

1 Solution Sheet 1

1.1 Problem 1.1

1.1.1 1.

Use fundamental theorem of algebra: a polynomial is a constant times a product of terms like $z - \lambda_k$, where λ_k are the roots. In this case, the roots are a times the quartic-root of -1 , hence this gives us:

$$z^4 + a^4 = (z - ae^{i\pi/4})(z - ae^{3i\pi/4})(z - ae^{5i\pi/4})(z - ae^{7i\pi/4})$$

We are only interested in the root $ae^{i\pi/4}$, thus we simplify the expression

$$\begin{aligned} \frac{z^3 \sin z}{z^4 + a^4} &= \frac{z^3 \sin z}{(z - ae^{3i\pi/4})(z - ae^{5i\pi/4})(z - ae^{7i\pi/4})} \frac{1}{z - ae^{i\pi/4}} \\ &= \frac{a^3 e^{3i\pi/4} \sin(ae^{i\pi/4})}{a^3 (e^{i\pi/4} - e^{3i\pi/4})(e^{i\pi/4} - e^{5i\pi/4})(e^{i\pi/4} - e^{7i\pi/4})} \frac{1}{z - ae^{i\pi/4}} + O(1) \end{aligned}$$

Therefore,

$$\operatorname{Res}_{z=ae^{i\pi/4}} \frac{z^3 \sin z}{z^4 + a^4} = \frac{e^{3i\pi/4} \sin(ae^{i\pi/4})}{e^{3i\pi/4} (1 - e^{i\pi/2})(1 - e^{i\pi})(1 - e^{3i\pi/2})} = \frac{\sin(ae^{i\pi/4})}{(1-i)(2)(1+i)} = \frac{\sin(ae^{i\pi/4})}{4}$$

Let's check our work: we compare the numerically calculated residue to the formula we have derived:

```
using ApproxFun, Plots, ComplexPhasePortrait, LinearAlgebra, DifferentialEquations
a = 2.0
γ = Circle(a*exp(im*π/4), 0.1)
f = Fun(z -> z^3*sin(z)/(z^4+a^4), γ)
sum(f)/(2π*im), sin(a*exp(im*π/4))/4

(0.5378838853348213 + 0.07544036746694016im, 0.5378838853348215 + 0.0754403674669402im)
```

2. We have

$$(z^2 - 1)^2 = (z - 1)^2(z + 1)^2$$

Thus this is a slightly more challenging since it has a double pole. But we can expand using Geometric series:

$$\frac{z+1}{(z^2-1)^2} = \frac{1}{(z-1)^2} \frac{1}{2-(1-z)} = \frac{1}{(z-1)^2} \frac{1}{2} (1+(1-z)/2 + O((1-z)^2)) = \frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} + O(1)$$

Thus the residue is the negative-first Laurent coefficient, namely $-\frac{1}{4}$.

We again check our work:

```

γ = Circle(1, 0.1)
f = Fun(z -> (z+1)/(z^2-1)^2, γ)
sum(f)/(2π*im) # almost equals -1/4

-0.25000000000000023 - 1.170965239236949e-16im

```

3.

$$\frac{z^2 e^z}{z^3 - a^3} = \frac{z^2 e^z}{(z-a)(z^2 + az + a^2)}$$

We thus need only evaluate the extra term at $z = a$:

$$\operatorname{Res}_{z=a} \frac{z^2 e^z}{z^3 - a^3} = \frac{e^a}{3}$$

Let's check:

```

a = 2.0

γ = Circle(a, 0.1)
f = Fun(z -> z^2*exp(z)/(z^3-a^3), γ)
sum(f)/(2π*im), exp(a)/3

(2.4630186996435506 + 4.676730094089873e-16im, 2.46301869964355)

```

1.2 Problem 1.2

1.2.1 1.

