1 Solution Sheet 5

1.1 Problem 1

1. Define $C_k^{(\alpha)}(z) = \mathcal{C}[L_k^{(\alpha)} \diamond^\alpha \mathrm{e}^{-\diamond}](z)$ and recall that

$$C_1(z) = \frac{\frac{1}{2\pi \mathrm{i}}\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x + (z-a_0)C_0(z)}{b_0} = -\frac{1}{2\pi\mathrm{i}} - (z-1)C_0(z)$$
$$= \frac{(z-1)\mathrm{e}^{-z}\mathrm{Ei}\,\mathrm{z} - 1}{2\pi\mathrm{i}}$$

We abbreviate $C_k^{(0)}(z)$ as $C_k(z)$ and we will also abbreviate $L_k^{(0)}$ as L_k.

Here we double check the formula, noting that $L_1(x) = \mathrm{e}^x \frac{\mathrm{d}}{\mathrm{d}x} x\mathrm{e}^{-x} = 1 - x$:

```
using ApproxFun, SingularIntegralEquations, Plots, QuadGK, LinearAlgebra,
SpecialFunctions

const ei_−_1 = let ζ = Fun(-100 .. -1)
    sum(exp(ζ)/ζ)
end
function ei(z)
    ζ = Fun(Segment(-1 , z))
    ei_−_1 + sum(exp(ζ)/ζ)
end

x = Fun(0..10)
w = exp(-x)
z = 1+im
cauchy((1-x)*w, z),((z-1)*exp(-z)*ei(z)-1)/(2π*im)
```

```
(0.018684644298457894 + 0.048361335653350004im, 0.01868392300262892 + 0.048
36848799089318im)
```

We now use these to determine the results with $\alpha = 1$. Note that:

$$C_0^{(1)}(z) = \mathcal{C}[\diamond \mathrm{e}^{-\diamond}](z) = C_0(z) - C_1(z) = \frac{-\mathrm{e}^{-z}\mathrm{Ei}\,\mathrm{z} - (\mathrm{z}-1)\mathrm{e}^{-z}\mathrm{Ei}\,\mathrm{z} + 1}{2\pi\mathrm{i}}$$

```
cauchy(x*w, z),(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1)/(2π*im)
```

```
(0.09210173751684986 - 0.029676691354892096im, 0.09210253209837325 - 0.0296
8456498826427im)
```

Therefore, we have

$$
\begin{aligned}
C_1^{(1)}(z) &= \frac{\frac{1}{2\pi\mathrm{i}}\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x + (z-a_0^{(1)})C_0^{(1)}(z)}{b_0^{(1)}} \\
&= \frac{\frac{1}{2\pi\mathrm{i}} + (z-2)C_0^{(1)}(z)}{-1} \\
&= \frac{1+(z-2)(-\mathrm{e}^{-z}\mathrm{Ei}\,\mathrm{z} - (\mathrm{z}-1)\mathrm{e}^{-\mathrm{z}}\mathrm{Ei}\,\mathrm{z}+1)}{-2\pi\mathrm{i}}
\end{aligned}
$$

Let's check the result using

$$
L_1^{(1)}(x) = x^{-1}\mathrm{e}^x \frac{\mathrm{d}}{\mathrm{d}x} x^2\mathrm{e}^{-x} = 2-x
$$

```
cauchy((2-x)*x*w, z),(1+(z-2)*(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1))/(-2π*im)
```

```
(0.0624250461619579 + 0.037297032364538574im, 0.06241796711010899 + 0.03736
784600525782im)
```

2.

We have

$$
\int_x^\infty L_2(x)\mathrm{e}^{-x}\mathrm{d}x = \frac{1}{2}x\mathrm{e}^{-x}L_1^{(1)}(x)
$$

Thus from lectures we have

$$
\frac{1}{\pi}\int_0^\infty L_2(x)\mathrm{e}^{-x}\log(z-x)\mathrm{d}x = \mathrm{i}\mathcal{C}[\diamond\mathrm{e}^{-\diamond}L_1^{(1)}](z)
$$

and therefore

$$
\frac{1}{\pi}\int_0^\infty L_2(x)\mathrm{e}^{-x}\log|z-x|\mathrm{d}x = \Im\mathcal{C}[\diamond\mathrm{e}^{-\diamond}L_1^{(1)}](z) = -\Re\frac{1+(z-2)(\mathrm{e}^{-\mathrm{z}}\mathrm{Ei}\,\mathrm{z}-(\mathrm{z}-1)\mathrm{e}^{-\mathrm{z}}\mathrm{Ei}\,\mathrm{z}+1)}{-2\pi}
$$

Let's check the result:

```
x = Fun(0 .. 100)
w = exp(-x)
z = 2+im
```

```
sum(1/2*(2 - 4x + x^2)*w*log(abs(z-x)))/π,real(sum(1/2*(2 - 4x + x^2)*w*log(z-x))/(π))
```

```
(0.0697232345397132, 0.06972323453971359)
```

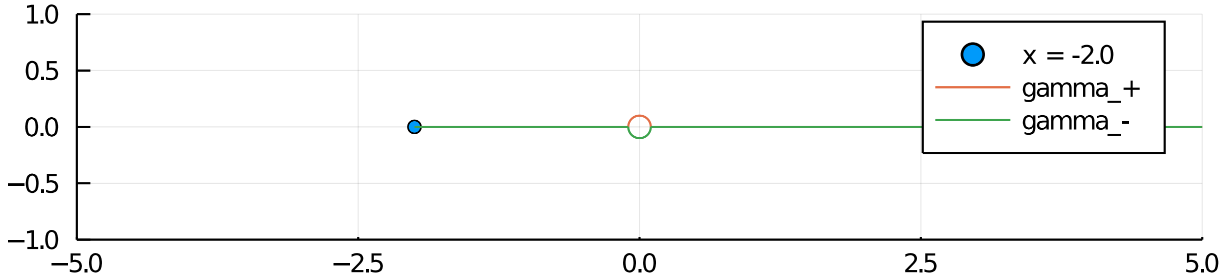
```
imag(cauchy((2-x)*x*w,z)),-real((1+(z-2)*(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1))/(-2π))
```

```
(0.06972323454064205, 0.06972323454061675)
```

1.1.1 Problem 2

Consider integration contours γ_{+x} and γ_{-x} that avoid 0 above and below:

```
x = -2.0
r = 0.1
γ_+_𝑥 = Segment(-2.0 , -r) ∪ Arc(0.,r, (π,0)) ∪ Segment(r , 100)
γ_−_𝑥 = Segment(-2.0 , -r) ∪ Arc(0.,r, (-π,0)) ∪ Segment(r , 100)
scatter([x],[0.0];label="x = $x",ratio = 1.0)
plot!(γ_+_𝑥 ; xlims=(-5,5), ylims=(-1,1), label="gamma_+")
plot!(γ_−_𝑥; xlims=(-5,5), ylims=(-1,1), label="gamma_-")
```



So that

$$\Gamma_{\pm}(\alpha, x) = \int_{\gamma_{\pm x}} \zeta^{\alpha-1} \mathrm{e}^{-\zeta} \mathrm{d}\zeta$$

Note that

$$\int_x^{-r} (\zeta_+^{\alpha-1} - \mathrm{e}^{2\mathrm{i}\pi\alpha} \zeta_-^{\alpha-1}) \mathrm{e}^{-\zeta} \mathrm{d}\zeta = 0$$

since $\zeta_+^{\alpha-1} = \mathrm{e}^{\pi\mathrm{i}(\alpha-1)} |\zeta|^{\alpha-1} = \mathrm{e}^{2\mathrm{i}\pi\alpha} \zeta_-^{\alpha-1}$. Furthermore, the integrals over the arcs tend to zero as $r \to 0$:

$$|\mathrm{i} r^\alpha \int_0^\pi \mathrm{e}^{-r\mathrm{e}^{\mathrm{i}\theta}} \mathrm{e}^{\mathrm{i}\theta\alpha} \mathrm{d}\theta| \leq r^\alpha \pi \mathrm{e}^r \to 0$$

and similarly on the lower arc. Thus we have

$$
\begin{aligned}
\Gamma_+(\alpha,x) - \mathrm{e}^{2\mathrm{i}\pi\alpha}\Gamma_-(\alpha,x) &= \lim_{r\to 0}\left(\int_{\gamma_{+x}} - \mathrm{e}^{2\mathrm{i}\pi\alpha}\int_{\gamma_{-x}}\right)\zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta \\
&= (1-\mathrm{e}^{2\mathrm{i}\pi\alpha})\int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x = (1-\mathrm{e}^{2\mathrm{i}\pi\alpha})\Gamma(\alpha)
\end{aligned}
$$

