Applied Complex Analysis (2021)

1 Problem sheet 5: Lectures 21-26

1.1 Problem 1

This problem considers Cauchy and Logarithmic transforms of Laguerre polynomials. Recall from lectures that

$$\mathcal{C}_{[0,\infty)}[\mathrm{e}^{-\diamond}](z) = -\frac{\mathrm{e}^{-z}\mathrm{Ei}\,\mathrm{z}}{2\pi\mathrm{i}}$$

for the exponential integral

$$\mathrm{Ei}\,\mathrm{z} = \int_{-\infty}^{\mathrm{z}} \frac{\mathrm{e}^{\zeta}}{\zeta}\mathrm{d}\zeta.$$

1. What is

$$\mathcal{C}_{[0,\infty)}[\diamond\mathrm{e}^{-\diamond}L_1^{(1)}(\diamond)](z) := \frac{1}{2\pi\mathrm{i}}\int_0^{\infty}\frac{x\mathrm{e}^{-x}L_1^{(1)}(x)}{x-z}\mathrm{d}x$$

in terms of $\mathrm{Ei}\,\mathrm{z}$?

2. What is

$$\frac{1}{\pi}\int_0^{\infty}\mathrm{e}^{-x}L_2(x)\log|z-x|\mathrm{d}x$$

in terms of the real and imaginary parts of $\mathrm{Ei}\,\mathrm{z}$?

1.2 Problem 2

Consider the incomplete Gamma function:

$$\Gamma(\alpha, z) = \int_z^{\infty}\zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta,$$

where the contour of integration is two straight line segments from z to 1 to ∞, hence this has a branch cut on $(-\infty, 0]$.

1. For $x < 0$ and $\alpha > 0$, show that

$$\Gamma_+(\alpha, x) - e^{2i\pi\alpha}\Gamma_-(\alpha, x) = (1 - e^{2i\pi\alpha})\Gamma(\alpha)$$

where $\Gamma(\alpha) = \Gamma(\alpha, 0) = \int_0^\infty x^{\alpha-1}e^{-x}dx$ is the Gamma function and

$$\Gamma_\pm(\alpha, x) = \lim_{\epsilon\to 0}\Gamma(\alpha, x \pm i\epsilon).$$

2. For $-1 < \alpha < 0$, express

$$\mathcal{C}_{[0,\infty)}[\diamond^\alpha e^{-\diamond}](z) = \frac{1}{2\pi i}\int_0^\infty \frac{x^\alpha e^{-x}}{x - z}dx$$

in terms of $\Gamma(-\alpha, -z)$ and $(-z)^\alpha e^z$ using Plemelj's lemma.

1.3 Problem 3

Define

$$a(z) = z^2 - 4 + z^{-2}.$$

1. What are the entries of $L[a(z)]^{-1}$?
2. Find the Wiener–Hopf factorisation

$$a(z) = \phi_+(z)\phi_-(z)$$

where $\phi_+(z)$ is analytic inside the unit circle and and $\phi_-(z)$ is analytic outside, with $\phi_-(\infty) = 1$.

3. Find the UL decomposition

$$T[a(z)] = UL$$

where U is upper-triangular with 1 on the diagonal and L is lower triangular.

4. What is $T[a(z)]^{-1}$?
5. What is $T[(z^2 + 3)/(z^2 + 2)]^{-1}$?

1.4 Problem 4

When the winding number is non-trivial, a Toeplitz operator can either be non-invertible or have multiple solutions. This problem sheet explores this.

1. What is the winding number of $a(z) = z$? Show that

$$T[z]\mathbf{u} = \mathbf{f}$$

only has a solution if f_0 (the first entry of $\mathbf{f}$) is zero.

2. What is the winding number of $a(z) = z^{-1}$? What are *all* solutions to

$$T[z^{-1}]\mathbf{u} = \mathbf{f}?$$

3. Show that if $a(z)$ has winding number κ it can be written as

$$a(z) = \phi_+(z) z^\kappa \phi_-(z)$$

What are $\phi_+(z)$ and $\phi_-(z)$ in terms of $\log(a(z)z^{-\kappa})$?

4. Show that if the winding number is κ there exists a

$$T[a(z)] = UPL$$

decomposition, where

$$P = T[z^\kappa]$$

is a permutation operator.

5. Find all solutions to

$$T[1/(2z^2+1)]\mathbf{u} = \mathbf{e}_0$$

1.5 Problem 5

This set of problems investigates the analyticity properties of the half-Fourier transform. Recall the definitions

$$u_{\mathrm{R}}(x) = \begin{cases} u(x) & x \geq 0 \\ 0 & \text{otherwise} \end{cases},$$
$$u_{\mathrm{L}}(x) = \begin{cases} u(x) & x < 0 \\ 0 & \text{otherwise} \end{cases},$$

The Fourier transform

$$\hat{u}(s) = \int_{-\infty}^{\infty} u(x) \mathrm{e}^{-\mathrm{i}xs} \mathrm{d}x,$$

and the inverse Fourier transform

$$u(x) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \hat{u}(s) \mathrm{e}^{\mathrm{i}sx} \mathrm{d}s$$

where the choice of γ is dictated by the analyticity of $\hat{u}(z)$.

1. Consider $f(x) = x$. Without computing it, in what strip, if any, is $\hat{f}(z)$ analytic? For what choice of γ, if any, does the inverse Fourier transform recover x?

2. Consider $f(x) = \frac{1}{1+\mathrm{e}^x}$. Without computing it, in what strip, if any, is $\hat{f}(z)$ analytic? For what choice of γ, if any, does the inverse Fourier transform recover x?

