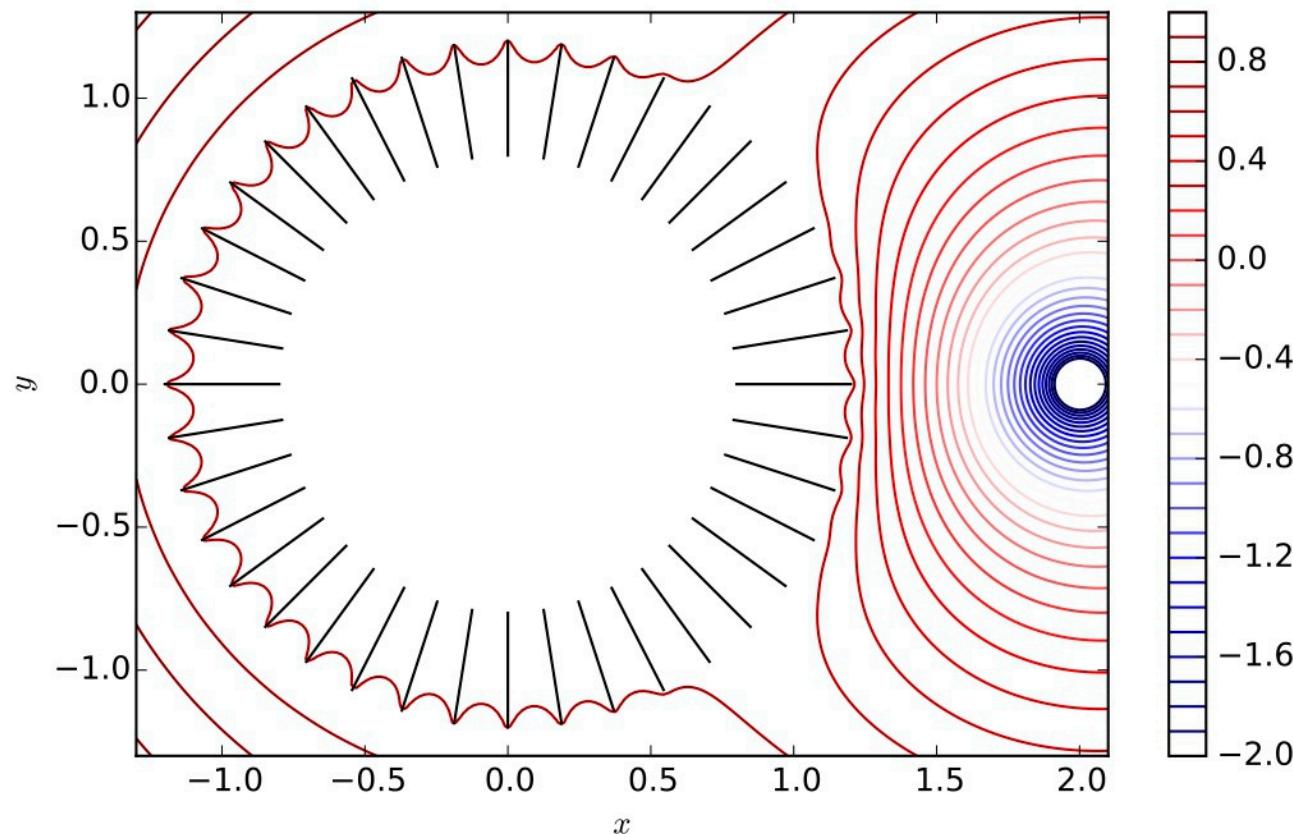


1 Lecture 16: Logarithmic singular integrals

The motivation behind this lecture is to calculate electrostatic potentials. An example is the Faraday cage: imagine a series of metal plates connected together so that they have the same charge. If configured to surround a region, this configuration will shield the interior from an external charge, see figure. Here, the coloured lines are equipotential lines, and there is a point source at $x = 2$, which corresponds to a forcing of $\log \|(x, y) - (2, 0)\| = \log |z - 2|$ where $z = x + iy$.



