

# Applied Complex Analysis (2021)

## 1 Lecture 17: Inverting logarithmic singular integrals and 2D electrostatic potentials

1. Solving a logarithmic singular integral equation
2. Application: electrostatic potentials in 2D
  - Potential arising from a point charge and a single plate

### 1.1 Solving logarithmic singular integral equations

Consider the problem of calculating  $u(x)$  such that

$$\frac{1}{\pi} \int_a^b u(t) \log|x-t| dt = f(x) \quad \text{for } a < x < b$$

To tackle this problem, recall that  $Lu(z) = \Re Mu(z)$  for

$$Mu(z) = \frac{1}{\pi} \int_a^b \log(z-t)u(t) dt = -2i \int_c^z \mathcal{C}u(\zeta) d\zeta + Mu(c)$$

where  $\mathcal{C} := \mathcal{C}_{[a,b]}$  and  $c$  is arbitrary, in fact we can choose it to be  $b$ . Then we have (assuming  $u$  is real-valued)

$$Lu(x) = \Re M^+ u(x) = 2\Im \int_b^x \mathcal{C}^+ u(t) dt = \int_b^x \mathcal{H}u(t) dt$$

where

$$\mathcal{H}u(x) = \frac{1}{\pi} \int_a^b \frac{u(t)}{t-x} dt$$

is the Hilbert transform. In other words, we have

$$\frac{d}{dx} Lu(x) = \mathcal{H}u(x).$$

Differentiating both sides of the equation for  $u$  therefore gives

$$\mathcal{H}u(x) = f'(x).$$

recall, we can express the solution as

$$u = \frac{-\mathcal{H}[\sqrt{b-\diamond}\sqrt{\diamond-a}f'] + D}{\sqrt{b-x}\sqrt{x-a}}$$

Here the constant  $D$  is not arbitrary: recall that

$$L[1/\sqrt{1-\diamond^2}](x) = -\log 2$$

hence varying  $D$  will vary  $u$  by a constant. To choose  $D$ , we use the fact that

$$f(c) = \frac{1}{\pi} \int_a^b u(t) \log |t-c| dt$$

for any choice of the point  $c$ .

## Demonstration

Let's do a numerical example:

```
using ApproxFun, SingularIntegralEquations, Plots
H(f) = -hilbert(f) # fix normalisation
H(f,x) = -hilbert(f,x)

x = Fun()
f = exp(x)
D = randn()

u_1 = -H(sqrt(1-x^2)*f')/sqrt(1-x^2)

u = u_1 + D/sqrt(1-x^2)

@show H(u, 0.1) - f'(0.1)
@show logkernel(u,0.1) - f(0.1) # didn't work
@show logkernel(u,0.2) - f(0.2); # but we are only off by a
constant

H(u, 0.1) - (f')(0.1) = 0.0
logkernel(u, 0.1) - f(0.1) = -1.2205937716368287
logkernel(u, 0.2) - f(0.2) = -1.2205937716368287
```

We thus need to choose the constant

```
# choose C so that
# logkernel(u_1, 0) + C*logkernel(1/sqrt(1-x^2), 0) == f(0)
C = (f(0) - logkernel(u_1, 0))/logkernel(1/sqrt(1-x^2), 0)
u = u_1 + C/sqrt(1-x^2)

@show H(u, 0.1) - f'(0.1)
@show logkernel(u,0.1) - f(0.1) # Works!
@show logkernel(u,0.2) - f(0.2); # And at all x!

H(u, 0.1) - (f')(0.1) = 0.0
logkernel(u, 0.1) - f(0.1) = -2.220446049250313e-16
logkernel(u, 0.2) - f(0.2) = 0.0
```

**Example 1** We now do an example which can be solved by hand. Find  $u(x)$  so that:

$$Lu(x) = x$$

for  $-1 < x < 1$ . Differentiating his becomes

$$\mathcal{H}u(x) = 1$$

which we solve via

$$u(x) = \frac{-\mathcal{H}[\sqrt{1 - \diamond^2}](x) + D}{\sqrt{1 - x^2}} = \frac{D - x}{\sqrt{1 - x^2}}$$

Note that

$$Lu(0) = 0$$

A symmetry argument shows that

$$L[\diamond/\sqrt{1 - \diamond^2}](0) = \frac{1}{\pi} \int_{-1}^1 \frac{t}{\sqrt{1 - t^2}} \log |t| dt = 0$$

which implies that  $D = 0$  and we have

$$u(x) = \frac{-x}{\sqrt{1 - x^2}}$$

We can confirm this:

```
x = Fun()
logkernel(-x/sqrt(1-x^2), 0.1) # approximately 0.1
0.1000000000000006
```

