

Applied Complex Analysis (2021)

1 Lecture 10: Branch cuts

We now discuss functions with branch cuts

1. Logarithm: $\log z$ with a cut on $(-\infty, 0]$
2. Powers: z^α with a cut on $(-\infty, 0]$
3. Combinations: $\sqrt{z-1}\sqrt{z+1}$ with a cut on $[-1, 1]$

This is a step towards Cauchy transforms on cuts, for recovering a holomorphic function from its behaviour on a cut. This lecture we discuss:

1. Complex logarithm
2. Algebraic powers
3. Inferring analyticity from continuity

1.1 Complex logarithm

One way to define the logarithm is as $\log |z| + i \arg z$. We find it more convenient in order to understand its behaviour to define it as an integral:

Definition (Complex Logarithm)

$$\log z := \int_1^z \frac{1}{\zeta} d\zeta$$

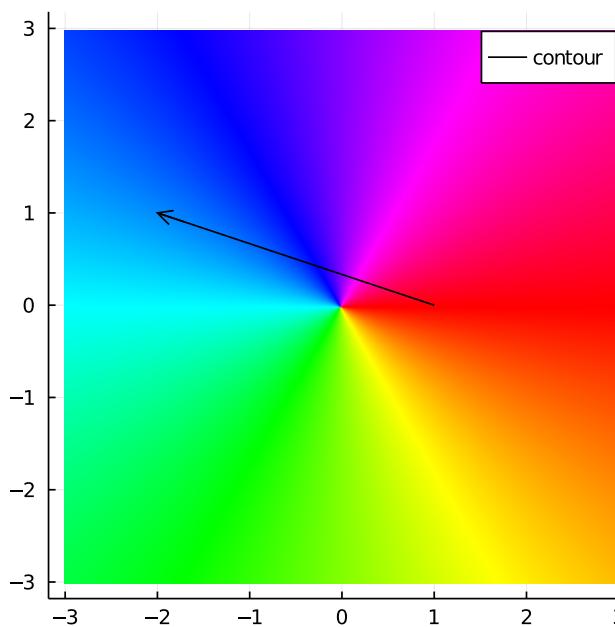
where the integral is understood to be on a straight line segment, that is

$$\log z := \int_{\gamma_z} \frac{1}{\zeta} d\zeta$$

where $\gamma_z(t) = 1 + (z - 1)t$ for $0 \leq t \leq 1$.

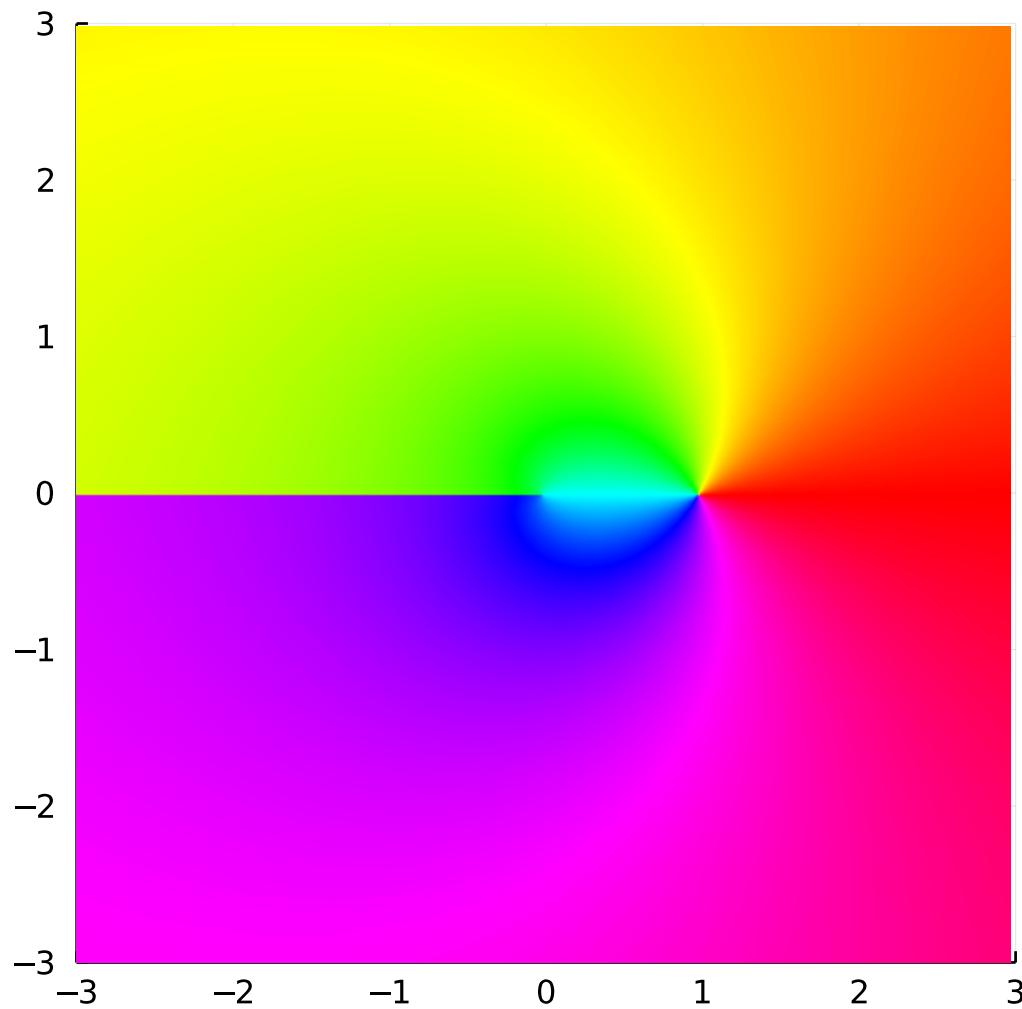
Demonstration this shows the integral path for a point z . We see how the path avoids the pole of ζ^{-1} at the origin:

```
using Plots, ComplexPhasePortrait, ApproxFun  
z = -2 + 1.0im  
phaseplot(-3..3, -3..3, ζ -> 1/ζ)  
t = 0:0.1:1  
γ = 1 .+ (z-1)*t  
plot!(real.(γ), imag.(γ); color=:black, label="contour", arrow=true)
```



