Applied Complex Analysis (2021)

1 Solution Sheet 5

1.1 Problem 1

1. Define $C_k^{(\alpha)}(z) = \mathcal{C}[L_k^{(\alpha)} \diamond^\alpha \mathrm{e}^{-\diamond}](z)$ and recall that

$$C_1(z) = \frac{\frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{e}^{-x}\mathrm{d}x + (z-a_0)C_0(z)}{b_0} = -\frac{1}{2\pi\mathrm{i}} - (z-1)C_0(z)$$
$$= \frac{(z-1)\mathrm{e}^{-z}\mathrm{Ei}\,\mathrm{z} - 1}{2\pi\mathrm{i}}$$

We abbreviate $C_k^{(0)}(z)$ as $C_k(z)$ and we will also abbreviate $L_k^{(0)}$ as L_k.

Here we double check the formula, noting that $L_1(x) = \mathrm{e}^x \frac{\mathrm{d}}{\mathrm{d}x} x\mathrm{e}^{-x} = 1 - x$:

```julia
using ApproxFun, SingularIntegralEquations, Plots, QuadGK,
LinearAlgebra, SpecialFunctions

const ei_−_1 = let ζ = Fun(-100 .. -1)
    sum(exp(ζ)/ζ)
end
function ei(z)
    ζ = Fun(Segment(-1 , z))
```

```
    ei_−_1 + sum(exp(ζ)/ζ)
end

x = Fun(0..10)
w = exp(-x)
z = 1+im
cauchy((1-x)*w, z),((z-1)*exp(-z)*ei(z)-1)/(2π*im)
```

```
(0.018684644298457894 + 0.048361335653350004im, 0.018683923002628 92 +
 0.048
36848799089318im)
```

We now use these to determine the results with $\alpha = 1$. Note that:

$$C_0^{(1)}(z) = \mathcal{C}[\diamond e^{-\diamond}](z) = C_0(z) - C_1(z) = \frac{-e^{-z}\mathrm{Ei}\, z - (z-1)e^{-z}\mathrm{Ei}\, z + 1}{2\pi i}$$

```
cauchy(x*w, z),(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1)/(2π*im)
```

```
(0.0921017375168 4986 - 0.029676691354892096im, 0.0921025320983 7325 -
0.0296
8456498826427im)
```

Therefore, we have

$$
\begin{aligned}
C_1^{(1)}(z) &= \frac{\frac{1}{2\pi \mathrm{i}}\int_0^\infty x\mathrm{e}^{-x}\mathrm{d}x + (z - a_0^{(1)})C_0^{(1)}(z)}{b_0^{(1)}} \\
&= \frac{\frac{1}{2\pi \mathrm{i}} + (z-2)C_0^{(1)}(z)}{-1} \\
&= \frac{1 + (z-2)(-\mathrm{e}^{-z}\mathrm{Ei}\,\mathrm{z} - (\mathrm{z}-1)\mathrm{e}^{-\mathrm{z}}\mathrm{Ei}\,\mathrm{z} + 1)}{-2\pi \mathrm{i}}
\end{aligned}
$$

Let's check the result using

$$
L_1^{(1)}(x) = x^{-1}\mathrm{e}^x \frac{\mathrm{d}}{\mathrm{d}x} x^2 \mathrm{e}^{-x} = 2 - x
$$

```
cauchy((2-x)*x*w,
z),(1+(z-2)*(-exp(-z)*ei(z)-(z-1)*exp(-z)*ei(z)+1))/(-2π*im)
```

```
(0.0624250461619579 + 0.03729703236453857 4im, 0.06241796711010899 +
0.03736
784600525782im)
```

2.

We have

$$
\int_x^\infty L_2(x)\mathrm{e}^{-x}\mathrm{d}x = \frac{1}{2} x\mathrm{e}^{-x} L_1^{(1)}(x)
$$

Thus from lectures we have

$$\frac{1}{2\pi \mathrm{i}}\int_0^\infty L_2(x)\mathrm{e}^{-x}\log(z-x)\mathrm{d}x = \mathrm{i}\mathcal{C}[\diamond \mathrm{e}^{-\diamond}L_1^{(1)}](z)$$

and therefore

$$\frac{1}{\pi}\int_0^\infty L_2(x)\mathrm{e}^{-x}\log|z-x|\mathrm{d}x = -\Im\mathcal{C}[\diamond \mathrm{e}^{-\diamond}L_1^{(1)}](z) = \Re\frac{1+(z-2)(\mathrm{e}^{-z}\mathrm{Ei}\,z-(\mathrm{z}-1)\mathrm{e}^{-\mathrm{z}}\mathrm{Ei}\,\mathrm{z}+}{-2\pi}$$

Let's check the result:

```
x = Fun(0 .. 100)
w = exp(-x)
z = 2+im

-sum(1/2*(2 - 4x + x^2)*w*log(abs(z-x)))/π,imag(sum(1/2*(2 - 4x +
x^2)*w*log(z-x))/(π*im))
```

```
(-0.0697232345397132, -0.0697232345397136)
```

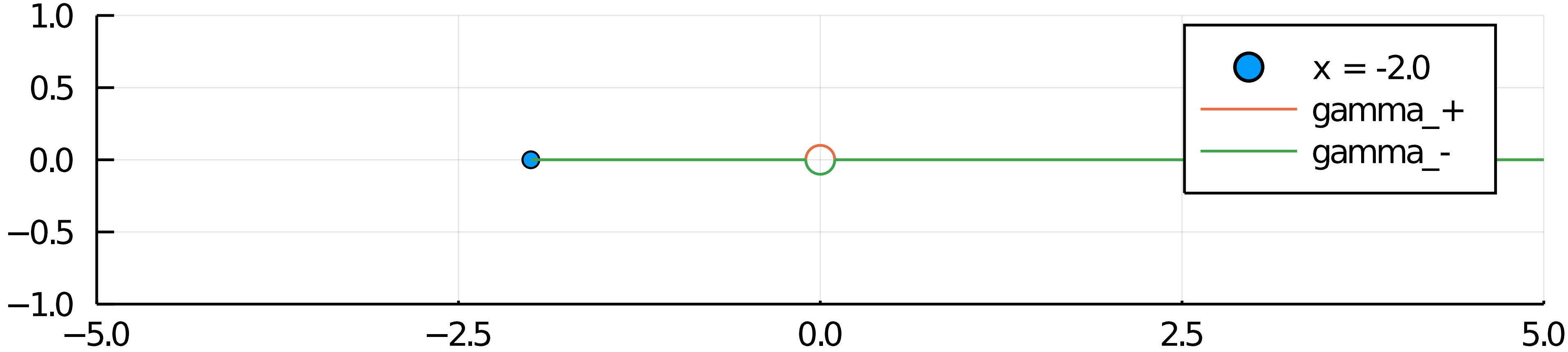
```
-imag(cauchy((2-x)*x*w,z)),real((1+(z-2)*(-exp(-z)*ei(z)-(z-1)*exp(-z)*e
```

```
(-0.06972323454064205, -0.06972323454061675)
```

1.1.1 Problem 2

Consider integration contours γ_{+x} and γ_{-x} that avoid 0 above and below:

```
x = -2.0
r = 0.1
γ_+_x = Segment(-2.0 , -r) ∪ Arc(0.,r, (π,0)) ∪ Segment(r , 100)
γ_-_x = Segment(-2.0 , -r) ∪ Arc(0.,r, (-π,0)) ∪ Segment(r , 100)
scatter([x],[0.0];label="x = $x",ratio = 1.0)
plot!(γ_+_x ; xlims=(-5,5), ylims=(-1,1), label="gamma_+")
plot!(γ_-_x; xlims=(-5,5), ylims=(-1,1), label="gamma_-")
```



So that

$$\Gamma_\pm(\alpha,x)=\int_{\gamma_{\pm x}}\zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta$$

Note that

$$\int_x^{-r}(\zeta_+^{\alpha-1}-\mathrm{e}^{2\mathrm{i}\pi\alpha}\zeta_-^{\alpha-1})\mathrm{e}^{-\zeta}\mathrm{d}\zeta=0$$

since $\zeta_+^{\alpha-1}=\mathrm{e}^{\pi\mathrm{i}(\alpha-1)}|\zeta|^{\alpha-1}=\mathrm{e}^{2\mathrm{i}\pi\alpha}\zeta_-^{\alpha-1}$. Furthermore, the integrals over the arcs tend to zero as $r\to 0$:

$$|\mathrm{i}r^\alpha\int_0^\pi \mathrm{e}^{-r\mathrm{e}^{\mathrm{i}\theta}}\mathrm{e}^{\mathrm{i}\theta\alpha}\mathrm{d}\theta|\le r^\alpha\pi\mathrm{e}^r\to 0$$

and similarly on the lower arc. Thus we have

$$\begin{aligned}\Gamma_+(\alpha,x)-\mathrm{e}^{2\mathrm{i}\pi\alpha}\Gamma_-(\alpha,x)&=\lim_{r\to0}\left(\int_{\gamma_{+x}}-\mathrm{e}^{2\mathrm{i}\pi\alpha}\int_{\gamma_{-x}}\right)\zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta\\&=(1-\mathrm{e}^{2\mathrm{i}\pi\alpha})\int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x=(1-\mathrm{e}^{2\mathrm{i}\pi\alpha})\Gamma(\alpha)\end{aligned}$$

Note that, for $0<\alpha<1$,

$$\psi(z)=z^{-\alpha}\mathrm{e}^z\Gamma(\alpha,z)$$

has the following properties:

1.

$$\psi(z)$$

decays as $z \to \infty$, via integration by parts:

$$z^{-\alpha}\mathrm{e}^{z}\int_z^\infty \zeta^{\alpha-1}\mathrm{e}^{-\zeta}\mathrm{d}\zeta = z^{-1} + z^{-\alpha}\int_z^\infty \zeta^{\alpha-2}\mathrm{e}^{z-\zeta}\mathrm{d}\zeta$$

and we have assuming z is bounded away from the negative real axis:

$$\left|\int_z^\infty \zeta^{\alpha-2}\mathrm{e}^{z-\zeta}\mathrm{d}\zeta\right| \leq \int_z^\infty |\zeta|^{\alpha-2}\mathrm{d}\zeta = \int_0^\infty |x+z|^{\alpha-2}\mathrm{d}x < \infty$$

(otherwise one would use a deformed contour).