Note that, for $0 < \alpha < 1$,

$$\psi(z) = z^{-\alpha}\mathrm{e}^{z}\Gamma(\alpha, z)$$

has the following properties:

1. $$\psi(z)$$

   decays as $z \to \infty$, via integration by parts:

$$z^{-\alpha}\mathrm{e}^{z}\int_z^\infty \zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta = z^{-1} + z^{-\alpha}\int_z^\infty \zeta^{\alpha-2}\mathrm{e}^{z-\zeta}\mathrm{d}\zeta$$

and we have assuming z is bounded away from the negative real axis:

$$\left|\int_z^\infty \zeta^{\alpha-2}\mathrm{e}^{z-\zeta}\mathrm{d}\zeta\right| \leq \int_z^\infty |\zeta|^{\alpha-2}\mathrm{d}\zeta = \int_0^\infty |x+z|^{\alpha-2}\mathrm{d}x < \infty$$

(otherwise one would use a deformed contour).

2. We have the subtractive jump:

$$
\begin{aligned}
\psi_+(x) - \psi_-(x) &= \mathrm{e}^{x}(x_+^{-\alpha}\Gamma_+(\alpha,x) - x_-^{-\alpha}\Gamma_-(\alpha,x)) \\
&= \mathrm{e}^{x}|x|^{\alpha}(\mathrm{e}^{-\mathrm{i}\pi\alpha}\Gamma_+(\alpha,z) - \mathrm{e}^{\mathrm{i}\pi\alpha}\Gamma_-(\alpha,x)) \\
&= \mathrm{e}^{x}|x|^{\alpha}\mathrm{e}^{-\mathrm{i}\pi\alpha}(1-\mathrm{e}^{2\mathrm{i}\pi\alpha})\Gamma(\alpha)
\end{aligned}
$$

We use these properties to verify that

$$\mathcal{C}[\diamond^{\alpha}\mathrm{e}^{-\diamond}](z) = \frac{1}{\Gamma(-\alpha)}\frac{(-z)^{\alpha}\mathrm{e}^{-z}\Gamma(-\alpha,-z)}{\breve{a}\mathrm{e}^{-\mathrm{i}\pi\alpha} - \mathrm{e}^{\mathrm{i}\pi\alpha}}$$

via Plemelj.

```
x = Fun(0 .. 20.0)
α = -0.1
z = 2.0+im
cauchy(x^α*exp(-x), z)
```

```
0.07199876331437781 + 0.05850612397322304im
```

```
Γ = (α,z) -> let ζ = z + Fun(0 .. 500.0)
    linesum(ζ^(α-1)*exp(-ζ))
end

-(-z)^α*exp(-z)Γ(-α,-z)/(gamma(-α)*(exp(im*π*α)-exp(-im*π*α)))
```

```
0.07199876331505128 + 0.05850612396048847im
```

1.2 Problem 3

1.2.1 Problem 3.1

We know that $L[a(z)]^{-1} = L[a(z)^{-1}]$ hence it's really about the Laurent series of $a(z)^{-1}$. We see that the roots of $a(z)$ satisfy

$$0 = z^2 a(z) = z^4 - 4z^2 + 1.$$

Using the quadratic formula with $w = z^2$ we have

$$w = 2 \pm \sqrt{3} \Rightarrow z = \pm\sqrt{2 \pm \sqrt{3}}.$$

Since $2 - \sqrt{3} < 1$ and $2 + \sqrt{3} > 1$ we have the factorisation

$$a(z) = \underbrace{z^2 - z_+}_{\phi_+(z)} \underbrace{1 - z_-/z^2}_{\phi_-(z)}$$

for $z_\pm = 2 \pm \sqrt{3}$. We can take the reciprocal of $\phi_\pm$ using Geometric series, that is

$$\begin{aligned}
\phi_+(z)^{-1} &= -\frac{1}{z_+}\frac{1}{1 - z^2/z_+} \\
&= -\frac{1}{z_+} - \frac{z^2}{z_+^2} - \frac{z^4}{z_+^3} - \cdots \\
\phi_-(z)^{-1} &= \frac{1}{1 - z_-/z^2} = 1 + \frac{z_-}{z^2} + \frac{z_-^2}{z^4} + \cdots
\end{aligned}$$

Thus we have

$$a(z)^{-1} = \phi_+(z)^{-1}\phi_-(z)^{-1} = \sum_{k=-\infty}^{\infty} b_{2k} z^{2k}$$

where for $k \geq 0$

$$b_{2k} = -\sum_{j=0}^{\infty} \frac{z_-^j}{z_+^{j+k+1}} = -\frac{z_+^{-k-1}}{1 - z_-/z_+}$$

and for $k < 0$

$$b_{2k} = -\sum_{j=0}^{\infty} \frac{z_-^{j-k}}{z_+^{j+1}} = -\frac{z_-^{-k}}{z_+ - z_-}$$

These give the diagonals of $L[a(z)^{-1}]$.

Verification

A *circulant matrix* is an effective approximation to a Laurent matrix (for reasons beyond the scope of this course, though it intuitively follows since the DFT diagonalises all circulant matrices):

```
using ToeplitzMatrices, ApproxFun, Plots, LinearAlgebra, ComplexPhasePortrait,
SingularIntegralEquations

n = 6
L = Circulant([-4; 1; zeros(n-3); 1])
```

```
6×@*(6 ToeplitzMatrices.Circulant(*@{Float64,Complex{Float64}}:
 -4.0   1.0   0.0   0.0   0.0   1.0
  1.0  -4.0   1.0   0.0   0.0   0.0
  0.0   1.0  -4.0   1.0   0.0   0.0
  0.0   0.0   1.0  -4.0   1.0   0.0
  0.0   0.0   0.0   1.0  -4.0   1.0
  1.0   0.0   0.0   0.0   1.0  -4.0
```

Taking n large, the entries inverse of L approximates the true inverse $L[a(z)]^{-1}$:

```
n = 1000
L = Circulant([-4; 1; zeros(n-3); 1])
inv(L)
```

```
1000×@*(1000 ToeplitzMatrices.Circulant(*@{Float64,Complex{Float64}}:
 -0.288675     -0.0773503    -0.0207259    ...@*( -0.0207259 -0.0773503-0.0773503
-0.288675 -0.0773503 -0.0055535 -0.0207259-0.0207259 -0.0773503 -0.288675 -0.00148806
-0.0055535-0.0055535 -0.0207259 -0.0773503 -0.000398723 -0.00148806-0.00148806 -0.0055535
-0.0207259 -0.000106838 -0.000398723-0.000398723 -0.00148806 -0.0055535 (*@...@*(
-2.8627e-5 -0.000106838-0.000106838 -0.000398723 -0.00148806 -7.67059e-6
-2.8627e-5-2.8627e-5 -0.000106838 -0.000398723 -2.05533e-6 -7.67059e-6-7.67059e-6
-2.8627e-5 -0.000106838 -5.50724e-7 -2.05533e-6-2.05533e-6 -7.67059e-6 -2.8627e-5
-1.47566e-7 -5.50724e-7(*@⋮@*( (*@⋱@*(-2.05533e-6 -5.50724e-7 -1.47566e-7 -2.8627e-5
-7.67059e-6-7.67059e-6 -2.05533e-6 -5.50724e-7 -0.000106838 -2.8627e-5-2.8627e-5
-7.67059e-6 -2.05533e-6 -0.000398723 -0.000106838-0.000106838 -2.8627e-5 -7.67059e-6
-0.00148806 -0.000398723-0.000398723 -0.000106838 -2.8627e-5 (*@...@*( -0.0055535
-0.00148806-0.00148806 -0.000398723 -0.000106838 -0.0207259 -0.0055535-0.0055535
-0.00148806 -0.000398723 -0.0773503 -0.0207259-0.0207259 -0.0055535 -0.00148806 -0.288675
-0.0773503-0.0773503 -0.0207259 -0.0055535 -0.0773503 -0.288675
```

We verify this approximates the true inverse we deduced above by comparing the first few entries:

```
zp = 2+sqrt(3)
zm = 2-sqrt(3)
-zp.^(-(0:4).-1) / (1-zm/zp), inv(L)[1,1:5]
```

```
([-0.28867513459481287, -0.07735026918962577, -0.020725942163690178, -0.0055
5349946513494, -0.001488055696849579], [-0.2886751345948129, -0.07735026918
962576, -0.020725942163690177, -0.005553499465134939, -0.001488055696849576
])
```

1.3 Problem 3.2

This part was solved as part of Problem 3.1.

