

# Applied Complex Analysis (2021)

## 1 Solution Sheet 2

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### 1.1 Problem 1

1.

We have

$$\sigma(A) \subseteq B(1, 3) \cup B(2, 3) \cup B(4, 1)$$

where  $B(z_0, r)$  is the ball of radius  $r$  around  $z_0$ .

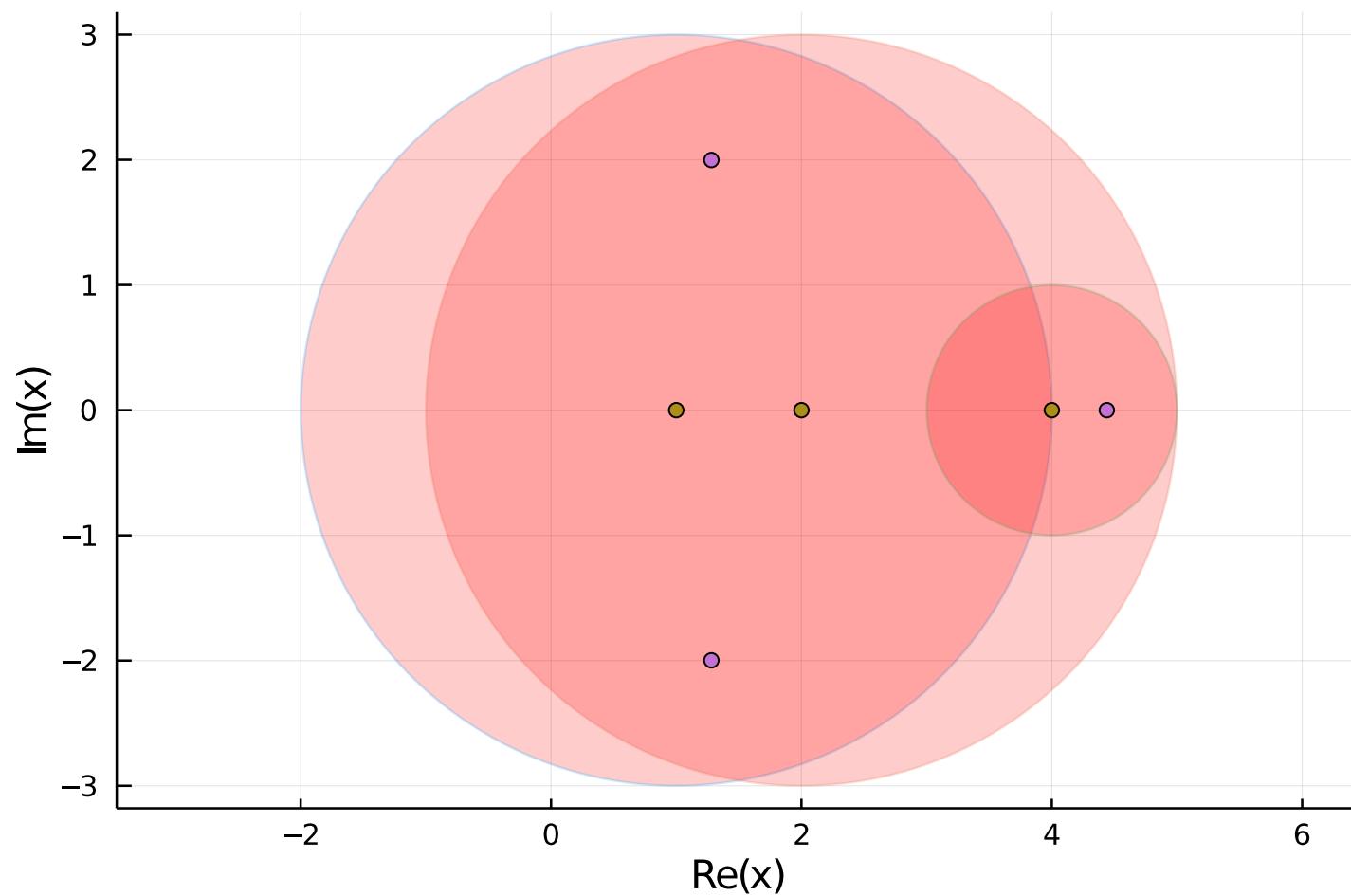
Here's a depiction:

```
using ApproxFun, Plots, ComplexPhasePortrait, LinearAlgebra,  
DifferentialEquations  
drawdisk!(z0, R) = plot!(θ-> real(z0) + R[1]*cos(θ), θ-> imag(z0) +  
R[1]*sin(θ), 0, 2π, fill=(0,:red), α = 0.2, legend=false)
```

```
A = [1 2 -1; -2 2 1; 0 1 4]
```