This is well-defined apart from $z \in (-\infty, 0]$, where there is a pole on the contour. This induces a *branch cut*: a jump in the value of the function, which can be clearly seen from a phase portrait:

```
phaseplot(-3..3, -3..3, z -> log(z))
```



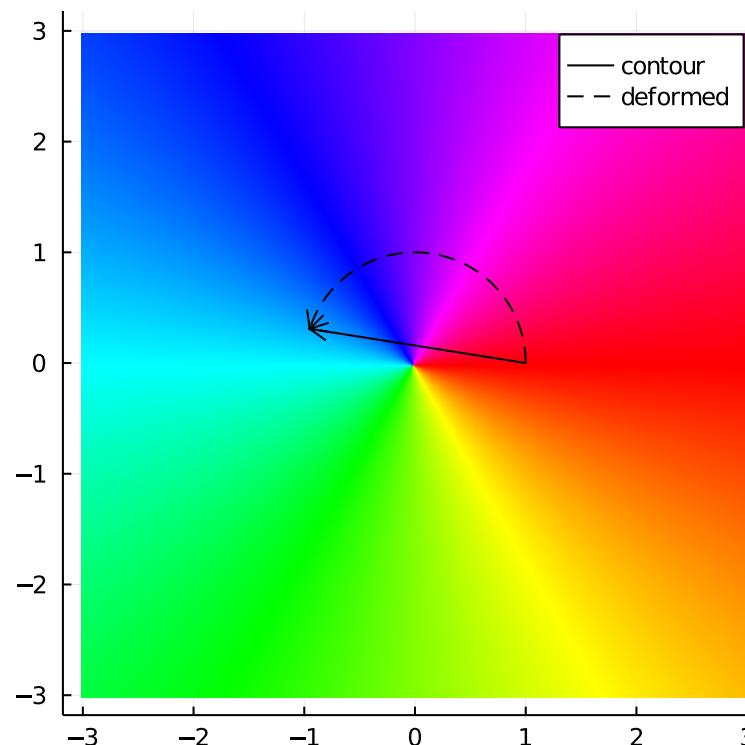
We see that the limits from above and below exist: we can define

$$\log_+ x := \lim_{\epsilon \rightarrow 0^+} \log(x + i\epsilon)$$

$$\log_- x := \lim_{\epsilon \rightarrow 0^+} \log(x - i\epsilon)$$

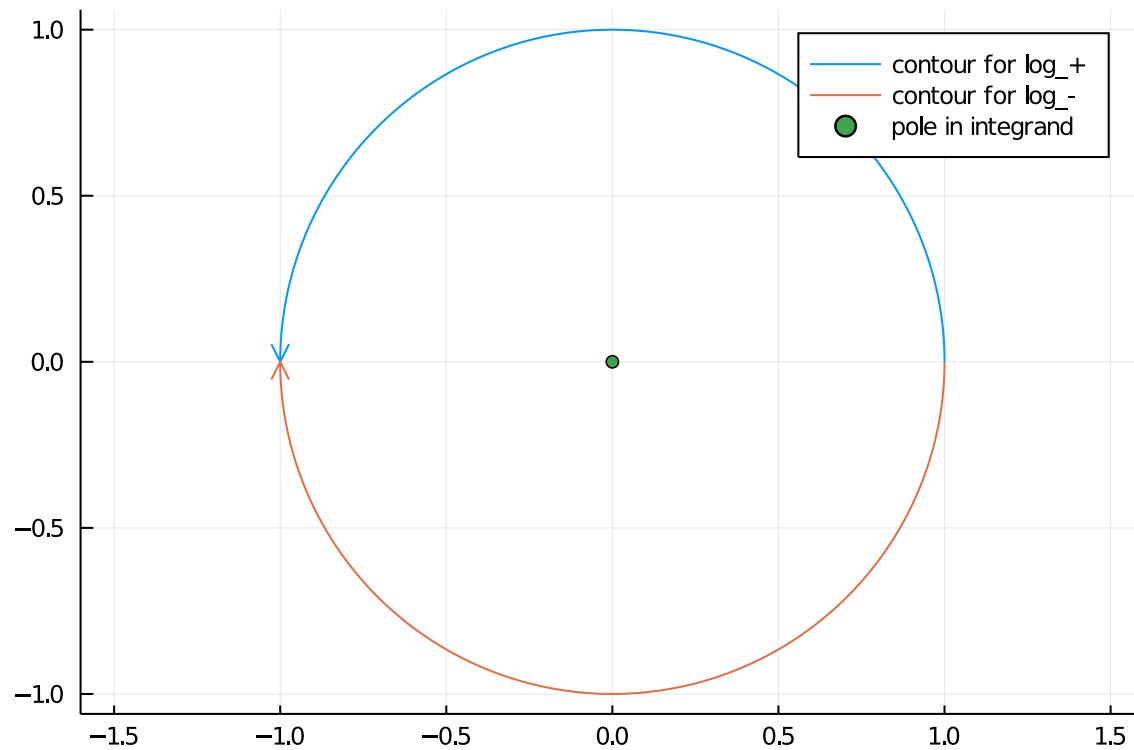
By deformation of contours, the value of the integrals is independent of the path. Here we calculate the integral on an arc:

```
θ = range(0, stop=0.9π, length=100)
a = exp.(im*θ)
z = exp(0.9*π*im)
γ = 1 .+ (z-1)*t
phaseplot(-3..3, -3..3, ζ -> 1/ζ)
plot!(real.(γ), imag.(γ); color=:black, label="contour", arrow=true)
plot!(real.(a), imag.(a), color=:black, linestyle=:dash,
label="deformed", arrow=true)
```



