# Applied Complex Analysis (2021)

## 1 Lecture 23: Riemann-Hilbert problems

Let $\Gamma$ be the unit circle or real line (or more generally, a set of general contours, but we won't pursue that in this course). Given functions $f$ and $g$ defined on $\Gamma$, a (scalar) Riemann–Hilbert problem consists of finding a function $\Psi(z)$ with left/right limits $\Psi_\pm(x) = \lim_{\epsilon \to 0} \Psi(x \pm \mathrm{i}\epsilon)$, satisfying the following conditions:

1. Analyticity: $\Psi(z)$ analytic in $\bar{\mathbb{C}} \backslash \Gamma$
2. Asymptotics: $\lim_{z\to\infty} \Psi(z) = C$
3. Regularity: $\Psi(z)$ has weaker than pole singularities everywhere
4. Jump: $\Psi_+(x) - g(x)\Psi_-(x) = f(x)$ for $x \in \Gamma$

Numerous applications! See [Trogdon & Olver 2015]. Here are some classical applications:

1. Ideal fluid flow
2. Solving integral equations via Weiner–Hopf factorization
3. Spectral analysis of Schrödinger operators

More recently, non-classical applications have arisen from integrable systems:

4. Solutions to Painlevé equations
5. Random matrix eigenvalue statistics
6. Asymptotics of orthogonal polynomials
7. Solving partial differential equations like the Korteweg–de Vries (KdV) equation describing shallow water waves

$$u_t + 6uu_x + u_{xxx} = 0$$

We tackle the solution to an RH problem similar to a differential equation: first find the homogeneous solution then use that to reduce inhomogeneous problems to something similar:

1. Homogeneous problems: $f = 0$
2. Inhomogeneous problems: $f \neq 0$

### 1.0.1 Homogeneous Riemann–Hilbert problems on the real line

Let's assume $f$ is zero and $C = 1$, that is we wish to solve

$$\Phi_+(\zeta) = g(\zeta)\Phi_-(\zeta) \qquad \text{and} \qquad \Phi(\infty) = 1$$

Formally, taking logs of both sides reduces this to a subtractive RH problem:

$$\log \Phi_+(\zeta) - \log \Phi_-(\zeta) \stackrel{?}{=} \log g(\zeta)$$

Assuming that $g(\zeta) \to 1$ as $\zeta \to \pm\infty$ at a sufficient rate, this motivates the guess

$$\Phi(z) = \mathrm{e}^{\mathcal{C}[\log g](z)}$$

Assuming $\log g(x)$ is "nice", we have guaranteed that this is the unique solution:

**Theorem (Homogeneous solution to RH problem)** Suppose $\log g(\zeta)$ satisfies the conditions of Plemelj on $\Gamma$ (the real line or unit circle), in particular, is continuously differentiable. Then $\Phi(z) = \mathrm{e}^{\mathcal{C}_\Gamma[\log g](z)}$ is the unique solution to the following RH problem:

1. Analyticity: $\Phi(z)$ is analytic off $\Gamma$
2. Asymptotics: $\lim_{z\to\infty} \Phi(z) = 1$
3. Regularity: $\Phi$ has weaker than pole singularities
4. Jump: $\Phi_+(\zeta) = g(\zeta)\Phi_-(\zeta)$ for $\zeta \in \Gamma$

**Proof** (1) follows from definition. (2) follows since $\mathcal{C}[\log g](z) \to 0$. And (4) follows via:

$$\Phi_+(\zeta) = \mathrm{e}^{\mathcal{C}_+[\log g](\zeta)} = \mathrm{e}^{\mathcal{C}_-[\log g](\zeta)+\log g(\zeta)} = \Phi_-(\zeta)g(\zeta)$$

To see uniqueness, observe that we can take the reciprocal of $\Phi$, as it is an exponential of something finite. Thus $\Phi(z)^{-1}$ is also analytic off $\mathbb{R}$. Therefore, if we have another solution $\tilde{\Phi}(z)$ we can consider $r(z) = \tilde{\Phi}(z)\Phi(z)^{-1}$ which satisfies:

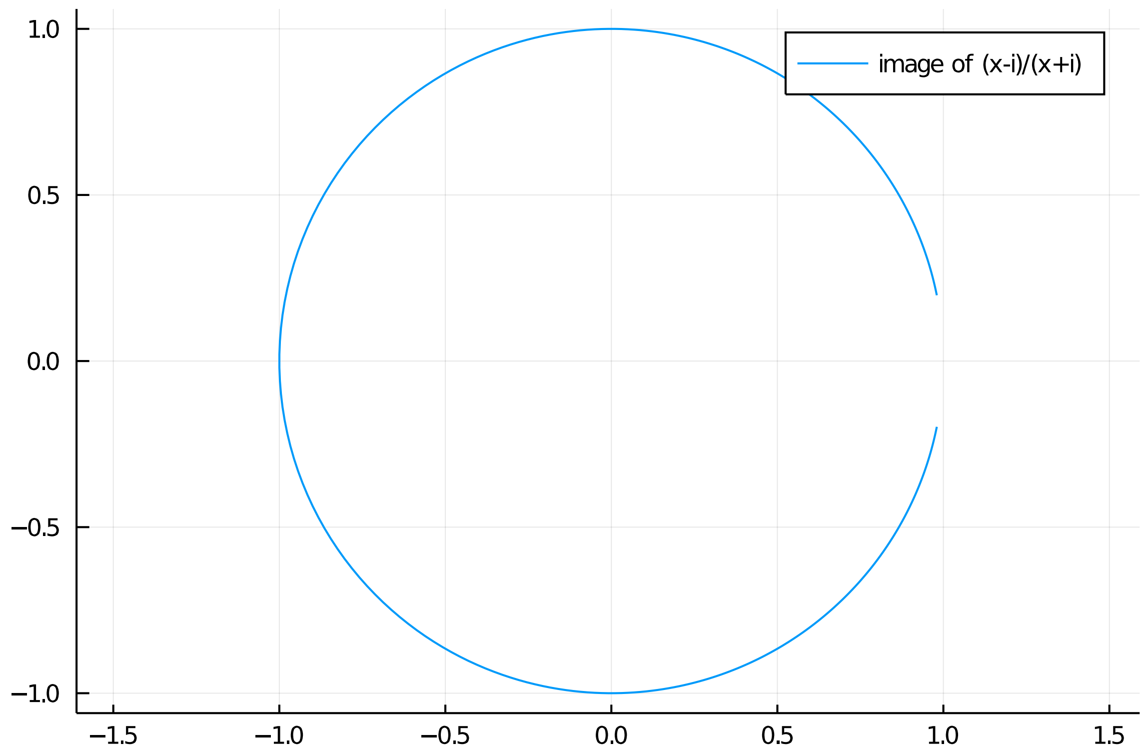
$$r_+(\zeta) = \frac{\tilde{\Phi}_+(\zeta)}{\Phi_+(\zeta)} = \frac{\tilde{\Phi}_-(\zeta)g(\zeta)}{\Phi_-(\zeta)g(\zeta)} = r_-(\zeta)$$

Hence $r(z)$ is entire. since both terms tend to $1$, it must be $r(z) = 1$.

