1 Lecture 12: Cauchy transforms and Plemelj's theorem

This lecture concerns the properties of the Cauchy transform:

Definition (Cauchy transform) For a contour γ and $f : \gamma \to \mathbb{C}$ define the *Cauchy transform* as

$$\mathcal{C}_\gamma f(z) := \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

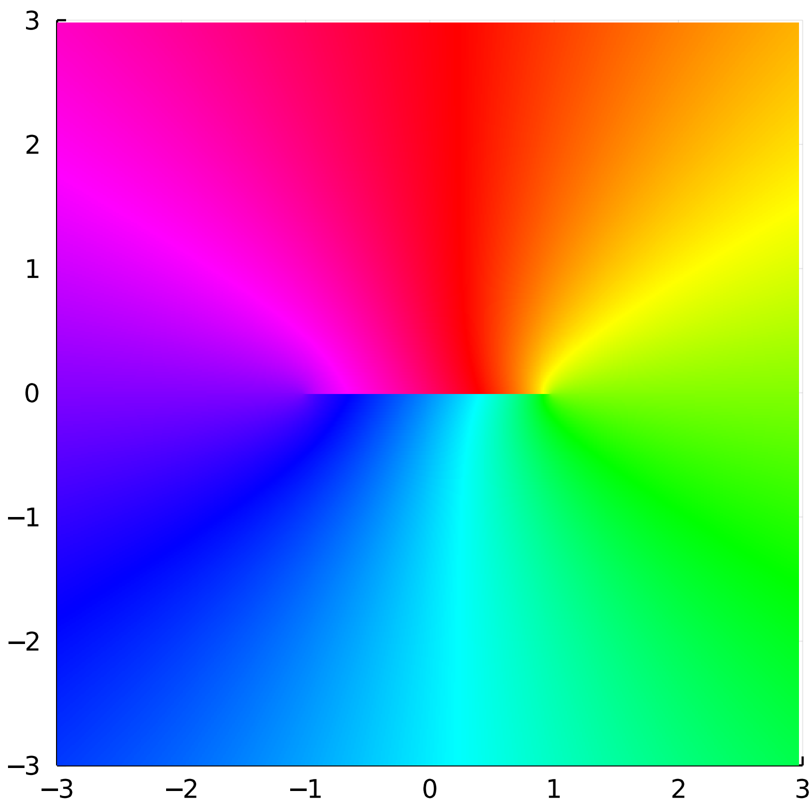
Unlike in Cauchy's integral formula, f need not be analytic and γ need not be closed.

We focus on the case of an interval $[a, b]$:

$$\mathcal{C}_{[a,b]} f(z) := \frac{1}{2\pi \mathrm{i}} \int_a^b \frac{f(x)}{x - z} \mathrm{d}x$$

Here is a phase portrait of the Cauchy transform of a simple function:

```
using ApproxFun, SingularIntegralEquations, ComplexPhasePortrait, Plots
x = Fun(-1 .. 1)
f = exp(x)*sqrt(1-x^2)
phaseplot(-3..3, -3..3, z -> cauchy(f,z))
```



What's evident here is that it has a jump on the contour. It turns out that the Cauchy transform has a very simple subtractive jump. Here we denote

$$\mathcal{C}^+_{[a,b]} f(x) = \lim_{\epsilon \to 0^+} \mathcal{C}_{[a,b]} f(x + \mathrm{i}\epsilon)$$
$$\mathcal{C}^-_{[a,b]} f(x) = \lim_{\epsilon \to 0^+} \mathcal{C}_{[a,b]} f(x - \mathrm{i}\epsilon)$$

that is the limit from above and below. For more complicated contours these would denote the limit from the left/right or in the case of a simple closed contour, interior/exterior.