1.4 Problem 3.3

Note that

$$
T[a(z)] = \begin{pmatrix} -4 & 0 & 1 & & & & \\ 0 & -4 & 0 & 1 & & & \\ 1 & 0 & -4 & 0 & 1 & & \\ & 1 & 0 & -4 & 0 & 1 & \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}
$$

The UL decomposition is $T[\phi_-]T[\phi_+]$, i.e., for $z_\pm = 2 \pm \sqrt{3}$,

$$
\underbrace{\begin{pmatrix} 1 & 0 & -z_- & & \\ & 1 & 0 & -z_- & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{U}
\underbrace{\begin{pmatrix} -z_+ & & & & \\ 0 & -z_+ & & & \\ 1 & 0 & -z_+ & & \\ & 1 & 0 & -z_+ & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{L}
$$

Verification

```
n = 10
U = Toeplitz([1; zeros(n-1)], [1; 0; -zm; zeros(n-3)])
L = Toeplitz([-zp; 0; 1; zeros(n-3)], [-zp; zeros(n-1)])
U*L
```

```
10×@*(10 Array(*@{Float64,2}:
 -4.0   0.0   1.0   0.0   0.0   0.0   0.0   0.0   0.0       0.0
  0.0  -4.0   0.0   1.0   0.0   0.0   0.0   0.0   0.0       0.0
  1.0   0.0  -4.0   0.0   1.0   0.0   0.0   0.0   0.0       0.0
  0.0   1.0   0.0  -4.0   0.0   1.0   0.0   0.0   0.0       0.0
  0.0   0.0   1.0   0.0  -4.0   0.0   1.0   0.0   0.0       0.0
  0.0   0.0   0.0   1.0   0.0  -4.0   0.0   1.0   0.0       0.0
  0.0   0.0   0.0   0.0   1.0   0.0  -4.0   0.0   1.0       0.0
  0.0   0.0   0.0   0.0   0.0   1.0   0.0  -4.0   0.0       1.0
  0.0   0.0   0.0   0.0   0.0   0.0   1.0   0.0  -3.73205   0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   1.0   0.0      -3.73205
```

1.5 Problem 3.4

We have (see Problem 3.1)

$$
T[a(z)]^{-1} = L^{-1}U^{-1} = \begin{pmatrix} -z_+^{-1} & & & & \\ 0 & -z_+^{-1} & & & \\ -z_+^{-2} & 0 & -z_+^{-1} & & \\ 0 & -z_+^{-2} & 0 & -z_+^{-1} & \\ -z_+^{-3} & 0 & -z_+^{-2} & 0 & -z_+^{-1} \\ & & \ddots & \ddots & \ddots \end{pmatrix}
\begin{pmatrix} 1 & 0 & z_- & 0 & z_-^2 & 0 & \cdots \\ & 1 & 0 & z_- & 0 & z_-^2 & \cdots \\ & & 1 & 0 & z_- & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \end{pmatrix}
$$

Verification For large `n` the entries of the inverse of Toeplitz matrix approximate those of the infinite-dimensional Toeplitz operator:

```
n = 1000
T = Toeplitz([-4; 0; 1; zeros(n-3)], [-4; 0; 1; zeros(n-3)])
inv(Matrix(T))
```

```
1000×@*(1000 Array(*@{Float64,2}:
 -0.267949      -0.0            -0.0717968     ...@*( -9.85983e-287 -0.00.0 -0.267949 -0.0
-0.0 -9.85983e-287-0.0717968 0.0 -0.287187 -3.94393e-286 -0.00.0 -0.0717968 0.0 -0.0
-3.94393e-286-0.0192379 0.0 -0.0769515 -1.47897e-285 -0.00.0 -0.0192379 0.0 (*@...@*( -0.0
-1.47897e-285-0.00515478 0.0 -0.0206191 -5.5215e-285 -0.00.0 -0.00515478 0.0 -0.0
-5.5215e-285-0.00138122 0.0 -0.00552487 -2.0607e-284 -0.00.0 -0.00138122 0.0 -0.0
-2.0607e-284(*@⋮@*( (*@⋱@*(0.0 -2.0607e-284 0.0 -0.0 -0.00138122-5.5215e-285 0.0
-2.2086e-284 -0.00515478 -0.00.0 -5.5215e-285 0.0 -0.0 -0.00515478-1.47897e-285 0.0
-5.9159e-285 -0.0192379 -0.00.0 -1.47897e-285 0.0 (*@...@*( -0.0 -0.0192379-3.94393e-286
 0.0 -1.57757e-285 -0.0717968 -0.00.0 -3.94393e-286 0.0 -0.0 -0.0717968-9.85983e-287 0.0
-3.94393e-286 -0.267949 -0.00.0 -9.85983e-287 0.0 0.0 -0.267949
```

