Applied Complex Analysis (2021)

1 Lecture 24: Laurent and Toeplitz operators

This lecture concerns the computation of the UL factorisation of Toeplitz operators, which are infinite matrices with constant diagonals. It turns out this can be reduced to solving an RH problem on the unit circle. We first discuss the case of inverting Laurent operators, which are bi-infinite matrices with constant diagonals.

1.1 Laurent operators

A *Laurent operator* is a bi-infinite matrix with constant diagonals. We associate it with a Laurent series on the unit circle called the *symbol*:

$$a(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

Then the corresponding Laurent operator is

$$L[a(z)] := \begin{pmatrix} \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ \ddots & a_1 & a_0 & a_{-1} & a_{-2} & a_{-3} & \ddots \\ \ddots & a_2 & a_1 & a_0 & a_{-1} & a_{-2} & \ddots \\ \ddots & a_3 & a_2 & a_1 & a_0 & a_{-1} & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

In other words, for $k, j \in \mathbb{Z}$ we have

$$\mathbf{e}_k^\top L[a(z)]\mathbf{e}_j = a_{k-j}$$

Note that $L[a(z)]\mathbf{b}$ is a discrete convolution:

$$\mathbf{e}_k^\top L[a(z)]\mathbf{b} = \sum_{j=-\infty}^{\infty} a_{k-j}b_j$$

and gives the Laurent coefficients of $a(z)b(z)$ where

$$b(z) = \sum_{k=-\infty}^{\infty} b_k z^k$$

This implies an algebraic structure: we have

$$L[a(z)]L[b(z)] = L[a(z)b(z)] = L[b(z)]L[a(z)]$$

in particular, provided $a(z) \neq 0$ on the unit circle,

$$L[a(z)]^{-1} = L[a(z)^{-1}]$$

1.1.1 Example: bi-infinite discrete Laplacian

The bi-infinite discrete Laplacian arises in simple models of crystal structure, though we can also think of it as a finite difference discretisation of the Laplacian on all of $\mathbb{R}$. That is, consider the bi-infinite continuous Helmholtz equation

$$u''(x) - k^2 u(x) = f(x)$$

for $x \in \mathbb{R}$. If we represent the second derivative by its finite-difference approximation at the grid $x_k = kh$,

$$\frac{u(x_{k-1}) - 2u(x_k) + u(x_{k+1})}{h^2}$$

That is, we have the discretisation

$$\Delta_h := \frac{1}{h^2} L[z - 2 + z^{-1}] = \frac{1}{h^2} \begin{pmatrix} \ddots & \ddots & & & \\ \ddots & -2 & 1 & & \\ & 1 & -2 & 1 & \\ & & 1 & -2 & \ddots \\ & & & \ddots & \ddots \end{pmatrix}$$

which gives us the system:

$$(\Delta_h - k^2 I)\mathbf{u} = \mathbf{f}$$

Noting that

$$\Delta_h - k^2 I = L[\underbrace{z/h^2 - k^2 - 2/h^2 + z^{-1}/h^2}_{a(z)}]$$

we know that we can invert this exactly as $L[a(z)^{-1}]$. To determite its Laurent series we do partial fraction expansion

$$a(z)^{-1} = \left[\frac{(z - z_-)(z - z_+)}{zh^2}\right]^{-1} = \frac{z_- h^2}{(z - z_-)(z_- - z_+)} + \frac{z_+ h^2}{(z - z_+)(z_+ - z_-)}$$

where

$$z_\pm = 1 + \frac{h^2 k^2}{2} \pm \frac{hk}{2}\sqrt{h^2 k^2 + 4}.$$

To find the Laurent series we use $|z_-| < 1$ so that

$$\frac{1}{z - z_-} = \frac{1}{z} + \frac{z_-}{z^2} + \frac{z_-^2}{z^3} + \cdots$$

and $|z_+| > 1$ so that

$$\frac{1}{z - z_+} = -\frac{1}{z_+} - \frac{z}{z_+^2} - \frac{z^2}{z_+^3} + \cdots ,$$

hence

$$a(z)^{-1} = \frac{h^2}{z_- - z_+} \left(\sum_{k=-\infty}^{-1} z_-^{-k} z^k + \sum_{k=0}^{\infty} z_+^{-k} z^k \right)$$

The entries of the Laurent series then give the entries of $L[a(z)^{-1}]$.

