Applied Complex Analysis (2021)

1 Solution Sheet 1

1.1 Problem 1.1

1.1.1 1.

Use fundamental theorem of algebra: a polynomial is a constant times a product of terms like $z - \lambda_k$, where λ_k are the roots. In this case, the roots are a times the quartic-root of -1, hence this gives us:

$$z^4 + a^4 = (z - a e^{i\pi/4})(z - a e^{3i\pi/4})(z - a e^{5i\pi/4})(z - a e^{7i\pi/4})$$

We are only interested in the root $a e^{i\pi/4}$, thus we simplify the expression

$$\begin{aligned}\frac{z^3 \sin z}{z^4 + a^4} &= \frac{z^3 \sin z}{(z - a e^{3i\pi/4})(z - a e^{5i\pi/4})(z - a e^{7i\pi/4})} \frac{1}{z - a e^{i\pi/4}} \\ &= \frac{a^3 e^{3i\pi/4} \sin(a e^{i\pi/4})}{a^3 (e^{i\pi/4} - e^{3i\pi/4})(e^{i\pi/4} - e^{5i\pi/4})(e^{i\pi/4} - e^{7i\pi/4})} \frac{1}{z - a e^{i\pi/4}} + O(1)\end{aligned}$$

Therefore,

$$\operatorname*{Res}_{z = a e^{i\pi/4}} \frac{z^3 \sin z}{z^4 + a^4} = \frac{e^{3i\pi/4} \sin(a e^{i\pi/4})}{e^{3i\pi/4}(1 - e^{\frac{i\pi}{2}})(1 - e^{i\pi})(1 - e^{3i\pi/2})} = \frac{\sin(a e^{i\pi/4})}{(1 - i)(2)(1 + i)} = \frac{\sin(a e^{i\pi/4})}{4}$$

Let's check our work: we compare the numerically calculated residue to the formula we have derived:

```julia
using ApproxFun, Plots, ComplexPhasePortrait, LinearAlgebra, DifferentialEquations
a = 2.0
γ = Circle(a*exp(im*π/4), 0.1)
f = Fun(z -> z^3*sin(z)/(z^4+a^4), γ)
sum(f)/(2π*im), sin(a*exp(im*π/4))/4
```

```
(0.5378838853348213 + 0.07544036746694016im, 0.5378838853348215 + 0.0754403
674669402im)
```

2. We have

$$(z^2 - 1)^2 = (z - 1)^2 (z + 1)^2$$

Thus this is a slightly more challenging since it has a double pole. But we can expand using Geometric series:

$$\frac{z+1}{(z^2-1)^2} = \frac{1}{(z-1)^2}\frac{1}{2-(1-z)} = \frac{1}{(z-1)^2}\frac{1}{2}(1+(1-z)/2+O(1-z)^2) = \frac{1}{2(z-1)^2} - \frac{1}{4(z-1)} + O(1)$$

Thus the residue is the negative-first Laurent coefficient, namely $-\frac{1}{4}$.

We again check our work:

```
γ = Circle(1, 0.1)
f = Fun(z -> (z+1)/(z^2-1)^2, γ)
sum(f)/(2π*im) # almost equals -1/4
```

```
-0.2500000000000023 - 1.170965239236949e-16im
```

3.

$$\frac{z^2 e^z}{z^3 - a^3} = \frac{z^2 e^z}{(z-a)(z^2+az+a^2)}$$

We thus need only evaluate the extra term at $z = a$:

$$\operatorname*{Res}_{z=a} \frac{z^2 e^z}{z^3 - a^3} = \frac{e^a}{3}$$

Let's check:

```
a = 2.0

γ = Circle(a, 0.1)
f = Fun(z -> z^2*exp(z)/(z^3-a^3), γ)
sum(f)/(2π*im), exp(a)/3
```

```
(2.4630186996435506 + 4.676730094089873e-16im, 2.46301869964355)
```

1.2 Problem 1.2

1.2.1 1.

Change of variables $z = e^{i\theta}$, $dz = ie^{i\theta}d\theta = izd\theta$, $\cos\theta = \frac{z+z^{-1}}{2}$ gives

$$\int_0^{2\pi} \frac{d\theta}{5 - 4\cos\theta} = -i\oint \frac{dz}{5z - 2z^2 - 2} = i\oint \frac{dz}{(z-2)(2z-1)} = -\pi \operatorname*{Res}_{z=1/2} \frac{1}{(z-2)(z-1/2)} = \frac{2}{3}\pi$$

```
θ = Fun(0 .. 2π)
sum(1/(5-4cos(θ))) , 2π/3
```

```
(2.0943951023931966, 2.0943951023931953)
```

1.2.2 2.