3. Consider $f(x) = \mathrm{e}^{2x}$.Without computing it, in what strip, if any, is $\widehat{f_{\mathrm{R}}}(z)$ analytic? For what choice of γ, if any, does the inverse Fourier transform recover x?

4. Consider $f(x) = x$. Without computing it, in what strip, if any, is $\widehat{f_{\mathrm{L}}}(z)$ analytic? For what choice of γ, if any, does the inverse Fourier transform recover x?

5. Calculate the Fourier transforms in the above problems and confirm your statements.

6. What is the Fourier transform of $\delta(x)$, i.e., the Dirac delta function satisfying

$$\int_{-\infty}^{\infty} \delta(x)g(x)\mathrm{d}x = g(0)$$

for smooth test functions g. Where is it analytic?

1.6 Problem 6

This set of problems considers extensions of the Wiener–Hopf method to functions that do not decay, degenerate integral equations, and to integro-differential equations. Please be precise on which contour the Riemann–Hilbert problem is solved on and the inverse Fourier transforms taken.

1. The function $u(x)$ is bounded by a polynomial for all $x \geq 0$, including as $x \to \infty$, and satisfies the integral equation

for $x \geq 0$,

$$u(x) + \frac{3}{2}\int_0^{\infty} \mathrm{e}^{-|x-t|}u(t)\mathrm{d}t = 1 + \alpha x$$

where α is a positive constant. Find $u(x)$ for $x \geq 0$. Hint: set up a Riemann–Hilbert problem on the contour $\mathbb{R} + \mathrm{i}\gamma$ where $-1 < \gamma < 0$ is arbitrary.

2. The function $u(x)$ is bounded by a polynomial for all $x \geq 0$, including as $x \to \infty$, and satisfies the integral equation

for $x \geq 0$,

$$\int_0^{\infty} \mathrm{e}^{-\alpha|x-t|}u(t)\mathrm{d}t = 1 + \alpha x$$

where α is a positive constant. Find $u(x)$ for $x \geq 0$. Hint: If you proceed na\"\i vely, we arrive at a Riemann–Hilbert problem of the form

$$\Phi_+(s) - g(s)\Phi_-(s) = f(s) \qquad \text{and} \qquad \lim_{z\to\infty}(\infty) = 0$$

but where $g(\infty) = 0$ instead of $g(\infty) = 1$. This is not in canonical form, but maybe this example is special. Try writing $\Phi(z) = \kappa(z)Y(z)$ as before but allowing different asymptotic behaviour in κ and Y in the different half planes in a way that they cancel out so that $\lim_{z\to\infty}\Phi(z) = 0$:

$$\kappa(z) = \begin{cases} O(z^{-1}) & \Im z > 0 \\ O(z) & \Im z < 0 \end{cases}$$

$$Y(z) = \begin{cases} O(1) & \Im z > 0 \\ O(z^{-2}) & \Im z < 0 \end{cases}.$$

3. Consider the integral equation

$$u(t) - \lambda \int_0^\infty \mathrm{e}^{-|x-t|} u(t)\mathrm{d}t = x$$

where $0 < \lambda < \frac{1}{2}$. Show that, for $x \geq 0$,

$$u(x) = A(\sinh\gamma x + \gamma\cosh\gamma x) + \frac{1}{\gamma^2}\left[x + \left(\gamma - \frac{1}{\gamma}\right)\sinh\gamma x\right]$$

where $\gamma^2 = 1 - 2\lambda$ and A is an arbitrary constant.

4. A bounded, smooth, function $u(x)$ satisfies the integro-differential equation

$$u''(x) - \frac{72}{5}\int_0^\infty \mathrm{e}^{-5|x-t|}u(t)\mathrm{d}t = 1 \qquad \text{for} \qquad x \geq 0$$

with $u(0) = 0$.

1.7 Problem 7

1. Rewrite the integral equation on the half line in the form:

$$u_\mathrm{R}''(x) - \frac{72}{5}\int_{-\infty}^\infty \mathrm{e}^{-5|x-t|}u(t)\mathrm{d}t = 1_\mathrm{R}(x) + \alpha\delta(x) + p_\mathrm{L}(x)$$

for $\alpha = u'(0)$ and a to-be-specified $p(x)$. Here δ is the Dirac delta function, that is, $\int_{-\infty}^\infty f(x)\delta(x)\mathrm{d}x = f(0)$.

2. Use integration by parts to determine that

$$\widehat{u_\mathrm{R}''}(s) = -u'(0) - s^2\hat{u}_\mathrm{R}(s).$$

What is $\hat{\delta}(s)$? Use these to translate the equation to Fourier space on a contour $s \in \mathbb{R} + \mathrm{i}\gamma$. What choices of γ are suitable?

3. Define $\Phi(z)$ in terms of $\widehat{p_{\mathrm{L}}}(z)$ and $\widehat{u_{\mathrm{R}}}(z)$ so that it satisfies the following (non-standard) RH problem

$$\Phi_+(s) - \frac{(s^2+9)(s^2+16)}{s^2+25}\Phi_-(s) = \alpha + \frac{1}{\mathrm{i}s}$$
$$\lim_{\substack{z\to\infty \\ \Im z > \gamma}} \Phi(z) = \alpha$$
$$\lim_{\substack{z\to\infty \\ \Im z < \gamma}} \Phi(z) = 0.$$

4. Solve the Riemann–Hilbert problem for Φ. Hint: write $\Phi(z) = \kappa(z)Y(z)$ where

$$\kappa(z) = \begin{cases} O(z) & \Im z > \gamma \\ O(z^{-1}) & \Im z < \gamma \end{cases},$$
$$Y(z) = O(z^{-1}).$$

Hint: $Y(z)$ does not depend on α in the lower-half plane.

5. Recover $u(x)$ by taking the inverse Fourier transform of $\Phi_-(s)$.