Theorem (Plemelj on the interval I) Suppose $(b-x)^\alpha (x-a)^\beta f(x)$ is differentiable on $[a,b]$, for $\alpha, \beta < 1$. Then the Cauchy transform has the following properties:

1. *Analyticity*: $\mathcal{C}_{[a,b]} f(z)$ is analytic in $\bar{\mathbb{C}} \backslash [a,b]$
2. *Decay*: $\mathcal{C}_{[a,b]} f(\infty) = 0$
3. *Jump*: It has the subtractive jump:

$$\mathcal{C}^+_{[a,b]} f(x) - \mathcal{C}^-_{[a,b]} f(x) = f(x) \qquad \text{for} \qquad a < x < b$$

4. *Regularity*: $\mathcal{C}_{[a,b]} f(z)$ has weaker than pole singularities at a and b

Sketch of Proof We show the proof for $[-1,1]$.

1. From the dominated convergence theorem, we know that $\mathcal{C} f(z)$ is complex-differentiable off $[-1,1]$:

$$\frac{\mathrm{d}}{\mathrm{d}z} \mathcal{C} f(z) = \frac{1}{2\pi \mathrm{i}} \int_{-1}^1 \frac{\mathrm{d}}{\mathrm{d}z} \frac{f(x)}{x-z} \mathrm{d}x = \frac{1}{2\pi \mathrm{i}} \int_{-1}^1 \frac{f(x)}{(x-z)^2} \mathrm{d}x$$

We know it is analytic at ∞ because

$$\mathcal{C} f(z^{-1}) = z \frac{1}{2\pi \mathrm{i}} \int_{-1}^1 \frac{f(x)}{zx - 1} \mathrm{d}x$$

is differentiable at zero.

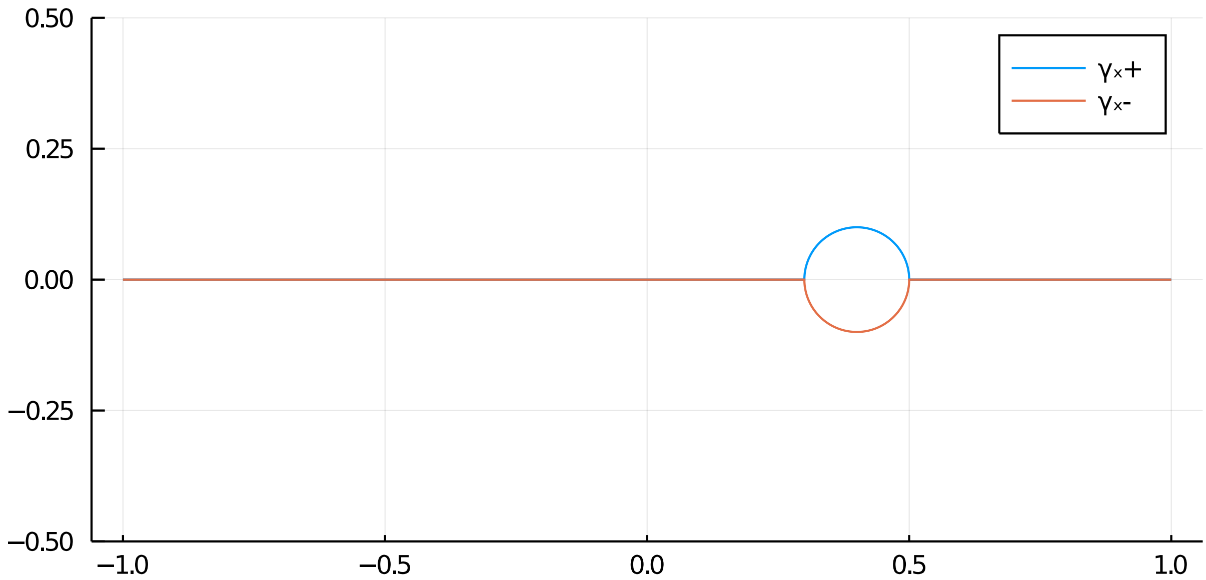
2.

$$\mathcal{C} f(\infty) = 0$$

follows from uniform convergence of $\frac{1}{z-x}$ to zero as $z \to \infty$.