This works all the way to the negative real axis. Thus we can calculate $\log_{\pm}(-1)$ using integrals over half circles:

```
plot(Arc(0.,1.,(π,0)); label="contour for log_+", arrow=true,
ratio=1.0)
plot!(Arc(0.,1.,(-π,0)); label="contour for log_-", arrow=true)
scatter!([0],[0]; label="pole in integrand")
```



Combining the two contours we have the *subtractive jump* (for any $x < 0$)

$$\log_+ x - \log_- x = \oint \frac{d\zeta}{\zeta} = 2\pi i$$

We can establish some properties. First we show that $\log z = -\log \frac{1}{z}$ by considering the change of variables $\zeta = \frac{1}{z}$. Because $\gamma_z(t)^{-1}$ stays uniformly in the lower-half plane, we can deform it to a straight contour, which gives us the result:

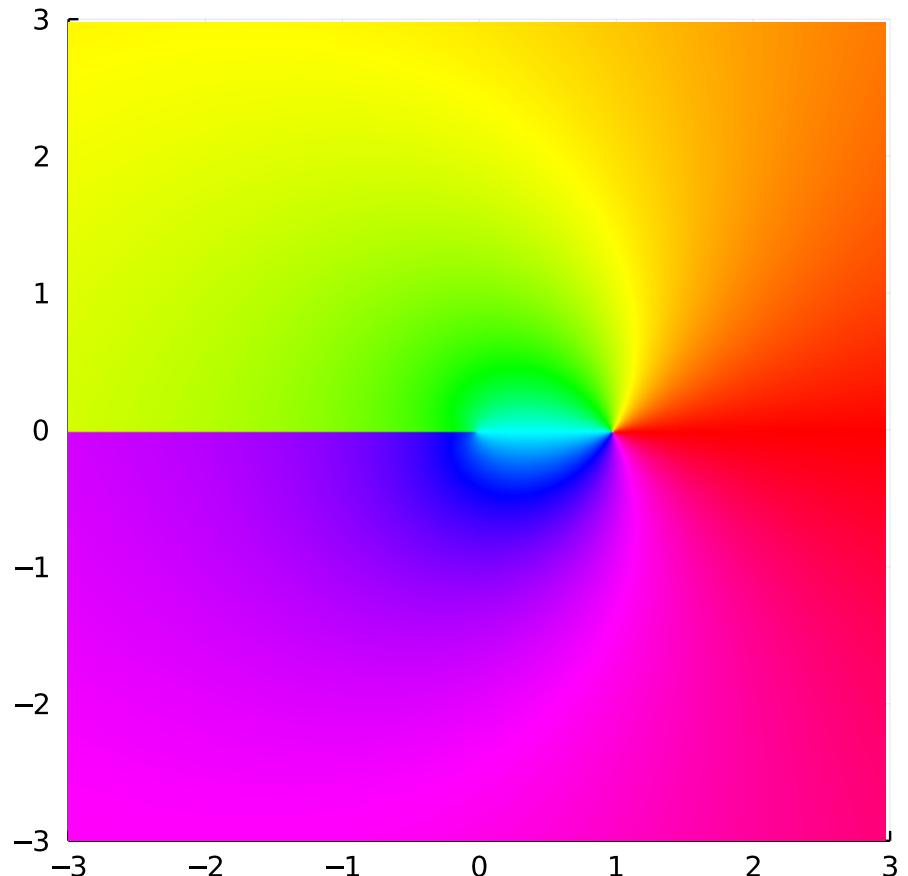
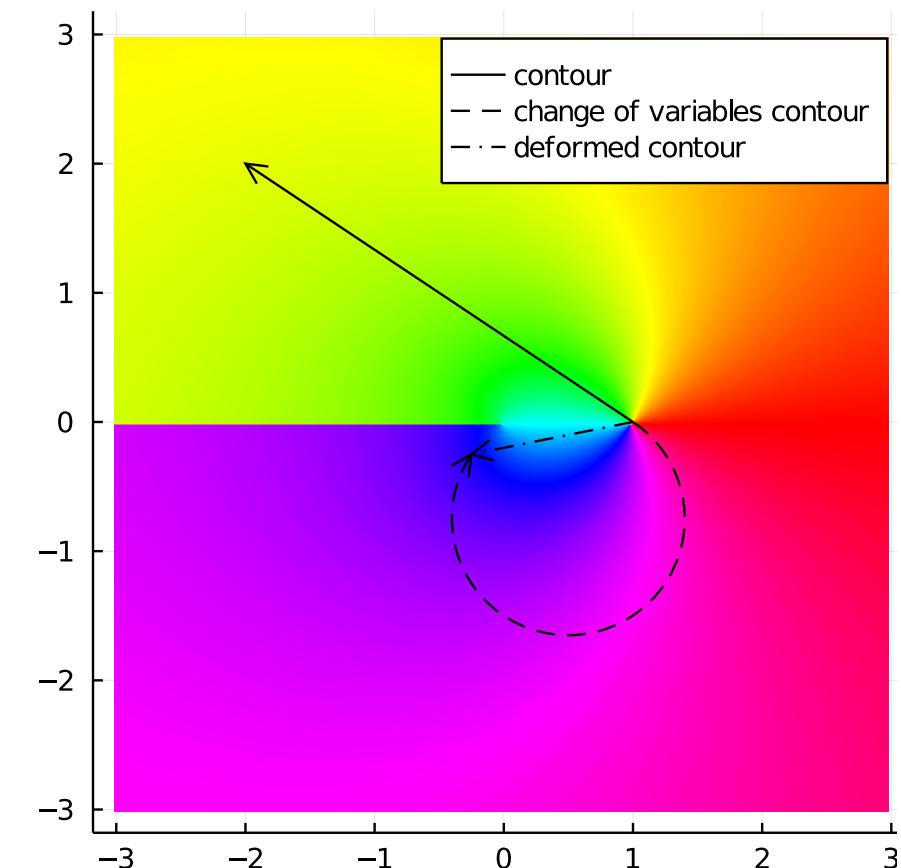
$$\log z = \int_{\gamma_z} \frac{d\zeta}{\zeta} = - \int_{\frac{1}{\zeta} \circ \gamma_z} \frac{ds}{s} = - \int_{\gamma_{z^{-1}}} \frac{ds}{s} = -\log z^{-1}$$

