Applied Complex Analysis (2021)

1 Lecture 25: Laplace transforms and half-Fourier transforms

A key tool in the Wiener–Hopf method will be the half-Fourier transforms

$$\int_{-\infty}^0 u(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t \qquad \text{and} \qquad \int_0^\infty u(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t$$

and the Laplace transform

$$\int_0^\infty u(t)\mathrm{e}^{-zt}\mathrm{d}t$$

in particular we are interested in the analyticity properties with respect to s/z.

Outline:

2. Analyticity properties of Fourier transforms
   - Inverse Fourier transform on shifted contours
3. Half-Fourier transforms
   - Inverting the Half-Fourier transform
   - Relationship to Laplace transform
4. Application: solving differential equations on the half-line

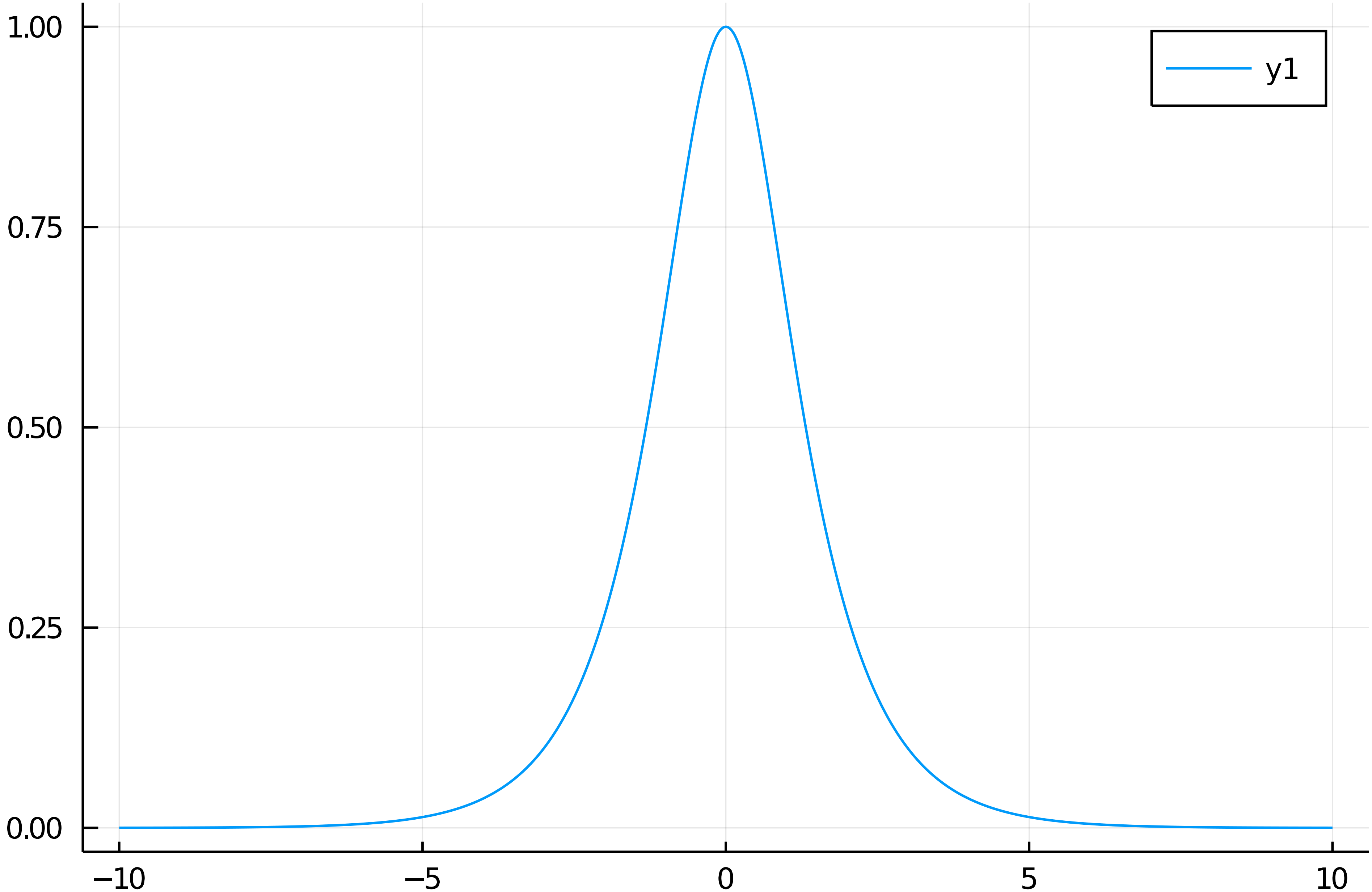
1.1 Analyticity properties of Fourier transforms