3. For the constant function, which is analytic, this follows by considering a contour γ_x^+ perturbed above x and γ_x^- perturbed below x, see the plot below:

```
x = 0.4
r = 0.1
tt = range(π,0.; length=100)
plot([-1.; x .+ r*cos.(tt);1.0], [0.; r*sin.(tt); 0.0];ylims=(-0.5,0.5),label="γ_x+",
ratio=1.0)
plot!([-1.; x .+ r*cos.(tt);1.0], [0.; -r*sin.(tt); 0.0];ylims=(-0.5,0.5),label="γ_x-")
```



Therefore, by the Cauchy integral formula we have

$$\mathcal{C}^+1(x) - \mathcal{C}^-1(x) = \frac{1}{2\pi\mathrm{i}} \int_{\gamma_x^-} \frac{1}{x-z} \mathrm{d}x - \frac{1}{2\pi\mathrm{i}} \int_{\gamma_x^+} \frac{1}{x-z} \mathrm{d}x = \frac{1}{2\pi\mathrm{i}} \oint \frac{1}{x-z} \mathrm{d}x = 1.$$

For other functions, we consider, for $z = x + \mathrm{i}\epsilon$,

$$\mathcal{C}f(z) = \frac{1}{2\pi\mathrm{i}} \int_{-1}^{1} \frac{f(t) - f(x)}{t - z} \mathrm{d}t + f(x)\mathcal{C}1(z)$$

For $\epsilon = 0$, the first integral exists because the singularity at $t = x$ is removable:

$$\lim_{t \to x} \frac{f(t) - f(x)}{t - x} = f'(x)$$

We leave it as an exercise (or see [Trogdon & Olver 2015, *Riemann-Hilbert Problems, Their Numerical Solution, and the Computation of Nonlinear Special Functions*, Lemma 2.7]) to show that $\int_{-1}^{1} \frac{f(t)-f(x)}{t-z} \mathrm{d}t$ converges to $\int_{-1}^{1} \frac{f(t)-f(x)}{t-x} \mathrm{d}t$ as $z \to x$. It follows that

$$\mathcal{C}^{\pm}f(x) = \frac{1}{2\pi\mathrm{i}} \int_{-1}^{1} \frac{f(t) - f(x)}{t - x} \mathrm{d}t + f(x)\mathcal{C}^{\pm}1(x)$$

and in particular

$$\mathcal{C}^+f(x) - \mathcal{C}^-f(x) = f(x)(\mathcal{C}^+1(x) - \mathcal{C}^-1(x)) = f(x)$$

