Applied Complex Analysis (2021)

1 Solution Sheet 3

1.1 Problem 1.1

1.1.1 1.

Take as an initial guess

$$\phi_1(z) = \frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}(1+z^2)}$$

This satisfies for $-1 < x < 1$

$$\phi_1^+(x) - \phi_1^-(x) = \frac{\sqrt{1-x^2}}{2(1+x^2)} - \frac{-\sqrt{1-x^2}}{2(1+x^2)} = \frac{\sqrt{1-x^2}}{1+x^2}$$

Further, as $z \to \infty$,

$$\phi_1(z) \sim \frac{z}{\mathrm{i}(1+z^2)} \to 0$$

The catch is that it has poles at $\pm\mathrm{i}$:

$$\phi_1(z) = -\frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{4}\frac{1}{z-\mathrm{i}} + O(1)$$
$$\phi_1(z) = \frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{4}\frac{1}{z+\mathrm{i}} + O(1)$$

Thus it follows that

$$\phi(z) = \phi_1(z) + \frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{4}\frac{1}{z-\mathrm{i}} - \frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{4}\frac{1}{z+\mathrm{i}}$$

is

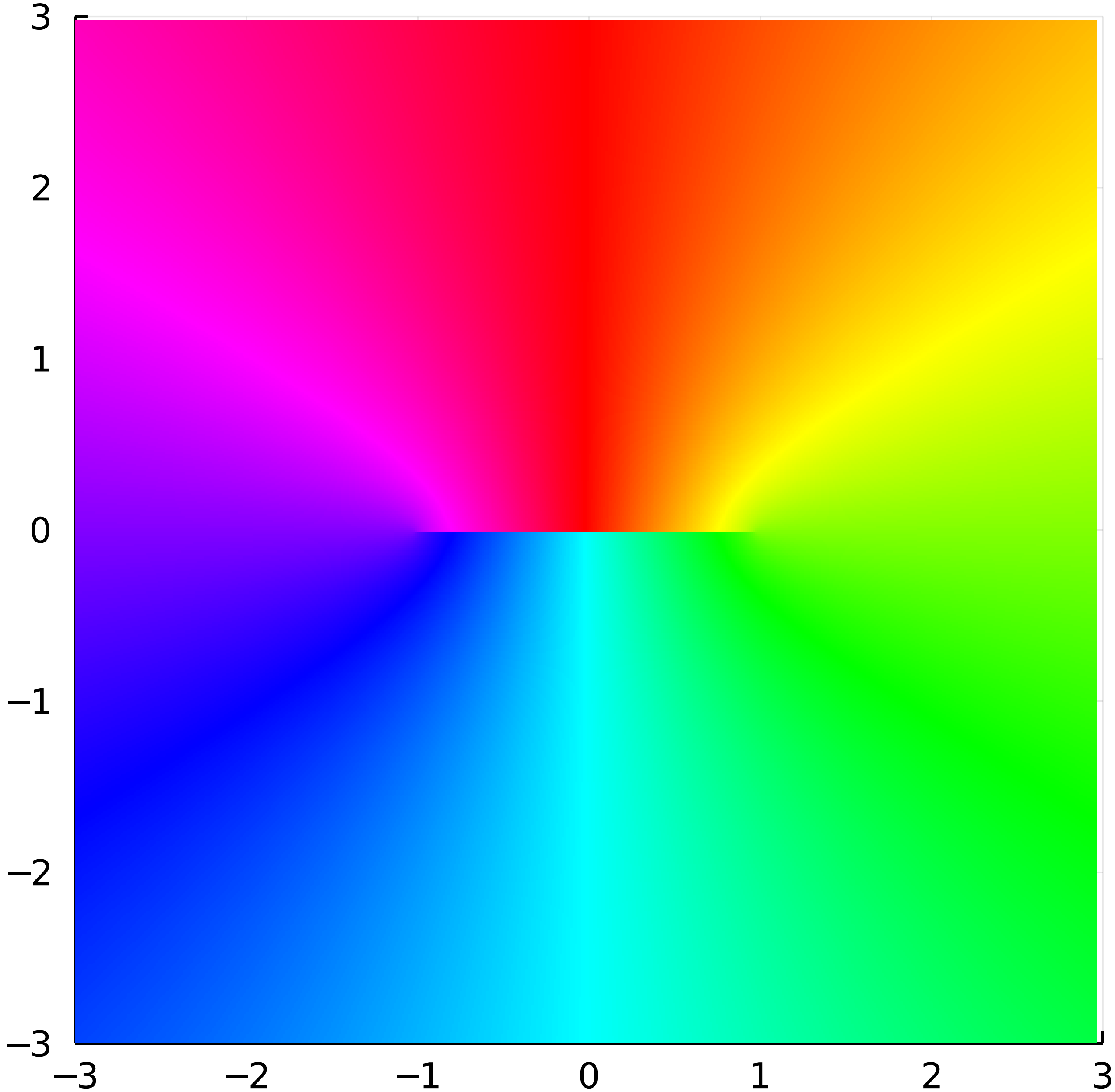
1. Analyticity: Analytic at $\pm\mathrm{i}$ and off $[-1,1]$
2. Decay: $\phi(\infty) = 0$
3. Regularity: Has weaker than pole singularities
4. Jump: Satisfies

$$\phi_+(x) - \phi_-(x) = \frac{\sqrt{1-x^2}}{1+x^2}$$

By Plemelj II, this must be the Cauchy transform.

Demonstration We will see experimentally that it correct. First we do a phase plot to make sure we satisfy (Analyticity):

```julia
using ComplexPhasePortrait, Plots, ApproxFun,
SingularIntegralEquations

H(f) = -hilbert(f)
H(f,x) = -hilbert(f,x)

φ = z -> sqrt(z-1)sqrt(z+1)/(2im*(1+z^2)) +
        sqrt(im-1)sqrt(im+1)/4*1/(z-im) -
        sqrt(-im-1)sqrt(-im+1)/4*1/(z+im)

phaseplot(-3..3, -3..3, φ)
```



We can also see from the phase plot (Regularity): we have weaker than pole singularities, otherwise we would have at least a full,counter clockwise colour wheel. We can check decay as well:

```
φ(200.0+200.0im)
```

```
0.0005177682933717976 + 0.0005177656125456 22im
```

Finally, we compare it numerically it to `cauchy(f, z)` which is implemented in SingularIntegralEquations.jl:

```
x = Fun()
φ(2.0+2.0im),cauchy(sqrt(1-x^2)/(1+x^2), 2.0+2.0im)
```

```
(0.05303535516221752 + 0.05036581190871381im, 0.05303535516221748 +
0.05036
581190871378im)
```

1.1.2 2.

Recall that

$$\psi(z) = \frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}}$$

satisfies

$$\psi_+(x) - \psi_-(x) = 1$$

Therefore, consider

$$\phi_1(z) = \frac{\psi(z)}{2+z}$$

This has the right jump, but has an extra pole at $z = -2$: for $x < -1$ we have

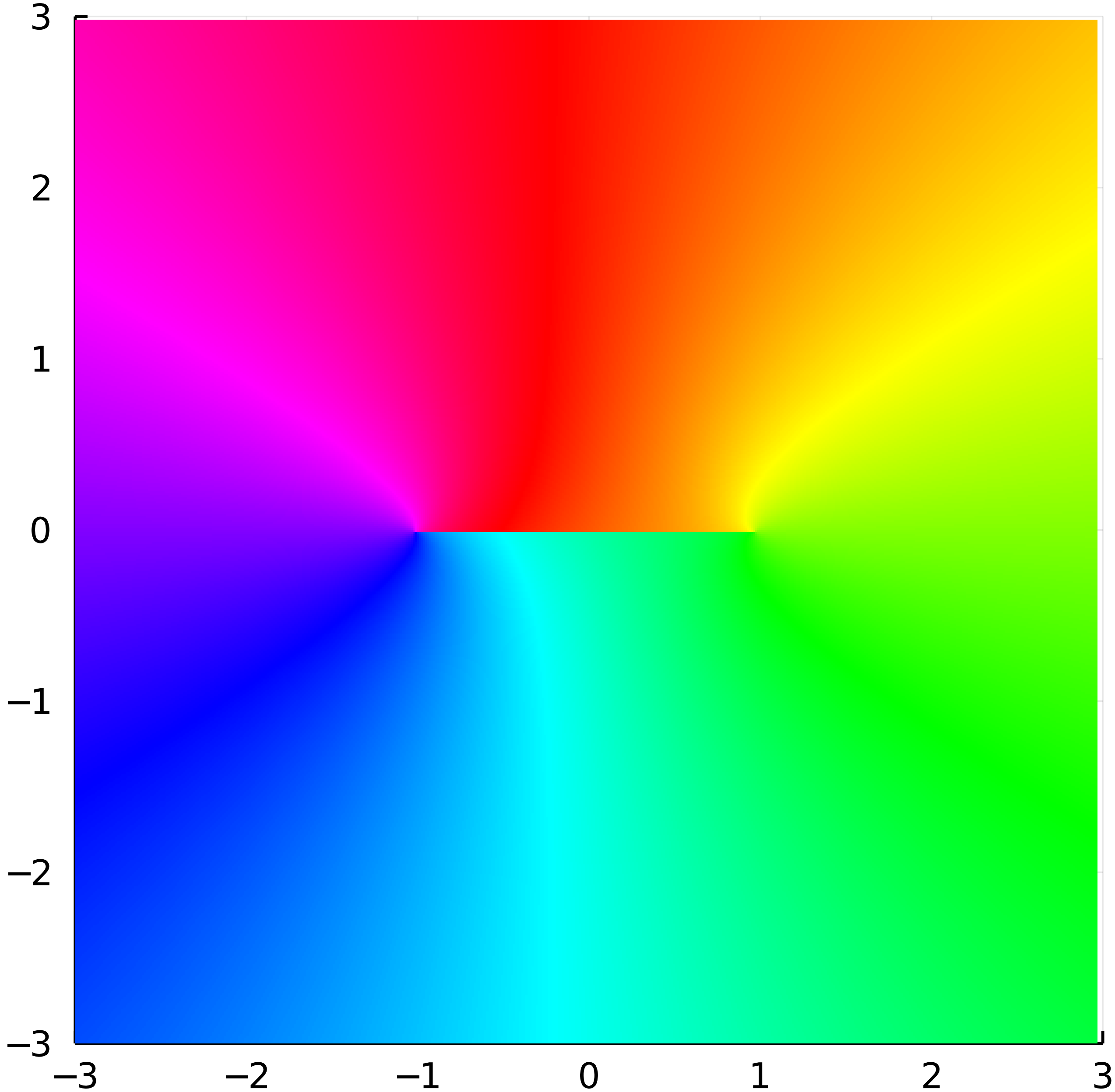
$$\phi_1(x) = \frac{\log_+(x-1) - \log_+(x+1)}{2\pi\mathrm{i}} \frac{1}{2+x} = \frac{\log(1-x) - \log(-1-x)}{2\pi\mathrm{i}} \frac{1}{2+x}$$

hence we arrive at the solution

$$\phi_1(z) - \frac{\log 3}{2\pi\mathrm{i}(2+z)}$$

We can verify that $\phi_1(\infty) = 0$.

```
φ =  z -> (log(z-1)-log(z+1)) / ((2π*im)*(2+z)) -
log(3)/(2π*im*(2+z))

phaseplot(-3..3, -3..3, φ)
```


3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

1.1.3 3.