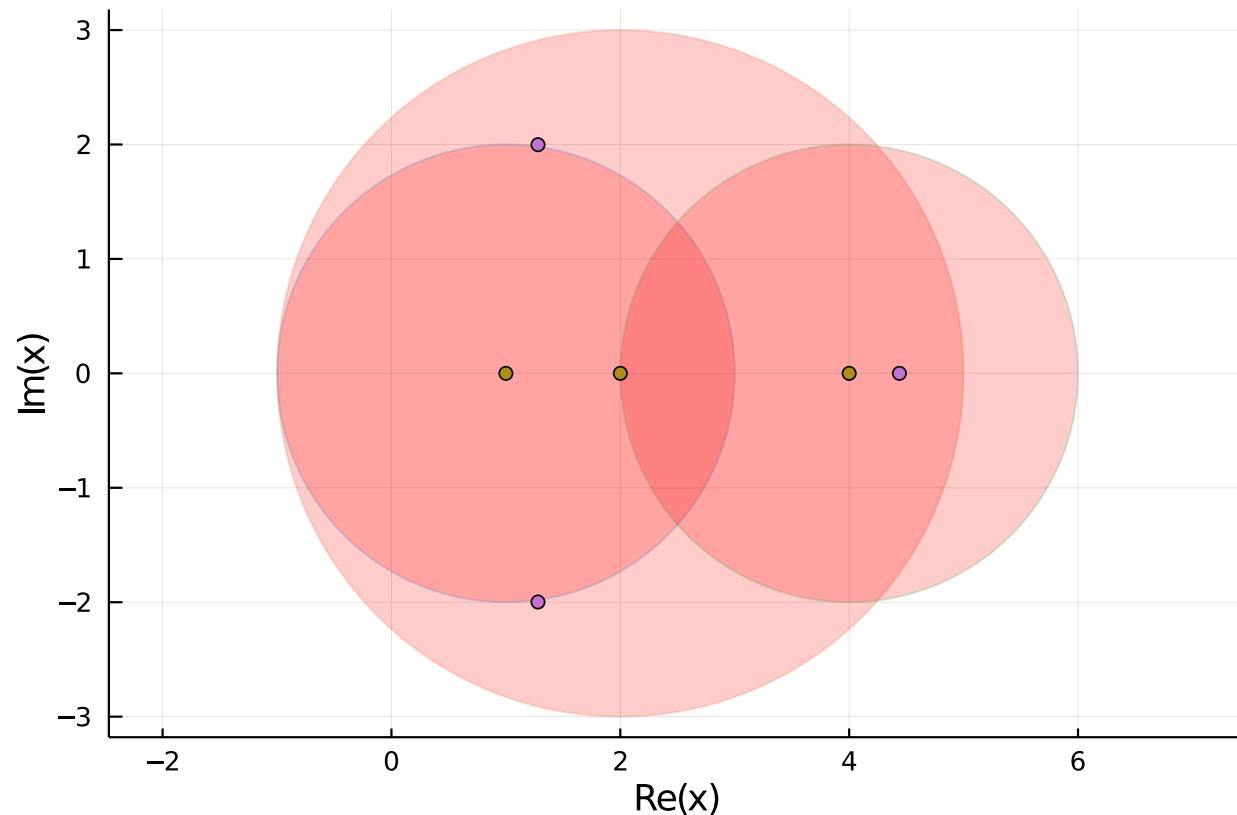
```
λ = eigvals(A)
```

```
p = plot()  
drawdisk!(1,3)  
drawdisk!(2,3)  
drawdisk!(4,1)  
scatter!(complex.(λ); label="eigenvalues",ratio=1.0)  
scatter!(complex.(diag(A)); label="diagonals")  
p
```