2. We have the subtractive jump:

$$\begin{aligned}
\psi_+(x) - \psi_-(x) &= \mathrm{e}^{x}(x_+^{-\alpha}\Gamma_+(\alpha,x) - x_-^{-\alpha}\Gamma_-(\alpha,x)) \\
&= \mathrm{e}^{x}|x|^{\alpha}(\mathrm{e}^{-\mathrm{i}\pi\alpha}\Gamma_+(\alpha,z) - \mathrm{e}^{\mathrm{i}\pi\alpha}\Gamma_-(\alpha,x)) \\
&= \mathrm{e}^{x}|x|^{\alpha}\mathrm{e}^{-\mathrm{i}\pi\alpha}(1-\mathrm{e}^{2\mathrm{i}\pi\alpha})
\end{aligned}$$

We use these properties to verify that

$$\mathcal{C}[\diamond^{\alpha}\mathrm{e}^{-\diamond}](z) = \frac{1}{\Gamma(-\alpha)}\frac{(-z)^{\alpha}\mathrm{e}^{-z}\Gamma(-\alpha,-z)}{\mathrm{e}^{-\mathrm{i}\pi\alpha} - \mathrm{e}^{\mathrm{i}\pi\alpha}}$$

via Plemelj.

```julia
x = Fun(0 .. 20.0)
α = -0.1
z = 2.0+im
cauchy(x^α*exp(-x), z)

Γ = (α,z) -> let ζ = z + Fun(0 .. 500.0)
    linesum(ζ^(α-1)*exp(-ζ))
end

-(-z)^α*exp(-z)Γ(-α,-z)/(gamma(-α)*(exp(im*π*α)-exp(-im*π*α)))
```

```
0.07199876331505128 + 0.05850612396048847im
```

1.2 Problem 3

1.2.1 Problem 3.1

We know that $L[a(z)]^{-1} = L[a(z)^{-1}]$ hence it's really about the Laurent series of $a(z)^{-1}$. We see that the roots of $a(z)$ satisfy

$$0 = z^2 a(z) = z^4 - 4z^2 + 1.$$

Using the quadratic formula with $w = z^2$ we have

$$w = 2 \pm \sqrt{3} \Rightarrow z = \pm\sqrt{2 \pm \sqrt{3}}.$$

Since $2 - \sqrt{3} < 1$ and $2 + \sqrt{3} > 1$ we have the factorisation

$$a(z) = \underbrace{z^2 - z_+}_{\phi_+(z)} \underbrace{1 - z_-/z^2}_{\phi_-(z)}$$

for $z_\pm = 2 \pm \sqrt{3}$. We can take the reciprocal of $\phi_\pm$ using Geometric series, that is

$$\begin{aligned}
\phi_+(z)^{-1} &= -\frac{1}{z_+}\frac{1}{1 - z^2/z_+} \\
&= -\frac{1}{z_+} - \frac{z^2}{z_+^2} - \frac{z^4}{z_+^3} - \cdots \\
\phi_-(z)^{-1} &= \frac{1}{1 - z_-/z^2} = 1 + \frac{z_-}{z^2} + \frac{z_-^2}{z^4} + \cdots
\end{aligned}$$

Thus we have

$$a(z)^{-1} = \phi_+(z)^{-1}\phi_-(z)^{-1} = \sum_{k=-\infty}^{\infty} b_{2k} z^{2k}$$

where for $k \geq 0$

$$b_{2k} = -\sum_{j=0}^{\infty} \frac{z_-^j}{z_+^{j+k+1}} = -\frac{z_+^{-k-1}}{1 - z_-/z_+}$$

and for $k < 0$

$$b_{2k} = -\sum_{j=0}^{\infty} \frac{z_-^{j-k}}{z_+^{j+1}} = -\frac{z_-^{-k}}{z_+ - z_-}$$

These give the diagonals of $L[a(z)^{-1}]$.

Verification

A *circulant matrix* is an effective approximation to a Laurent matrix (for reasons beyond the scope of this course, though it intuitively follows since the DFT diagonalises all circulant matrices):

```
using ToeplitzMatrices, ApproxFun, Plots, LinearAlgebra,
ComplexPhasePortrait, SingularIntegralEquations


n = 6
L = Circulant([-4; 1; zeros(n-3); 1])
```

```
6×@*(6 ToeplitzMatrices.Circulant(*@{Float64,Complex{Float64}}:
 -4.0   1.0   0.0   0.0   0.0   1.0
  1.0  -4.0   1.0   0.0   0.0   0.0
  0.0   1.0  -4.0   1.0   0.0   0.0
```

```
  0.0   0.0   1.0  -4.0   1.0   0.0
  0.0   0.0   0.0   1.0  -4.0   1.0
  1.0   0.0   0.0   0.0   1.0  -4.0
```

Taking n large, the entries inverse of L approximates the true inverse $L[a(z)]^{-1}$:

```
n = 1000
L = Circulant([-4; 1; zeros(n-3); 1])
inv(L)
```

```
1000×@*(1000
ToeplitzMatrices.Circulant(*@{Float64,Complex{Float64}}:
 -0.288675      -0.0773503     -0.0207259     ...@*( -0.0207259
-0.0773503-0.0773503 -0.288675 -0.0773503 -0.0055535
-0.0207259-0.0207259 -0.0773503 -0.288675 -0.00148806
-0.0055535-0.0055535 -0.0207259 -0.0773503 -0.000398723
-0.00148806-0.00148806 -0.0055535 -0.0207259 -0.000106838
-0.000398723-0.000398723 -0.00148806 -0.0055535 (*@...@*( -2.8627e-5
-0.000106838-0.000106838 -0.000398723 -0.00148806 -7.67059e-6
-2.8627e-5-2.8627e-5 -0.000106838 -0.000398723 -2.05533e-6
-7.67059e-6-7.67059e-6 -2.8627e-5 -0.000106838 -5.50724e-7
-2.05533e-6-2.05533e-6 -7.67059e-6 -2.8627e-5 -1.47566e-7
-5.50724e-7(*@:@*( (*@⋅⋅.@*(-2.05533e-6 -5.50724e-7 -1.47566e-7
-2.8627e-5 -7.67059e-6-7.67059e-6 -2.05533e-6 -5.50724e-7
-0.000106838 -2.8627e-5-2.8627e-5 -7.67059e-6 -2.05533e-6
```

```
-0.000398723 -0.000106838-0.000106838 -2.8627e-5 -7.67059e-6
-0.00148806 -0.000398723-0.000398723 -0.000106838 -2.8627e-5 (*@...@*(
-0.0055535 -0.00148806-0.00148806 -0.000398723 -0.000106838
-0.0207259 -0.0055535-0.0055535 -0.00148806 -0.000398723 -0.0773503
-0.0207259-0.0207259 -0.0055535 -0.00148806 -0.288675
-0.0773503-0.0773503 -0.0207259 -0.0055535 -0.0773503 -0.288675
```

We verify this approximates the true inverse we deduced above by comparing the first few entries:

```
zp = 2+sqrt(3)
zm = 2-sqrt(3)
-zp.^(-(0:4).-1) / (1-zm/zp), inv(L)[1,1:5]
```

```
([-0.28867513459481287, -0.07735026918962577, -0.020725942163690183, -0.0055
5349946513494, -0.001488055696849579], [-0.288675134594813, -0.07735026918
962576, -0.020725942163690177, -0.005553499465134939, -0.001488055696849576
])
```

1.3 Problem 3.2

This part was solved as part of Problem 3.1.

