Applied Complex Analysis (2021)

1 Mastery solutions

$$f(x) = \sum_{k=0}^{\infty} c_k T_k(x)$$

$$c_k = \frac{\langle f, T_k\rangle}{\langle T_k, T_k\rangle}$$

$$\langle f, g\rangle = \int_{-1}^{1} \frac{f(x)g(x)}{\sqrt{1-x^2}} \mathrm{d}x$$

$$f_n(x) = \sum_{k=0}^{n-1} c_k T_k(x)$$

$$p_n(x) = \sum_{k=0}^{n-1} a_k T_k(x)$$

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right)$$

$$T_{(2j+1)n}(x_k) = \cos\left((2j+1)n\frac{2k-1}{2n}\pi\right) = 0, \qquad j \geq 0.$$

$$\begin{aligned} T_{2jn\pm m}(x_k) &= \cos\left((2jn \pm m)\frac{2k-1}{2n}\pi\right) \\ &= (-1)^j T_m(x_k) \end{aligned}$$

$$\begin{aligned} f(x) = &\sum_{m=0}^{n-1} c_m T_m(x) + \sum_{j=0}^{\infty} c_{(2j+1)n} T_{(2j+1)n}(x) + \sum_{j=1}^{\infty} c_{2jn} T_{2jn}(x) \\ &+ \sum_{m=1}^{n-1}\sum_{j=1}^{\infty} [c_{2jn+m} T_{2jn+m}(x) + c_{2jn-m} T_{2jn-m}(x)] \end{aligned}$$

$$\begin{aligned} f(x_k) &= \sum_{m=0}^{n-1} c_m T_m(x_k) + \sum_{j=1}^{\infty} c_{2jn}(-1)^j T_0(x) + \sum_{m=1}^{n-1} T_m(x_k) \sum_{j=1}^{\infty} (-1)^j \left[c_{2jn+m} + c_{2jn-m}\right] \\ &= p_n(x_k) \\ &= \sum_{m=0}^{n-1} a_m T_m(x_k) \end{aligned}$$

$$a_0 = \sum_{j=0}^{\infty} c_{2jn}(-1)^j$$

for $1 \leq m \leq n-1$

$$a_m = c_m + \sum_{j=1}^{\infty} (-1)^j \left[c_{2jn+m} + c_{2jn-m}\right]$$

$$\begin{aligned}
p_n(x) - f(x) &= \sum_{k=0}^{n-1} a_k T_k(x) - \sum_{k=0}^{\infty} c_k T_k(x) \\
&= \sum_{k=0}^{n-1} \left(a_k - c_k\right) T_k(x) - \sum_{k=n}^{\infty} c_k T_k(x) \\
&= \sum_{k=0}^{n-1} \left(a_k - c_k\right) T_k(x) - \left(f(x) - f_n(x)\right)
\end{aligned}$$