2. We get  $\sigma(A) \subseteq B(1, 2) \cup B(2, 3) \cup B(4, 2)$

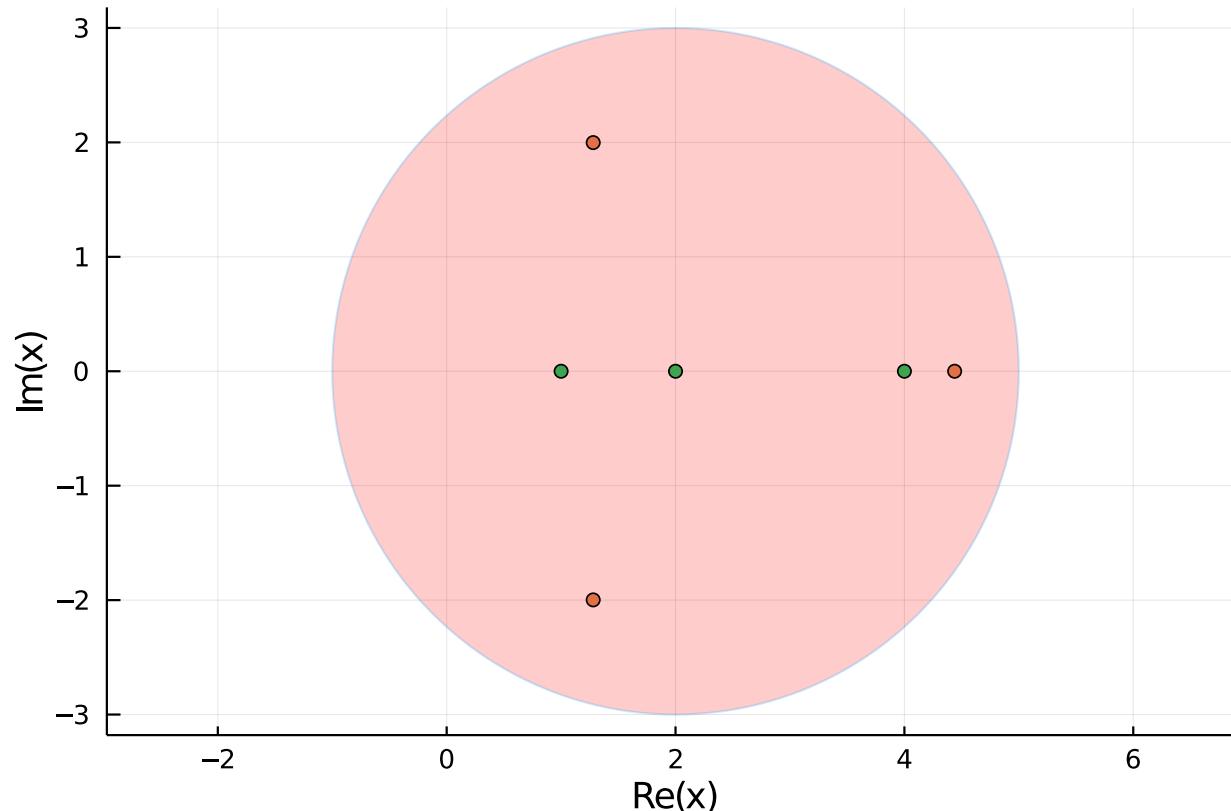
```
λ = eigvals(A)
p = plot()
drawdisk!(1,2)
drawdisk!(2,3)
drawdisk!(4,2)
scatter!(complex.(λ); label="eigenvalues")
scatter!(complex.(diag(A)); label="diagonals", ratio=1.0)
p
```



3. Because the spectrum live in the intersection of the two estimates, the sharpest bound is

$$\sigma(A) \subseteq B(2, 3)$$

```
λ = eigvals(A)
p = plot()
drawdisk!(2,3)
scatter!(complex.(λ); label="eigenvalues")
scatter!(complex.(diag(A)); label="diagonals",ratio=1.0)
```



Thus we can take  $2 + 3e^{i\theta}$  as the contour.

## 1.2 Problem 2

1.

Note that in the scalar case  $u'' = au$  we have the solution

$$u(t) = u_0 \cosh \sqrt{a}t + v_0 \frac{\sinh \sqrt{a}t}{\sqrt{a}}$$

Write

$$A = Q \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} Q^\top$$

where  $\lambda_k > 0$  and then the solution has the form, where  $\gamma$  is a contour surrounding the eigenvalues and to the right of zero:

$$\begin{aligned}
\mathbf{u}(t) &= Q \begin{pmatrix} \cosh \sqrt{\lambda_1} t & & \\ & \ddots & \\ & & \cosh \sqrt{\lambda_n} t \end{pmatrix} Q^\top \mathbf{u}_0 + Q \begin{pmatrix} \frac{\sinh \sqrt{\lambda_1} t}{\sqrt{\lambda_1}} & & \\ & \ddots & \\ & & \frac{\sinh \sqrt{\lambda_n} t}{\sqrt{\lambda_n}} \end{pmatrix} Q^\top \mathbf{v}_0 \\
&= \frac{1}{2\pi i} Q \begin{pmatrix} \oint_{\gamma} \frac{\cosh \sqrt{z} t dz}{z - \lambda_1} & & \\ & \ddots & \\ & & \oint_{\gamma} \frac{\cosh \sqrt{z} t dz}{z - \lambda_n} \end{pmatrix} Q^\top \mathbf{u}_0 \\
&\quad + \frac{1}{2\pi i} Q \begin{pmatrix} \oint_{\gamma} \frac{\sinh \sqrt{z} t}{\sqrt{z}} \frac{dz}{z - \lambda_1} & & \\ & \ddots & \\ & & \oint_{\gamma} \frac{\sinh \sqrt{z} t}{\sqrt{z}} \frac{dz}{z - \lambda_n} \end{pmatrix} Q^\top \mathbf{v}_0 \\
&= \frac{1}{2\pi i} \oint_{\gamma} \cosh \sqrt{z} t Q \begin{pmatrix} (z - \lambda_1)^{-1} & & \\ & \ddots & \\ & & (z - \lambda_n)^{-1} \end{pmatrix} Q^\top \mathbf{u}_0 dz \\
&\quad + \frac{1}{2\pi i} \oint_{\gamma} \frac{\sinh \sqrt{z} t}{\sqrt{z}} Q \begin{pmatrix} (z - \lambda_1)^{-1} & & \\ & \ddots & \\ & & (z - \lambda_n)^{-1} \end{pmatrix} Q^\top \mathbf{v}_0 dz \\
&= \frac{1}{2\pi i} \oint_{\gamma} \cosh \sqrt{z} t (zI - A)^{-1} \mathbf{u}_0 dz + \frac{1}{2\pi i} \oint_{\gamma} \frac{\sinh \sqrt{z} t}{\sqrt{z}} (zI - A)^{-1} \mathbf{v}_0 dz
\end{aligned}$$

Here we verify the formulae numerically:

```
n = 5
A = randn(n,n)
A = A + A' + 10I
```

```
λ, Q = eigen(A)
```

```
λ
```

```
5-element Array{Float64,1}:
```

```
7.474242825312149
8.40196998720244
11.200557022982455
12.811011606308103
15.643625647665232
```

```
norm(A - Q*Diagonal(λ)*Q')
```

```
1.7060853784660773e-14
```

Time-stepping solution:

```
u_0 = randn(n)
v_0 = randn(n)
uv = solve(ODEProblem((uv,_,t) -> [uv[n+1:end]; A*uv[1:n]], [u_0;
v_0], (0.,2.)); reltol=1E-10);
```

```
t = 2.0
uv(t)[1:n]

5-element Array{Float64,1}:
-288.6451624526969
1043.2769593567198
115.6190946303411
-228.26666410841935
97.37054579190372
```

Solution via diagonalization:

```
Q*Diagonal(cosh.(sqrt.(λ) .* t))*Q'*u_0 + Q*Diagonal(sinh.(sqrt.(λ)
.* t) ./ sqrt.(λ))*Q'*v_0
```

```
5-element Array{Float64,1}:
-288.6451623500332
1043.2769592553739
115.6190946080924
-228.26666407820585
97.37054577842983
```