This matches our construction:

```
li = zeros(n) # inv(L) coefficients
li[1:2:end] = -zp.^(-(1:(n ÷ 2)))
Li = Toeplitz(li, [li[1]; zeros(n-1)])
```

```
1000×@*(1000 ToeplitzMatrices.Toeplitz(*@{Float64,Complex{Float64}}:
 -0.267949       0.0            ...@*( 0.0 0.0 0.00.0 -0.267949 0.0 0.0 0.0-0.0717968 0.0
0.0 0.0 0.00.0 -0.0717968 0.0 0.0 0.0-0.0192379 0.0 0.0 0.0 0.00.0 -0.0192379 (*@...@*(
0.0 0.0 0.0-0.00515478 0.0 0.0 0.0 0.00.0 -0.00515478 0.0 0.0 0.0-0.00138122 0.0 0.0 0.0
0.00.0 -0.00138122 0.0 0.0 0.0(*@⋮@*( (*@⋱@*(0.0 -2.06071e-284 0.0 0.0
0.0-5.52165e-285 0.0 0.0 0.0 0.00.0 -5.52165e-285 0.0 0.0 0.0-1.47952e-285 0.0 0.0 0.0
0.00.0 -1.47952e-285 (*@...@*( 0.0 0.0 0.0-3.96437e-286 0.0 0.0 0.0 0.00.0 -3.96437e-286
-0.267949 0.0 0.0-1.06225e-286 0.0 0.0 -0.267949 0.00.0 -1.06225e-286 -0.0717968 0.0
-0.267949
```

```
ui = zeros(n) # inv(U) coefficients
ui[1:2:end] = zm.^(0:(n ÷ 2)-1)
Ui = Toeplitz([ui[1]; zeros(n-1)], ui)
```

```
1000×@*(1000 ToeplitzMatrices.Toeplitz(*@{Float64,Complex{Float64}}:
 1.0  0.0  0.267949  0.0       0.0717968  ...@*( 3.96437e-286 0.00.0 1.0 0.0 0.267949 0.0
0.0 3.96437e-2860.0 0.0 1.0 0.0 0.267949 1.47952e-285 0.00.0 0.0 0.0 1.0 0.0 0.0
1.47952e-2850.0 0.0 0.0 0.0 1.0 5.52165e-285 0.00.0 0.0 0.0 0.0 0.0 (*@...@*( 0.0
5.52165e-2850.0 0.0 0.0 0.0 0.0 2.06071e-284 0.00.0 0.0 0.0 0.0 0.0 0.0 2.06071e-2840.0
0.0 0.0 0.0 0.0 7.69067e-284 0.00.0 0.0 0.0 0.0 0.0 0.0 7.69067e-284(*@⋮@*( (*@⋱@*(0.0
0.0 0.0 0.0 0.0 0.0 0.005154780.0 0.0 0.0 0.0 0.0 0.0192379 0.00.0 0.0 0.0 0.0 0.0 0.0
0.01923790.0 0.0 0.0 0.0 0.0 0.0717968 0.00.0 0.0 0.0 0.0 0.0 (*@...@*( 0.0 0.07179680.0
0.0 0.0 0.0 0.0 0.267949 0.00.0 0.0 0.0 0.0 0.0 0.0 0.2679490.0 0.0 0.0 0.0 0.0 1.0
0.00.0 0.0 0.0 0.0 0.0 0.0 1.0
```

```
Li*Ui
```

```
1000×@*(1000 Array(*@{Float64,2}:
 -0.267949       0.0            -0.0717968     ...@*( -1.06225e-286 0.00.0 -0.267949 0.0
0.0 -1.06225e-286-0.0717968 0.0 -0.287187 -4.249e-286 0.00.0 -0.0717968 0.0 0.0
-4.249e-286-0.0192379 0.0 -0.0769515 -1.59337e-285 0.00.0 -0.0192379 0.0 (*@...@*( 0.0
-1.59337e-285-0.00515478 0.0 -0.0206191 -5.94859e-285 0.00.0 -0.00515478 0.0 0.0
-5.94859e-285-0.00138122 0.0 -0.00552487 -2.2201e-284 0.00.0 -0.00138122 0.0 0.0
-2.2201e-284(*@⋮@*( (*@⋱@*(0.0 -2.06071e-284 0.0 0.0 -0.00148806-5.52165e-285 0.0
-2.20866e-284 -0.0055535 0.00.0 -5.52165e-285 0.0 0.0 -0.0055535-1.47952e-285 0.0
-5.91809e-285 -0.0207259 0.00.0 -1.47952e-285 0.0 (*@...@*( 0.0 -0.0207259-3.96437e-286
 0.0 -1.58575e-285 -0.0773503 0.00.0 -3.96437e-286 0.0 0.0 -0.0773503-1.06225e-286 0.0
-4.249e-286 -0.288675 0.00.0 -1.06225e-286 0.0 0.0 -0.288675
```

Note the inverse of a Toeplitz operator/matrix is not Toeplitz, unlike the case of a Laurent operator / Circulant matrix.

1.5.1 Problem 3.5

This is somewhat a trick question as $a(z) = (z^2+3)/(z^2+2)$ is analytic inside the unit circle, so $T[a(z)]$ is lower triangular and therefore

$$T[a(z)]^{-1} = T[a(z)^{-1}] = T[(z^2+2)/(z^2+3)]$$

]

1.6 Problem 4

1.6.1 Problem 4.1

It is 1 since we go around the origin once. The easiest way to see this is by direct inspection, we want to solve:

$$\underbrace{\begin{pmatrix} 0 & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}}_{T[z]} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix}$$

But the first row is always zero. If $f_0 = 0$ we therefore have the solution $u_n = f_{n+1}$.

1.6.2 Problem 4.2

The winding number is -1. We want to solve:

$$\underbrace{\begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}}_{T[z^{-1}]} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix}$$

Now we have the solution for any constant c $u_0 = c, u_n = f_{n-1}$. In other words, $\mathbf{e}_0$ is in the kernel.

1.7 Problem 4.3

If $a(z)$ has winding number κ then $z^{-k}a(z)$ has trivial winding number. Therefore we have

$$z^{-k}a(z) = \phi_+(z)\phi_-(z)$$

As usual we can now take logarithms to deduce:

$$\log(a(z)z^{-k}) = \log\phi_+(z) + \log\phi_-(z)$$

which by Plemelj implies

$$\phi_+(z) = \mathrm{e}^{\mathcal{C}[\log(\diamond^{-k} a)](z)}$$
$$\phi_+(z) \ = \mathrm{e}^{-\mathcal{C}[\log(\diamond^{-k} a)](z)}$$

1.8 Problem 4.4

Note that for $\kappa \geq 0$ that P is lower triangular Toeplitz, therefore we have using the algebraic properties of triangular Toeplitz

$$T[\phi_-]T[z^\kappa]T[\phi_+] = T[\phi_-]T[z^\kappa \phi_+] = T[\phi_- z^\kappa \phi_+] = T[a(z)]$$

When $\kappa \leq 0$ then P is upper triangular Toeplitz and so

$$T[\phi_-]T[z^\kappa]T[\phi_+] = T[\phi_- z^\kappa]T[\phi_+] = T[\phi_- z^\kappa \phi_+] = T[a(z)].$$

1.9 Problem 4.5

The first question is: what is the winding number? The straightforward way to compute is via residue calculus. That is, if we calculate

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}} \oint_a \frac{1}{z}\mathrm{d}z = \frac{1}{2\pi\mathrm{i}} \oint_C \frac{a'(z)}{a(z)}\mathrm{d}z = \frac{-1}{\pi\mathrm{i}} \oint_C \frac{z}{z^2+1/2}\mathrm{d}z \\ = -2\left(\operatorname*{Res}_{z=-\frac{\mathrm{i}}{\sqrt{2}}} + \operatorname*{Res}_{z=\frac{\mathrm{i}}{\sqrt{2}}}\right) \frac{z}{z^2+1/2} = -2.\end{aligned}$$

This is also intuitive since z^2 clearly goes around the origin twice counterclockwise, so does $2z^2+1$ as the shift by 1 is not enough to change anything, therefore $(2z^2+1)^{-1}$ goes around twice clockwise.

Note that

$$z^2 a(z) = \underbrace{\frac{z^2}{2z^2+1}}_{\phi_-(z)}$$

Is already analytic outside the unit circle so we have $L = I$ and thus the factorisation

$$T[a(z)] = \underbrace{T[\phi_-]}_{U}\underbrace{T[z^{-2}]}_{P}$$

From the Laurent expansion

$$\phi_-(z)^{-1} = 2 + 1/z^2$$

We can compute

$$U^{-1}\mathbf{e}_0 = T[\phi_-^{-1}]\mathbf{e}_0 = 2\mathbf{e}_0$$

The kernel of P is $\mathbf{e}_0$ and $\mathbf{e}_1$. Thus putting everything together we get the rather boring answer

$$\begin{pmatrix} c \\ d \\ 2 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

where c and d are arbitrary constants.

1.10 Problem 5

1.10.1 Problem 5.1

To be analytic at all we need decay at either $\pm\infty$, this has neither so is not defined.

1.10.2 Problem 5.2

It has exponential decay in the right-half plane, therefore

$$\mathrm{e}^{\gamma x} f(x) = \frac{\mathrm{e}^{\gamma x}}{1 + \mathrm{e}^x}$$

has exponential decay at both $\pm\infty$, provided $0 < \gamma < 1$. Therefore, we can take the strip $0 < \Im s < 1$. (Note in each case the contour for the inverse Fourier transform can be any contour in the domain of analyticity.)