Here's a plot of the relevant contours:

```
phaseplot(-3..3, -3..3, z -> log(z))
z = -2 + 2im
γ = (z,t) -> 1 + t*(z-1)
tt = range(0,stop=1,length=100)
plot!(real.(γ.(z,tt)), imag.(γ.(z,tt)); color=:black,
label="contour", arrow=true)
plot!(real.(1 ./ γ.(z,tt)), imag.(1 ./ γ.(z,tt)); color=:black,
linestyle=:dash, arrow=true, label="change of variables contour")
plot!(real.(γ.(1/z,tt)), imag.(γ.(1/z,tt)); color=:black,
linestyle=:dashdot, arrow=true, label="deformed contour")
```

Here we see by looking at the phase plot that the two functions match:

```
phaseplot(-3..3, -3..3, z -> -log(1/z))
```



1.2 Algebraic powers

We define algebraic powers in terms of logarithms, which gives us what we need to know about their jumps.

Definition (algebraic power)

$$z^\alpha := e^{\alpha \log z}$$

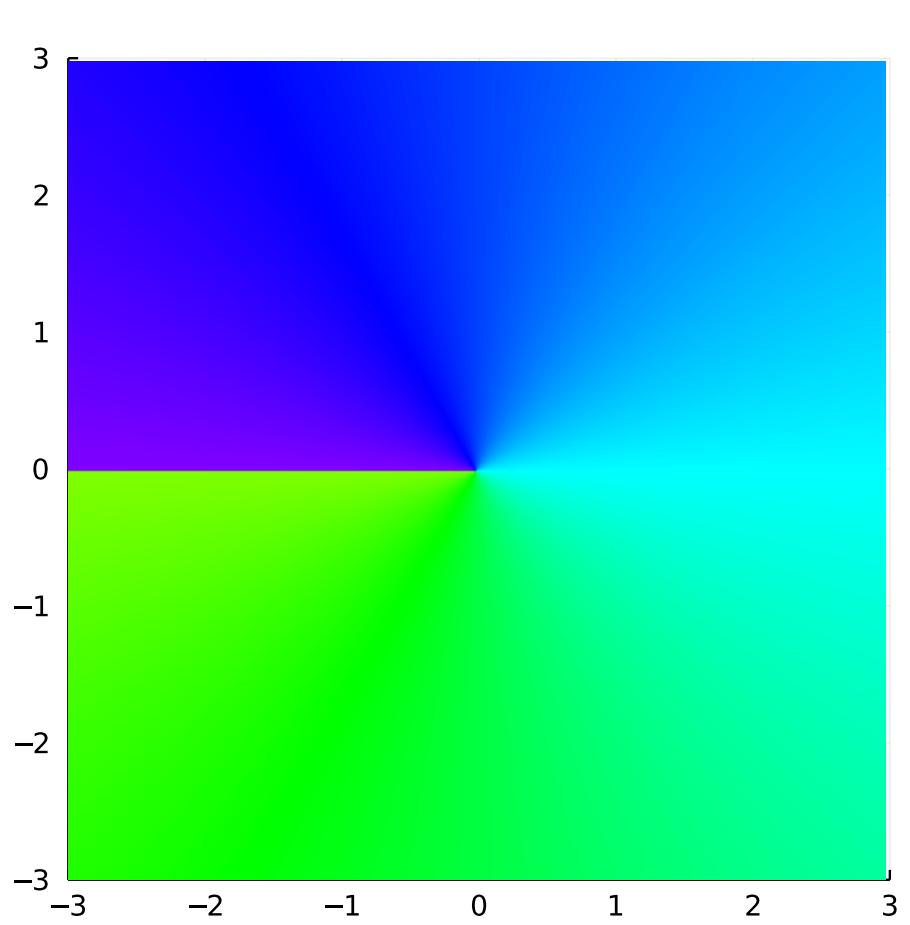
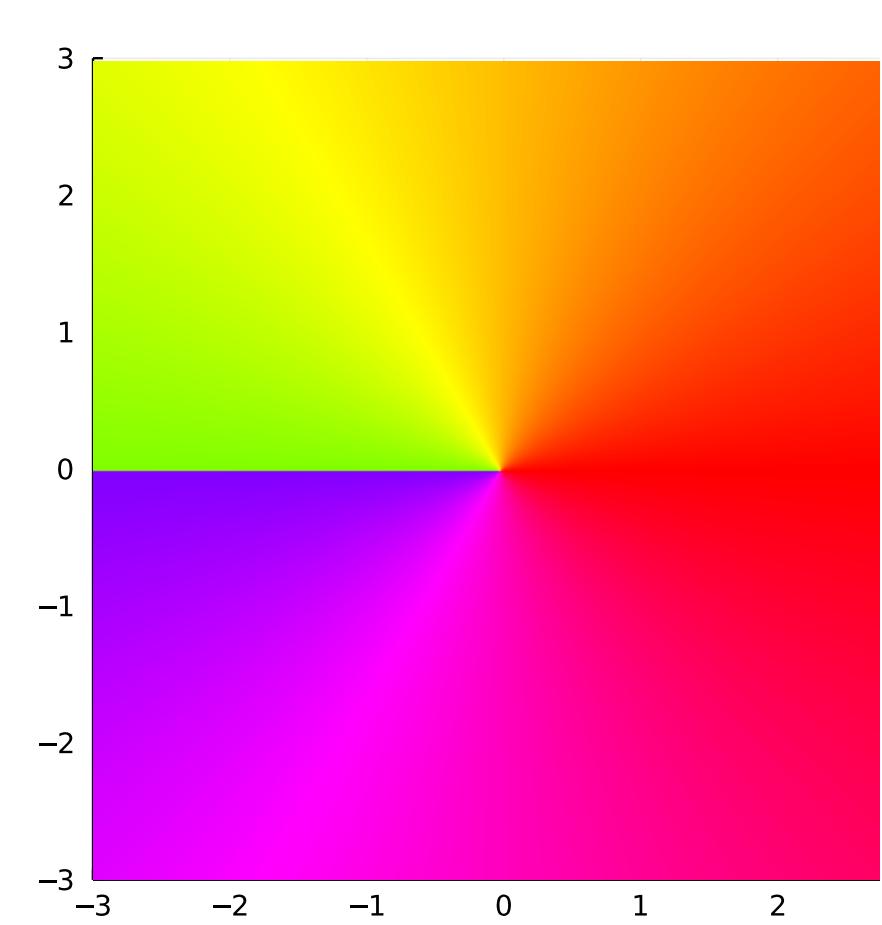
Note, for example, when $\alpha = 1/2$, $\sqrt{z} \equiv z^{1/2}$ is only one solution to $y^2 = z$.