1.1 Logarithmic singular integrals

From the Green's function of the Laplacian, it is natural to consider logarithmic singular integrals over a contour

$$L_\gamma f(z) = \frac{1}{\pi} \int_{\gamma} f(\zeta) \log |\zeta - z| ds$$

where ds is the arclength differential (as in line integrals, not complex analysis) and f is real-valued. We focus on intervals where the arclength differential is just the standard one:

$$L_{a,b} f(z) = \frac{1}{\pi} \int_a^b f(t) \log |z - t| dt$$

Note that off $[a, b]$, $u(x, y) \equiv u(x + iy) = Lf(x + iy)$ solves Laplace's equation, and as the integrand has an integrable singularity it is continuous on $[a, b]$:

```
using ApproxFun, SingularIntegralEquations, Plots
```

```
t = Fun()
```

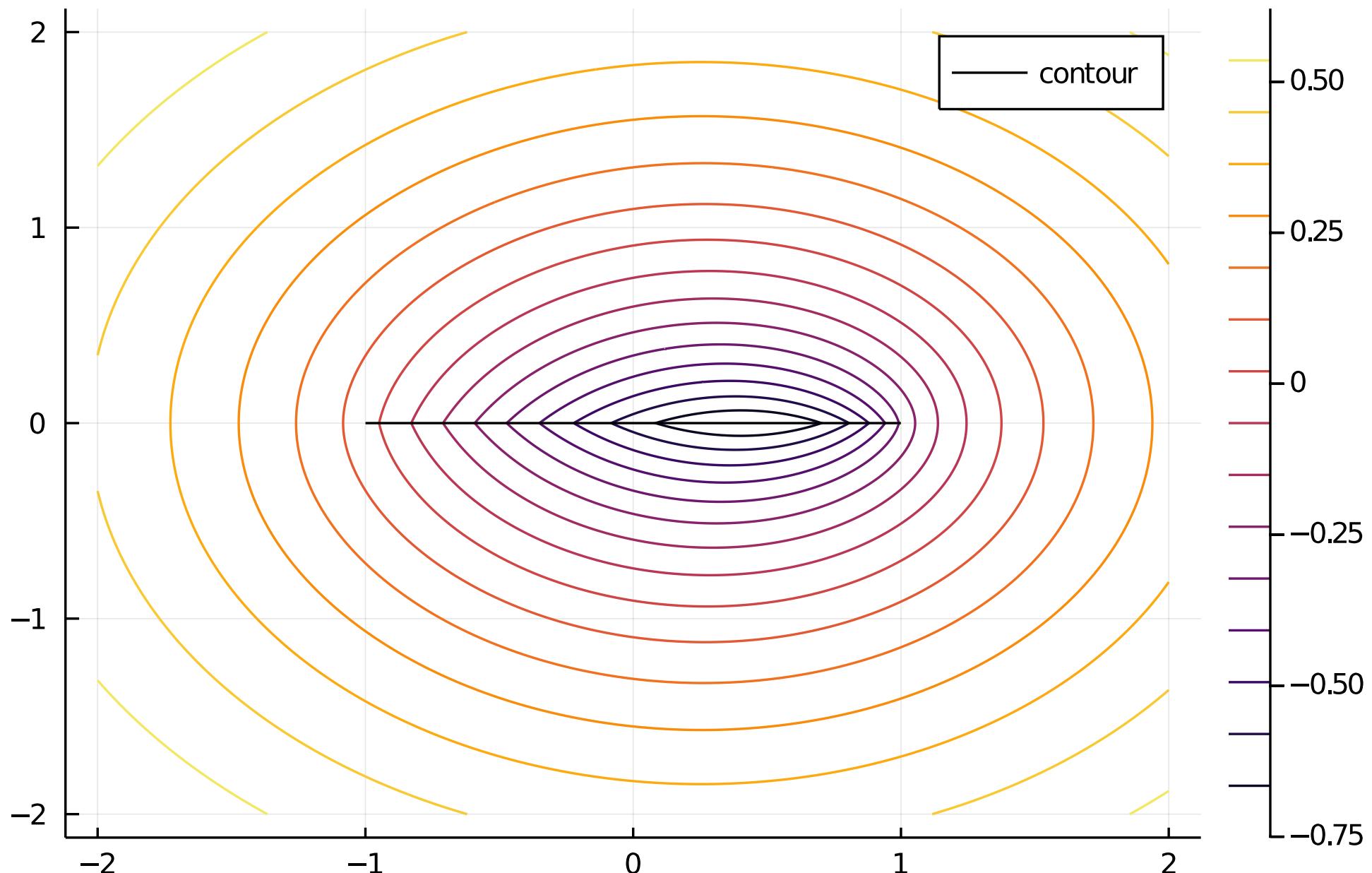
```
f = sqrt(1-t^2)*exp(t)
```

```
u = z -> logkernel(f, z) # logkernel(f, z) calculates 1/pi * \int
f(t)*log|t-z| dt
```

```
xx = yy = -2:0.01:2
```