Use $\cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{z^2+z^{-2}}{2}$ to get

$$\int_0^{2\pi} \frac{\cos 2\theta \mathrm{d}\theta}{5+4\cos\theta} = -\frac{\mathrm{i}}{4}\oint \frac{(z^4+1)\mathrm{d}z}{z^2(z+1/2)(z+2)} = \frac{\pi}{2}\left(\operatorname*{Res}_{z=-1/2} + \operatorname*{Res}_{z=0}\right)\frac{z^4+1}{z^2(z+2)(z+1/2)} = \frac{\pi}{6}$$

```
θ = Fun(0 .. 2π)
sum(cos(2θ)/(5+4cos(θ))), π/6
```

```
(0.5235987755982991, 0.5235987755982988)
```

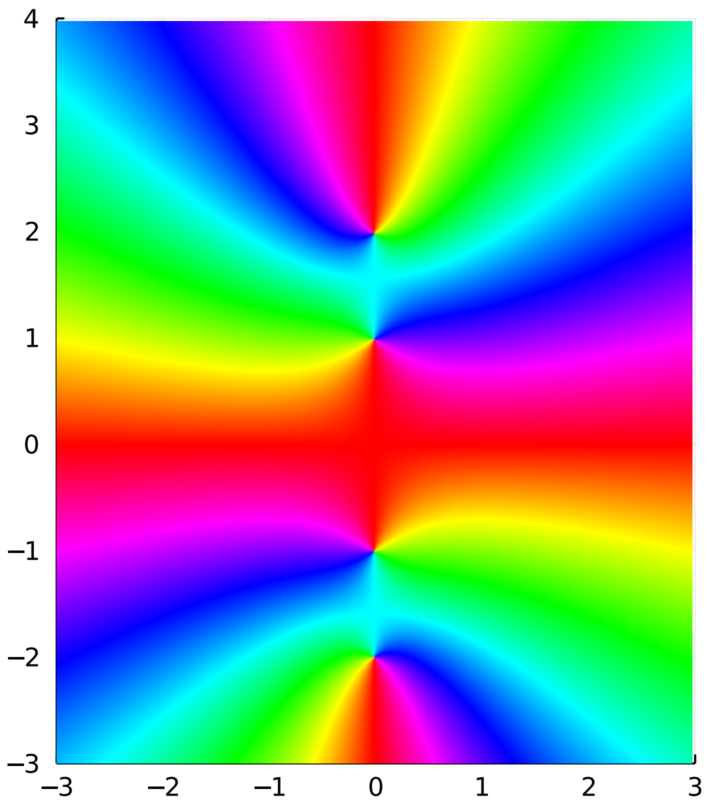
1.2.3 3.

Because the integrand is analytic and $O(z^{-2})$ in the upper half plane, we can use the residue theorem in the upper half plane using

$$\frac{1}{(z^2+1)(z^2+4)} = \frac{1}{(z+\mathrm{i})(z-\mathrm{i})(z+2\mathrm{i})(z-2\mathrm{i})}$$

This has two poles in the upper half plane:

```
phaseplot(-3..3, -3..4, z-> 1/((z^2+1)*(z^2+4)))
```



$$\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} \mathrm{d}x = 2\pi \mathrm{i} \left(\operatorname*{Res}_{z=\mathrm{i}} + \operatorname*{Res}_{z=2\mathrm{i}} \right) \frac{1}{(z^2+1)(z^2+4)}$$
$$= 2\pi \mathrm{i} \left(\frac{1}{2\mathrm{i}3\mathrm{i}(-\mathrm{i})} + \frac{1}{3\mathrm{i}\mathrm{i}4\mathrm{i}} \right) = \pi/6$$

We can check the result numerically:

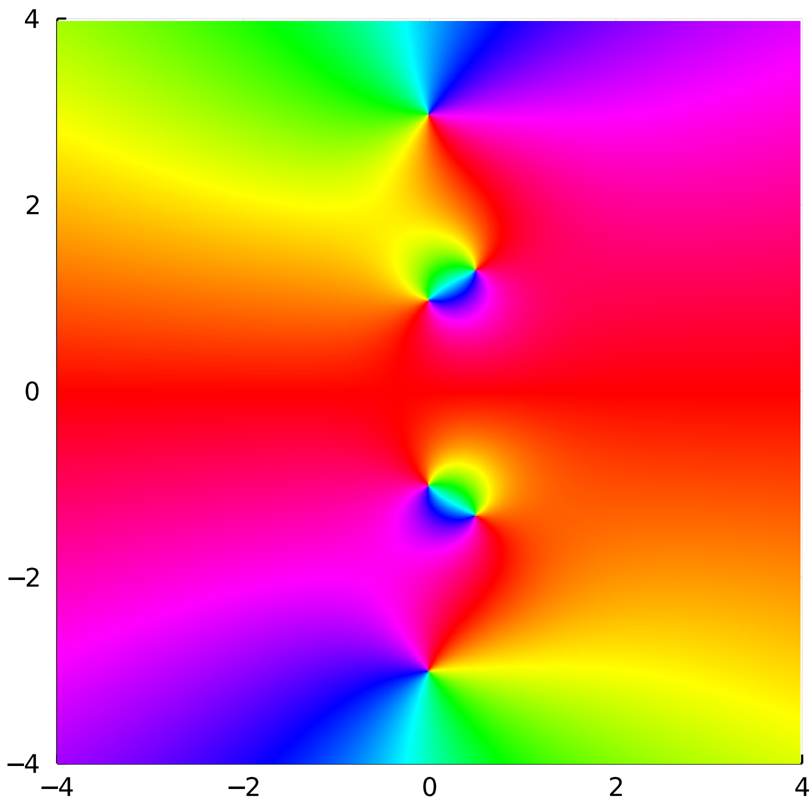
```
x = Fun( Line())
sum(1/((x^2+1)*(x^2+4))), π/6
```

```
(0.5235987755982988, 0.5235987755982988)
```

1.2.4 4.

Again, decays like $O(z^{-2})$ in upper half plane so we can use residue calculus. This integrand has poles at $z = \mathrm{i}$ and $z = 3\mathrm{i}$:

```
phaseplot(-4..4, -4..4, z-> (z^2 - z + 2) / (z^4 + 10z^2 +9))
```



The residues are $(-1-\mathrm{i})/16$ and $(3-7\mathrm{i})/48$ giving the answer

$$\frac{5\pi}{12}$$

which we check numerically:

```
f = x -> (x^2 - x + 2) / (x^4 + 10x^2 +9)
sum(Fun(f,-10_000..10_000)), 5π/12
```

```
(1.3087969390010812, 1.3089969389957472)
```

1.2.5 5.

$$\int_{-\infty}^{\infty} \frac{1}{x+\mathrm{i}} \mathrm{d}x$$

Trick question: it's undefined because the integral doesn't decay fast enough. But what if I had asked for

$$⨍_{-\infty}^{\infty} \frac{1}{x+\mathrm{i}} \mathrm{d}x?$$

We can't use residue theorem since it doesn't decay fast enough, but we can use, with a contour $C_R = \{R\mathrm{e}^{\mathrm{i}\theta} : 0 \leq \theta \leq \pi\}$

$$\oint_{[-M,M]\cup C_R} \frac{1}{z+\mathrm{i}} = 0$$

Further, by direct substitution, we have

$$\int_{C_R} \frac{1}{z+\mathrm{i}} \mathrm{d}z = \mathrm{i} \int_0^{\pi} R \frac{\mathrm{e}^{\mathrm{i}\theta}}{R\mathrm{e}^{\mathrm{i}\theta}+\mathrm{i}} \mathrm{d}\theta$$

Letting $R \to \infty$, the integrand tends to one uniformly hence

$$\int_{C_R} \frac{1}{z+\mathrm{i}} \mathrm{d}z \to \mathrm{i} \int_0^{\pi} \mathrm{d}\theta = \mathrm{i}\pi.$$

Therefore, we have

$$⨍_{-\infty}^{\infty} \frac{1}{x+\mathrm{i}} \mathrm{d}x = \lim_{M\to\infty} \int_{-M}^{M} \frac{1}{x+\mathrm{i}} \mathrm{d}x = -\mathrm{i}\pi.$$