Here are phase plots showing that \sqrt{z} also has a branch cut on $(-\infty, 0]$ on both of its branches:

$\alpha = 0.5$

`phaseplot(-3..3, -3..3, z -> z^\alpha)`

`phaseplot(-3..3, -3..3, z -> -z^\alpha)`



On the branch cut along $(-\infty, 0]$ it has the jump:

$$\frac{x_+^\alpha}{x_-^\alpha} = e^{\alpha(\log_+ x - \log_- x)} = e^{2\pi i \alpha}$$

In particular,

$$\sqrt{x_+} = -\sqrt{x_-} = i\sqrt{|x|}$$

These are *multiplicative jumps*. We also have a *subtractive jump*:

$$\begin{aligned} x_+^\alpha - x_-^\alpha &= e^{\alpha \log_+ x} - e^{\alpha \log_- x} = e^{\alpha \log(-x) + i\pi\alpha} - e^{\alpha \log(-x) - i\pi\alpha} \\ &= 2i(-x)^\alpha \sin \pi\alpha \end{aligned}$$

and an *additive jump*:

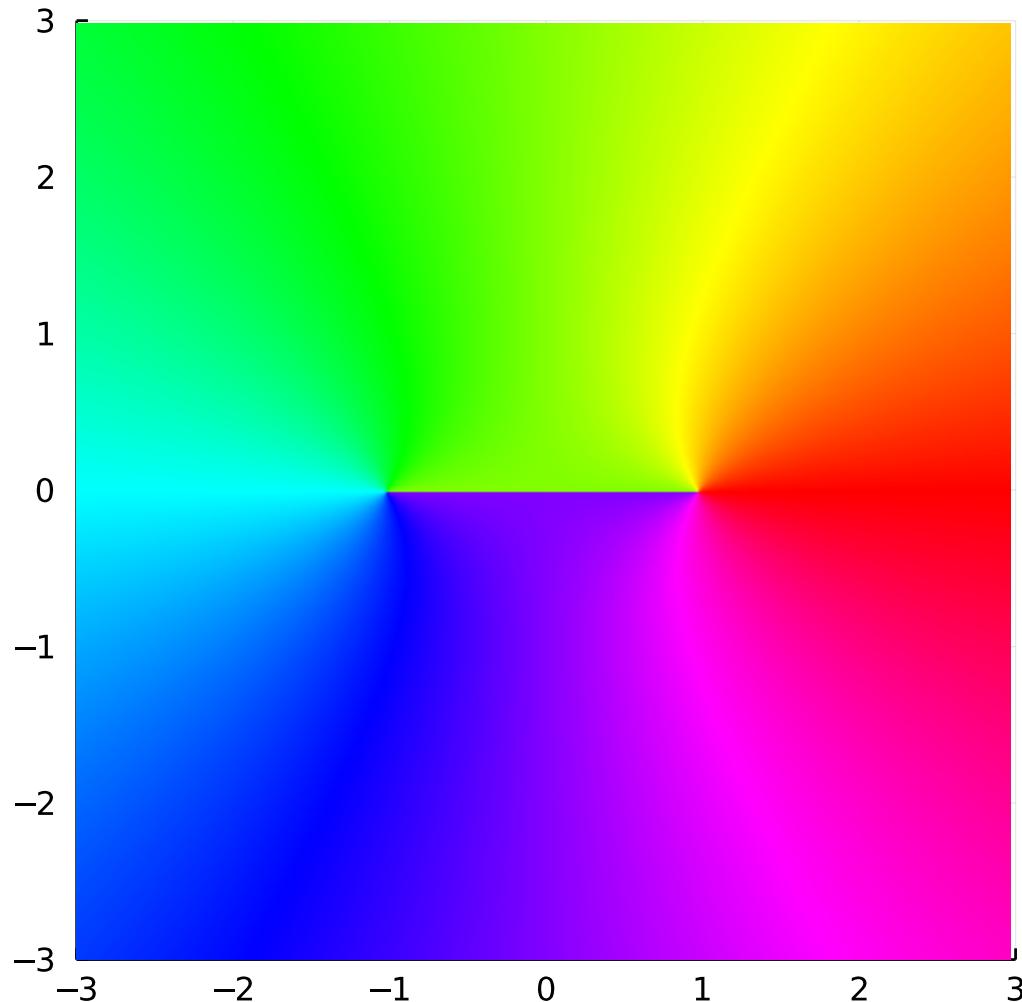
$$x_+^\alpha + x_-^\alpha = 2(-x)^\alpha \cos \pi\alpha$$

