Compare two ways of getting the 2D uniform Ideal Fluid Flow

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Introduction

We have seen in the lecture that we can get the 2D ideal fluid flow with the help of Cauchy Transform (CT) and Hilbert Transform (HT). Normally, people usually conformal mapping (CM) to retort spaces. And once we know the complex potential in one space, we can substitute the mapping and get the potential in the other space.

In this project, we will work out the potential flow of several famous fluid models, using both methods. These models start from the simple circle and build up to the final case of the complicated aerofoil.

Then I will do two special cases in which two methods are limited.

1.1. Flow around a simple circle with interior bound. Flow around a rotating circle with interior bound. (Magnus effect.)

1.2. Flow around a segment with an angle of attack.

1.3. Flow around complex geometry like aerofoils.

2.1. Flow around two obstacles. (CM is limited here.)

2.2. Flow around a corner, which has infinite bounds. (CT/HT is limited here.)

Background

In fluid dynamics, we derived that if a 2D flow is irrotational and incompressible, then there exist a complex potential w(z)=$\varphi(x, y) + i\psi(x, y)$ where φ is the velocity potential and ψ is the stream function.

In this way, any ideal fluid flow around a motionless rigid body can be think as the complex function w(z) that satisfies:

1. w(z) is analytic everywhere outside the body contour S;
2. the flow satisfies the impermeability condition on the body contour;
   i.e.: Im{w(z)} = constant on the body contour S

3. the free-stream condition

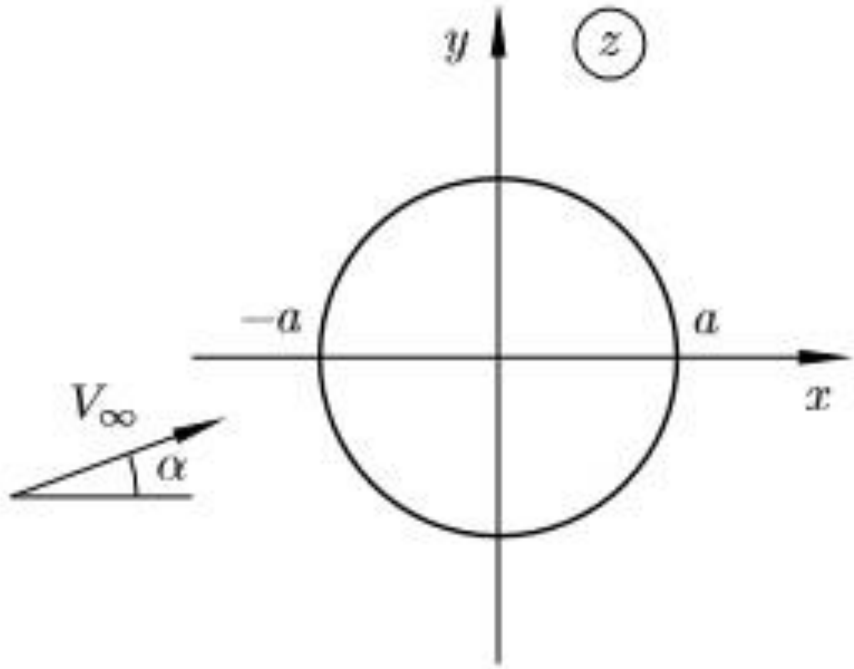
$$\frac{dw}{dz} = u_\infty - iv_\infty \qquad at\ z = \infty$$

where u_∞, v_∞ are the components of the far away velocity vector.

Note: With these three conditions, the complex potential is uniquely determined up to an added constant.

Now we consider several cases of uniform ideal flow and aim to solve for the streamline function.

Case 1.1. Flow around a simple circle with interior bound.



In this case, we find the streamline function by finding the complex potential w(z), which satisfies the criterions:

1. w(z) is analytic for {z: |z|>a};
2. Im{w(z)} = constant on {z: |z|=a}

3. the free-stream condition

$$\frac{dw}{dz} = u_\infty - iv_\infty \qquad at\ z = \infty$$

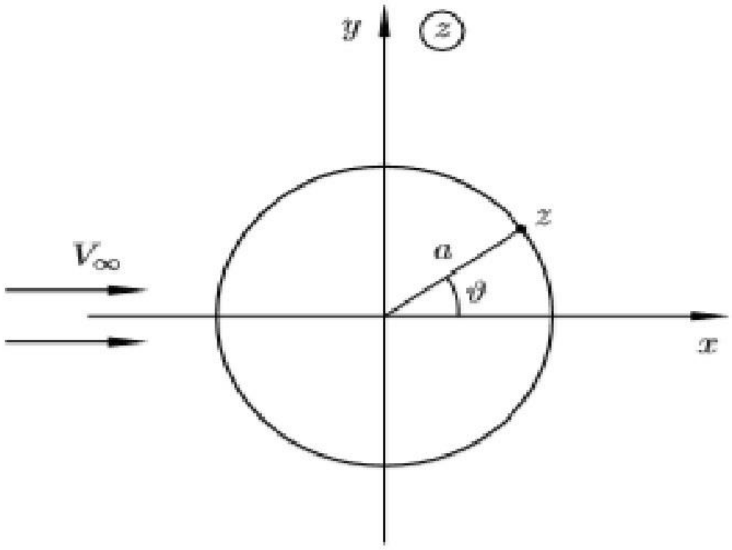
where u_∞, v_∞ are the components of the far away velocity vector V_∞.

We call this boundary value problem 1.1.

1. Conformal mapping method:

We derived in fluid dynamics that the potential flow past a cylinder has potential

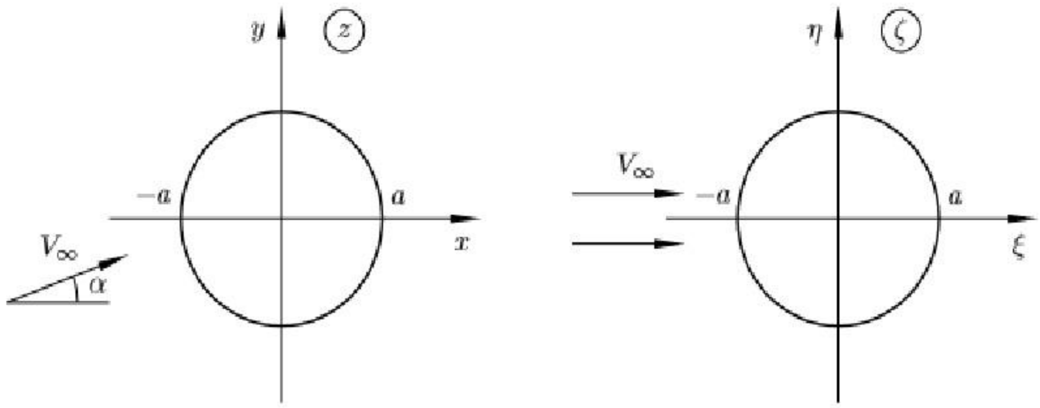
$$w(z) = V_\infty (z + \frac{a^2}{z})$$



To generalize the flow to the case with an angle of attack, we can use the conformal mapping to map back to the case above by

$$\varsigma = e^{-i\alpha} z$$

because if we rotate z-plane by the angle, two planes are identical.



Rotation of the cylinder through angle α.

We know that the complex potential in the ς-plane is:

$$W(\varsigma) = V_{\infty}(\varsigma + \frac{a^2}{\varsigma})$$

So, simply plug in the substitution, we can get the complex potential in the z-plane:

$$w(z) = V_{\infty}(e^{-i\alpha}z + \frac{a^2}{e^{-i\alpha}z})$$

2. Cauchy/Hilbert transform method:

We consider the Cauchy transform over the circle T= {z: |z|=a}:

$$C_T f(z) = \frac{1}{2\pi i}\oint \frac{f(t)}{t-z}dt$$

By the Plemelj formula on the circle, we have:

1) $C_T f(z)$ analytic in C\T;
2) $C_T^+ f(t) - C_T^- f(t) = f(t)$ for t on T;
3) $C_T f(\infty) = 0$;

where $C_T^- f(t)$ is the limit from outside of the circle.

We consider w(z)= $V_{\infty}(e^{-i\alpha}z + C_T f(t))$ and solve the unknown function f s.t. w(z) satisfies the three criterions.

Criterion 1 is satisfied since $C_T f(z)$ analytic in C\T;

Criterion 3 is satisfied since $V_{\infty}e^{-i\alpha}z$ satisfies trivially and $C_T f(\infty) = 0$.

Criterion 2 is equivalent to say that Im{w(z)} = 0 on {z: |z|=a}, since w(z) is determined up to an added constant. Thus,

$$0=\text{Im}\left((e^{-i\alpha}z+C_T^- f(t)\right))$$

$$= \text{Im}\left((cos\alpha - isin\alpha)z + C_T^- f(t)\right)$$

$$=a^2(\cos\alpha\frac{t-\frac{1}{t}}{2i} - sin\alpha\frac{t+\frac{1}{t}}{2}) + Im(C_T^- f(t))$$

So, $Im(C_T^- f(t))$= -$a^2\cos\alpha\frac{t-\frac{1}{t}}{2i} + a^2 sin\alpha\frac{t+\frac{1}{t}}{2}$.

