

## 1 Lecture 23: Riemann-Hilbert problems

Let  $\Gamma$  be the unit circle or real line (or more generally, a set of general contours, but we won't pursue that in this course). Given functions  $f$  and  $g$  defined on  $\Gamma$ , a (scalar) Riemann-Hilbert problem consists of finding a function  $\Psi(z)$  with left/right limits  $\Psi_{\pm}(x) = \lim_{\epsilon \rightarrow 0} \Psi(x \pm i\epsilon)$ , satisfying the following conditions:

1. Analyticity:  $\Psi(z)$  analytic in  $\bar{\mathbb{C}} \setminus \Gamma$
2. Asymptotics:  $\lim_{z \rightarrow \infty} \Psi(z) = C$
3. Regularity:  $\Psi(z)$  has weaker than pole singularities everywhere
4. Jump:  $\Psi_+(x) - g(x)\Psi_-(x) = f(x)$  for  $x \in \Gamma$

Numerous applications! See [\[Trogdon & Olver 2015\]](#). Here are some classical applications:

1. Ideal fluid flow
2. Solving integral equations via Wiener-Hopf factorization
3. Spectral analysis of Schrödinger operators

More recently, non-classical applications have arisen from integrable systems:

4. Solutions to Painlevé equations
5. Random matrix eigenvalue statistics
6. Asymptotics of orthogonal polynomials
7. Solving partial differential equations like the Korteweg-de Vries (KdV) equation describing shallow water waves

$$u_t + 6uu_x + u_{xxx} = 0$$

We tackle the solution to an RH problem similar to a differential equation: first find the homogeneous solution then use that to reduce inhomogeneous problems to something similar:

1. Homogeneous problems:  $f = 0$
2. Inhomogeneous problems:  $f \neq 0$

### 1.0.1 Homogeneous RiemannHilbert problems on the real line

Let's assume  $f$  is zero and  $C = 1$ , that is we wish to solve

$$\Phi_+(\zeta) = g(\zeta)\Phi_-(\zeta) \quad \text{and} \quad \Phi(\infty) = 1$$

Formally, taking logs of both sides reduces this to a subtractive RH problem:

$$\log \Phi_+(\zeta) - \log \Phi_-(\zeta) \stackrel{?}{=} \log g(\zeta)$$

Assuming that  $g(\zeta) \rightarrow 1$  as  $\zeta \rightarrow \pm\infty$  at a sufficient rate, this motivates the guess

$$\Phi(z) = e^{\mathcal{C}[\log g](z)}$$

Assuming  $\log g(x)$  is "nice", we have guaranteed that this is the unique solution:

**Theorem (Homogeneous solution to RH problem)** Suppose  $\log g(\zeta)$  satisfies the conditions of Plemelj on  $\Gamma$  (the real line or unit circle), in particular, is continuously differentiable. Then  $\Phi(z) = e^{\mathcal{C}_\Gamma[\log g](z)}$  is the unique solution to the following RH problem:

1. Analyticity:  $\Phi(z)$  is analytic off  $\Gamma$
2. Asymptotics:  $\lim_{z \rightarrow \infty} \Phi(z) = 1$
3. Regularity:  $\Phi$  has weaker than pole singularities
4. Jump:  $\Phi_+(\zeta) = g(\zeta)\Phi_-(\zeta)$  for  $\zeta \in \Gamma$

**Proof** (1) follows from definition. (2) follows since  $\mathcal{C}[\log g](z) \rightarrow 0$ . And (4) follows via:

$$\Phi_+(\zeta) = e^{\mathcal{C}_+[\log g](\zeta)} = e^{\mathcal{C}_-[\log g](\zeta) + \log g(\zeta)} = \Phi_-(\zeta)g(\zeta)$$

To see uniqueness, observe that we can take the reciprocal of  $\Phi$ , as it is an exponential of something finite. Thus  $\Phi(z)^{-1}$  is also analytic off  $\mathbb{R}$ . Therefore, if we have another solution  $\tilde{\Phi}(z)$  we can consider  $r(z) = \tilde{\Phi}(z)\Phi(z)^{-1}$  which satisfies:

$$r_+(\zeta) = \frac{\tilde{\Phi}_+(\zeta)}{\Phi_+(\zeta)} = \frac{\tilde{\Phi}_-(\zeta)g(\zeta)}{\Phi_-(\zeta)g(\zeta)} = r_-(\zeta)$$