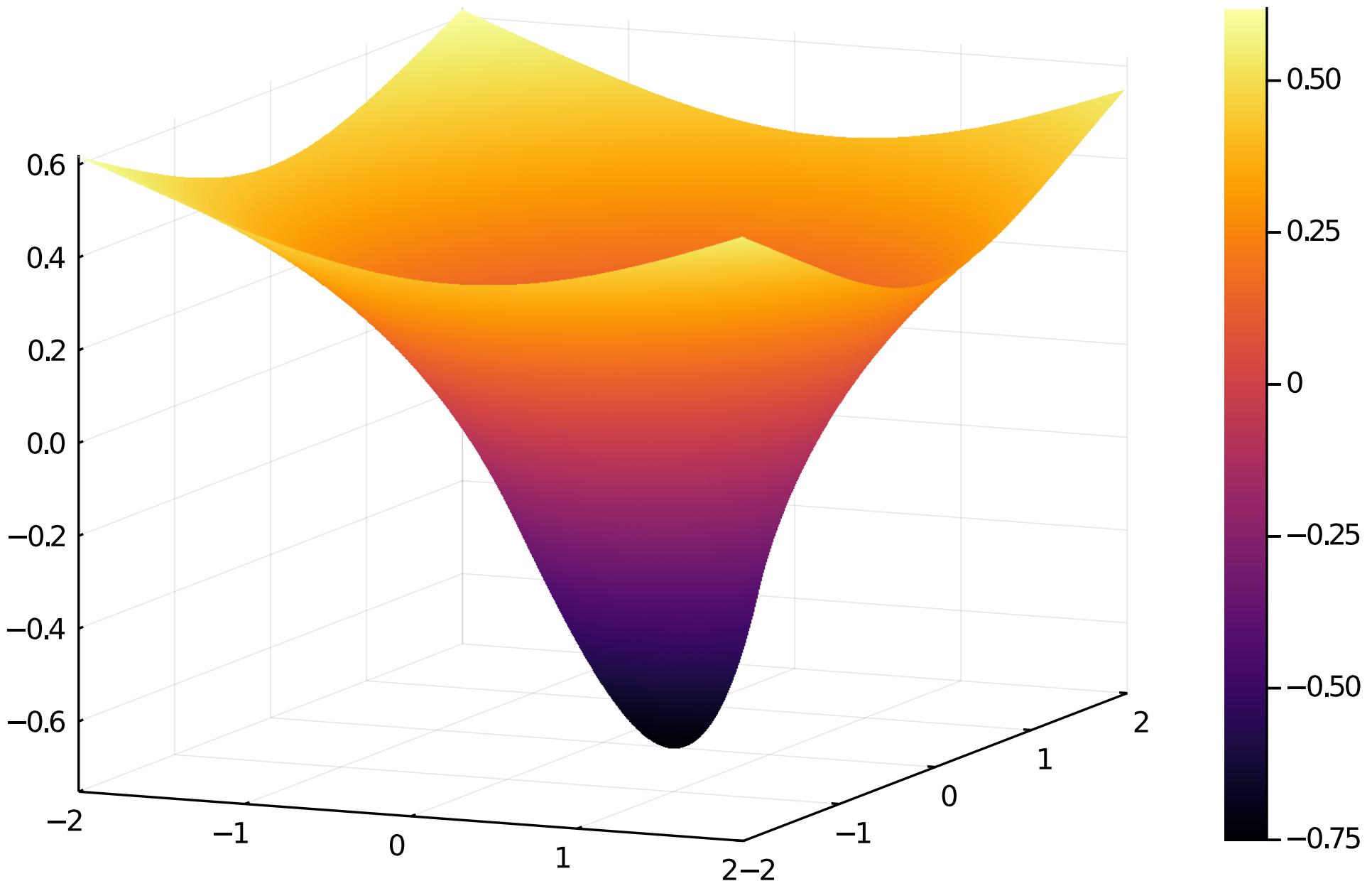
```
U = u.(xx' .+ im*yy)
```

```
contour(xx, yy, U)
plot!(domain(t); color=:black, label="contour")
```



Or this can also be seen from a surface plot:

`surface(xx, yy, U)`



For $z \notin (-\infty, b]$ the fact that u is harmonic (solves Laplace's equation) can also be seen since u is the real part of an analytic function:

$$u(z) = \Re Mf(z) \quad \text{for} \quad Mf(z) := \frac{1}{\pi} \int_a^b f(t) \log(z - t) dt$$

Note that the integrand avoids the branch cut of $\log z$. To extend this to $z \in (-\infty, a]$ (or more generally, $z \notin [a, \infty)$), we can use the alternative expression

$$u(z) = \Re \tilde{M}f(z) \quad \text{for} \quad \tilde{M}f(z) := \frac{1}{\pi} \int_a^b f(t) \log(t - z) dt$$

which follows from $\log|z - t| = \log|t - z|$.