Solving this by the hint in the problem sheet 2 Problem 3.3, we have:

$$f(t)= a^2\ (\cos\alpha+i\sin\alpha)\frac{-1}{t}$$

So,

$$w(z)= V_\infty(e^{-i\alpha}z + C_T[e^{i\alpha}\frac{-a^2}{t}](z))$$

$$=V_\infty(e^{-i\alpha}z - e^{i\alpha}\frac{-a^2}{z})$$

$$=V_\infty(e^{-i\alpha}z + \frac{a^2}{ze^{-i\alpha}})$$

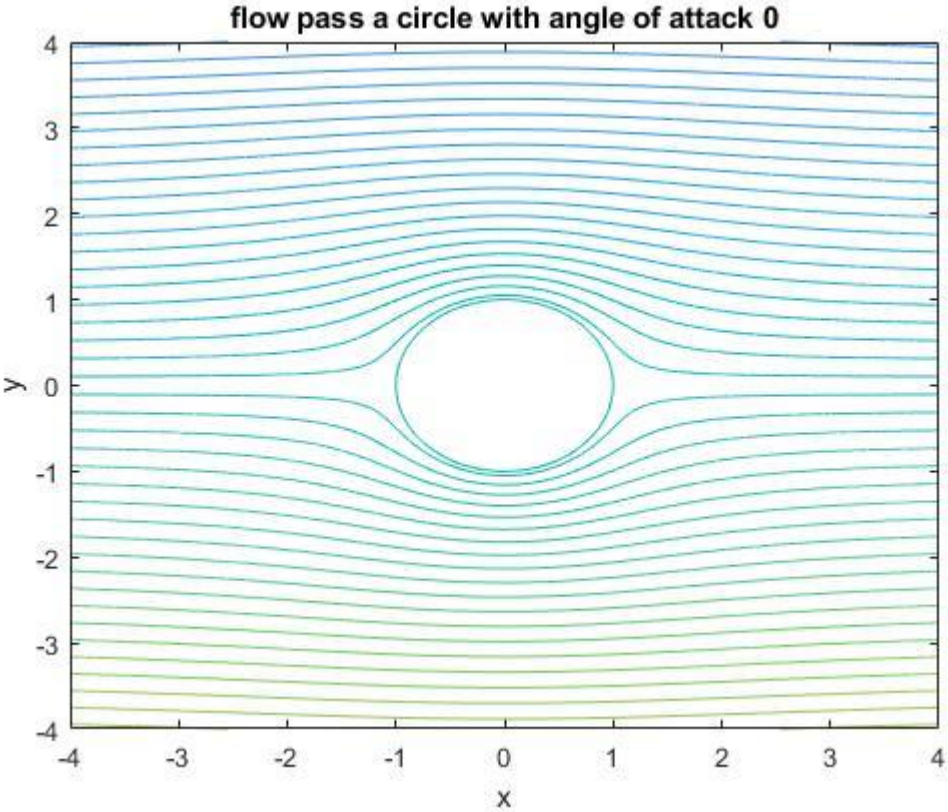
exactly the same as we got before.

Then we can get the streamline function: $\psi(x,y) = Im(w)$.

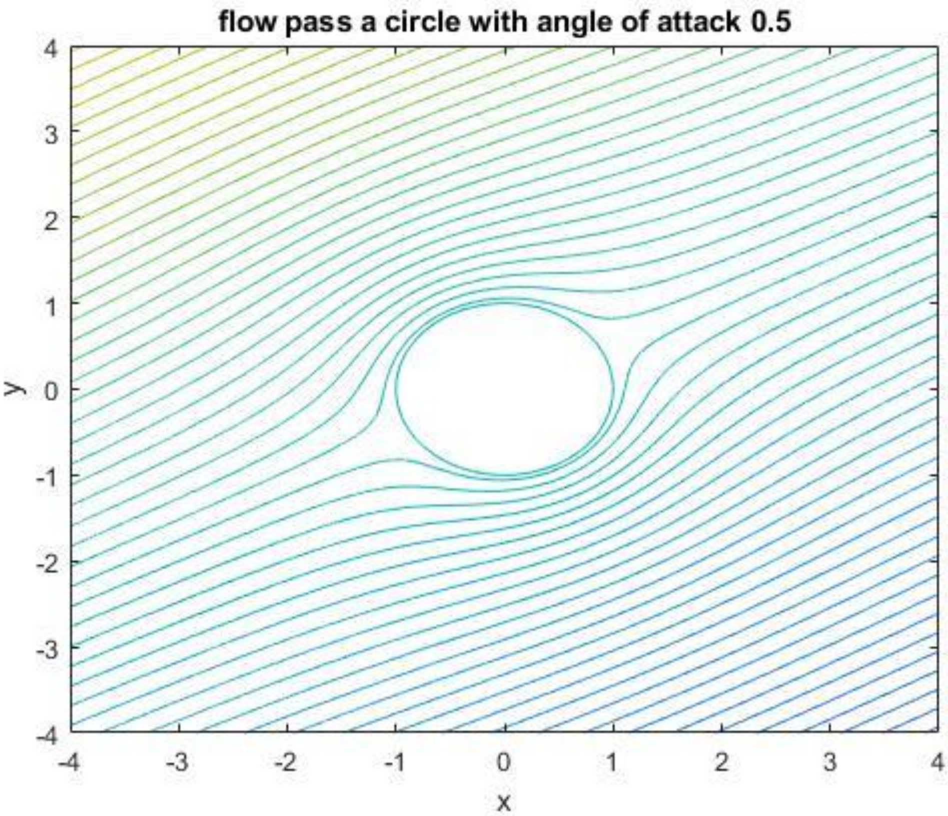
Plots obtained by Matlab:

```
clear
syms x; syms y;
ezplot((x).^2+(y).^2-1.^2)
hold on

theta=0.5;
Theta = [0:0.01:2*pi];
R = [1.0:0.001:10.0];
[ThetaG,RG] = meshgrid(Theta,R);
X = RG.*cos(ThetaG);
Y = RG.*sin(ThetaG);
Z=imag(exp(-i*theta) .* (X+i.*Y) + exp(i*theta) .* ((X+i.*Y).^(-1)));
contour(X,Y,Z,100)
ylim([-4,4]);
xlim([-4,4]);
hold off

title('flow pass a circle with angle of attack 0.5')
```

Here is the case a=1, α=0:



Here is the case a=1, α=0.5:



Remark: if we consider the circle to be rotating with angular velocity Ω, then the criterions are not changed.

Ie: boundary value problem 1.1 may defines a unique solution for the motionless body, but doesn't define a unique solution in general.

For example, in the Cauchy Transform:

Criterion 2 is changed to Im{w(z)} = C on {z: |z|=a}. Thus,

$$C=\mathrm{Im}\,((e^{-i\alpha}z + C_T^- f(t)))$$

$$=a^2(\cos\alpha\frac{t-\frac{1}{t}}{2i} - sin\alpha\frac{t+\frac{1}{t}}{2}) + Im(C_T^- f(t))$$

⇨ $$Im(C_T^- f(t)) = C - a^2\cos\alpha\frac{t-\frac{1}{t}}{2i} + a^2 sin\alpha\frac{t+\frac{1}{t}}{2}.$$

So, if we add a term to the imaginary part of w(z), which satisfies:

1) Analytic outside the circle;
2) Derivative decays at infinity;
3) Cauchy transform is constant on the circle;

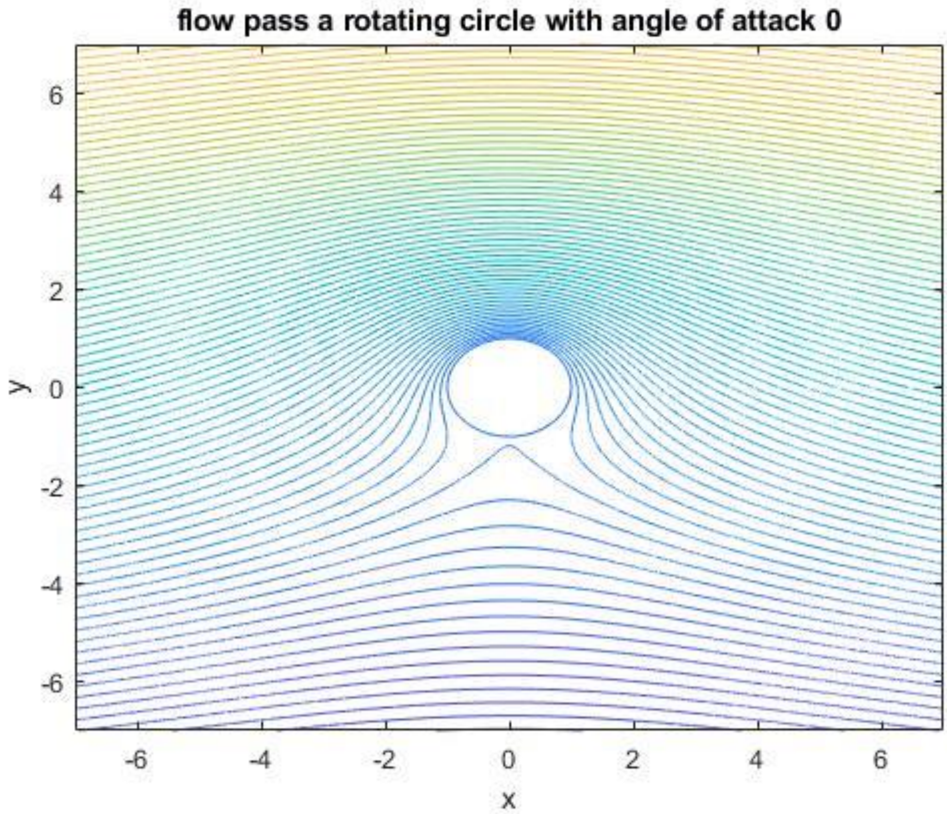
Then it still solves the boundary value problem 1.1.

Except adding a constant, adding a log term also satisfies the above.

