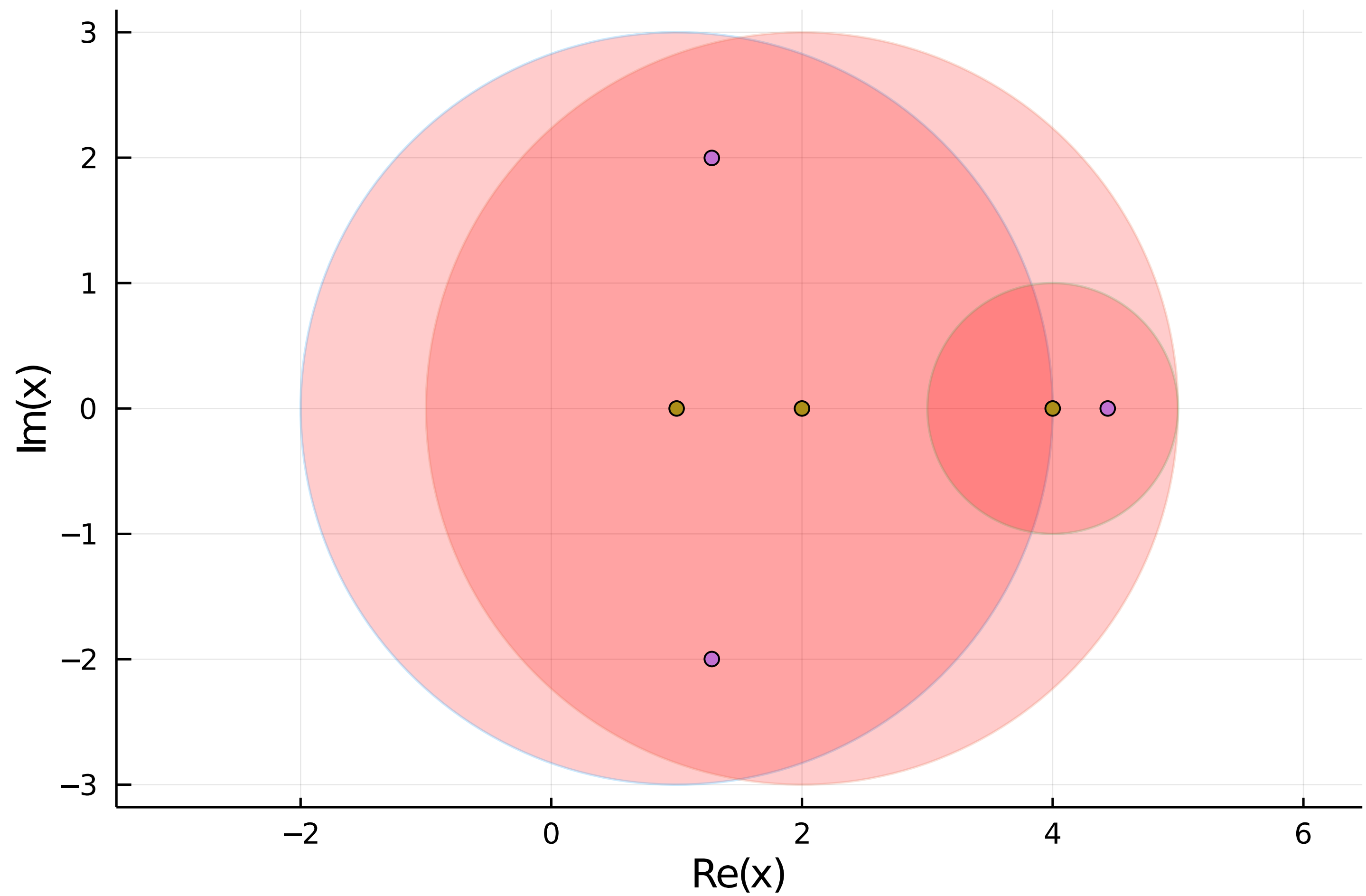
$$\sigma(A) \subseteq B(1,3) \cup B(2,3) \cup B(4,1)$$

where $B(z_0, r)$ is the ball of radius r around z_0.

