Applied Complex Analysis (2021)

1 Lecture 3: Laurent series and residue calculus

Here we cover:

1. Fourier and Laurent series
2. Contour integrals and Laurent coefficients
3. Isolated singularities
    - Residue at a point
4. Contour integrals in domains with multiple holes
    - The residue theorem
    - Calculating integrals

1.1 Fourier and Laurent series

Definition (Fourier series) On $[-\pi, \pi)$, *Fourier series* is an expansion of the form

$$g(\theta) = \sum_{k=-\infty}^{\infty} g_k \mathrm{e}^{ik\theta}$$

where

$$g_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) \mathrm{e}^{-ik\theta} d\theta$$

Definition (Laurent series) In the complex plane, Laurent series around z_0 is an expansion of the form

$$f(z) = \sum_{k=-\infty}^{\infty} f_k (z - z_0)^k$$

Lemma (Fourier series = Laurent series) On a circle in the complex plane

$$C_r = \{z_0 + re^{i\theta} : -\pi \leq \theta < \pi\},$$

Laurent series of $f(z)$ around z_0 converges for $z \in C_r$ if the Fourier series of $g(\theta) = f(z_0 + re^{i\theta})$ converges, and the coefficients are given by

$$f_k = \frac{g_k}{r^k} = \frac{1}{2\pi i} \oint_{C_r} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}} d\zeta.$$

Proof This follows immediately from the change of variables $z = r\mathrm{e}^{i\theta} + z_0$. ■

As we shall see in a future lecture, analytic properties of f can be used to show decaying properties in g:

1.2 Contour integrals and Laurent coefficients

In this module, we will *always* think of Laurent series as living on a circle $C_r(z_0) = \{z : |z - z_0| = r\}$. That is,

$$f(z) \approx \sum_{k=-\infty}^{\infty} f_k (z - z_0)^k$$

for $z \in C_r(z_0)$.

Proposition (Residue on a circle) Suppose the Laurent series is absolutely summable on $C_r(z_0)$. Then

$$\oint_{C_r(z_0)} f(z) dz = 2\pi \mathrm{i} f_{-1}$$

We refer to f_{-1} as the residue over $C_r(z_0)$:

$$\operatorname*{Res}_{C_r(z_0)} f(z) := f_{-1} = \frac{1}{2\pi \mathrm{i}} \int_{C_r} f(z) \mathrm{d}z.$$

Example For all $0 < r < \infty$,

$$\oint_{C_r} \frac{1}{z} dz = 2\pi \mathrm{i}$$

When f is holomorphic in a neighbourhood of the circle, we can extend it to an annulus (like Taylor series and disks):

Proposition (Laurent series in an annulus) Suppose f is holomorphic in an open annulus $A_{\rho R}(z_0) = \{z : \rho < |z - z_0| < R\}$. Then the Laurent series converges uniformly in any closed annulus inside $A_{\rho R}$

Proposition (Residue on a circle) holds true regardless of the radius, that is, the definition of f_{-1} only depends on the annulus of analyticity:

$$\operatorname*{Res}_{A_{\rho R}(z_0)} f(z) := f_{-1} = \frac{1}{2\pi\mathrm{i}} \int_{C_r} f(z)\mathrm{d}z.$$

for any $\rho < r < R$.

1.3 Isolated singularities

Definition (isolated singularity) f has an *isolated singularity at* z_0 if it is holomorphic in an open annulus with inner radius 0:

$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}.$$

Definition (Removable singularity) f has a *removable singularity at* z_0 if it has an isolated singularity at z_0 and all negative terms in the Laurent series in $A_{0R}(z_0)$ are zero:

$$f(z) = f_0 + f_1(z - z_0) + f_2(z - z_0)^2 + \cdots$$

Equivalently, f has a removable singularity at z_0 if it is bounded as $z \to z_0$.