1.2 Evaluating logarithmic singular integrals

To evaluate $Lf(z)$ we evaluate $Mf(z)$. Following our claim that the right way to represent analytic functions is by their behaviour at singularities/branch cuts, we will accomplish this by investigating the jumps of $Mf(z)$. This allows us to reduce it to computing a Cauchy transform.

Lemma (Jumps of M) Suppose f satisfies the conditions of Plemelj. Then $Mf(z)$ satisfies the following

1. Analyticity: M is analytic off $(-\infty, b]$
2. Regularity: M has weaker than pole singularities
3. Asymptotics:

$$Mf(z) = \frac{1}{\pi} \int_a^b f(t) dt \log z + o(1) \quad \text{as } z \rightarrow \infty$$

4. Jump: for $x < a$ we have

$$M^+f(x) - M^-f(x) = 2i \int_a^b f(t) dt$$

and for $a < x < b$ we have

$$M^+f(x) - M^-f(x) = 2i \int_x^b f(t) dt$$

Proof

Analyticity property (1) follows immediately from definition: we have for $z \notin (-\infty, b]$

$$\frac{d}{dz} Mf(z) = \frac{1}{\pi} \int_a^b f(t) \frac{1}{z-t} dt = -2i\mathcal{C}_{[a,b]}f(z)$$

The regularity property (2) follows from the expression as an antiderivative: we know $\mathcal{C}f(z)$ has weaker than pole singularities and integrating only makes them better. To derive the asymptotics (3) we use

$$\log(z-t) = \log z + \log(1-t/z)$$

Finally we get to the jumps. For any point $c > b$ we write

$$Mf(z) = -2i \int_c^z \mathcal{C}f(\zeta)d\zeta + Mf(c)$$

For $x < a$ we can deform the contour just above/below the interval giving us

$$M^\pm f(x) = Mf(c) - 2i \int_c^b \mathcal{C}f(t)dt - 2i \int_b^a \mathcal{C}^\pm f(t)dt - 2i \int_a^x \mathcal{C}f(t)dt$$

so that

$$(M^+ - M^-)f(x) = -2i \int_b^a (\mathcal{C}^+ - \mathcal{C}^-)f(t)dt = 2i \int_a^b f(t)dt$$

Similarly for $a < x < b$ we have

$$M^\pm f(x) = Mf(c) - 2i \int_c^b \mathcal{C}f(t)dt - 2i \int_b^x \mathcal{C}^\pm f(t)dt$$

which gives

$$(M^+ - M^-)f(x) = -2i \int_b^x (\mathcal{C}^+ - \mathcal{C}^-)f(t)dt = 2i \int_x^b f(t)dt.$$



We can construct a function that satisfies the same 4 properties using the Cauchy transform. By uniqueness they must be the same.

Theorem (Log kernel as Cauchy) Suppose f satisfies the conditions of Plemelj and define the (negative) indefinite integral of f via

$$F(x) := \int_x^b f(t)dt.$$

Then

$$Mf(z) = \frac{\log(z-a)}{\pi} \int_a^b f(t)dt + 2i\mathcal{C}_{[a,b]}F(z)$$

Proof

Define

$$\phi(z) := \frac{\log(z-a)}{\pi} \int_a^b f(t)dt + 2i\mathcal{C}_{[a,b]}F(z)$$

This satisfies conditions (1–4) above. Therefore $\phi(z) - Mf(z)$ is continuous hence analytic on $(-\infty, a)$ and (a, b) , has weaker than pole singularities at a and b so analytic there too: it is entire. The logarithmic growth at infinity cancels therefore it is zero by Liouville.