Here's a depiction:

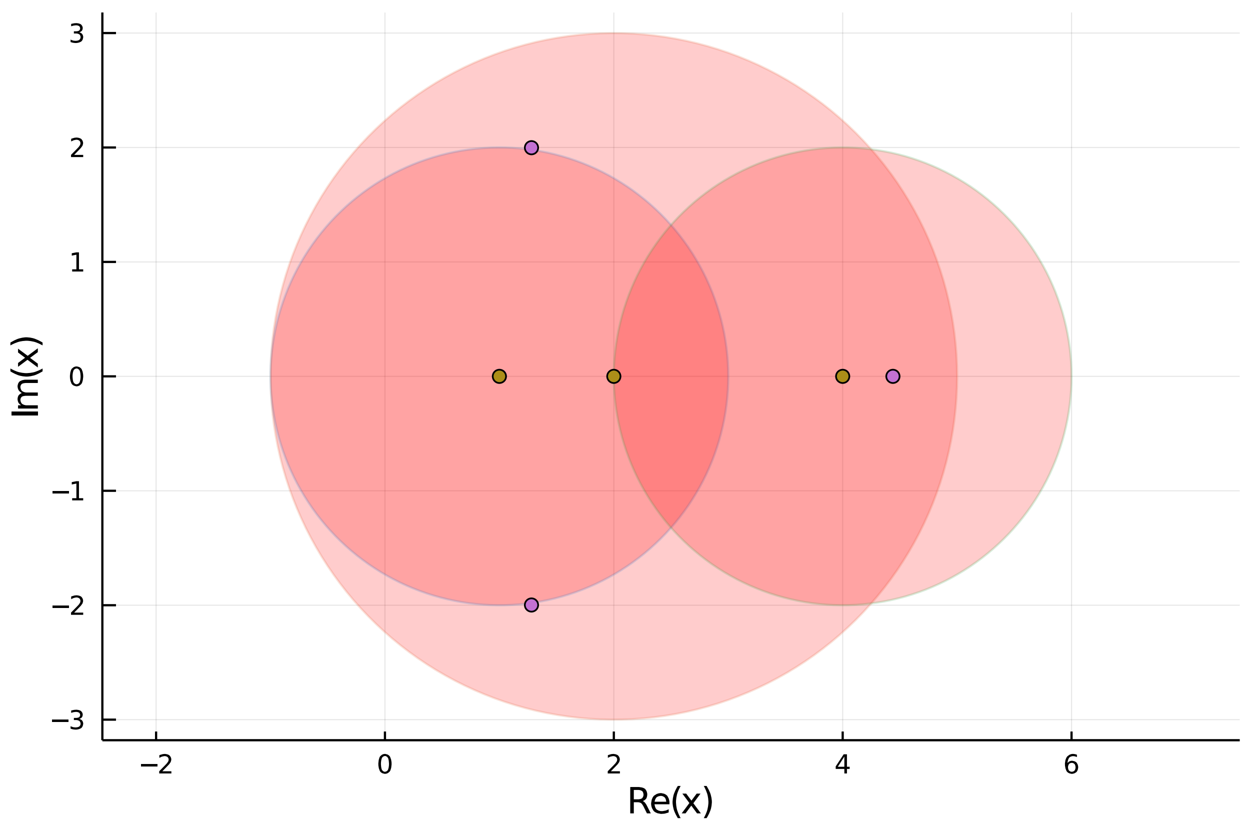
```
using ApproxFun, Plots, ComplexPhasePortrait, LinearAlgebra,
DifferentialEquations
drawdisk!(z0, R) = plot!(θ-> real(z0) + R[1]*cos(θ), θ-> imag(z0) +
R[1]*sin(θ), 0, 2π, fill=(0,:red), α = 0.2, legend=false)

A = [1 2 -1; -2 2 1; 0 1 4]

λ = eigvals(A)
```

```julia
p = plot()
drawdisk!(1,3)
drawdisk!(2,3)
drawdisk!(4,1)
scatter!(complex.(λ); label="eigenvalues",ratio=1.0)
scatter!(complex.(diag(A)); label="diagonals")
p
```