1.2 Toeplitz operators

A *Toeplitz operator* is an infinite-dimensional matrix with constant diagonals. As in a Laurent operator, we associate it with a Laurent series on the unit circle called the *symbol*:

$$a(z) = \sum_{k=-\infty}^{\infty} a_k z^k$$

Then the corresponding Toeplitz operator is

$$T[a(z)] := \begin{pmatrix} a_0 & a_{-1} & a_{-2} & a_{-3} & \cdots \\ a_1 & a_0 & a_{-1} & a_{-2} & \cdots \\ a_2 & a_1 & a_0 & a_{-1} & \cdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

In other words, for $k, j \in \{0, 1, 2, 3, \dots\}$ we have

$$\mathbf{e}_k^\top T[a(z)] \mathbf{e}_j = a_{k-j}$$

Note that $T[a(z)]\mathbf{b}$ is a half-discrete convolution:

$$\mathbf{e}_k^\top L[a(z)]\mathbf{b} = \sum_{j=0}^{\infty} a_{k-j}b_j$$

1.2.1 Algebraic properties

Unlike Laurent operators, we do not have a trivial algebraic structure: in general

$$T[a(z)]T[b(z)] \neq T[a(z)b(z)] \neq T[b(z)]T[a(z)]$$

For example, if $a(z) = z$ and $b(z) = z^{-1}$ so that $T[a(z)b(z)] = I$ we have

$$\underbrace{\begin{pmatrix} 0 & & & \\ 1 & & & \\ & 1 & & \\ & & \ddots & \end{pmatrix}}_{T[a(z)]} \underbrace{\begin{pmatrix} 0 & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}}_{T[b(z)]} = \begin{pmatrix} 0 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix} \neq I$$

On the other hand $T[b(z)]T[a(z)] = I$, and we will see that the analyticity properties are what ensure this, and thence lead to a RH problem.

The easiest way to understand the algebraic properties is to note that a Toeplitz operator is a sub-component of a Laurent operator, that is,

$$L[a(z)] = \begin{pmatrix} \times & \times \\ \times & T[a(z)] \end{pmatrix}$$

(Here $\times$ means we don't use those entries.) In the special case where

$$a(z) = \sum_{k=0}^{\infty} a_k z^k \qquad \text{and} \qquad b(z) = \sum_{k=0}^{\infty} b_k z^k$$

are analytic, $L[a(z)]$, $L[b(z)]$, and $L[a(z)b(z)]$ are lower-triangular, hence

$$\begin{pmatrix} \times & \\ \times & T[a(z)b(z)] \end{pmatrix} = L[a(z)b(z)] = L[a(z)]L[b(z)] = \begin{pmatrix} \times & \\ \times & T[a(z)] \end{pmatrix} \begin{pmatrix} \times & \\ \times & T[b(z)] \end{pmatrix}$$

$$= \begin{pmatrix} \times & \\ \times & T[a(z)]T[b(z)] \end{pmatrix}$$

Similarly, if

$$a(z) = \sum_{k=-\infty}^{0} a_k z^k \qquad \text{and} \qquad b(z) = \sum_{k=-\infty}^{0} b_k z^k$$

are analytic outside the unit circle, $L[a(z)]$, $L[b(z)]$, and $L[a(z)b(z)]$ are upper-triangular, hence

$$\begin{pmatrix} \times & \times \\ & T[a(z)b(z)] \end{pmatrix} = L[a(z)b(z)] = L[a(z)]L[b(z)] = \begin{pmatrix} \times & \times \\ & T[a(z)] \end{pmatrix}\begin{pmatrix} \times & \times \\ & T[b(z)] \end{pmatrix}$$

$$= \begin{pmatrix} \times & \times \\ & T[a(z)]T[b(z)] \end{pmatrix}$$

In both cases, if $a(z)$ is analytic inside/outside and has no roots, then the above imply that

$$T[a(z)]^{-1} = T[a(z)^{-1}]$$

Finally, if

$$a(z) = \sum_{k=-\infty}^{0} a_k z^k \qquad \text{and} \qquad b(z) = \sum_{k=0}^{\infty} b_k z^k$$

that is $a(z)$ is analytic outside and $b(z)$ is analytic inside the unit circle, then we have $L[a(z)]$ is upper triangular, $L[b(z)]$ is lower triangular, and

$$\begin{pmatrix} \times & \times \\ \times & T[a(z)b(z)] \end{pmatrix} = L[a(z)b(z)] = L[a(z)]L[b(z)] = \begin{pmatrix} \times & \times \\ & T[a(z)] \end{pmatrix}\begin{pmatrix} \times & \\ \times & T[b(z)] \end{pmatrix}$$