We first calculate the Cauchy transform of $f(x) = x/\sqrt{1-x^2}$:

$$\phi(z) = \frac{\mathrm{i}z}{2\sqrt{z-1}\sqrt{z+1}} - \frac{\mathrm{i}}{2}$$

This vanishes at ∞ and has the correct jump. We then have

$$\mathrm{i}\mathcal{H}f(x) = \phi^+(x) + \phi^-(x) = -\mathrm{i}$$

This implies that (note the sign)

$$\int_{-1}^{1} \frac{t}{(t-x)\sqrt{1-t^2}}\mathrm{d}t = -\pi\mathcal{H}f(x) = \pi$$

```
f = x/sqrt(1-x^2)
-π*H(f, 0.1)
```

```
3.141592653589793
```

1.2 Problem 1.2

1.2.1 1.2.1

From Problem 1.1 part 3, we have a solution:

$$\phi(z) = -\frac{z}{2\sqrt{z-1}\sqrt{z+1}} + \frac{1}{2}$$

All other solutions are then of the form:

$$\phi(z) + \frac{C}{\sqrt{z-1}\sqrt{z+1}}$$

```
C = randn()
φ = z -> -z/(2*sqrt(z-1)*sqrt(z+1))+1/2 + C/(sqrt(z-1)*sqrt(z+1))
φ(0.1+0.0im)+φ(0.1-0.0im), φ(1E8)
```

```
(1.0 + 0.0im, -1.782207108835652e-8)
```

1.2.2 1.2.2

$$\psi(z) = -2\phi(z) + 1 = \frac{z}{\sqrt{z-1}\sqrt{z+1}}$$

satisfies

$$\psi_+(x) + \psi_-(x) = 0, \qquad \psi(\infty) = 1$$

```
C = randn()
φ = z -> z/(sqrt(z-1)*sqrt(z+1)) + C/(sqrt(z-1)*sqrt(z+1))

φ(0.1+0.0im)+φ(0.1-0.0im),φ(1E9)
```

```
(0.0 + 0.0im, 0.9999999993222088)
```

1.2.3 1.2.3

For $f(x) = \sqrt{1-x^2}$, we use the formula

$$\phi(z) = \frac{\mathrm{i}}{\sqrt{z-1}\sqrt{z+1}}\mathcal{C}[\sqrt{1-\diamond^2} f](z) + \frac{C}{\sqrt{z-1}\sqrt{z+1}} = \frac{\mathrm{i}}{\sqrt{z-1}\sqrt{z+1}}\mathcal{C}[1-\diamond^2](z) + \frac{}{\sqrt{z -}}$$

We already know $\mathcal{C}1(z)$, and we can deduce $\mathcal{C}[\diamond^2]$ as follows: try

$$\phi_1(z) = z^2 \mathcal{C}1(z) = z^2 \frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}}$$

this has the right jump, but blows up at ∞ like:

$$x^2(\log(x-1) - \log(x+1)) = x^2(\log(1-1/x) - \log(1+1/x)) = -2x + O(x^{-1})$$

using

$$\log z = (z-1) - \frac{1}{2}(z-1)^2 + O(z-1)^3$$

Thus we have

$$\mathcal{C}[\diamond^2](z) = \frac{z^2(\log(z-1) - \log(z+1)) + 2z}{2\pi\mathrm{i}}$$

and

$$\phi(z) = \frac{\mathrm{i}}{\sqrt{z-1}\sqrt{z+1}} \frac{(1-z^2)(\log(z-1)-\log(z+1))-2z}{2\pi\mathrm{i}} + \frac{C}{\sqrt{z-1}\sqrt{z+1}}$$

Demonstration Here we see that the Cauchy transform of x^2 has the correct formula:

```
z = 2.0+2.0im
cauchy(x^2, z),(z^2*(log(z-1)-log(z+1))+2z)/(2π*im)
```

```
(0.0283222937395961 + 0.024377589786690298im, 0.028322293739596032 +
0.0243
7758978669024im)
```

We now see that ϕ has the right jumps:

```
C = randn()
φ = z -> im/(sqrt(z-1)*sqrt(z+1)) *
((1-z^2)*(log(z-1)-log(z+1))-2z)/(2π*im) + C/(sqrt(z-1)sqrt(z+1))

φ(0.1+0.0im) + φ(0.1-0.0im) - sqrt(1-0.1^2)
```

```
1.1102230246251565e-16 + 0.0im
```

Finally, it vanishes at infinity:

```
φ(1E5)
```

```
-6.764623255449203e-7 - 0.0im
```

1.2.4 1.2.4

Let $f(x) = \frac{1}{1+x^2}$. From Problem 1.1 part 1 we know

$$\mathcal{C}\left[\frac{\sqrt{1-\diamond^2}}{1+\diamond^2}\right](z) = \frac{\sqrt{z-1}\sqrt{z+1}}{2\mathrm{i}(1+z^2)} + \frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}}{4}\frac{1}{z-\mathrm{i}} - \frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}}{4}\frac{1}{z+\mathrm{i}}$$

hence from the solution formula we have

$$\phi(z) = \frac{1}{2(1+z^2)} + \frac{\sqrt{\mathrm{i}-1}\sqrt{\mathrm{i}+1}\mathrm{i}}{4\sqrt{z-1}\sqrt{z+1}}\frac{1}{z-\mathrm{i}} - \frac{\sqrt{-\mathrm{i}-1}\sqrt{-\mathrm{i}+1}\mathrm{i}}{4\sqrt{z-1}\sqrt{z+1}}\frac{1}{z+\mathrm{i}} + \frac{C}{\sqrt{z-1}\sqrt{z+1}}$$

But we want something stronger: that $\phi(z) = O(z^{-2})$. To accomplish this, we need to choose C. Fortunately, I made the problem easy as every term apart from the last one is already $O(z^{-2})$, so choose $C = 0$:

```
φ = z -> 1/(2*(1+z^2)) +
        sqrt(im-1)sqrt(im+1)*im/(4sqrt(z-1)sqrt(z+1))*1/(z-im) -
        sqrt(-im-1)sqrt(-im+1)*im/(4sqrt(z-1)sqrt(z+1))*1/(z+im)


φ(1E5)*1E5
```

```
-2.071067812011923e-6 + 0.0im
```

We see also that it has the right jump:

```
φ(0.1+0.0im) + φ(0.1-0.0im),1/(1+0.1^2)
```

```
(0.9900990099009901 + 0.0im, 0.9900990099009901)
```

1.3 Problem 1.3

1. From the Hilbert formula, we know that the general solution of $\mathcal{H}u = f$ is

$$u(x) = \frac{-1}{\sqrt{1-x^2}}\mathcal{H}\left[f(\diamond)\sqrt{1-\diamond^2}\right](x) - \frac{C}{\sqrt{1-x^2}}$$

Plugging in $f(x) = x/\sqrt{1-x^2}$ means we need to calculate

$$\mathcal{H}\left[\diamond\right](x)$$

We do so by first finding the Cauchy transform. Consider

$$\phi_1(z) = z\mathcal{C}1(z) = z\frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}}$$

This has the right jump:

$$\phi_1^+(x) - \phi_1^-(x) = x$$

but doesn't decay at ∞:

$$x\frac{\log(x-1)-\log(x+1)}{2\pi\mathrm{i}} = x\frac{\log x + \log(1-1/x) - \log x - \log(1+1/x)}{2\pi\mathrm{i}}$$
$$= -x\frac{2}{x2\pi\mathrm{i}} = -\frac{1}{\mathrm{i}\pi}$$

But this means that

$$\phi(z) = \phi_1(z) + \frac{1}{\mathrm{i}\pi} = z\frac{\log(z-1)-\log(z+1)}{2\pi\mathrm{i}} + \frac{1}{\mathrm{i}\pi}$$

Decays and has the right jump, hence is $\mathcal{C}[\diamond](z)$.

```
t = Fun()
z = 2.0+2.0im
cauchy(t, z), z*(log(z-1)-log(z+1))/(2π*im) + 1/(im*π)
```

```
(0.013174970881571602 - 0.0009861759882264494im, 0.013174970881571569
 - 0.0
009861759882264232im)
```

Therefore, we have

$$\mathcal{H}[\diamond](x) = -\mathrm{i}(\mathcal{C}^+ + \mathcal{C}^-)\diamond(x) = -x\frac{\log(1-x)-\log(1+x)}{\pi} - \frac{2}{\pi}$$

```
x = 0.1
H(t,x), -x*(log(1-x)-log(1+x))/(π) - 2/π
```

```
(-0.6302322257442835, -0.6302322257442834)
```

Therefore, we get

$$u(x) = -\frac{x(\log(1-x) - \log(1+x)) + 2}{\pi\sqrt{1-x^2}} - \frac{C}{\sqrt{1-x^2}}$$

This can be verified in Mathematica via

```
NIntegrate[-((
    x (Log[1 - x] - Log[1 + x]) +
     2)/(Pi Sqrt[1 - x^2] (x - 0.1))), {x, -1, 0.1, 1},
  PrincipalValue -> True]/Pi
```

2. Following the procedure of multiplying $\mathcal{C}[\sqrt{1-\diamond^2}](z)$ by $1/(2+z)$ and subtracting off the pole at $z=-2$, we first find:

$$\mathcal{C}\left[\frac{\sqrt{1-\diamond^2}}{2+\diamond}\right](z) = \frac{\sqrt{z-1}\sqrt{z+1}-z}{2\mathrm{i}(2+z)} - \frac{\sqrt{-2-1}_+\sqrt{-2+1}_+ + 2}{2\mathrm{i}(2+z)} = \frac{\sqrt{z-1}\sqrt{z+1}-z}{2\mathrm{i}(2+z)} + \frac{}{2}$$

```
t = Fun()
z = 2.0+2.0im
cauchy(sqrt(1-t^2)/(2+t), z), (sqrt(z-1)sqrt(z+1)-z)/(2im*(2+z)) +
(sqrt(3)-2)/(2im*(z+2))
```

```
(0.03230545315801244 + 0.03244969518322382 6im, 0.03230545315801245 5 +
 0.032
44969518322378 4im)
```

Therefore, calculating $-\mathrm{i}(\mathcal{C}^+ + \mathcal{C}^-)$ we find that

$$\mathcal{H}\left[\frac{\sqrt{1-\diamond^2}}{2+\diamond}\right](z) = \frac{x}{2+x} - \frac{\sqrt{3}-2}{2+x}$$

```
x = 0.1
H(sqrt(1-t^2)/(2+t), x), x/(2+x)-(sqrt(3)-2)/(2+x)
```

```
(0.17521390115767735, 0.17521390115767752)
```