1.4 Problem 3.3

Note that

$$T[a(z)] = \begin{pmatrix} -4 & 0 & 1 & & & \\ 0 & -4 & 0 & 1 & & \\ 1 & 0 & -4 & 0 & 1 & \\ & 1 & 0 & -4 & 0 & 1 \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

The UL decomposition is $T[\phi_-]T[\phi_+]$, i.e., for $z_\pm = 2 \pm \sqrt{3}$,

$$\underbrace{\begin{pmatrix} 1 & 0 & -z_- & & \\ & 1 & 0 & -z_- & \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{U} \underbrace{\begin{pmatrix} -z_+ & & & \\ 0 & -z_+ & & \\ 1 & 0 & -z_+ & \\ & 1 & 0 & -z_+ \\ & & \ddots & \ddots & \ddots \end{pmatrix}}_{L}$$

Verification

```
n = 10
U = Toeplitz([1; zeros(n-1)], [1; 0; -zm; zeros(n-3)])
L = Toeplitz([-zp; 0; 1; zeros(n-3)], [-zp; zeros(n-1)])
U*L
```

```
10×@*(10 Array(*@{Float64,2}:
 -4.0   0.0   1.0   0.0   0.0   0.0   0.0   0.0   0.0        0.0
  0.0  -4.0   0.0   1.0   0.0   0.0   0.0   0.0   0.0        0.0
  1.0   0.0  -4.0   0.0   1.0   0.0   0.0   0.0   0.0        0.0
  0.0   1.0   0.0  -4.0   0.0   1.0   0.0   0.0   0.0        0.0
  0.0   0.0   1.0   0.0  -4.0   0.0   1.0   0.0   0.0        0.0
  0.0   0.0   0.0   1.0   0.0  -4.0   0.0   1.0   0.0        0.0
  0.0   0.0   0.0   0.0   1.0   0.0  -4.0   0.0   1.0        0.0
  0.0   0.0   0.0   0.0   0.0   1.0   0.0  -4.0   0.0        1.0
  0.0   0.0   0.0   0.0   0.0   0.0   1.0   0.0  -3.73205    0.0
  0.0   0.0   0.0   0.0   0.0   0.0   0.0   1.0   0.0       -3.73205
```

1.5 Problem 3.4

We have (see Problem 3.1)

$$T[a(z)]^{-1} = L^{-1}U^{-1} = \begin{pmatrix} -z_+^{-1} & & & & \\ 0 & -z_+^{-1} & & & \\ -z_+^{-2} & 0 & -z_+^{-1} & & \\ 0 & -z_+^{-2} & 0 & -z_+^{-1} & \\ -z_+^{-3} & 0 & -z_+^{-2} & 0 & -z_+^{-1} \\ & & \ddots & \ddots & \ddots \end{pmatrix} \begin{pmatrix} 1 & 0 & z_- & 0 & z_-^2 & 0 & \cdots \\ & 1 & 0 & z_- & 0 & z_-^2 & \cdots \\ & & 1 & 0 & z_- & 0 & \cdots \\ & & & \ddots & \ddots & \ddots & \end{pmatrix}$$

Verification For large `n` the entries of the inverse of Toeplitz matrix approximate those of the infinite-dimensional Toeplitz operator:

```
n = 1000
T = Toeplitz([-4; 0; 1; zeros(n-3)], [-4; 0; 1; zeros(n-3)])
inv(Matrix(T))
```

```
1000×@*(1000 Array(*@{Float64,2}:
 -0.267949       -0.0                -0.0717968      ...@*( -9.85983e-287
-0.00.0 -0.267949 -0.0 -0.0 -9.85983e-287-0.0717968 0.0 -0.287187
-3.94393e-286 -0.00.0 -0.0717968 0.0 -0.0 -3.94393e-286-0.0192379 0.0
-0.0769515 -1.47897e-285 -0.00.0 -0.0192379 0.0 (*@...@*( -0.0
-1.47897e-285-0.00515478 0.0 -0.0206191 -5.5215e-285 -0.00.0
-0.00515478 0.0 -0.0 -5.5215e-285-0.00138122 0.0 -0.00552487
-2.0607e-284 -0.00.0 -0.00138122 0.0 -0.0 -2.0607e-284(*@:@*(
(*@⋅ ⋅ .@*(0.0 -2.0607e-284 0.0 -0.0 -0.00138122-5.5215e-285 0.0
-2.2086e-284 -0.00515478 -0.00.0 -5.5215e-285 0.0 -0.0
-0.00515478-1.47897e-285 0.0 -5.9159e-285 -0.0192379 -0.00.0
-1.47897e-285 0.0 (*@...@*( -0.0 -0.0192379-3.94393e-286 0.0
-1.57757e-285 -0.0717968 -0.00.0 -3.94393e-286 0.0 -0.0
-0.0717968-9.85983e-287 0.0 -3.94393e-286 -0.267949 -0.00.0
-9.85983e-287 0.0 0.0 -0.267949
```

This matches our construction:

```
li = zeros(n) # inv(L) coefficients
```

```julia
li[1:2:end] = -zp.^(-(1:(n ÷ 2)))
Li = Toeplitz(li, [li[1]; zeros(n-1)])
```

```
1000×@*(1000 ToeplitzMatrices.Toeplitz(*@{Float64,Complex{Float64}}:
 -0.267949        0.0              ...@*( 0.0 0.0 0.00.0 -0.267949 0.0 0.0
0.0-0.0717968 0.0 0.0 0.0 0.00.0 -0.0717968 0.0 0.0 0.0-0.0192379 0.0
0.0 0.0 0.00.0 -0.0192379 (*@..@*( 0.0 0.0 0.0-0.00515478 0.0 0.0 0.0
0.00.0 -0.00515478 0.0 0.0 0.0-0.00138122 0.0 0.0 0.0 0.00.0
-0.00138122 0.0 0.0 0.0(*@:@*( (*@⋅⋅.@*(0.0 -2.06071e-284 0.0 0.0
0.0-5.52165e-285 0.0 0.0 0.0 0.00.0 -5.52165e-285 0.0 0.0
0.0-1.47952e-285 0.0 0.0 0.0 0.00.0 -1.47952e-285 (*@..@*( 0.0 0.0
0.0-3.96437e-286 0.0 0.0 0.0 0.00.0 -3.96437e-286 -0.267949 0.0
0.0-1.06225e-286 0.0 0.0 -0.267949 0.00.0 -1.06225e-286 -0.0717968
0.0 -0.267949
```

```julia
ui = zeros(n) # inv(U) coefficients
ui[1:2:end] = zm.^(0:(n ÷ 2)-1)
Ui = Toeplitz([ui[1]; zeros(n-1)], ui)
```

```
1000×@*(1000 ToeplitzMatrices.Toeplitz(*@{Float64,Complex{Float64}}:
 1.0   0.0   0.267949   0.0         0.0717968   ...@*( 3.96437e-286 0.00.0
1.0 0.0 0.267949 0.0 0.0 3.96437e-2860.0 0.0 1.0 0.0 0.267949
1.47952e-285 0.00.0 0.0 0.0 1.0 0.0 0.0 1.47952e-2850.0 0.0 0.0 0.0
1.0 5.52165e-285 0.00.0 0.0 0.0 0.0 0.0 (*@..@*( 0.0 5.52165e-2850.0
0.0 0.0 0.0 0.0 2.06071e-284 0.00.0 0.0 0.0 0.0 0.0 0.0
```

```
2.06071e-2840.0 0.0 0.0 0.0 0.0 7.69067e-284 0.00.0 0.0 0.0 0.0 0.0
0.0 7.69067e-284(*@:@*( (*@⋅⋅.@*(0.0 0.0 0.0 0.0 0.0 0.0
0.005154780.0 0.0 0.0 0.0 0.0 0.0192379 0.00.0 0.0 0.0 0.0 0.0 0.0
0.01923790.0 0.0 0.0 0.0 0.0 0.0717968 0.00.0 0.0 0.0 0.0 0.0
(*@...@*( 0.0 0.07179680.0 0.0 0.0 0.0 0.0 0.267949 0.00.0 0.0 0.0 0.0
0.0 0.0 0.2679490.0 0.0 0.0 0.0 0.0 1.0 0.00.0 0.0 0.0 0.0 0.0 0.0
1.0
```

```
Li*Ui
```

```
1000×@*(1000 Array(*@{Float64,2}:
 -0.267949       0.0             -0.0717968     ...@*( -1.06225e-286
0.00.0 -0.267949 0.0 0.0 -1.06225e-286-0.0717968 0.0 -0.287187
-4.249e-286 0.00.0 -0.0717968 0.0 0.0 -4.249e-286-0.0192379 0.0
-0.0769515 -1.59337e-285 0.00.0 -0.0192379 0.0 (*@...@*( 0.0
-1.59337e-285-0.00515478 0.0 -0.0206191 -5.94859e-285 0.00.0
-0.00515478 0.0 0.0 -5.94859e-285-0.00138122 0.0 -0.00552487
-2.2201e-284 0.00.0 -0.00138122 0.0 0.0 -2.2201e-284(*@:@*(
(*@⋅⋅.@*(0.0 -2.06071e-284 0.0 0.0 -0.00148806-5.52165e-285 0.0
-2.20866e-284 -0.0055535 0.00.0 -5.52165e-285 0.0 0.0
-0.0055535-1.47952e-285 0.0 -5.91809e-285 -0.0207259 0.00.0
-1.47952e-285 0.0 (*@...@*( 0.0 -0.0207259-3.96437e-286 0.0
-1.58575e-285 -0.0773503 0.00.0 -3.96437e-286 0.0 0.0
-0.0773503-1.06225e-286 0.0 -4.249e-286 -0.288675 0.00.0
```

```
-1.06225e-286 0.0 0.0 -0.288675
```

Note the inverse of a Toeplitz operator/matrix is not Toeplitz, unlike the case of a Laurent operator / Circulant matrix.