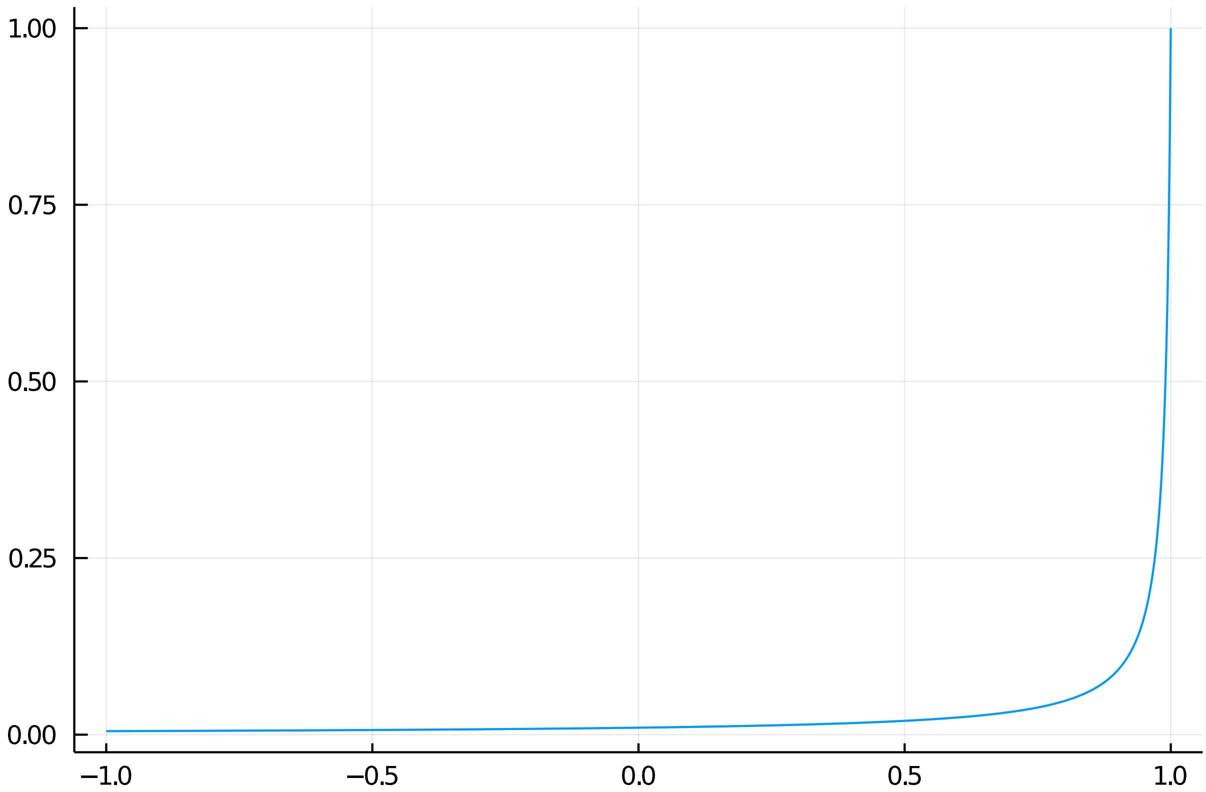
4. We show that it has a weaker than pole singularity at $+1$, with -1 following by the same argument. First note that f is absolutely integrable.

If we assume we approach 1 at an angle of $-\pi + \delta \leq \theta \leq \pi - \delta$, the uniform convergence of $(z-1)\mathcal{C}f(z)$ to zero follows from observing that $\frac{z-1}{z-t}$ can be made arbitrarily small in a larger and larger interval. This is easiest to see for real $x > 1$, where for $1 \leq x \leq 1 + \epsilon^2$ we have

$$\left|\frac{x-1}{x-t}\right| \leq \epsilon$$

for all $t \leq 1 + \epsilon^2 - \epsilon$, or more generously, $t \leq 1 - \epsilon$. Here is a plot of $\frac{x-1}{x-t}$ showing that it is small on an increasing portion of the interval as $x \to 1$ from the right:

```
x = 1 + 0.01
tt = range(-1.,1.; length=1000)
plot(tt, abs.((x - 1) ./ (x .- tt)); legend=false)
```



Therefore,

$$|(x-1)\mathcal{C}f(x)| \leq \frac{1}{2\pi}\int_{-1}^{1-\epsilon} |f(t)| \left|\frac{x-1}{x-t}\right| \mathrm{d}t + \int_{1-\epsilon}^{1} |f(t)|\mathrm{d}t \leq \epsilon \int_{-1}^{1} |f(t)|\mathrm{d}t + \int_{1-\epsilon}^{1} |f(t)|\mathrm{d}t$$

Both terms tends to zero as $\epsilon \to 0$, hence so does $|(x-1)\mathcal{C}f(x)|$. To extend this to the interval itself (that is, $\delta = 0$), we use the stronger requirement that $(1-x)^\alpha(1+x)^\beta f(x)$ is differentiable. For $\alpha = \beta = 0$, this follows from the expression in condition (3) and the fact that (found via direct integration)

$$\mathcal{C}1(z) = \frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}}$$

has only logarithmic singularities, and $f(x)$ is bounded.

