Applied Complex Analysis (2021)

1 Lecture 10: Branch cuts

We now discuss functions with branch cuts

1. Logarithm: $\log z$ with a cut on $(-\infty, 0]$
2. Powers: z^α with a cut on $(-\infty, 0]$
3. Combinations: $\sqrt{z-1}\sqrt{z+1}$ with a cut on $[-1, 1]$

This is a step towards Cauchy transforms on cuts, for recovering a holomorphic function from its behaviour on a cut. This lecture we discuss:

1. Complex logarithm
2. Algebraic powers
3. Inferring analyticity from continuity

1.1 Complex logarithm

One way to define the logarithm is as $\log |z| + \mathrm{i} \arg z$. We find it more convenient in order to understand its behaviour to define it as an integral:

Definition (Complex Logarithm)

$$\log z := \int_1^z \frac{1}{\zeta} \mathrm{d}\zeta$$

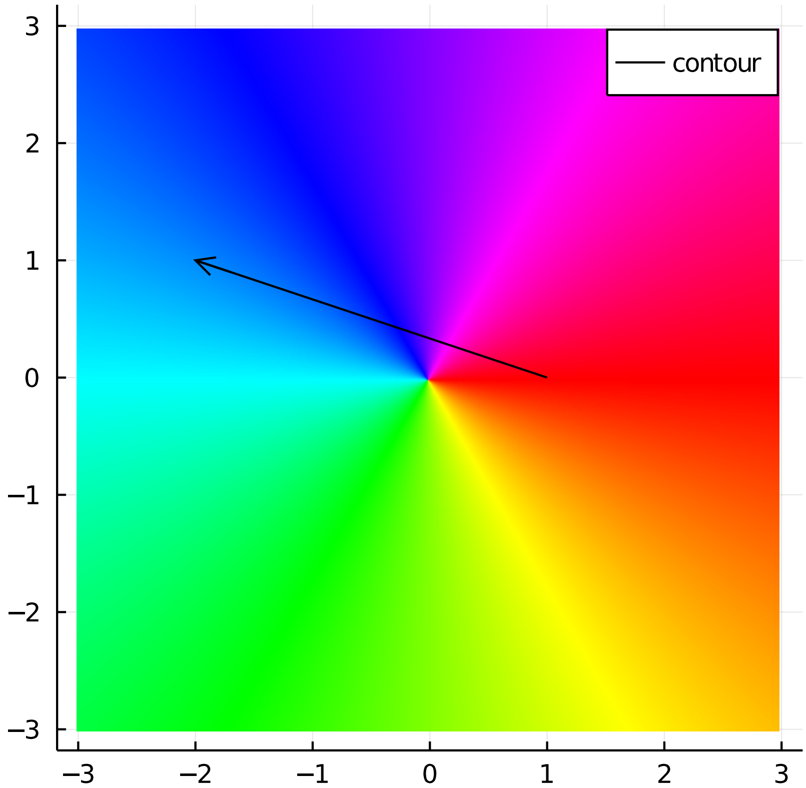
where the integral is understood to be on a straight line segment, that is

$$\log z := \int_{\gamma_z} \frac{1}{\zeta} d\zeta$$

where $\gamma_z(t) = 1 + (z-1)t$ for $0 \leq t \leq 1$.

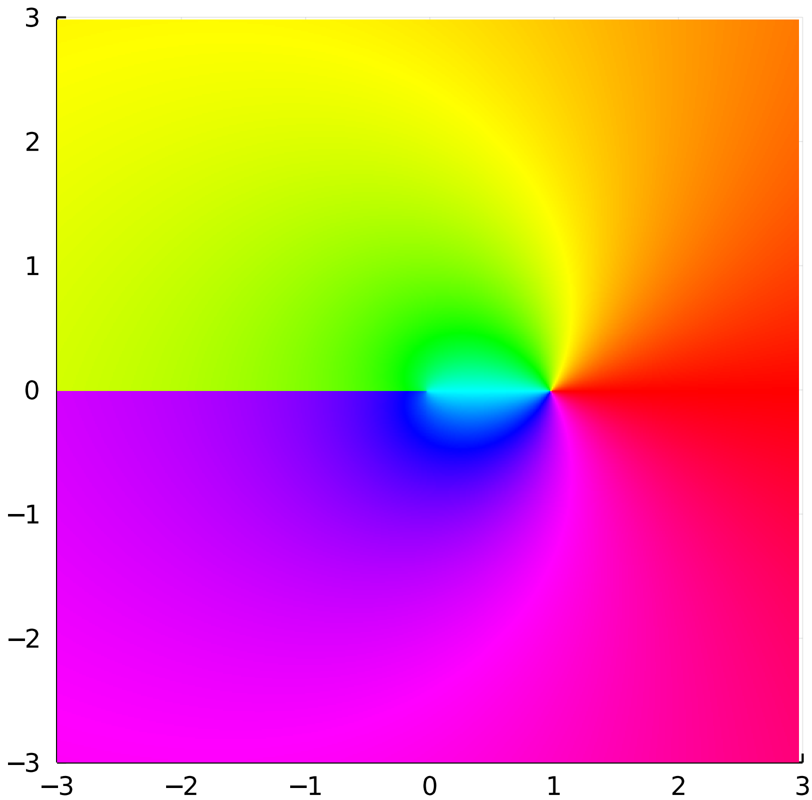
Demonstration this shows the integral path for a point z. We see how the path avoids the pole of ζ^{-1} at the origin:

```
using Plots, ComplexPhasePortrait, ApproxFun
z = -2 + 1.0im
phaseplot(-3..3, -3..3, ζ -> 1/ζ)
t = 0:0.1:1
γ = 1 .+ (z-1)*t
plot!(real.(γ), imag.(γ); color=:black, label="contour", arrow=true)
```



