1 Lecture 11: Representing analytic functions by their behaviour near singularities

A *key* theme in complex analysis is representing functions by their behaviour near singularities. A simple example of this is a partial fraction expansion: a rational function $p(z)/q(z)$ can be expressed as a sum of its behaviour near poles and infinity. This is more complicated, but doable in a systematic manner for functions with branch cuts. In this lecture we:

1. Derive partial fraction expansions using Cauchy's integral formula
2. Recover functions such as $\sqrt{z-1}\sqrt{z+1}$ from their behaviour on the branch cut

1.1 Partial fraction expansion

Theorem (Cauchy's integral representation around holes) Let $D \subset \mathbb{C}$ be a domain with g holes (i.e., genus g). Suppose f is holomorphic in and on the boundary of D. Given g simple closed negatively oriented contours surrounding the holes $\gamma_1, \ldots, \gamma_g$ and a simple closed positively oriented contour γ_∞ surrounding the outer boundary of D, we have for $z \in D$,

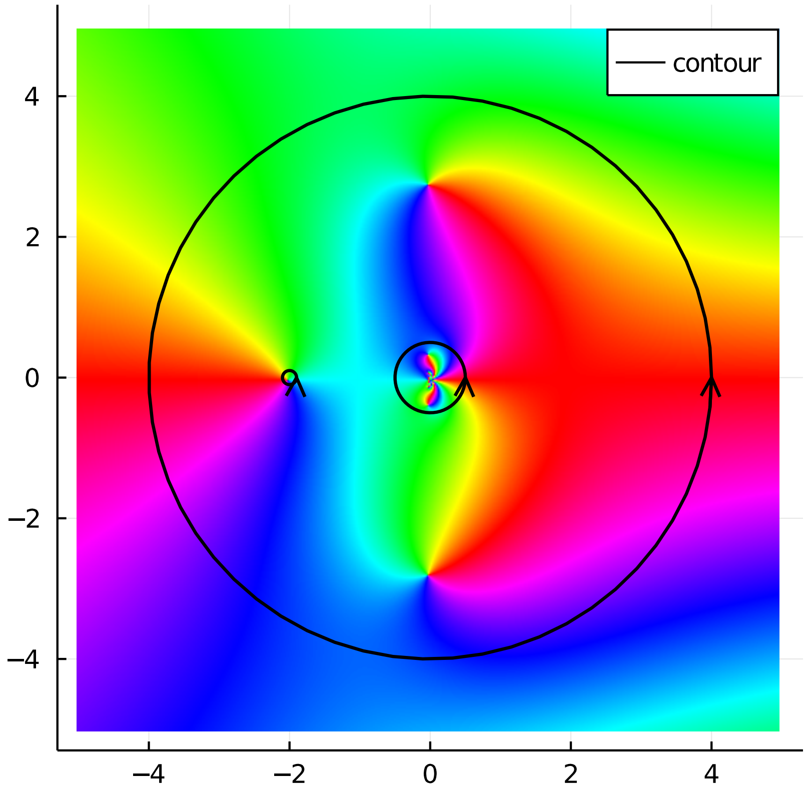
$$f(z) = \frac{1}{2\pi\mathrm{i}}\left[\sum_{k=1}^{g}\oint_{\gamma_k} + \oint_{\gamma_\infty}\right]\frac{f(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

Here is an example. Consider

$$f(z) = (\mathrm{e}^{1/z} + \mathrm{e}^{z})/(z(z+2))$$

which has an essential singularity at 0 and ∞ and a simple pole at -2. We can recover f from contours around each singularity:

```
using ApproxFun, ComplexPhasePortrait, Plots
f = z -> (exp(1/z) + exp(z))/(z*(z+2))
Γ = Circle(0.0, 4.0) ∪ Circle(0.0,0.5,false) ∪ Circle(-2.0,0.1,false)
phaseplot(-5..5, -5..5, f)
plot!(Γ; color=:black, label="contour", arrow=true, linewidth=1.5)
```



Cauchy's integral formula is still valid:

```
ζ = Fun(Γ)
z = 2.0+1.0im
sum(f.(ζ)/(ζ - z))/(2π*im), f(z)
```

```
(0.8671607060038516 + 0.10261889457156094im, 0.8671607060038514 + 0.1026188
9457156062im)
```