Thus the general solution is

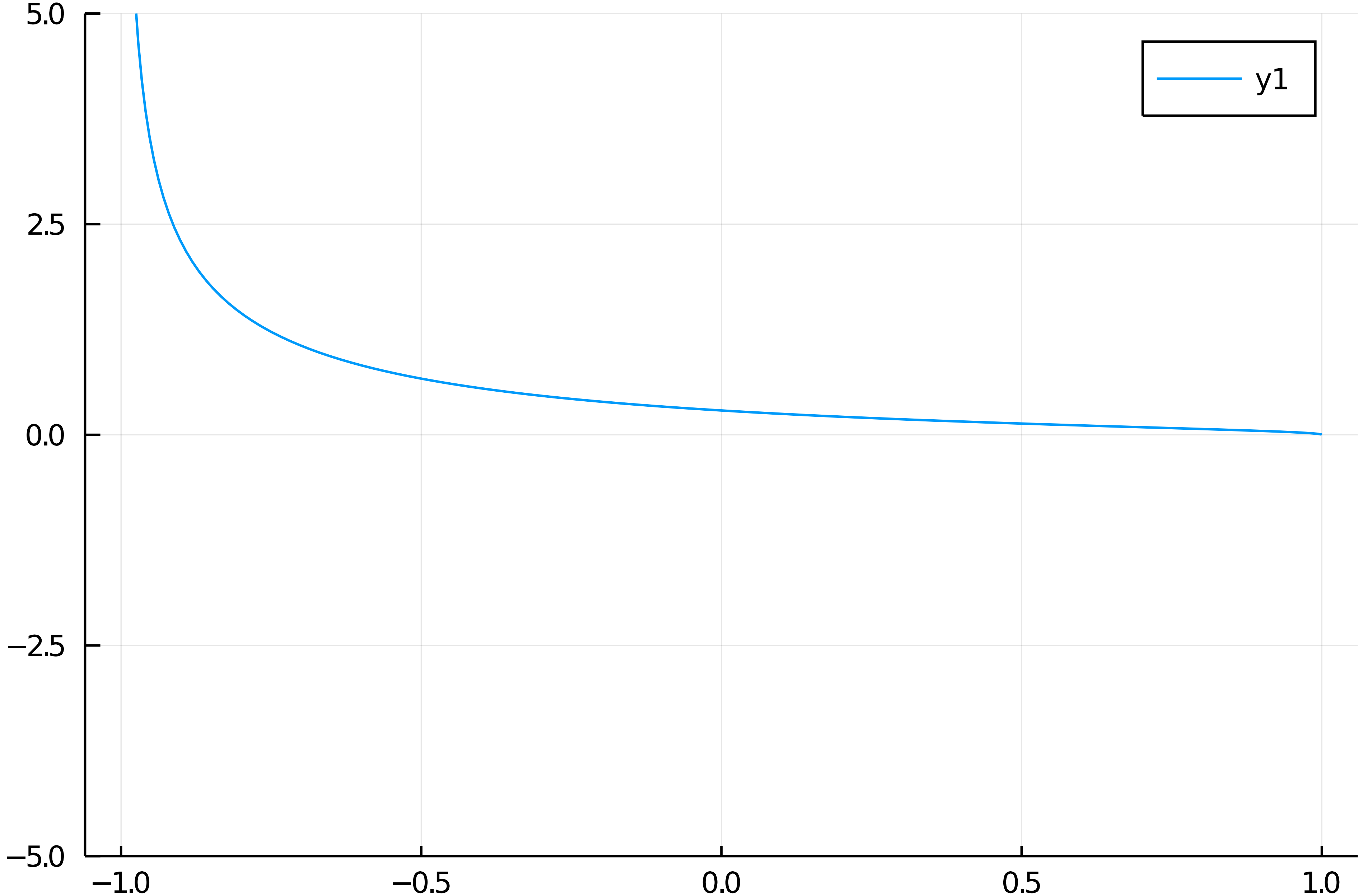
$$u(x) = -\frac{1}{\sqrt{1-x^2}}\left(\frac{x}{2+x} - \frac{\sqrt{3}-2}{2+x} + C\right)$$

We need to choose C so this is bounded at the right-endpoint: In other words,

$$u(x) = -\frac{1}{\sqrt{1-x^2}}\left(\frac{x}{2+x} - \frac{\sqrt{3}-2}{2+x} + \frac{\sqrt{3}-3}{3}\right)$$

```
u = -(t/(2+t)-(sqrt(3)-2)/(2+t)+(sqrt(3)-3)/3)/sqrt(1-t^2)
```

```julia
plot(u; ylims=(-5,5))
```




```julia
x = 0.1
```

```
H(u,x) , 1/(2+x)
```

```
(0.47619047619047533, 0.47619047619047616)
```

1.4 Problem 2.1

Doing the change of variables $\zeta = bs$ we have

$$\log(ab) = \int_1^{ab} \frac{\mathrm{d}\zeta}{\zeta} = \int_{1/b}^{a} \frac{\mathrm{d}s}{s}$$

if γ does not surround the origin, we have

$$0 = \oint_\gamma \frac{\mathrm{d}s}{s} = \left[\int_1^{1/b} + \int_{1/b}^{a} + \int_a^1\right] \frac{\mathrm{d}s}{s}$$

which implies

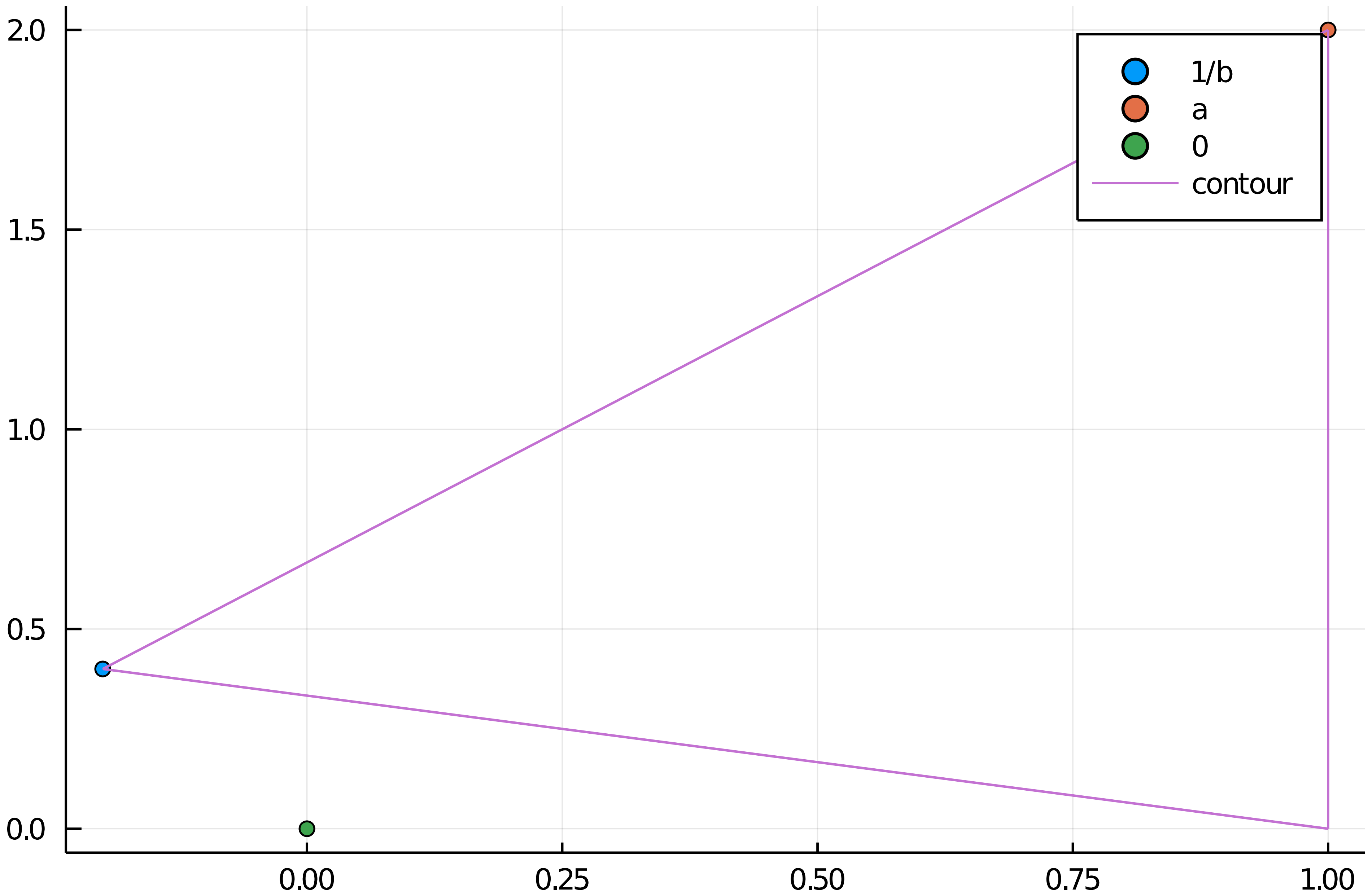
$$\log(ab) = \left[-\int_a^1 - \int_1^{1/b}\right] \frac{\mathrm{d}s}{s} = \log a - \log\frac{1}{b} = \log a + \log b$$

Here's a picture:

```
a = 1.0+2.0im
b = -1.0-2.0im
```

```julia
@show log(a*b)
@show log(a) + log(b)

scatter([real(1/b)], [imag(1/b)]; label="1/b")
scatter!([real(a)], [imag(a)]; label="a")
scatter!([0.0], [0.0]; label="0")
plot!(Segment(1, 1/b) ∪ Segment(1/b, a) ∪ Segment(a, 1);
label="contour")
```

```
log(a * b) = 1.6094379124341003 - 0.9272952180016122im
log(a) + log(b) = 1.6094379124341003 - 0.9272952180016123im
```



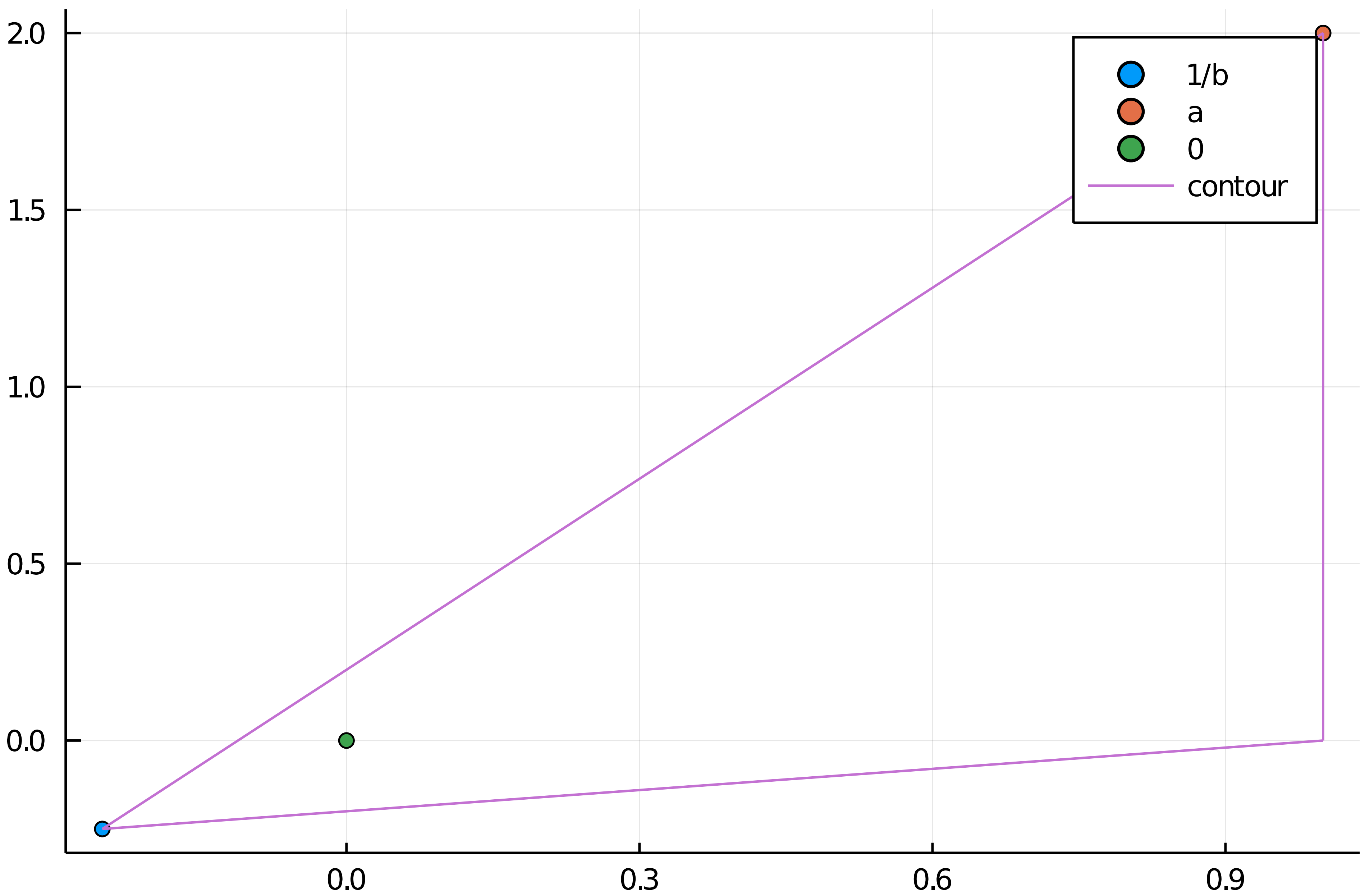
If it surrounds the origin counbter-clockwise, that is, it has positive orientation, we have $2\pi\mathrm{i} = \oint_\gamma \frac{\mathrm{d}s}{s}$, which shoes that