Change of variables $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta = izd\theta$, $\cos \theta = \frac{z+z^{-1}}{2}$ gives

$$\int_0^{2\pi} \frac{d\theta}{5-4\cos \theta} = -i \oint \frac{dz}{5z-2z^2-2} = i \oint \frac{dz}{(z-2)(2z-1)} = -\pi \operatorname{Res}_{z=1/2} \frac{1}{(z-2)(z-1/2)} = \frac{2}{3}\pi$$

```

θ = Fun(0 .. 2π)
sum(1/(5-4cos(θ))) , 2π/3

(2.0943951023931966, 2.0943951023931953)

```

1.2.2 2.

Use $\cos 2\theta = \frac{e^{2i\theta} + e^{-2i\theta}}{2} = \frac{z^2 + z^{-2}}{2}$ to get

$$\int_0^{2\pi} \frac{\cos 2\theta d\theta}{5 + 4\cos \theta} = -\frac{i}{4} \oint \frac{(z^4 + 1)dz}{z^2(z + 1/2)(z + 2)} = \frac{\pi}{2} \left(\text{Res}_{z=-1/2} + \text{Res}_{z=0} \right) \frac{z^4 + 1}{z^2(z + 2)(z + 1/2)} = \frac{\pi}{6}$$

```

θ = Fun(0 .. 2π)
sum(cos(2θ)/(5+4cos(θ))), π/6

(0.5235987755982991, 0.5235987755982988)

```

1.2.3 3.

Because the integrand is analytic and $O(z^{-2})$ in the upper half plane, we can use the residue theorem in the upper half plane using

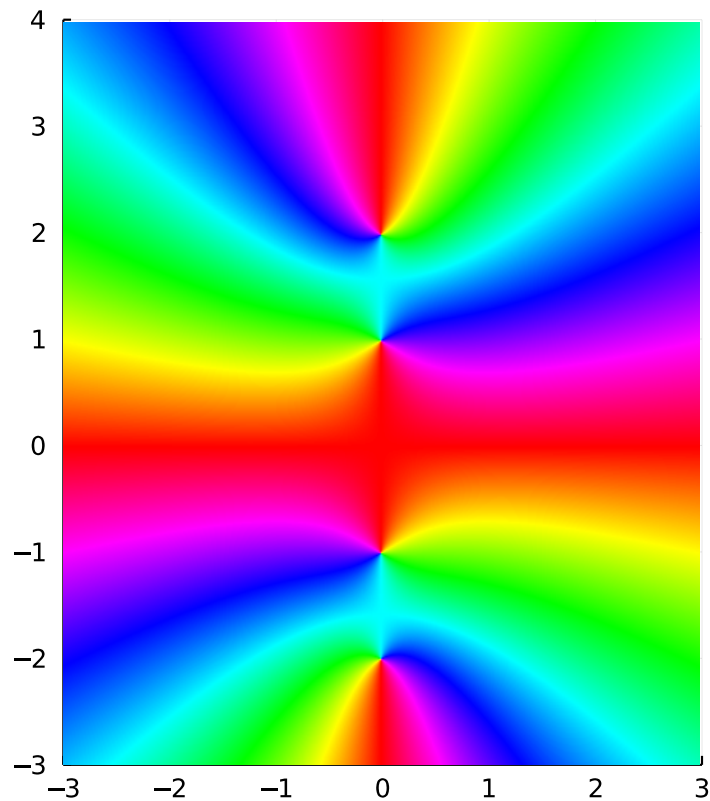
$$\frac{1}{(z^2 + 1)(z^2 + 4)} = \frac{1}{(z + i)(z - i)(z + 2i)(z - 2i)}$$

This has two poles in the upper half plane:

```

phaseplot(-3..3, -3..4, z-> 1/((z^2+1)*(z^2+4)))

```



$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} dx &= 2\pi i \left(\text{Res}_{z=i} + \text{Res}_{z=2i} \right) \frac{1}{(z^2+1)(z^2+4)} \\ &= 2\pi i \left(\frac{1}{2i3i(-i)} + \frac{1}{3ii4i} \right) = \pi/6\end{aligned}$$