$$= \begin{pmatrix} \times & \times \\ \times & T[a(z)]T[b(z)] \end{pmatrix}$$

1.2.2 Wiener-Hopf factorisation

This presents a procedure for inverting Toeplitz operators: find $\phi_+(z)$ analytic inside the unit circle, and $\phi_-(z)$ is analytic outside with $\phi_-(\infty) = 1$ satisfying

$$a(z) = \phi_+(z)\phi_-(z)$$

so that we have the UL decomposition

$$T[a(z)] = T[\phi_-(z)]T[\phi_+(z)]$$

This is equivalent to solving the RH problem

$$\phi_+(z) = a(z)\phi_-(z)^{-1}$$

assuming $\phi_-(z)$ has no zeros. As outlined last lecture, this is satisfied if the winding number of $a(z)$ is trivial. In particular, we have the solution

$$\phi_\pm(z) = e^{\pm \mathcal{C}[\log a](z)}$$

though when $a(z)$ is rational it is more feasible to compute the roots and factorise appropriately.

1.2.3   Example: Finite difference on half-line

We consider a semi-infinite discrete Laplacian. This arises naturally from a finite-difference discretisation with zero Dirichlet conditions, e.g., for $x > 0$ we have

$$u''(x) - k^2 u(x) = f(x) \qquad \text{and} \qquad u(0) = 0$$

We discretise at $x_k = hk$ for $k = 0, 1, \dots$. The finite-difference gives us

$$u''(x_1) \approx \frac{u(x_0) - 2u(x_1) + u(x_2)}{h^2} = \frac{-2u(x_1) + u(x_2)}{h^2}$$
$$u''(x_k) \approx \frac{u(x_{k-1}) - 2u(x_k) + u(x_{k+1})}{h^2}$$

That is, we have the discretisation

$$\Delta_h := \frac{1}{h^2} T[z - 2 + z^{-1}] = \frac{1}{h^2} \begin{pmatrix} -2 & 1 & & \\ 1 & -2 & 1 & \\ & 1 & -2 & \ddots \\ & & \ddots & \ddots \end{pmatrix}$$

which gives us the system:

$$(\Delta_h - k^2 I)\mathbf{u} = \mathbf{f}$$

where

$$\mathbf{u} = \begin{pmatrix} u(x_1) \\ u(x_2) \\ \vdots \end{pmatrix}, \qquad \mathbf{f} = \begin{pmatrix} f(x_1) \\ f(x_2) \\ \vdots \end{pmatrix}.$$

Noting that

$$\Delta_h - k^2 I = T[\underbrace{z/h^2 - k^2 - 2/h^2 + z^{-1}/h^2}_{a(z)}]$$

we know that we can invert this exactly by finding the Wiener–Hopf factorisation of $a(z)$, namely,

$$a(z) = \underbrace{(1 - z_-/z)}_{\phi_-(z)} \underbrace{h^{-1}(z - z_+)}_{\phi_+(z)}$$

where $z_\pm$ were defined above. Therefore

$$T[a(z)] = T[\phi_-(z)]T[\phi_+(z)]$$

This implies that

$$T[a(z)]^{-1} = T[\phi_+(z)]^{-1}T[\phi_-(z)]^{-1} = T[\phi_+(z)^{-1}]T[\phi_-(z)^{-1}]$$

We can use Geometric series to compute

$$\phi_+(z)^{-1} = \frac{h}{z - z_+} = -\frac{h}{z_+} - \frac{hz}{z_+^2} - \frac{hz^2}{z_+^3} - \cdots$$

$$\phi_-(z)^{-1} = \frac{1}{1 - z_-/z} = 1 + \frac{z_-}{z} + \frac{z_-^2}{z^2} + \cdots$$

The Laurent coefficients tell us the entries of $T[\phi_\pm(z)^{-1}]$.