Consider the Fourier transform of

$$\operatorname{sech} x = \frac{2}{\mathrm{e}^x + \mathrm{e}^{-x}}$$

This function has exponential decay in both directions:

```
using Plots, ApproxFun, OscillatoryIntegrals, ComplexPhasePortrait,
LinearAlgebra
xx = -10:0.01:10
plot(xx,sech.(xx))
```



Now the Fourier transform of $\operatorname{sech} x$ is

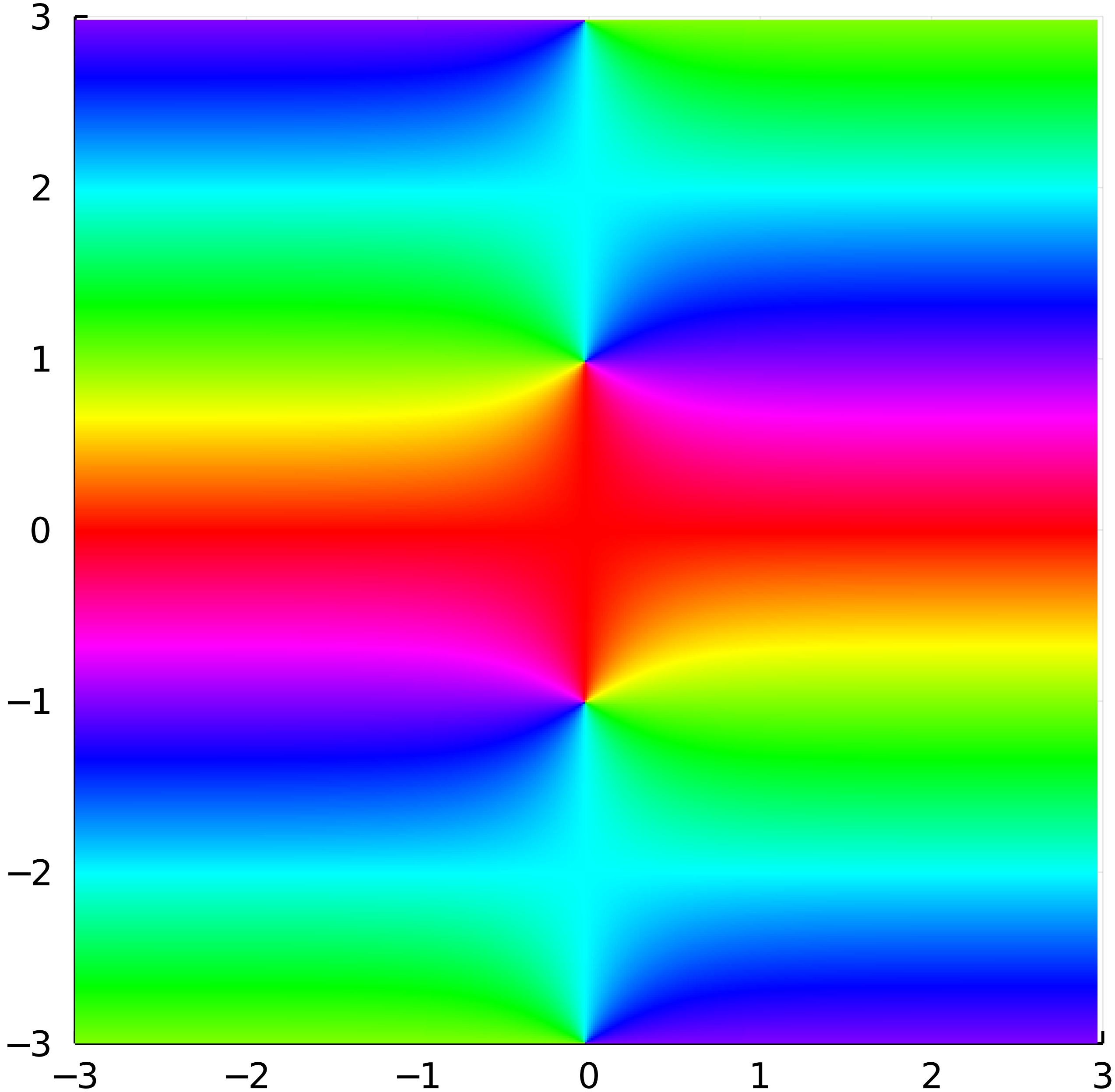
$$\mathcal{F}\mathrm{sech}\,(s) = \int_{-\infty}^{\infty} \mathrm{sech}\, t\, \mathrm{e}^{-\mathrm{i}st} \mathrm{d}t = \pi \mathrm{sech}\, \frac{\pi s}{2}$$