1.3 Examples of logarithmic singular integrals

We look at 3 examples of computing logarithmic singular integrals: $x/\sqrt{1-x^2}$, 1, and $1/\sqrt{1-x^2}$

1.3.1 Example 1

Consider $f(x) = x/\sqrt{1-x^2}$ and we want to compute

$$Lf(z) = \frac{1}{\pi} \int_{-1}^1 \frac{t}{\sqrt{1-t^2}} \log|t-z| dt = \Re \underbrace{\frac{1}{\pi} \int_{-1}^1 \frac{t}{\sqrt{1-t^2}} \log(z-t) dt}_{Mf(z)}$$

We have

$$\frac{d}{dx} \sqrt{1-x^2} = -f(x)$$

hence

$$F(x) = \int_x^1 f(t) dt = \sqrt{1-x^2}.$$

and in particular $F(-1) = 0$. Recall that

$$\mathcal{C}F(z) = \frac{\sqrt{z-1}\sqrt{z+1}-z}{2i}$$

Thus the result from last lecture gives

$$Mf(z) = \frac{1}{\pi} \int_{-1}^1 f(t) dt \log(z+1) + 2i\mathcal{C}F(z) = \sqrt{z-1}\sqrt{z+1} - z$$

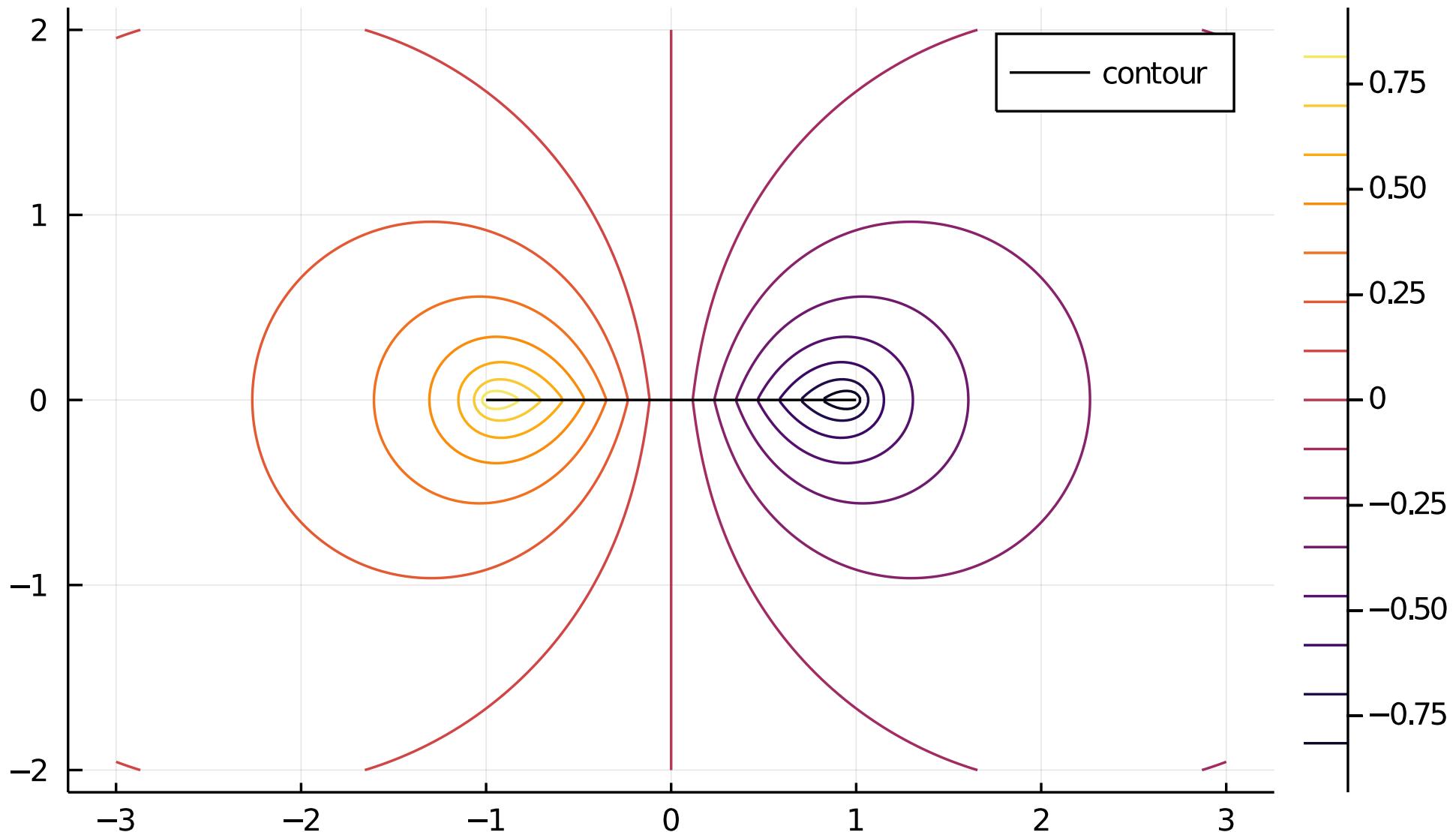
Let's double check the formula:

```
using ApproxFun, SingularIntegralEquations, Plots
x = Fun()
f = x/sqrt(1-x^2)
Mf = z -> sqrt(z-1)sqrt(z+1) - z
Lf = z -> real(Mf(z))
z = 0.1+0.1im
logkernel(f,z), Lf(z)

(-0.09000049991252067, -0.09000049991252063)
```