**Example 2** Find  $u(x)$  so that:

$$\frac{1}{\pi} \int_{-1}^1 u(t) \log |x - t| dt = 1.$$

Differentiating, we know that

$$\int_{-1}^1 \frac{u(t)}{x - t} dt = 0$$

hence  $u(x)$  must be of the form  $D/\sqrt{1 - x^2}$ . Since we know

$$L[1/\sqrt{1 - \diamond^2}](x) = -\log 2$$

we have  $D = -1/\log 2$ . Let's check:

```
x = Fun()
u = -1/(log(2)*sqrt(1-x^2))
logkernel(u, 0.1) # approximately 1
```

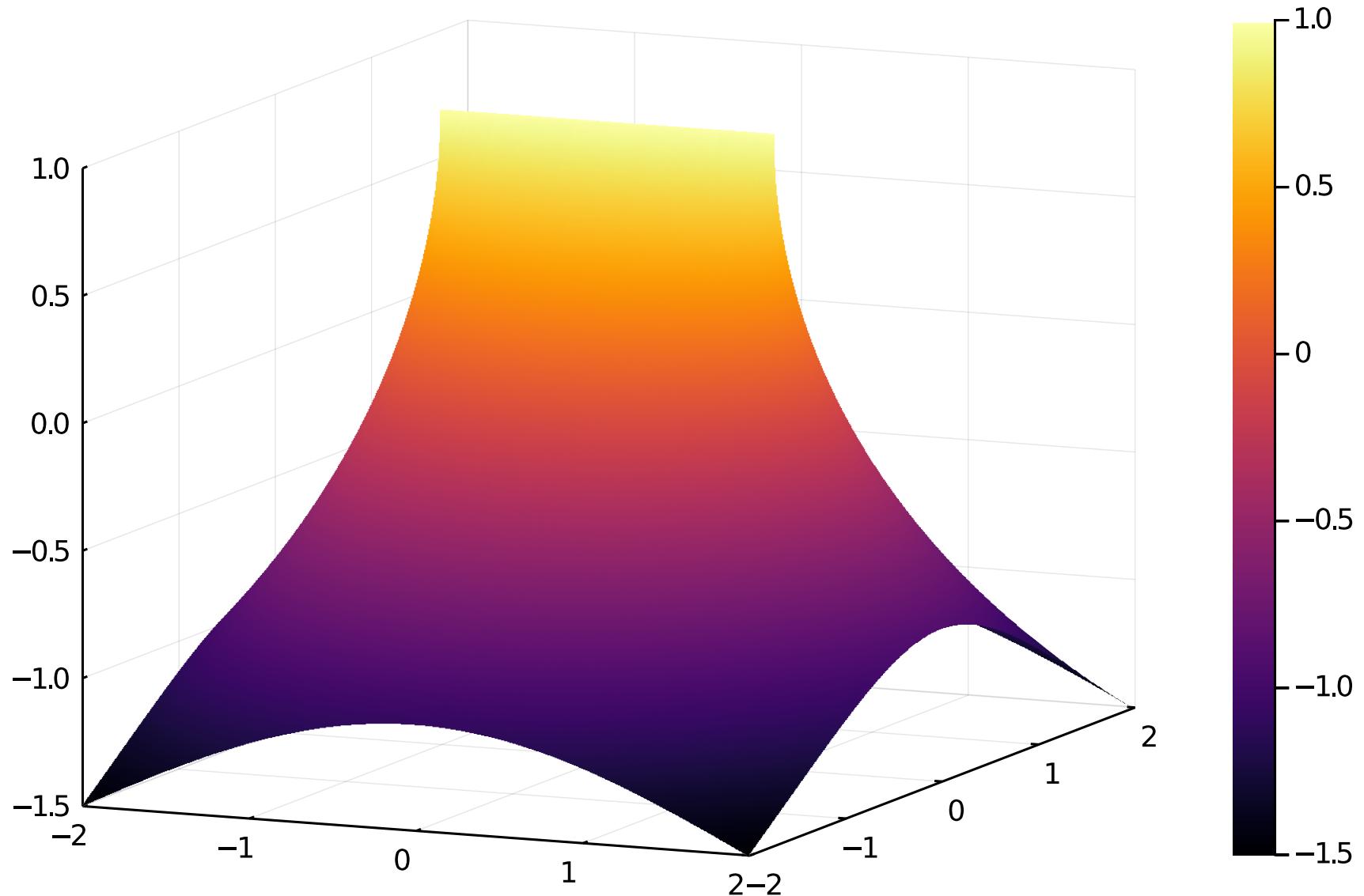
1.0

Physically, this solution gives us the potential field

$$Lu(z)$$

corresponding to holding a metal plate at constant potential:

```
v = z -> logkernel(u, z)
xx = yy = -2:0.01:2
V = v.(xx' .+ im*yy)
surface(xx, yy, V)
```