Solution via elliptic integrals. We chose the ellipse to surround all the spectrum of our particular  $A$  with eigenvalues:

```
periodic_rule(n) = 2π/n*(0:(n-1)), 2π/n*ones(n)
function ellipse_rule(n, a, b)
```

```

θ = periodic_rule(n)[1]
a*cos.(θ) + b*im*sin.(θ), 2π/n*(-a*sin.(θ) + im*b*cos.(θ))
end
function ellipse_f(f, A, n, z_0, a, b)
    z,w = ellipse_rule(n,a,b)
    z .+= z_0
    ret = zero(A)
    for j=1:n
        ret += w[j]*f(z[j])*inv(z[j]*I - A)
    end
    ret/(2π*im)
end

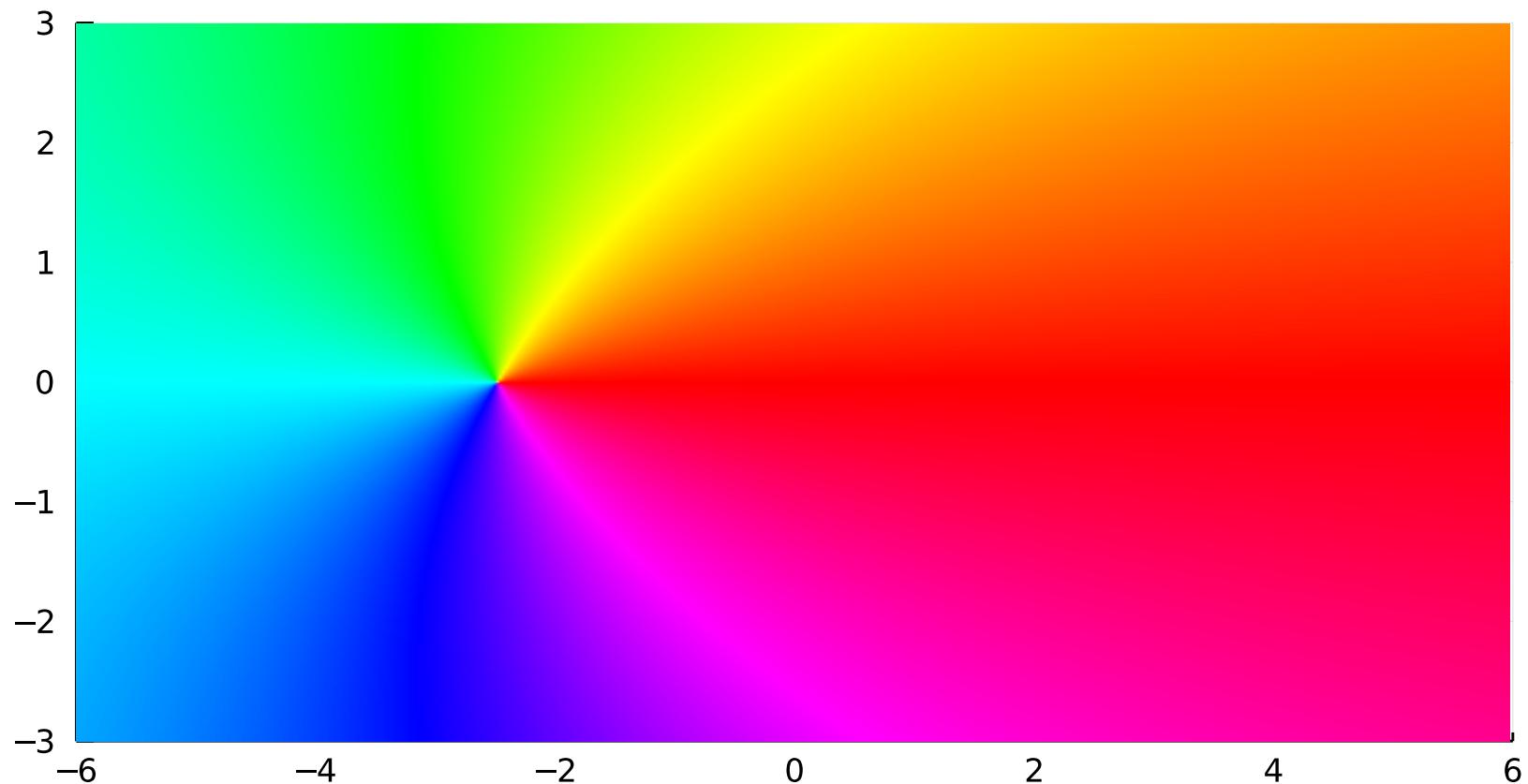
n = 50
ellipse_f(z -> cosh(sqrt(z)*t), A, n, 10.0, 7.0, 2.0)*u_0 +
    ellipse_f(z -> sinh(sqrt(z)*t)/sqrt(z), A, n, 10.0, 7.0, 2.0)*v_0
5-element Array{Complex{Float64},1}:
-288.6443320135477 + 1.0338431970895553e-14im
  1043.27679357605 + 8.483556655112882e-14im
  115.61922544738485 - 1.3389188081129559e-14im
-228.2668036755129 + 6.902268756464615e-15im
  97.37045481662241 - 9.896654266268307e-15im