1.5.1 Problem 3.5

This is somewhat a trick question as $a(z) = (z^2 + 3)/(z^2 + 2)$ is analytic inside the unit circle, so $T[a(z)]$ is lower triangular and therefore

$$T[a(z)]^{-1} = T[a(z)^{-1}] = T[(z^2 + 2)/(z^2 + 3)]$$

1.6 Problem 4

1.6.1 Problem 4.1

It is 1 since we go around the origin once. The easiest way to see this is by direct inspection, we want to solve:

$$\underbrace{\begin{pmatrix} 0 & & & \\ 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & \ddots \end{pmatrix}}_{T[z]} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix}$$

But the first row is always zero. If $f_0 = 0$ we therefore have the solution $u_n = f_{n+1}$.

1.6.2 Problem 4.2

The winding number is -1. We want to solve:

$$\underbrace{\begin{pmatrix} 0 & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}}_{T[z^{-1}]} \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix} = \begin{pmatrix} f_0 \\ f_1 \\ \vdots \end{pmatrix}$$

Now we have the solution for any constant c $u_0 = c, u_n = f_{n-1}$. In other words, $\mathbf{e}_0$ is in the kernel.

1.7 Problem 4.3

If $a(z)$ has winding number κ then $z^{-k}a(z)$ has trivial winding number. Therefore we have

$$z^{-k}a(z) = \phi_+(z)\phi_-(z)$$

As usual we can now take logarithms to deduce:

$$\log(a(z)z^{-k}) = \log\phi_+(z) + \log\phi_-(z)$$

which by Plemelj implies

$$\phi_+(z) = \mathrm{e}^{\mathcal{C}_+[\log(\diamond^{-k}a)](z)}$$
$$\phi_+(z) \ = \mathrm{e}^{-\mathcal{C}_-[\log(\diamond^{-k}a)](z)}$$

1.8 Problem 4.4

Note that for $\kappa \geq 0$ that P is lower triangular Toeplitz, therefore we have using the algebraic properties of triangular Toeplitz

$$T[\phi_-]T[z^\kappa]T[\phi_+] = T[\phi_-]T[z^\kappa\phi_+] = T[\phi_- z^\kappa\phi_+] = T[a(z)]$$

When $\kappa \leq 0$ then P is upper triangular Toeplitz and so

$$T[\phi_-]T[z^\kappa]T[\phi_+] = T[\phi_- z^\kappa]T[\phi_+] = T[\phi_- z^\kappa\phi_+] = T[a(z)].$$

1.9 Problem 4.5

The first question is: what is the winding number? The straightforward way to compute is via residue calculus. That is, if we calculate

$$\frac{1}{2\pi\mathrm{i}}\oint_a \frac{1}{z}\mathrm{d}z = \frac{1}{2\pi\mathrm{i}}\oint_C \frac{a'(z)}{a(z)}\mathrm{d}z = \frac{-1}{\pi\mathrm{i}}\oint_C \frac{z}{z^2+1/2}\mathrm{d}z$$
$$= -2\left(\operatorname*{Res}_{z=-\frac{\mathrm{i}}{\sqrt{2}}} + \operatorname*{Res}_{z=\frac{\mathrm{i}}{\sqrt{2}}}\right)\frac{z}{z^2+1/2} = -2.$$

This is also intuitive since z^2 clearly goes around the origin twice counterclockwise, so does $2z^2+1$ as the shift by 1 is not enough to change anything, therefore $(2z^2+1)^{-1}$ goes around twice clockwise.

Note that

$$z^2a(z) = \underbrace{\frac{z^2}{2z^2+1}}_{\phi_-(z)}$$

Is already analytic outside the unit circle so we have $L = I$ and thus the factorisation

$$T[a(z)] = \underbrace{T[\phi_-]}_{U}\underbrace{T[z^{-2}]}_{P}$$

From the Laurent expansion

$$\phi_-(z)^{-1} = 2 + 1/z^2$$

We can compute

$$U^{-1}\mathbf{e}_0 = T[\phi_-^{-1}]\mathbf{e}_0 = 2\mathbf{e}_0$$

The kernel of P is $\mathbf{e}_0$ and $\mathbf{e}_1$. Thus putting everything together we get the rather boring answer

$$\begin{pmatrix} c \\ d \\ 2 \\ 0 \\ 0 \\ 0 \\ \vdots \end{pmatrix}$$

where c and d are arbitrary constants.

1.10 Problem 5

1.10.1 Problem 5.1

To be analytic at all we need decay at either $\pm\infty$, this has neither so is not defined.

1.10.2 Problem 5.2

It has exponential decay in the right-half plane, therefore

$$\mathrm{e}^{\gamma x} f(x) = \frac{\mathrm{e}^{\gamma x}}{1 + \mathrm{e}^x}$$

has exponential decay at both $\pm\infty$, provided $0 < \gamma < 1$. Therefore, we can take the strip $0 < \Im s < 1$. (Note in each case the contour for the inverse Fourier transform can be any contour in the domain of analyticity.)

We can verify this by exact computation using Residue calculus: for $0 < \Im s < 1$, we can integrate over a rectangle to get:

$$\left(\int_{-R}^{R} + \int_{R}^{2\mathrm{i}\pi+R} + \int_{2\mathrm{i}\pi+R}^{2\mathrm{i}\pi-R} + \int_{2\mathrm{i}\pi-R}^{-R}\right) \frac{\mathrm{e}^{-\mathrm{i}sx}}{1+\mathrm{e}^x} \mathrm{d}x = 2\pi\mathrm{i} \operatorname*{Res}_{z=\mathrm{i}\pi} \frac{\mathrm{e}^{-\mathrm{i}sz}}{1+\mathrm{e}^z} = -2\pi\mathrm{i}\mathrm{e}^{\pi s}$$

Note that

$$\frac{\mathrm{e}^{-\mathrm{i}s(R+\mathrm{i}t)}}{1+\mathrm{e}^{R+\mathrm{i}t}} = \frac{\mathrm{e}^{-\mathrm{i}R\Re s + R\Im s + t}}{1+\mathrm{e}^{R+\mathrm{i}t}} \to 0$$

and

$$\frac{\mathrm{e}^{-\mathrm{i}s(-R+\mathrm{i}t)}}{1+\mathrm{e}^{R+\mathrm{i}t}} = \frac{\mathrm{e}^{\mathrm{i}R\Re s - R\Im s + t}}{1+\mathrm{e}^{R+\mathrm{i}t}} \to 0$$

uniformly in t as $R \to \infty$, hence we deduce that

$$\left(\int_{-\infty}^{\infty} + \int_{2\mathrm{i}\pi+\infty}^{2\mathrm{i}\pi-\infty}\right) \frac{\mathrm{e}^{-\mathrm{i}sx}}{1+\mathrm{e}^x} \mathrm{d}x = -2\pi\mathrm{i}\mathrm{e}^{\pi s}$$

Now note that

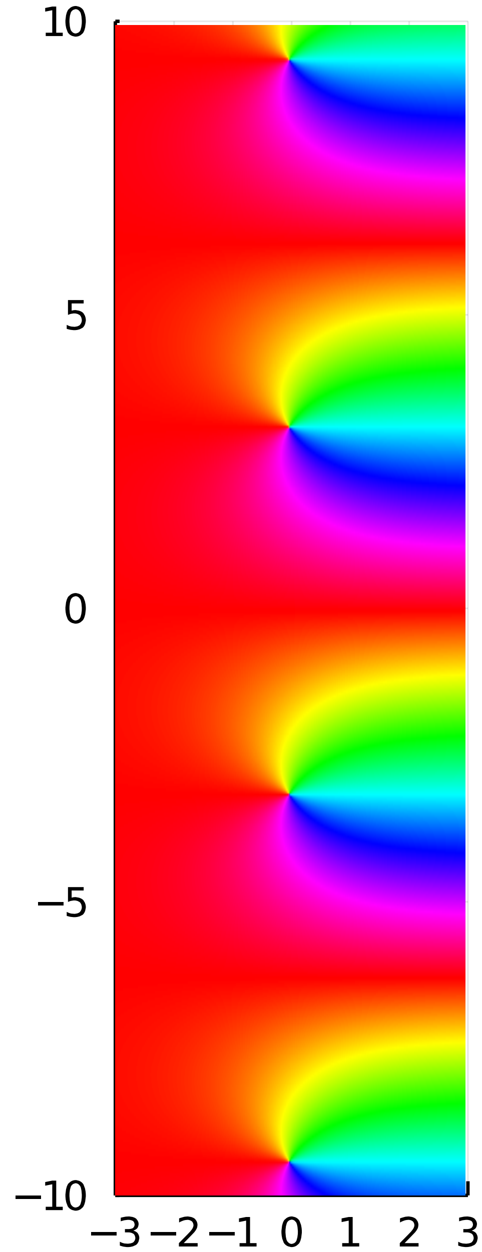
$$\int_{2\mathrm{i}\pi+\infty}^{2\mathrm{i}\pi-\infty} \frac{\mathrm{e}^{-\mathrm{i}st}}{1+\mathrm{e}^t}\mathrm{d}t = \int_{\infty}^{-\infty} \frac{\mathrm{e}^{-\mathrm{i}s(x+2\mathrm{i}\pi)}}{1+\mathrm{e}^x}\mathrm{d}x = -\mathrm{e}^{2\pi s}\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}sx}}{1+\mathrm{e}^x}\mathrm{d}x$$