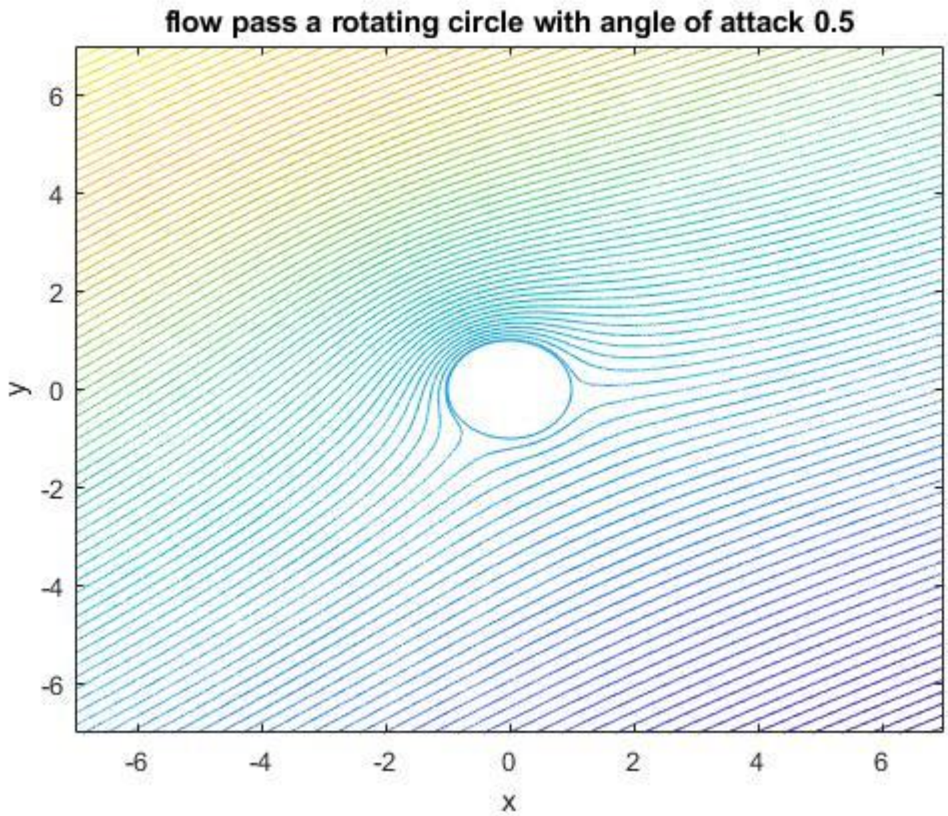
$$w(z) = V_\infty(e^{-i\alpha}z + \frac{a^2}{ze^{-i\alpha}}) + iElnz$$

where E is a free parameter. By verifying the no-slip condition in fluid dynamics, can reveal that parameter E indeed relates on the angular velocity Ω. And thus we obtained a family of solutions to the problem 1.1.

Here is the case a=1, α=0, E=2:



Here is the case a=1, α=0.5, E=1:

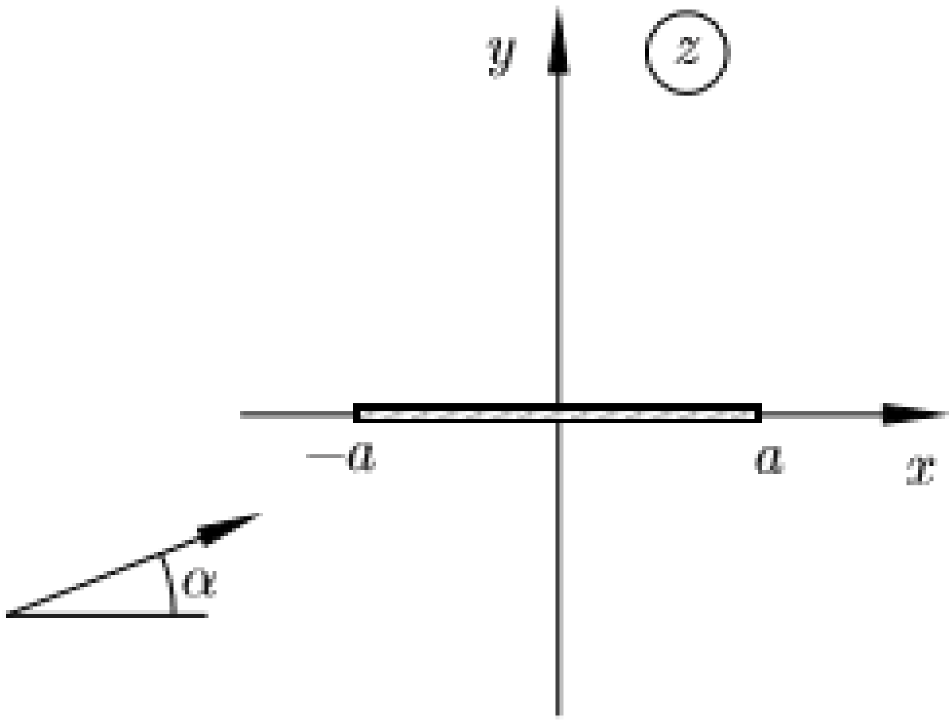


Plots obtained by Matlab:

```
clear
syms x; syms y;
ezplot((x).^2+(y).^2-1.^2)
hold on

theta=0.5;
E=1;
Theta = [0:0.01:2*pi];
R = [1.0:0.001:10.0];
[ThetaG,RG] = meshgrid(Theta,R);
X = RG.*cos(ThetaG);
Y = RG.*sin(ThetaG);
Z=imag(exp(-i*theta) .* (X+i.*Y) + exp(i*theta) .* ((X+i.*Y).^(-
1))+i*E*log(X+i.*Y));
contour(X,Y,Z,100)
ylim([-4,4]);
xlim([-4,4]);
hold off

title('flow pass a rotating circle with angle of attack 0.5')
```

Remark: The asymmetric flow below would generate an upward pressure force, which is called the Magnus effect. And it has a lot of applications in real life. And the bigger the Ω, the bigger the force.

Case 1.2. Flow around a segment with an angle of attack.



In this case, we find the streamline function by finding the complex potential w(z), which satisfies the criterions:

1. w(z) is analytic off {x: -a < x < a};
2. Im{w(z)} = 0 on {x: -a < x < a}
3. the free-stream condition

$$\frac{dw}{dz} = u_\infty - iv_\infty \qquad at\ z = \infty$$

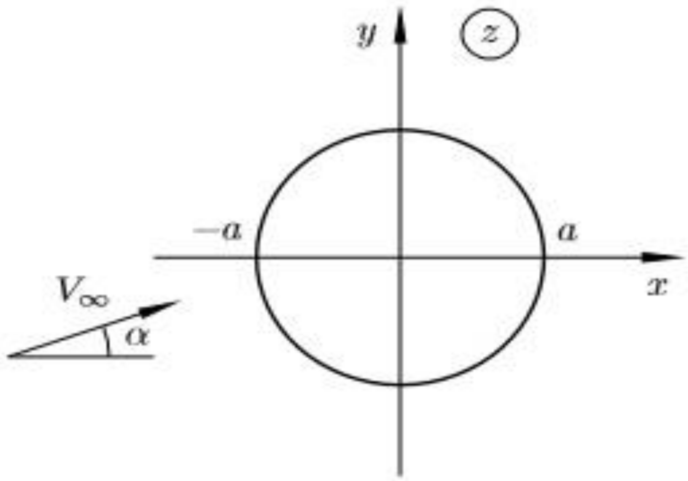
where u_∞, v_∞ are the components of the far away velocity vector V_∞.

We call this boundary value problem 1.2.

1. Conformal mapping method:

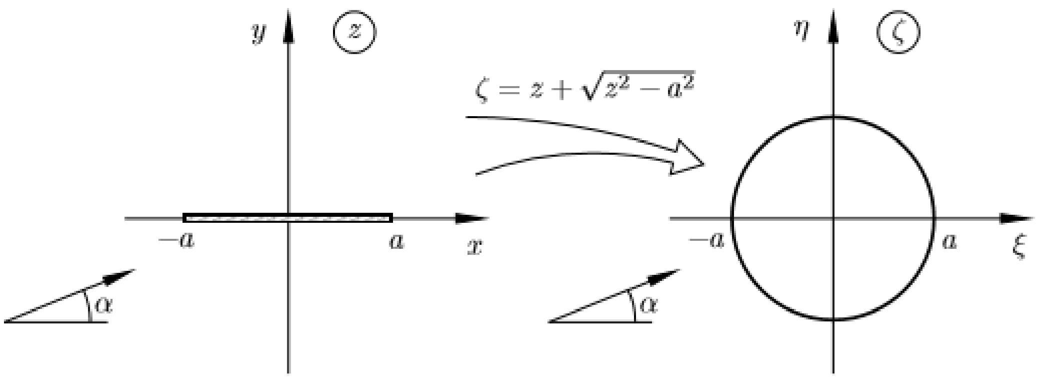
We derived in fluid dynamics that the potential flow past a cylinder has potential

$$w(z) = V_\infty\left(e^{-i\alpha}z + \frac{a^2}{e^{-i\alpha}z}\right)$$



To generalize the flow to the case with an angle of attack, we can use the conformal mapping to map back to the case above by

$$\varsigma = z + \sqrt{z^2 - a^2}$$



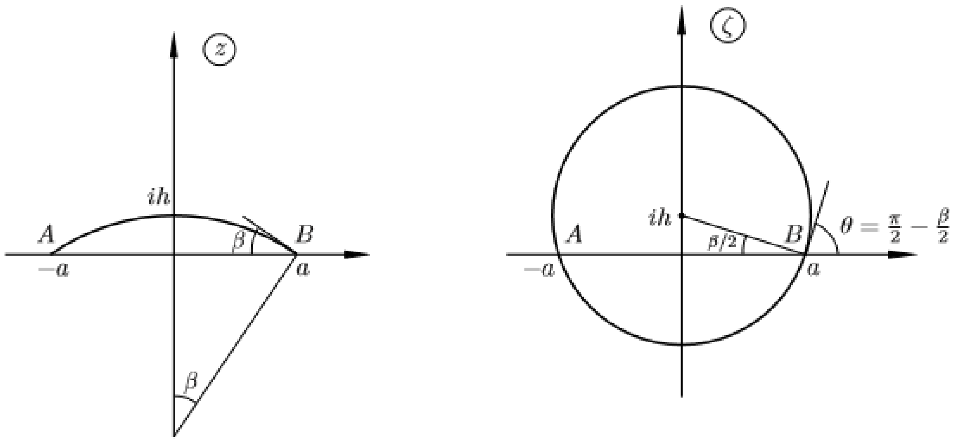
(a) Physical plane.

(b) Auxiliary plane.