Here is a plot of the solution:

```
xx = range(-3,3,length=300); yy = range(-2,2,length=200)
Z = Lf.(xx' .+ im*yy)
contour(xx,yy,Z;ratio=1.0)
plot!(-1..1; color=:black, label="contour")
```



1.3.2 Example 2

Consider $f(x) = 1$ and we want to compute

$$Lf(z) = \frac{1}{\pi} \int_{-1}^1 \log|t-z|dt = \Re \underbrace{\frac{1}{\pi} \int_{-1}^1 \log(z-t)dt}_{Mf(z)}$$

We have

$$F(x) = \int_x^1 f(t)dt = 1 - x.$$

We can determine the Cauchy transform by considering $1 - z$ times the known Cauchy transform

$$\mathcal{C}1(z) = \frac{\log(z-1) - \log(z+1)}{2\pi i} = \frac{i}{\pi z} + O(z^{-3})$$

so that

$$\mathcal{C}F(z) = (1-z) \frac{\log(z-1) - \log(z+1)}{2\pi i} + \frac{i}{\pi}$$

Thus the result from last lecture gives

$$\begin{aligned}
M1(z) &= \frac{1}{\pi} \int_{-1}^1 dt \log(z+1) + 2i\mathcal{C}F(z) = \frac{2}{\pi} \log(z+1) + (1-z) \frac{\log(z-1) - \log(z+1)}{\pi} - \frac{2}{\pi} \\
&= \frac{(1-z)\log(z-1) + (1+z)\log(z+1) - 2}{\pi}
\end{aligned}$$

For $-1 < x < 1$ real this gives a particularly simple formula:

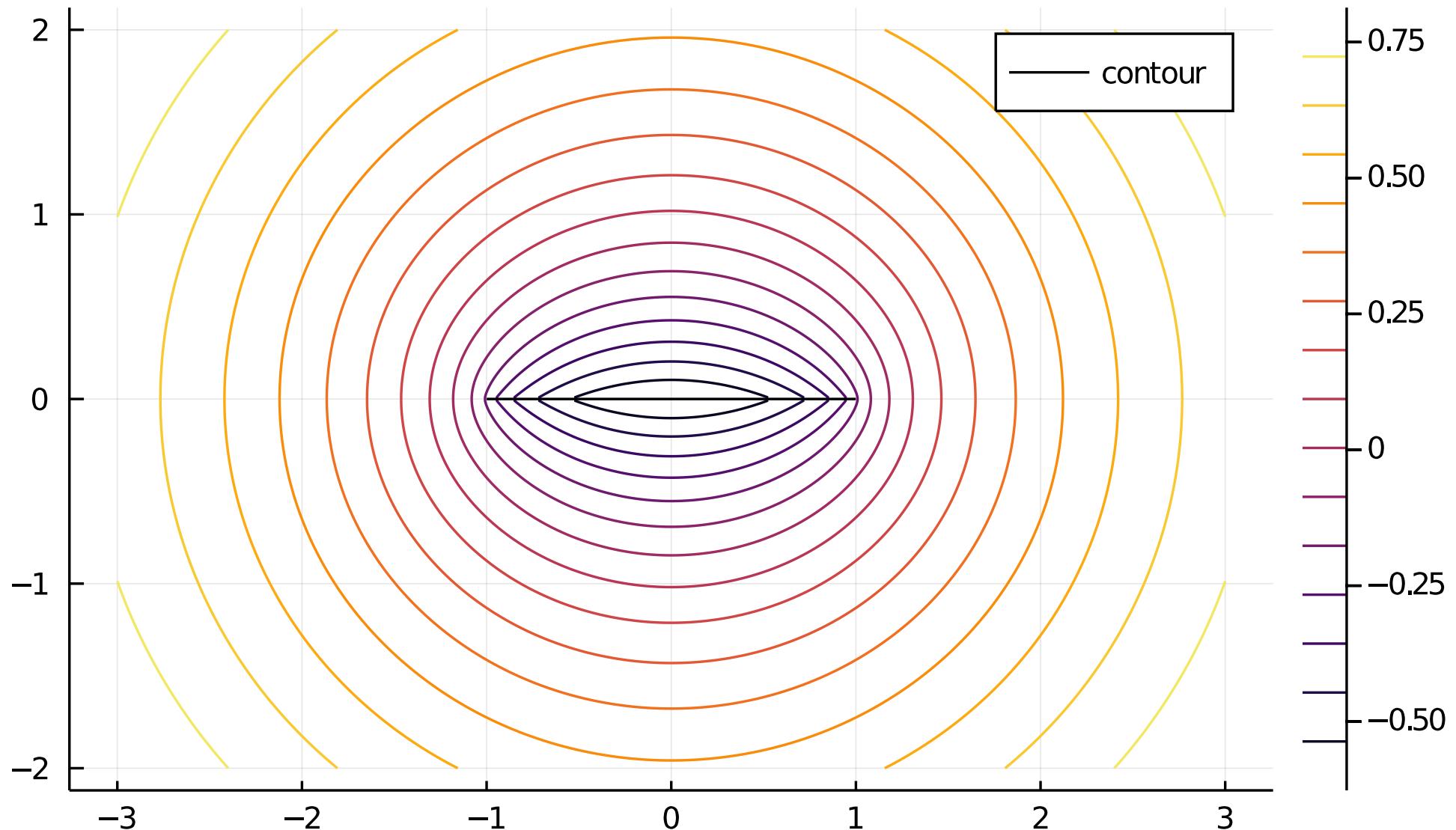
$$L1(x) = \Re M^+ 1(x) = \frac{(1-x)\log(1-x) + (1+x)\log(x+1) - 2}{\pi}$$