This is well-defined apart from $z \in (-\infty, 0]$, where there is a pole on the contour. This induces a *branch cut*: a jump in the value of the function, which can be clearly seen from a phase portrait:

```
phaseplot(-3..3, -3..3, z -> log(z))
```

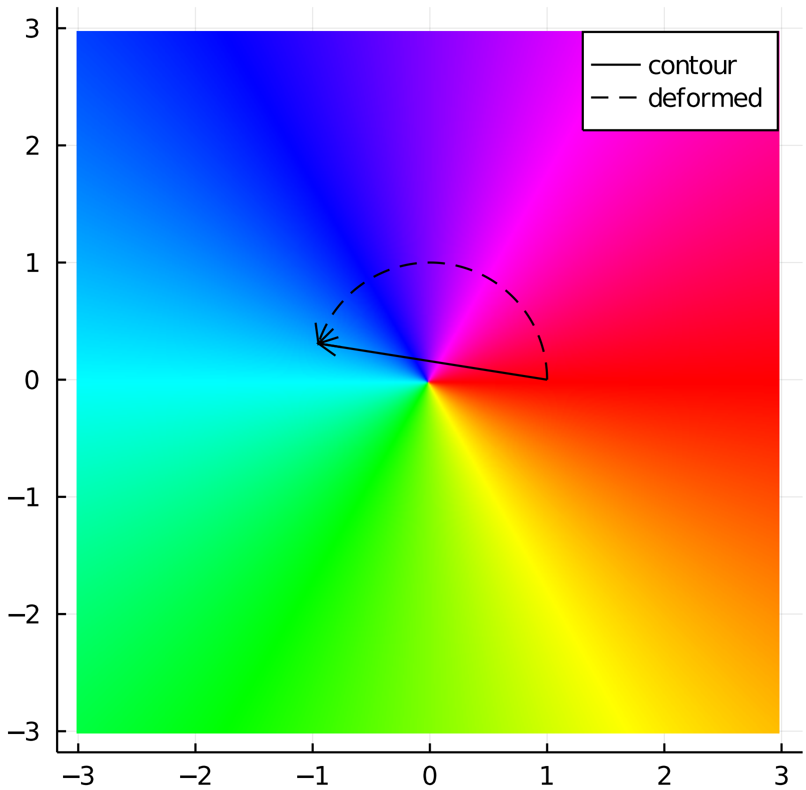


We see that the limits from above and below exist: we can define

$$\log_+ x := \lim_{\epsilon \to 0^+} \log(x + \mathrm{i}\epsilon)$$
$$\log_- x := \lim_{\epsilon \to 0^+} \log(x - \mathrm{i}\epsilon)$$