`Indeed:`

```
x = Fun(-1000 .. 1000)
sum(1/(x+im))
```

```
1.1102230246251565e-15 - 3.1395926542565897im
```

1.2.6 6.

$$\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1} \mathrm{d}x = \Im \int_{-\infty}^{\infty} \frac{\mathrm{e}^{2\mathrm{i}x}}{x^2+x+1} \mathrm{d}x$$

This can be deformed in the upper half plane with a pole at $\frac{-1+\mathrm{i}\sqrt{3}}{2}$, using residue calculus gives us

$$-\frac{2\pi}{\sqrt{3}} \frac{\sin 1}{\mathrm{e}^{\sqrt{3}}}$$

```
x = Fun(-100 .. 100)
sum(sin(2x)/(x^2+x+1)),-2π/sqrt(3) * sin(1)/exp(sqrt(3))
```

```
(-0.5400548830723215, -0.5400553569742235)
```

1.2.7 7.

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4} \mathrm{d}x = \Re \int_{-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}x}}{x^2+4} \mathrm{d}x$$

and residue calculus gives $\frac{\pi}{2\mathrm{e}^2}$

```
M = 200
x = Fun(-M .. M)
sum(cos(x)/(x^2+4)),π/(2*exp(2)) # converges if we make M even bigger
```

```
(0.21254026836701112, 0.21258416579381814)
```

1.2.8 8.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1} \mathrm{d}x = \frac{\pi}{\mathrm{e}}$$

using Residue calculus. You need to appeal to Jordan's lemma to argue that it can still be done even with only $O(x^{-1})$ decay.

```
M = 10_000
x = Fun(-M .. M)
sum(x*sin(x)/(x^2+1)),π/exp(1) # Converges if we make M even bigger
```

```
(1.1559177936504117, 1.1557273497909217)
```

1.2.9 9.

$$\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} \mathrm{d}x \qquad \text{where} \qquad a, b > 0$$

We have for $f(x) = \frac{\mathrm{e}^{\mathrm{i}ax} - \mathrm{e}^{\mathrm{i}bx}}{x^2}$

$$\Re f(x) = \frac{\cos ax - \cos bx}{x^2}$$

Note that, since $\cos x = 1 + x^2/2 + O(x^4)$, the integrand is fine near zero:

$$\frac{\cos ax - \cos bx}{x^2} = \frac{(a-b)}{2} + O(x^2)$$

But $f(x)$ has a pole:

$$\frac{\mathrm{e}^{\mathrm{i}ax} - \mathrm{e}^{\mathrm{i}bx}}{x^2} = \frac{\mathrm{i}(a-b)}{x} + O(1)$$

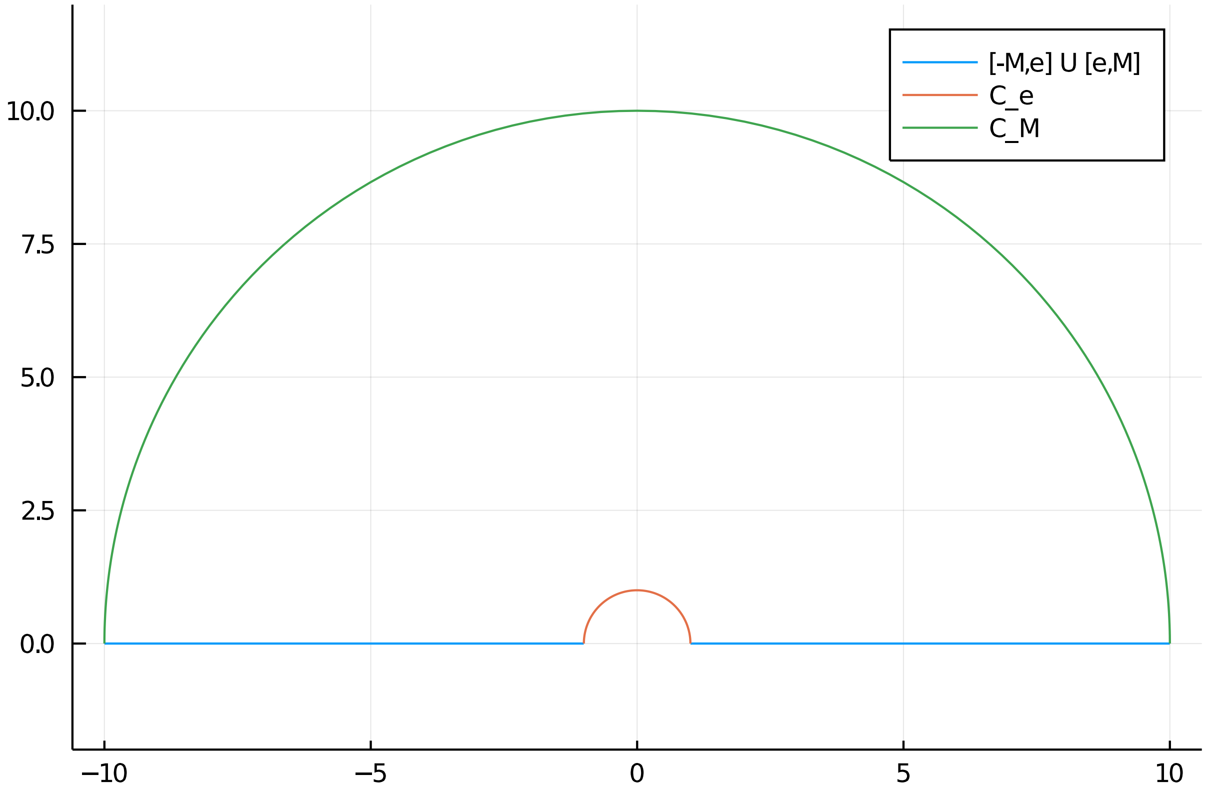
To rectify this, we need to be a bit more careful. First note that