We can double check this:

```
f = Fun(1,-1..1)
x = 0.1
logkernel(f,x), ((1-x)*log(1-x) + (1+x)*log(x+1) - 2)/π
(-0.6334313470059203, -0.6334313470059203)
```

Here is a plot of the solution:

```
Lf = z -> real((1-z)*log(z-1) + (1+z)*log(z+1) - 2)/π
Z = Lf.(xx' .+ im*yy)
contour(xx,yy,Z;ratio=1.0)
plot!(-1..1; color=:black, label="contour")
```



1.3.3 Example 3

Consider $f(x) = 1/\sqrt{1-x^2}$. We have

$$F(x) = \arccos x = 2\arctan \frac{\sqrt{1-x}}{\sqrt{1+x}}$$

where the second version can be verified by differentiation, using $\arctan' x = 1/(1+x^2)$. One can determine M by indefinite integration of

$$\mathcal{C}f(z) = \frac{i}{2\sqrt{z-1}\sqrt{z+1}}$$

but we prefer to start with an ansatz and verify the solution. Namely, consider

$$\phi(z) := \frac{\log(\sqrt{z-1} + \sqrt{z+1})}{i}.$$

First, this is analytic off $(-\infty, 1]$, as for z in the upper-half plane we have $\sqrt{z-1}$ and $\sqrt{z+1}$ are in the upper-right quadrant: we never cross the branch cut of $\log z$. Similar argument holds for z in the lower-half plane. On $x \in (-\infty, -1]$ it has the jump

$$\begin{aligned} \phi_+(x) - \phi_-(x) &= \frac{\log(i(\sqrt{1-x} + \sqrt{-x-1})) - \log(-i(\sqrt{1-x} + \sqrt{-x-1}))}{i} \\ &= 2\arg(i(\sqrt{1-x} + \sqrt{-x-1})) = \pi \end{aligned}$$

For $x \in (-1, 1)$ we have

$$\begin{aligned}\phi_+(x) - \phi_-(x) &= \frac{\log(i\sqrt{1-x} + \sqrt{1-x}) - \log(-i\sqrt{1-x} + \sqrt{1-x})}{i} \\ &= 2 \arg(i\sqrt{1-x} + \sqrt{-x-1}) = 2\arctan \frac{\sqrt{1-x}}{\sqrt{1+x}} = F(x).\end{aligned}$$