■ *Remark:* The singularities of $\mathcal{C}_{[a,b]}f(z)$ at $z = a, b$ are analysed in detail in Lemma 7.2.2, Ablowitz & Fokas, *Complex Variables: Introduction and Applications*. Roughly speaking, if f is smooth on $[a, b]$, then $\mathcal{C}_{[a,b]}f(z)$ will have logarithmic singularities at $z = a, b$ (e.g., $\mathcal{C}1(z)$). If f has an algebraic branch point at a or b of order α, then so has $\mathcal{C}_{[a,b]}f(z)$. For example, if $f(x) = g(x)/\sqrt{1-x^2}$ and g is smooth on $[-1, 1]$, then f has branch points of order $-1/2$ at ± 1 and

$$\mathcal{C}_{[a,b]}f(z) = O\left((1-z)^{-1/2}\right), \qquad z \to 1,$$

and

$$\mathcal{C}_{[a,b]}f(z) = O\left((1+z)^{-1/2}\right), \qquad z \to -1.$$

We can use the previous results, combined with Liouville's theorem, to show that the function satisfying (1)-(4) is in fact unique:

Theorem (Liouville) If f is entire and bounded in $\mathbb{C}$, then f must be constant.

Theorem (Plemelj on the interval II) Suppose $\phi(z)$ satisfies the following properties:

1. *Analyticity*: $\phi(z)$ is analytic in $\bar{\mathbb{C}}\backslash[a, b]$
2. *Decay*: $\phi(\infty) = 0$
3. *Jump*: It has the subtractive jump:

$$\phi^+(x) - \phi^-(x) = f(x) \qquad \text{for} \qquad a < x < b$$

where $(b-x)^\alpha(x-a)^\beta f(x)$ is differentiable in $[a, b]$ for $\alpha, \beta < 1$.

4. *Regularity*: $\phi(z)$ has weaker than pole singularities at a and b

Then $\phi(z) = \mathcal{C}_{[a,b]}f(z)$.

Sketch of Proof Consider

$$A(z) = \phi(z) - \mathcal{C}_{[a,b]}f(z)$$

This is continuous (hence analytic) on (a, b) as

$$A^+(x) - A^-(x) = \phi^+(x) - \phi^-(x) - \mathcal{C}^+_{[a,b]}f(x) + \mathcal{C}^-_{[a,b]}f(x) = f(x) - f(x) = 0$$

Also, A has weaker than pole singularities at a and b, hence is analytic there as well: it's entire. Only entire functions that are bounded are constant, since it vanishes at ∞ the constant must be zero.

■

Example 1 We can use this theorem to prove the following relationships (using $\diamond$ for the dummy variable):

$$\frac{1}{\sqrt{z-1}\sqrt{z+1}} = -2\mathrm{i}\mathcal{C}\left[\frac{1}{\sqrt{1-\diamond^2}}\right](z) = -\frac{1}{\pi}\int_{-1}^{1}\frac{\mathrm{d}x}{\sqrt{1-x^2}(x-z)}$$

(1) follows because the jumps cancel. (2 and 4) are immediate. (3) follows from a simple calculation.

Example 2 Now consider a problem of reducing

$$\phi(z) = \sqrt{z-1}\sqrt{z+1}$$

to its behaviour near its singularities. It has two singularities: it blows up at ∞ and has a branch cut on $[-1, 1]$

We can subtract out the singularity at infinity first to determine

$$\phi(z) = z + 2\mathrm{i}\mathcal{C}[\sqrt{1-\diamond^2}](z)$$

Note this works because, as $z \to \infty$, we have

$$\phi(z) = z(\sqrt{1-1/z}\sqrt{1+1/z}) = z(1+O(1/z))(1+O(1/z)) = z + O(1/z)$$

hence $\phi(z) - z$ vanishes at ∞. This is an example of summing over the behaviour at each singularity to recover the function (in this case, ϕ has a singularity along the cut $[-1, 1]$ and polynomial growth at ∞).