In particular, for $x < 0$,

$$\begin{aligned} \sqrt{x_+} - \sqrt{x_-} &= 2i\sqrt{-x} \\ \sqrt{x_+} + \sqrt{x_-} &= 0 \end{aligned}$$

Let's look at another example: $\varphi(z) = \sqrt{z-1}\sqrt{z+1}$. Each square root term induces a jump: one on $(-\infty, -1]$ and one on $(-\infty, 1]$. Surprisingly these jumps cancel out, in fact (as we explain below) φ is analytic off $[-1, 1]$, as can be seen from the phase portrait:

```
φ = z → sqrt(z-1)*sqrt(z+1)  
phaseplot(-3..3, -3..3, φ)
```



For $-1 < x < 1$ we have the multiplicative jump:

$$\varphi_+(x) = \sqrt{x-1}_+ \sqrt{x+1} = -\sqrt{x-1}_- \sqrt{x+1} = -\varphi_-(x)$$

which gives the additive jump

$$\varphi_+(x) + \varphi_-(x) = 0$$

But we also have a *subtractive jump*:

$$\varphi_+(x) - \varphi_-(x) = (\sqrt{x-1}_+ - \sqrt{x-1}_-) \sqrt{x+1} = 2i\sqrt{1-x}\sqrt{x+1} = 2i\sqrt{1-x^2}$$

For $x < -1$ we actually have continuity:

$$\varphi_+(x) = \sqrt{x-1}_+ \sqrt{x+1}_+ = (-\sqrt{x-1}_-)(-\sqrt{x+1}_-) = \varphi_-(x)$$

This feature is what we use to show analyticity.

2 Inferred analyticity

Here we review properties of inferring analyticity from continuity.

Theorem (continuity on a curve implies analyticity) Let D be a domain and $\gamma \subset D$ a contour. Suppose f is analytic in $D \setminus \gamma$, and continuous on the interior of γ . Then f is analytic in $D \setminus \{\gamma(a), \gamma(b)\}$.

Sketch of Proof

For simplicity, suppose D is a circle of radius 2 and γ is the interval $[-1, 1]$. For any z off the interval, we can write

$$f(z) = \frac{1}{2\pi i} \oint_{\Gamma_x} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where Γ_x is a simple closed contour that surrounds z and passes through x in both directions:

$z = 0.2 + 0.2i$

$x = 0.1$

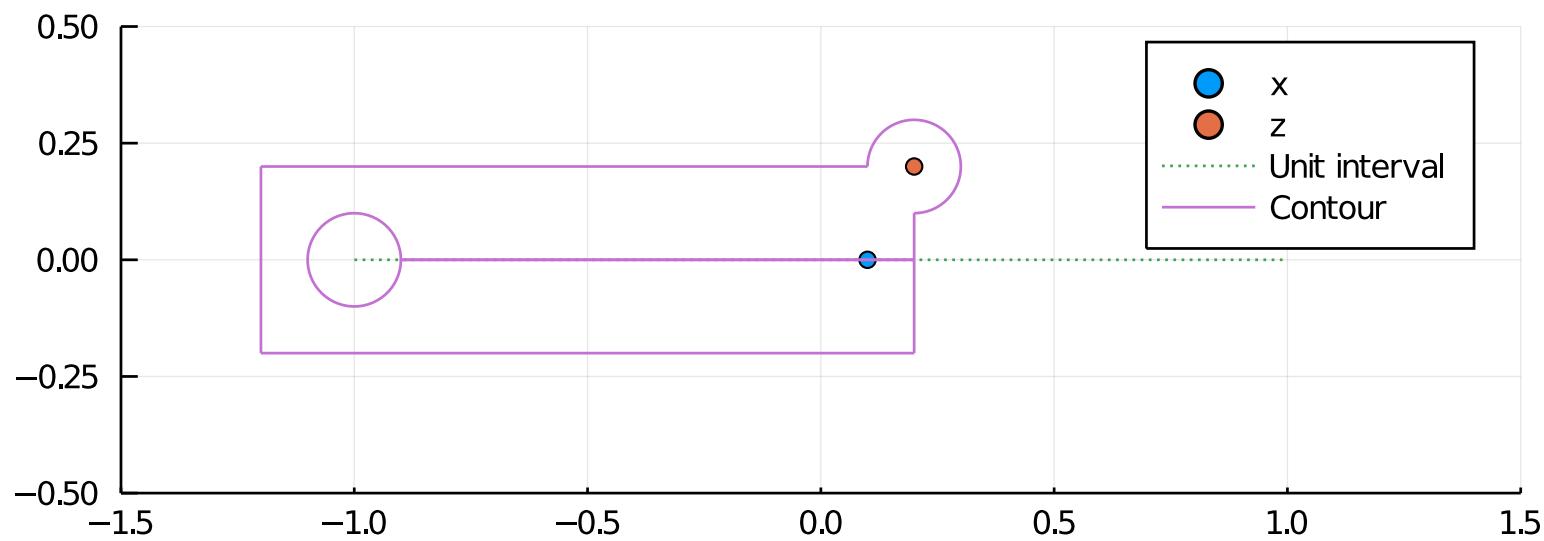
$\varepsilon = 0.001$

```
scatter([x], [0.]; label="x", xlims=(-1.5, 1.5),
       ylims=(-0.5, 0.5), ratio=1.0)
scatter!([real(z)], [imag(z)]; label="z")
plot!(-1..1; label="Unit interval", linestyle=:dot)
Γ_x = Arc(z, 0.1, (-π/2, π)) ∪ Segment(0.2+0.1im, 0.2+0.0im) ∪
      Segment(0.2+0.0im, -0.9+0.0im) ∪
```

```

Circle(-1.0, 0.1) ∪ Segment(-0.9 -0.0im, 0.2 -0.0im) ∪
Segment(0.2-0.0im, 0.2 - 0.2im) ∪
Segment(0.2 - 0.2im, -1.2-0.2im) ∪ Segment(-1.2 -0.2im, -1.2+
0.2im) ∪
Segment(-1.2+ 0.2im, 0.1+0.2im)
plot!(Γ_x; label="Contour")

