Applied Complex Analysis (2021)

1 Problem sheet 2

1.1 Problem 1

Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

1. Use Gershgorin's theorem to bound the eigenvalues of A.
2. Recall that the eigenvalues of A and $A^\top$ are the same. Use this fact to find a bound on the eigenvalues based on the absolute *column* sums.
3. Design a circular contour surrounding the spectrum of A.

1.2 Problem 2

1. Given $A \in \mathbb{R}^{n\times n}$ that is symmetric positive definite (that is, all eigenvalues of A are real and greater than zero) and $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}^n$, write a contour integral solution to the second-order linear constant coefficient ODE:

$$\begin{aligned} \mathbf{u}''(t) &= A\mathbf{u}(t) \\ \mathbf{u}(0) &= \mathbf{u}_0 \\ \mathbf{u}'(0) &= \mathbf{v}_0 \end{aligned}$$

2. Was the restriction to symmetric positive definite matrices necessary? Why or why not?

1.3 Problem 3

1. Suppose that $g(\theta)$ has absolutely summable Fourier coefficients, that is,

$$g(\theta) = \sum_{k=-\infty}^{\infty} g_k \mathrm{e}^{\mathrm{i}k\theta} \qquad \text{where} \qquad \sum_{k=-\infty}^{\infty} |g_k| < \infty.$$

Show that the periodic trapezium rule satisfies

$$\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) = \cdots + g_{-2n} + g_{-n} + g_0 + g_n + g_{2n} + \cdots$$

where $\theta_j = \frac{2\pi j}{n}$. Hint: use the geometric series to simplify $\sum_{j=0}^{n-1} \mathrm{e}^{\mathrm{i}k\theta_j}$.

2. Suppose that $g(\theta) = f(e^{i\theta})$ where $f(z)$ is holomorphic in an annulus $\{z : R^{-1} < |z| < R\}$. Prove that the periodic trapezium rule converges exponentially fast:

$$\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) \to \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{d}\theta.$$

3. Find an upper bound for the error

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) - \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\mathrm{d}\theta\right|$$

for $g(\theta) = \frac{1}{2-\cos\theta}$.