$$\log(ab) = 2\pi\mathrm{i} - \left[\int_a^1 + \int_1^{1/b}\right] \frac{\mathrm{d}s}{s} = \log a + \log b + 2\pi\mathrm{i}$$

and a similar result when counter clockwise.

```
a = 1.0+2.0im
b = -2.0+2.0im

@show log(a*b)
@show log(a) + log(b) - 2π*im
scatter([real(1/b)], [imag(1/b)]; label="1/b")
scatter!([real(a)], [imag(a)]; label="a")
scatter!([0.0], [0.0]; label="0")
plot!(Segment(1, 1/b) ∪ Segment(1/b, a) ∪ Segment(a, 1);
label="contour")
```

```
log(a * b) = 1.8444397270569681 - 2.819842099193151im
(log(a) + log(b)) - (2π@*() * im = 1.84443972705968 -
2.819842099193151im
```



If the contour passes through the origin, there are three possibility:

1. $[a, 1]$ contains zero, hence $a < 0$

2.

$$[1, 1/b]$$

contains zero, hence $b < 0$

3.

$$[1/b, a]$$

contains zero, which can only be true if $ab < 0$ by considering the equation of the line segment.

1. In the case where $a < 0$ and $b < 0$ (and hence $ab > 0$), perturbing a above and b below or vice versa avoids γ winding around zero, so we have

$$\log(ab) = \log_+ a + \log_- b = \log_- a + \log_+ b = \log_+ a + \log_+ b - 2\pi\mathrm{i} = \log_- a + \log_- b + 2\pi\mathrm{i}$$

```
a = -2.0
b = -3.0

@show log(a*b)
@show log(a+0.0im) + log(b-0.0im)
@show log(a-0.0im) + log(b+0.0im)
@show log(a-0.0im) + log(b-0.0im) + 2π*im
@show log(a+0.0im) + log(b+0.0im) - 2π*im;
```

```
log(a * b) = 1.791759469228055
log(a + 0.0im) + log(b - 0.0im) = 1.791759469228055 + 0.0im
log(a - 0.0im) + log(b + 0.0im) = 1.791759469228055 + 0.0im
log(a - 0.0im) + log(b - 0.0im) + (2π@*() * im = 1.791759469228055 +
0.0im(log(a + 0.0im) + log(b + 0.0im)) - (2(*@π@*() * im =
1.791759469228055 + 0.0im
```

In the case where $a < 0$ and $b > 0$, then $ab < 0$, but we can perturb a above/below to get

$$\log_{\pm}(ab) = \log_{\pm} a + \log b$$

(and by symmetry, the equivalent holds for $b < 0$ and $a > 0$.)

```
a = -2.0
b = 3.0

@show log(a*b +0.0im)
@show log(a+0.0im) + log(b);

@show log(a*b -0.0im)
@show log(a-0.0im) + log(b);
```

```
log(a * b + 0.0im) = 1.791759469228055 + 3.141592653589793im
log(a + 0.0im) + log(b) = 1.791759469228055 + 3.141592653589793im
log(a * b - 0.0im) = 1.791759469228055 - 3.141592653589793im
```

```
log(a - 0.0im) + log(b) = 1.791759469228055 - 3.141592653589793im
```

In the case where $a < 0$, if $\Im b > 0$ we can perturb a below so that γ does not contain zero, giving us

$$\log(ab) = \log\_ a + \log b$$

similarly, if $\Im b < 0$ we can perturb a above.

```
a = -2.0
b = 3.0 + im

@show log(a*b)
@show log(a-0.0im) + log(b);

b = 3.0 + im;
@show log(a*b)
@show log(a+0.0im) + log(b);
```

```
log(a * b) = 1.844397270569681 - 2.819842099193151im
log(a - 0.0im) + log(b) = 1.844397270569683 - 2.819842099193151im
log(a * b) = 1.844397270569681 - 2.819842099193151im
log(a + 0.0im) + log(b) = 1.844397270569683 + 3.463343207986435 2im
```

2. In this case, swap the role of a and b and use the answers for $a < 0$.
3. Finally, we have the case $ab < 0$ and neither a nor b is real. Note that

$$ab = (a_x + \mathrm{i}a_y)(b_x + \mathrm{i}b_y) = a_x b_x - a_y b_y + \mathrm{i}(a_x b_y + a_y b_x)$$

It follows if $b_x > 0$ we have

$$(ab)_+ = a_+ b$$

and if $b_x < 0$ we have

$$(ab)_+ = a_- b$$

```
a = -1.0 + 1.0im
b = 1.0 + 1.0im
@show log(a*b +  eps()im)
@show log((a+eps()im)*b)

a = 1.0 + 1.0im
b = -1.0 + 1.0im
@show log(a*b +  eps()im)
@show log((a-eps()im)*b)
```

```
log(a * b + eps() * im) = 0.6931471805599453 + 3.141592653589793im
log((a + eps() * im) * b) = 0.6931471805599453 + 3.141592653589793im
log(a * b + eps() * im) = 0.6931471805599453 + 3.141592653589793im
log((a - eps() * im) * b) = 0.6931471805599452 + 3.141592653589793im
```

```
0.6931471805599452 + 3.141592653589793im
```

We can use this perturbation to reduce to the previous cases. For example, if $a = 1 + \mathrm{i}$ and $b = -1 + \mathrm{i}$, pertubing ab above causes a to be perturbed above, which causes the contour to surround the origin clockwise, hence we have

$$\log_+(ab) = \log(a)_+b = \log ab - 2\pi\mathrm{i}$$

```
a = 1.0 + 1.0im
b = -1.0 + 1.0im
@show log(a*b -  eps()im)
@show log(a)+log(b)-2π*im;
```

```
log(a * b - eps() * im) = 0.6931471805599453 - 3.141592653589793im
(log(a) + log(b)) - (2π@*() * im = 0.6931471805599453 -
3.141592653589793im
```

1.5 Problem 2.2

Use the contour $\gamma(t) = 1 + t(z-1)$ to reduce it to a normal integral: $\overline{\log z} = \overline{\int_1^z \frac{1}{\zeta} \mathrm{d}\zeta} = \overline{\int_0^1 \frac{(z-1)}{1+(z-1)t} \mathrm{d}t} = \int_0^1 \frac{(\bar{z}-1)}{1+(\bar{z}-1)t} \mathrm{d}t = \int_1^{\bar{z}} \frac{\mathrm{d}\zeta}{\zeta} = \log \bar{z}$. We then have, since the contour from 1 to $1/(\bar{z})$ to z never surrounds the origin since both $\Im z$ and $\Im 1/(\bar{z})$ have the same sign, we have

$$2\Re \log z = \log z + \overline{\log z} = \log z + \log \bar{z} = \log z\bar{z} = \log |z|^2 = 2\log |z|$$

On the other hand, we have, where the contour of integration is chosen to be to the right of zero and then we do the change of variables $\zeta = |z|e^{i\theta}$

$$2\Im \log z = \log z - \log \bar{z} = \int_{\bar{z}}^{z} \frac{d\zeta}{\zeta} = i\int_{-\arg z}^{\arg z} d\theta = 2i\arg z$$

1.6 Problem 2.3

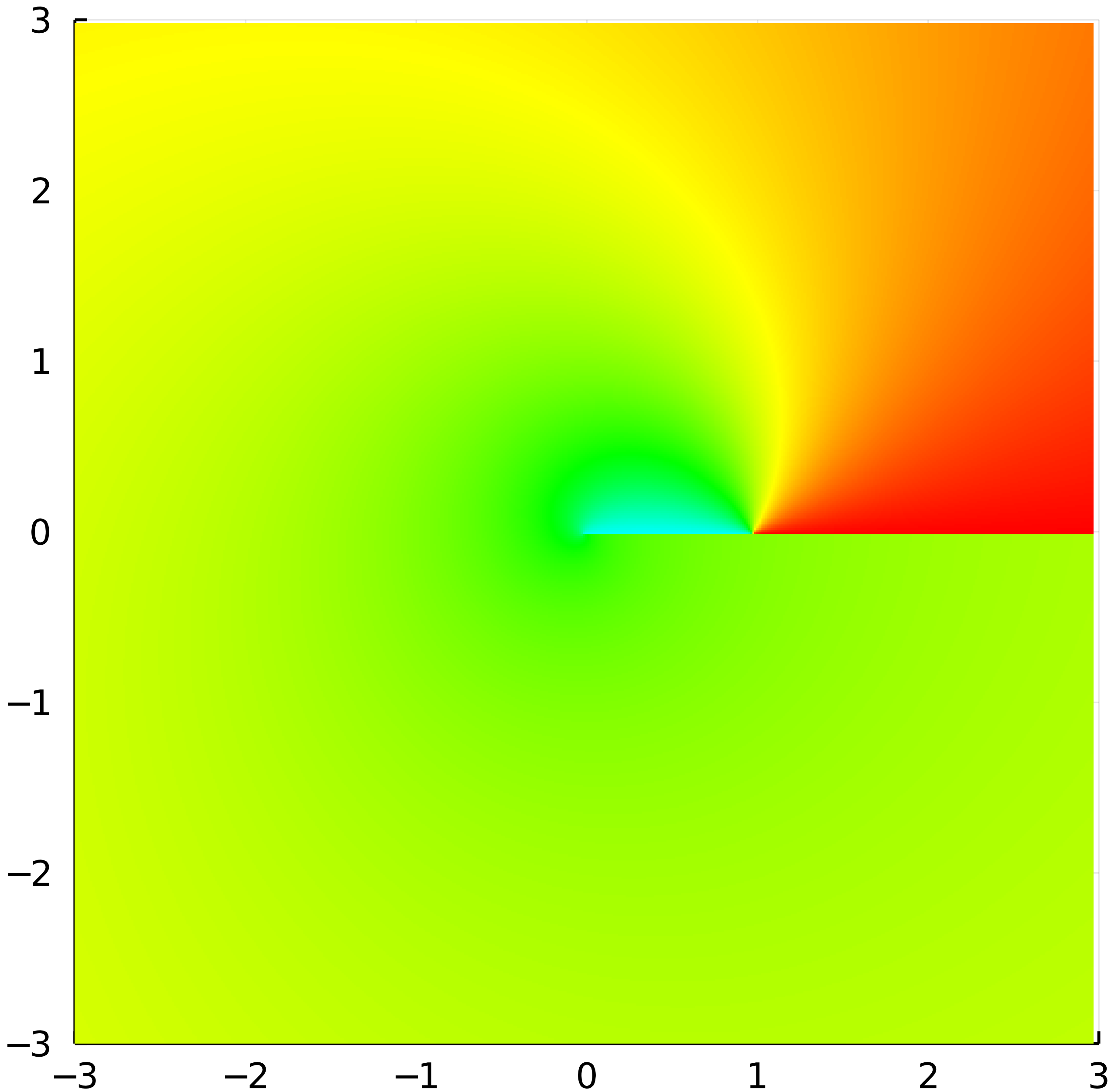
We first show that it is analytic on $(-\infty, 0)$. To do this, we need to show that the limit from above equals the limit from below: for $x < 0$ we have $\log_1^+ x - \log_1^- x = \log_+ x - \log_- x - 2\pi i = 0$ Then for $x > 0$ and using $\log_1^{\pm}(x) = \lim_{\epsilon\to 0} \log(x \pm i\epsilon)$ we find

$$\log_1^+(x) - \log_1^- x = \log x - \log x - 2\pi i = -2\pi i$$

Demonstration Here we see that the following is the analytic continuation:

```
log1 = z -> begin
    if imag(z) > 0
        log(z)
    elseif imag(z) == 0 && real(z) < 0
        log(z + 0.0im)
    elseif imag(z) < 0
        log(z) + 2π*im
```

```
    else
            error("log1 not defined on real axis")
    end
end

phaseplot(-3..3, -3..3, log1)
```