Transformation of a flat plate onto a circle.

Because famous Joukovskii transformation maps an arc to a full circle:

$$\frac{z-a}{z+a} = (\frac{\varsigma - a}{\varsigma + a})^2$$



(a) z-plane. (b) ζ-plane.

$$ie: \varsigma = z \pm \sqrt{z^2 - a^2}$$

Note that the map with '+' maps points inside the arc to the points inside the circle, and the map with '–' maps the points inside the arc to the points outside the circle. We choose '+'.

And if we take h=0, then in z-plane, it's a segment, while in ς-plane, it's a circle located at the origin with radius a.

We know that the complex potential in the ς-plane is:

$$W(\varsigma) = V_\infty (e^{-i\alpha}\varsigma + \frac{a^2}{e^{-i\alpha}\varsigma})$$

So, simply plug in the substitution, we can get the complex potential in the z-plane:

$$w(z) = \frac{V_\infty}{2}(e^{-i\alpha}(z + \sqrt{z^2 - a^2}) + \frac{a^2}{e^{-i\alpha}(z + \sqrt{z^2 - a^2})})$$

$$= \frac{V_\infty}{2}(e^{-i\alpha}(z+\sqrt{z^2-a^2})+e^{i\alpha}(z-\sqrt{z^2-a^2}))$$

$$= \frac{V_\infty}{2}(2zcos\alpha - i2\sqrt{z^2-a^2}sin\alpha)$$

(Note: in conformal mapping, simply plug in the substitution doesn't necessarily satisfy the third criterion. Usually, one need to check and rescale by a factor. In this case, we need to scale the V_∞ by $\frac{1}{2}$.)

2. Cauchy/Hilbert transform method:

We consider the Cauchy transform over the branch cut γ = {x: -a < x < a};

$$C_\gamma f(z) = \frac{1}{2\pi i}\int_{-a}^{a}\frac{f(t)}{t-z}dt$$

By the Plemelj formula on the branch cut, we have:

1) $C_\gamma f(z)$ analytic in C\ γ;
2) $C_\gamma^+ f(x) - C_\gamma^- f(x) = f(x)$ for x on γ;
3) $C_\gamma f(\infty) = 0$;
4) $C_\gamma f(z)$ has weaker than pole singularity on γ.

We consider w(z)= $V_\infty(e^{-i\alpha}z + C_\gamma f(t))$ and solve the unknown function f s.t. w(z) satisfies the three criterions.

Criterion 1 is satisfied since $C_\gamma f(z)$ analytic in C\ γ;

Criterion 3 is satisfied since $V_\infty e^{-i\alpha}z$ satisfies trivially and $C_\gamma f(\infty) = 0$.

Criterion 2 is to say that Im{w(t)} = 0 on γ. Thus,

$$0=\text{Im}\left(\left(e^{-i\alpha}t + C_\gamma^{\pm} f(t)\right)\right)$$

$$= \text{Im}\left((cos\alpha - isin\alpha)t + C_\gamma^{\pm} f(t)\right)$$

$$=-tsin\alpha + Im(C_\gamma^{\pm} f(t))$$

In problem 1.2, we have $C_\gamma^{+} f(t) = C_\gamma^{-} f(t)$, which means f(t) has to be real. And thus $Im(C_\gamma^{\pm} f(t))$=$Im(C_\gamma f(t))$=$\frac{H_\gamma f(t)}{2}$.

So,

$$H_\gamma f(t)\text{=}2tsin\alpha$$

Solving this by the formula in the lecture, we have:

$$\text{f(t)= }2\text{sin}\alpha\sqrt{a^2 - t^2} + \frac{C}{\sqrt{a^2-t^2}} = 2\text{sin}\alpha\sqrt{a^2 - t^2}$$

where we choose c=0 to make it bounded at $\pm$a.

So,

$$\text{w(z)= } V_\infty\left(e^{-i\alpha}z + C_\gamma[f](z)\right)$$

$$=V_\infty\left(e^{-i\alpha}z + \text{sin}\alpha\,\frac{\sqrt{z-a}\sqrt{z+a}-z}{i}\right)$$

$$=V_\infty\left(zcos\alpha - i\sqrt{z-a}\sqrt{z+a}\,\text{sin}\alpha\right)$$

exactly the same as we got before.

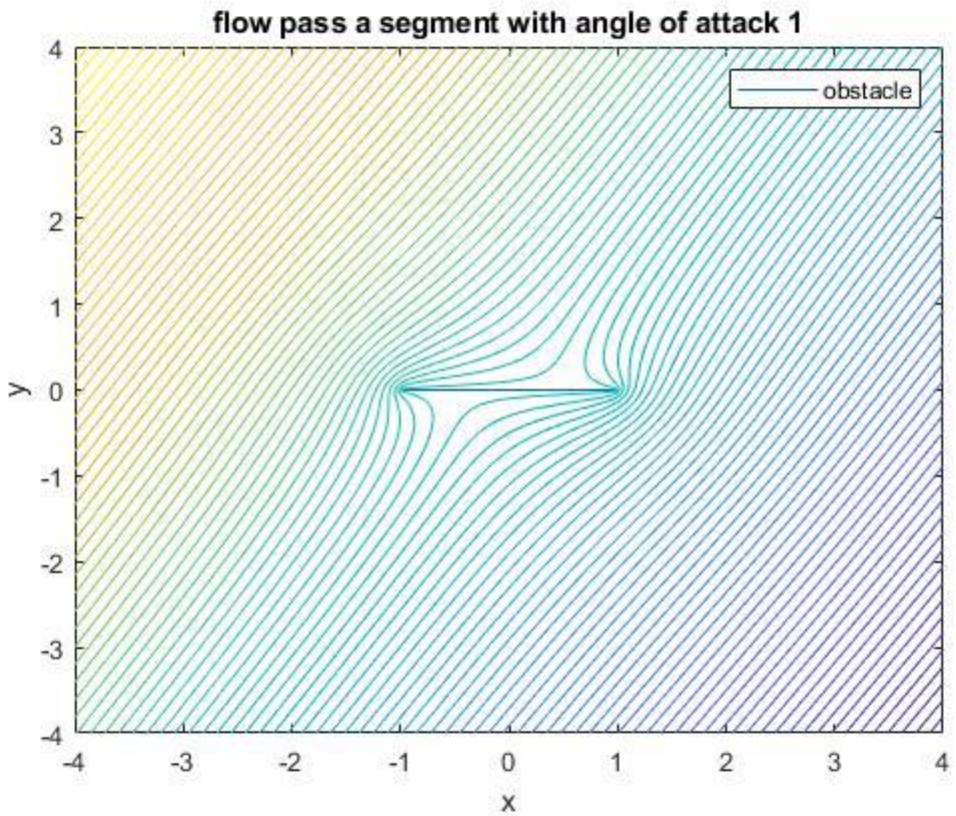
Then we can get the streamline function: $\psi(x,y) = Im(w)$.

Plots obtained by Matlab:

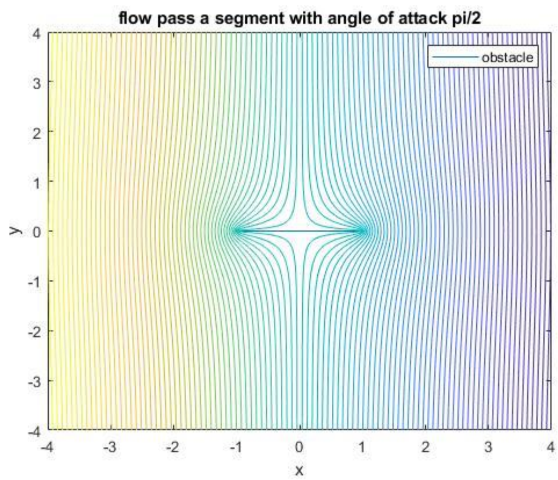
```
clear
plot([-1,1],[0,0])
hold on

theta=1;
X = [-4.0:0.01:4.0];
Y = [-4.005:0.01:4.005];
[X,Y] = meshgrid(X,Y);
Z=cos(theta)* Y-sin(theta)*real(sqrt(X+i*Y-1).*sqrt(X+i*Y+1));
contour(X,Y,Z,200)
ylim([-4,4]);
```

```
xlim([-4,4]);
hold off

title('flow pass a segment with angle of attack 1')
xlabel('x')
ylabel('y')
legend('obstacle')
```

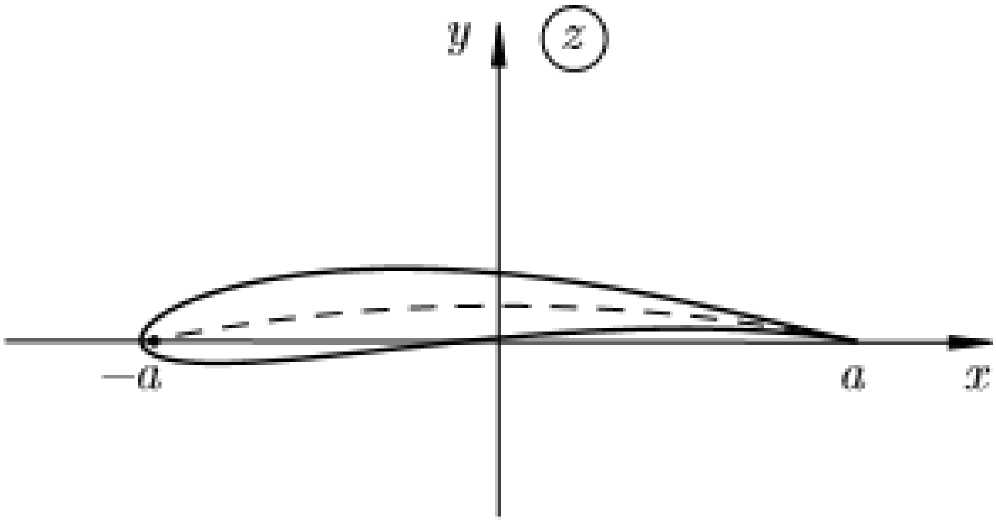
Here is the case a=1, α=1:



Here is the case a=1, α=pi/2:



Case 1.3. Flow around an aerofoil.