Now we specialise to the case where we have a rational function

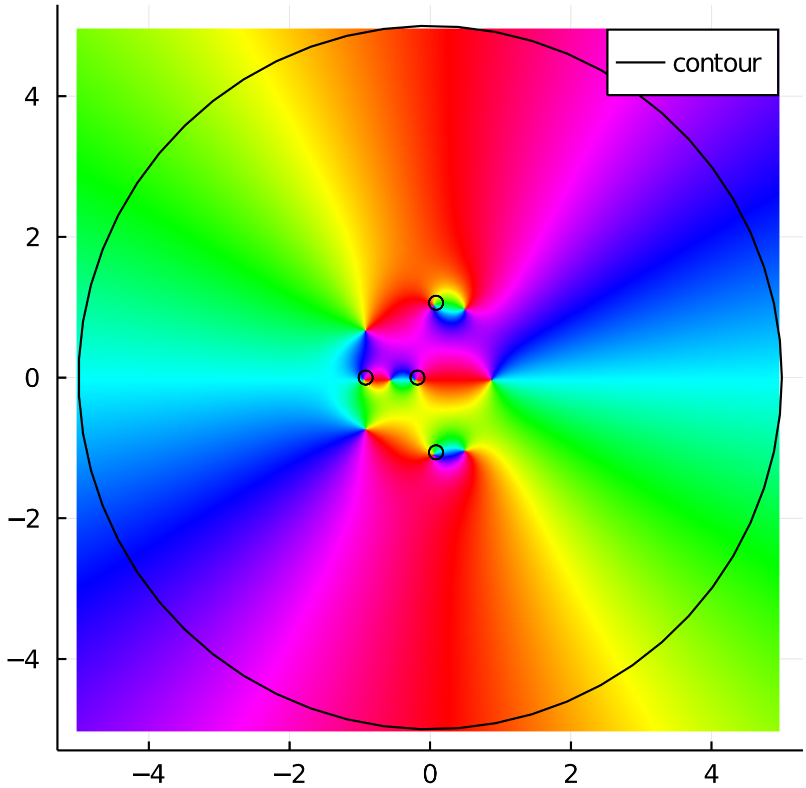
$$r(z) = \frac{p(z)}{q(z)}$$

where p, q are both polynomials. This is analytic everywhere apart from the roots of q, which we enumerate $\lambda_1, \ldots, \lambda_g$. If we integrate over negatively oriented circles around each root:

```
n = 7
m = 5
p = Fun(Taylor(), randn(n))
q = Fun(Taylor(), randn(m))
λ = complexroots(q)

Γ = Circle(0.0, 5.0)
for λ in λ
    global Γ
    Γ = Γ ∪ Circle(λ, 0.1, false)
end
r = z -> extrapolate(p,z)/extrapolate(q,z)

phaseplot(-5..5, -5..5, r)
plot!(Γ; color=:black, label="contour")
```



we recover the function:

```
ζ = Fun(Γ)
z = 2.0+2.0im
sum(r.(ζ)/(ζ - z))/(2π*im) , r(z)
```

```
(2.2559772942300795 - 9.350942597392965im, 2.25597729423008 - 9.35094259739
2963im)
```

But now we can use the residue theorem to simplify the integrals!

Near the jth root we have the Laurent series

$$r(z) = r^j_{-N_j}(z-\lambda_j)^{-N_j} + \cdots + r^j_{-1}(z-\lambda_j)^{-1} + r^j_0 + r^j_1(z-\lambda_j) + \cdots$$

where N_j is the order of the zero of $q(z)$ at λ_j.

Then it follows that

$$\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_j}\frac{r(\zeta)}{z-\zeta}\mathrm{d}\zeta = r^j_{-N_j}(z-\lambda_j)^{-N} + \cdots + r^j_{-1}(z-\lambda_j)^{-1}$$

for z outside the contour γ_j.

Similarly, for the contour around infinity γ_∞, if we have the Laurent series

$$r(z) = \cdots + r^\infty_{-1}z^{-1} + r^\infty_0 + r^\infty_1 z + \cdots + r^\infty_{N_0}z^{N_0}$$

where N_∞ is the degree of $p(z)$ minus the degree of $q(z)$. Then we have

$$\frac{1}{2\pi\mathrm{i}}\oint_{\gamma_\infty}\frac{r(\zeta)}{\zeta-z}\mathrm{d}\zeta = r^\infty_0 + r^\infty_1 z + \cdots + r^\infty_{N_\infty}z^{N_\infty}.$$

Thus we have the expansion summing over the behaviour near each singularity that holds for all z:

$$r(z) = \sum_{k=0}^{N_\infty} r_k^\infty z^k + \sum_{j=1}^{d} \sum_{k=-N_j}^{-1} r_k^j (z-\lambda_j)^k$$

Example When we only have simple poles and no polynomial growth at ∞, this has a simple form in terms of residues:

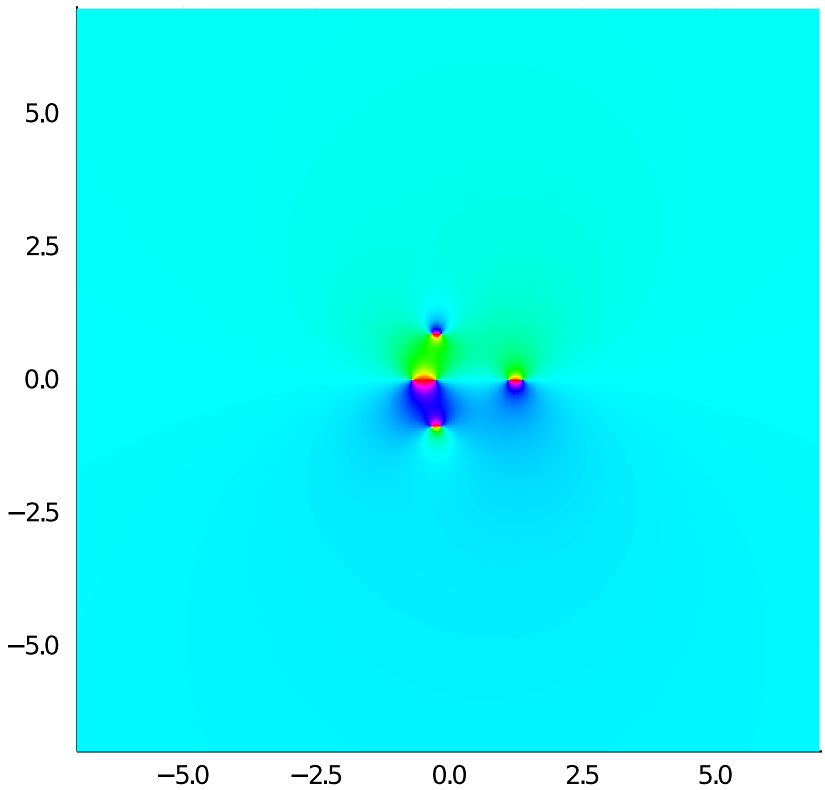
$$r(z) = r(\infty) + \sum_{j=1}^{d} (z-\lambda_j)^{-1} \operatorname*{Res}_{z=\lambda_j} r(z)$$

Here we demonstrate it on a random polynomial:

```
n = 5
m = 5
p = Fun(Taylor(), randn(n))
q = Fun(Taylor(), randn(m))
λ = complexroots(q)

r = z -> extrapolate(p,z)/extrapolate(q,z)

phaseplot(-7..7, -7..7, r)
```



This constructs `r_2` as the partial fraction expansion of `r`:

```
res = extrapolate.(p,λ)./extrapolate.(q',λ)
r∞ = p.coefficients[n]/q.coefficients[m]

r_2 = z -> r∞ + sum(res.*(z .- λ).^(-1))
```

```
z = 0.1+0.2im
r(z) - r_2(z) # we match to high accuracy
```

```
-4.440892098500626e-16 + 4.440892098500626e-16im
```

1.2 Recovering analytic functions

We now consider the above approach for 2 examples with branch cuts.

Example 1

Consider $\phi(z) = \log(z-1) - \log(z+1)$. For $x < -1$ the branch cuts cancel and we have

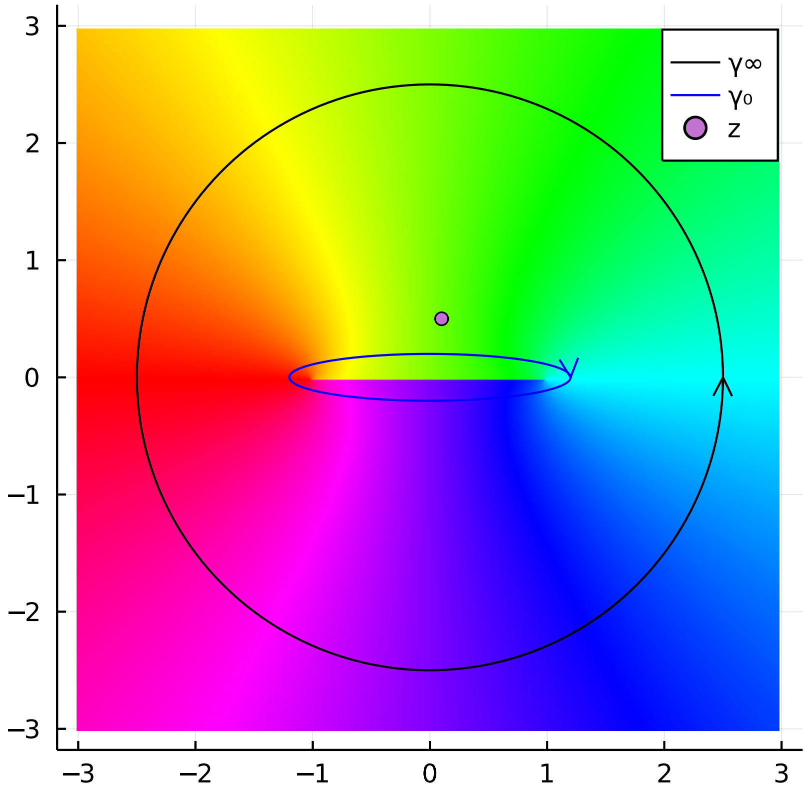
$$\phi_+(x) = \lim_{\epsilon \to 0^+} \phi(x+\mathrm{i}\epsilon) = \log_+(x-1) - \log_+(x+1) = \log|x-1| + \mathrm{i}\pi - \log|x+1| - \mathrm{i}\pi = \log(1-x) - \log(-1-x).$$