An example would be a function like $f(z) = (\mathrm{e}^z - 1)/z$, which is analytic for $z \neq 0$ but is not defined for $z = 0$. However, this singularity is artificial: we have from its Taylor series that

$$f(z) = (\mathrm{e}^z - 1)/z = (z + z^2/2! + \cdots)/z = 1 + z/2! + \cdots$$

hence $f(z) \to 1$. We can therefore remove the singularity by taking f_0, the zeroth Laurent coefficient:

$$\tilde{f}(z) = \begin{cases} (\mathrm{e}^z - 1)/z & z \neq 0 \\ 1 & z = 0 \end{cases}$$

This construction is general:

Proposition (Removing a removable singularity) If f has a removable singularity at z_0 and f_0 is the zeroth Laurent coefficient

$$f_0 := \frac{1}{2\pi \mathrm{i}} \oint_{C_r(z_0)} \frac{f(z)}{z - z_0} \mathrm{d}z$$

for r sufficiently small, then

$$\tilde{f}(z) = \begin{cases} f_0 & z = z_0 \\ f(z) & 0 < |z - z_0| < R \end{cases}$$

is analytic in the disk $B_R(z_0) = \{z : |z - z_0| < R\}$, with a convergent Taylor series. Hence the term 'removable singularity'.

Definition (simple pole) f has a *simple pole at* z_0 if it is holomorphic in

$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}$$

with only one negative term in the Laurent series in $A_{0R}(z_0)$:

$$f(z) = \frac{f_{-1}}{z - z_0} + f_0 + f_1(z - z_0) + \cdots$$

where $f_{-1} \neq 0$.

Definition (higher order pole) f has a *pole of order* N *at* z_0 if it is holomorphic in

$$A_{0R}(z_0) = \{z : 0 < |z - z_0| < R\}$$

with only N negative coefficients in the Laurent series:

$$f(z) = \frac{f_{-N}}{(z - z_0)^N} + \frac{f_{1-N}}{(z - z_0)^{N-1}} + \cdots + \frac{f_{-1}}{z - z_0} + f_0 + f_1(z - z_0) + \cdots$$

where $f_{-N} \neq 0$.

Definition (essential singularity) f has an *essential singularity at* z_0 if it is holomorphic in $A_{0R}(z_0)$ and has an infinite number of negative Laurent coefficients.

An essential singularity is complicated but isolated, hence we can still calculate the integrals using Residue calculus.

1.3.1 Residue at a point

Definition (Residue at a point) Suppose f has an isolated singularity at z_0, and is analytic in the annulus $A_{0R}(z_0)$ for some $R > 0$. Then we define the *residue at* z_0 as

$$\operatorname*{Res}_{z=z_0} f(z) = \operatorname*{Res}_{A_{0R}} f(z) = f_{-1}$$

where f_{-1} is the first negative coefficient of the Laurent series in $A_{0R}(z_0)$, that is, integrating over C_r for any $0 < r < R$.

Proposition (Residue of ratio of analytic functions with simple pole) Suppose

$$f(z) = \frac{A(z)}{B(z)}$$

and A, B are analytic/holomorphic in a disk of radius R around z_0 and that B has only a single zero at z_0:

$$\begin{aligned} A(z) &= A_0 + A_1(z - z_0) + \cdots \\ B(z) &= B_1(z - z_0) + \cdots \end{aligned}$$

Then $\operatorname*{Res}_{z=z_0} f(z) = \frac{A_0}{B_1}$

Exercise (Residue of ratio of analytic functions with higher order poles) What is the residue at z_0 if B has a higher order zero: $B(z) = B_N(z - z_0)^N + \cdots$?