In this case, we find the streamline function by finding the complex potential w(z), which satisfies the criterions:

1. w(z) is analytic outside the contour;
2. Im{w(z)} = 0 on the contour;
3. the free-stream condition

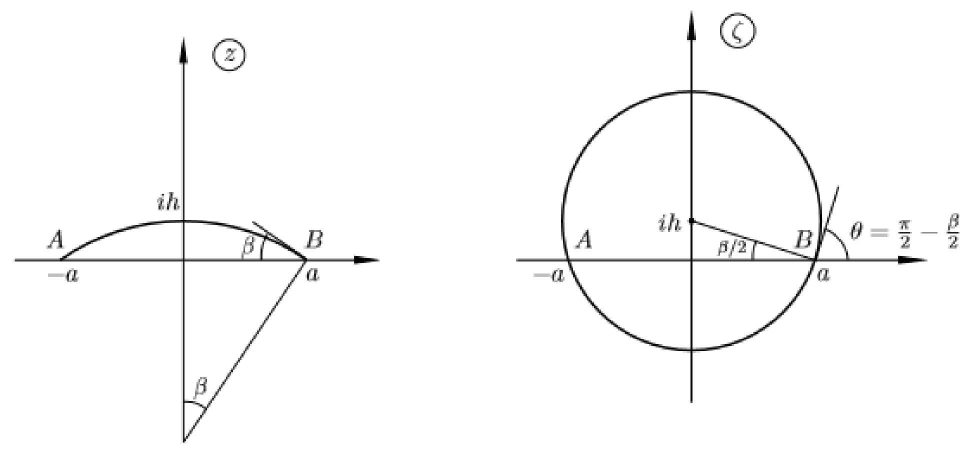
$$\frac{dw}{dz} = u_\infty - iv_\infty \qquad at\ z = \infty$$

where u_∞, v_∞ are the components of the far away velocity vector V_∞.

We call this boundary value problem 1.3.

1. Conformal mapping method:

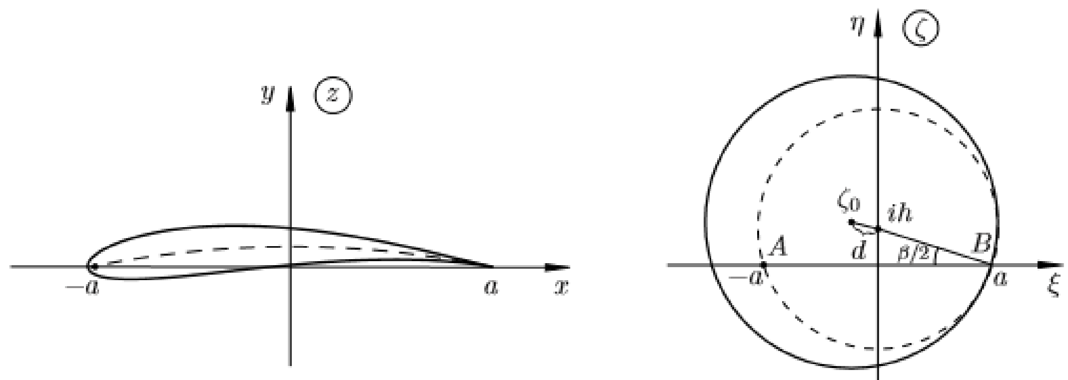
We still use the Joukovskii conformal mapping.



(a) z-plane. (b) ζ-plane.

Because famous Joukovskii transformation is the map between an arc and a full circle:

$$\frac{z-a}{z+a} = (\frac{\varsigma - a}{\varsigma + a})^2$$



(a) z-plane. (b) ζ-plane.

By the transformation, the smaller (dashed) circle is mapped to the dashed arc on the left plane. And we draw a circle which is tangent to the dashed circle at B. Then the map of the bigger (solid) circle should also be tangent to the dashed arc, as we can see on the left. The other points on two circles doesn't match, so their distance is also mapped to a nonzero distance. And as a result, we get the shape on the left, which is called 'Joukovskii aerofoil'.

Note there are three parameters in the graph. They are: h, d and a.

The solid circle has radius R=d+$\sqrt{h^2+a^2}$, and centre at ς_0, so the flow passes the solid circle is:

$$W(\varsigma) = V_\infty((\varsigma - \varsigma_0)e^{-i\alpha} + \frac{R^2}{(\varsigma - \varsigma_0)e^{-i\alpha}})$$

Recall the map from ς-plane to z-plane is

$$\varsigma = z + \sqrt{z^2 - a^2}$$

So, simply plug in the substitution, we can get the complex potential in the z-plane:

$$w(z) = \frac{V_\infty}{2}((z + \sqrt{z^2 - a^2} - \varsigma_0)e^{-i\alpha} + \frac{R^2}{(z + \sqrt{z^2 - a^2} - \varsigma_0)e^{-i\alpha}})$$

$$= \frac{V_\infty}{2}((z + \sqrt{z^2 - a^2} - \varsigma_0)e^{-i\alpha} + e^{i\alpha}(z - \sqrt{z^2 - a^2} - \varsigma_0))$$

$$= \frac{V_\infty}{2}(2(z - \varsigma_0)cos\alpha - 2isin\alpha\sqrt{z^2 - a^2})$$

(Remember in conformal mapping, condition three is not automatically satisfied. In this case, after some calculation, it turns out that we need to scale the V_∞ by $\frac{1}{2}$.)

2. Cauchy/Hilbert transform method:

We consider the Cauchy transform over the aerofoil contour $\mathbb{\Gamma}$. First, let the Cauchy transform over the solid circle γ be

$$C_\gamma g(z) = \frac{1}{2\pi i}\int_\gamma \frac{g(t)}{t - z}dt$$

Then by the theorem in mastery material,

$$C_{\mathbb{\Gamma}}g(z) = \sum_{i=1}^{2} C_{\gamma}[g^{\circ}J](J^{-1}(z)_i) - '\mathit{the\ behavior\ at\ infinity}'$$

where J is the Joukovskii transformation from the circle to the aerofoil.

$$J: \qquad z = \frac{1}{2}(\varsigma + \frac{1}{\varsigma})$$

Remember that it has two preimages in the circle-plane. '+' means the preimage outside the circle, and '-' means the preimage inside the circle

$$J^{-1}: \qquad \varsigma = z \pm \sqrt{z^2 - a^2}$$

The Plemelj formula will be held on the contour $\mathbb{\Gamma}$, we have:

1) $C_{\gamma}g(z)$ analytic in C\ $\mathbb{\Gamma}$;
2) $C_{\gamma}^{+}g(x) - C_{\gamma}^{-}g(x) = g(x)$ for x on γ;
3) $C_{\gamma}g(\infty) = 0$;

Because of the above four properties, we can again consider w(z)= $V_{\infty}(e^{-i\alpha}z + C_{\mathbb{\Gamma}}g(t))$ as the complex potential around the aerofoil. Since one can check that w(z) satisfies the three criterion of problem 1.3.

Remember in the circle case, f(t) was: (R radius)

$$\text{f(t)}= R^2\, e^{i\alpha}\, \frac{-1}{t}$$

So, here it is reasonable to have g(t)= $J^{\circ}f^{\circ}J^{-1}(t)$. Can think as g maps t to $\varsigma - plane$ first, then have f on it, and then maps back to $Z - plane$. I.e.:

$$C_{\mathbb{T}}g(z) = C_\gamma[g^\circ J]\left(z+\sqrt{z^2-a^2}\right) + C_\gamma[g^\circ J]\left(z-\sqrt{z^2-a^2}\right) - C_\gamma[g^\circ J](0)$$

$$= C_\gamma[J^\circ f]\left(z+\sqrt{z^2-a^2}\right) + C_\gamma[J^\circ f]\left(z-\sqrt{z^2-a^2}\right) - C_\gamma[J^\circ f](0)$$

Now, we calculate $C_\gamma[J^\circ f](z)$:

$$C_\gamma[J^\circ f](z) = C_\gamma\left[-\frac{1}{2}(\frac{R^2e^{i\alpha}}{t}+\frac{t}{R^2e^{i\alpha}})\right] \overset{claim}{=} \begin{cases} \frac{-z}{2R^2e^{i\alpha}} & inside\ \gamma \\ \frac{R^2e^{i\alpha}}{2z} & outside\ \gamma \end{cases}$$

because

1. analytic everywhere off γ;

2.decays;

3.right jumps: (inner limit – outer limit)

$$\frac{-z}{2R^2e^{i\alpha}} - \frac{R^2e^{i\alpha}}{2z} = -\frac{1}{2}(\frac{R^2e^{i\alpha}}{z}+\frac{z}{R^2e^{i\alpha}})$$

And remember, here, γ is generally a circle with radius R, located at ς_0. So, we do one more transformation.

$$C_\gamma[J^\circ f](z) = \begin{cases} \frac{-(z-\varsigma_0)}{2R^2e^{i\alpha}} & inside\ \gamma \\ \frac{R^2e^{i\alpha}}{2(z-\varsigma_0)} & outside\ \gamma \end{cases}$$

Note: when $\varsigma_0 = 0$, R=1, γ matches with the circle at the origin.