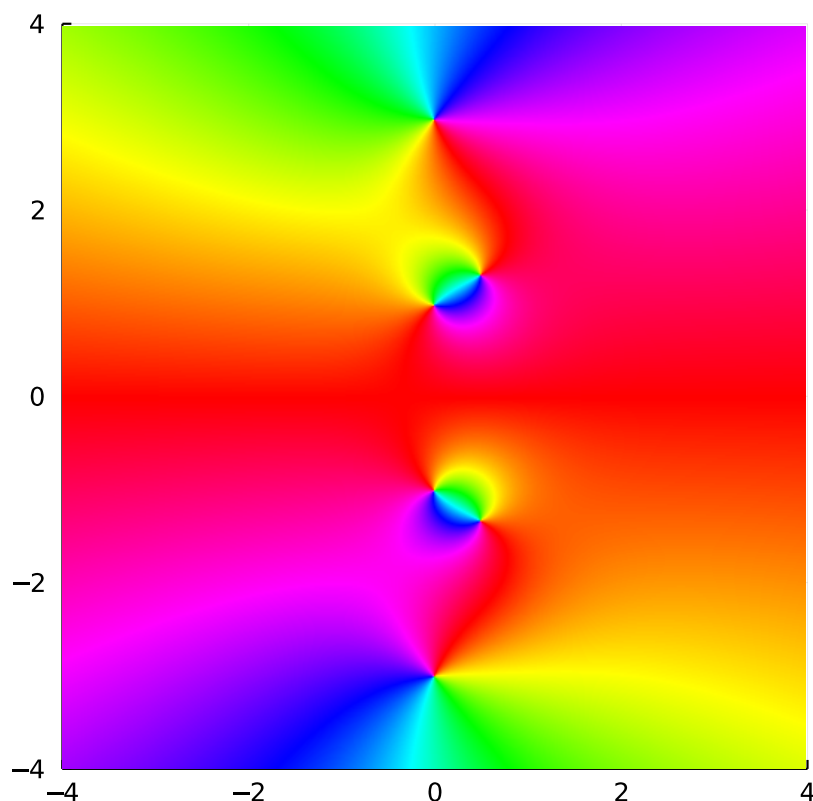
We can check the result numerically:

```
x = Fun( Line())
sum(1/((x^2+1)*(x^2+4))), pi/6
(0.5235987755982988, 0.5235987755982988)
```

1.2.4 4.

Again, decays like $O(z^{-2})$ in upper half plane so we can use residue calculus. This integrand has poles at $z = i$ and $z = 3i$:

```
phaseplot(-4..4, -4..4, z-> (z^2 - z + 2) / (z^4 + 10z^2 + 9))
```



The residues are $(-1 - i)/16$ and $(3 - 7i)/48$ giving the answer

$$\frac{5\pi}{12}$$

which we check numerically:

```
f = x -> (x^2 - x + 2) / (x^4 + 10x^2 + 9)
sum(Fun(f, -10.000..10.000)), 5pi/12
(1.3087969390010812, 1.3089969389957472)
```

1.2.5 5.

$$\int_{-\infty}^{\infty} \frac{1}{x+i} dx$$

Trick question: it's undefined because the integral doesn't decay fast enough. But what if I had asked for

$$\oint_{-\infty}^{\infty} \frac{1}{x+i} dx?$$

We can't use residue theorem since it doesn't decay fast enough, but we can use, with a contour $C_R = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}$

$$\oint_{[-M,M] \cup C_R} \frac{1}{z+i} = 0$$

Further, by direct substitution, we have

$$\int_{C_R} \frac{1}{z+i} dz = i \int_0^\pi R \frac{e^{i\theta}}{Re^{i\theta} + i} d\theta$$

Letting $R \rightarrow \infty$, the integrand tends to one uniformly hence

$$\int_{C_R} \frac{1}{z+i} dz \rightarrow i \int_0^\pi d\theta = i\pi.$$

Therefore, we have

$$\int_{-\infty}^{\infty} \frac{1}{x+i} dx = \lim_{M \rightarrow \infty} \int_{-M}^M \frac{1}{x+i} dx = -i\pi.$$

Indeed:

```
x = Fun(-1000 .. 1000)
sum(1/(x+im))
```

```
1.1102230246251565e-15 - 3.1395926542565897im
```

1.2.6 6.