Demonstration We want to factorise and invert $T[a(z)]$:

```
using ToeplitzMatrices, LinearAlgebra
h = k = 1
n = 10
Δ = 1/h^2 * Toeplitz([-2; 1; zeros(n-2)], [-2; 1; zeros(n-2)])
T = Δ - k^2 * I
```

10×@*(10 Array(*@{Float64,2}:

-3.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
1.0	-3.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	1.0	-3.0	1.0	0.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	1.0	-3.0	1.0	0.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	1.0	-3.0	1.0	0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	1.0	-3.0	1.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	1.0	-3.0	1.0	0.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	1.0	-3.0	1.0	0.0
0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0	-3.0	1.0

```
    0.0    0.0    0.0    0.0    0.0    0.0    0.0    0.0    1.0   -3.0
```

We can find the UL decomposition as follows

```
z_p = 1 + (h^2 * k^2)/2 + (h*k)/2 * sqrt(h^2*k^2+4)
z_m = 1 + (h^2 * k^2)/2 - (h*k)/2 * sqrt(h^2*k^2+4)


z = exp(0.1im)
(1-z_p/z)*(inv(h) * (z-z_m))
z/h^2 - 2/h^2 + 1/(z*h^2) - k^2


U = Toeplitz([1; zeros(n-1)], [1; -z_m; zeros(n-2)])

10×@*(10 Toeplitz(*@{Float64,Complex{Float64}}:
  1.0  -0.381966    0.0             0.0             ...@*( 0.0 0.0 0.00.0 1.0
-0.381966 0.0 0.0 0.0 0.00.0 0.0 1.0 -0.381966 0.0 0.0 0.00.0 0.0 0.0
1.0 0.0 0.0 0.00.0 0.0 0.0 0.0 0.0 0.0 0.00.0 0.0 0.0 0.0 (*@...@*(
0.0 0.0 0.00.0 0.0 0.0 0.0 -0.381966 0.0 0.00.0 0.0 0.0 0.0 1.0
-0.381966 0.00.0 0.0 0.0 0.0 0.0 1.0 -0.3819660.0 0.0 0.0 0.0 0.0 0.0
1.0

L = Toeplitz([-z_p/h; 1/h; zeros(n-2)], [-z_p/h; zeros(n-1)])

10×@*(10 Toeplitz(*@{Float64,Complex{Float64}}:
  -2.61803    0.0         0.0             0.0             ...@*( 0.0 0.0 0.01.0
-2.61803 0.0 0.0 0.0 0.0 0.00.0 1.0 -2.61803 0.0 0.0 0.0 0.00.0 0.0
```

1.0 -2.61803 0.0 0.0 0.00.0 0.0 0.0 1.0 0.0 0.0 0.00.0 0.0 0.0 0.0
(*@...@*(0.0 0.0 0.00.0 0.0 0.0 0.0 0.0 0.0 0.00.0 0.0 0.0 0.0
-2.61803 0.0 0.00.0 0.0 0.0 0.0 1.0 -2.61803 0.00.0 0.0 0.0 0.0 0.0
1.0 -2.61803

U*L

10×@*(10 Array(*@{Float64,2}:
 -3.0   1.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  1.0   -3.0   1.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   1.0   -3.0   1.0   0.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   1.0   -3.0   1.0   0.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   1.0   -3.0   1.0   0.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   1.0   -3.0   1.0   0.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   1.0   -3.0   1.0   0.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   1.0   -3.0   1.0   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   1.0   -3.0   1.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   0.0   1.0   -2.61803

This equals T, apart from the bottom right entry, which is off as these are finite-dimensional matrices.

We can then invert U and L, to construct the inverse of T:

```
Ui = Toeplitz([1; zeros(n-1)], z_m.^(0:n-1))
Li = -h*Toeplitz(z_p.^(-(1:n)), [1/z_p; zeros(n-1)])
```

```
Ti = Li*Ui

10×@*(10 Array(*@{Float64,2}:
 -0.381966      -0.145898      -0.0557281     ...@*( -0.00017307
-6.6107e-5-0.145898 -0.437694 -0.167184 -0.000519211
-0.000198321-0.0557281 -0.167184 -0.445825 -0.00138456
-0.000528856-0.0212862 -0.0638587 -0.17029 -0.00363448
-0.00138825-0.00813062 -0.0243919 -0.065045 -0.00951886
-0.00363588-0.00310562 -0.00931686 -0.024845 (*@...@*( -0.0249221
-0.0095194-0.00118624 -0.00355872 -0.00948993 -0.0652475
-0.0249223-0.000453104 -0.00135931 -0.00362483 -0.17082
-0.0652476-0.00017307 -0.000519211 -0.00138456 -0.447214
-0.17082-6.6107e-5 -0.000198321 -0.000528856 -0.17082 -0.447214
```

This is an inverse (apart from truncation error)

```
norm((Ti*T - I)[1:end-1,1:end-1])
```

```
4.22804213279984e-16
```