So,

$$C_{\mathbb{\Gamma}}g(z) = C_{\gamma}[J^{\circ}f]\left(z+\sqrt{z^2-a^2}\right) + C_{\gamma}[J^{\circ}f]\left(z-\sqrt{z^2-a^2}\right) - 0$$

$$= \frac{-\left(z-\sqrt{z^2-a^2}-\varsigma_0\right)}{2R^2e^{i\alpha}} + \frac{R^2e^{i\alpha}}{2\left(z+\sqrt{z^2-a^2}-\varsigma_0\right)}$$

$$= \frac{(e^{i\alpha}-e^{-i\alpha})R^2}{2\left(z+\sqrt{z^2-a^2}-\varsigma_0\right)}$$

$$w(z) = V_{\infty}\left(e^{-i\alpha}(z-\varsigma_0) + C_{\mathbb{\Gamma}}g(z)\right)$$

$$= V_{\infty}\left(e^{-i\alpha}(z-\varsigma_0) + isin\alpha\frac{R^2}{z+\sqrt{z^2-a^2}-\varsigma_0}\right)$$

$$= V_{\infty}((z-\varsigma_0)cos\alpha - isin\alpha\sqrt{z^2-a^2})$$

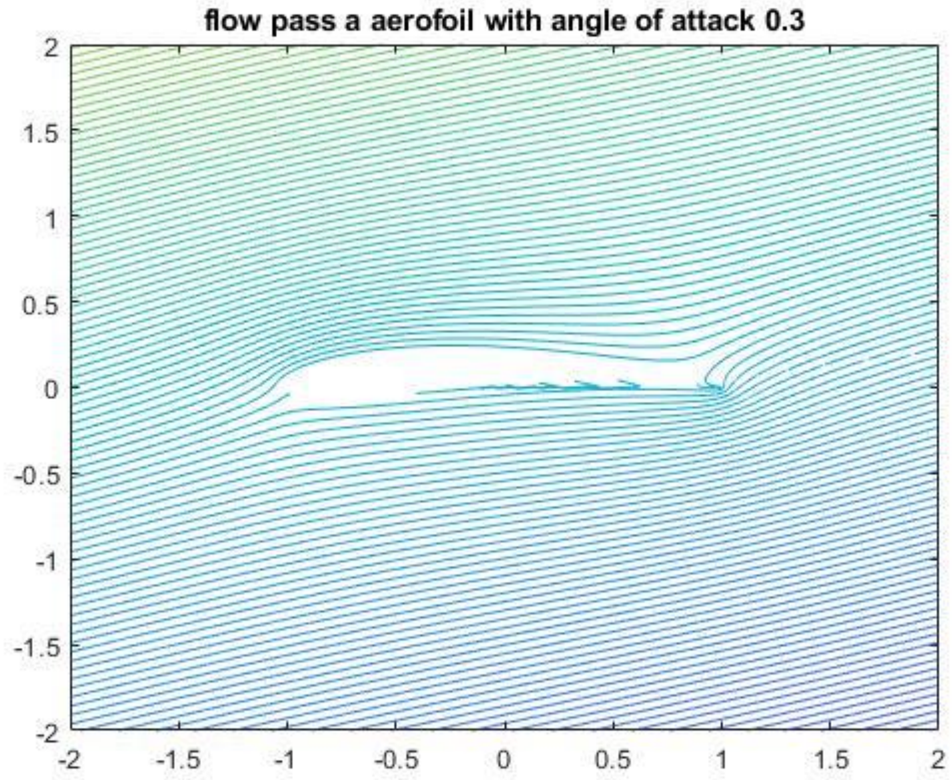
which is exactly the same as we got by conformal mapping.

Then we can get the streamline function: $\psi(x,y) = Im(w)$.

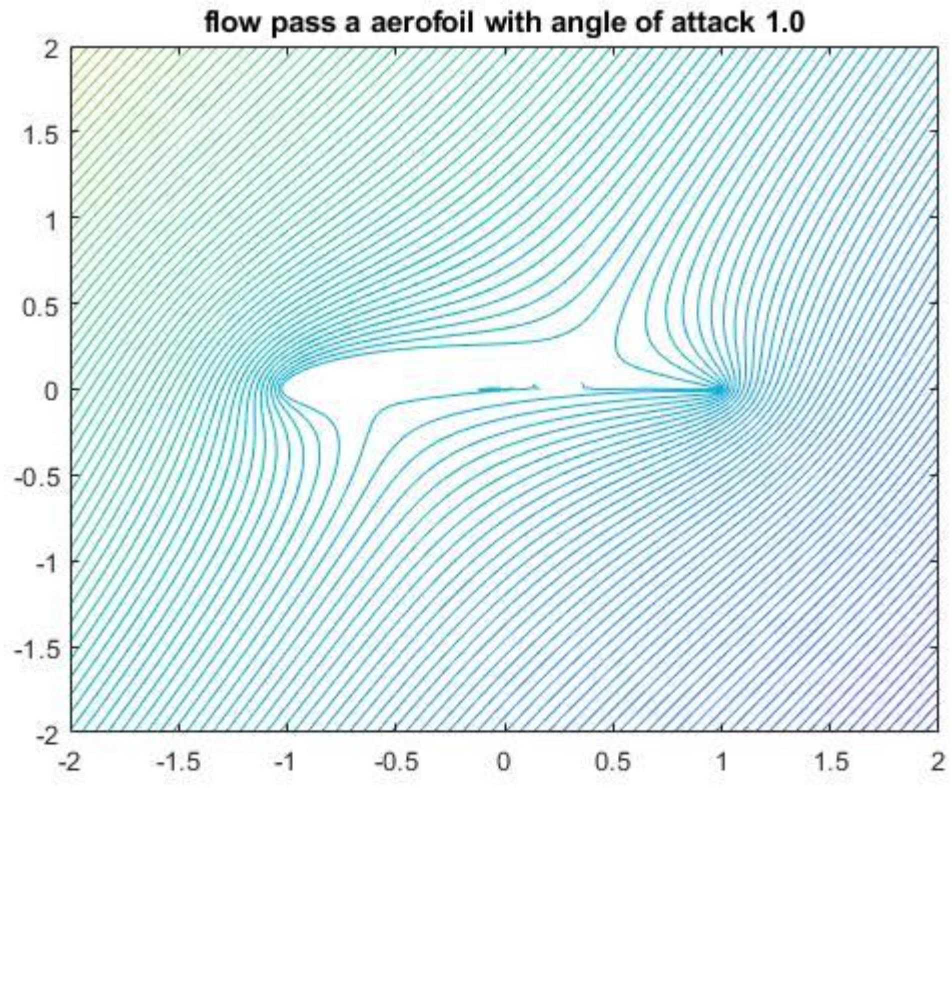
Plots obtained by Matlab:

```
clear
theta=1.0;
ra=sqrt(1.1^2+0.11^2);
Theta = [0:0.01:2*pi];
R = [ra+0.01:0.01:10.0];
[ThetaG,RG] = meshgrid(Theta,R);
zeta0=-0.1+i*0.11;
Z =
(RG.*(cos(ThetaG)+i*sin(ThetaG)+zeta0)+(1./(RG.*(cos(ThetaG)+i*sin(ThetaG)+
zeta0)))))./2;
X= real(Z);
Y= imag(Z);
Z=imag(exp(-i*theta) .* ((Z-zeta0)+sqrt(X+i*Y+1).*sqrt(X+i*Y-1)) + ra^2 *
exp(i*theta) .* (((Z-zeta0)+sqrt(X+i*Y+1).*sqrt(X+i*Y-1)).^(-1)));
contour(X,Y,Z,200)
ylim([-2,2]);
xlim([-2,2]);

title('flow pass a aerofoil with angle of attack 1.0')
```

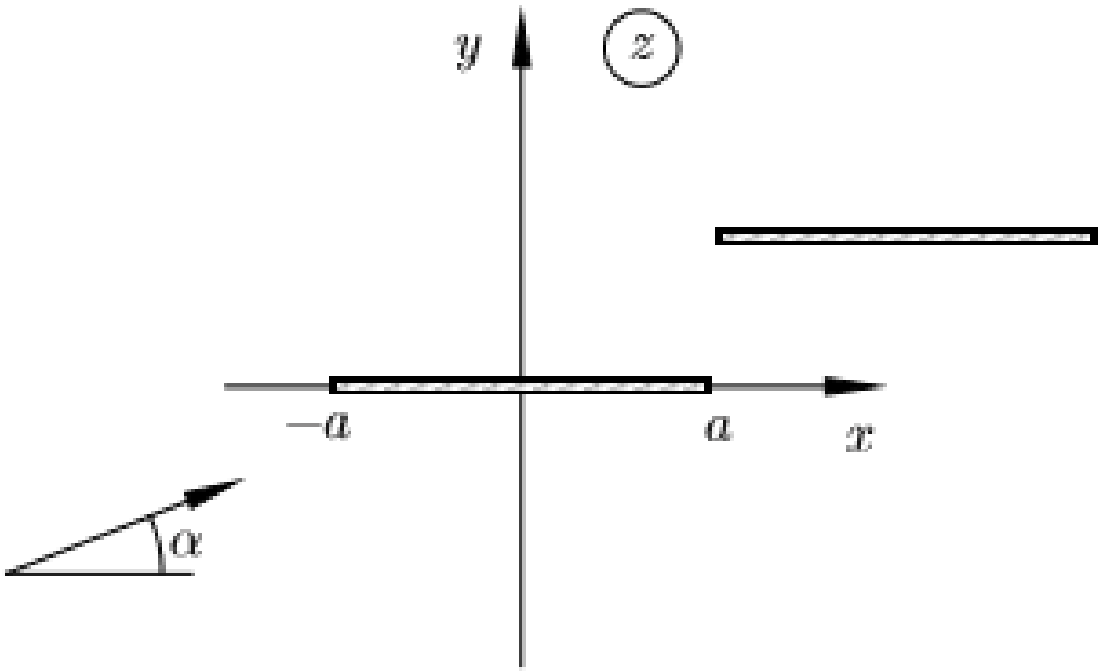
Here is the case a=1, d=h=0.1, α=0.3:



Here is the case a=1, d=h=0.1, α=1.0:



Case 2.1. Flow around two segments with an angle of attack.