We can verify this by exact computation using Residue calculus: for $0 < \Im s < 1$, we can integrate over a rectangle to get:

$$\left(\int_{-R}^{R} + \int_{R}^{2\mathrm{i}\pi+R} + \int_{2\mathrm{i}\pi+R}^{2\mathrm{i}\pi-R} + \int_{2\mathrm{i}\pi-R}^{-R}\right) \frac{\mathrm{e}^{-\mathrm{i}sx}}{1+\mathrm{e}^x}\mathrm{d}x = 2\pi\mathrm{i}\operatorname*{Res}_{z=\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}sz}}{1+\mathrm{e}^z} = -2\pi\mathrm{i}\mathrm{e}^{\pi s}$$

Note that

$$\frac{\mathrm{e}^{-\mathrm{i}s(R+\mathrm{i}t)}}{1+\mathrm{e}^{R+\mathrm{i}t}} = \frac{\mathrm{e}^{-\mathrm{i}R\Re s + R\Im s + t}}{1+\mathrm{e}^{R+\mathrm{i}t}} \to 0$$

and

$$\frac{\mathrm{e}^{-\mathrm{i}s(-R+\mathrm{i}t)}}{1+\mathrm{e}^{R+\mathrm{i}t}} = \frac{\mathrm{e}^{\mathrm{i}R\Re s - R\Im s + t}}{1+\mathrm{e}^{R+\mathrm{i}t}} \to 0$$

uniformly in t as $R \to \infty$, hence we deduce that

$$\left(\int_{-\infty}^{\infty} + \int_{2i\pi+\infty}^{2i\pi-\infty}\right) \frac{e^{-isx}}{1+e^{x}} dx = -2\pi i e^{\pi s}$$

Now note that

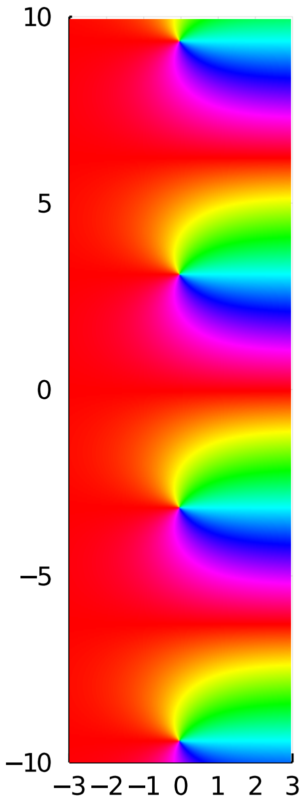
$$\int_{2i\pi+\infty}^{2i\pi-\infty} \frac{e^{-ist}}{1+e^{t}} dt = \int_{\infty}^{-\infty} \frac{e^{-is(x+2i\pi)}}{1+e^{x}} dx = -e^{2\pi s} \int_{-\infty}^{\infty} \frac{e^{-isx}}{1+e^{x}} dx$$

Therefore, we have

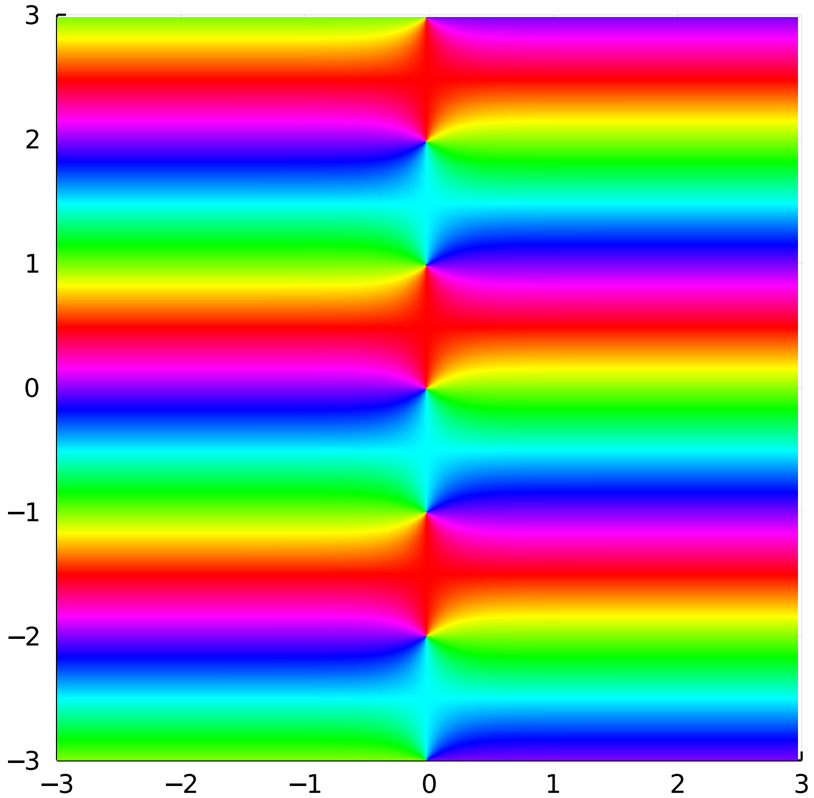
$$\int_{-\infty}^{\infty} \frac{e^{-isx}}{1+e^{x}} dx = -2i\pi \frac{e^{\pi s}}{1-e^{2\pi s}} = i\pi \operatorname{csch} \pi x$$

which has poles at 0 and i:

```
phaseplot(-3..3, -10..10, z -> 1/(1+exp(z))) #integrand
```




```
phaseplot(-3..3, -3..3, z -> im*π*csch(π*z)) # transform
```



1.10.3 Problem 5.3

Here $e^{\gamma x} f(x) = e^{(\gamma+2)x}$ has decay at $+\infty$ proved $\gamma < -2$, hence we have the strip $\Im s < -2$. Indeed, its Fourier transform is

$$-\frac{\mathrm{i}}{2\mathrm{i} + s}$$

by integration by parts.

1.10.4 Problem 5.4

Here it's $\Im s > 0$: unlike 1.1, we now have decay at $x \to \infty$ since $f_{\mathrm{L}}(x)$ is identically zero. It's Fourier transform is determinable by integration-by-parts:

$$\hat{f}(s) = \int_{-\infty}^{0} x e^{-\mathrm{i}sx} \mathrm{d}x = \frac{1}{\mathrm{i}s} \int_{-\infty}^{0} e^{-\mathrm{i}sx} \mathrm{d}x = \frac{1}{s^2}$$

1.10.5 Problem 5.5

The Fourier transforms are given above.

1.10.6 Problem 5.6

$$\int_{-\infty}^{\infty} \delta(x) \mathrm{e}^{\mathrm{i}sx} \mathrm{d}x = 1$$

It's actually an entire function, but non-decaying. This is hinting at the relationship between smoothness of a function and decay of its Fourier transform, and vice-versa: since $\delta(x)$ "decays" to all orders, we expect its Fourier transform to be entire, but since its not smooth at all, we expect no decay, so on a formal level we can predict the analyticity properties.

1.11 Problem 6

1.11.1 Problem 6.1

Note that

$$K(z) = \frac{3}{2} \mathrm{e}^{-|x|} \Rightarrow \hat{K}(s) = \frac{3}{1+s^2}$$

Provided $-1 < \Im s < 1$, and

$$\widehat{f}_{\mathrm{R}}(s) = -\frac{\mathrm{i}}{s} - \frac{\alpha}{s^2}$$

for $\Im s < 0$. Define

$$h(s) = -\widehat{f}_{\mathrm{R}}(s) = \frac{\mathrm{i}}{s} + \frac{\alpha}{s^2}$$

```
α = 0.3
x = Fun(0..100)
f = 1  + α*x
h = s -> (im/s + α/s^2)

γ = -0.5 # we take the Fourier transform on R + im*γ
s = -0.5 + im*γ

-sum(f*exp(-im*s*x)) , h(s)
```

```
(-0.9999999999999862 - 1.5999999999999914im, -1.0 - 1.6im)
```

Transforming the equation, we have

$$\Phi_+(s) - (1 + \hat{K}(s))\Phi_-(s) = \frac{\mathrm{i}}{s} + \frac{\alpha}{s^2}$$

where

$$1 + \hat{K}(s) = \frac{4+s^2}{1+s^2} = \frac{(s-2\mathrm{i})(s+2\mathrm{i})}{(s+\mathrm{i})(s-\mathrm{i})}$$

This is very close to the example we did in lectures, so we already know the homogenous solution:

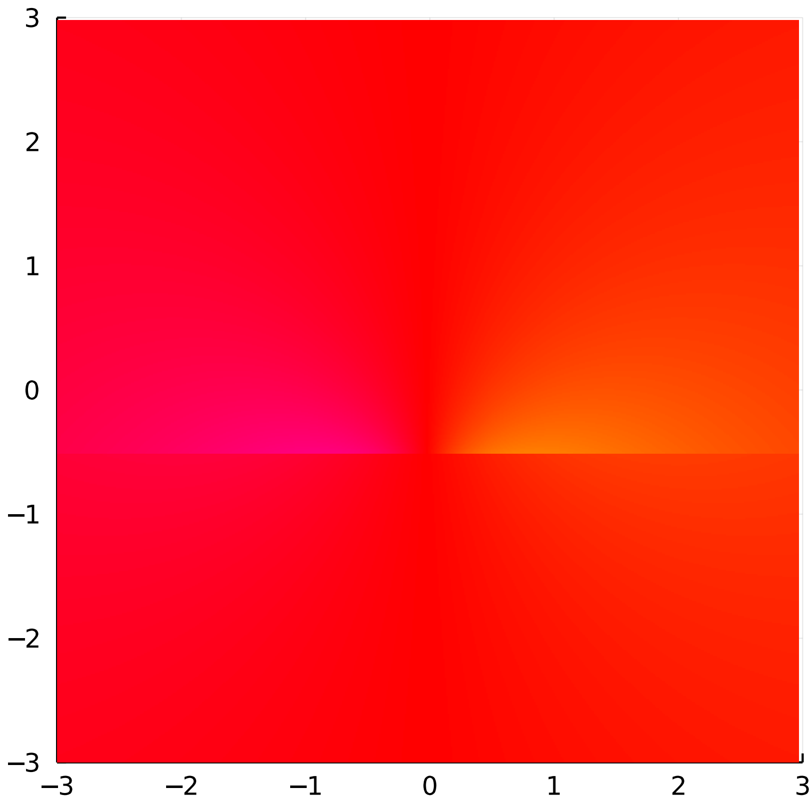
$$\kappa(z) = \begin{cases} \frac{z+2\mathrm{i}}{z+\mathrm{i}} & \Im z > \gamma \\ \frac{z-\mathrm{i}}{z-2\mathrm{i}} & \Im z < \gamma \end{cases}$$

which is valid for $-1 < \gamma < 0$.

```
g = s -> (4+s^2)/(1+s^2)

κ = z -> imag(z) > γ ? (z+im*2)/(z+im) :
                       (z-im)/(z-im*2)


phaseplot(-3..3, -3..3, κ)
```




```
s = 0.1 + γ*im
κp = κ(s + eps()*im)
κm = κ(s - eps()*im)

κp - κm*g(s)
```

```
-1.3322676295501878e-15 - 2.7755575615628914e-16im
```

We thus get the RH problem

$$Y_+(s) - Y_-(s) = h(s)/\kappa_+(s) = (\frac{\mathrm{i}}{s} + \frac{\alpha}{s^2})\frac{s+\mathrm{i}}{s+2\mathrm{i}}$$

We see this has poles at 0 and $-2\mathrm{i}$, so using partial fraction expansion we get

$$(\frac{\mathrm{i}}{s} + \frac{\alpha}{s^2})\frac{s+\mathrm{i}}{s+\sqrt{3}\mathrm{i}} = \frac{\alpha}{2s^2} - \frac{\mathrm{i}(\alpha-2)}{4s} + \frac{\mathrm{i}(2+\alpha)}{4(s+2\mathrm{i})}$$

Therefore, splitting the poles between those above and below γ, we have

$$Y(z) = \begin{cases} \frac{\mathrm{i}(2+\alpha)}{4(z+2\mathrm{i})} & \Im z > \gamma \\ -\frac{\alpha}{2z^2} + \frac{\mathrm{i}(\alpha-2)}{4z} & \Im z < \gamma \end{cases}$$

```
s = 0.1 + γ*im
Y = z -> imag(z) > γ ? im*(2+α)/(4*(z+2im)) :
                       - α/(2z^2) + im*(α-2)/(4z)


Yp = Y(s + eps()*im)
Ym = Y(s - eps()*im)

Yp - Ym  , h(s)/κp
```

```
(-0.9682149028643237 + 0.4107975074619046im, -0.9682149028643242 + 0.410797
507461905im)
```

We therefore have

$$\Phi(z) = \kappa(z)Y(z) = \begin{cases} \frac{\mathrm{i}(2+\alpha)}{4(z+\mathrm{i})} & \Im z > \gamma \\ (-\frac{\alpha}{2z^2} + \frac{\mathrm{i}(\alpha-2)}{4z})\frac{z-\mathrm{i}}{z-2\mathrm{i}} & \Im z < \gamma \end{cases}$$

```
Φ = z -> imag(z) > γ ? im*(2+α)/(4*(z+im)) :
                       (-α/(2z^2) + im*(α-2)/(4z))*(z-im)/(z-2im)
Φp = Φ(s+eps()im)
Φm = Φ(s-eps()im)

Φp - Φm*g(s) , h(s)
```

```
(-2.9881656804733714 + 0.8284023668639042im, -2.9881656804733723 + 0.828402
3668639053im)
```

Finally, we recover the solution by inverting Φ_-, using Residue calculus in the upper half plane: for $x > 0$ we have

$$u(x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma}(-\frac{\alpha}{2z^2} + \frac{\mathrm{i}(\alpha-2)}{4z})\frac{z-\mathrm{i}}{z-2\mathrm{i}}\mathrm{e}^{\mathrm{i}zx}\mathrm{d}z$$
$$= \mathrm{i}(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=2\mathrm{i}})(-\frac{\alpha}{2z^2} + \frac{\mathrm{i}(\alpha-2)}{4z})\frac{z-\mathrm{i}}{z-2\mathrm{i}}\mathrm{e}^{\mathrm{i}zx} = \frac{1+x\alpha}{4} - \frac{\alpha+1}{4}\mathrm{e}^{-2x}$$

Did it work? yes:

```
t = Fun(0 .. 50)
u = (1+t*α)/4 - (α-1)/4*exp(-2t)
x = 0.1
u(x) + 3/2*sum(exp(-abs(t-x))*u) , f(x)
```

```
(1.0300000000000025, 1.03)
```

1.11.2 Problem 6.2

Setting up the problem as above, we arrive at a degenerate RH problem:

$$\Phi_+(s) - g(s)\Phi_-(s) = h(s)$$

where

$$g(s) = \widehat{K}(s) = \frac{2\alpha}{\alpha^2 + s^2} = \frac{2\alpha}{(s - \mathrm{i}\alpha)(s + \mathrm{i}\alpha)}$$

and

$$h(s) = \frac{\mathrm{i}}{s} + \frac{\alpha}{s^2} = \mathrm{i}\frac{s - \mathrm{i}\alpha}{s^2}$$

Given α, we need to solve the Riemann-Hilbert problem on a contour $\mathbb{R}+\mathrm{i}\gamma$ with $-\alpha < \gamma < 0$. Suppose we allow $\kappa_-(s) \sim s$ to have growth, then we can write

$$\kappa(z) = \begin{cases} \frac{1}{z+\mathrm{i}\alpha} & \Im z > \gamma \\ \frac{z-\mathrm{i}\alpha}{2\alpha} & \Im z < \gamma \end{cases}$$