2. We get $\sigma(A) \subseteq B(1,2) \cup B(2,3) \cup B(4,2)$

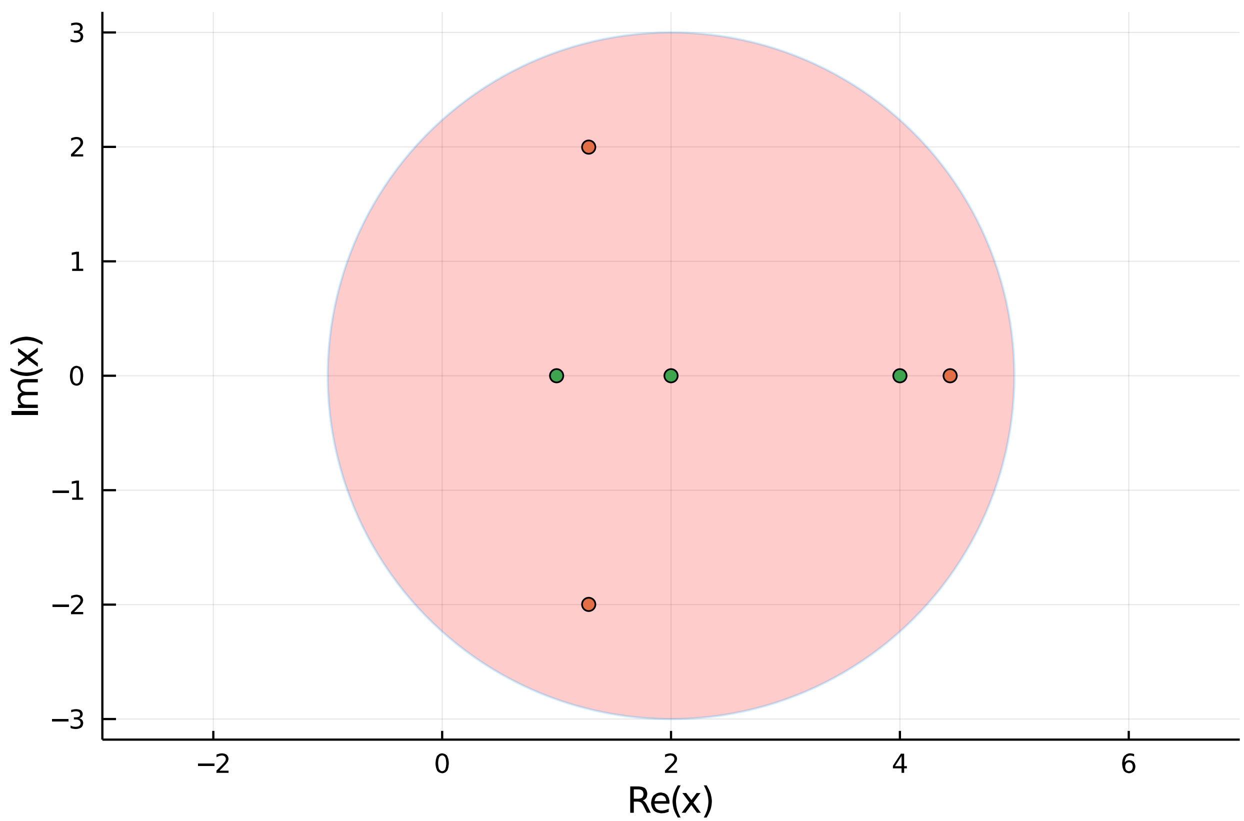
```
λ = eigvals(A)
p = plot()
drawdisk!(1,2)
drawdisk!(2,3)
drawdisk!(4,2)
scatter!(complex.(λ); label="eigenvalues")
scatter!(complex.(diag(A)); label="diagonals", ratio=1.0)
p
```



3. Because the spectrum live in the intersection of the two estimates, the sharpest bound is

$$\sigma(A) \subseteq B(2,3)$$

```
λ = eigvals(A)
p = plot()
drawdisk!(2,3)
scatter!(complex.(λ); label="eigenvalues")
scatter!(complex.(diag(A)); label="diagonals",ratio=1.0)
```



Thus we can take $2 + 3e^{i\theta}$ as the contour.

1.2 Problem 2

1.

Note that in the scalar case $u'' = au$ we have the solution

$$u(t) = u_0 \cosh \sqrt{a}t + v_0 \frac{\sinh \sqrt{a}t}{\sqrt{a}}$$

Write

$$A = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^\top$$

where $\lambda_k > 0$ and then the solution has the form, where γ is a contour surrounding the eigenvalues and to the right of zero:

$$
\begin{aligned}
\mathbf{u}(t) &= Q \begin{pmatrix} \cosh\sqrt{\lambda_1}t & & \\ & \ddots & \\ & & \cosh\sqrt{\lambda_n}t \end{pmatrix} Q^\top \mathbf{u}_0 + Q \begin{pmatrix} \frac{\sinh\sqrt{\lambda_1}t}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{\sinh\sqrt{\lambda_n}t}{\sqrt{\lambda_n}} \end{pmatrix} Q^\top \mathbf{v}_0 \\
&= \frac{1}{2\pi\mathrm{i}} Q \begin{pmatrix} \oint_\gamma \frac{\cosh\sqrt{z}t\,\mathrm{d}z}{z-\lambda_1} & & \\ & \ddots & \\ & & \oint_\gamma \frac{\cosh\sqrt{z}t\,\mathrm{d}z}{z-\lambda_n} \end{pmatrix} Q^\top \mathbf{u}_0 \\
&\quad + \frac{1}{2\pi\mathrm{i}} Q \begin{pmatrix} \oint_\gamma \frac{\sinh\sqrt{z}t}{\sqrt{z}} \frac{\mathrm{d}z}{z-\lambda_1} & & \\ & \ddots & \\ & & \oint_\gamma \frac{\sinh\sqrt{z}t}{\sqrt{z}} \frac{\mathrm{d}z}{z-\lambda_n} \end{pmatrix} Q^\top \mathbf{v}_0 \\
&= \frac{1}{2\pi\mathrm{i}} \oint_\gamma \cosh\sqrt{z}t\, Q \begin{pmatrix} (z-\lambda_1)^{-1} & & \\ & \ddots & \\ & & (z-\lambda_n)^{-1} \end{pmatrix} Q^\top \mathbf{u}_0 \mathrm{d}z \\
&\quad + \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\sinh\sqrt{z}t}{\sqrt{z}} Q \begin{pmatrix} (z-\lambda_1)^{-1} & & \\ & \ddots & \\ & & (z-\lambda_n)^{-1} \end{pmatrix} Q^\top \mathbf{v}_0 \mathrm{d}z \\
&= \frac{1}{2\pi\mathrm{i}} \oint_\gamma \cosh\sqrt{z}t (zI - A)^{-1} \mathbf{u}_0 \mathrm{d}z + \frac{1}{2\pi\mathrm{i}} \oint_\gamma \frac{\sinh\sqrt{z}t}{\sqrt{z}} (zI - A)^{-1} \mathbf{v}_0 \mathrm{d}z
\end{aligned}
$$

Here we verify the formulae numerically:

```julia
n = 5
A = randn(n,n)
A = A+ A' + 10I

λ, Q = eigen(A)
λ
```

```
5-element Array{Float64,1}:
  7.474242825312149
  8.40196998720244
 11.200557022982455
 12.81101606308103
 15.643625647665232
```

```julia
norm(A - Q*Diagonal(λ)*Q')
```

```
1.7060853784660773e-14
```

Time-stepping solution:

```julia
u_0 = randn(n)
v_0 = randn(n)
uv = solve(ODEProblem((uv,_,t) -> [uv[n+1:end]; A*uv[1:n]], [u_0;
v_0], (0.,2.)); reltol=1E-10);
```

```
t = 2.0
uv(t)[1:n]
```

```
5-element Array{Float64,1}:
 -288.6451624526969
 1043.2769593567198
  115.6190946303411
 -228.2666410841935
   97.37054579190372
```

Solution via diagonalization:

```
Q*Diagonal(cosh.(sqrt.(λ) .* t))*Q'*u_0 + Q*Diagonal(sinh.(sqrt.(λ)
.* t) ./ sqrt.(λ))*Q'*v_0
```

```
5-element Array{Float64,1}:
 -288.6451623500332
 1043.2769592553739
  115.6190946080924
 -228.2666407820585
   97.37054577842983
```

Solution via elliptic integrals. We chose the ellipse to surround all the spectrum of our particular A with eigenvalues:

```
periodic_rule(n) = 2π/n*(0:(n-1)), 2π/n*ones(n)
function ellipse_rule(n, a, b)
```

```
    θ = periodic_rule(n)[1]
    a*cos.(θ) + b*im*sin.(θ), 2π/n*(-a*sin.(θ) + im*b*cos.(θ))
end
function ellipse_f(f, A, n, z_0, a, b)
    z,w = ellipse_rule(n,a,b)
    z .+= z_0
    ret = zero(A)
    for j=1:n
        ret += w[j]*f(z[j])*inv(z[j]*I - A)
    end
    ret/(2π*im)
end

n = 50
ellipse_f(z -> cosh(sqrt(z)*t), A, n, 10.0, 7.0, 2.0)*u_0 +
    ellipse_f(z -> sinh(sqrt(z)*t)/sqrt(z), A, n, 10.0, 7.0, 2.0)*v_0
```