Therefore, we have

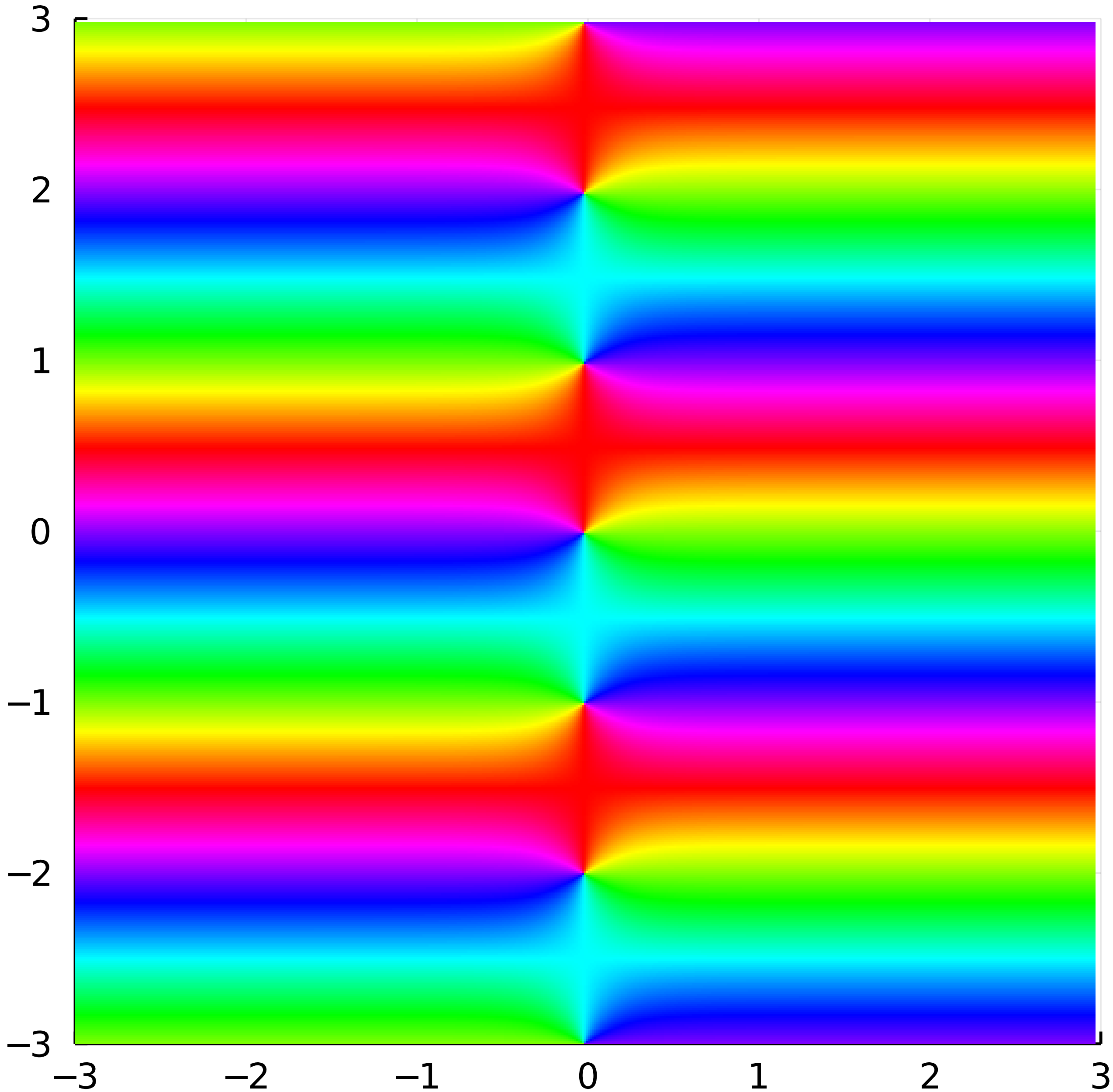
$$\int_{-\infty}^{\infty} \frac{\mathrm{e}^{-\mathrm{i}sx}}{1+\mathrm{e}^x}\mathrm{d}x = -2\mathrm{i}\pi\frac{\mathrm{e}^{\pi s}}{1-\mathrm{e}^{2\pi s}} = \mathrm{i}\pi\operatorname{csch}\pi x$$

which has poles at 0 and i:

```
phaseplot(-3..3, -10..10, z -> 1/(1+exp(z))) #integrand
```




```
phaseplot(-3..3, -3..3, z -> im*π*csch(π*z)) # transform
```


3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

1.10.3 Problem 5.3

Here $\mathrm{e}^{\gamma x} f(x) = \mathrm{e}^{(\gamma+2)x}$ has decay at $+\infty$ proved $\gamma < -2$, hence we have the strip $\Im s < -2$. Indeed, its Fourier transform is

$$-\frac{\mathrm{i}}{2\mathrm{i} + s}$$

by integration by parts.

1.10.4 Problem 5.4

Here it's $\Im s > 0$: unlike 1.1, we now have decay at $x \to \infty$ since $f_{\mathrm{L}}(x)$ is identically zero. It's Fourier transform is determinable by integration-by-parts:

$$\hat{f}(s) = \int_{-\infty}^{0} x\mathrm{e}^{-\mathrm{i}sx}\mathrm{d}x = \frac{1}{\mathrm{i}s}\int_{-\infty}^{0} \mathrm{e}^{-\mathrm{i}sx}\mathrm{d}x = \frac{1}{s^2}$$

1.10.5 Problem 5.5

The Fourier transforms are given above.

1.10.6 Problem 5.6

$$\int_{-\infty}^{\infty} \delta(x)\mathrm{e}^{\mathrm{i}sx}\mathrm{d}x = 1$$

It's actually an entire function, but non-decaying. This is hinting at the relationship between smoothness of a function and decay of its Fourier transform, and vice-versa: since $\delta(x)$ "decays" to all orders, we expect its Fourier transform to be entire, but since its not smooth at all, we expect no decay, so on a formal level we can predict the analyticity properties.

1.11 Problem 6

1.11.1 Problem 6.1

Note that

$$K(z) = \frac{3}{2}\mathrm{e}^{-|x|} \Rightarrow \hat{K}(s) = \frac{3}{1+s^2}$$

Provided $-1 < \Im s < 1$, and

$$\widehat{f_{\mathrm{R}}}(s) = -\frac{\mathrm{i}}{s} - \frac{\alpha}{s^2}$$

for $\Im s < 0$. Define

$$h(s) = -\widehat{f_{\mathrm{R}}}(s) = \frac{\mathrm{i}}{s} + \frac{\alpha}{s^2}$$

```julia
α = 0.3
x = Fun(0..100)
f = 1   + α*x
h = s -> (im/s + α/s^2)

γ = -0.5 # we take the Fourier transform on R + im*γ
s = -0.5 + im*γ

-sum(f*exp(-im*s*x)) , h(s)
```

```
(-0.9999999999999862 - 1.5999999999999914im, -1.0 - 1.6im)
```

Transforming the equation, we have

$$\Phi_+(s) - (1 + \hat{K}(s))\Phi_-(s) = \frac{\mathrm{i}}{s} + \frac{\alpha}{s^2}$$

where

$$1 + \hat{K}(s) = \frac{4 + s^2}{1 + s^2} = \frac{(s - 2\mathrm{i})(s + 2\mathrm{i})}{(s + \mathrm{i})(s - \mathrm{i})}$$

This is very close to the example we did in lectures, so we already know the homogenous solution:

$$\kappa(z) = \begin{cases} \frac{z+2\mathrm{i}}{z+\mathrm{i}} & \Im z > \gamma \\ \frac{z-\mathrm{i}}{z-2\mathrm{i}} & \Im z < \gamma \end{cases}$$