Because $\phi(z) - z$ decays, we can now deploy Plemelj II to determine:

$$\phi(z) - z = \mathcal{C}[\phi_+ - \phi_-](z)$$

where

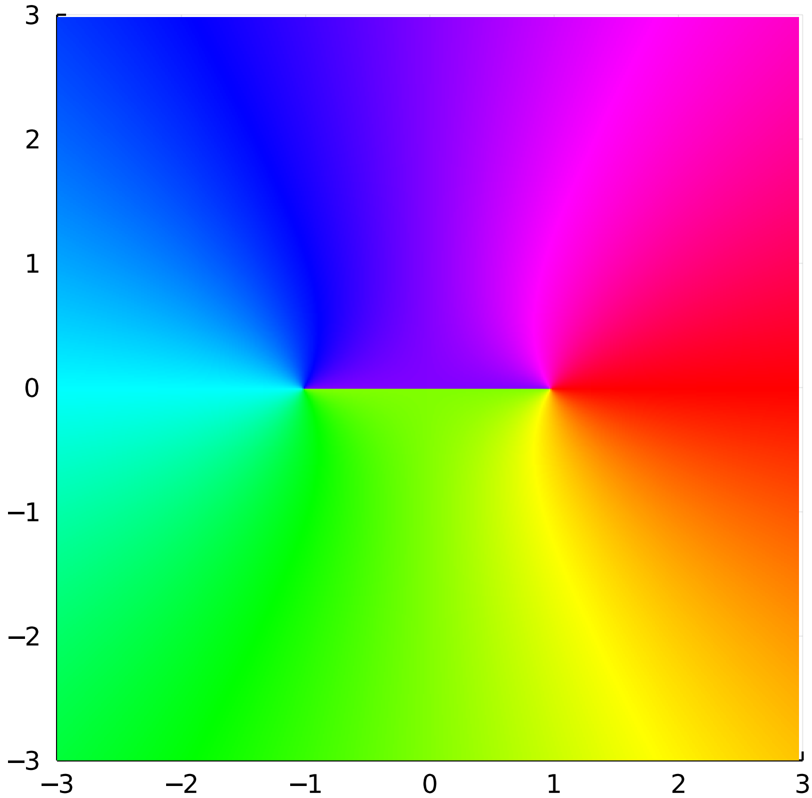
$$\phi_+(x) - \phi_-(x) = 2\mathrm{i}\sqrt{1-x^2}$$

Example 3 Finally, we have the following (also verifiable using indefinite integration):

$$\frac{\log(z-1) - \log(z+1)}{2\pi\mathrm{i}} = \mathcal{C}[1](z) = \frac{1}{2\pi\mathrm{i}} \int_{-1}^{1} \frac{\mathrm{d}x}{x-z}$$

Demonstration From the phase plot we see it has a branch cut on $[-1, 1]$:

```
κ = z -> 1/(sqrt(z-1)sqrt(z+1))
phaseplot(-3..3, -3..3, κ)
```



On the branch there is the expected jump:

```julia
x = 0.1
κ(x + 0.0im) - κ(x - 0.0im) , -2im/sqrt(1-x^2)
```

```
(0.0 - 2.010075630518424im, 0.0 - 2.010075630518424im)
```

For $x < -1$ the branch cut is removable: we have continuity and therefore analyticity:

```julia
x = -2.3
κ(x + 0.0im) - κ(x - 0.0im)
```