```

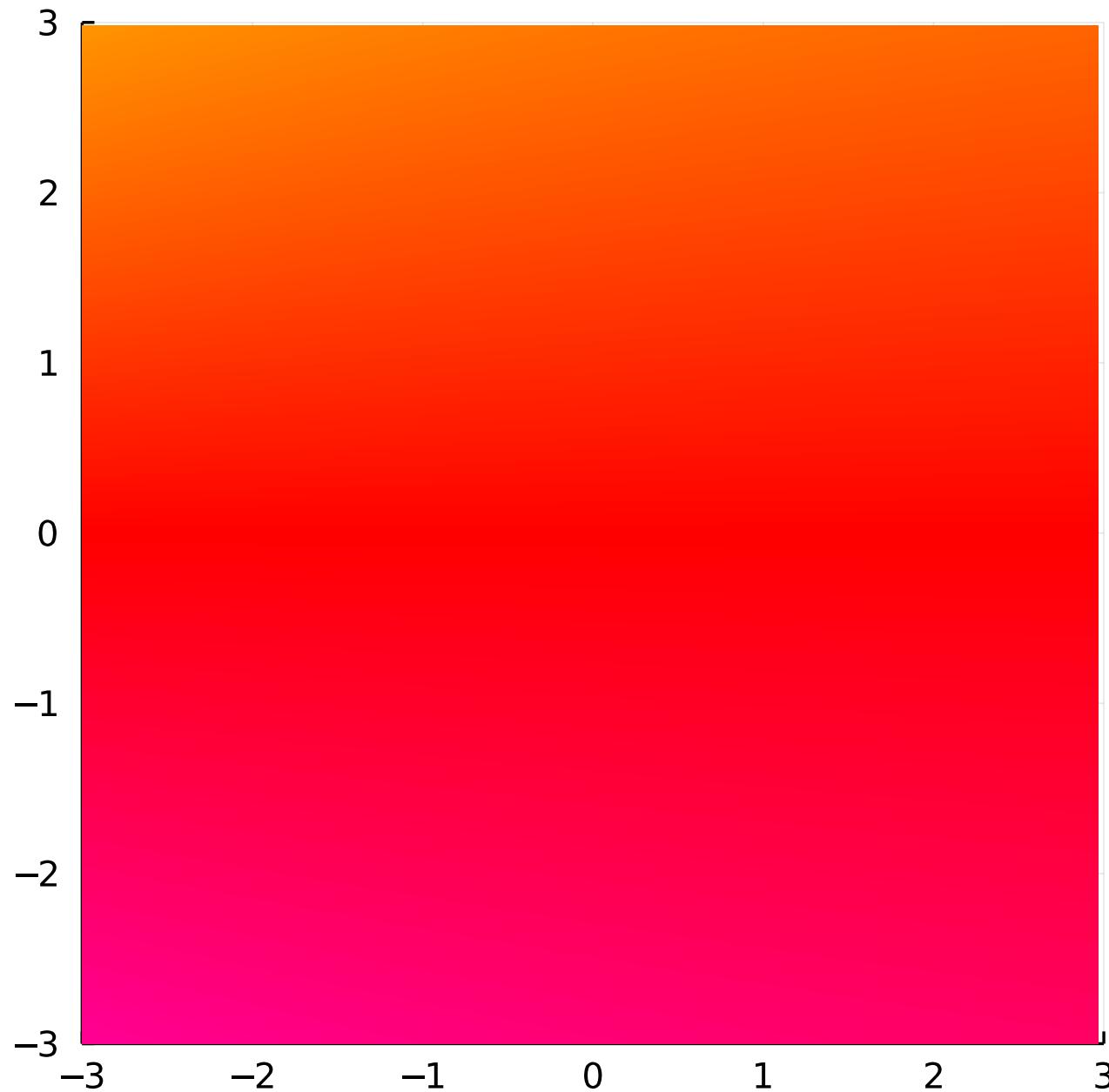
2.

I put the restriction in because of the  $\sqrt{z}$  term, which look like it is not analytic on  $(-\infty, 0]$ . However, this restriction was NOT necessary, since in fact  $\cosh \sqrt{z}t$  and  $\frac{\sinh \sqrt{z}t}{\sqrt{z}}$  are entire:

```
phaseplot(-6..6, -3..3, z -> cosh(sqrt(z)))
```



```
phaseplot(-3..3, -3..3, z -> sinh(sqrt(z))/sqrt(z))
```



This follows from Taylor series, though I prefer the following argument: we have

$$\cosh z = \frac{e^z + e^{-z}}{2}$$

hence  $\cosh it = \cos t = \cosh -it$ . Therefore, on the possible branch cut  $\cosh \sqrt{z}$  is continuous (hence analytic):

$$\cosh \sqrt{x}_+ = \cosh i\sqrt{|x|} = \cosh -i\sqrt{|x|} = \cosh \sqrt{x}_-$$

Similarly,

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

implies  $\sinh it = i \sin t = -i \sin(-t) = -\sinh -it$ , which gives us continuity:

$$\frac{\sinh \sqrt{x}_+}{\sqrt{x}_+} = \frac{\sinh i\sqrt{|x|}}{i\sqrt{|x|}} = \frac{\sinh(-i\sqrt{|x|})}{-i\sqrt{|x|}} = \frac{\sinh \sqrt{x}_-}{\sqrt{x}_-}$$

furthermore, they are both bounded at zero, hence analytic there too.