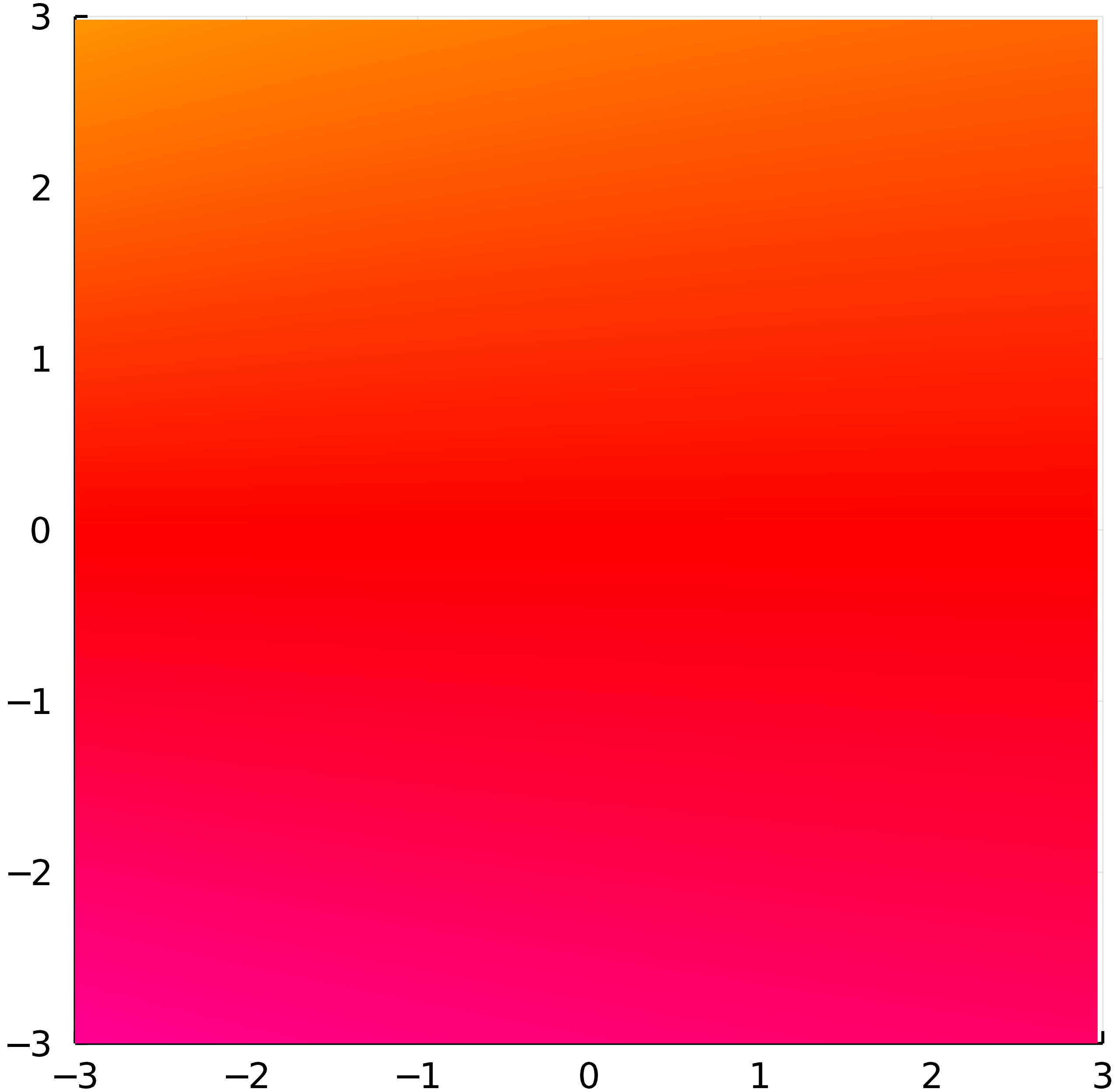
```
5-element Array{Complex{Float64},1}:
 -288.6443320135477 + 1.0338431970895553e-14im
   1043.27679357605 + 8.483556655112882e-14im
  115.6192254473848 - 1.3389188081129559e-14im
 -228.2668036755129 + 6.902268756464615e-15im
   97.3704548166224 - 9.896654266268307e-15im
```

2.

I put the restriction in because of the $\sqrt{z}$ term, which look like it is not analytic on $(-\infty, 0]$. However, this restriction was NOT necessary, since in fact $\cosh\sqrt{z}t$ and $\frac{\sinh\sqrt{z}t}{\sqrt{z}}$ are entire:

```
phaseplot(-6..6, -3..3, z -> cosh(sqrt(z)))
```

```
phaseplot(-3..3, -3..3, z -> sinh(sqrt(z))/sqrt(z))
```



This follows from Taylor series, though I prefer the following argument: we have

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

hence $\cosh it = \cos t = \cosh -it$. Therefore, on the possible branch cut $\cosh \sqrt{z}$ is continuous (hence analytic):

$$\cosh \sqrt{x}_+ = \cosh i\sqrt{|x|} = \cosh -i\sqrt{|x|} = \cosh \sqrt{x}_-$$

Similarly,

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

implies $\sinh it = i \sin t = -i \sin(-t) = -\sinh -it$, which gives us continuity:

$$\frac{\sinh \sqrt{x}_+}{\sqrt{x}_+} = \frac{\sinh i\sqrt{|x|}}{i\sqrt{|x|}} = \frac{\sinh(-i\sqrt{|x|})}{-i\sqrt{|x|}} = \frac{\sinh \sqrt{x}_-}{\sqrt{x}_-}$$

furthermore, they are both bounded at zero, hence analytic there too.