This is calculated via Residue theorem with a bit of work. Note that

$$\pi \mathrm{sech}\, \frac{\pi z}{2} = \frac{2\pi}{\mathrm{e}^{\frac{\pi z}{2}} + \mathrm{e}^{-\frac{\pi z}{2}}}$$

is analytic for $-1 < \Im z < 1$.

```
phaseplot(-3..3, -3..3, z -> π*sech(π*z/2))
```



This is because of the expontial decay.

Theorem (Analyticity of Fourier transforms) Suppose $|f(x)\mathrm{e}^{\gamma x}| < |M(x)|$ where M

is absolutely integrable for all $a < \gamma < b$. Then

$$\widehat{f}(z) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}zt}\mathrm{d}t$$

is analytic for $a < \Im z < b$.

Proof Let $z = s + \mathrm{i}\gamma$ and note that $|f(t)\mathrm{e}^{-\mathrm{i}zt}| = |f(t)\mathrm{e}^{\gamma t}|$. Thus for $a < \gamma < b$, we can exchange differentiation and integration to get

$$\frac{\mathrm{d}\widehat{f}}{\mathrm{d}z} = -\mathrm{i}z\hat{f}(z)$$

■

Remark We don't need f to be analytic at all! Decay in f gives analyticity.

In the case of $\operatorname{sech} x$, we get exponential decay in both directions: that is $\operatorname{sech} x\mathrm{e}^{|\gamma|x}$ is absolutely integrable for $\gamma < 1$.

Another example is $\mathrm{e}^{-x^2/2}$, which is absolutely integrable for any γ. Therefore, it's Fourier transform is in fact entire:

$$\mathcal{F}[\mathrm{e}^{-\diamond^2/2}](z) = \sqrt{2\pi}\mathrm{e}^{-\frac{z^2}{2}}$$

1.1.1 Inverse Fourier transform on shifted contours

A neglected fact of the Fourier transform is that we can think of $\hat{f}(z)$ living on any line $(-\infty + \mathrm{i}\gamma, \infty + \mathrm{i}\gamma)$, and in fact we can recover f from the Fourier transform only on this line. This works even if $\hat{f}(s)$ is not defined on the real-axis, the real-axis is NOT special!

Theorem Suppose $f(x)\mathrm{e}^{\gamma x}$ is square integrable. Then

$$f(x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \widehat{f}(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}\mathrm{d}\zeta$$

Proof Note for $g(x) = f(x)\mathrm{e}^{\gamma x}$

$$\widehat{g}(s) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\gamma t - \mathrm{i}st}\mathrm{d}t = \widehat{f}(s + \mathrm{i}\gamma).$$

Therefore we have

$$\begin{aligned}
\mathrm{e}^{\gamma x} f(x) = g(x) = \mathcal{F}^{-1}\widehat{g}(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{g}(s)\mathrm{e}^{\mathrm{i}xs}\mathrm{d}s = \frac{1}{2\pi}\int_{-\infty}^{\infty} \widehat{f}(s+\mathrm{i}\gamma)\mathrm{e}^{\mathrm{i}xs}\mathrm{d}s \\
&= \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \widehat{f}(\zeta)\mathrm{e}^{\mathrm{i}x(\zeta-\mathrm{i}\gamma)}\mathrm{d}\zeta \\
&= \frac{1}{2\pi}\mathrm{e}^{\gamma x}\int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \widehat{f}(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}\mathrm{d}\zeta
\end{aligned}$$

Which shows the result by cancelling out $\mathrm{e}^{\gamma x}$.