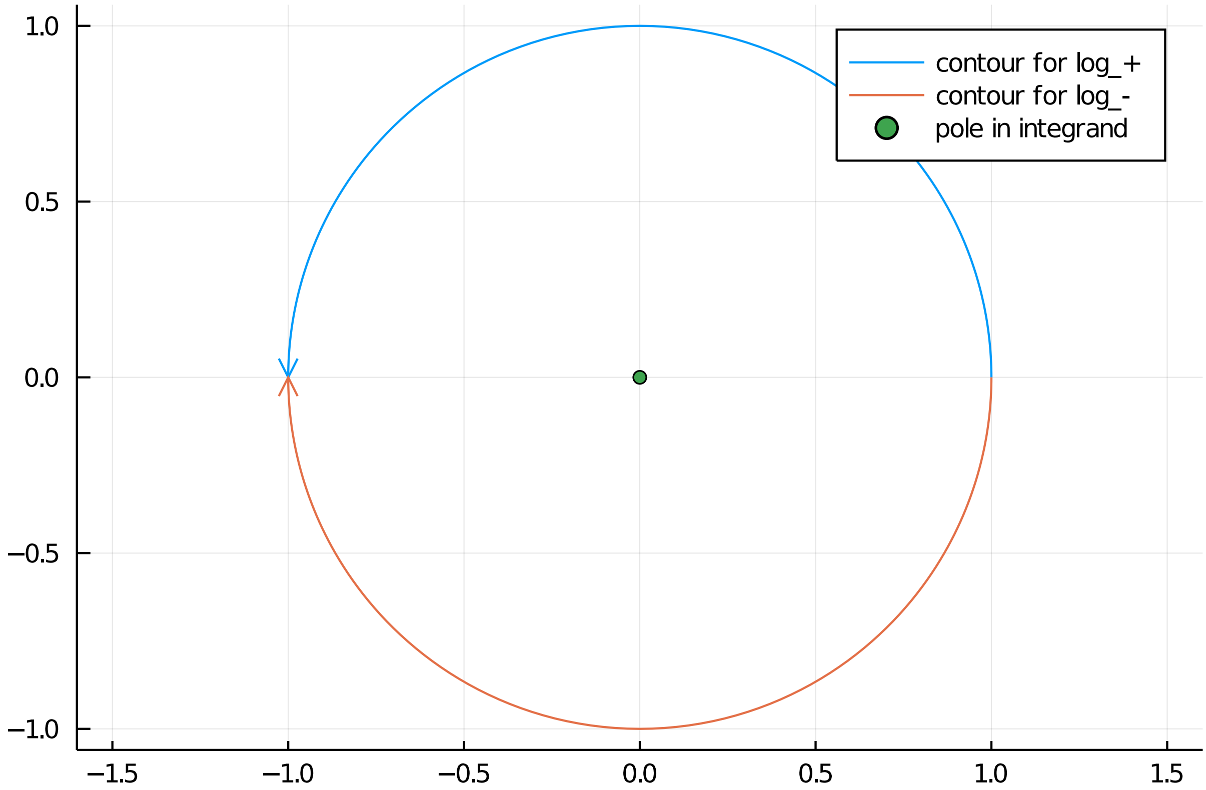
By deformation of contours, the value of the integrals is independent of the path. Here we calculate the integral on an arc:

```
θ = range(0, stop=0.9π, length=100)
a = exp.(im*θ)

z = exp(0.9*π*im)

γ = 1 .+ (z-1)*t
phaseplot(-3..3, -3..3, ζ -> 1/ζ)
plot!(real.(γ), imag.(γ); color=:black, label="contour", arrow=true)
plot!(real.(a), imag.(a), color=:black, linestyle=:dash, label="deformed", arrow=true)
```



This works all the way to the negative real axis. Thus we can calculate $\log_\pm(-1)$ using integrals over half circles:

```
plot(Arc(0.,1.,(π,0)); label="contour for log_+", arrow=true, ratio=1.0)
plot!(Arc(0.,1.,(-π,0)); label="contour for log_-", arrow=true)
scatter!([0],[0]; label="pole in integrand")
```



Combining the two contours we have the *subtractive jump* (for any $x < 0$)

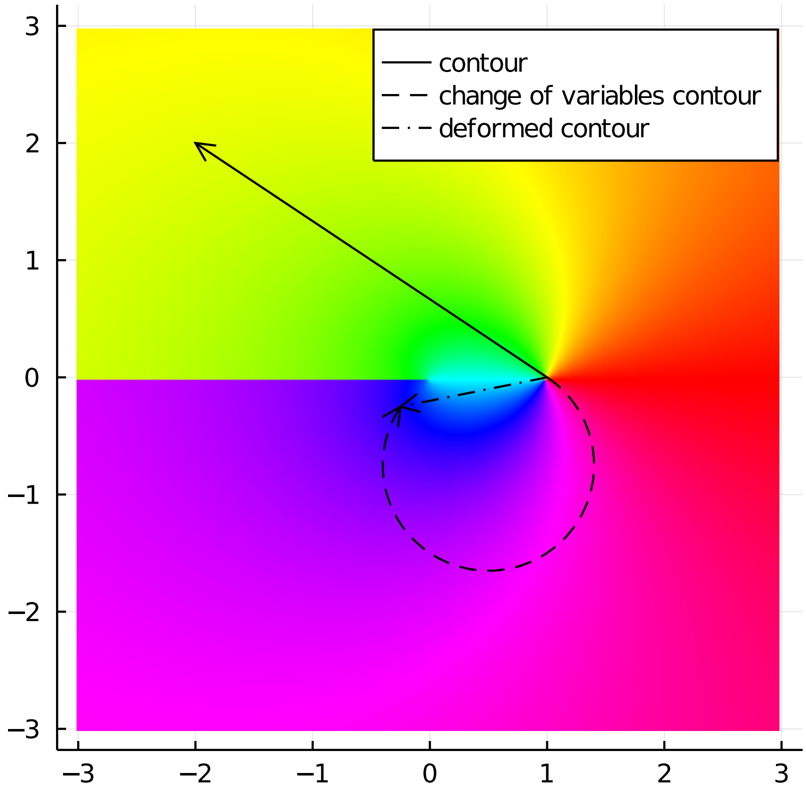
$$\log_+ x - \log_- x = \oint \frac{\mathrm{d}\zeta}{\zeta} = 2\pi\mathrm{i}$$