3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

1.7 Problem 3.1

1. Because it's absolutely integrable, we can exchange derivatives and integrals to determine

$$\frac{\mathrm{d}\mathcal{C}f}{\mathrm{d}z} = \frac{1}{2\pi\mathrm{i}}\oint \frac{f(\zeta)}{(\zeta - z)^2}\mathrm{d}\zeta$$

2. There are two different possible approaches:

   – the subtract and add back in technique: since f is analytic for z near ζ, we can write

$$\mathcal{C}f(z) = \frac{1}{2\pi\mathrm{i}}\oint \frac{f(\zeta) - f(z)}{\zeta - z}\mathrm{d}\zeta + f(z)\mathcal{C}1(z)$$

Therefore

$$\mathcal{C}^+ f(\zeta) - \mathcal{C}^- f(\zeta) = f(\zeta)(\mathcal{C}^+ 1(\zeta) - \mathcal{C}^- 1(\zeta))$$

But we know (using Cauchy's integral formula / Residue calculus)

$$\mathcal{C}1(z) = \begin{cases} 1 & |z| < 1 \\ 0 & |z| > 1 \end{cases}$$

hence $(\mathcal{C}^+ - \mathcal{C}^-)1(\zeta) = 1$

- Since f is analytic, we have for any radius $R > 1$ but inside the annulus

$$\mathcal{C}^+ f(\zeta) = \frac{1}{2\pi\mathrm{i}} \oint_{|\zeta|=R} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Similarly, for $\mathcal{C}^- f(\zeta)$ with any radius $r < 1$ but inside the annulus. Therfore,

$$\mathcal{C}^+ f(\zeta) - \mathcal{C}^- f(\zeta) = \frac{1}{2\pi\mathrm{i}} \left[\oint_{|\zeta|=R} - \oint_{|\zeta|=r} \right] \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Deforming the contour and using Cauchy-integral formula gives the result.

3. This follow since $\frac{1}{\zeta - z} \to 0$ uniformly.

1.8 Problem 3.2

Suppose we have another solution ϕ and consider $\psi(z) = \phi(z) - \mathcal{C}f(z)$. Then on the circle we have

$$\psi_+(\zeta) - \psi_-(\zeta) = \phi_+(\zeta) - \mathcal{C}_+ f(\zeta) - \phi_-(\zeta) + \mathcal{C}_+ f(\zeta) = f(\zeta) - f(\zeta) = 0$$

Thus ψ is entire, and since it decays at infinity, it must be zero by Liouville's theorem.

1.9 Problem 3.3

When $k \geq 0$, we have from 3.1 and 3.2

$$\mathcal{C}[\diamond^k](z) = \begin{cases} z^k & |z| < 1 \\ 0 & |z| > 1 \end{cases}$$

when $k < 0$ since $\mathcal{C}[\diamond^k]^+(\zeta) - \mathcal{C}[\diamond^k]^-(\zeta) = \zeta^k - 0 = \zeta^k$. we similarly have

$$\mathcal{C}[\diamond^k](z) = \begin{cases} 0 & |z| < 1 \\ -z^k & |z| > 1 \end{cases}$$

Therefore,

$$\Im \mathcal{C}^-[\diamond^k](\zeta) = \begin{cases} 0 & k \geq 0 \\ -\frac{\zeta^k - \zeta^{-k}}{2\mathrm{i}} & k < 0 \end{cases}$$

and

$$\Re \mathcal{C}^-[\diamond^k](\zeta) = \begin{cases} 0 & k \geq 0 \\ -\frac{\zeta^k + \zeta^{-k}}{2} & k < 0 \end{cases}$$

1.10 Problem 3.4

Express the solution outside the circle as

$$v(x, y) = \Im(\mathrm{e}^{-\mathrm{i}\theta} z + \mathcal{C} f(z))$$

for a to-be-determined f. On the circle, this reduces to

$$\Im \mathcal{C}^- f(\zeta) = -\cos\theta \frac{\zeta - \zeta^{-1}}{2\mathrm{i}} + \sin\theta \frac{\zeta + \zeta^{-1}}{2}$$

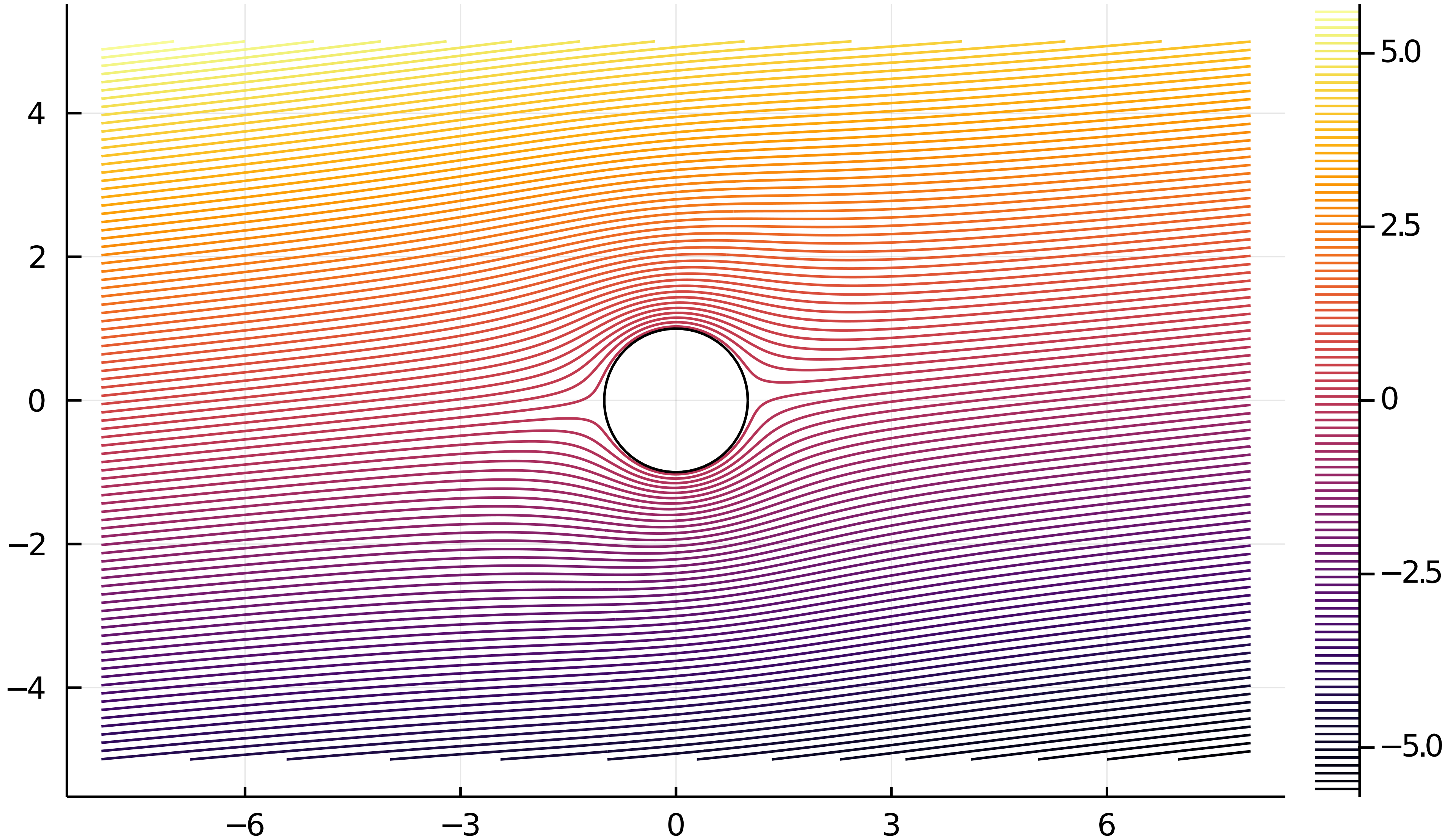
Unlike the real case, we can include imaginary coefficients, thus the solution is

$$f(\zeta) = (\cos\theta + \mathrm{i}\sin\theta)\zeta^{-1}$$

and thus the full solution is

$$v(x, y) = \Im(\mathrm{e}^{-\mathrm{i}\theta} z - \mathrm{e}^{\mathrm{i}\theta} z^{-1}))$$

```
θ = 0.1
v = (x,y) -> x^2 + y^2 < 1 ? 0 : imag(exp(-im*θ) * (x+im*y) +
exp(im*θ) * (x+im*y)^(-1))

xx = -8:0.01:8; yy = -5:0.01:5

contour(xx, yy, v.(xx', yy); nlevels=100, ratio=1.0)
plot!(Circle(); color=:black, legend=false)
```



Problem 4.1

$$z^\alpha$$

has the limits $z_\pm^\alpha = \mathrm{e}^{\pm \mathrm{i}\pi\alpha}|z|^\alpha$, thus choose $\alpha = -\frac{\theta}{2\pi}$ where if we take $0 < \theta < 2\pi$ we have $0 < \alpha < 1$ (the case $\theta = 0$ and $\theta = \pi$ are covered by the Cauchy transform, that is). Then consider

$$\kappa(z) = (z-1)^{-\alpha}(z+1)^{\alpha-1}$$

which has weaker than pole singularities and satisfies $\kappa(z) \sim z^{-1}$. For $-1 < x < 1$ it has the right jump

$$\begin{aligned}\kappa_+(x) &= (x-1)_+^{-\alpha}(x+1)^{\alpha-1} = \mathrm{e}^{-\mathrm{i}\pi\alpha}(1-x)^{-\alpha}(x+1)^{\alpha-1} = \mathrm{e}^{-2\mathrm{i}\pi\alpha}(x-1)_-^{-\alpha}(x+1)^{\alpha-1} \\ &= \mathrm{e}^{-2\mathrm{i}\pi\alpha}\kappa_-(x) = \mathrm{e}^{\mathrm{i}\theta}\kappa_-(x)\end{aligned}$$

and for $x < -1$ it has the jump

$$\kappa_+(x) = (x-1)_+^{-\alpha}(x+1)_+^{\alpha-1} = \mathrm{e}^{-\mathrm{i}\pi\alpha}\mathrm{e}^{\mathrm{i}\pi(\alpha-1)}(1-x)^{-\alpha}(-1-x)^{\alpha-1} = \kappa_-(x)$$

hence κ is analytic.