$$\int_{\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} \mathrm{d}x = \lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} + \int_{\epsilon}^{\infty} \right) \frac{\cos ax - \cos bx}{x^2} \mathrm{d}x = \Re \fint_{-\infty}^{\infty} f(x) \mathrm{d}x$$

Then we construct a contour avoiding zero as follows:

```julia
M = 10
ε = 1.0

plot(Segment(-M, -ε) ∪ Segment(ε, M);label="[-M,e] U [e,M]", ratio = 1.0)
plot!(Arc(0.,ε, (π,0.)); label="C_e")
plot!(Arc(0., M, (0,π)); label = "C_M")
```



Note that $\oint_\gamma f(z)\mathrm{d}z = 0$,

$$\int_{C_\epsilon} \frac{\mathrm{e}^{iaz} - \mathrm{e}^{ibz}}{z^2}\mathrm{d}z = \int_\pi^0 \frac{(b-a)\mathrm{e}^{\mathrm{i}\theta} + O(\epsilon)}{\mathrm{e}^{\mathrm{i}\theta}}\mathrm{d}\theta \to (a-b)\pi$$

Also, as the integrand is $O(z^{-2})$ the integral over C_M vanishes as $M \to \infty$. We therefore get

$$⨍ f(x)\mathrm{d}x = (b-a)\pi$$

```julia
ε =0.001
M = 1_000.0
x = Fun(Segment(-M , -ε) ∪ Segment(ε, M))
a = 2.3; b = 3.8
sum((cos(a*x) - cos(b*x))/x^2),π*(b-a) # Converges if we make M bigger
```

```
(4.703250780477666, 4.71238898038469)
```

1.2.10 10.

Use binomial formula

$$\begin{aligned}\int_0^{2\pi}(\cos\theta)^n \mathrm{d}\theta &= \frac{1}{2^n \mathrm{i}}\oint (z+z^{-1})^n \frac{\mathrm{d}z}{z} \\ &= \frac{1}{2^n \mathrm{i}}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\oint z^k z^{k-n}\frac{\mathrm{d}z}{z} \\ &= \frac{1}{2^n \mathrm{i}}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\oint z^{2k-n-1}\mathrm{d}z\end{aligned}$$

We only have a residue of $2k-n-1=-1$, that is, if $2k=n$. If n is odd, this can't happen (duh! the integral is symmetric with respect to θ). If it's even, then we have

$$\int_0^{2\pi}(\cos\theta)^n \mathrm{d}\theta = \frac{\pi}{2^{n-1}}\frac{n!}{2(n/2)!}$$

```
θ = Fun(0 .. 2π)
n = 4;
sum(cos(θ)^n), π*factorial(1.0n)/(2^(n-1)*2*factorial(n/2))
```

```
(2.356194490192349, 2.356194490192345)
```

1.3 Problem 2.1

By integrating around a rectangular contour with vertices at $\pm R$ and $\pi\mathrm{i}\pm R$ and letting $R\to\infty$, show that:

$$\int_0^\infty \operatorname{sech} x \mathrm{d}x = \frac{\pi}{2}$$

where $\operatorname{sech} x = \frac{2}{\mathrm{e}^{-x}+\mathrm{e}^{x}}$.