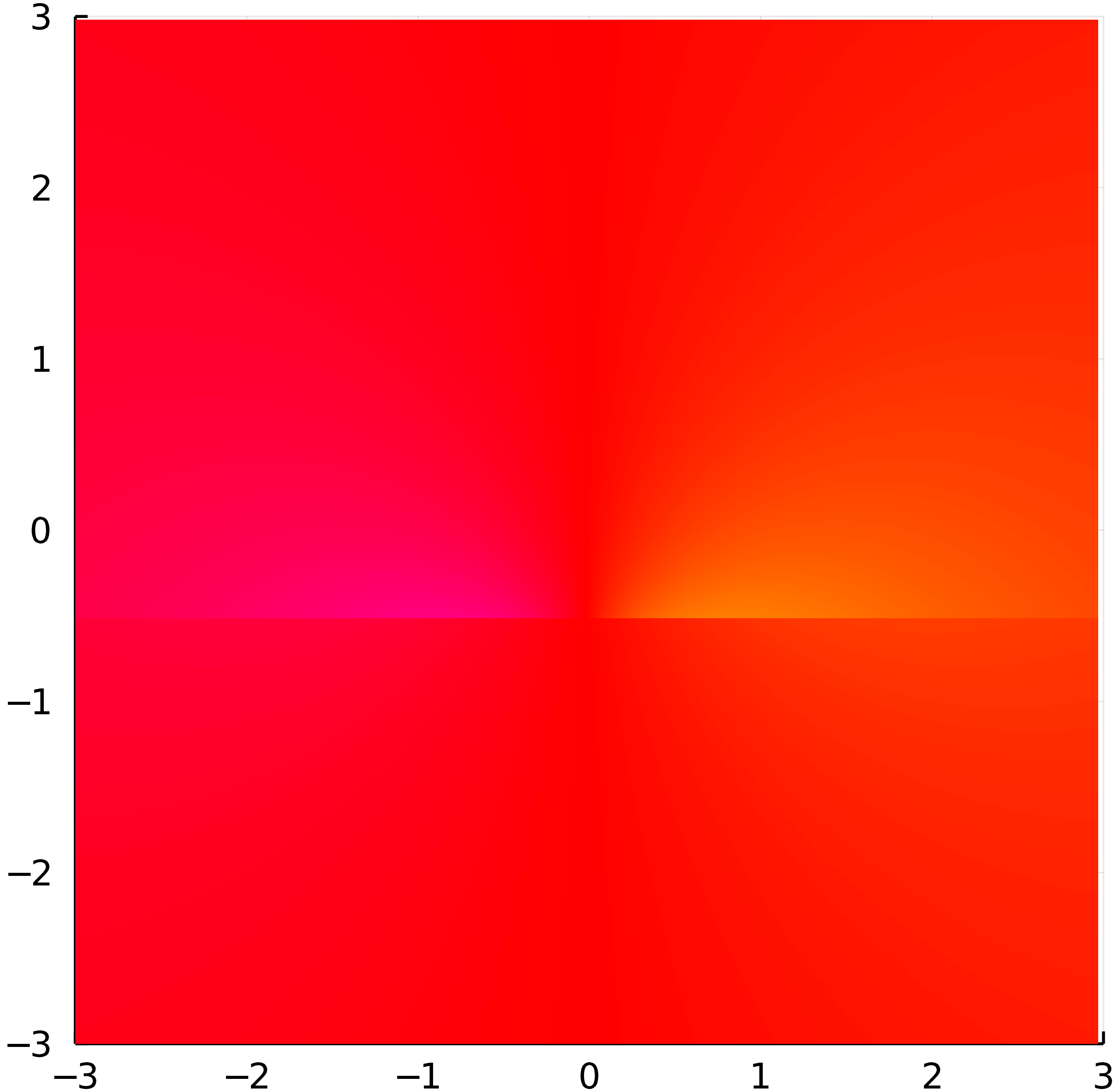
which is valid for $-1 < \gamma < 0$.

```
g = s -> (4+s^2)/(1+s^2)

κ = z -> imag(z) > γ ? (z+im*2)/(z+im) :
                        (z-im)/(z-im*2)



phaseplot(-3..3, -3..3, κ)
```




```
s = 0.1 + γ*im
κp = κ(s + eps()*im)
κm = κ(s - eps()*im)
```

```
κp - κm*g(s)
```

```
-1.3322676295501878e-15 - 2.7755575615628914e-16im
```

We thus get the RH problem

$$Y_+(s) - Y_-(s) = h(s)/\kappa_+(s) = (\frac{\mathrm{i}}{s} + \frac{\alpha}{s^2})\frac{s+\mathrm{i}}{s+2\mathrm{i}}$$

We see this has poles at 0 and $-2\mathrm{i}$, so using partial fraction expansion we get

$$(\frac{\mathrm{i}}{s} + \frac{\alpha}{s^2})\frac{s+\mathrm{i}}{s+\sqrt{3}\mathrm{i}} = \frac{\alpha}{2s^2} - \frac{\mathrm{i}(\alpha-2)}{4s} + \frac{\mathrm{i}(2+\alpha)}{4(s+2\mathrm{i})}$$

Therefore, splitting the poles between those above and below γ, we have

$$Y(z) = \begin{cases} \frac{\mathrm{i}(2+\alpha)}{4(z+2\mathrm{i})} & \Im z > \gamma \\ -\frac{\alpha}{2z^2} + \frac{\mathrm{i}(\alpha-2)}{4z} & \Im z < \gamma \end{cases}$$

```
s = 0.1 + γ*im
Y = z -> imag(z) > γ ? im*(2+α)/(4*(z+2im)) :
                        - α/(2z^2) + im*(α-2)/(4z)



Yp = Y(s + eps()*im)
```

```
Ym = Y(s - eps()*im)

Yp - Ym  , h(s)/κp
```

```
(-0.9682149028643237 + 0.4107975074619046im, -0.9682149028643242 +
0.410797
507461905im)
```

We therefore have

$$\Phi(z) = \kappa(z)Y(z) = \begin{cases} \frac{\mathrm{i}(2+\alpha)}{4(z+\mathrm{i})} & \Im z > \gamma \\ \left(-\frac{\alpha}{2z^2} + \frac{\mathrm{i}(\alpha-2)}{4z}\right)\frac{z-\mathrm{i}}{z-2\mathrm{i}} & \Im z < \gamma \end{cases}$$

```
Φ = z -> imag(z) > γ ? im*(2+α)/(4*(z+im)) :
                         (-α/(2z^2) + im*(α-2)/(4z))*(z-im)/(z-2im)
Φp = Φ(s+eps()im)
Φm = Φ(s-eps()im)

Φp - Φm*g(s) , h(s)
```

```
(-2.9881656804733714 + 0.8284023668639042im, -2.9881656804733723 +
0.828402
3668639053im)
```

Finally, we recover the solution by inverting Φ_-, using Residue calculus in the upper half plane: for $x > 0$ we have

$$u(x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma}(-\frac{\alpha}{2z^2}+\frac{\mathrm{i}(\alpha-2)}{4z})\frac{z-\mathrm{i}}{z-2\mathrm{i}}\mathrm{e}^{\mathrm{i}zx}\mathrm{d}z$$
$$= \mathrm{i}(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=2\mathrm{i}})(-\frac{\alpha}{2z^2}+\frac{\mathrm{i}(\alpha-2)}{4z})\frac{z-\mathrm{i}}{z-2\mathrm{i}}\mathrm{e}^{\mathrm{i}zx} = \frac{1+x\alpha}{4} - \frac{\alpha+1}{4}\mathrm{e}^{-2x}$$

Did it work? yes:

```
t = Fun(0 .. 50)
u = (1+t*α)/4 - (α-1)/4*exp(-2t)
x = 0.1
u(x) + 3/2*sum(exp(-abs(t-x))*u) , f(x)
```

```
(1.0300000000000025, 1.03)
```

1.11.2 Problem 6.2

Setting up the problem as above, we arrive at a degenerate RH problem:

$$\Phi_+(s) - g(s)\Phi_-(s) = h(s)$$

where

$$g(s) = \widehat{K}(s) = \frac{2\alpha}{\alpha^2+s^2} = \frac{2\alpha}{(s-\mathrm{i}\alpha)(s+\mathrm{i}\alpha)}$$

and

$$h(s) = \frac{\mathrm{i}}{s} + \frac{\alpha}{s^2} = \mathrm{i}\frac{s - \mathrm{i}\alpha}{s^2}$$

Suppose we allow $\kappa_-(s) \sim s$ to have growth, then we can write

$$\kappa(z) = \begin{cases} \frac{1}{z+\mathrm{i}\alpha} & \Im z > \gamma \\ \frac{z-\mathrm{i}\alpha}{2\alpha} & \Im z < \gamma \end{cases}$$