In this case, we find the streamline function by finding the complex potential w(z), which satisfies the criterions:

1. w(z) is analytic off {x: -a < x < a} and {x+i: a < x < 2a};
2. Im{w(z)} = constants on {x: -a < x < a} and {x+i: a < x < 2a};
3. the free-stream condition

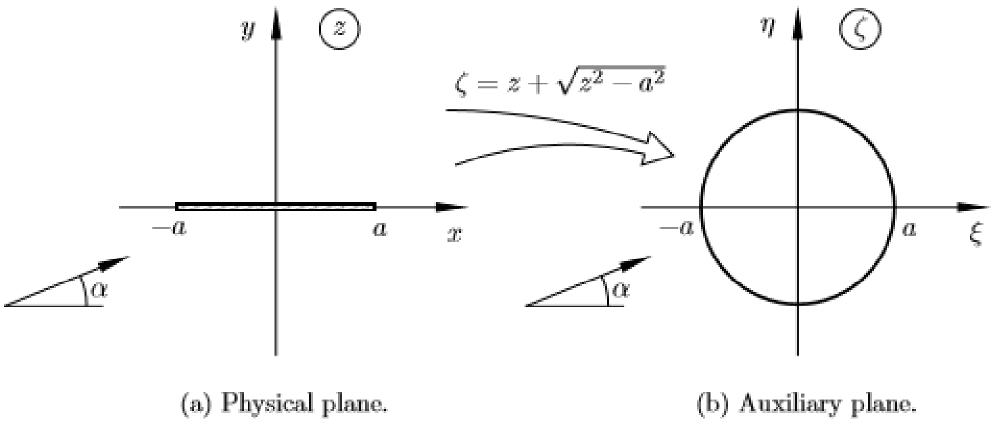
$$\frac{dw}{dz} = u_\infty - iv_\infty \qquad at\ z = \infty$$

where u_∞, v_∞ are the components of the far away velocity vector V_∞.

We call this boundary value problem 2.1.

1. Conformal mapping method:

Remember in case two, we used the Joukovskii transformation to map one segment onto a circle located at the origin.



Transformation of a flat plate onto a circle.

Generally, conformal mapping converts line to line or circle and distort the space elsewhere. So, I think it's impossible to map two separated straight lines onto two separated complete circles. Conformal mapping is limited here.

2. Cauchy/Hilbert transform method:

We consider the Cauchy transform over the branch cut γ = {x: -a < x < a} U {x+i: a < x < 3a};

$$C_\gamma f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(t)}{t-z} dt$$

By the Plemelj formula on the branch cut, we have:

1) $C_\gamma f(z)$ analytic in C\ γ;
2) $C_\gamma^+ f(x) - C_\gamma^- f(x) = f(x)$ for x on γ;

3) $C_\gamma f(\infty) = 0$;
4) $C_\gamma f(z)$ has weaker than pole singularity on γ.

We consider w(z)= $V_\infty(e^{-i\alpha}z + C_\gamma f(t))$ and solve the unknown function f s.t. w(z) satisfies the three criterions.

Criterion 1 is satisfied since $C_\gamma f(z)$ analytic in C\ γ;

Criterion 3 is satisfied since $V_\infty e^{-i\alpha}z$ satisfies trivially and $C_\gamma f(\infty) = 0$.

Criterion 2 is to say that Im{w(t)} = 0 on γ. Thus,

$$0=\text{Im}\left((e^{-i\alpha}t + C_\gamma^\pm f(t))\right)$$

$$= \text{Im}\left((cos\alpha - isin\alpha)t + C_\gamma^\pm f(t)\right)$$

$$=-tsin\alpha + Im(C_\gamma^\pm f(t))$$

In problem 1.2, we have $C_\gamma^+ f(t) = C_\gamma^- f(t)$, which means f(t) has to be real. And thus $Im(C_\gamma^\pm f(t))$=$Im(C_\gamma f(t))$=$\frac{H_\gamma f(t)}{2}$.

So,

$$H_\gamma f(t)\text{=}2tsin\alpha$$

Solving this similarly to the procedure in the lecture:

$$\text{f(z)=} \frac{-H\left[2x\sin\alpha\sqrt{3a+i-z}\sqrt{a+i-z}\sqrt{a-z}\sqrt{-a-z}\right](z)+Dz+C}{\sqrt{3a+i-z}\sqrt{a+i-z}\sqrt{a-z}\sqrt{-a-z}}$$

To calculate $H\left[2x\sin\alpha\sqrt{3a+i-z}\sqrt{a+i-z}\sqrt{a-z}\sqrt{-a-z}\right]$(z):

$$\text{C}[x\sqrt{3a+i-z}\sqrt{a+i-z}\sqrt{a-z}\sqrt{-a-z}]\text{(z)}$$

$$= \frac{Z\sqrt{z-3a-i}\sqrt{z-a-i}\sqrt{z-a}\sqrt{z+a}-z^3+\frac{5}{2}z}{2i}$$

$$\Rightarrow \text{H}[x\sqrt{3a+i-z}\sqrt{a+i-z}\sqrt{a-z}\sqrt{-a-z}]\text{(z)}$$

$$= -i\,(C^{+} + C^{-}) = z^3 - \frac{5}{2}z$$

$$\Rightarrow f(z) = 2\sin\alpha \frac{-z^3+Dz+C}{\sqrt{3a+i-z}\sqrt{a+i-z}\sqrt{a-z}\sqrt{-a-z}}$$

$$= 2\sin\alpha(\sqrt{3a+i-z}\sqrt{a+i-z} + \sqrt{a-z}\sqrt{-a-z})$$

where I chose suitable C and D to make it bounded at four end pts.

$$\Rightarrow C_\gamma f(z) = -i\sin\alpha(\sqrt{z-a}\sqrt{z+a} + \sqrt{z-3a-i}\sqrt{z-a-i} - 2z)$$

So,

$$w(z) = V_\infty(e^{-i\alpha}z + C_\gamma[f](z))$$

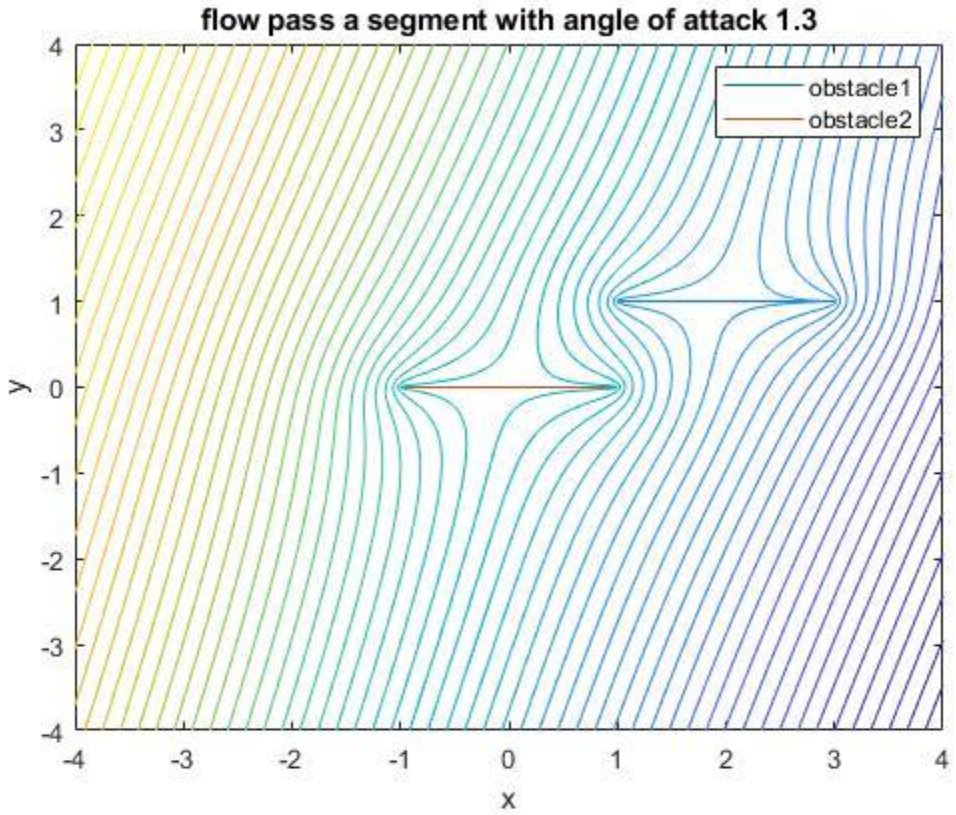
$$= V_\infty(e^{-i\alpha}z - i\sin\alpha(\sqrt{z-a}\sqrt{z+a} + \sqrt{z-3a-i}\sqrt{z-a-i} - 2z)).$$

Then we can get the streamline function: $\psi(x,y) = Im(w)$.