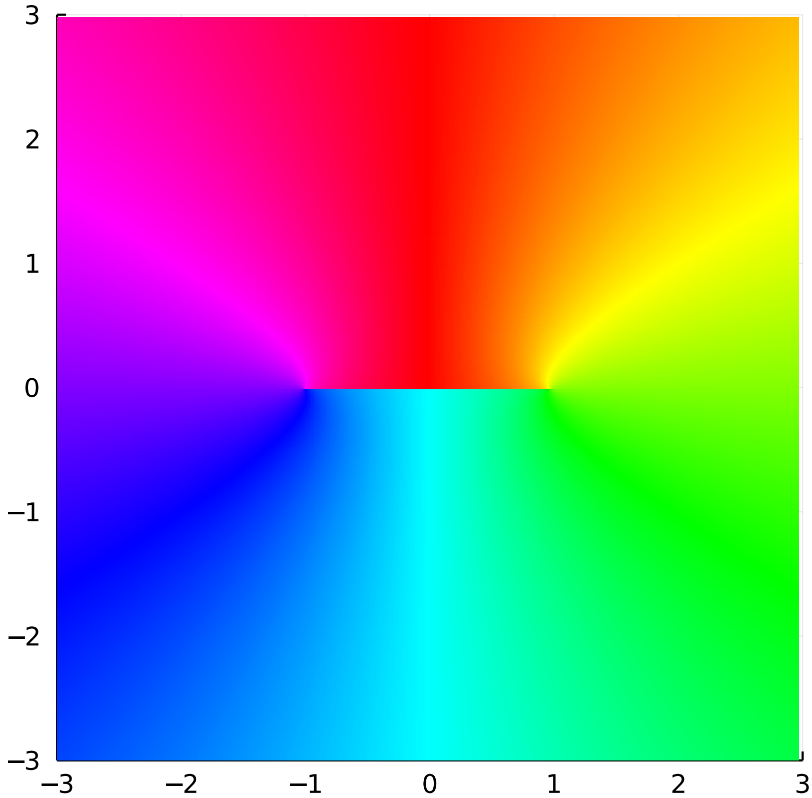
```
0.0 - 0.0im
```

3. $\mu(z) = \frac{\log(z-1)-\log(z+1)}{2\pi \mathrm{i}}$ is the unique function analytic in $\mathbb{C}\backslash[-1,1]$ with weaker than pole singularities at ± 1 satisfying $\mu(\infty) = 0$ and

$$\mu_+(x) - \mu_-(x) = 1 \qquad \text{for} \qquad -1 < x < 1.$$

Demonstration Here we see from the phase plot of μ that it has a branch cut on $[-1, 1]$:

```julia
μ = z -> (log(z-1) - log(z+1))/(2π*im)
phaseplot(-3..3, -3..3, μ)
```



For $-1 < x < 1$ we have the jump 1:

```
x = 0.3
μ(x + 0.0im) - μ(x - 0.0im)
```

```
1.0 + 0.0im
```

For $x < -1$ we see that the branch cuts cancel and we have continuity:

```
x = -4.3
μ(x + 0.0im) - μ(x - 0.0im)
```

```
0.0 + 0.0im
```

Remark As an aside, these integrals are computationally difficult because of the singularity in the integrand, hence standard integration methods become slow as z approaches the interval. There are other specialised routines (as implemented in `cauchy(f,z)`) that are much more efficient:

```
using BenchmarkTools
z = 0.1 +0.0001im
x = Fun()
f = exp(x)
# μs is micro seconds (1E-6 seconds) while ms is milliseconds (1E-3 seconds)
@btime cauchy(f, z ) # specialised routine
@btime sum(f/(x-z))/(2π*im) # standard quadrature
```

```
10.799 μ@*(s (18 allocations: 1.64 KiB)324.779 ms (305 allocations: 207.54
MiB)0.5525638794334992 - 0.3181012137711561im
```

We can evaluate the limit from above and below using the specialised routine, where standard quadrature breaks down. Here we see numerically that we recover f from taking the difference:

```
cauchy(f, 0.1+0.0im)-cauchy(f, 0.1-0.0im) , f(0.1)
```

```
(1.1051709180756475 + 0.0im, 1.1051709180756475)
```