Applied Complex Analysis (2021)

1 Problem sheet 1

1.1 Problem 1.1

Compute the residues of

1. $$\operatorname*{Res}_{z=ae^{i\pi/4}} \frac{z^3 \sin z}{z^4 + a^4} \qquad \text{where} \qquad a > 0$$

2. $$\operatorname*{Res}_{z=1} \frac{z+1}{(z^2-1)^2}$$

3. $$\operatorname*{Res}_{z=a} \frac{z^2 e^z}{z^3 - a^3} \qquad \text{where} \qquad a \neq 0$$

1.2 Problem 1.2

Use contour integration to find the values of

1. $$\int_0^{2\pi} \frac{1}{5 - 4\cos\theta} \mathrm{d}\theta$$

2. $$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\cos\theta} \mathrm{d}\theta$$

3. $$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} \mathrm{d}x$$

4. $$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} \mathrm{d}x$$

5.

$$\int_{-\infty}^{\infty} \frac{1}{x+\mathrm{i}} \mathrm{d}x$$

6.

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1} \mathrm{d}x$$

7.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4} \mathrm{d}x$$

8.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} \mathrm{d}x$$

9.

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} \mathrm{d}x \qquad \text{where} \qquad a, b > 0$$

10.

$$\int_0^{2\pi} (\cos\theta)^n \mathrm{d}\theta \qquad \text{where} \qquad n = 0, 1, 2, \ldots \qquad \text{(Hint: consider even and odd } n \text{ separately.)}$$

1.3 Problem 2.1

By integrating around a rectangular contour with vertices at $\pm R$ and $\pi\mathrm{i} \pm R$ and letting $R \to \infty$, show that:

$$\int_0^{\infty} \operatorname{sech} x \mathrm{d}x = \frac{\pi}{2}$$

where $\operatorname{sech} x = \frac{2}{\mathrm{e}^{-x}+\mathrm{e}^{x}}$.

1.4 Problem 2.2

Show that the Fourier transform of $\operatorname{sech} x$ satisfies

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \operatorname{sech} x \mathrm{d}x = \pi \operatorname{sech} \frac{\pi k}{2}$$