so that

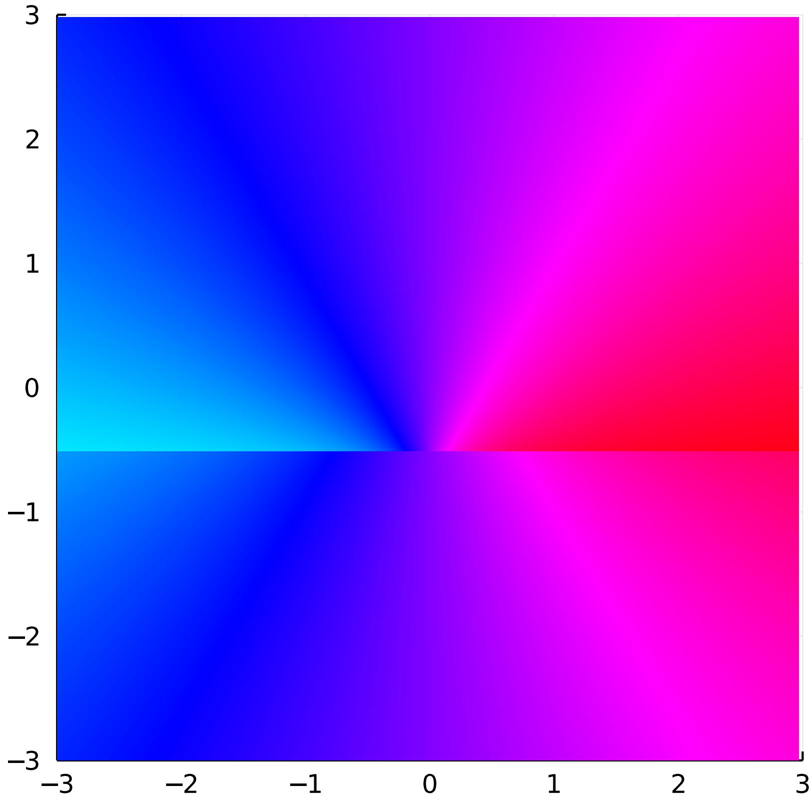
$$\kappa_+(s) = \kappa_-(s)g(s)$$

```
α = 0.8

g = s -> (2α)/(α^2+s^2)
h = s -> (im/s + α/s^2)

κ = z -> imag(z) > γ ? 1/(z + im*α) :
                        (z-im*α)/(2α)

phaseplot(-3..3, -3..3, κ)
```




```
s = 0.1 + γ*im
κp = κ(s + eps()*im)
κm = κ(s - eps()*im)

κp , κm*g(s)
```

```
(0.9999999999999983 - 2.999999999999998im, 0.9999999999999997 - 3.0im)
```

Then we have

$$h(s)/\kappa_+(s) = \mathrm{i}\frac{s^2+\alpha^2}{s^2} = \mathrm{i} + \mathrm{i}\frac{\alpha^2}{s^2}$$

and then we can write

$$Y(z) = \begin{cases} \mathrm{i} & \Im z > \gamma \\ -\frac{\mathrm{i}\alpha^2}{z^2} & \Im z < \gamma \end{cases}$$

```
s = 0.1 + γ*im
Y = z -> imag(z) > γ ? im :
                    -im*α^2/s^2


Yp = Y(s + eps()*im)
Ym = Y(s - eps()*im)

Yp - Ym  , h(s)/κp
```

```
(-0.9467455621301777 - 1.2721893491124265im, -0.946745562130178 - 1.2721893
491124274im)
```

Putting things together, we get

$$\Phi(z) = \kappa(z)Y(z) = \begin{cases} \frac{\mathrm{i}}{z+\mathrm{i}\alpha} & \Im z > \gamma \\ -\mathrm{i}\frac{\alpha^2}{z^2}\frac{z-\mathrm{i}\alpha}{2\alpha} & \Im z < \gamma \end{cases}$$

```
Φ = z -> imag(z) > γ ? im/(z + im*α)  :
                     -im*α^2/z^2* (z-im*α)/(2α)
Φp = Φ(s+eps()im)
Φm = Φ(s-eps()im)

Φp - Φm*g(s) , h(s)
```

```
(-4.763313609467453 + 1.5680473372781032im, -4.763313609467456 + 1.56804733
72781065im)
```

We now invert the Fourier transform of $\Phi_-(s)$ using Jordan's lemma:

$$u(x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \Phi_-(s)\mathrm{e}^{\mathrm{i}sx}\mathrm{d}s = \frac{\alpha}{2}\operatorname*{Res}_{z=0}\frac{z-\mathrm{i}\alpha}{z^2}\mathrm{e}^{\mathrm{i}zx} = \frac{\alpha}{2}(1+x\alpha)$$

```
t = Fun(0 .. 200)

u = α*(1+t*α)/2

x = 0.1

sum(exp(-α*abs(t-x))*u)  , (1 + α*x)
```

```
(1.0799999999999645, 1.08)
```

1.12 Problem 6.3

1. From the same logic as 2.2, we know we need to solve

$$\Phi_+(s) - g(s)\Phi_-(s) = h(s)$$

where

$$g(s) = 1 - \frac{2\lambda}{s^2+1} = \frac{s^2+1-2\lambda}{s^2+1} = \frac{(s-\mathrm{i}\gamma)(s+\mathrm{i}\gamma)}{(s+\mathrm{i})(s-\mathrm{i})}$$

and

$$h(s) = \frac{1}{s^2}$$

where $-1 < \Im s < 0$, let's say $\Im s = \delta$ because I annoyingly used γ in the statement of the problem. Writing $s = t + \mathrm{i}\delta$, we see that

$$g(s) = \frac{t^2 + 2\mathrm{i}\delta t - \delta^2 + \gamma^2}{s^2+1}$$

By ensuring its real part is positive, this has trivial winding number provided $\gamma^2 = 1-2\lambda > 0$, which is true for $0 < \lambda < \frac{1}{2}$, and restricting the contour s lives on to be $-\gamma < \delta < 0$. Factorizing the kernel we get

$$\kappa(z) = \begin{cases} \frac{z+\mathrm{i}\gamma}{z+\mathrm{i}} & \Im z > \delta \\ \frac{z-\mathrm{i}}{z-\mathrm{i}\gamma} & \Im z < \delta \end{cases}$$

Thus we want to solve

$$Y_+(s) - Y_-(s) = h(s)\kappa_+(s)^{-1} = \frac{s+\mathrm{i}}{s+\mathrm{i}\gamma}\frac{1}{s^2} = \frac{1}{\gamma s^2} - \frac{\mathrm{i}(\gamma-1)}{\gamma^2 s} + \frac{\mathrm{i}}{\gamma^2}\frac{\gamma-1}{s+\mathrm{i}\gamma}$$

Which has solution, (since $\delta > -\gamma$),

$$Y(z) = \begin{cases} \frac{\mathrm{i}}{\gamma^2}\frac{\gamma-1}{z+\mathrm{i}\gamma} & \Im z > \delta \\ \frac{\mathrm{i}(\gamma-1)}{\gamma^2 z} - \frac{1}{\gamma z^2} & \Im z < \delta \end{cases}$$

We thus get

$$\Phi_-(z) = \left(\frac{\mathrm{i}(\gamma-1)}{\gamma^2 z} - \frac{1}{\gamma z^2}\right)\frac{z-\mathrm{i}}{z-\mathrm{i}\gamma}$$

and Jordan's lemma gives us

$$u(x) = \frac{x}{\gamma^2} - \mathrm{e}^{-x\gamma}(\gamma-1)/\gamma^2$$

```
t = Fun(0 .. 200)
λ = 0.1
γ = sqrt(1-2λ)

u = t/γ^2 - exp(-t*γ)*(γ-1)/γ^2

x = 0.1

u(x) - λ*sum(exp(-abs(t-x))*u) , x
```

```
(0.09999999999999876, 0.1)
```

Oddly, this is definitely a solution, but not in the form the question asked for. To get the other solution, consider now the bad winding number case of $-1 < \delta < -\gamma$. Motivated by 2.2, what if we allow κ to have different behaviour? Consider

$$\kappa(z) = \begin{cases} \frac{1}{z+\mathrm{i}} & \Im z > \delta \\ \frac{(z-\mathrm{i})}{(z-\mathrm{i}\gamma)(z+\mathrm{i}\gamma)} & \Im z < \delta \end{cases}$$

Chosen so that both κ_+ and κ_+^{-1} are analytic.