Here's a numerical example:

```
n = 5
A = randn(n,n)
λ, V = eigen(A)
λ
```

```
5-element Array{Complex{Float64},1}:
 -1.7721772573359895 + 0.0im
 -0.7138876142550249 + 0.0im
  0.5464654673558129 - 0.8279272378886993im
  0.5464654673558129 + 0.8279272378886993im
  2.471420139077356 + 0.0im
```

```
norm(A - V*Diagonal(λ)*inv(V))
```

```
5.2634945993010825e-15
```

```
u_0 = randn(n)
```

```
v_0 = randn(n)
```

```
uv = solve(ODEProblem((uv,_,t) -> [uv[n+1:end]; A*uv[1:n]], [u_0;
v_0], (0.,2.)); reltol=1E-10);
```

```
t = 2.0
```

```
uv(t)[1:n]
```

```
5-element Array{Float64,1}:
```

```
7.1401654391437095
6.873041006972097
-5.853476755795754
1.564633322228491
-1.0479652241742514
```

```
V*Diagonal(cosh.(sqrt.(\lambda) .* t))*inv(V)*u_0 +
V*Diagonal(sinh.(sqrt.(\lambda) .* t) ./ sqrt.(\lambda))*inv(V)*v_0
```

5-element Array{Complex{Float64},1}:

```
7.14016543836947 + 8.564739318812795e-17im
6.873041006687994 + 1.0004265220412661e-16im
-5.853476755584865 - 1.2117587861480155e-16im
1.5646333220833089 + 4.499356215805608e-18im
-1.0479652242083004 + 8.668666210552416e-17im
```

Here's the solution using an elliptic integral:

```
n = 100
ellipse_f(z -> cosh(sqrt(z)*t), A, n, 0.0, 3.0, 3.0)*u_0 +
    ellipse_f(z -> sinh(sqrt(z)*t)/sqrt(z), A, n, 0.0, 3.0, 3.0)*v_0
```

5-element Array{Complex{Float64},1}:

```
7.140165464464756 + 1.026246911352165e-15im
6.873041015788506 - 7.656795438333267e-16im
-5.853476763092647 - 2.7605971357564325e-16im
1.564633327596869 - 2.62286035872063e-16im
-1.0479652229387915 + 1.3148223160134344e-17im
```

### 1.3 Problem 3

1.

We have

$$\frac{1}{n} \sum_{j=0}^{n-1} g(\theta_j) = \sum_{k=0}^{\infty} g_k \frac{1}{n} \sum_{j=0}^{n-1} e^{ik\theta_j}$$

Define the  $n$ -th root of unity as  $\omega = e^{\frac{2\pi i}{n}}$  (that is  $\omega^n = 1$ ), and simplify

$$\sum_{j=0}^{n-1} e^{ik\theta_j} = \sum_{j=0}^{n-1} e^{\frac{2\pi j ik}{n}} = \sum_{j=0}^{n-1} \omega^{kj} = \sum_{j=0}^{n-1} (\omega^k)^j$$

If  $k$  is a multiple of  $n$ , then  $\omega^k = 1$ , and this sum is equal to  $n$ . If  $k$  is not a multiple of  $n$ , use Geometric series:

$$\sum_{j=0}^{n-1} (\omega^k)^j = \frac{\omega^{nk} - 1}{\omega^k - 1} = \frac{1^k - 1}{\omega^k - 1} = 0$$

2.

From lecture 4, we have  $|f_k| \leq M_r r^{-|k|}$  for any  $1 \leq r < R$ , where  $M_r$  is the supremum of  $f$  in an annulus  $\{z : r^{-1} < |z| < r\}$ . Thus from the previous part we have (using geometric series)

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} g(\theta_j) - \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta \right| \leq \sum_{K=1}^{\infty} |f_{Kn}| + |f_{-Kn}| \leq 2M \sum_{K=1}^{\infty} r^{-Kn} = 2M_r \frac{r^{-n}}{1 - r^{-n}}.$$

This is an upper bound that decays exponentially fast.

3.

Note that  $f(z) = 2z/(4z - z^2 - 1)$  satisfies  $f(e^{i\theta}) = g(\theta)$ . This has two poles at  $2 \pm \sqrt{3}$ :