■

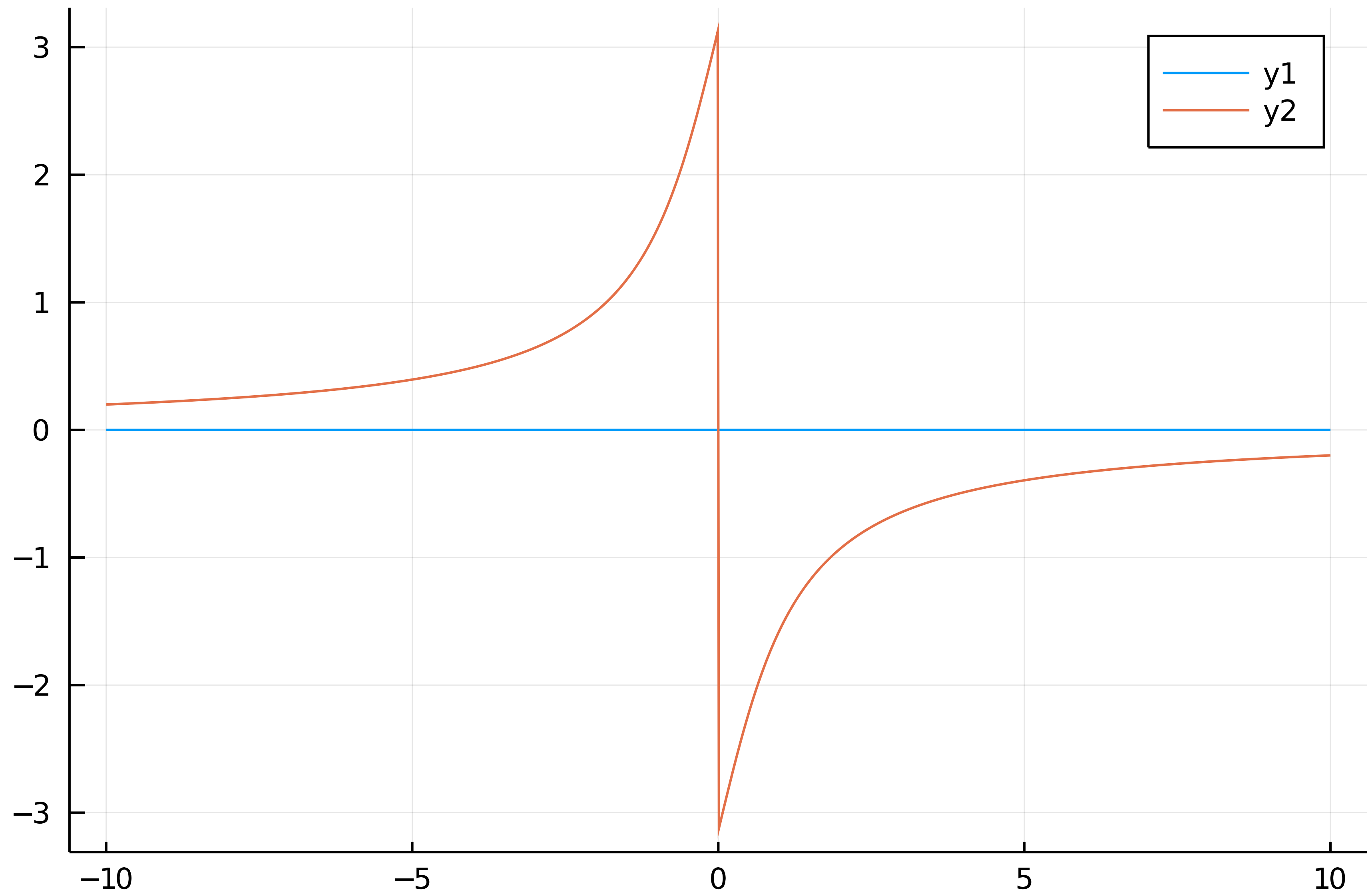
When is $\log g(\zeta)$ nice? For the real line it is necessary that $g(x) = 1 + O(x^{-1})$ at $x \to \pm\infty$. We also need to worry about the image: for example, $g(\zeta) \neq 0$ is required to avoid a singularity. We also need the winding number of the image of $g(\zeta)$ to be zero: otherwise, $\log g(\zeta)$ will extend to another sheet and be discontinuous. For example, if $g(z) = \frac{z-\mathrm{i}}{z+\mathrm{i}}$ it satisfies the right asymptotics, but surrounds the origin:

```
using Plots, ComplexPhasePortrait
g = x -> (x-im)/(x+im)
xx = range(-10.,10.; length=1000)
plot(real.(g.(xx)), imag.(g.(xx)); label="image of (x-i)/(x+i)",
ratio=1.0)
```



Therefore, $\log g(x)$ has a branch cut if we use the standard branch, which breaks the continuity requirement:

```
plot(xx, real.(log.(g.(xx))))
plot!(xx, imag.(log.(g.(xx))))
```