1.2 Half-Fourier transforms

Consider now

$$\int_0^\infty f(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t$$

This is in fact the Fourier transform of f extended to the negative real axis by zero:

$$\int_{-\infty}^\infty \begin{cases} f(t) & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \mathrm{e}^{-\mathrm{i}st}\mathrm{d}t$$

To make sure we remember the domain of definition, we introduce the notation:

$$f_{\mathrm{R}}(x) = \begin{cases} f(t) & t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and

$$f_{\mathrm{L}}(x) = \begin{cases} f(t) & t < 0 \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$\widehat{f_{\mathrm{R}}}(s) = \int_0^\infty f(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t \qquad \text{and} \qquad \widehat{f_{\mathrm{L}}}(s) = \int_{-\infty}^0 f(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t$$

Because it is identically zero on the negative real axis, we immediately get the following:

Corollary (analyticity of Half-Fourier transform) Suppose $f(x)$ is bounded for $x \geq 0$. Then $\widehat{f_{\mathrm{R}}}(z)$ is analytic in the lower half plane

$$\mathbb{H}_+ = \{z : \Im z < 0\}.$$

More generally, f can even have exponential decay: if $f(x)\mathrm{e}^{\gamma x}$ is bounded then $\widehat{f_{\mathrm{R}}}(z)$ is analytic in $\{z : \Im z < \gamma\}$. As before, the same inversion formula follows:

Corollary (inverting Half-Fourier transform) Suppose $f(x)\mathrm{e}^{\gamma x}$ is square integrable for $x \geq 0$. Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}M}^{\infty+\mathrm{i}M} \widehat{f_{\mathrm{R}}}(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}\mathrm{d}\zeta$$

for any choice of $-\infty < M \leq \gamma$.

Example Consider $f(x) = x\mathrm{e}^{-x}$ for $0 \leq x < \infty$. Note that $f(x)\mathrm{e}^{\gamma x}$ is square integrable for any $\gamma < 1$, and we have

$$\widehat{f_{\mathrm{R}}}(z) = \int_0^\infty t\mathrm{e}^{-t-\mathrm{i}zt}\mathrm{d}t = \frac{1}{(1+\mathrm{i}z)^2} = -\frac{1}{(z-\mathrm{i})^2}$$

is analytic for $\Im z < 1$. Thus for any $M < 1$ we have

$$
\begin{aligned}
f(x) &= \frac{1}{2\pi}\int_{-\infty+\mathrm{i}M}^{\infty+\mathrm{i}M} \widehat{f_{\mathrm{R}}}(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}\mathrm{d}\zeta \\
&= \frac{1}{2\pi}\int_{-\infty+\mathrm{i}M}^{\infty+\mathrm{i}M} \frac{1}{(1+\mathrm{i}\zeta)^2}\mathrm{e}^{\mathrm{i}x\zeta}\mathrm{d}\zeta
\end{aligned}
$$

Since $x > 0$, we can use Residue calculus in the upper-half plane, which confirms the result:

$$
\operatorname*{Res}_{z=\mathrm{i}} \frac{\mathrm{e}^{\mathrm{i}xz}}{(1+\mathrm{i}z)^2} = \operatorname*{Res}_{z=\mathrm{i}} \frac{\mathrm{e}^{-x} + \mathrm{i}x\mathrm{e}^{-x}(z-\mathrm{i}) + O(z-\mathrm{i})^2}{-(z-\mathrm{i})^2} = -\mathrm{i}x\mathrm{e}^{-x}.
$$

Example Consider $f(x) = x$. This function is not square-integrable, but we have $f(x)\mathrm{e}^{\gamma x}$ is square integrable for any $\gamma < 0$, and we find for $\Im z < 0$

$$
\widehat{f_{\mathrm{R}}}(z) = -\frac{1}{z^2}
$$

Thus we can still use the result to say, for any $M < 0$,

$$
f(x) = -\frac{1}{2\pi}\int_{-\infty+\mathrm{i}M}^{\infty+\mathrm{i}M} \frac{1}{\zeta^2}\mathrm{e}^{\mathrm{i}x\zeta}\mathrm{d}\zeta
$$

Note that the results have corresponding analogues for f_{L}:

Corollary (analyticity of left Half-Fourier transform) Suppose $f(x)\mathrm{e}^{\gamma x}$ is bounded for $x \le 0$. Then $\widehat{f_{\mathrm{L}}}(z)$ is analytic for $\{z : \Im z > \gamma\}$.

