Applied Complex Analysis (2021)

1 Problem sheet 3

1.1 Problem 1.1

Use the Plemelj formulae to calculate the following:

1.
$$\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{(1+t^2)(t-z)} \mathrm{d}t$$
for $z \notin [-1,1]$.

2.
$$\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{1}{(t-z)(2+t)} \mathrm{d}t$$
for $z \notin [-1,1]$.

3.
$$⨍_{-1}^{1} \frac{t}{(t-x)\sqrt{1-t^2}} \mathrm{d}t$$
for $-1 < x < 1$.

1.2 Problem 1.2

Find all solutions $\phi(z)$ analytic on $\bar{\mathbb{C}} \backslash [-1,1]$ with weaker than pole singularities satisfying the following, where $-1 < x < 1$:

1.
$$\phi_+(x) + \phi_-(x) = 1$$
and $\phi(\infty) = 0$

2.
$$\phi_+(x) + \phi_-(x) = 0$$
and $\phi(\infty) = 1$

3.
$$\phi_+(x) + \phi_-(x) = \sqrt{1-x^2}$$
and $\phi(\infty) = 0$

4.
$$\phi_+(x) + \phi_-(x) = \frac{1}{x^2+1}$$
$\phi(\infty) = 0$ and $\lim_{z\to\infty} z\phi(z) = 0$.

1.3 Problem 1.3

Use Plemelj formulae to find all solutions $u(x)$ defined on $[-1,1]$ to the following, where $-1 < x < 1$:

1. $$\frac{1}{\pi}\fint_{-1}^{1} \frac{u(t)}{t-x}\mathrm{d}t = \frac{x}{\sqrt{1-x^2}}$$
   .

2. $$\frac{1}{\pi}\fint_{-1}^{1} \frac{u(t)}{t-x}\mathrm{d}t = \frac{1}{2+x}$$
   where u is bounded at the right-endpoint.

In the following problems, use only the definitions

$$\log z = \int_1^z \frac{1}{\zeta}\mathrm{d}\zeta \qquad \text{for} \qquad z \notin (-\infty, 0]$$
$$\log_\pm x = \log(x \pm \mathrm{i}\epsilon) \qquad \text{for} \qquad x \in (-\infty, 0]$$

For example, do not use $\log z = \log|z| + \mathrm{i}\arg z$ as we need to prove it first! You can use the result from lectures that $\log z^{-1} = -\log z$ and $\log_\pm x = \log|x| \pm \mathrm{i}\pi$.

1.4 Problem 2.1

Show that $\log(ab) = \log a + \log b$ provided that the closed contour defined by the oriented line segments $\gamma = [1, 1/b] \cup [1/b, a] \cup [a, 1]$ does not surround the origin. Show that if γ surrounds the origin counter-clockwise then

$$\log(ab) = \log a + \log b + 2\pi\mathrm{i}$$

and if γ surrounds the origin clockwise then

$$\log(ab) = \log a + \log b - 2\pi\mathrm{i}$$

Use $\log_\pm x$ to express the equivalent formulae for all cases where γ passes through the origin.

1.5 Problem 2.2

Show that $\overline{\log z} = \log \bar{z}$. Use this to show that $\log z = \log|z| + \mathrm{i}\arg z$. (Hint: deform along an arc for the imaginary part.)

1.6 Problem 2.3

Consider $\log_1 z$ defined off $[0, \infty)$ by

$$\log_1 z := \begin{cases} \log z & \text{if } \Im z > 0 \\ \log_+ z & \text{if } \Im z = 0 \text{ and } z < 0 \\ \log z + 2\pi \mathrm{i} & \text{if } \Im z < 0 \end{cases}$$

show that $\log_1 z$ is analytic in $\mathbb{C} \backslash [0, \infty)$ and for $x > 0$

$$\lim_{\epsilon \to 0} \log_1(x - \mathrm{i}\epsilon) = \lim_{\epsilon \to 0} \log_1(x + \mathrm{i}\epsilon) + 2\pi \mathrm{i}.$$

(This is the analytic continuation of $\log z$ over its branch cut.)

Consider the Cauchy transform over the unit circle $\mathbb{T} = \{z : |z| = 1\}$:

$$\mathcal{C}_{\mathbb{T}} f(z) = \frac{1}{2\pi \mathrm{i}} \oint \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Denote the limit from the left/right (inside/outside) as

$$\mathcal{C}_{\mathbb{T}}^{+} f(\zeta) = \lim_{\epsilon \to 0} \mathcal{C}_{\mathbb{T}} f((1 - \epsilon)\zeta)$$
$$\mathcal{C}_{\mathbb{T}}^{-} f(\zeta) = \lim_{\epsilon \to 0} \mathcal{C}_{\mathbb{T}} f((1 + \epsilon)\zeta)$$

1.7 Problem 3.1

Assuming that f is analytic in an annulus containing $\mathbb{T}$, show that $\mathcal{C}_{\mathbb{T}} f(z)$ satisfies the following (Plemelj formulae on the circle):

1. $$\mathcal{C}_{\mathbb{T}} f(z)$$
   is analytic in $\bar{\mathbb{C}} \backslash \mathbb{T}$

2. $$\mathcal{C}_{\mathbb{T}}^{+} f(\zeta) - \mathcal{C}_{\mathbb{T}}^{-} f(\zeta) = f(\zeta)$$

3. $$\mathcal{C}_{\mathbb{T}} f(\infty) = 0$$
   .