We can establish some properties. First we show that $\log z = -\log \frac{1}{z}$ by considering the change of variables $\zeta = \frac{1}{s}$. Because $\gamma_z(t)^{-1}$ stays uniformly in the lower-half plane, we can deform it to a straight contour, which gives us the result:

$$\log z = \int_{\gamma_z} \frac{\mathrm{d}\zeta}{\zeta} = -\int_{\frac{1}{\zeta}\circ\gamma_z} \frac{\mathrm{d}s}{s} = -\int_{\gamma_{z^{-1}}} \frac{\mathrm{d}s}{s} = -\log z^{-1}$$

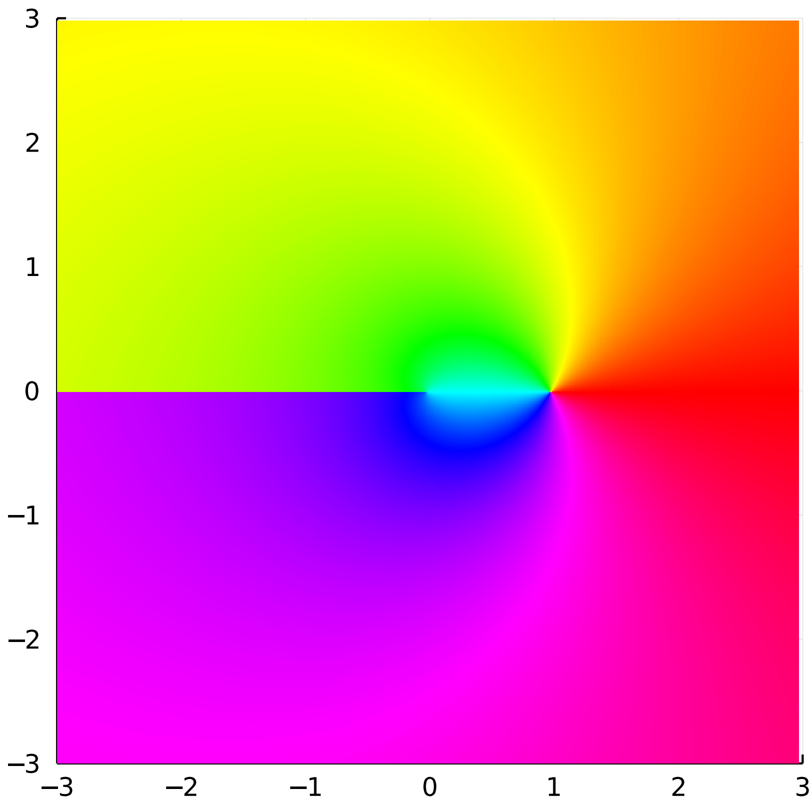
Here's a plot of the relevant contours:

```
phaseplot(-3..3, -3..3, z -> log(z))
z = -2 + 2im
γ = (z,t) -> 1 + t*(z-1)
tt = range(0,stop=1,length=100)
plot!(real.(γ.(z,tt)), imag.(γ.(z,tt)); color=:black, label="contour", arrow=true)
plot!(real.(1 ./ γ.(z,tt)), imag.(1 ./ γ.(z,tt)); color=:black, linestyle=:dash,
arrow=true, label="change of variables contour")
plot!(real.(γ.(1/z,tt)), imag.(γ.(1/z,tt)); color=:black, linestyle=:dashdot,
arrow=true, label="deformed contour")
```



Here we see by looking at the phase plot that the two functions match:

```
phaseplot(-3..3, -3..3, z -> -log(1/z))
```



1.2 Algebraic powers

We define algebraic powers in terms of logarithms, which gives us what we need to know about their jumps.

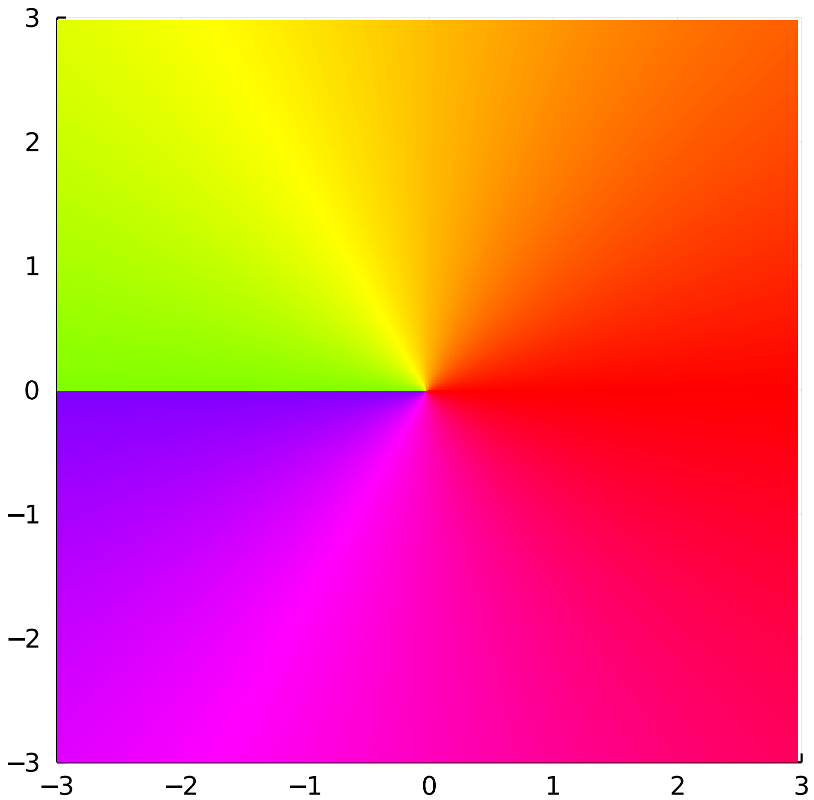
Definition (algebraic power)