## 1.2 Application: Potential arising from a point charge and a single plate

Now imagine we put a point source at  $x = 2$ , and a metal plate on  $[-1, 1]$ . We know the potential on the plate must be constant, but we don't know what constant. This is equivalent to the following problem (see e.g. [Chapman, Hewett & Trefethen 2015]):

$$\begin{aligned} v_{xx} + v_{yy} &= 0 && \text{for } z \text{ off } [-1, 1] \text{ and } 2 \\ v(z) &\sim \log |z - 2| + O(1) && \text{for } z \rightarrow 2 \\ v(z) &\sim \log |z| + o(1) && \text{for } z \rightarrow \infty \\ v(x) &= \kappa && \text{for } -1 < x < 1 \end{aligned}$$

where  $\kappa$  is an unknown constant. We write the solution as

$$v(z) = \frac{1}{\pi} \int_{-1}^1 u(t) \log |t - z| dt + \log |z - 2|$$

for a to-be-determined  $u$ . On  $-1 < x < 1$  this satisfies

$$\frac{1}{\pi} \int_{-1}^1 u(t) \log |t - x| dt = \kappa - \log(2 - x)$$

We can solve this equation explicitly. By differentiating we see that  $u$  satisfies the following:

$$\mathcal{H}u(x) = f'(x) = \frac{1}{2 - x}$$

We now use the inverse Hilbert formula to determine:

$$u(x) = \frac{1}{\sqrt{1-x^2}} \mathcal{H}\left[\frac{1}{\diamond - 2} \sqrt{1-\diamond^2}\right](x) + \frac{D}{\sqrt{1-x^2}}$$

Using the usual procedure of taking an obvious ansatz and subtracting off the singularities at poles and  $\infty$  we find that:

$$\mathcal{C}\left[\frac{\sqrt{1-\diamond^2}}{x-2}\right](z) = \underbrace{\frac{\sqrt{z-1}\sqrt{z+1}}{2i(z-2)}}_{\text{ansatz}} - \underbrace{\frac{\sqrt{3}}{2i(z-2)}}_{\text{remove pole near } z=2} - \underbrace{\frac{1}{2i}}_{\text{remove constant at } \infty}$$

This implies that

$$\mathcal{H}\left[\frac{1}{\diamond - 2} \sqrt{1-\diamond^2}\right](x) = -i(C^+ + C^-)\left[\frac{1}{\diamond - 2} \sqrt{1-\diamond^2}\right](x) = \frac{\sqrt{3}}{x-2} + 1$$

in other words,

$$u = \frac{\sqrt{3}}{(x-2)\sqrt{1-x^2}} + \frac{D}{\sqrt{1-x^2}}$$

We still need to determine  $D$ . This is chosen so that we tend to  $\log|z|$  near  $\infty$ . In particular, we know that

$$\frac{1}{\pi} \int_{-1}^1 \log|z-x| u(x) dx \sim \frac{\int_{-1}^1 u(x) dx}{\pi} \log|z|$$

so we need to choose  $D$  so that  $\int_{-1}^1 u(x) dx = 0$  and only the  $\log|z-2|$  singularity appears near  $\infty$ . We first note that

$$\begin{aligned} C\left[\frac{1}{(\diamond - 2)\sqrt{1-\diamond^2}}\right](z) &= \underbrace{\frac{i}{2\sqrt{z-1}\sqrt{z+1}(z-2)}}_{\text{ansatz}} - \underbrace{\frac{i}{2\sqrt{3}(z-2)}}_{\text{remove pole}} \\ &= -\frac{i}{2\sqrt{3}z} + O(z^{-2}) \end{aligned}$$

near  $\infty$ . Since

$$Cw(z) = \frac{\int_{-1}^1 w(x) dx}{-2\pi iz} + O(z^{-2})$$

we can infer the integral from the Cauchy transform and find that

$$\int_{-1}^1 \frac{1}{(x-2)\sqrt{1-x^2}} dx = -\frac{\pi}{\sqrt{3}}$$

On the other hand,

$$\int_{-1}^1 \frac{dx}{\sqrt{1-x^2}} = \pi$$

Thus we require  $D = 1$  and have the solution:

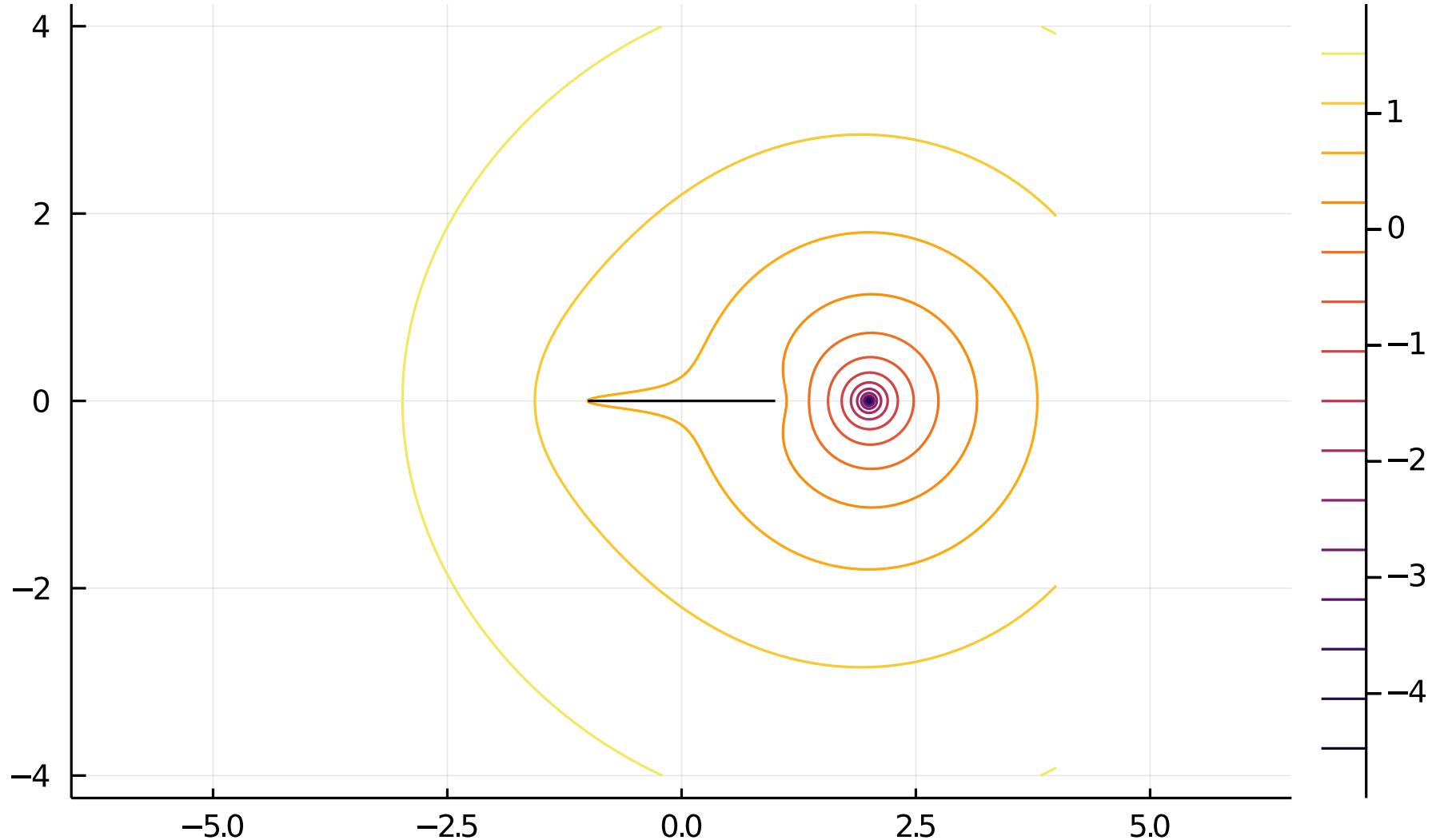
$$u(x) = \frac{\sqrt{3}}{(x-2)\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$$

*Demonstration*

```
x = Fun()
u = sqrt(3)/((x-2)*sqrt(1-x^2)) + 1/sqrt(1-x^2)
v = z -> logkernel(u, z) + log(abs(z-2))

xx = yy = -4:0.011:4
V = v.(xx' .+ im*yy)

contour(xx, yy, V;ratio=1.0)
plot!(domain(x); color=:black, legend=false)
```



We can compute  $\kappa$  numerically:

$v(0.2)$

0.6238107163648714