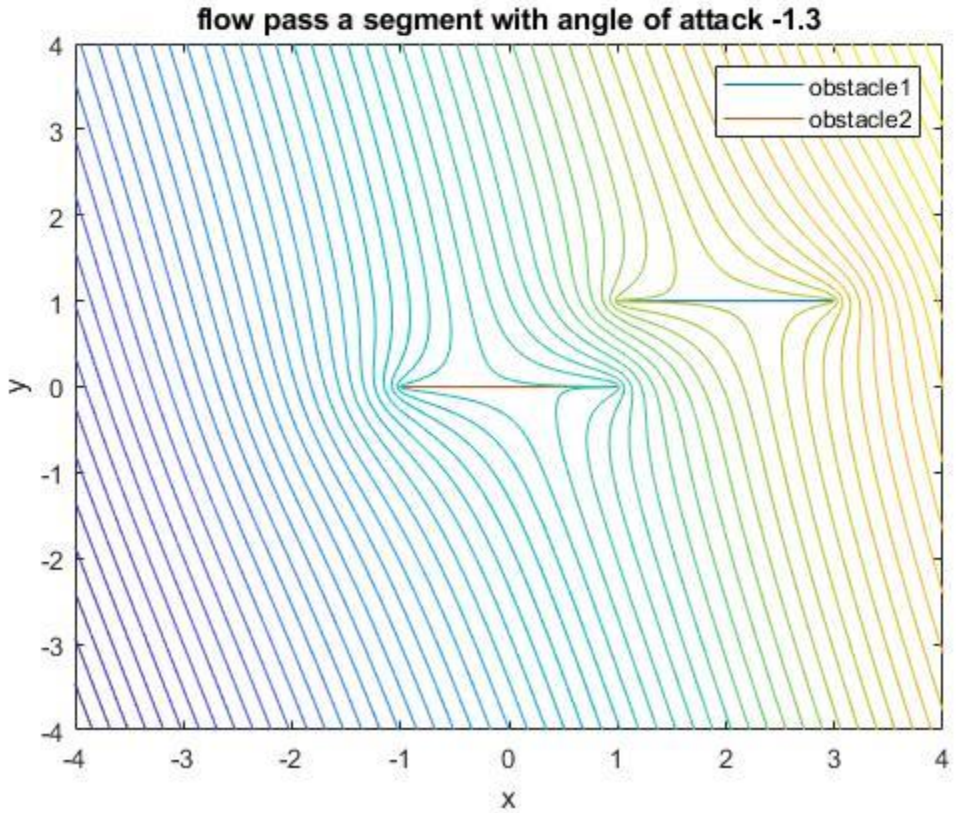
Plots obtained by Matlab:

```
clear
plot([1,3],[1,1])
hold on
plot([-1,1],[0,0])

theta=1.3;
X = [-4.0:0.01:4.0];
Y = [-4.005:0.01:4.005];
[X,Y] = meshgrid(X,Y);
Z=imag(exp(-i*theta)*(X+i*Y)-i*sin(theta)*(((X+i*Y-
1).^(1/2)).*((X+i*Y+1).^(1/2))+((X+i*Y-3-i).^(1/2)).*((X+i*Y-1-i).^(1/2))-
2*(X+i*Y)));

contour(X,Y,Z,100)
ylim([-4,4]);
xlim([-4,4]);
hold off

title('flow pass a segment with angle of attack 1.3')
xlabel('x')
ylabel('y')
legend('obstacle1','obstacle2')
```

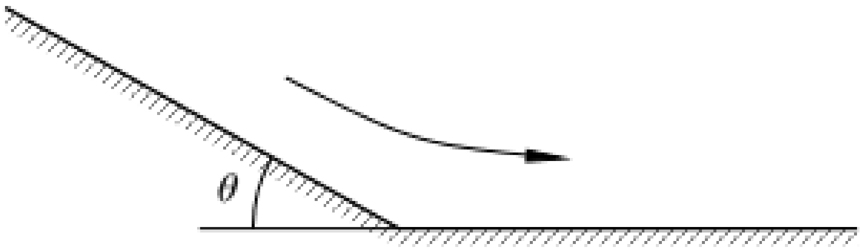
Here is the case a=1, α=1.3:



Here is the case a=1, α=-1.3:



Case 2.2. Flow around a corner with infinite bounds.



In this case, we find the streamline function by finding the complex potential w(z), which satisfies the criterions:

1. w(z) is analytic in {r$e^{i\vartheta}$: 0 < ϑ < $\pi - \theta, , r > 0$};
2. Im{w(z)} = 0 on {r$e^{i\vartheta}$: ϑ = $\pi - \theta$ or 0, $r > 0$}
3. the free-stream condition

$$\frac{dw}{dz} = u_\infty - iv_\infty \qquad at\ z = \infty$$

where u_∞, v_∞ are the components of the far away velocity vector V_∞.

We call this boundary value problem 2.2.

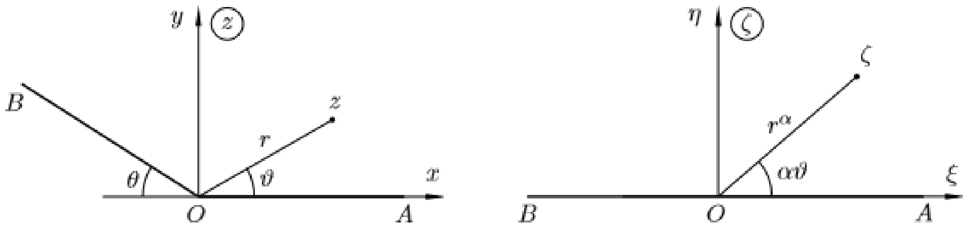
1. Conformal mapping method:

We derived in fluid dynamics that the potential of a uniform flow parallel to the x-axis is

$$w(\varsigma) = V_\infty \varsigma$$

To generalize the flow to the case around a corner, we use the map

$$\varsigma = z^{\alpha} = (Re^{i\vartheta})^{\alpha} = R^{\alpha}e^{i\alpha\vartheta}$$



(a) z-plane. (b) ζ-plane.

We can see that $\alpha\vartheta$=0 at $\vartheta = 0$, so we only have to choose α=$\frac{\pi}{\pi-\theta}$ to make $\alpha\vartheta = \pi$ at $\vartheta = \pi - \theta$.

We know that the complex potential in the ς-plane is:

$$w(\varsigma) = V_{\infty}\varsigma$$

Plug into the substitution, we get in the z-plane:

$$w(z) = V_{\infty}z^{\frac{\pi}{\pi-\theta}}$$

Then we can get the streamline function: $\psi(x, y) = Im(w)$.

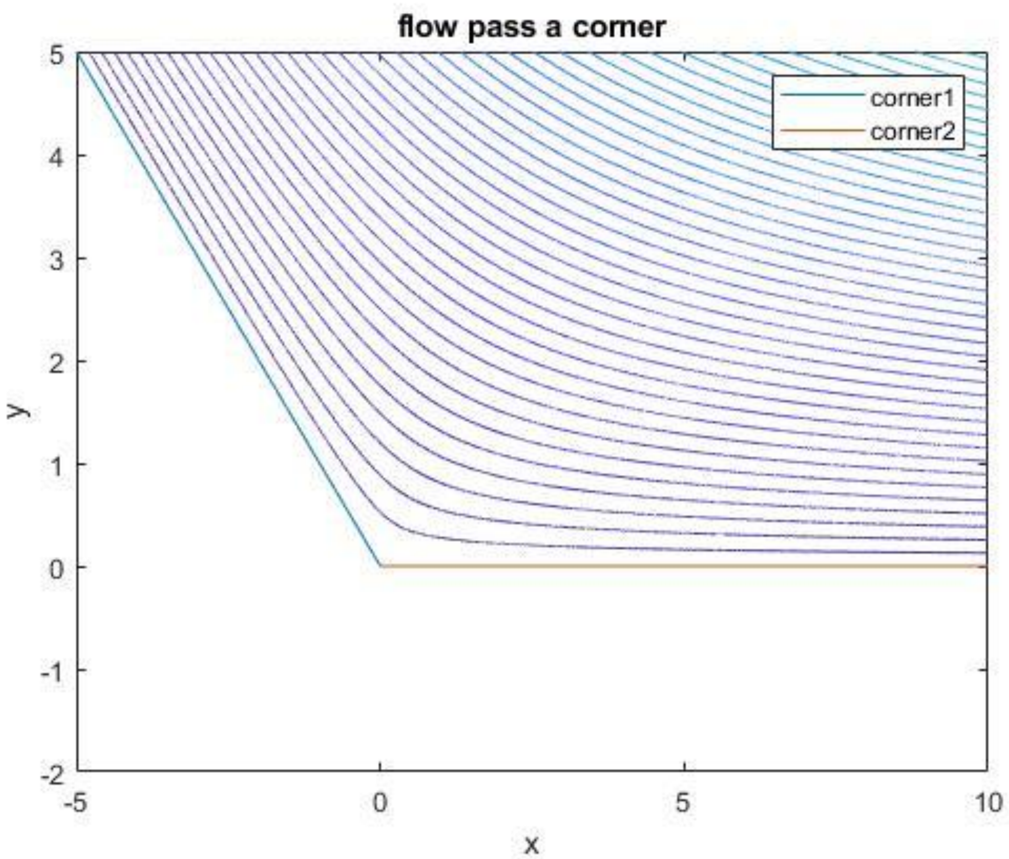
2. Cauchy/Hilbert transform method:

For α be a rational multiple of π, we can use the monomial map formulae in the mastery materials. Since we know how Cauchy transform looks like on R, we can work out the Cauchy transform on this inclined wall analytically. However, for general α, the method doesn't work. So, the Cauchy transform is limited in this case.

Plots obtained by Matlab:

```
clear
plot([-1,1],[0,0])
hold on

theta=1;
X = [-4.0:0.01:4.0];
Y = [-4.005:0.01:4.005];
[X,Y] = meshgrid(X,Y);
Z=cos(theta)* Y-sin(theta)*real(sqrt(X+i*Y-1).*sqrt(X+i*Y+1));
contour(X,Y,Z,200)
ylim([-4,4]);
xlim([-4,4]);
hold off

title('flow pass a segment with angle of attack 1')
xlabel('x')
ylabel('y')
legend('obstacle')
```

Here is the case θ=pi/4:



Conclusion:

In this project, we can see that:

1. Generally, both methods work very well in most cases.
2. For the conformal mapping (CM), finding the right conformal map would be the key point and usually requires some algebra. Once we have the map and the complex potential in ς-plane, we can simply substitute into the complex potential in ς-plane and we'll get the complex potential in Z-plane.
3. For the Cauchy Transform (CT), it usually requires computing invert transforms of CT/HT which would be the difficult step in general.
4. The shortcoming of CM is that:
   1) It is hard to map one obstacle to two unconnected obstacles in general. Like the case 2.1.
5. The shortcomings of CT are that:
   1) It is hard to deal with infinite boundary cases. Like the case 2.1.
   2) And usually, CT would be analytic on both sides of the obstacle (anywhere off the branch cut), however, sometimes analyticity is not posed or guaranteed inside the obstacle. Like the case 2.1.
   3) Sometimes it is hard to specify the branch cut, then we also need the conformal map and map back to a simpler branch cut. Like the case 1.3

Following the result, we can draw the conclusion that conformal mapping is a more straightforward and easy-to-do method if we have the map. Practically, people usually use computer to find the conformal map numerically. So, that won't be an issue. CM is favoured in practice. However, as we showed, in a few cases, CT would be an easy way.