Here's a numerical example:

```
n = 5
A = randn(n,n)
λ, V = eigen(A)
λ
```

```
5-element Array{Complex{Float64},1}:
 -1.772177257335989 5 + 0.0im
```

```
norm(A - V*Diagonal(λ)*inv(V))
```

```
5.2634945993010825e-15
```

```
u_0 = randn(n)
v_0 = randn(n)
uv = solve(ODEProblem((uv,_,t) -> [uv[n+1:end]; A*uv[1:n]], [u_0;
v_0], (0.,2.)); reltol=1E-10);

t = 2.0
uv(t)[1:n]
```

```
5-element Array{Float64,1}:
  7.140165439143709 5
  6.87304100697209 7
 -5.85347675579575 4
  1.56463332222849 1
 -1.047965224174251 4
```

```
V*Diagonal(cosh.(sqrt.(λ) .* t))*inv(V)*u_0 +
   V*Diagonal(sinh.(sqrt.(λ) .* t) ./ sqrt.(λ))*inv(V)*v_0
```

```
5-element Array{Complex{Float64},1}:
    7.14016543836947 + 8.564739318812795e-17im
   6.873041006687994 + 1.0004265220412661e-16im
  -5.853476755584865 - 1.2117587861480155e-16im
  1.5646333220833089 + 4.499356215805608e-18im
 -1.0479652242083004 + 8.668666210552416e-17im
```

Here's the solution using an elliptic integral:

```
n = 100
ellipse_f(z -> cosh(sqrt(z)*t), A, n, 0.0, 3.0, 3.0)*u_0 +
    ellipse_f(z -> sinh(sqrt(z)*t)/sqrt(z), A, n, 0.0, 3.0, 3.0)*v_0
```

```
5-element Array{Complex{Float64},1}:
   7.140165464464756 + 1.026246911352165e-15im
   6.873041015788506 - 7.656795438333267e-16im
  -5.853476763092647 - 2.7605971357564325e-16im
   1.564633275968693 - 2.622860358720763e-16im
 -1.0479652229387915 + 1.3148223160134344e-17im
```

1.3 Problem 3

1.

We have

$$\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) = \sum_{k=0}^{\infty} g_k \frac{1}{n}\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j}$$

Define the n-th root of unity as $\omega = e^{\frac{2\pi \mathrm{i}}{n}}$ (that is $\omega^n = 1$), and simplify

$$\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j} = \sum_{j=0}^{n-1} e^{\frac{2\pi j \mathrm{i} k}{n}} = \sum_{j=0}^{n-1} \omega^{kj} = \sum_{j=0}^{n-1} (\omega^k)^j$$

If k is a multiple of n, then $\omega^k = 1$, and this sum is equal to n. If k is not a multiple of n, use Geometric series:

$$\sum_{j=0}^{n-1} (\omega^k)^j = \frac{\omega^{nk} - 1}{\omega^k - 1} = \frac{1^k - 1}{\omega^k - 1} = 0$$

2.

From lecture 4, we have $|f_k| \leq M_r r^{-|k|}$ for any $1 \leq r < R$, where M_r is the supremum of f in an annulus $\{z : r^{-1} < |z| < r\}$. Thus from the previous part we have (using geometric series)

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) - \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{d}\theta\right| \leq \sum_{K=1}^{\infty} |f_{Kn}| + |f_{-Kn}| \leq 2M \sum_{K=1}^{\infty} r^{-Kn} = 2M_r \frac{r^{-n}}{1 - r^{-n}}.$$

This is an upper bound that decays exponentially fast.

3.