1.8 Problem 3.2

Show that it is the unique function $\phi(z)$ satisfying

1.
$$\phi(z)$$
is analytic in $\bar{\mathbb{C}}\backslash\mathbb{T}$

2.
$$\phi^+(\zeta) - \phi^-(\zeta) = f(\zeta)$$
where f is analytic in an annulus containing $\mathbb{T}$,

3.
$$\phi(\infty) = 0$$
.

where

$$\phi^+(\zeta) = \lim_{\epsilon \to 0} \phi((1-\epsilon)\zeta),$$
$$\phi^-(\zeta) = \lim_{\epsilon \to 0} \phi((1+\epsilon)\zeta)$$

You can assume that $\phi^{\pm}(\zeta)$ converges uniformly.

1.9 Problem 3.3

What is $\mathcal{C}_{\mathbb{T}}[\diamond^k](z)$? What about $\Re\mathcal{C}_{\mathbb{T}}^-[\diamond^k](\zeta)$ and $\Im\mathcal{C}_{\mathbb{T}}^-[\diamond^k](\zeta)$? Express your answers separately for negative and non-positive k and justify the answers using 3.1 and 3.2.

1.10 Problem 3.4

Construct the solution to ideal fluid flow around a circle, that is, to find a function $v(x, y)$ satisfying

1.
$$v_{xx} + v_{yy} = 0$$
for $x^2 + y^2 > 1$.

2.
$$v(x, y) \sim y \cos\theta - x \sin\theta$$

3.
$$v(x, y) = 0$$
for $x^2 + y^2 = 1$.

We investigate the solutions to $\phi^+(x) - c\phi^-(x) = f(x)$ on $[-1, 1]$.

1.11 Problem 4.1

Describe all solutions $\kappa(z)$ to

1. $$\kappa(z)$$
   is analytic off $[-1, 1]$.
2. $$\kappa(\infty) = 0$$
   .
3. $$\kappa$$
   has weaker than pole singularities at ± 1.
4. $\kappa^+(x) - \mathrm{e}^{\mathrm{i}\theta}\kappa^-(x) = 0$ for $-1 < x < 1$.

Show that your answer satisfies all three properties.

1.12 Problem 4.2

Construct all solutions $\phi(z)$ to

1. $$\phi(z)$$
   is analytic off $[-1, 1]$.
2. $$\phi(\infty) = 0$$
   .
3. $$\phi$$
   has weaker than pole singularities at ± 1.
4. $$\phi^+(x) - \mathrm{e}^{\mathrm{i}\theta}\phi^-(x) = f(x)$$
   .

in terms of a Cauchy transform involving f. You can assume that f is smooth (infinitely-differentiable on $[-1, 1]$), and use the fact that

$$\mathcal{C}_{[-1,1]}[(1 - \diamond)^\alpha (1 + \diamond)^\beta f(\diamond)](z)$$

is bounded for all z if $\alpha, \beta > 0$ when f is smooth.

1.13 Problem 4.3

Let $c \in \mathbb{C}$. Repeat Problem 4.1 and Problem 4.2 for $\phi^+(x) - c\phi^-(x) = f(x)$.

We investigate the solutions to $\phi^+(x) + \phi^-(x) = f(x)$ on two intervals $(-1, -a) \cup (a, 1)$, where $0 < a < 1$.

1.14 Problem 5.1

Show that $\kappa(z) = \frac{1}{\sqrt{z-1}\sqrt{z-a}\sqrt{z+a}\sqrt{z+1}}$ satisfies the following properties:

1. $$\kappa(z)$$
   is analytic off $[-1, -a] \cup [1, a]$.
2. $$\kappa^+(x) + \kappa^-(x) = 0$$
   for $a < |x| < 1$.
3. $$\kappa$$
   has weaker than pole singularities everywhere.
4. $$\kappa(\infty) = 0$$
   .

1.15 Problem 5.2

Describe all solutions $\psi(z)$ satisfying:

1. $$\psi(z)$$
   is analytic off $[-1, -a] \cup [1, a]$.
2. $$\psi^+(x) + \psi^-(x) = 0$$
   for $a < |x| < 1$.
3. $$\psi$$
   has weaker than pole singularities everywhere.
4. $$\psi(\infty) = 0$$
   .

1.16 Problem 5.3

Assuming $f(x)$ is smooth (infinitely differentiable on $[-1,-a] \cup [a,1]$), express in terms of a Cauchy transform involving f all solutions $\phi(z)$ to the following:

1. $$\phi(z)$$

   is analytic off $[-1,-a] \cup [1,a]$.

2. $$\phi^+(x) + \phi^-(x) = f(x)$$

   for $a < |x| < 1$.

3. $$\phi$$

   has weaker than pole singularities everywhere.

4. $$\phi(\infty) = 0$$

   .

1.17 Problem 6.1

Describe the limiting distribution of electric charges under the potential $V(x) = x^4$, that is, the limit of

$$\frac{\mathrm{d}^2\lambda_k}{\mathrm{d}t^2} + \frac{\mathrm{d}\lambda_k}{\mathrm{d}t} = \sum_{\substack{j=1 \\ j\neq k}}^{N} \frac{1}{\lambda_k - \lambda_j} - V'(\lambda_k)$$

for $k = 1, \ldots, N$ as $N \to \infty$. (Hint: scale λ_k appropriately.)