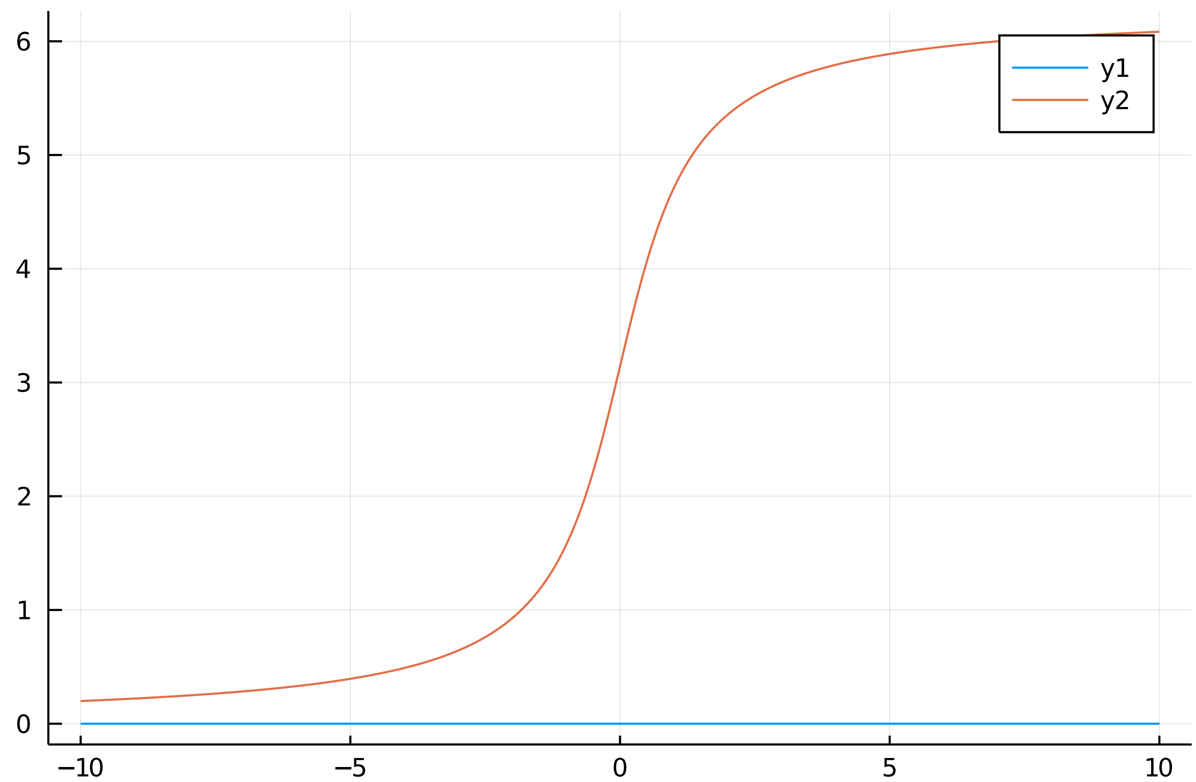


We could have analytically continued $\log g(z)$ using

$$\log_1 z = \begin{cases} \log z & \Im z > 0 \\ \log_+ z & z < 0 \\ \log z + 2\pi \mathrm{i} & \Im z < 0 \end{cases}$$

But then $\lim_{x \to +\infty} \log_1 g(x) = 2\pi\mathrm{i}$:

```
log_1 = z -> imag(z) > 0 ? log(z) : log(z)+2π*im
plot(xx, real.(log_1.(g.(xx))))
plot!(xx, imag.(log_1.(g.(xx))))
```



**Example** Consider

$$g(x) = \frac{x^2+3}{x^2+1} = 1 + O(x^{-1})$$

Before we do anything *Verify that the winding number is zero*.

We provide two methods for calculating $\Phi$: one guesses the solution, the other uses the solution formula.