Similarly

$$\phi_-(x) = \log(1-x) - \log(-1-x) = \phi_+(x)$$

i.e., we are continuous on the branch cut (with $\phi(x) := \phi_+(x)$) and therefore analytic. Thus $\phi(z)$ is analytic off $[-1,1]$ which can be seen clearly from a phase portrait. Using the corollary above we can recover f from integrating over two contours: γ_∞ surrounding ∞ and γ_0 surrounding the branch cut, with z in-between:

```
φ = z -> log(z-1) - log(z+1)
phaseplot(-3..3, -3..3, φ)
θ = range(0,2π; length=200)
plot!(2.5cos.(θ), 2.5sin.(θ); color=:black, label="γ∞", arrow=true)
plot!(1.2cos.(θ), 0.2sin.(-θ); color=:blue, label="γ_0", arrow=true)
scatter!([0.1],[0.5]; label="z")
```



That is, we have

$$\phi(z) = \frac{1}{2\pi\mathrm{i}}\left[\oint_{\gamma_0} + \oint_{\gamma_\infty}\right]\frac{\phi(\zeta)}{\zeta - z}\mathrm{d}\zeta$$

Note that $\phi(z)$ is analytic at ∞ because it has a convergent Taylor expansion in inverse powers of z. For $|z| > 1$,

$$\phi(z) = -2\sum_{k=0}^{\infty}\frac{1}{(2k+1)z^{2k+1}},$$

hence $\phi(\infty) = 0$. We can also show that $\phi(z)$ is analytic at infinity by showing that $\phi(z^{-1})$ is analytic at $z = 0$. It follows from Cauchy's theorem (exterior) that

$$\oint_{\gamma_\infty}\frac{\phi(\zeta)}{\zeta - z}\mathrm{d}\zeta = 0$$

as the integrand decays like $O(\zeta^{-2})$.

We are left with the integral on γ_0. We can think of it as a rectangular contour with contours $[-1-\epsilon-\mathrm{i}\epsilon, -1-\epsilon+\mathrm{i}\epsilon, 1+\epsilon+\mathrm{i}\epsilon, 1+\epsilon-\mathrm{i}\epsilon]$. Letting $\epsilon \to 0$, on the contour above $\phi(z)$ tends to

$$\lim_{\epsilon\to 0}\phi(x+\mathrm{i}\epsilon) = \phi_+(x)$$

and similar to the contour below. Since ϕ only has logarithmic singularities this limit can be done safely. Thus we end up with the expression

$$\phi(z) = \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_0} \frac{\phi(\zeta)}{\zeta - z} \mathrm{d}\zeta = \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\phi_+(x) - \phi_-(x)}{x - z} \mathrm{d}x = \int_{-1}^{1} \frac{1}{x - z} \mathrm{d}x.$$

Example 2

We repeat the above procedure with $\phi(z) = \sqrt{z-1}\sqrt{z+1}$. Again this is analytic off $[-1, 1]$ and we can express it as integrals over γ_0 and γ_∞. Now it grows like z at ∞,

$$\phi(z) = z + O(z^{-1}),$$

hence we have (as above)

$$\frac{1}{2\pi \mathrm{i}} \oint_{\gamma_\infty} \frac{\phi(\zeta)}{\zeta - z} \mathrm{d}\zeta = z.$$

The integral over the contour γ_0 can be collapsed. On the jump $-1 < x < 1$ we have

$$\phi_+(x) = \sqrt{x-1}_+\sqrt{x+1} = \mathrm{i}\sqrt{|x-1|}\sqrt{x+1} = \mathrm{i}\sqrt{1-x}\sqrt{x+1} = \mathrm{i}\sqrt{1-x^2}$$

while $\phi_-(x) = -\phi_+(x) = -\mathrm{i}\sqrt{1-x^2}$. We thus have

$$\begin{aligned}\phi(z) &= z + \frac{1}{2\pi \mathrm{i}} \oint_{\gamma_0} \frac{\phi(\zeta)}{\zeta - z} \mathrm{d}\zeta = z + \frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\phi_+(x) - \phi_-(x)}{x - z} \mathrm{d}x \\ &= z + \frac{1}{\pi} \int_{-1}^{1} \frac{\sqrt{1-x^2}}{x - z} \mathrm{d}x.\end{aligned}$$