$$z^\alpha := \mathrm{e}^{\alpha \log z}$$

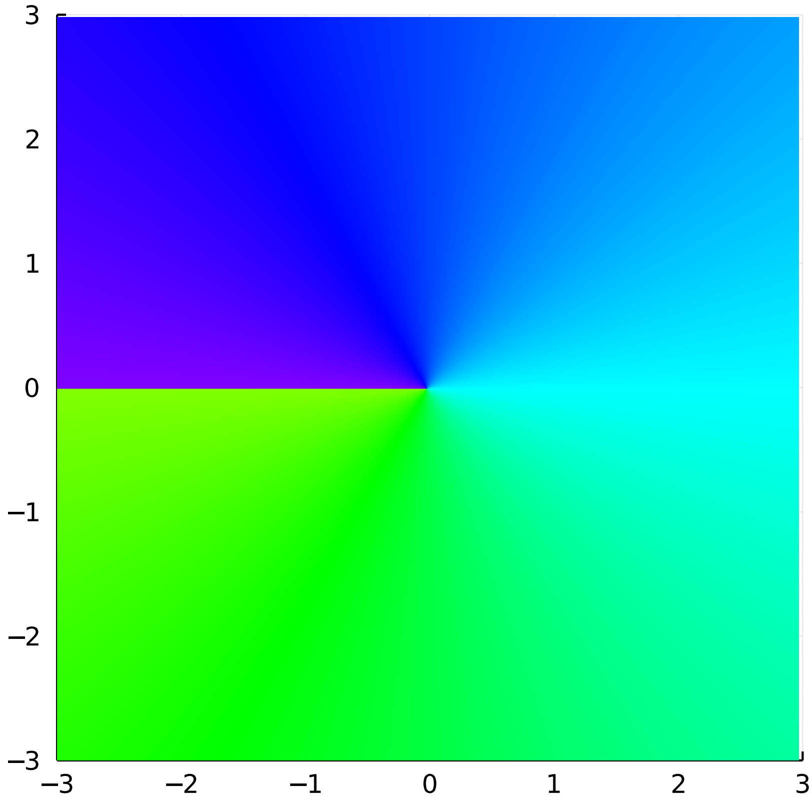
Note, for example, when $\alpha = 1/2$, $\sqrt{z} \equiv z^{1/2}$ is only one solution to $y^2 = z$.

Here are phase plots showing that $\sqrt{z}$ also has a branch cut on $(-\infty, 0]$ on both of its branches:

```
α = 0.5
phaseplot(-3..3, -3..3, z -> z^α)
```




```
phaseplot(-3..3, -3..3, z -> -z^α)
```



On the branch cut along $(-\infty, 0]$ it has the jump:

$$\frac{x_+^\alpha}{x_-^\alpha} = e^{\alpha(\log_+ x - \log_- x)} = e^{2\pi i \alpha}$$

In particular,

$$\sqrt{x}_+ = -\sqrt{x}_- = i\sqrt{|x|}$$

These are *multiplicative jumps*. We also have a *subtractive jump*:

$$\begin{aligned} x_+^\alpha - x_-^\alpha &= e^{\alpha \log_+ x} - e^{\alpha \log_- x} = e^{\alpha \log(-x) + i\pi\alpha} - e^{\alpha \log(-x) - i\pi\alpha} \\ &= 2i(-x)^\alpha \sin \pi\alpha \end{aligned}$$

and an *additive jump*:

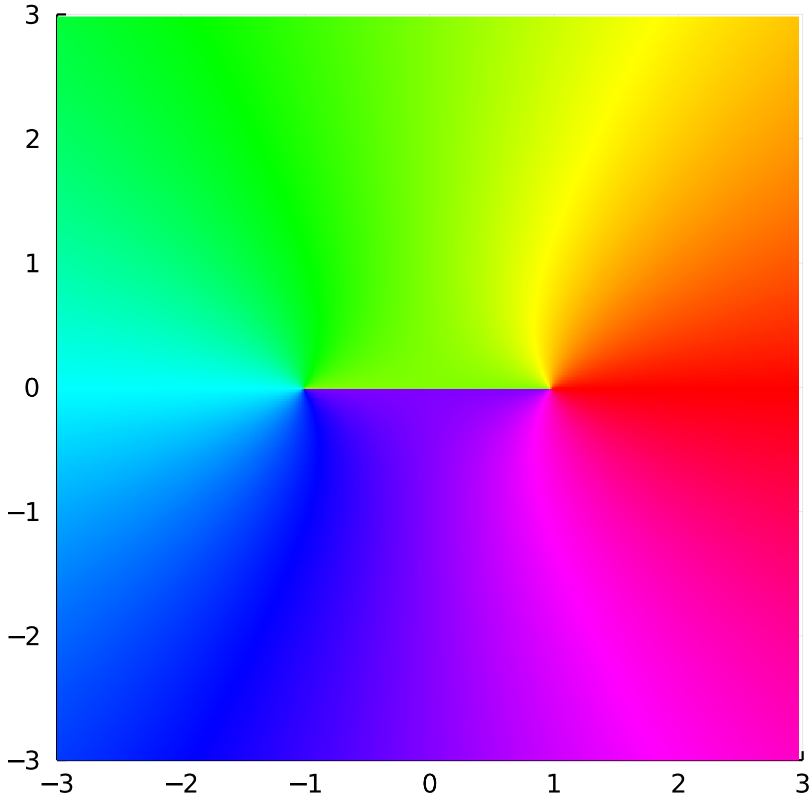
$$x_+^\alpha + x_-^\alpha = 2(-x)^\alpha \cos \pi\alpha$$

In particular, for $x < 0$,

$$\begin{aligned} \sqrt{x}_+ - \sqrt{x}_- &= 2i\sqrt{-x} \\ \sqrt{x}_+ + \sqrt{x}_- &= 0 \end{aligned}$$

Let's look at another example: $\varphi(z) = \sqrt{z-1}\sqrt{z+1}$. Each square root term induces a jump: one on $(-\infty, -1]$ and one on $(-\infty, 1]$. Surprisingly these jumps cancel out, in fact (as we explain below) φ is analytic off $[-1, 1]$, as can be seen from the phase portrait:

```
φ = z -> sqrt(z-1)*sqrt(z+1)
phaseplot(-3..3, -3..3, φ)
```



For $-1 < x < 1$ we have the multiplicative jump:

$$\varphi_+(x) = \sqrt{x-1}_+ \sqrt{x+1} = -\sqrt{x-1}_- \sqrt{x+1} = -\varphi_-(x)$$

which gives the additive jump

$$\varphi_+(x) + \varphi_-(x) = 0$$

But we also have a *subtractive jump*:

$$\varphi_+(x) - \varphi_-(x) = (\sqrt{x-1}_+ - \sqrt{x-1}_-)\sqrt{x+1} = 2\mathrm{i}\sqrt{1-x}\sqrt{x+1} = 2\mathrm{i}\sqrt{1-x^2}$$

For $x < -1$ we actually have continuity:

$$\varphi_+(x) = \sqrt{x-1}_+ \sqrt{x+1}_+ = (-\sqrt{x-1}_-)(-\sqrt{x+1}_-) = \varphi_-(x)$$

This feature is what we use to show analyticity.

2 Inferred analyticity

Here we review properties of inferring analyticity from continuity.

Theorem (continuity on a curve implies analyticity) Let D be a domain and $\gamma \subset D$ a contour. Suppose f is analytic in $D \backslash \gamma$, and continuous on the interior of γ. Then f is analytic in $D \backslash \{\gamma(a), \gamma(b)\}$.

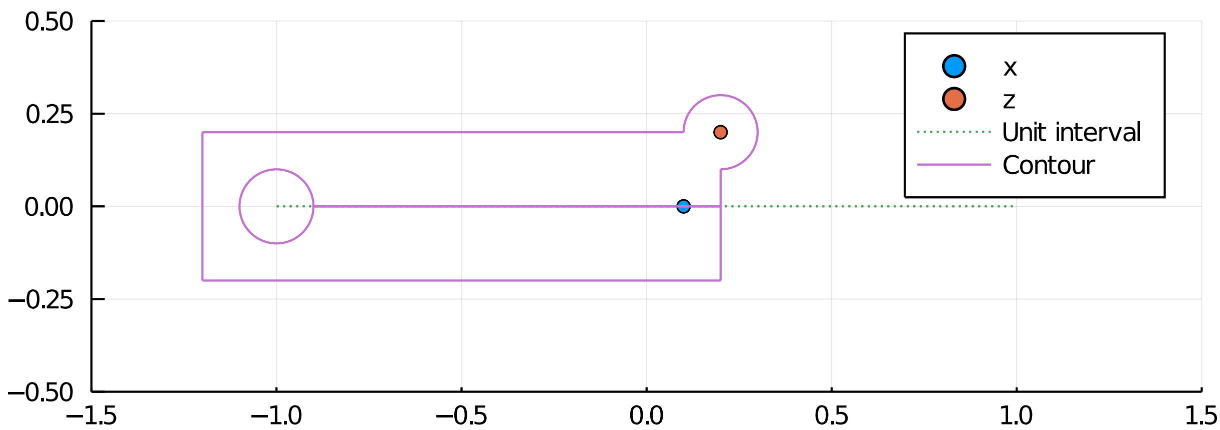
Sketch of Proof

For simplicity, suppose D is a circle of radius 2 and γ is the interval $[-1,1]$. For any z off the interval, we can write