*Method 1 (Guess and check / kernel factorization)* If we can guess the solution, we can check it satisfies the right criteria. Factoring $g$ we see immediately that

$$g(x) = \left(\frac{x+\sqrt{3}\mathrm{i}}{x+\mathrm{i}}\right)\left(\frac{x-\sqrt{3}\mathrm{i}}{x-\mathrm{i}}\right)$$

Note that the first factor is analytic in the upper half-plane. The second factor is analytic in the lower half-plane *and* we can take its reciprocal. Therefore we can guess the solution is

$$\Phi(z) = \begin{cases} \frac{z+\sqrt{3}\mathrm{i}}{z+\mathrm{i}} & \Im z > 0 \\ \frac{z-\mathrm{i}}{z-\sqrt{3}\mathrm{i}} & \Im z < 0 \end{cases}$$

This satisfies the four conditions:

1. Analyticity: $\Phi$ is analytic off $\mathbb{R}$
2. Asymptotics: $\lim_{z\to\infty} \Phi(z) = 1$
3. Weaker than pole singularities
4. It has the right jump

$$g(x)\Phi_-(x) = \frac{x^2+3}{x^2+1}\frac{x-\mathrm{i}}{x-\sqrt{3}\mathrm{i}} = \frac{x+\sqrt{3}\mathrm{i}}{x+\mathrm{i}} = \Phi_+(x)$$

This function is indeed analytic off the real line.

*Method 2 (evaluate explicit formula)* This is real valued and positive, hence the winding number of its image is zero. We have

$$\log g(x) = \log(\frac{x+\sqrt{3}\mathrm{i}}{x+\mathrm{i}}\frac{x-\sqrt{3}\mathrm{i}}{x-\mathrm{i}})$$

Because they are complex conjugates, we know $\log a\bar{a} = \log a + \log \bar{a}$ as $[1, \bar{a}, a]$ lies in the same half plane for $a = \frac{s+\sqrt{3}\mathrm{i}}{s+\mathrm{i}}$, therefore we can expand:

$$\log g(x) = \log\frac{x+\sqrt{3}\mathrm{i}}{x+\mathrm{i}} + \log\frac{x-\sqrt{3}\mathrm{i}}{x-\mathrm{i}}$$

Now we note that $\log\frac{x+3\mathrm{i}}{x+\mathrm{i}}$ is analytic in the upper-half plane, therefore it's Cauchy transform, by Plemelj, is

$$\mathcal{C}\left[\log\frac{x+\sqrt{3}\mathrm{i}}{x+\mathrm{i}}\right](z) = \begin{cases}\log\frac{z+\sqrt{3}\mathrm{i}}{z+\mathrm{i}} & \Im z > 0\\ 0 & \Im z < 0\end{cases}$$

Similarly,

$$\mathcal{C}\left[\log\frac{x-\sqrt{3}\mathrm{i}}{x-\mathrm{i}}\right](z) = \begin{cases}-\log\frac{z-\sqrt{3}\mathrm{i}}{z-\mathrm{i}} & \Im z < 0\\ 0 & \Im z > 0\end{cases}$$

We thus get:

$$\Phi(z) = \mathrm{e}^{\mathcal{C}\log g(z)} = \mathrm{e}^{\begin{cases}\log\frac{z+\sqrt{3}\mathrm{i}}{z+\mathrm{i}} & \Im z > 0\\ -\log\frac{z-\sqrt{3}\mathrm{i}}{z-\mathrm{i}} & \Im z < 0\end{cases}} = \begin{cases}\frac{z+\sqrt{3}\mathrm{i}}{z+\mathrm{i}} & \Im z > 0\\ \frac{z-\mathrm{i}}{z-\sqrt{3}\mathrm{i}} & \Im z < 0\end{cases}$$

### 1.0.2 Inhomogeneous Riemann–Hilbert problem

Consider now the Riemann–Hilbert problem with zero at infinity:

$$\Psi_+(x) - g(x)\Psi_-(x) = f(x) \qquad \text{and} \qquad \Psi(\infty) = 0$$

Consider writing $\Psi(z) = \Phi(z)Y(z)$. Then we can reduce the Riemann–Hilbert problem to a subtractive problem:

$$\Psi_+(x) - g(x)\Psi_-(x) = \Phi_+(x)(Y_+(x) - Y_-(x)) = f(x) \qquad \text{and} \qquad Y(\infty) = 0$$

Thus once we have $\Phi$, we can determine $Y$ as a Cauchy transform, and thence construct $\Psi$.