```



Because f is continuous at x , we have

$$f_+(x) = f_-(x) = f(x)$$

where

$$f_{\pm}(x) = \lim_{\epsilon \rightarrow 0} f(x \pm i\epsilon)$$

Therefore, the two integrals along $[-1, 1]$ cancel out and we get:

$$f(z) = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_x} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $\tilde{\Gamma}_x$ is Γ_x with the contour on the interval removed:

`z = 0.2+0.2im`

`x = 0.1`

`scatter([x], [0.]; label="x", xlims=(-1.5, 1.5),`

`ylims=(-0.5, 0.5), ratio=1.0)`

`scatter!([real(z)], [imag(z)]; label="z")`

`plot!(-1..1; label="Unit interval", linestyle=:dot)`

`Γt_x = Arc(z, 0.1, (-π/2, π)) ∪ Segment(0.2+0.1im, 0.2 -0.2im) ∪`

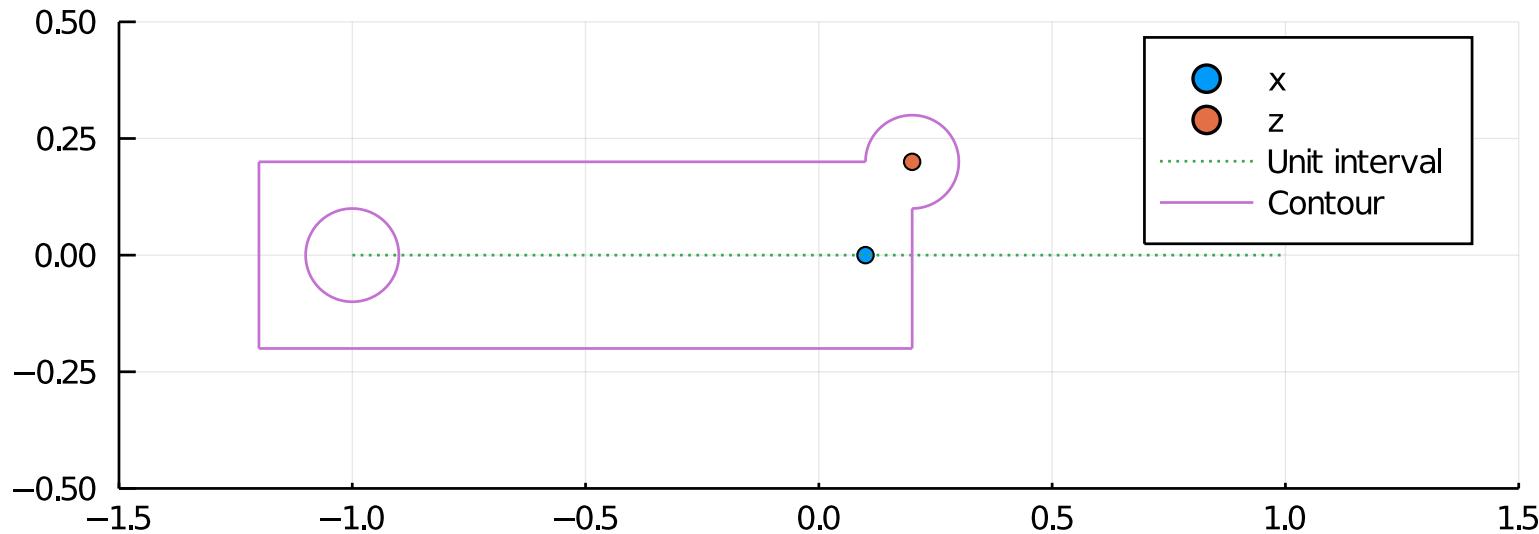
`Circle(-1.0, 0.1) ∪`

`Segment(0.2 - 0.2im, -1.2-0.2im) ∪ Segment(-1.2 -0.2im, -1.2+0.2im) ∪`

```

Segment(-1.2+ 0.2im, 0.1+0.2im)
plot!(Γt_x; label="Contour")

```



This integral expression holds for all z inside the contour $\tilde{\Gamma}_x$ but off the interval. But it therefore holds true for $f(x) = f_+(x) = f_-(x)$ by taking limits. Thus $f(x) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_x} \frac{f(\zeta)}{\zeta - x} d\zeta$ hence f is analytic at x . ■

In an upcoming lecture on the Cauchy transform, we'll encounter a function that has isolated singularities that are weaker than poles (according to the definitions in previous lectures, this is a contradiction in terms). We'll then need the following result which shows that we can analytically continue the function to such singularities via the Cauchy integral formula.