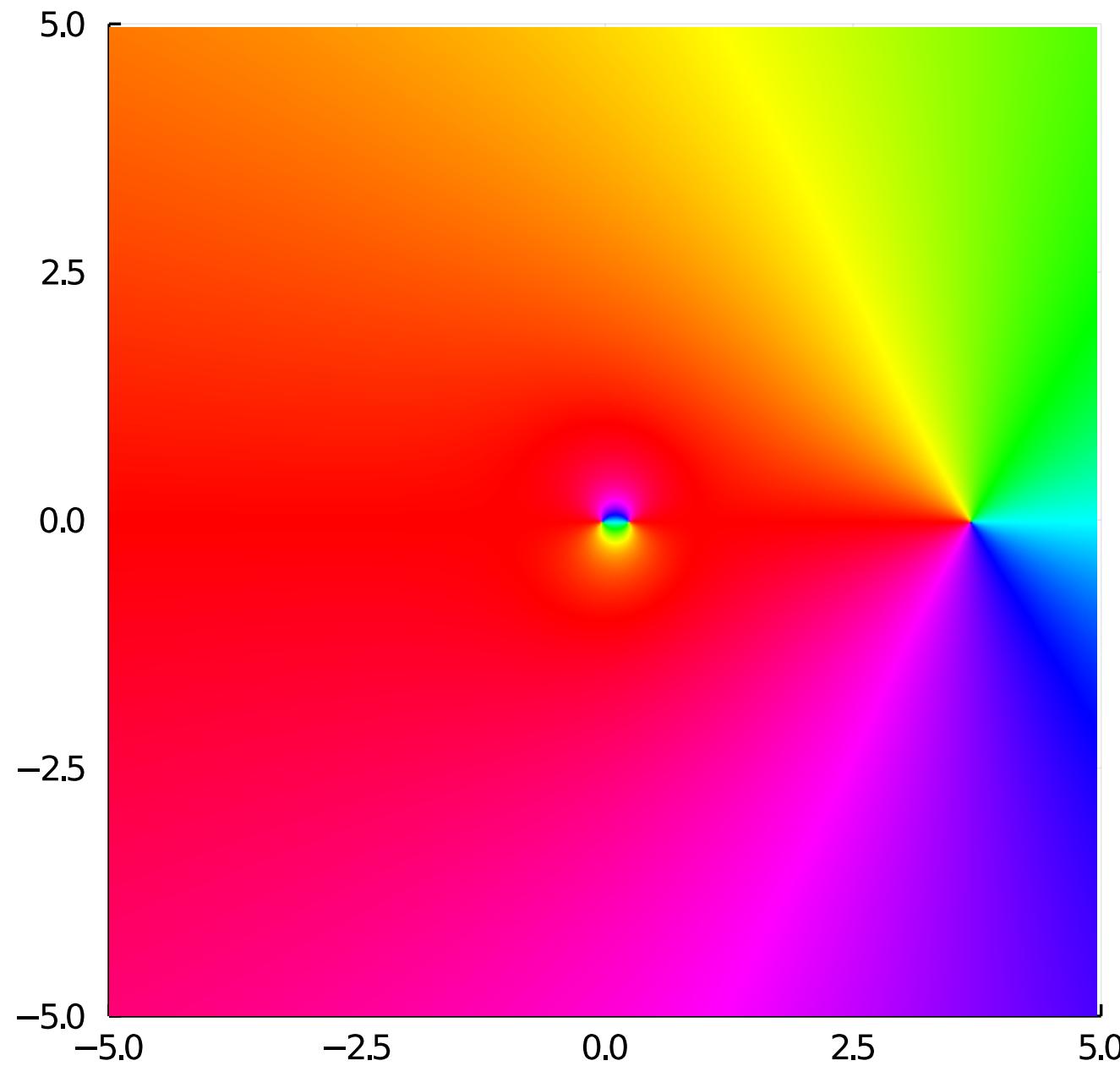
`f = z -> 2z/(4z-z^2-1)`

`f(exp(0.1im)) - 1/(2-cos(0.1))`

`1.1102230246251565e-16 + 0.0im`

Here's a phase plot showing the location of poles:

`phaseplot(-5..5, -5..5, f)`



$2+\sqrt{3}$ ,  $2-\sqrt{3}$ ,  $1/(2+\sqrt{3})$

(3.732050807568877, 0.2679491924311228, 0.2679491924311227)

Note that  $2 + \sqrt{3} = 1/(2 - \sqrt{3})$  so in the previous result, we take  $R = 2 + \sqrt{3}$ . For any  $1 \leq r < 2 + \sqrt{3}$  we have

$$M_r = \frac{2}{4 - r - r^{-1}}$$

Thus we get the upper bounds

$$\frac{4}{4 - r - r^{-1}} \frac{r^{-n}}{1 - r^{-n}}$$

Let's see how sharp it is:

```
periodic_rule(n) = 2π/n*(0:(n-1)), 2π/n*ones(n)
```

```
g = θ → 1/(2 - cos(θ))
```

```
Q = sum(Fun(g, 0 .. 2π))
```

```
err = Float64[ (
    (θ, w) = periodic_rule(n);
    sum(w.*g.(θ)) - Q
) for n=1:30];
```

```
N = length(err)
```

```
scatter(abs.(err), yscale=:log10, label="true error",
title="Trapezium error bounds", legend=:bottomleft)
r = 1.1
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label
= "r = $r")
r = 3.0
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label
= "r = $r")
r = 2+sqrt(3) - 0.01
scatter!(4/(4-r-inv(r)) .* r.^(-(1:N)) ./ (1 .- r.^(-(1:N))), label
= "r = $r")
```

# Trapezium error bounds