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_{\Gamma_x} \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

where Γ_x is a simple closed contour that surrounds z and passes through x in both directions:

```
z = 0.2+0.2im
x = 0.1
ε = 0.001
scatter([x],[0.]; label="x", xlims=(-1.5,1.5), ylims=(-0.5,0.5),ratio=1.0)
scatter!([real(z)],[imag(z)]; label="z")
plot!(-1..1; label="Unit interval", linestyle=:dot)

Γ_x =  Arc(z, 0.1, (-π/2,π)) ∪ Segment(0.2+0.1im,0.2 +0.0im) ∪ Segment(0.2  +0.0im,
-0.9 +0.0im) ∪
     Circle(-1.0, 0.1) ∪ Segment(-0.9 -0.0im, 0.2  -0.0im) ∪ Segment(0.2-0.0im, 0.2 -
0.2im) ∪
      Segment(0.2 - 0.2im, -1.2-0.2im) ∪ Segment(-1.2 -0.2im, -1.2+ 0.2im) ∪
        Segment(-1.2+ 0.2im, 0.1+0.2im)

plot!(Γ_x; label="Contour")
```



Because f is continuous at x, we have

$$f_+(x) = f_-(x) = f(x)$$

where

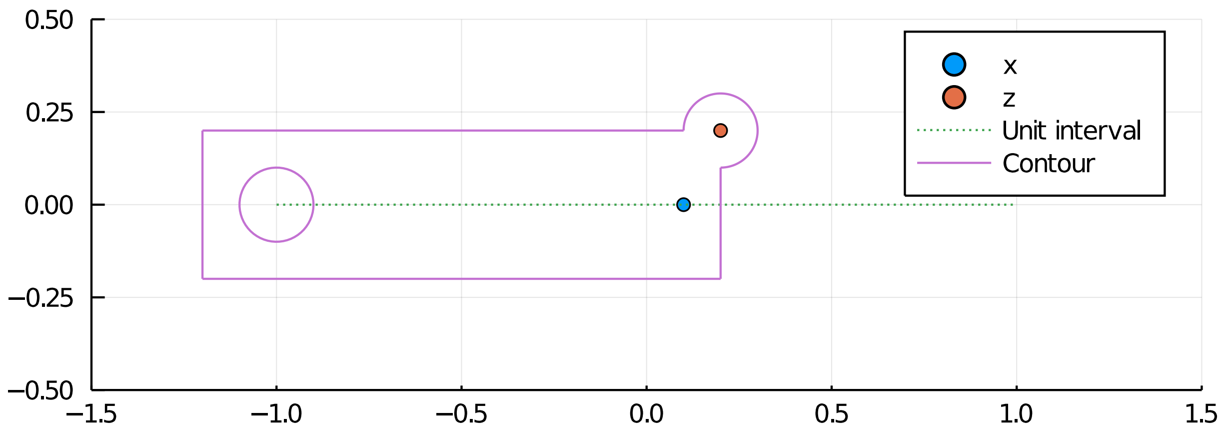
$$f_\pm(x) = \lim_{\epsilon \to 0} f(x \pm i\epsilon)$$

Therefore, the two integrals along $[-1, 1]$ cancel out and we get:

$$f(z) = \frac{1}{2\pi i} \oint_{\tilde{\Gamma}_x} \frac{f(\zeta)}{\zeta - z} d\zeta$$

where $\tilde{\Gamma}_x$ is Γ_x with the contour on the interval removed:

```
z = 0.2+0.2im
x = 0.1
ε = 0.001
scatter([x],[0.]; label="x", xlims=(-1.5,1.5), ylims=(-0.5,0.5),ratio=1.0)
scatter!([real(z)],[imag(z)]; label="z")
plot!(-1..1; label="Unit interval", linestyle=:dot)

Γt_x =  Arc(z, 0.1, (-π/2,π)) ∪ Segment(0.2+0.1im,0.2 -0.2im) ∪    Circle(-1.0, 0.1) ∪
      Segment(0.2 - 0.2im, -1.2-0.2im) ∪ Segment(-1.2 -0.2im, -1.2+ 0.2im) ∪
        Segment(-1.2+ 0.2im, 0.1+0.2im)

plot!(Γt_x; label="Contour")
```



This integral expression holds for all z inside the contour $\tilde{\Gamma}_x$ but off the interval. But it therefore holds true for $f(x) = f_+(x) = f_-(x)$ by taking limits. Thus $f(x) = \frac{1}{2\pi i} \int_{\tilde{\Gamma}_x} \frac{f(\zeta)}{\zeta - x} d\zeta$ hence f is analytic at x.