Corollary (inverting left Half-Fourier transform) Suppose $f(x)\mathrm{e}^{\gamma x}$ is square integrable for $x < 0$. Then

$$f(x) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}M}^{\infty+\mathrm{i}M} \widehat{f_{\mathrm{L}}}(\zeta)\mathrm{e}^{\mathrm{i}x\zeta}\mathrm{d}\zeta$$

for any choice of $\gamma \le M < \infty$.

1.2.1 Laplace transforms

Now consider the Laplace transform

$$\check{f}(z) = \int_0^\infty f(t)\mathrm{e}^{-zt}\mathrm{d}t$$

but this is just the half Fourier transform evaluated on the negative imaginary axis!

$$\check{f}(z) = \widehat{f_{\mathrm{R}}}(-\mathrm{i}z)$$

Thus if $f(x)\mathrm{e}^{\gamma x}$ is square integrable, then $\check{f}(z)$ is well-defined for $\Re z \ge \gamma$.

NEVER think of the Laplace transform as a real-valued object: it only makes sense as a complex object. This is seen from the inverse Laplace transform

$$f(x) = \frac{1}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty-M}^{\mathrm{i}\infty-M} \check{f}(\zeta)\mathrm{e}^{\zeta x}\mathrm{d}\zeta$$

which is of course just the inverse Fourier transform in disguise.

1.3 Application: solving differential equations on the half-line

Consider the following ODE for $x \geq 0$:

$$u''(x) + 2u'(x) + u(x) = f(x)$$

with initial conditions $u(0) = u'(0) = 0$. Note that we have by integration-by-parts

$$\check{u'}(z) = \int_0^\infty u'(t)\mathrm{e}^{-zt}\mathrm{d}t = u(0) + z\int_0^\infty u(t)\mathrm{e}^{-zt}\mathrm{d}t = u(0) + z\check{u}(z)$$
$$\check{u''}(z) = u'(0) + z\check{u'}(z) = u'(0) + zu(0) + z^2\check{u}(0)$$

Thus taking into account the initial conditions, are equation in Laplace space becomes

$$(z^2 + 2z + 1)\check{u}(z) = \check{f}(z)$$

Hence we have

$$\check{u}(z) = \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty-M}^{\mathrm{i}\infty-M} \frac{\check{f}(\zeta)}{\zeta^2 + 2\zeta + 1}\mathrm{e}^{x\zeta}\mathrm{d}\zeta$$

Consider the case $f(x) = x$, so that

$$\check{f}(z) = \frac{1}{z^2}$$

Here we need $M < 0$ hence we are integrating on a contour in the right-half plane. Using Residue calculus, we have

$$\begin{aligned} u(x) &= \left(\operatorname*{Res}_{z=-1} + \operatorname*{Res}_{z=0}\right) \frac{\mathrm{e}^{zx}}{z^2(z+1)^2} \\ &= \operatorname*{Res}_{z=-1} \frac{\mathrm{e}^{-x} + \mathrm{e}^{-x}(x+2)(z+1) + O(z+1)^2}{(z+1)^2} + \operatorname*{Res}_{z=0} \frac{1 + (x-2)z + O(z)^2}{z^2} \\ &= (x+2)\mathrm{e}^{-x} + x - 2 \end{aligned}$$

1.4 Laplace transform of rational functions

We now consider the question of calculating Laplace transforms (or equivalently, half-Fourier transforms)

$$\check{f}(s) = \int_0^\infty f(t)\mathrm{e}^{-st}\mathrm{d}t$$

where f is rational. We're going to do something seemingly crazy: we'll first calculate the Cauchy transform

$$\mathcal{C}[f\mathrm{e}^{-s\diamond}](z) = \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{f(t)\mathrm{e}^{-st}}{t-z}\mathrm{d}t$$

so that

$$\check{f}(s) = -2\pi\mathrm{i} \lim_{z\to\infty} z\mathcal{C}[f\mathrm{e}^{-s\diamond}](z)$$

Note that the exponential decay in the integrand allows us to use Plemelj's lemma: if we find a function $\phi(z)$ such that