We need to show this times a constant spans the entire space. Suppose we have another solution $\tilde\kappa$ and consider $r(z) = \frac{\tilde\kappa(z)}{\kappa(z)}$. Note by construction that κ has no zeros. Then

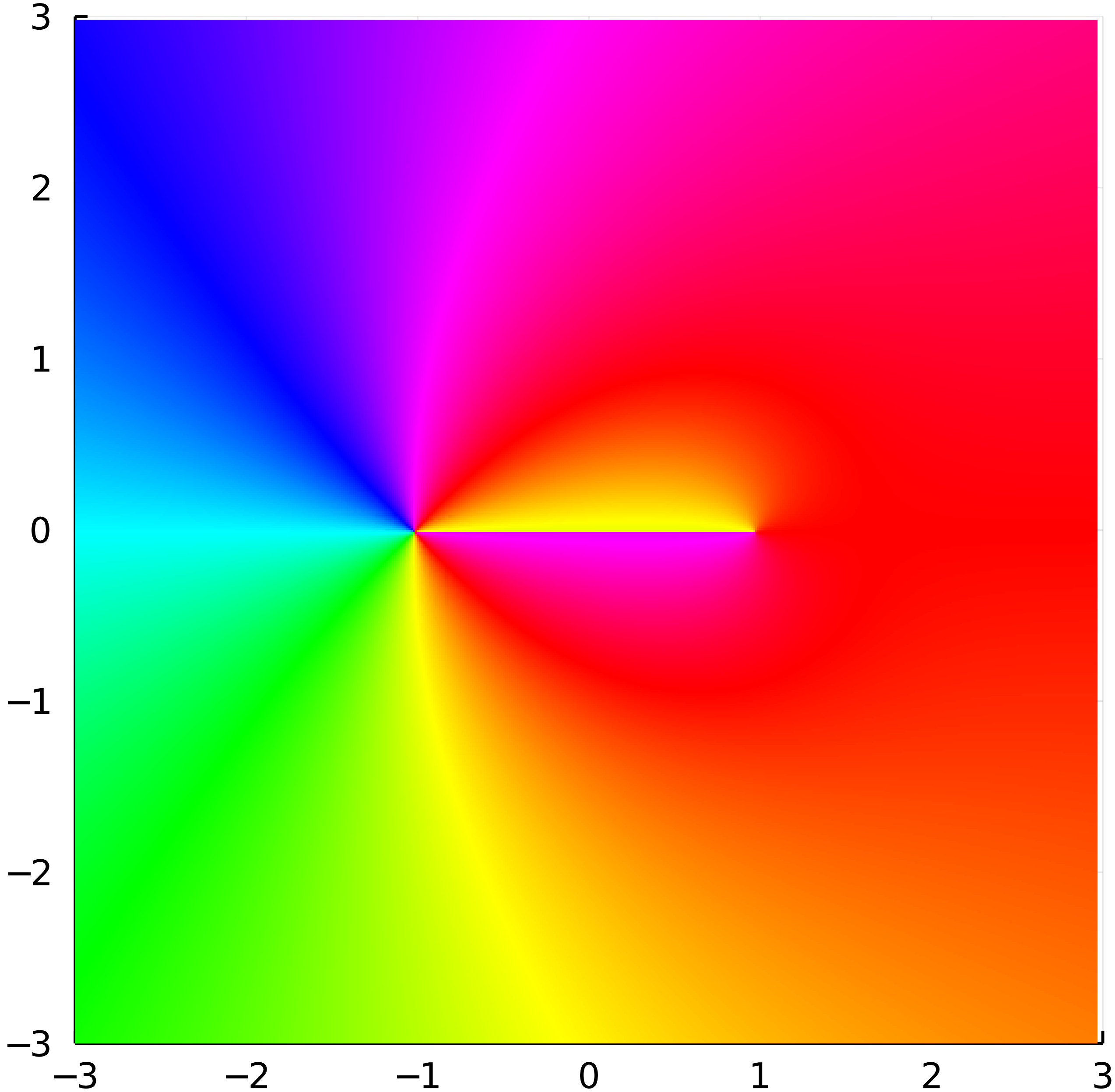
$$r_+(x) = \frac{\tilde\kappa_+(x)}{\kappa_+(x)} = \frac{\tilde\kappa_-(x)}{\kappa_-(x)} = r_-(x)$$

hence r is analytic on $(-1,1)$. It has weaker than pole singularities because $\kappa(z)^{-1}$ is actually bounded at ± 1. Therefore r is bounded and entire, and thus must be a constant $r(z) \equiv r$, and thence $\tilde{\kappa}(z) = r\kappa(z)$.

```
θ =2.3
α = -θ/(2π)
κ = z -> (z-1)^(-α)*(z+1)^(α-1)
κ(0.1+0.0im) - exp(im*θ)*κ(0.1-0.0im),κ(100.0)
```

```
(0.0 + 1.1102230246251565e-16im, 0.009828765985323333)
```

```
phaseplot(-3..3, -3..3, κ)
```



3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

1.11 Problem 4.2

We want to mimic the solution of $\phi_+(x) + \phi_-(x)$. So take

$$\phi(z) = \kappa(z)\mathcal{C}\left[\frac{f}{\kappa_+}\right](z) = \mathrm{e}^{-\mathrm{i}\theta/2}(z-1)^{-\alpha}(z+1)^{\alpha-1}\mathcal{C}[f(1-x)^{\alpha}(1+x)^{1-\alpha}](z)$$

This has the jump

$$\begin{aligned}\phi_+(x) - \mathrm{e}^{\mathrm{i}\theta}\phi_-(x) &= \kappa_+(z)\mathcal{C}_+\left[\frac{f}{\kappa_+}\right](x) - \mathrm{e}^{\mathrm{i}\theta}\kappa_-(z)\mathcal{C}_-\left[\frac{f}{\kappa_+}\right](x) \\ &= \kappa_+(x)\left(\mathcal{C}_+\left[\frac{f}{\kappa_+}\right](x) - \mathcal{C}_-\left[\frac{f}{\kappa_+}\right](x)\right) = f(x)\end{aligned}$$

Thus the general solution is $\phi(z) + C\kappa(z)$.

```
θ =2.3
α = -θ/(2π)
κ = z -> (z-1)^(-α)*(z+1)^(α-1)

x = Fun()
κ_+ = exp(im*θ/2)*(1-x)^(-α)*(x+1)^(α-1)

f = Fun(exp)
```

```
z = 2+im
φ = z -> κ(z)*cauchy(f/κ_+, z)
```

```
φ(0.1+0.0im)-exp(im*θ)*φ(0.1-0.0im) - f(0.1)
```

```
-1.3322676295501878e-15 - 2.220446049250313e-16im
```

1.12 Problem 4.3

Note for $x < 0$

$$x_+^{\mathrm{i}\beta} = \mathrm{e}^{\mathrm{i}\beta \log_+ x} = \mathrm{e}^{\mathrm{i}\beta \log_- x - 2\pi\beta} = \mathrm{e}^{-2\pi\beta} x_-^{\mathrm{i}\beta}$$

```
β =  2.3;
x = -2.0
(x+0.0im)^(im*β) - exp(-2π*β)*(x-0.0im)^(im*β)
```

```
-3.3881317890172014e-21 + 1.0842021724855044e-19im
```

We actually have bounded (oscillatory) growth near zero since

$$|\mathrm{e}^{\mathrm{i}\beta \log z}| = |\mathrm{e}^{\mathrm{i}\beta \log |z|} \mathrm{e}^{-\beta \arg z}| = \mathrm{e}^{-\beta \arg z}$$

Thus if we write $c = r\mathrm{e}^{\mathrm{i}\theta}$ for $0 < \theta < 2\pi$ and define $\alpha = -\frac{\theta}{2\pi} + \mathrm{i}\frac{\log r}{2\pi}$ we can write the solution to 4.1 as

$$\kappa(z) = (z-1)^{-\alpha}(z+1)^{\alpha-1}$$

The same arguments as before then proceed. and the solution to 4.3 is

$$\phi(z) = \kappa(z)\mathcal{C}\left[\frac{f}{\kappa_+}\right](z) + C\kappa(z)$$

```
θ =2.3
α = -θ/(2π)
κ = z -> (z-1)^(-α)*(z+1)^(α-1)
κ(0.1+0.0im) - exp(im*θ)*κ(0.1-0.0im)
```

```
0.0 + 1.1102230246251565e-16im
```

```
r = 2.4
θ = 2.1
c = r*exp(im*θ)

α = -θ/(2π) + im*log(r)/(2π)

κ = z -> (z-1)^(-α)*(z+1)^(α-1)

κ(0.1+0.0im)-c*κ(0.1-0.0im)
```

```
2.220446049250313e-16 + 2.220446049250313e-16im
```

1.13 Problem 5.1

1. It is a product of functions analytic off $(-\infty, 1]$ hence is analytic off $(-\infty, 1]$, and we just have to check that it has no jump on $(-\infty, -1)$ and $(-a, a)$. This follows via, for $x < -1$:

$$\kappa_+(x) = \frac{1}{\mathrm{i}^4\sqrt{1-x}\sqrt{-1-x}\sqrt{a-x}\sqrt{-a-x}} = \frac{1}{(-\mathrm{i})^4\sqrt{1-x}\sqrt{-1-x}\sqrt{a-x}\sqrt{-a-x}} =$$

and for $-a < x < a$ we have

$$\kappa_+(x) = \frac{1}{\mathrm{i}^2\sqrt{1-x}\sqrt{x+1}\sqrt{a-x}\sqrt{x+a}} = \frac{1}{(-\mathrm{i})^2\sqrt{1-x}\sqrt{1+x}\sqrt{a-x}\sqrt{-a-x}} = \kappa_-(x)$$

2. This follows via the usual arguments: for $a < x < 1$ we have:

$$\kappa_+(x) = \frac{1}{\mathrm{i}\sqrt{1-x}\sqrt{x+1}\sqrt{x-a}\sqrt{x+a}} = -\kappa_-(x)$$

and for $-1 < x < -a$ we have

$$\kappa_+(x) = \frac{1}{\mathrm{i}^3\sqrt{1-x}\sqrt{x+1}\sqrt{x-a}\sqrt{x+a}} = -\kappa_-(x)$$

3. This has at most square singularitie4s which are weaker than poles

4.

$$\kappa(z) = \frac{1}{z^2\sqrt{1-1/z}\sqrt{1-a/z}\sqrt{1+a/z}\sqrt{1+1/z}} \sim \frac{1}{z^2} \to 0$$

.

1.14 Problem 5.2

Ah, this is a trick question! Note that $z\kappa(z) \sim z^{-1} = O(z)$ and satisfies all the other properties. Thus consider any other solution $\tilde{\kappa}(z)$ and write

$$r(z) = \frac{\tilde{\kappa}(z)}{\kappa(z)}$$

This has trivial jumps and hence is entire: for example, on $(a, 1)$ we have

$$r_+(x) = \frac{\tilde{\kappa}_+(x)}{\kappa_+(x)} = \frac{-\tilde{\kappa}_-(x)}{-\kappa_-(x)} = r_-(x)$$

But since $\kappa \sim O(z^{-2})$ we only know that κ has at most $O(z)$ growth, hence it can be any first degree polynomial. Therefore, the space of all solutions is in fact two-dimensional: $\psi(z) = (A + Bz)\kappa(z)$.

