Applied Complex Analysis (2021)

1 Mastery Sheet

This problem sheet is about the relationship between truncated Chebyshev expansions and Chebyshev interpolants. If a function f is analytic on $[-1,1]$, it can be expanded as a Chebyshev series:

$$f = \sum_{k=0}^{\infty} c_k T_k(x).$$

A truncated Chebyshev expansion $f_n(x)$ is an approximation to f obtained by truncating the Chebyshev series after n terms

$$\sum_{k=0}^{n-1} c_k T_k(x) =: f_n(x),$$

see Lecture 19. A Chebyshev interpolant $p_n(x)$ interpolates f at the roots of $T_n(x)$,

$$x_k = \cos\left(\frac{2k-1}{2n}\pi\right), \qquad k = 1, \ldots, n,$$

i.e.,

$$p_n(x_k) = f(x_k), \qquad k = 1, \ldots, n.$$

Since p_n is a polynomial of degree $n-1$,

$$p_n(x) = \sum_{k=0}^{n-1} a_k T_k(x)$$

1. Give a formula for the coefficients c_k.

2. Show that for $k = 1, \ldots, n$,

$$T_m(x_k) = -T_{2n\pm m}(x_k) = T_{4n\pm m}(x_k) = -T_{6n\pm m}(x_k) = T_{8n\pm m}(x_k) = -T_{10n\pm m}(x_k) = \cdots,$$

for $0 \le m \le n$.

3. Show that

$$a_0 = \sum_{j=0}^{\infty} c_{2jn}(-1)^j$$

and for $1 \le m \le n-1$

$$a_m = c_m + \sum_{j=1}^{\infty} (-1)^j \left[c_{2jn+m} + c_{2jn-m}\right].$$

4. Show that

$$p_n(x) - f(x) = \sum_{k=0}^{n-1} (a_k - c_k)\, T_k(x) - (f(x) - f_n(x)).$$

5. Obtain an upper bound on $|p_n(x) - f(x)|$ for $x \in [-1, 1]$.