so that

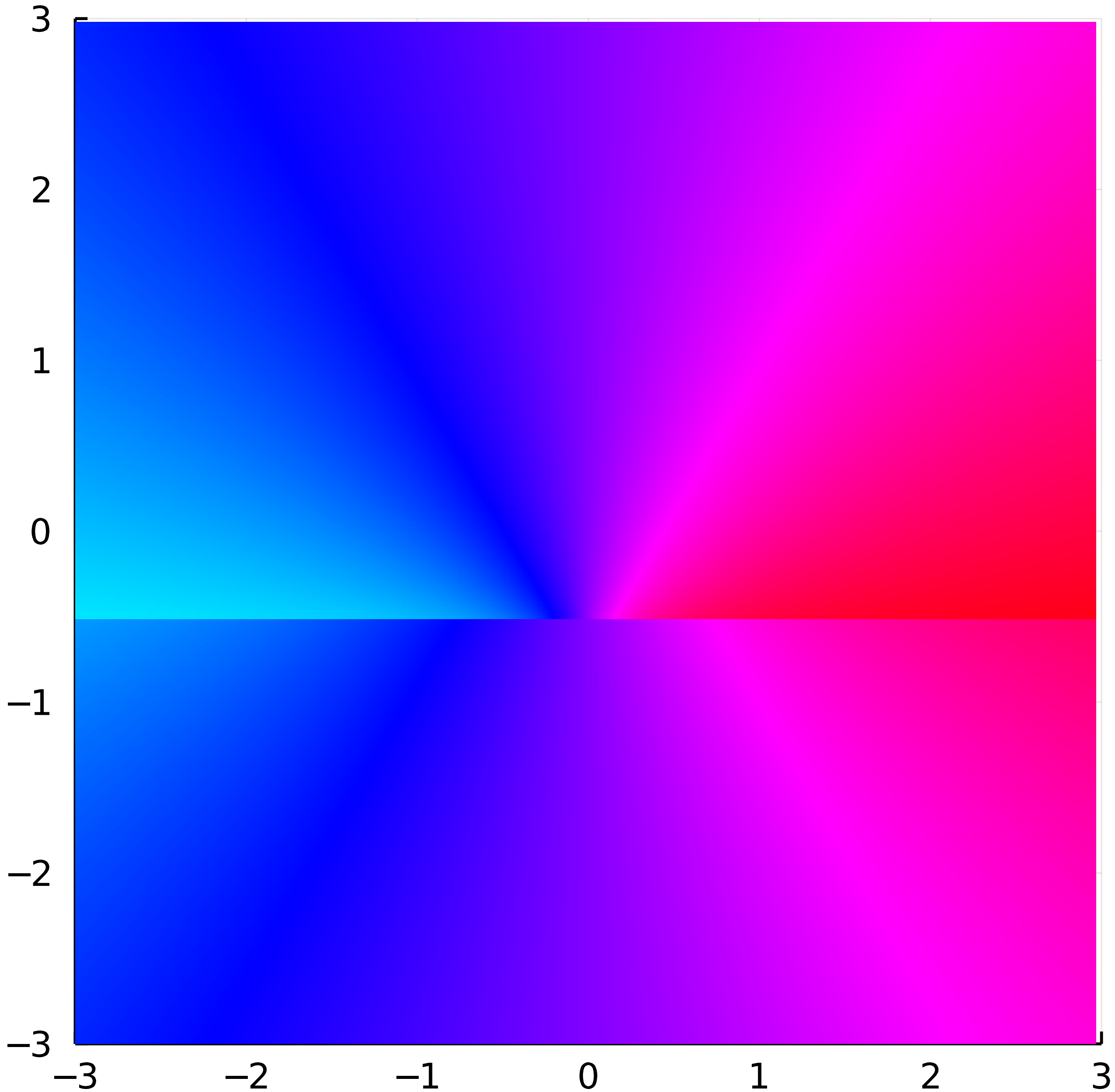
$$\kappa_+(s) = \kappa_-(s) g(s)$$

```
α = 0.8

g = s -> (2α)/(α^2+s^2)
h = s -> (im/s + α/s^2)

κ = z -> imag(z) > γ ? 1/(z + im*α) :
                            (z-im*α)/(2α)

phaseplot(-3..3, -3..3, κ)
```




```
s = 0.1 + γ*im
κp = κ(s + eps()*im)
κm = κ(s - eps()*im)
```

```
κp , κm*g(s)
```

```
(0.9999999999999983 - 2.9999999999999998im, 0.9999999999999997 - 3.0im
)
```

Then we have

$$h(s)/\kappa_+(s) = \mathrm{i}\frac{s^2+\alpha^2}{s^2} = \mathrm{i} + \mathrm{i}\frac{\alpha^2}{s^2}$$

and then we can write

$$Y(z) = \begin{cases} \mathrm{i} & \Im z > \gamma \\ -\frac{\mathrm{i}\alpha^2}{z^2} & \Im z < \gamma \end{cases}$$

```
s = 0.1 + γ*im
Y = z -> imag(z) > γ ? im :
                        -im*α^2/s^2



Yp = Y(s + eps()*im)
Ym = Y(s - eps()*im)

Yp - Ym  , h(s)/κp
```

```
(-0.9467455621301777 - 1.2721893491124265im, -0.946745562130178 -
1.2721893
491124274im)
```

Putting things together, we get

$$\Phi(z) = \kappa(z)Y(z) = \begin{cases} \frac{\mathrm{i}}{z+\mathrm{i}\alpha} & \Im z > \gamma \\ -\mathrm{i}\frac{\alpha^2}{z^2}\frac{z-\mathrm{i}\alpha}{2\alpha} & \Im z < \gamma \end{cases}$$

```
Φ = z -> imag(z) > γ ? im/(z + im*α)  :
                       -im*α^2/z^2* (z-im*α)/(2α)
Φp = Φ(s+eps()im)
Φm = Φ(s-eps()im)

Φp - Φm*g(s) , h(s)
```

```
(-4.763313609467453 + 1.568047337278103 2im, -4.763313609467456 +
1.56804733
72781065im)
```

We now invert the Fourier transform of $\Phi_-(s)$ using Jordan's lemma:

$$u(x) = \frac{1}{2\pi}\int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \Phi_-(s)\mathrm{e}^{\mathrm{i}sx}\mathrm{d}s = \frac{\alpha}{2}\operatorname*{Res}_{z=0}\frac{z-\mathrm{i}\alpha}{z^2}\mathrm{e}^{\mathrm{i}zx} = \frac{\alpha}{2}(1+x\alpha)$$

```
t = Fun(0 .. 200)
```

```
u = α*(1+t*α)/2
```

```
x = 0.1
```

```
sum(exp(-α*abs(t-x))*u)  , (1 + α*x)
```

```
(1.0799999999999645, 1.08)
```

1.12 Problem 6.3

1. From the same logic as 2.2, we know we need to solve

$$\Phi_+(s) - g(s)\Phi_-(s) = h(s)$$

where

$$g(s) = 1 - \frac{2\lambda}{s^2+1} = \frac{s^2+1-2\lambda}{s^2+1} = \frac{(s-\mathrm{i}\gamma)(s+\mathrm{i}\gamma)}{(s+\mathrm{i})(s-\mathrm{i})}$$

and

$$h(s) = \frac{1}{s^2}$$

where $-1 < \Im s < 0$, let's say $\Im s = \delta$ because I annoyingly used γ in the statement of the problem. Writing $s = t + \mathrm{i}\delta$, we see that

$$g(s) = \frac{t^2 + 2\mathrm{i}\delta t - \delta^2 + \gamma^2}{s^2 + 1}$$

By ensuring its real part is positive, this has trivial winding number provided $\gamma^2 = 1 - 2\lambda > 0$, which is true for $0 < \lambda < \frac{1}{2}$, and restricting the contour s lives on to be $-\gamma < \delta < 0$. Factorizing the kernel we get

$$\kappa(z) = \begin{cases} \frac{z+\mathrm{i}\gamma}{z+\mathrm{i}} & \Im z > \delta \\ \frac{z-\mathrm{i}}{z-\mathrm{i}\gamma} & \Im z < \delta \end{cases}$$

Thus we want to solve

$$Y_+(s) - Y_-(s) = h(s)\kappa_+(s)^{-1} = \frac{s+\mathrm{i}}{s+\mathrm{i}\gamma}\frac{1}{s^2} = \frac{1}{\gamma s^2} - \frac{\mathrm{i}(\gamma-1)}{\gamma^2 s} + \frac{\mathrm{i}}{\gamma^2}\frac{\gamma-1}{s+\mathrm{i}\gamma}$$

Which has solution, (since $\delta > -\gamma$),

$$Y(z) = \begin{cases} \frac{\mathrm{i}}{\gamma^2}\frac{\gamma-1}{z+\mathrm{i}\gamma} & \Im z > \delta \\ \frac{\mathrm{i}(\gamma-1)}{\gamma^2 z} - \frac{1}{\gamma z^2} & \Im z < \delta \end{cases}$$

We thus get

$$\Phi_-(z) = (\frac{\mathrm{i}(\gamma-1)}{\gamma^2 z} - \frac{1}{\gamma z^2})\frac{z-\mathrm{i}}{z-\mathrm{i}\gamma}$$

and Jordan's lemma gives us

$$u(x) = \frac{x}{\gamma^2} - \mathrm{e}^{-x\gamma}(\gamma-1)/\gamma^2$$

```
t = Fun(0 .. 200)
λ = 0.1
γ = sqrt(1-2λ)

u = t/γ^2 - exp(-t*γ)*(γ-1)/γ^2

x = 0.1

u(x) - λ*sum(exp(-abs(t-x))*u) , x
```

```
(0.09999999999999876, 0.1)
```

Oddly, this is definitely a solution, but not in the form the question asked for. To get the other solution, consider now the bad winding number case of $-1 < \delta < -\gamma$. Motivated by 2.2, what if we allow κ to have different behaviour? Consider

$$\kappa(z) = \begin{cases} \frac{1}{z+\mathrm{i}} & \Im z > \delta \\ \frac{(z-\mathrm{i})}{(z-\mathrm{i}\gamma)(z+\mathrm{i}\gamma)} & \Im z < \delta \end{cases}$$

Chosen so that both κ_+ and κ_+^{-1} are analytic.