■ In an upcoming lecture on the Cauchy transform, we'll encounter a function that has isolated singularities that are weaker than poles (according to the definitions in previous lectures, this is a contradiction in terms). We'll then need the following result which shows that we can analytically continue the function to such singularities via the Cauchy integral formula. **Theorem (weaker than pole singularity implies analyticity)** Suppose f is analytic in $D\backslash\{z_0\}$ and has a weaker than pole singularity at z_0:

$$\lim_{z\to z_0}(z-z_0)f(z)=0$$

holds uniformly. Then f is analytic at z_0. (More precisely: f can be analytically continued to z_0.)

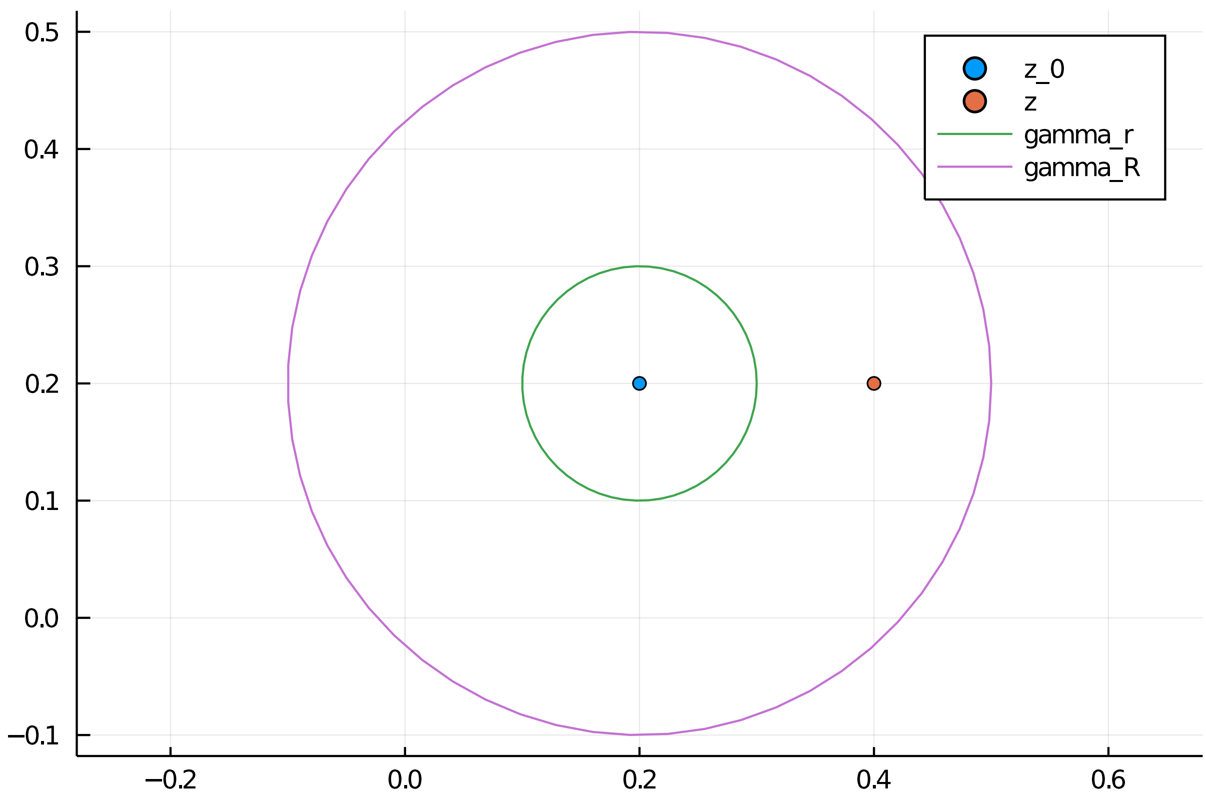
Proof

Around z_0 is an annulus A_{R0} inside which f is analytic. Consider z in A_{R0} and a positively oriented circle γ_r of radius r with $|r|<|z-z_0|$. Then we have

$$f(z)=\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_R}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta-\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_r}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta.$$

here's a plot:

```
z_0 = 0.2 +0.2im
z = 0.4 +0.2im

scatter([real(z_0)],[imag(z_0)]; label="z_0")
scatter!([real(z)],[imag(z)]; label="z")
plot!(Circle(z_0, 0.1); label="gamma_r",ratio=1.0)
plot!(Circle(z_0, 0.3); label="gamma_R")
```



But we have

$$\left|\oint_{\gamma_r}\frac{f(\zeta)}{\zeta-z}\mathrm{d}\zeta\right|\leq 2\pi r\sup_{\zeta\in\gamma_r}\left|\frac{f(\zeta)}{\zeta-z}\right|\leq 2\pi r\frac{1}{|z_0-z|-r}\sup_{\zeta\in\gamma_r}|f(\zeta)|$$

which tends to zero as $r \to 0$.

■