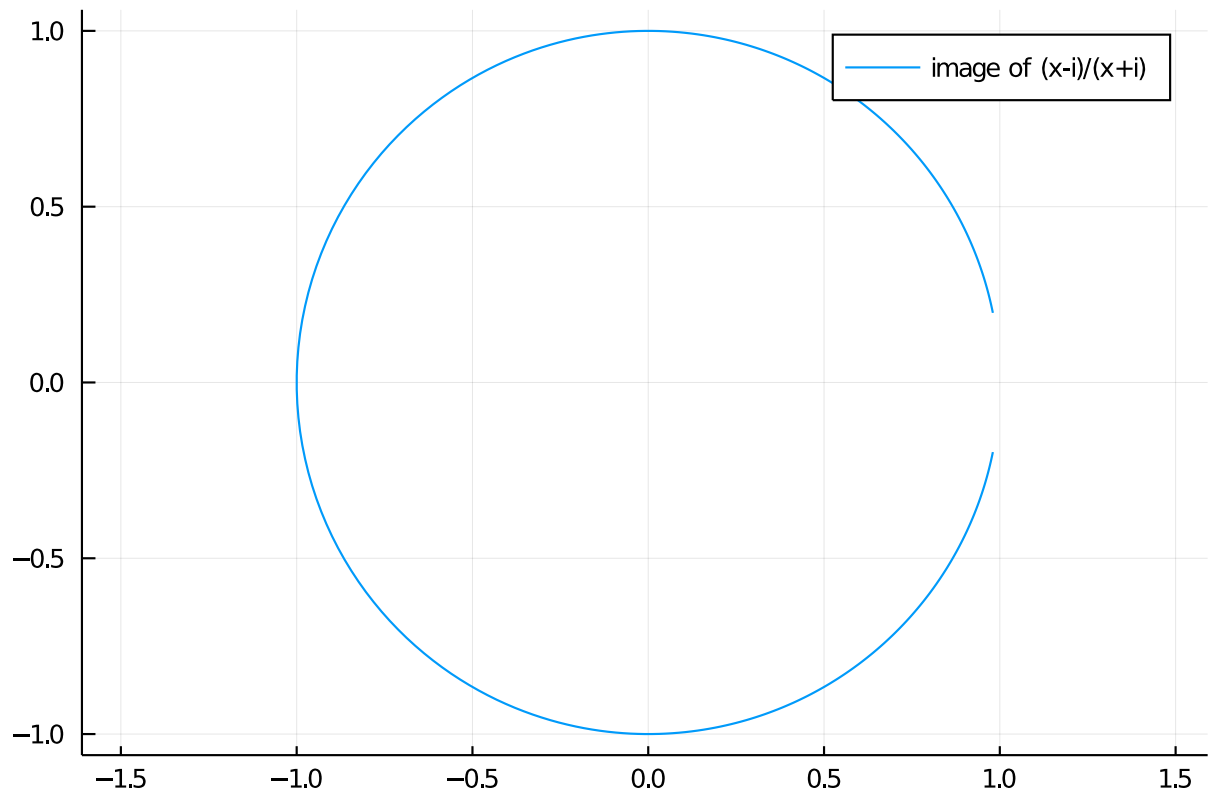
Hence  $r(z)$  is entire. since both terms tend to 1, it must be  $r(z) = 1$ .

■

When is  $\log g(\zeta)$  nice? For the real line it is necessary that  $g(x) = 1 + O(x^{-1})$  at  $x \rightarrow \pm\infty$ . We also need to worry about the image: for example,  $g(\zeta) \neq 0$  is required to avoid a singularity. We also need the winding number of the image of  $g(\zeta)$  to be zero: otherwise,  $\log g(\zeta)$  will extend to another sheet and be discontinuous. For example, if  $g(z) = \frac{z-i}{z+i}$  it satisfies the right asymptotics, but surrounds the origin:

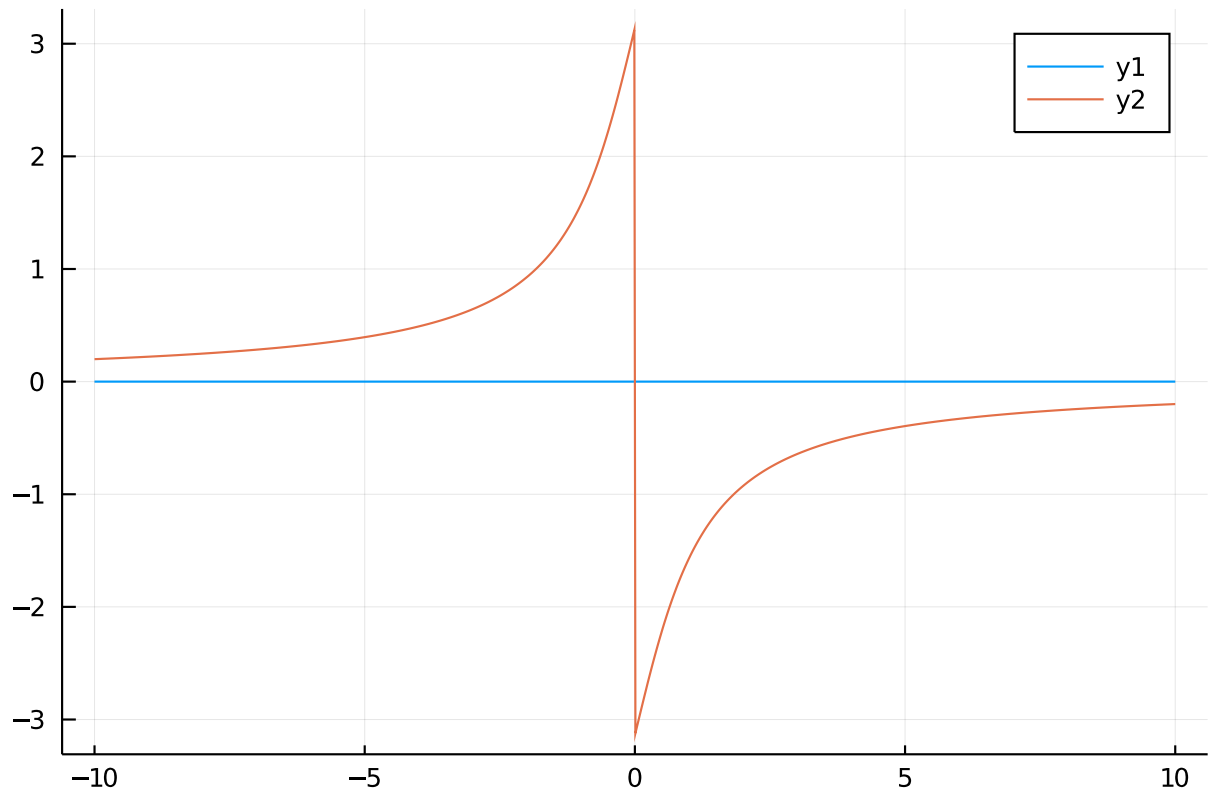
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using Plots, ComplexPhasePortrait
g = x -> (x-im)/(x+im)

xx = range(-10.,10.; length=1000)
plot(real.(g.xx), imag.(g.xx); label="image of (x-i)/(x+i)", ratio=1.0)
```



Therefore,  $\log g(x)$  has a branch cut if we use the standard branch, which breaks the continuity requirement:

```
plot(xx, real.(log.(g.(xx))))
plot!(xx, imag.(log.(g.(xx))))
```

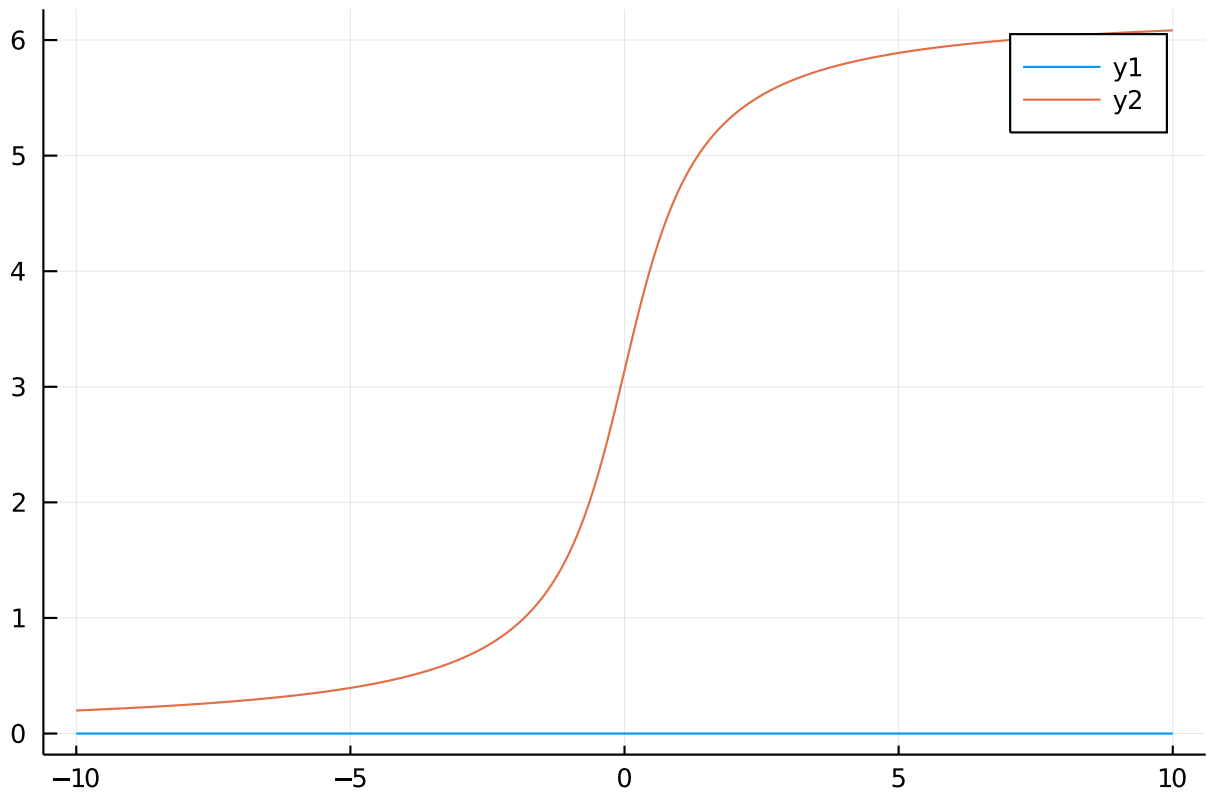


We could have analytically continued  $\log g(z)$  using

$$\log_1 z = \begin{cases} \log z & \Re z > 0 \\ \log_+ z & z < 0 \\ \log z + 2\pi i & \Im z < 0 \end{cases}$$

But then  $\lim_{x \rightarrow +\infty} \log_1 g(x) = 2\pi i$ :