Thus we want to solve

$$Y_+(s) - Y_-(s) = h(s)\kappa_+(s)^{-1} = \frac{s+\mathrm{i}}{s^2} = \frac{1}{s} + \frac{\mathrm{i}}{s^2}$$

but now we only need $Y_+(s) = O(1)$ and $Y_-(s) = O(1)$. Here is where the non-uniqueness comes in, as we can add an arbitrary constant:

$$Y(z) = \begin{cases} A & \Im z > 0 \\ A - \frac{1}{z} - \frac{\mathrm{i}}{z^2} & \Im z < 0 \end{cases}$$

Thus we have

$$\Phi_-(z) = Y_-(z)\kappa_-(z) = -(A + \frac{1}{z} + \frac{\mathrm{i}}{z^2})\frac{(z-\mathrm{i})}{(z-\mathrm{i}\gamma)(z+\mathrm{i}\gamma)}$$

Using Jordan's lemma, and now since $\delta < -\gamma$, we get

$$\begin{aligned} u(x) &= \mathrm{i}(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=\mathrm{i}\gamma} + \operatorname*{Res}_{z=-\mathrm{i}\gamma})\Phi_-(z)\mathrm{e}^{\mathrm{i}xz} \\ &= \frac{x}{\gamma^2} - \mathrm{e}^{-x\gamma}(\frac{\gamma^2-1}{2\gamma^3} + \frac{\gamma-1}{2\gamma^3}A) - \mathrm{e}^{x\gamma}(\frac{1-\gamma^2}{2\gamma^3} + \frac{\gamma+1}{2\gamma^3}A) \\ &= \frac{x}{\gamma^2} + \frac{\mathrm{e}^{x\gamma}-\mathrm{e}^{-x\gamma}}{2}\frac{\gamma-\gamma^{-1}}{2\gamma^2} - \frac{A}{\gamma^3}(\frac{\mathrm{e}^{x\gamma}-\mathrm{e}^{-x\gamma}}{2} + \gamma\frac{\mathrm{e}^{x\gamma}+\mathrm{e}^{-x\gamma}}{2}) \end{aligned}$$

Redefining A and using the definition of sinh and cosh gives the form in the assignment.

What's the moral of the story?

1. Different choices of contours can give different solutions
2. When the winding number is non-trivial, the solution may not be unique

1.12.1 Problem 6.4

1. Integrating by parts we have

$$\begin{aligned} \widehat{u'_{\mathrm{R}}}(s) &= \mathrm{i}s\widehat{u_{\mathrm{R}}}(s) - u(0) = \mathrm{i}s\widehat{u_{\mathrm{R}}}(s) \\ \widehat{u''_{\mathrm{R}}}(s) &= \mathrm{i}s\widehat{u'_{\mathrm{R}}}(s) - u'(0) = -s^2\widehat{u_{\mathrm{R}}}(s) - u'(0) \end{aligned}$$

2. Our integral equation when cast on the whole real line is:

$$u''_{\mathrm{R}}(x) - \frac{72}{5}\int_{-\infty}^{\infty} \mathrm{e}^{-5|x-t|}u_{\mathrm{R}}(t)\mathrm{d}t = 1_{\mathrm{R}}(x) + p_{\mathrm{L}}(x)$$

where

$$p(x) = \frac{72}{5}\int_{-\infty}^{\infty} \mathrm{e}^{-5|x-t|}u_{\mathrm{R}}(t)\mathrm{d}t = \frac{72}{5}\int_{0}^{\infty} \mathrm{e}^{-5|x-t|}u_{\mathrm{R}}(t)\mathrm{d}t.$$

Note that, for $-5 < \Im s < 5$,

$$\hat{K}(s) = \frac{10}{s^2+25}$$

provided s is in the lower half plane,

$$\widehat{1_{\mathrm{R}}}(s) = \int_0^\infty \mathrm{e}^{-\mathrm{i}sx}\mathrm{d}x = \frac{1}{\mathrm{i}s}$$

Thus our integral equation in frequency space is

$$\begin{aligned}
-\alpha - s^2\widehat{u_{\mathrm{R}}}(s) - \frac{72}{5}\widehat{K}(s)\widehat{u_{\mathrm{R}}}(s) &= \widehat{p_{\mathrm{L}}}(s) + \widehat{1_{\mathrm{R}}}(s) \\
\Phi_+(s) - (s^2 + \frac{144}{s^2+25})\Phi_-(s) &= \alpha + \frac{1}{\mathrm{i}s} \\
\Phi_+(s) - \frac{(s^2+9)(s^2+16)}{s^2+25}\Phi_-(s) &= \alpha + \frac{1}{\mathrm{i}s}
\end{aligned}$$

where $s \in \mathbb{R} + \mathrm{i}\gamma$ for any $-5 < \gamma < 0$.

3. We can factorize this to construct $g(s)$ as

$$g(s) = \kappa_+(s)\kappa_-(s)^{-1} = \frac{(s+3\mathrm{i})(s+4\mathrm{i})}{s+5\mathrm{i}}\frac{(s-3\mathrm{i})(s-4\mathrm{i})}{s-5\mathrm{i}}$$

```
κ = z -> imag(z) > γ ?
        (z+3im)*(z+4im)/(z+5im) :
        (z-5im)/((z-3im)*(z-4im))

γ  = -1.0
s = 0.1+γ*im
g = s -> (s^2+9)*(s^2+16)/(s^2+25)

κ(s+eps()im) , g(s)κ(s-eps()im)
```

```
(0.08750780762023733 + 1.4996876951905058im, 0.08750780762023738 + 1.499687
695190506im)
```

Writing $\Phi(z) = \kappa(z)Y(z)$ we get the subtractive RH problem

$$Y_+(s) - Y_-(s) = \frac{h(s)}{\kappa_+(s)} = (\alpha + \frac{1}{\mathrm{i}s})\frac{s+5\mathrm{i}}{(s+3\mathrm{i})(s+4\mathrm{i})}$$

We use partial fraction expansion to write

$$\frac{h(s)}{\kappa_+(s)} = -\frac{\alpha+1/4}{s+4\mathrm{i}} + \frac{2/3+2\alpha}{s+3\mathrm{i}} - \frac{5}{12s}$$

Therefore we have

$$Y(z) = \begin{cases} -\frac{\alpha+1/4}{s+4\mathrm{i}} + \frac{2/3+2\alpha}{s+3\mathrm{i}} & \Im z > \gamma \\ \frac{5}{12s} & \Im z < \gamma \end{cases}$$

and hence

$$\Phi(z) = \begin{cases} \frac{(z+3\mathrm{i})(z+4\mathrm{i})}{z+5\mathrm{i}}(-\frac{\alpha+1/4}{z+4\mathrm{i}} + (2/3+2\alpha)/(z+3\mathrm{i})) & \Im z > \gamma \\ \frac{z-5\mathrm{i}}{(z-3\mathrm{i})(z-4\mathrm{i})}\frac{5}{12z} & \Im z < \gamma \end{cases}$$

We can now invert the Fourier transform of

$$\Phi_-(s) = \frac{s - 5\mathrm{i}}{(s - 3\mathrm{i})(s - 4\mathrm{i})} \frac{5}{12s}$$

This actually decays so fast that we don't need Jordan's lemma to justify here. This has three poles above our contour, so we sum over each residue to get

$$u(x) = \mathrm{i}(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=3\mathrm{i}} + \operatorname*{Res}_{z=4\mathrm{i}})\mathrm{e}^{\mathrm{i}zx} \frac{z - 5\mathrm{i}}{(z - 3\mathrm{i})(z - 4\mathrm{i})} \frac{5}{12z} = -\frac{25}{144} - \frac{5\mathrm{e}^{-4x}}{48} + \frac{5\mathrm{e}^{-3x}}{18}$$

Here's we check the solution:

```
t = Fun(0 .. 200)

u = -25/144 - 5exp(-4t)/48 + 5exp(-3t)/18
x = 1.1
u''(x) - 72/5*sum(exp(-5abs(x-t))*u)
```

```
1.0000000000000175
```

Here we check the jump of Y:

```
α = u'(0)

h = s -> α + 1/(im*s)
Y = z -> imag(z) > γ ?
       (-(α+1/4)/(z+4im) + (2/3 + 2α)/(z+3im)) :
         5/(12z)

Y(s+eps()im) - Y(s-eps()im) , h(s)/κ(s + eps()im)
```

```
(-0.04356060486688343 - 0.38490963026239366im, -0.04356060486688346 - 0.384
9096302623938im)
```

Here we check the jump of Φ:

```
γ  = -1.0

Φ = z -> imag(z) > γ ?
        (z+3im)*(z+4im)/(z+5im) * (-(α+1/4)/(z+4im) + (2/3 + 2α)/(z+3im)) :
        (z-5im)/((z-3im)*(z-4im)) * 5/(12z)

Φ(s + eps()*im) - g(s)*Φ(s - eps()*im) , h(s)
```

```
(0.5734323432343266 - 0.099009900990099im, 0.5734323432343267 - 0.099009900
99009901im)
```