1.15 Problem 5.3

Here we mimick the usual solution techniques and propose:

$$\phi(z) = \kappa(z)\mathcal{C}\left[\frac{f}{\kappa_+}\right](z) + (A + Bz)\kappa(z)$$

A quick check confirms it has the right jumps:

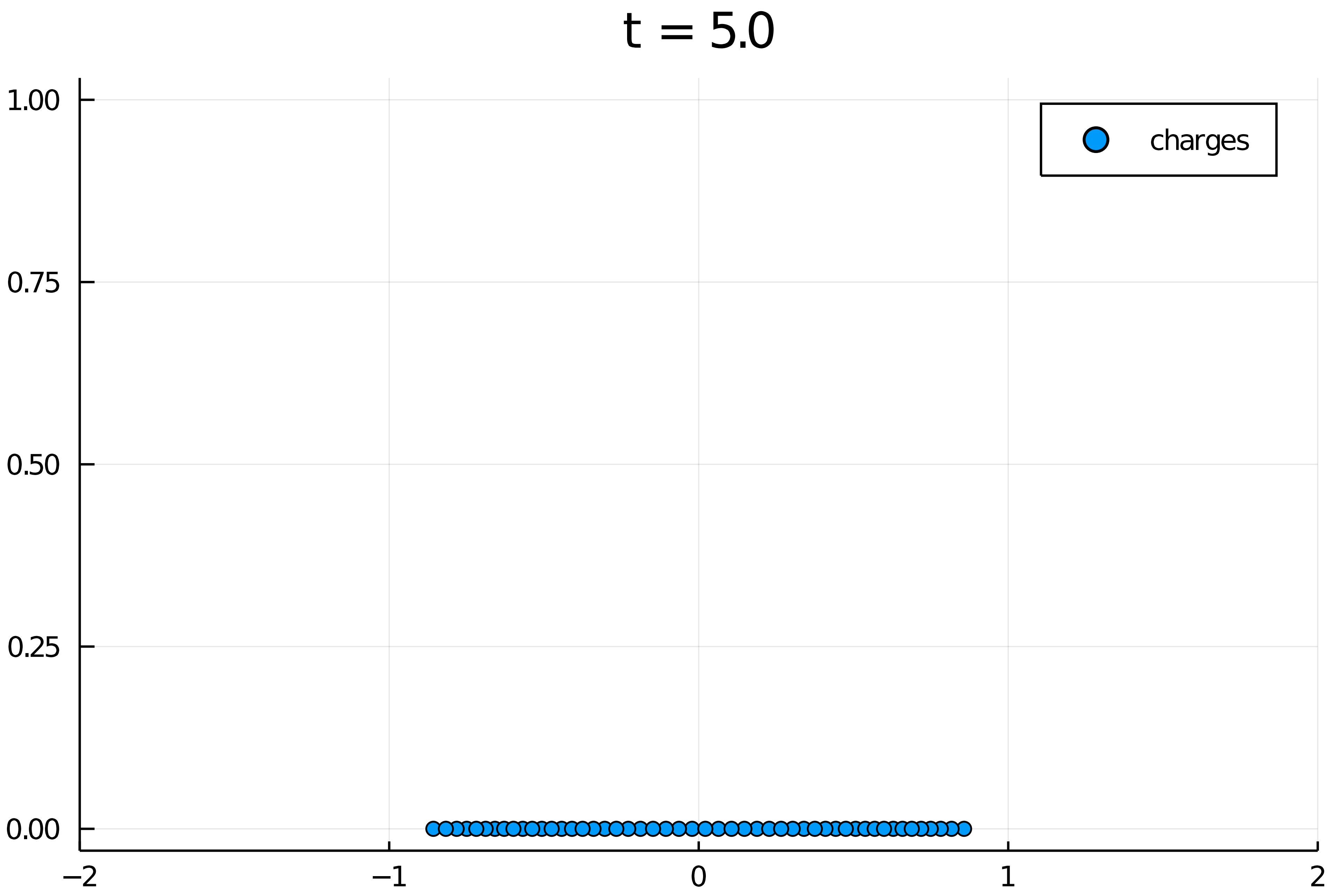
$$\begin{aligned}
\phi_+(x) &= \kappa_+(x)\mathcal{C}_+\left[\frac{f}{\kappa_+}\right](x) + (A + Bx)\kappa_+(x) \\
&= \kappa_+(x)\left(\frac{f(x)}{\kappa_+(x)} + \mathcal{C}_-\left[\frac{f}{\kappa_+}\right](x)\right) - (A + Bx)\kappa_-(x) = -\phi_-(x) + f(x)
\end{aligned}$$

1.16 Problem 6.1

Let's first do a plot and histogram. Here we see the right scaling is $N^{1/4}$, using a simplified model without the second derivative (we expect this to go to the same distribution):

```
using DifferentialEquations

V = x -> x^4
Vp = x -> 4x^3

N = 50
λ_0 = randn(N)  # initial location

prob = ODEProblem(function(λ,_,t)
```

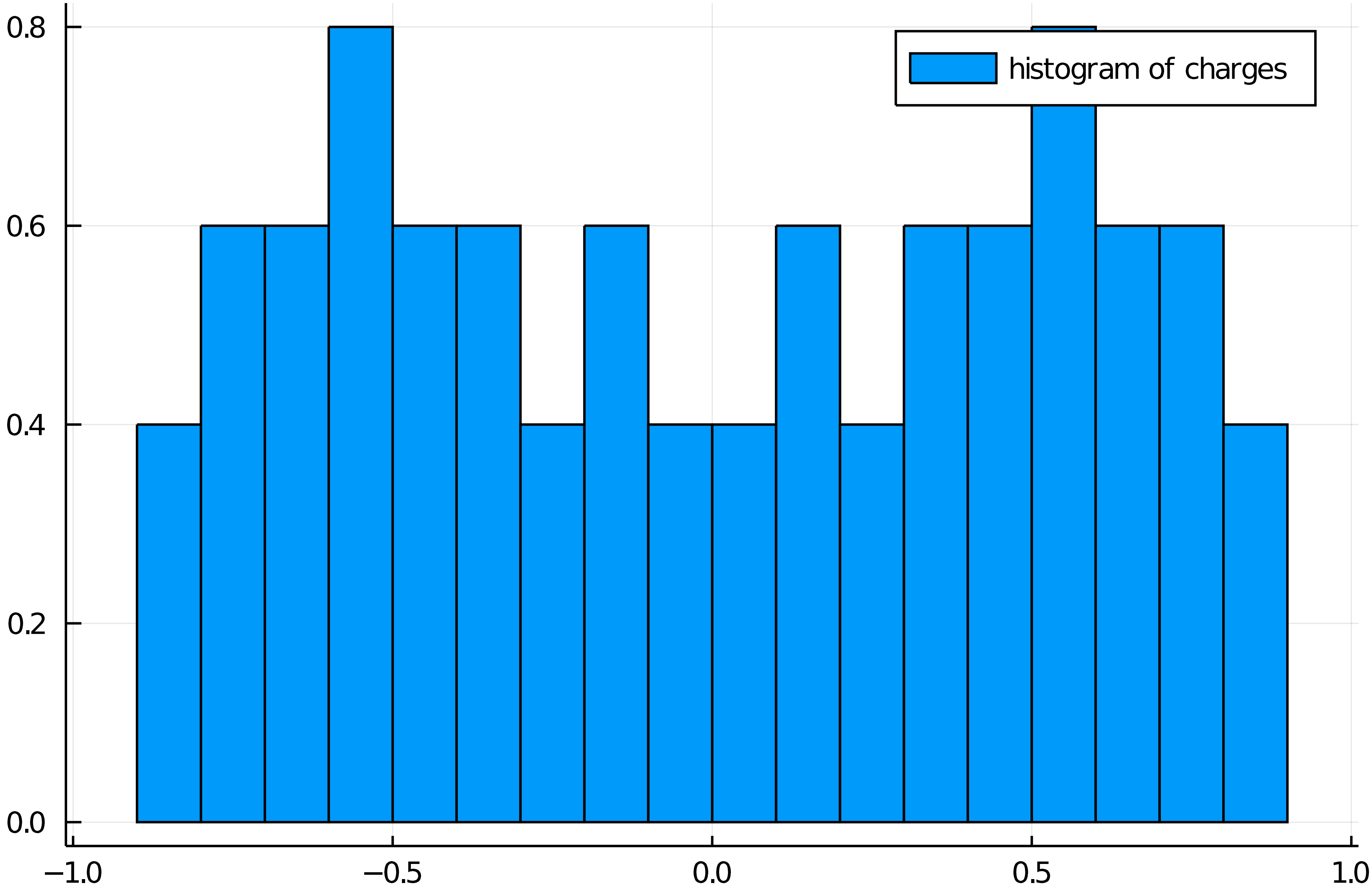
```
        [sum(1 ./ (λ[k] .- λ[[1:k-1;k+1:end]])) - Vp(λ[k]) for k=1:N]
         end, λ_0, (0.0, 5.0))
λ = solve(prob; reltol=1E-6);

t = 5.0
scatter(λ(t)/N^(1/4) ,zeros(N); label="charges", xlims=(-2,2),
title="t = $t")
```



The limiting distribution has the following form:

```
histogram(λ(t)/N^(1/4); nbins=30, normalize=true, label="histogram
```

```
of charges")
```



We want to solve

$$\frac{\mathrm{d}^2\lambda_k}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\lambda_k}{\mathrm{d}t} = \sum_{\substack{j=1\\j\neq k}}^{N} \frac{1}{\lambda_k - \lambda_j} - 4\lambda_k^3$$

Rescale via $\mu_k = \frac{\lambda_k}{N^{1/4}}$ gives

$$0 = N^{1/4}\left[\frac{\mathrm{d}^2\lambda_k}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\lambda_k}{\mathrm{d}t}\right] = N^{-1/4}\sum_{\substack{j=1\\j\neq k}}^{N} \frac{1}{\mu_k - \mu_j} - 4N^{3/4}\mu_k^3$$

or in other words

$$0 = N^{-1/2}\left[\frac{\mathrm{d}^2\lambda_k}{\mathrm{d}t^2} + \gamma\frac{\mathrm{d}\lambda_k}{\mathrm{d}t}\right] = \frac{1}{N}\sum_{\substack{j=1\\j\neq k}}^{N} \frac{1}{\mu_k - \mu_j} - 4\mu_k^3$$

We can now formally let $N \to \infty$ to get our equation

$$⨍_{-b}^{b} \frac{w(t)}{x-t}\mathrm{d}t = 4x^3$$

where I've used symmetry to assume that the interval is symmetric. We want to find w and b so that this equations holds true and w is a bounded probability density:

1.

$$w(x) > 0$$

for $-b < x < b$

2.

$$\int w(x)\mathrm{d}x = 1$$

3.

$$w$$

is bounded

Our equation is equivalent to

$$\mathcal{H}_{[-b,b]}w(x) = \frac{4x^3}{\pi}$$

recall the inversion formula

$$u(x) = \frac{-1}{\sqrt{b^2 - x^2}}\mathcal{H}\left[f(\diamond)\sqrt{b^2 - \diamond^2}\right](x) - \frac{C}{\sqrt{b^2 - x^2}}$$

In our case $f(x) = \frac{4x^3}{\pi}$ and we use

$$\sqrt{z-b}\sqrt{z+b} = z\sqrt{1-b/z}\sqrt{1+b/z} = z - \frac{b^2}{2z} - \frac{b^4}{8z^3} + O(z^{-4})$$

to determine

$$2\mathrm{i}\mathcal{C}\left[\diamond^3\sqrt{b^2-\diamond^2}\right](x) = z^3(\sqrt{z-b}\sqrt{z+b} - z + \frac{b^2}{2z} + \frac{b^4}{8z^3}) = z^3\sqrt{z-b}\sqrt{z+b} - z^4 + \frac{b^2z^2}{2} + \frac{b^4}{8}$$

```
b = 5
x = Fun(-b .. b)
(2im)cauchy(x^3*sqrt(b^2-x^2),z)
```

```
71.6687198577313 + 40.09051049243315im
```