```
log_1 = z -> imag(z) > 0 ? log(z) : log(z)+2*pi*im
plot(xx, real.(log_1.(g.(xx))))
plot!(xx, imag.(log_1.(g.(xx))))
```



**Example** Consider

$$g(x) = \frac{x^2 + 3}{x^2 + 1} = 1 + O(x^{-1})$$

Before we do anything *Verify that the winding number is zero.*

We provide two methods for calculating  $\Phi$ : one guesses the solution, the other uses the solution formula.

*Method 1 (Guess and check / kernel factorization)* If we can guess the solution, we can check it satisfies the right criteria. Factoring  $g$  we see immediately that

$$g(x) = \left( \frac{x + \sqrt{3}i}{x + i} \right) \left( \frac{x - \sqrt{3}i}{x - i} \right)$$

Note that the first term is analytic in the upper half plane. The second term is analytic in the lower half plane *and* we can take its reciprocal. Therefore we can guess the solution is

$$\Phi(z) = \begin{cases} \frac{z+\sqrt{3}i}{z+i} & \Im z > 0 \\ \frac{z-i}{z-\sqrt{3}i} & \Im z < 0 \end{cases}$$

This satisfies the four conditions:

1. Analyticity:  $\Phi$  is analytic off  $\mathbb{R}$
2. Asymptotics:  $\lim_{z \rightarrow \infty} \Phi(z) = 1$
3. Weaker than pole singularities
4. It has the right jump

$$g(x)\Phi_-(x) = \frac{x^2+3}{x^2+1} \frac{x-i}{x-\sqrt{3}i} = \frac{x+\sqrt{3}i}{x+i} = \Phi_+(x)$$

This function is indeed analytic off the real line.

*Method 2 (evaluate explicit formula)* This is real valued and positive, hence the winding number of its image is zero. We have

$$\log g(x) = \log\left(\frac{x+\sqrt{3}i}{x+i} \frac{x-\sqrt{3}i}{x-i}\right)$$

Because they are complex conjugates, we know  $\log a\bar{a} = \log a + \log \bar{a}$  as  $[1, \bar{a}, a]$  lies in the same half plane for  $a = \frac{s+\sqrt{3}i}{s+i}$ , therefore we can expand:

$$\log g(x) = \log \frac{x+\sqrt{3}i}{x+i} + \log \frac{x-\sqrt{3}i}{x-i}$$

Now we note that  $\log \frac{x+\sqrt{3}i}{x+i}$  is analytic in the upper-half plane, therefore its Cauchy transform, by Plemelj, is

$$\mathcal{C} \left[ \log \frac{x+\sqrt{3}i}{x+i} \right] (z) = \begin{cases} \log \frac{z+\sqrt{3}i}{z+i} & \Im z < 0 \\ 0 & \Im z > 0 \end{cases}$$

Similarly,

$$\mathcal{C} \left[ \log \frac{x-\sqrt{3}i}{x-i} \right] (z) = \begin{cases} -\log \frac{z-\sqrt{3}i}{z-i} & \Im z > 0 \\ 0 & \Im z < 0 \end{cases}$$

We thus get:

$$\Phi(z) = e^{C \log g(z)} = e^{\begin{cases} \log \frac{z+\sqrt{3}i}{z+i} & \Im z > 0 \\ -\log \frac{z-\sqrt{3}i}{z-i} & \Im z < 0 \end{cases}} = \begin{cases} \frac{z+\sqrt{3}i}{z+i} & \Im z > 0 \\ \frac{z-i}{z-\sqrt{3}i} & \Im z < 0 \end{cases}$$

### 1.0.2 Inhomogeneous RiemannHilbert problem

Consider now the RiemannHilbert problem with zero at infinity:

$$\Psi_+(x) - g(x)\Psi_-(x) = f(x) \quad \text{and} \quad \Psi(\infty) = 0$$

Consider writing  $\Psi(z) = \Phi(z)Y(z)$ . Then we can reduce the RiemannHilbert problem to a subtractive problem:

$$\Psi_+(x) - g(x)\Psi_-(x) = \Phi_+(x)(Y_+(x) - Y_-(x)) = f(x) \quad \text{and} \quad Y(\infty) = 0$$

Thus once we have  $\Phi$ , we can determine  $Y$  as a Cauchy transform, and thence construct  $\Psi$ .

What if we don't have decay? Just add in a constant times  $\Phi$ :

**Corollary** Suppose  $\log g$  satisfies the conditions of Plemelj's theorem. Then

$$\Psi(z) = \Phi(z)\mathcal{C}_{\mathbb{R}} \left[ \frac{f}{\Phi_+} \right] (z) + D\Phi(z)$$

is the unique solution to

$$\Psi_+(\zeta) - g(\zeta)\Psi_-(\zeta) = f(\zeta) \quad \text{and} \quad \Psi(\infty) = D$$

*Example* Suppose  $f(x) = \frac{i}{i-x}$ .

To decompose this as a sum of things analytic in half planes, we just use partial fraction expansion!

$$\begin{aligned} \frac{i}{i-x} \frac{x+i}{x+i\sqrt{3}} &= \frac{i}{i-x} \frac{2i}{i(1+\sqrt{3})} + \frac{i}{i(1+\sqrt{3})} \frac{i(1-\sqrt{3})}{x+i\sqrt{3}} \\ &= \underbrace{\frac{-2i}{x-i} \frac{1}{1+\sqrt{3}}}_{-Y_-(x)} + \underbrace{\frac{i}{1+\sqrt{3}} \frac{1-\sqrt{3}}{x+i\sqrt{3}}}_{Y_+(x)} \end{aligned}$$

Thus we get

$$Y(z) = \begin{cases} \frac{i}{1+\sqrt{3}} \frac{1-\sqrt{3}}{z+i\sqrt{3}} & \Im z > 0 \\ \frac{2i}{z-i} \frac{1}{1+\sqrt{3}} & \Im z < 0 \end{cases}$$

Let's double check: We thus have the solution:

$$\begin{aligned} \Psi(z) &= \Phi(z)\mathcal{C}_{\mathbb{R}} \left[ \frac{f}{\Phi_+} \right] (z) = \begin{cases} \frac{i}{1+\sqrt{3}} \frac{1-\sqrt{3}}{z+i\sqrt{3}} \frac{z+\sqrt{3}i}{z+i} & \Im z > 0 \\ \frac{2i}{z-i} \frac{1}{1+\sqrt{3}} \frac{z-i}{z-\sqrt{3}i} & \Im z < 0 \end{cases} \\ &= \begin{cases} \frac{i}{1+\sqrt{3}} \frac{1-\sqrt{3}}{z+i} & \Im z > 0 \\ \frac{2i}{1+\sqrt{3}} \frac{1}{z-\sqrt{3}i} & \Im z < 0 \end{cases} \end{aligned}$$

Let's verify it's the right thing:

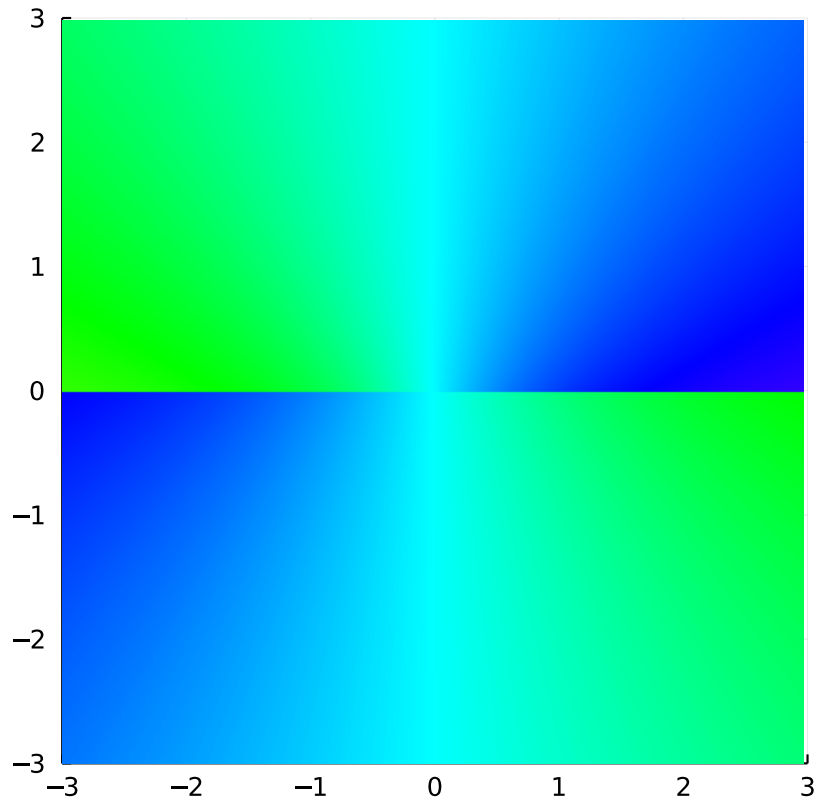
1. It's analytic off  $\mathbb{R}$

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Ψ = z -> imag(z) > 0 ? im*(1-sqrt(3))/(1+sqrt(3))/(z+im) :
      2im/((z-sqrt(3)*im)*(1+sqrt(3)))

phaseplot(-3..3, -3..3, Ψ)

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2. It goes to zero at infinity

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Ψ(300.0+300.0im)

-0.0004465795146304169 - 0.0004450958617578906im

```

3. It satisfies the right jump:

$$\begin{aligned}
 \Psi_+(x) - g(x)\Psi_-(x) &= \frac{i}{1+\sqrt{3}} \frac{1-\sqrt{3}}{x+i} - \frac{x^2+3}{x^2+1} \frac{2i}{1+\sqrt{3}} \frac{1}{x-\sqrt{3}i} \\
 &= \frac{i(x-i)}{1+\sqrt{3}} \frac{1-\sqrt{3}}{x^2+1} - \frac{x+\sqrt{3}i}{x^2+1} \frac{2i}{1+\sqrt{3}} \\
 &= \frac{1}{x^2+1} \frac{1}{1+\sqrt{3}} \left( i(1-\sqrt{3})x + 1 - \sqrt{3} - 2ix + 2\sqrt{3} \right) \\
 &= \frac{1}{x^2+1} (-ix + 1) = \frac{i}{i-x}
 \end{aligned}$$

```

f = x -> im/(im-x)
g = x -> (x^2+3)/(x^2+1)
Ψ(0.1+eps()im) - Ψ(0.1-eps()im)*g(0.1) - f(0.1)

-1.1102230246251565e-16 + 2.7755575615628914e-17im

```