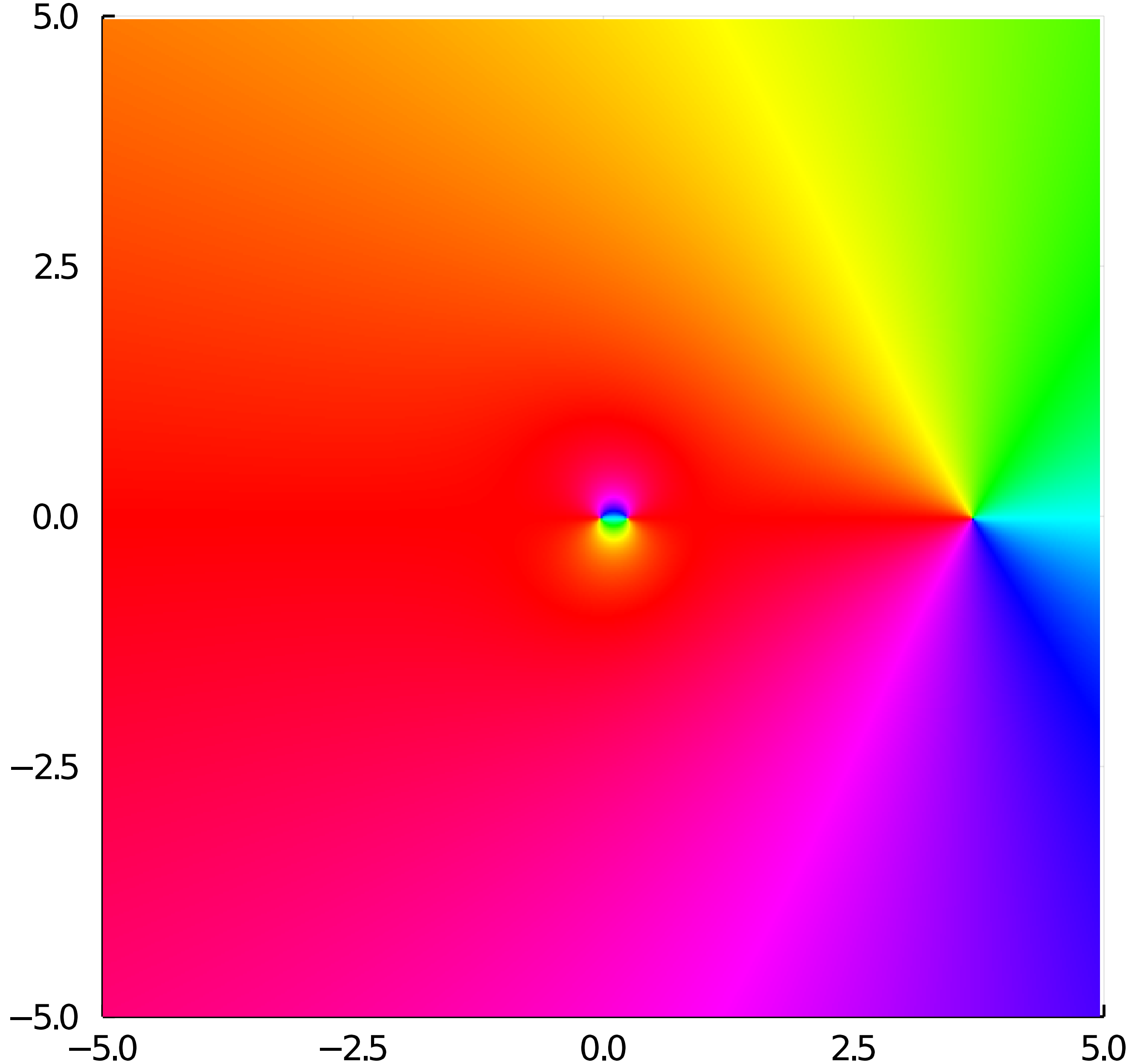
Note that $f(z) = 2z/(4z - z^2 - 1)$ satisfies $f(e^{i\theta}) = g(\theta)$. This has two poles at $2 \pm \sqrt{3}$:

```
f = z -> 2z/(4z-z^2-1)
f(exp(0.1im)) - 1/(2-cos(0.1))
```

```
1.1102230246251565e-16 + 0.0im
```

Here's a phase plot showing the location of poles:

```
phaseplot(-5..5, -5..5, f)
```




```
2+sqrt(3), 2- sqrt(3), 1/(2+sqrt(3))
```

```
(3.732050807568877, 0.2679491924311228, 0.2679491924311227)
```