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + x + 1} dx = \Im \int_{-\infty}^{\infty} \frac{e^{2ix}}{x^2 + x + 1} dx$$

This can be deformed in the upper half plane with a pole at $\frac{-1+i\sqrt{3}}{2}$, using residue calculus gives us

$$-\frac{2\pi \sin 1}{\sqrt{3} e^{\sqrt{3}}}$$

```
x = Fun(-100 .. 100)
sum(sin(2x)/(x^2+x+1)), -2pi/sqrt(3) * sin(1)/exp(sqrt(3))
```

```
(-0.5400548830723215, -0.5400553569742235)
```

1.2.7 7.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 4} dx = \Re \int_{-\infty}^{\infty} \frac{e^{ix}}{x^2 + 4} dx$$

and residue calculus gives $\frac{\pi}{2e^2}$

```
M = 200
x = Fun(-M .. M)
sum(cos(x)/(x^2+4)), pi/(2*exp(2)) # converges if we make M even bigger

(0.21254026836701112, 0.21258416579381814)
```

1.2.8 8.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} dx = \frac{\pi}{e}$$

using Residue calculus. You need to appeal to Jordan's lemma to argue that it can still be done even with only $O(x^{-1})$ decay.

```
M = 10_000
x = Fun(-M .. M)
sum(x*sin(x)/(x^2+1)), pi/exp(1) # Converges if we make M even bigger

(1.1559177936504117, 1.1557273497909217)
```

1.2.9 9.

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx \quad \text{where} \quad a, b > 0$$

We have for $f(x) = \frac{e^{iax} - e^{ibx}}{x^2}$

$$\Re f(x) = \frac{\cos ax - \cos bx}{x^2}$$

Note that, since $\cos x = 1 + x^2/2 + O(x^4)$, the integrand is fine near zero:

$$\frac{\cos ax - \cos bx}{x^2} = \frac{(a - b)}{2} + O(x^2)$$

But $f(x)$ has a pole:

$$\frac{e^{iax} - e^{ibx}}{x^2} = \frac{i(a - b)}{x} + O(1)$$

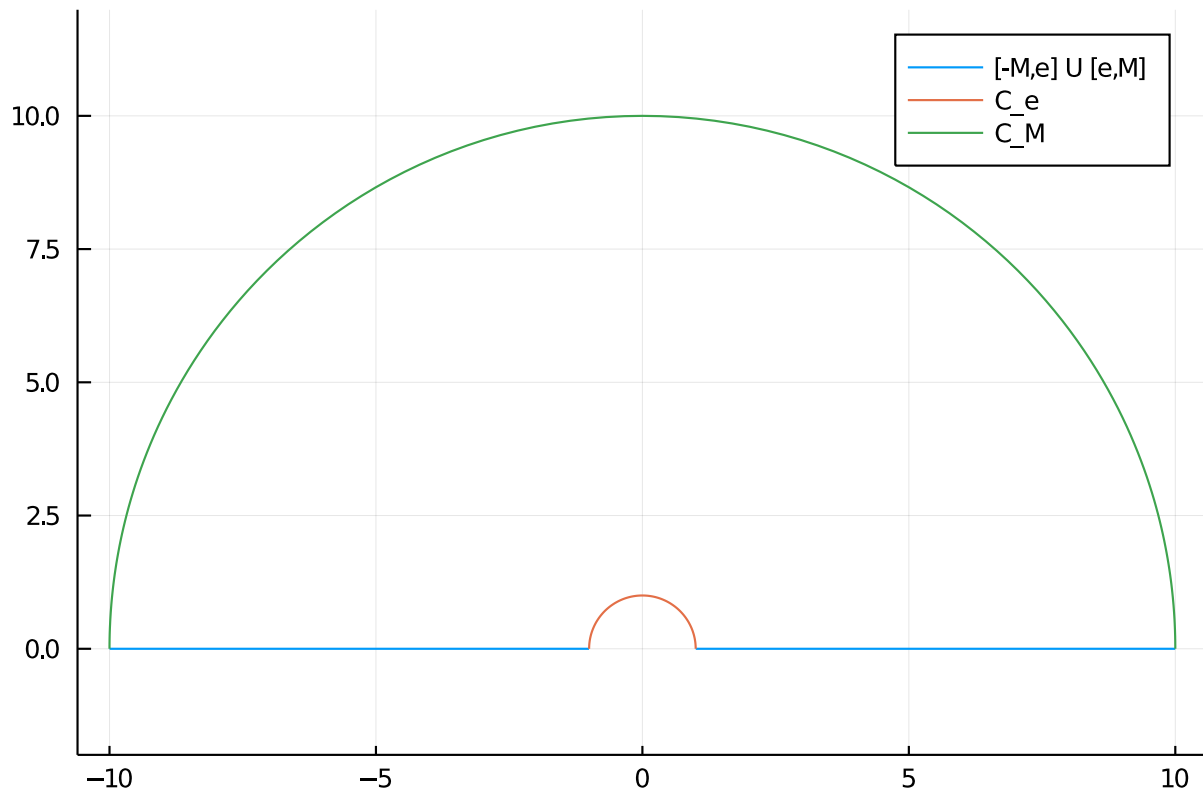
To rectify this, we need to be a bit more careful. First note that

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx = \lim_{\epsilon \rightarrow 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{\cos ax - \cos bx}{x^2} dx = \Re \int_{-\infty}^{\infty} f(x) dx$$

Then we construct a contour avoiding zero as follows:

```
M = 10
ε = 1.0
```

```
plot(Segment(-M, -ε) ∪ Segment(ε, M);label="[-M,e] ∪ [e,M]", ratio = 1.0)
plot!(Arc(0.,ε, (π,0.))); label="C_e"
plot!(Arc(0., M, (0,π))); label = "C_M"
```



Note that $\oint_{\gamma} f(z)dz = 0$,

$$\int_{C_\epsilon} \frac{e^{iaz} - e^{ibz}}{z^2} dz = \int_{\pi}^0 \frac{(b-a)e^{i\theta} + O(\epsilon)}{e^{i\theta}} d\theta \rightarrow (a-b)\pi$$

Also, as the integrand is $O(z^{-2})$ the integral over C_M vanishes as $M \rightarrow \infty$. We therefore get

$$\oint f(x)dx = (b-a)\pi$$

```
ε =0.001
M = 1_000.0
x = Fun(Segment(-M , -ε) ∪ Segment(ε, M))
a = 2.3; b = 3.8
sum((cos(a*x) - cos(b*x))/x^2),π*(b-a) # Converges if we make M bigger

(4.703250780477666, 4.71238898038469)
```

1.2.10 10.

Use binomial formula

$$\begin{aligned}
\int_0^{2\pi} (\cos \theta)^n d\theta &= \frac{1}{2^{n+1}} \oint (z + z^{-1})^n \frac{dz}{z} \\
&= \frac{1}{2^{n+1}} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \oint z^k z^{k-n} \frac{dz}{z} \\
&= \frac{1}{2^{n+1}} \sum_{k=0}^n \frac{n!}{k!(n-k)!} \oint z^{2k-n-1} dz
\end{aligned}$$