Finally we have

$$\phi(z) = \frac{\log z}{2i} - i \log 2 + o(1)$$

We therefore have from Plemelj that

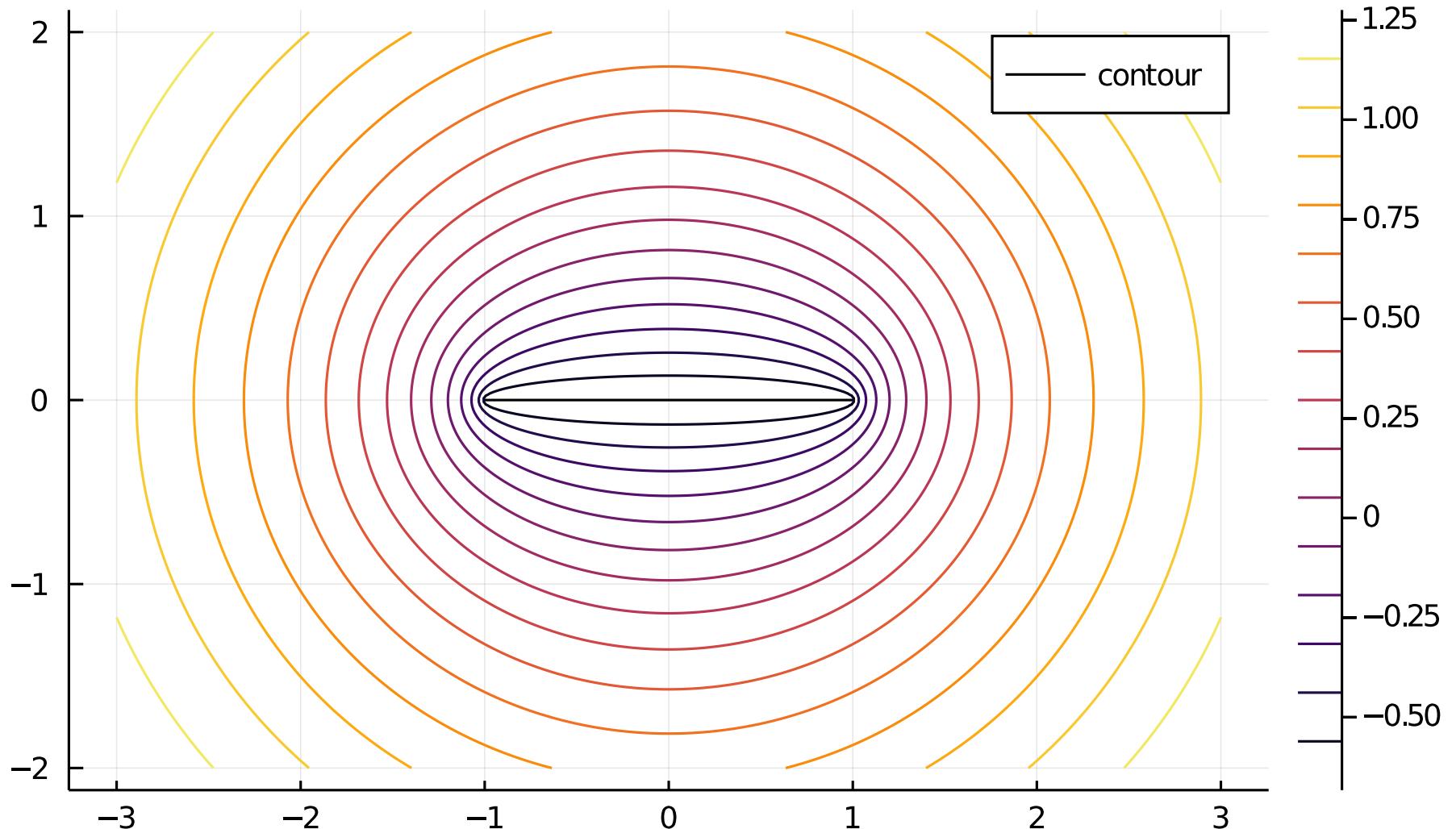
$$\mathcal{C}F(z) = \phi(z) - \frac{\log(z+1)}{2i} + i \log 2$$

and

$$Mf(z) = \frac{F(-1)}{\pi} \log(z+1) + 2i\mathcal{C}F(z) = 2 \log(\sqrt{z-1} + \sqrt{z+1}) - 2 \log 2$$

Here is a plot of the solution:

```
Lf = z -> real(2log(sqrt(z-1) + sqrt(z+1))) - 2log(2))
Z = Lf.(xx' .+ im*yy)
contour(xx,yy,Z;ratio=1.0)
plot!(-1..1; color=:black, label="contour")
```



Note that it is clearly visible that it is constant on the contour, which follows from:

$$\begin{aligned}
 Lf(x) &= \Re Mf(x) = 2\Re \log(i\sqrt{1-x} + \sqrt{1+x}) - 2\log 2 = 2\log \sqrt{1-x+1+x} - 2\log 2 \\
 &= -\log 2
 \end{aligned}$$