1.4 Contour integrals on domains with multiple holes

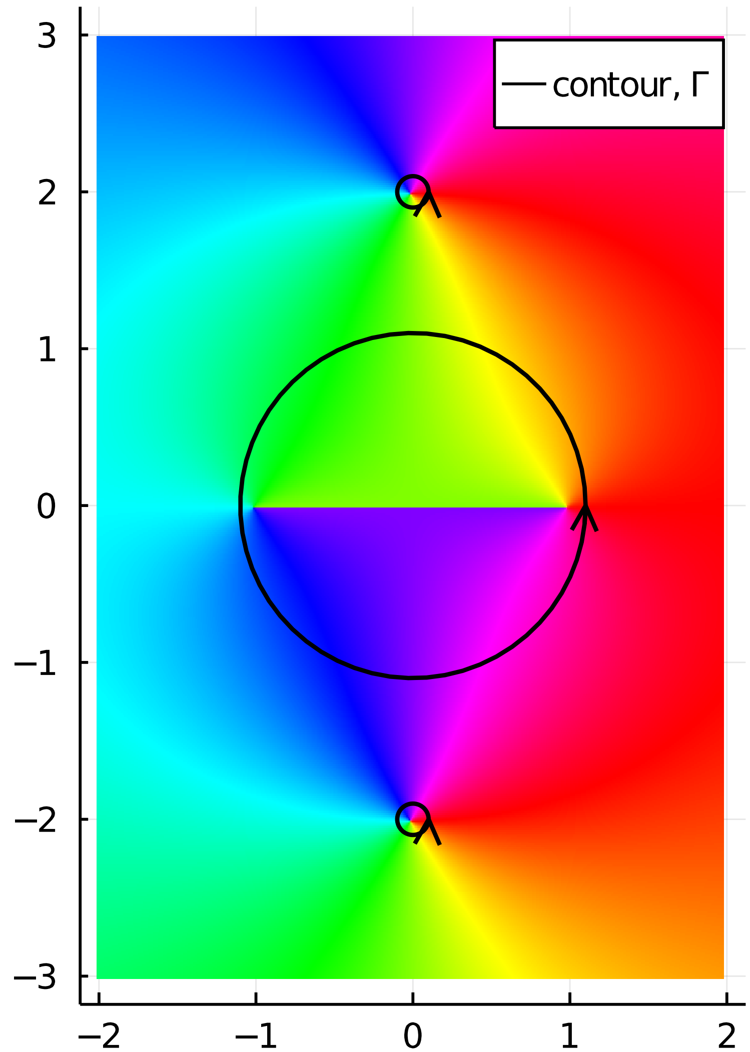
Consider the following example:

$$\frac{\sqrt{z-1}\sqrt{z+1}}{z^2+4}$$

We still have the contour integral over a circle, and so *Proposition (Residue on a circle)* still holds true for $r > 2$. But we can also deform the contour into three contours:

```
using ApproxFun, Plots, ComplexPhasePortrait
f = z -> sqrt(z-1)sqrt(z+1)/(z^2+4)

Γ = Circle(1.1) ∪ Circle(2.0im,0.1) ∪ Circle(-2.0im,0.1)
phaseplot(-2..2, -3..3, f)
plot!(Γ; color=:black, label="contour, Γ", arrow=true,
linewidth=1.5)
```




```
sum(Fun(f, Circle(2.1))), sum(Fun(f, Γ))
```

```
(-6.139435942503095e-16 + 6.283185307179586im, -1.1518915294168 3e-15
+ 6.28
3185307179586im)
```

Thus we can sum over three residues.

1.4.1 Residue theorem

Theorem (Cauchy's Residue Theorem) Let f be holomorphic inside and on a simple closed, positively oriented contour γ except at isolated points $z_1, \ldots, z_r$ inside γ. Then