Theorem (weaker than pole singularity implies analyticity) Suppose f is analytic in $D \setminus \{z_0\}$ and has a weaker than pole singularity at z_0 :

$$\lim_{z \rightarrow z_0} (z - z_0)f(z) = 0$$

holds uniformly. Then f is analytic at z_0 . (More precisely: f can be analytically continued to z_0 .)

Proof

Around z_0 is an annulus A_{R0} inside which f is analytic. Consider z in A_{R0} and a positively oriented circle γ_r of radius r with $|r| < |z - z_0|$. Then we have

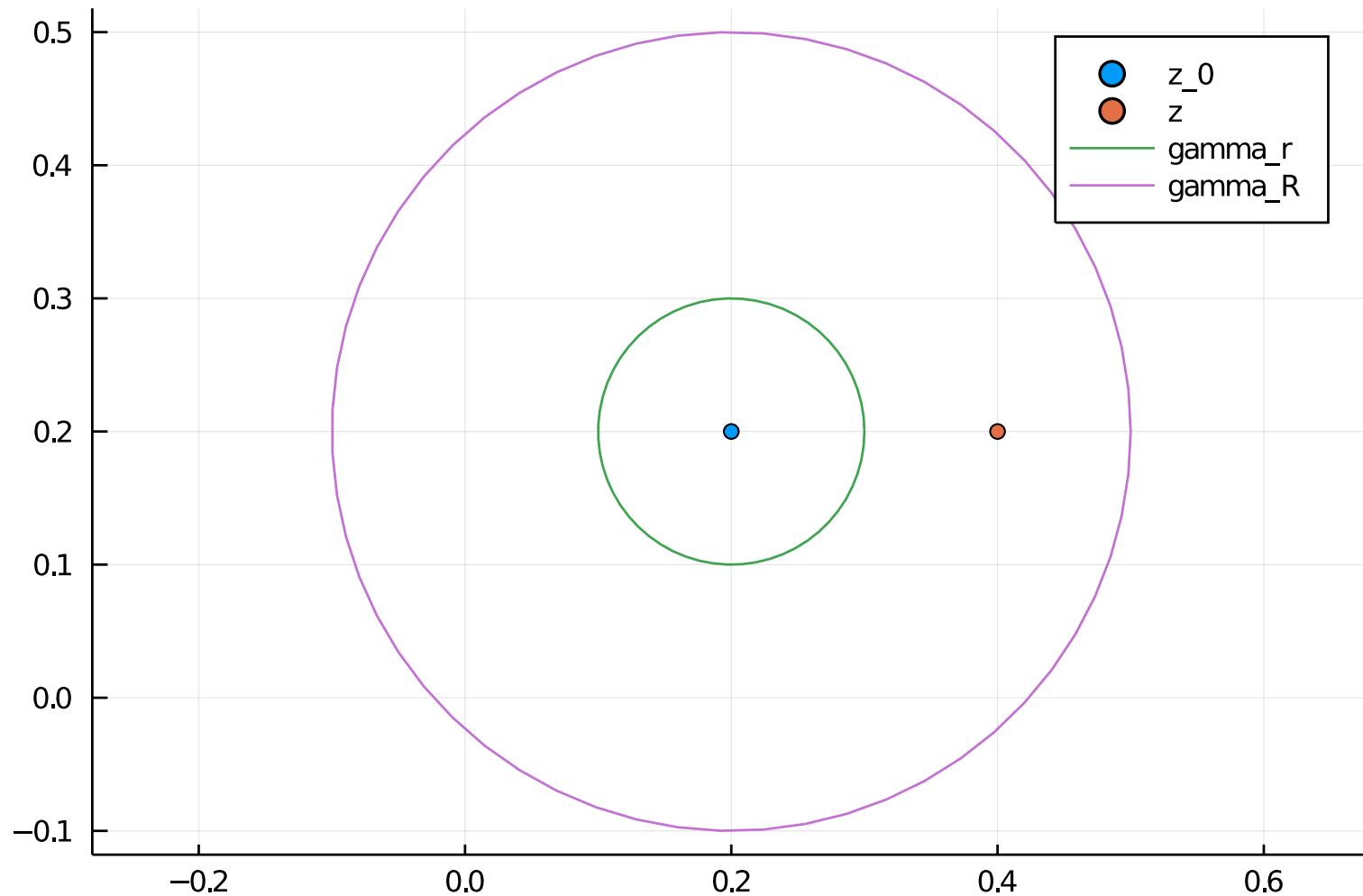
$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(\zeta)}{\zeta - z} d\zeta - \frac{1}{2\pi i} \oint_{\gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

here's a plot:

```
z_0 = 0.2 +0.2im
```

```
z = 0.4 +0.2im
```

```
scatter([real(z_0)], [imag(z_0)]; label="z_0")
scatter!([real(z)], [imag(z)]; label="z")
plot!(Circle(z_0, 0.1); label="gamma_r", ratio=1.0)
plot!(Circle(z_0, 0.3); label="gamma_R")
```



But we have

$$\left| \oint_{\gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta \right| \leq 2\pi r \sup_{\zeta \in \gamma_r} \left| \frac{f(\zeta)}{\zeta - z} \right| \leq 2\pi r \frac{1}{|z_0 - z| - r} \sup_{\zeta \in \gamma_r} |f(\zeta)|$$

which tends to zero as $r \rightarrow 0$.