What if we don't have decay? Just add in a constant times $\Phi$:

**Corollary** Suppose $\log g$ satisfies the conditions of Plemelj's theorem. Then

$$\Psi(z) = \Phi(z)\mathcal{C}_{\mathbb{R}}\left[\frac{f}{\Phi_+}\right](z) + D\Phi(z)$$

is the unique solution to

$$\Psi_+(\zeta) - g(\zeta)\Psi_-(\zeta) = f(\zeta) \qquad \text{and} \qquad \Psi(\infty) = D$$

*Example* Suppose $f(x) = \frac{\mathrm{i}}{\mathrm{i}-x}$.

To decompose this as a sum of things analytic in half planes, we just use partial fraction expansion!

$$\begin{aligned}\frac{\mathrm{i}}{\mathrm{i}-x}\frac{x+\mathrm{i}}{x+\mathrm{i}\sqrt{3}} &= \frac{\mathrm{i}}{\mathrm{i}-x}\frac{2\mathrm{i}}{\mathrm{i}(1+\sqrt{3})} + \frac{\mathrm{i}}{\mathrm{i}(1+\sqrt{3})}\frac{\mathrm{i}(1-\sqrt{3})}{x+\mathrm{i}\sqrt{3}} \\ &= \underbrace{\frac{-2\mathrm{i}}{x-\mathrm{i}}\frac{1}{1+\sqrt{3}}}_{-Y_-(x)} + \underbrace{\frac{\mathrm{i}}{1+\sqrt{3}}\frac{1-\sqrt{3}}{x+\mathrm{i}\sqrt{3}}}_{Y_+(x)}\end{aligned}$$

Thus we get

$$Y(z) = \begin{cases} \frac{\mathrm{i}}{1+\sqrt{3}}\frac{1-\sqrt{3}}{z+\mathrm{i}\sqrt{3}} & \Im z > 0 \\ \frac{2\mathrm{i}}{z-\mathrm{i}}\frac{1}{1+\sqrt{3}} & \Im z < 0 \end{cases}$$

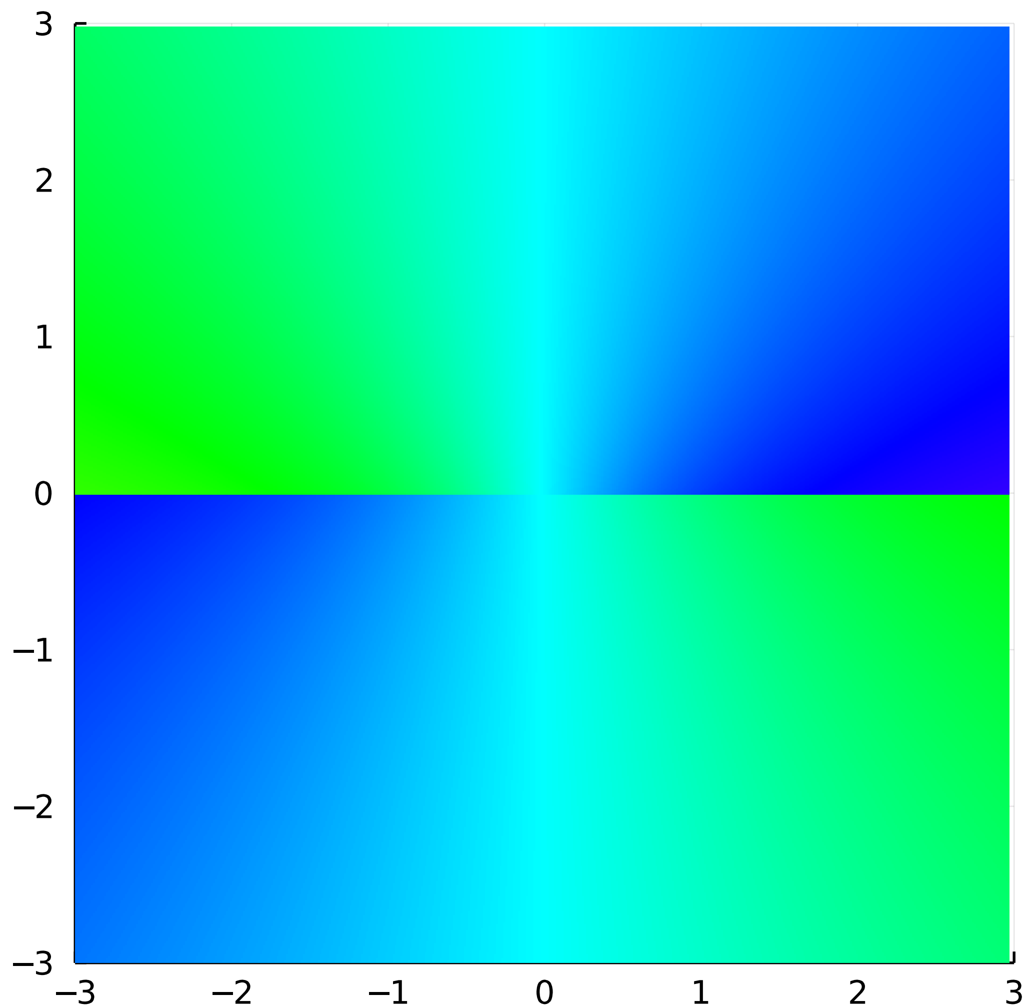
Let's double check: We thus have the solution:

$$\begin{aligned}\Psi(z) = \Phi(z)\mathcal{C}_{\mathbb{R}}\left[\frac{f}{\Phi_+}\right](z) &= \begin{cases} \frac{\mathrm{i}}{1+\sqrt{3}}\frac{1-\sqrt{3}}{z+\mathrm{i}\sqrt{3}}\frac{z+\sqrt{3}\mathrm{i}}{z+\mathrm{i}} & \Im z > 0 \\ \frac{2\mathrm{i}}{z-\mathrm{i}}\frac{1}{1+\sqrt{3}}\frac{z-\mathrm{i}}{z-\sqrt{3}\mathrm{i}} & \Im z < 0 \end{cases} \\ &= \begin{cases} \frac{\mathrm{i}}{1+\sqrt{3}}\frac{1-\sqrt{3}}{z+\mathrm{i}} & \Im z > 0 \\ \frac{2\mathrm{i}}{1+\sqrt{3}}\frac{1}{z-\sqrt{3}\mathrm{i}} & \Im z < 0 \end{cases}\end{aligned}$$

Let's verify it's the right thing:

1. It's analytic off $\mathbb{R}$

```
Ψ =  z -> imag(z) > 0 ? im*(1-sqrt(3))/(1+sqrt(3))/(z+im) :
                         2im/((z-sqrt(3)*im)*(1+sqrt(3)))
phaseplot(-3..3, -3..3, Ψ)
```



2. It goes to zero at infinity

```
Ψ(300.0+300.0im)
```

```
-0.00044657951463O4169 - 0.00044509586175789O6im
```

3. It satisfies the right jump:

$$
\begin{aligned}
\Psi_+(x) - g(x)\Psi_-(x) &= \frac{\mathrm{i}}{1+\sqrt{3}}\frac{1-\sqrt{3}}{x+\mathrm{i}} - \frac{x^2+3}{x^2+1}\frac{2\mathrm{i}}{1+\sqrt{3}}\frac{1}{x-\sqrt{3}\mathrm{i}} \\
&= \frac{\mathrm{i}(x-\mathrm{i})}{1+\sqrt{3}}\frac{1-\sqrt{3}}{x^2+1} - \frac{x+\sqrt{3}\mathrm{i}}{x^2+1}\frac{2\mathrm{i}}{1+\sqrt{3}} \\
&= \frac{1}{x^2+1}\frac{1}{1+\sqrt{3}}\left(\mathrm{i}(1-\sqrt{3})x + 1 - \sqrt{3} - 2\mathrm{i}x + 2\sqrt{3}\right) \\
&= \frac{1}{x^2+1}(-\mathrm{i}x+1) = \frac{\mathrm{i}}{\mathrm{i}-x}
\end{aligned}
$$

```
f = x -> im/(im-x)
g = x -> (x^2+3)/(x^2+1)
Ψ(0.1+eps()im) - Ψ(0.1-eps()im)*g(0.1) - f(0.1)
```

```
-1.1102230246251565e-16 + 2.7755575615628914e-17im
```