Recall $\operatorname{sech} x = \frac{2}{\mathrm{e}^{-x}+\mathrm{e}^{x}}$. This shows that $\operatorname{sech}(-x) = \operatorname{sech} x$ But we also have

$$\operatorname{sech}(x+\mathrm{i}\pi) = \frac{2}{\mathrm{e}^{-x-\mathrm{i}\pi}+\mathrm{e}^{x+\mathrm{i}\pi}} = \frac{2}{-\mathrm{e}^{-x}-\mathrm{e}^{x}} = -\operatorname{sech} x$$

Thus we have

$$4\int_0^\infty \operatorname{sech} x \mathrm{d}x = \left[\int_{-\infty}^{\infty} + \int_{\infty+\mathrm{i}\pi}^{-\infty+\mathrm{i}\pi}\right]\operatorname{sech} z \mathrm{d}z$$

We can approximate this using

$$\left[\int_{-R}^{R} + \int_{R}^{R+\mathrm{i}\pi} + \int_{R+\mathrm{i}\pi}^{-R+\mathrm{i}\pi} + \int_{-R+\mathrm{i}\pi}^{-R}\right]\operatorname{sech} z \mathrm{d}z = 2\pi\mathrm{i}\operatorname*{Res}_{z=\frac{\mathrm{i}\pi}{2}}\operatorname{sech} z = 2\pi$$

since, for $z_0 = \frac{\mathrm{i}\pi}{2}$, we have

$$\operatorname{sech} z = \frac{1}{\cos \mathrm{i}z} = \frac{1}{-\mathrm{i}\sin \mathrm{i}z_0(z-z_0)+O(z-z_0)^2} = -\frac{\mathrm{i}}{(z-z_0)} + O(1)$$

Finally, we need to show that the limit as $R\to\infty$ tends to the right value. In this case, it follows since

$$\left|\int_R^{R+\mathrm{i}\pi} \operatorname{sech} z \mathrm{d}z\right| \leq \frac{2\pi \mathrm{e}^{-R}}{1-\mathrm{e}^{-2R}} \to 0$$

(and by symmetry for $\int_{-R+\mathrm{i}\pi}^{-R}\cdot$.)

1.4 Problem 2.2

Show that the Fourier transform of $\operatorname{sech} x$ satisfies

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \operatorname{sech} x \mathrm{d}x = \pi \operatorname{sech} \frac{\pi k}{2}$$

Define

$$f(z) = \mathrm{e}^{\mathrm{i}kz} \operatorname{sech} z = \frac{2\mathrm{e}^{(1+\mathrm{i}k)z}}{\mathrm{e}^{2z}+1}$$

In this case, we have the symmetry

$$f(x+\mathrm{i}\pi) = -\mathrm{e}^{-k\pi}\mathrm{e}^{\mathrm{i}kx} \operatorname{sech} x = -\mathrm{e}^{-k\pi} f(x)$$

```
k = 2.0
f = z -> exp(im*k*z)*sech(z)
-exp(-k*π)f(2.0),f(2.0+im*π)
```

```
(0.00032444937189257726 + 0.00037565438782217884im, 0.0003244493718925772 +
0.00037565438782217884im)
```

In other words, we have

$$(1+\mathrm{e}^{-k\pi}) \int_{-\infty}^{\infty} f(x)\mathrm{d}x = \left(\int_{-\infty}^{\infty} + \int_{\infty+\mathrm{i}\pi}^{-\infty+\mathrm{i}\pi}\right) f(z)\mathrm{d}z$$

By similar logic as above, we can show that the integral over the rectangular contour converges to this.

Again, the only pole inside is at $z = \frac{\mathrm{i}\pi}{2}$, where the residue is $-\mathrm{i}\mathrm{e}^{\frac{-\pi k}{2}}$. Thus we have

$$\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \frac{2\pi \mathrm{e}^{\frac{-\pi k}{2}}}{1+\mathrm{e}^{-k\pi}} = \pi \operatorname{sech} \frac{\pi k}{2}$$