Thus we want to solve

$$Y_+(s) - Y_-(s) = h(s)\kappa_+(s)^{-1} = \frac{s+\mathrm{i}}{s^2} = \frac{1}{s} + \frac{\mathrm{i}}{s^2}$$

but now we only need $Y_+(s) = O(1)$ and $Y_-(s) = O(1)$. Here is where the non-uniqueness comes in, as we can add an arbitrary constant:

$$Y(z) = \begin{cases} A & \Im z > 0 \\ A - \frac{1}{z} - \frac{\mathrm{i}}{z^2} & \Im z < 0 \end{cases}$$

Thus we have

$$\Phi_-(z) = Y_-(z)\kappa_-(z) = -(A + \frac{1}{z} + \frac{\mathrm{i}}{z^2})\frac{(z-\mathrm{i})}{(z-\mathrm{i}\gamma)(z+\mathrm{i}\gamma)}$$

Using Jordan's lemma, and now since $\delta < -\gamma$, we get

$$
\begin{aligned}
u(x) &= \mathrm{i}(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=\mathrm{i}\gamma} + \operatorname*{Res}_{z=-\mathrm{i}\gamma})\Phi_-(z)\mathrm{e}^{\mathrm{i}xz} \\
&= \frac{x}{\gamma^2} - \mathrm{e}^{-x\gamma}(\frac{\gamma^2-1}{2\gamma^3} + \frac{\gamma-1}{2\gamma^3}A) - \mathrm{e}^{x\gamma}(\frac{1-\gamma^2}{2\gamma^3} + \frac{\gamma+1}{2\gamma^3}A) \\
&= \frac{x}{\gamma^2} + \frac{\mathrm{e}^{x\gamma}-\mathrm{e}^{-x\gamma}}{2}\frac{\gamma-\gamma^{-1}}{2\gamma^2} - \frac{A}{\gamma^3}(\frac{\mathrm{e}^{x\gamma}-\mathrm{e}^{-x\gamma}}{2} + \gamma\frac{\mathrm{e}^{x\gamma}+\mathrm{e}^{-x\gamma}}{2})
\end{aligned}
$$

Redefining A and using the definition of $\sinh$ and $\cosh$ gives the form in the assignment.

What's the moral of the story?

1. Different choices of contours can give different solutions
2. When the winding number is non-trivial, the solution may not be unique

1.12.1 Problem 6.4

1. Integrating by parts we have

$$
\begin{aligned}
\widehat{u'_{\mathrm{R}}}(s) &= \mathrm{i}s\widehat{u_{\mathrm{R}}}(s) - u(0) = \mathrm{i}s\widehat{u_{\mathrm{R}}}(s) \\
\widehat{u''_{\mathrm{R}}}(s) &= \mathrm{i}s\widehat{u'_{\mathrm{R}}}(s) - u'(0) = -s^2\widehat{u_{\mathrm{R}}}(s) - u'(0)
\end{aligned}
$$

2. Our integral equation when cast on the whole real line is:

$$u_{\mathrm{R}}''(x) - \frac{72}{5}\int_{-\infty}^{\infty} \mathrm{e}^{-5|x-t|}u_{\mathrm{R}}(t)\mathrm{d}t = 1_{\mathrm{R}}(x) + p_{\mathrm{L}}(x)$$

where

$$p(x) = \frac{72}{5}\int_{-\infty}^{\infty} \mathrm{e}^{-5|x-t|}u_{\mathrm{R}}(t)\mathrm{d}t = \frac{72}{5}\int_{0}^{\infty} \mathrm{e}^{-5|x-t|}u_{\mathrm{R}}(t)\mathrm{d}t.$$

Note that, for $-5 < \Im s < 5$,

$$\hat{K}(s) = \frac{10}{s^2+25}$$

provided s is in the lower half plane,

$$\widehat{1_{\mathrm{R}}}(s) = \int_{0}^{\infty} \mathrm{e}^{-\mathrm{i}sx}\mathrm{d}x = \frac{1}{\mathrm{i}s}$$

Thus our integral equation in frequency space is

$$-\alpha - s^2\widehat{u_{\mathrm{R}}}(s) - \frac{72}{5}\widehat{K}(s)\widehat{u_{\mathrm{R}}}(s) = \widehat{p_{\mathrm{L}}}(s) + \widehat{1_{\mathrm{R}}}(s)$$

$$\Phi_+(s) - (s^2 + \frac{144}{s^2+25})\Phi_-(s) = \alpha + \frac{1}{\mathrm{i}s}$$

$$\Phi_+(s) - \frac{(s^2+9)(s^2+16)}{s^2+25}\Phi_-(s) = \alpha + \frac{1}{\mathrm{i}s}$$

where $s \in \mathbb{R} + \mathrm{i}\gamma$ for any $-5 < \gamma < 0$.

3. We can factorize this to construct $g(s)$ as

$$g(s) = \kappa_+(s)\kappa_-(s)^{-1} = \frac{(s+3\mathrm{i})(s+4\mathrm{i})}{s+5\mathrm{i}}\frac{(s-3\mathrm{i})(s-4\mathrm{i})}{s-5\mathrm{i}}$$

```
κ = z -> imag(z) > γ ?
        (z+3im)*(z+4im)/(z+5im) :
        (z-5im)/((z-3im)*(z-4im))


γ  = -1.0
s = 0.1+γ*im
g = s -> (s^2+9)*(s^2+16)/(s^2+25)

κ(s+eps()im) , g(s)κ(s-eps()im)
```

```
(0.08750780762023733 + 1.4996876951905058im, 0.08750780762023738 +
1.499687
695190506im)
```

Writing $\Phi(z) = \kappa(z)Y(z)$ we get the subtractive RH problem

$$Y_+(s) - Y_-(s) = \frac{h(s)}{\kappa_+(s)} = (\alpha + \frac{1}{\mathrm{i}s})\frac{s+5\mathrm{i}}{(s+3\mathrm{i})(s+4\mathrm{i})}$$

We use partial fraction expansion to write

$$\frac{h(s)}{\kappa_+(s)} = -\frac{\alpha + 1/4}{s + 4\mathrm{i}} + \frac{2/3 + 2\alpha}{s + 3\mathrm{i}} - \frac{5}{12s}$$

Therefore we have

$$Y(z) = \begin{cases} -\frac{\alpha+1/4}{s+4\mathrm{i}} + \frac{2/3+2\alpha}{s+3\mathrm{i}} & \Im z > \gamma \\ \frac{5}{12s} & \Im z < \gamma \end{cases}$$

and hence

$$\Phi(z) = \begin{cases} \frac{(z+3\mathrm{i})(z+4\mathrm{i})}{z+5\mathrm{i}}(-\frac{\alpha+1/4}{z+4\mathrm{i}} + (2/3 + 2\alpha)/(z + 3\mathrm{i})) & \Im z > \gamma \\ \frac{z-5\mathrm{i}}{(z-3\mathrm{i})(z-4\mathrm{i})}\frac{5}{12z} & \Im z < \gamma \end{cases}$$

We can now invert the Fourier transform of

$$\Phi_-(s) = \frac{s - 5\mathrm{i}}{(s - 3\mathrm{i})(s - 4\mathrm{i})}\frac{5}{12s}$$

This actually decays so fast that we don't need Jordan's lemma to justify here. This has three poles above our contour, so we sum over each residue to get

$$u(x) = \mathrm{i}(\operatorname*{Res}_{z=0} + \operatorname*{Res}_{z=3\mathrm{i}} + \operatorname*{Res}_{z=4\mathrm{i}})\mathrm{e}^{\mathrm{i}zx}\frac{z - 5\mathrm{i}}{(z - 3\mathrm{i})(z - 4\mathrm{i})}\frac{5}{12z} = -\frac{25}{144} - \frac{5\mathrm{e}^{-4x}}{48} + \frac{5\mathrm{e}^{-3x}}{18}$$

Here's we check the solution:

```
t = Fun(0 .. 200)
```

```
u = -25/144 - 5exp(-4t)/48 + 5exp(-3t)/18
x = 1.1
u''(x) - 72/5*sum(exp(-5abs(x-t))*u)
```

```
1.000000000000175
```

Here we check the jump of Y:

```
α = u'(0)

h = s -> α + 1/(im*s)
Y = z -> imag(z) > γ ?
        (-(α+1/4)/(z+4im) + (2/3 + 2α)/(z+3im)) :
          5/(12z)

Y(s+eps()im) - Y(s-eps()im) , h(s)/κ(s + eps()im)
```

```
(-0.04356060486688343 - 0.3849096302623936 6im, -0.04356060486688346 -
 0.384
9096302623938im)
```

Here we check the jump of Φ:

```
γ  = -1.0

Φ = z -> imag(z) > γ ?
```

```
        (z+3im)*(z+4im)/(z+5im) * (-(α+1/4)/(z+4im) + (2/3 +
2α)/(z+3im)) :
        (z-5im)/((z-3im)*(z-4im)) * 5/(12z)
```

```
Φ(s + eps()*im) - g(s)*Φ(s - eps()*im) , h(s)
```

```
(0.5734323432343266 - 0.099009900990099im, 0.5734323432343267 -
0.099009900
99009901im)
```