$$\oint_\gamma f(z)dz = 2\pi i \sum_{j=1}^{r} \operatorname*{Res}_{z=z_j} f(z)$$

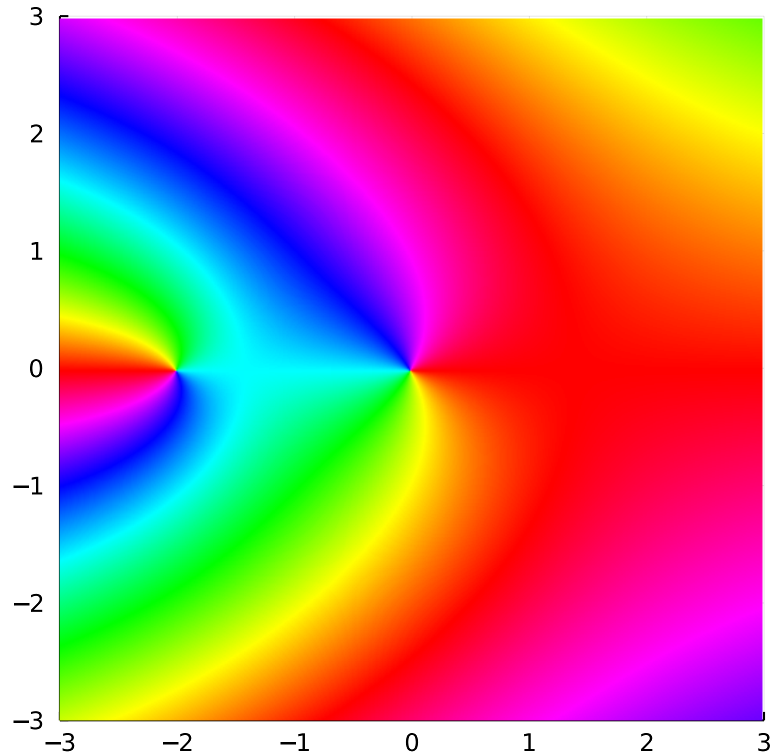
1.4.2 Calculating integrals

We can use the Residue theorem to calculate "hard" integrals. First, two trivial examples: consider

$$f(z) = \frac{\mathrm{e}^z}{z(z+2)}$$

This has two simple poles, one at $z = 0$ and one at $z = -2$, as seen clearly from the phase portrait:

```
f = z -> exp(z)/(z*(z+2))
phaseplot(-3..3, -3..3, f)
```



Consider integrating over C_3, a circle of radius 3 centred at the origin. The residues are

$$\operatorname*{Res}_{z=0} f(z) = \operatorname*{Res}_{z=0} \frac{1}{z}\frac{\mathrm{e}^z}{z+2} = \frac{1}{2}$$
$$\operatorname*{Res}_{z=-2} f(z) = \operatorname*{Res}_{z=-2} \frac{1}{z+2}\frac{\mathrm{e}^z}{z} = -\frac{\mathrm{e}^{-2}}{2}$$

Thus the integral must be equal to $2\pi\mathrm{i}(1/2 - \mathrm{e}^{-2}/2)$. This matches a numerical approximation of the integral

```
sum(Fun(f, Circle(3.0))), 2π*im*(1/2 - exp(-2)/2)
```

```
(-4.61073616575903e-16 + 2.7164243220021564im, 0.0 +
2.716424322002157im)
```

A more complicated example is

$$f(z) = \frac{\mathrm{e}^z}{z^2(z+2)}$$

which has a double pole at $z = 0$. We find the residue by expanding in Taylor series:

$$\frac{1}{z^2}\frac{\mathrm{e}^z}{z+2} = \frac{1}{2z^2}(1 + z + O(z^2))(1 - z/2 + O(z^2)) = \frac{1}{2z^2} + \frac{1}{4z} + O(1)$$

That is,

$$\operatorname*{Res}_{z=0} f(z) = \frac{1}{4} \qquad \text{and} \qquad \operatorname*{Res}_{z=-2} f(z) = \frac{\mathrm{e}^{-2}}{4}$$

Again, residue calculus matches the numerical computation:

```
sum(Fun(z -> exp(z)/(z^2*(z+2)), Circle(3.0))), 2*pi*im * (1/4 +
exp(-2)/4)
```

```
(-2.6019610263896015e-16 + 1.7833804925887144im, 0.0 +
1.78338049258715im)
```