Therefore,

$$u(x) = \frac{4\mathrm{i}}{\pi\sqrt{b^2-x^2}}(\mathcal{C}^+ + \mathcal{C}^-)\left[\diamond^3\sqrt{b^2-\diamond^2}\right](x) - \frac{C}{\sqrt{b^2-x^2}}$$
$$\frac{4}{\pi\sqrt{b^2-x^2}}(-x^4 + \frac{b^2x^2}{2} + \frac{b^4}{8}) - \frac{C}{\sqrt{b^2-x^2}}$$

We choose C so this is bounded, in particular, we get get the solution

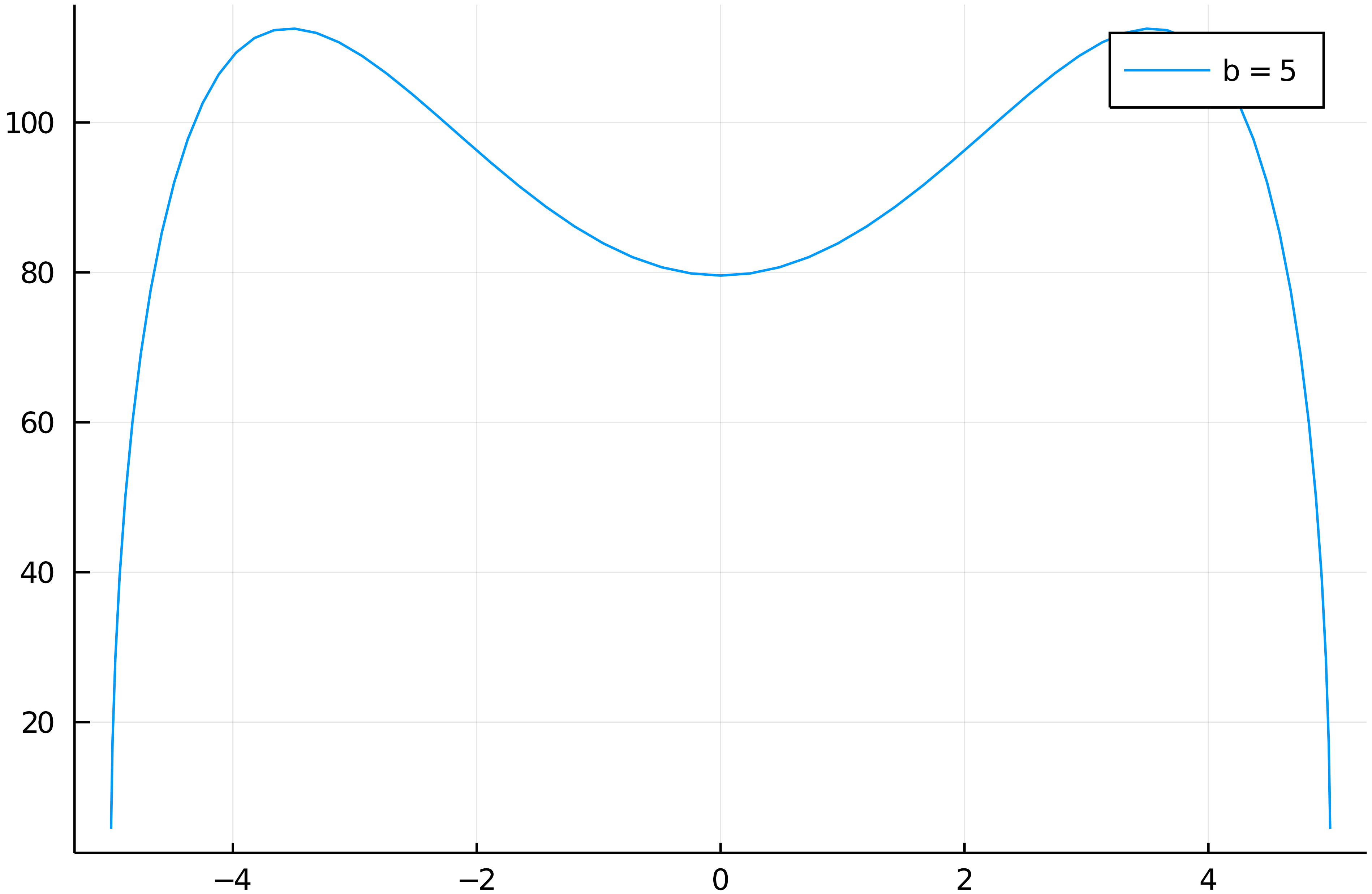
$$u(x) = \frac{4}{\pi\sqrt{b^2-x^2}}(-x^4 + \frac{b^2x^2}{2} + \frac{b^4}{2})$$

```
u = 4/(π*sqrt(b^2-x^2))*(-x^4 + b^2*x^2/2 + b^4/2)
H(u, 0.1),4*0.1^3/π
```

```
(0.001273239544735292, 0.001273239544735163)
```

At least it looks right, we just need to get the right b:

```
plot(u; label = "b = $b")
```



We want to choose b now so that this integrates to 1. For example, this choice of b is horrible:

```
sum(u)
```

```
937.5
```

There's a nice trick: If $-2\pi\mathrm{i}\mathcal{C}u(z) \sim \frac{1}{z}$ then $\int_{-b}^{b} u(x)\mathrm{d}x = 1$. We know since

$$\frac{1}{\sqrt{1+z}} = 1 - \frac{z}{2} + \frac{3z^2}{8} - \frac{5z^3}{16} + O(z^4)$$

$$\begin{aligned}
\frac{-z^4 + \frac{b^2 z^2}{2} + \frac{b^4}{2}}{\sqrt{z-b}\sqrt{z+b}} &= \frac{-z^3 + \frac{b^2 z}{2} + \frac{b^4}{2z}}{\sqrt{1-b/z}\sqrt{1+b/z}} \\
&= (-z^3 + \frac{b^2 z}{2})(1 + \frac{b}{2z} + \frac{3b^2}{8z^2} + \frac{5b^3}{16z^3})(1 - \frac{b}{2z} + \frac{3b^2}{8z^2} - \frac{5b^3}{16z^3}) + O(z^{-1}) \\
&= -z^3 + \left(\frac{b^2}{2} + \frac{b^2}{4} + \frac{b^2}{2} - \frac{3b^2}{8} - \frac{3b^2}{8}\right) z + O(z^{-1}) \\
&= -z^3 + O(z^{-1})
\end{aligned}$$

Thus we know

$$\mathcal{C}u(z) = \frac{2\mathrm{i}}{\pi} z^3 + \frac{2\mathrm{i}}{\pi} \frac{-z^4 + \frac{b^2 z^2}{2} + \frac{b^4}{2}}{\sqrt{z-b}\sqrt{z+b}}$$

```julia
cauchy(u, z)
```

```julia
z =100.0im;
2im/π*z^3 + 2im/π*(-z^4 + b^2*z^2/2 +b^4/2)/(sqrt(z-b)sqrt(z+b))
```

```
1.4908359390683472 - 0.0im
```

taking this one term further we find

$$\frac{-z^4 + \frac{b^2 z^2}{2} + \frac{b^4}{2}}{\sqrt{z-b}\sqrt{z+b}} = -z^3 + \frac{3b^4}{8z} + O(z^{-2})$$

Hence we want to choose b so that

$$-2\mathrm{i}\pi\mathcal{C}u(z) = \frac{3b^4}{2z} \sim \frac{1}{z}$$

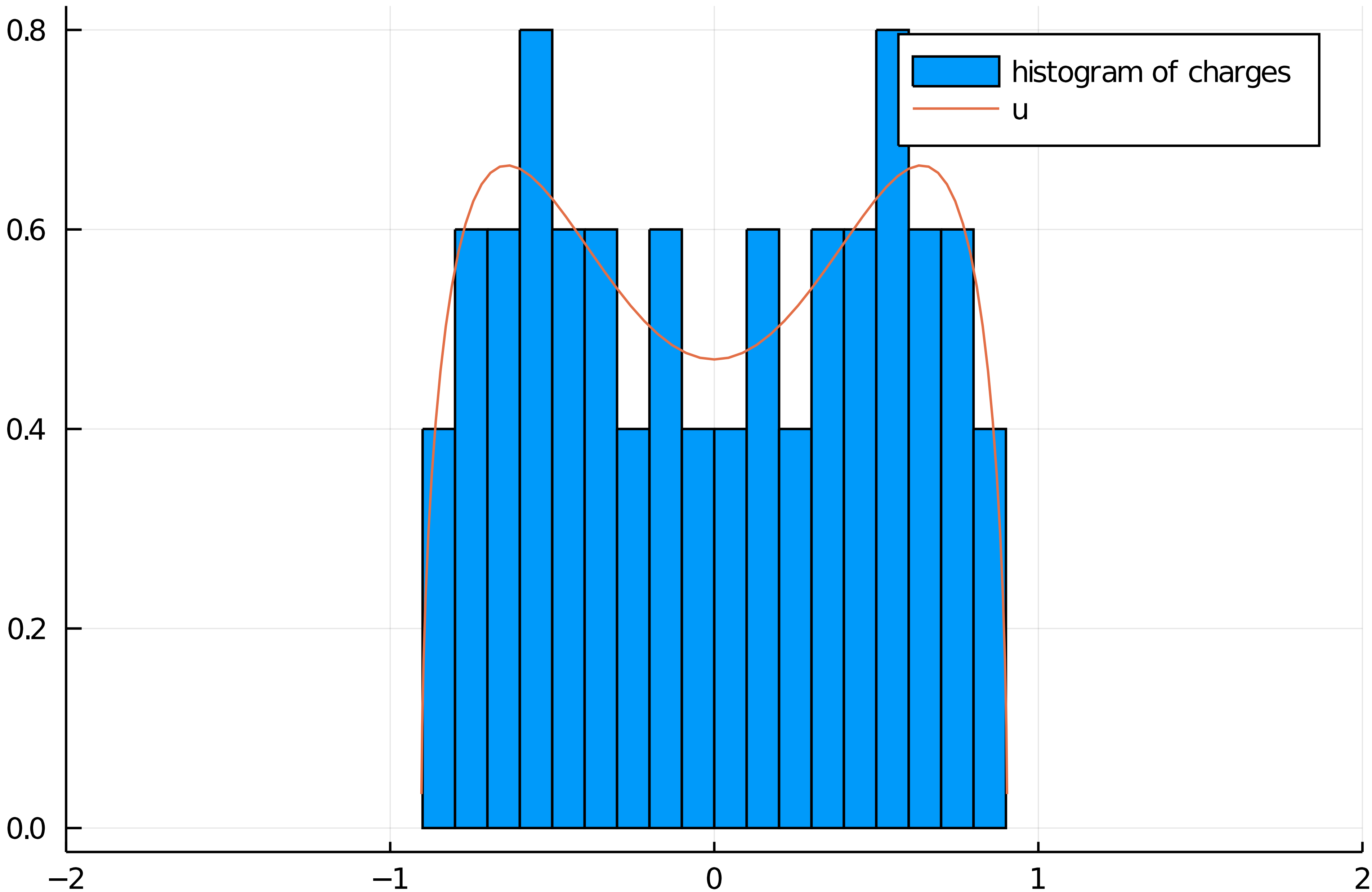
in other words, $b = (\frac{2}{3})^{1/4}$

```julia
b = (2/3)^(1/4)
x = Fun(-b .. b)
u = 4/(π*sqrt(b^2-x^2))*(-x^4 + b^2*x^2/2 + b^4/2)
sum(u)
```

```
0.9999999999999997
```

And it worked!

```julia
histogram(λ(5.0)/N^(1/4); nbins=25, normalize=true ,
label="histogram of charges")
plot!(u; label="u",xlims=(-2,2))
```


histogram of charges
u
0.8
0.6
0.4
0.2
0.0
−2
−1
0
1
2