Applied Complex Analysis (2021)

1 Revision questions

1.1 Question 1

Consider the solution of the following Laplace's equation, using $z = x + \mathrm{i}y$:

1. $$v_{xx} + v_{yy} = 0$$

   for $z \notin [-1,1] \cup \{\pm 2\}$,

2. $$v(x,y) = \pm \log |z \mp 2| + O(1)$$

   as $z \to \pm 2$,

3. $$v(x,y) = o(1)$$

   as $z \to \infty$, and

4. $$v(x,0) = \kappa$$

   for $-1 < x < 1$ where κ is an unknown constant.

This equation models the potential field of two unit charges of opposite sign at ± 2 with a metal sheet that has no net charge placed on $[-1,1]$.

(a) By writing

$$v(x,y) = \int_{-1}^{1} u(t) \log |t - z| \mathrm{d}t + \log |z - 2| - \log |z + 2|,$$

show that the problem of finding $v(x,y)$ can be reformulated as finding $u(x)$ such that

$$\int_{-1}^{1} u(t) \log |t - x| \mathrm{d}t = f(x),$$

where

$$\int_{-1}^{1} u(x) \mathrm{d}x = 0.$$

What is $f(x)$ in this equation? Explain why $v(x,y)$ will thereby satisfy the required four conditions.

(b) Find $u(x)$. Hint: reduce the problem to one of inverting the Hilbert transform.

(c) What is the value of κ?

1.2 Question 2

The Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}x^n + O(x^{n-1}),$$

where $\alpha > -1$, are orthogonal with respect to

$$\langle f, g \rangle_\alpha = \int_0^\infty f(x)g(x)x^\alpha \mathrm{e}^{-x}\mathrm{d}x,$$

and they satisfy

$$xL_n^{(\alpha)}(x) = -(n+\alpha)L_{n-1}^{(\alpha)}(x) + (2n+\alpha+1)L_n^{(\alpha)}(x) - (n+1)L_{n+1}^{(\alpha)}(x)$$

and

$$L_n^{(\alpha)}(x) = L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x).$$

(a) Show that

$$\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} = -L_{n-1}^{(\alpha+1)}(x).$$

(b) Let

$$\mathbf{L}^{(\alpha)} = \begin{pmatrix} L_0^{(\alpha)}(x) \\ L_1^{(\alpha)}(x) \\ \vdots \end{pmatrix}.$$

Give operators J, D and S such that

$$x\mathbf{L}^{(\alpha)} = J\mathbf{L}^{(\alpha)}, \quad \frac{\mathrm{d}}{\mathrm{d}x}\mathbf{L}^{(\alpha)} = D\mathbf{L}^{(\alpha)}, \quad \mathbf{L}^{(\alpha)} = S\mathbf{L}^{(\alpha+1)}.$$

(c) Suppose $u(x)$ has a weighted Laguerre expansion for $x \in [0, \infty)$,

$$u(x) = \sum_{k=0}^{\infty} \mathrm{e}^{-x/2}L_k(x)u_k = \mathrm{e}^{-x/2}\mathbf{L}^\intercal\mathbf{u}, \qquad \mathbf{u} = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix},$$

where we abbreviate $L_k^{(0)}(x)$ as $L_k(x)$ and $\mathbf{L}^{(0)}$ as $\mathbf{L}$. Use the operators J, D and S to represent the ordinary differential operator

$$u'(x) - xu(x) \qquad \text{for} \qquad x \geq 0$$

as an operator on the coefficients of u, where the range of the operator is specified in $\mathrm{e}^{-x/2}\left(\mathbf{L}^{(1)}\right)^\intercal$.

1.3 Question 3

Let $u(x)$ solve the integral equation

$$\int_0^\infty K(t-x)u(t)\mathrm{d}t = f(x) \qquad \text{for} \qquad x \geq 0,$$

where

$$K(x) = \mathrm{e}^{-|x|} \qquad \text{and} \qquad f(x) = 2 - \mathrm{e}^{-x}.$$

We will use the notations

$$g_{\mathrm{L}}(x) := \begin{cases} g(x) & x < 0 \\ 0 & x \geq 0 \end{cases}, \qquad g_{\mathrm{R}}(x) := \begin{cases} 0 & x < 0 \\ g(x) & x \geq 0 \end{cases},$$

and the Fourier transform

$$\hat{f}(s) := \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t.$$

(a) What are the regions of analyticity of $\hat{K}(s)$, and $\widehat{f_{\mathrm{R}}}(s)$? Assuming that $|u(x)|$ is bounded, what is the region of analyticity of $\widehat{u_{\mathrm{R}}}(s)$? Justify your answers without explicit calculation.

(b) Show that the Fourier transforms satisfy

$$\hat{K}(s) = \frac{2}{1+s^2} \qquad \text{and} \qquad \widehat{f_{\mathrm{R}}}(s) = \frac{2+\mathrm{i}s}{\mathrm{i}s - s^2}.$$

(c) For the integral equation above, set up a RiemannHilbert problem of the form

$$\Phi_+(s) - g(s)\Phi_-(s) = h(s) \qquad \text{for} \qquad s \in (-\infty, \infty) + \mathrm{i}\delta,$$

where $\Phi_+(s)$ is analytic above $(-\infty,\infty)+\mathrm{i}\delta$, $\Phi_-(s)$ is analytic below $(-\infty,\infty)+\mathrm{i}\delta$, $\Phi_\pm(s)$ decay at infinity, and

$$g(s) = \frac{2}{1+s^2}.$$

Explain the choice of δ and the definition of $\Phi_\pm(s)$, $g(s)$ and $h(s)$ in terms of the Fourier transforms of u, f, and K.

(d) Is $g(s)$ degenerate? Explain why or why not.

(e) Find a solution to the homogeneous RiemannHilbert problem

$$\kappa_+(s) = g(s)\kappa_-(s) \qquad \text{for} \qquad s \in (-\infty,\infty) + \mathrm{i}\delta$$

such that $\kappa_+(s) = o(1)$ and $\kappa_-(s) = s + O(1)$ as $s \to \infty$, where δ is the same constant as in (c).

(f) Determine $u(x)$.