We only have a residue of $2k - n - 1 = -1$, that is, if $2k = n$. If n is odd, this can't happen (duh! the integral is symmetric with respect to θ). If it's even, then we have

$$\int_0^{2\pi} (\cos \theta)^n d\theta = \frac{\pi}{2^{n-1}} \frac{n!}{((n/2)!)^2}$$

```

θ = Fun(0 .. 2π)
n = 10;
sum(cos(θ)^n), π*factorial(1.0n)/(2^(n-1)*factorial(n/2)^2)

(1.5462526341887322, 1.5462526341887264)

```

1.3 Problem 2.1

By integrating around a rectangular contour with vertices at $\pm R$ and $\pi i \pm R$ and letting $R \rightarrow \infty$, show that:

$$\int_0^\infty \operatorname{sech} x dx = \frac{\pi}{2}$$

where $\operatorname{sech} x = \frac{2}{e^{-x} + e^x}$.

Recall $\operatorname{sech} x = \frac{2}{e^{-x} + e^x}$. This shows that $\operatorname{sech}(-x) = \operatorname{sech} x$. But we also have

$$\operatorname{sech}(x + i\pi) = \frac{2}{e^{-x-i\pi} + e^{x+i\pi}} = \frac{2}{-e^{-x} - e^x} = -\operatorname{sech} x$$

Thus we have

$$4 \int_0^\infty \operatorname{sech} x dx = \left[\int_{-\infty}^\infty + \int_{\infty+i\pi}^{-\infty+i\pi} \right] \operatorname{sech} z dz$$

We can approximate this using

$$\left[\int_{-R}^R + \int_R^{R+i\pi} + \int_{R+i\pi}^{-R+i\pi} + \int_{-R+i\pi}^{-R} \right] \operatorname{sech} z dz = 2\pi i \operatorname{Res}_{z=\frac{i\pi}{2}} \operatorname{sech} z = 2\pi$$

since, for $z_0 = \frac{i\pi}{2}$, we have

$$\operatorname{sech} z = \frac{1}{\cos iz} = \frac{1}{-i \sin iz_0(z - z_0) + O(z - z_0)^2} = -\frac{i}{(z - z_0)} + O(1)$$

Finally, we need to show that the limit as $R \rightarrow \infty$ tends to the right value. In this case, it follows since

$$\left| \int_R^{R+i\pi} \operatorname{sech} z dz \right| \leq \frac{2\pi e^{-R}}{1 - e^{-2R}} \rightarrow 0$$

(and by symmetry for $\int_{-R+i\pi}^{-R}$.)

1.4 Problem 2.2

Show that the Fourier transform of $\operatorname{sech} x$ satisfies

$$\int_{-\infty}^{\infty} e^{ikx} \operatorname{sech} x dx = \pi \operatorname{sech} \frac{\pi k}{2}$$

Define

$$f(z) = e^{ikz} \operatorname{sech} z = \frac{2e^{(1+ik)z}}{e^{2z} + 1}$$

In this case, we have the symmetry

$$f(x + i\pi) = -e^{-k\pi} e^{ikx} \operatorname{sech} x = -e^{-k\pi} f(x)$$

`k = 2.0`

`f = z -> exp(im*k*z)*sech(z)`
`-exp(-k*pi)*f(2.0), f(2.0+im*pi)`

`(0.00032444937189257726 + 0.0003756543878221788im, 0.0003244493718925772 +`
`0.00037565438782217884im)`

In other words, we have

$$(1 + e^{-k\pi}) \int_{-\infty}^{\infty} f(x) dx = \left(\int_{-\infty}^{\infty} + \int_{\infty+i\pi}^{-\infty+i\pi} \right) f(z) dz$$

By similar logic as above, we can show that the integral over the rectangular contour converges to this.

Again, the only pole inside is at $z = \frac{i\pi}{2}$, where the residue is $-ie^{\frac{-\pi k}{2}}$. Thus we have

$$\int_{-\infty}^{\infty} f(x) dx = \frac{2\pi e^{\frac{-\pi k}{2}}}{1 + e^{-k\pi}} = \pi \operatorname{sech} \frac{\pi k}{2}$$