1. $$\phi(z)$$

   is analytic off $[0, \infty)$

2. $$\lim_{z\to\infty} \phi(z) = 0$$

   for any angle of approach

3. $$\phi$$

   has weaker than pole singularities at 0

4. $$\phi_+(x) - \phi_-(x) = f(x)\mathrm{e}^{-sx}$$

Then we have calculated the Cauchy transform:

$$\phi(z) = \mathcal{C}[f\mathrm{e}^{-s\diamond}](z).$$

Let's start with $f(x) = 1$ and $s = 1$, that is, what is the Cauchy transform of e^{-x}? Consider the exponential integral:

$$\mathrm{Ei}(z) = \int_{-\infty}^{z} \frac{\mathrm{e}^{\zeta}}{\zeta} \mathrm{d}\zeta$$

Without loss of generality, the contour of integration is

$$(\infty, -1) \cup [-1, z)$$

that is, a straight line to -1 and a straightline from -1 to z. Thus we have a branch cut on $[0, \infty)$ which has the jump

$$\mathrm{Ei}^{+}(x) - \mathrm{Ei}^{-}(x) = -\oint \frac{\mathrm{e}^{\zeta}}{\zeta} \mathrm{d}\zeta = -2\pi\mathrm{i}$$

where

$$\mathrm{Ei}^{\pm}(x) = \lim_{\epsilon \to 0} \mathrm{Ei}(x \pm \mathrm{i}\epsilon)$$

Consider

$$\phi(z) = -\frac{\mathrm{e}^{-z}\mathrm{Ei}(z)}{2\pi\mathrm{i}}$$

1. This is analytic off $[0, \infty)$

2. Integrating by parts we have decay at ∞ in all directions:

$$\mathrm{e}^{-z}\mathrm{Ei}(z) = \frac{1}{z} - \int_{-\infty}^{z} \frac{\mathrm{e}^{\zeta - z}}{\zeta^2} \mathrm{d}\zeta = O(z^{-1})$$

3.

$$\phi$$

has a logarithmic singularity at 0

4.

$$\phi_+(x) - \phi_-(x) = \mathrm{e}^{-x}.$$

Thus

$$\mathcal{C}[\mathrm{e}^{-s\diamond}](z) = \phi(sz) = -\frac{\mathrm{e}^{-sz}\mathrm{Ei}(sz)}{2\pi\mathrm{i}}$$

Let's make sure I didn't make a mistake. Here we first define Ei:

```
const ei_−_1 = let ζ = Fun(-50 .. -1)
    sum(exp(ζ)/ζ)
end
function ei(z)
    ζ = Fun(Segment(-1 , z))
    ei_−_1 + sum(exp(ζ)/ζ)
end
```

```
φ = (z) -> -exp(-z)*ei(z)/(2π*im)
```

```
#3 (generic function with 1 method)
```

The expression matches the Cauchy transform:

```
t = Fun(0 .. 50)
s = 2.0
z = 2.0+2.0im
sum(exp(-s*t)/(t-z))/(2π*im), φ(s*z)
```

```
(0.025348537106990083 + 0.017828329563678146im, 0.025348537106990053 + 0.0178
28329563677806im)
```

We then recover the Laplace transform by taking the limit:

$$\check{1}(s) = -2\pi\mathrm{i} \lim_{z\to\infty} z\phi(sz) = \lim_{z\to\infty} \frac{z}{sz} = \frac{1}{s}$$

What about rational f? Do the same trick of subtracting off the singularities. For example, consider $f(z) = 1/(z+1)$. Then

$$\frac{\phi(sz) - \phi(-s)}{z+1}$$

satisfies all the necessary properties.

```
t = Fun(0 .. 50)
s = 2.0
z = 2.0+2.0im
f = 1/(t+1)
sum(exp(-s*t)*f/(t-z))/(2π*im), (φ(s*z)-φ(-s))/(z+1)
```

```
(0.017439749335766613 + 0.013485355450154743im, 0.017439749335766627 +
 0.013
4853554501541 4im)
```

Therefore,

$$\check{f}(s) = -2\pi\mathrm{i} \lim_{z\to\infty} z\frac{\phi(sz) - \phi(-s)}{z+1} = 2\pi\mathrm{i}\phi(-s) = -\mathrm{e}^s\mathrm{Ei}(-s)$$

```
sum(exp(-s*t)*f), 2π*im*φ(-s), -exp(s)*ei(-s)
```

```
(0.36132861688822454, 0.361328616882152 + 0.0im, 0.361328616882152)
```