Note that $2+\sqrt{3} = 1/(2-\sqrt{3})$ so in the previous result, we take $R = 2+\sqrt{3}$. For any $1 \leq r < 2+\sqrt{3}$ we have

$$M_r = \frac{2}{4 - r - r^{-1}}$$

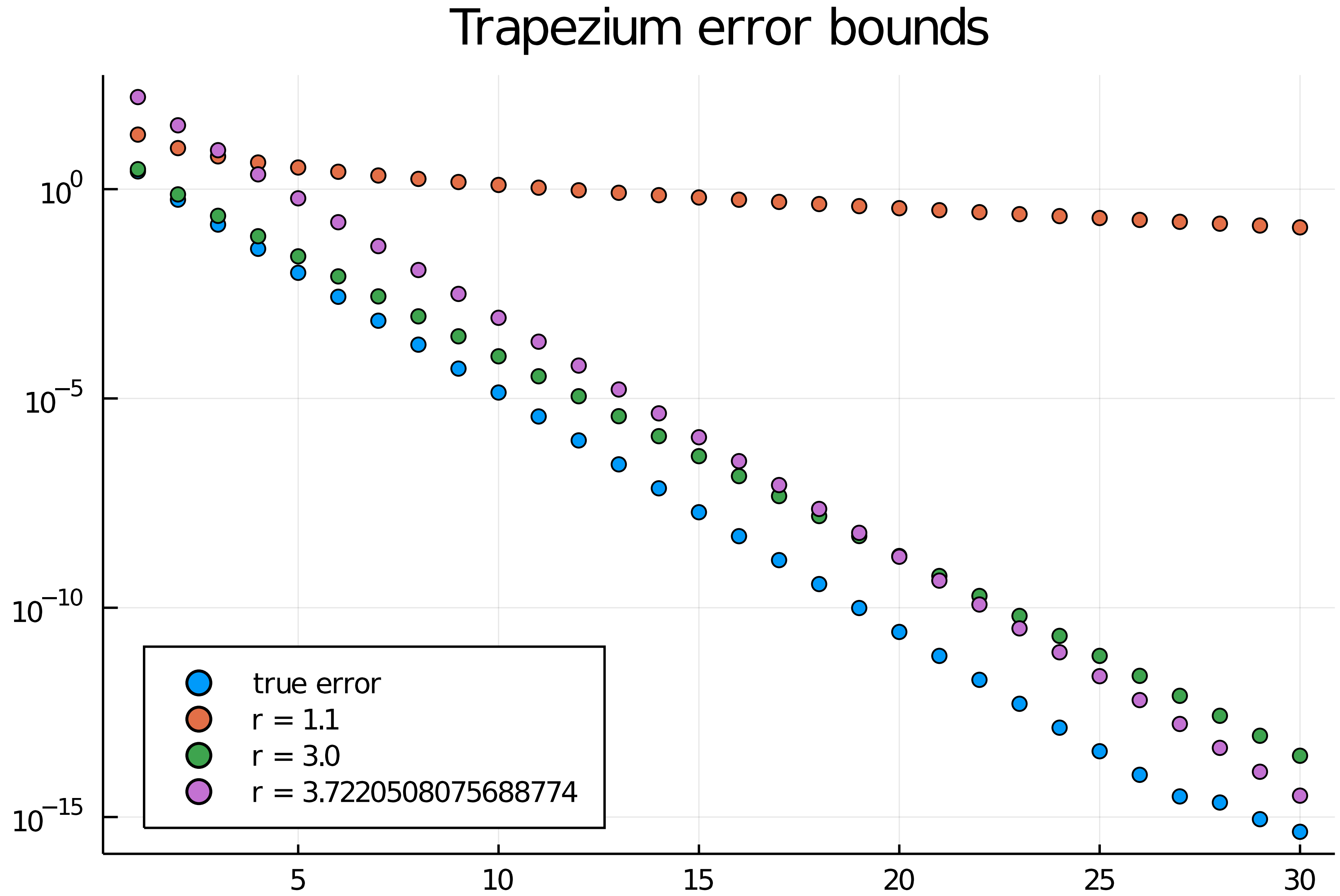
Thus we get the upper bounds

$$\frac{4}{4 - r - r^{-1}} \frac{r^{-n}}{1 - r^{-n}}$$

Let's see how sharp it is:

```
periodic_rule(n) = 2π/n*(0:(n-1)), 2π/n*ones(n)

g = θ -> 1/(2 - cos(θ))

Q = sum(Fun(g, 0 .. 2π))

err = Float64[ (
            (θ, w) = periodic_rule(n);
            sum(w.*g.(θ)) - Q
        ) for n=1:30];

N = length(err)
```

```julia
scatter(abs.(err), yscale=:log10, label="true error",
title="Trapezium error bounds", legend=:bottomleft)
r = 1.1
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label
= "r = $r")
r = 3.0
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label
= "r = $r")
r = 2+sqrt(3)  - 0.01
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label
= "r = $r")
```



Trapezium error bounds
10^{0}
10^{-5}
10^{-10}
10^{-15}
5
10
15
20
25
30
true error
r = 1.1
r = 3.0
r = 3.7220508075688774