

**Problem Sheet 2**

**Problem 1.1** Use the Plemelj formulæ to calculate the following:

- 1)  $\frac{1}{2\pi i} \int_{-1}^1 \frac{\sqrt{1-t^2}}{(1+t^2)(t-z)} dt$  for  $z \notin [-1, 1]$ .
- 2)  $\frac{1}{2\pi i} \int_{-1}^1 \frac{1}{(t-z)(2+t)} dt$  for  $z \notin [-1, 1]$ .
- 3)  $\int_{-1}^1 \frac{t}{(t-x)\sqrt{1-t^2}} dt$  for  $-1 < x < 1$

**Problem 1.2** Find all solutions  $\phi(z)$  analytic on  $\mathbb{C} \setminus [-1, 1]$  with weaker than pole singularities satisfying the following, where  $-1 < x < 1$ :

- 1)  $\phi_+(x) + \phi_-(x) = 1$  and  $\phi(\infty) = 0$
- 2)  $\phi_+(x) + \phi_-(x) = 0$  and  $\phi(\infty) = 1$
- 3)  $\phi_+(x) + \phi_-(x) = \sqrt{1-x^2}$  and  $\phi(\infty) = 0$
- 4)  $\phi_+(x) + \phi_-(x) = \frac{1}{x^2+1}$ ,  $\phi(\infty) = 0$  and  $\lim_{z \rightarrow \infty} z\phi(z) = 0$ .

**Problem 1.3** Use Plemelj formulæ to find all solutions  $u(x)$  defined on  $[-1, 1]$  to the following, where  $-1 < x < 1$ :

- 1)  $\frac{1}{\pi} \int_{-1}^1 \frac{u(t)}{t-x} dt = \frac{x}{\sqrt{1-x^2}}$ .
- 2)  $\frac{1}{\pi} \int_{-1}^1 \frac{u(t)}{t-x} dt = \frac{1}{2+x}$  where  $u$  is bounded at the right-endpoint.

In the following problems, use only the definitions

$$\log z = \int_1^z \frac{1}{\zeta} d\zeta \quad \text{for } z \notin (-\infty, 0]$$

$$\log_{\pm} x = \log(x \pm i\epsilon) \quad \text{for } x \in (-\infty, 0]$$

For example, do not use  $\log z = \log |z| + i \arg z$  as we need to prove it first! You can use the result from lectures that  $\log z^{-1} = -\log z$  and  $\log_{\pm} x = \log |x| \pm i\pi$ .

**Problem 2.1** Show that  $\log(ab) = \log a + \log b$  provided that the closed contour defined by the oriented line segments  $\gamma = [1, 1/b] \cup [1/b, a] \cup [a, 1]$  does not surround the origin. Show that if  $\gamma$  surrounds the origin counter-clockwise then

$$\log(ab) = \log a + \log b + 2\pi i$$

and if  $\gamma$  surrounds the origin clockwise then

$$\log(ab) = \log a + \log b - 2\pi i$$

Use  $\log_{\pm} x$  to express the equivalent formulæ for all cases where  $\gamma$  passes through the origin.

**Problem 2.2** Show that  $\overline{\log z} = \log \bar{z}$ . Use this to show that  $\log z = \log |z| + i \arg z$ . (Hint: deform along an arc for the imaginary part.)

**Problem 2.3** Consider  $\log_1 z$  defined off  $[0, \infty)$  by

$$\log_1 z := \begin{cases} \log z & \text{if } \operatorname{Im} z > 0 \\ \log_+ z & \text{if } \operatorname{Im} z = 0 \text{ and } z < 0 \\ \log z + 2\pi i & \text{if } \operatorname{Im} z < 0 \end{cases}$$

show that  $\log_1 z$  is analytic in  $\mathbb{C} \setminus [0, \infty)$  and for  $x > 0$

$$\lim_{\epsilon \rightarrow 0} \log_1(x - i\epsilon) = \lim_{\epsilon \rightarrow 0} \log_1(x + i\epsilon) + 2\pi i.$$

(This is the analytic continuation of  $\log z$  over its branch cut.)

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Consider the Cauchy transform over the unit circle  $\mathbb{T} = \{z : |z| = 1\}$ :

$$\mathcal{C}_{\mathbb{T}} f(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{\zeta - z} d\zeta$$

Denote the limit from the left/right (inside/outside) as

$$\mathcal{C}_{\mathbb{T}}^+ f(\zeta) = \lim_{\epsilon \rightarrow 0} \mathcal{C}_{\mathbb{T}} f((1 - \epsilon)\zeta)$$

$$\mathcal{C}_{\mathbb{T}}^- f(\zeta) = \lim_{\epsilon \rightarrow 0} \mathcal{C}_{\mathbb{T}} f((1 + \epsilon)\zeta)$$

**Problem 3.1** Assuming that  $f$  is analytic in an annulus containing  $\mathbb{T}$ , show that  $\mathcal{C}_{\mathbb{T}} f(z)$  satisfies the following (Plemelj formulæ on the circle):

- 1)  $\mathcal{C}_{\mathbb{T}} f(z)$  is analytic in  $\bar{\mathbb{C}} \setminus \mathbb{T}$
- 2)  $\mathcal{C}_{\mathbb{T}}^+ f(\zeta) - \mathcal{C}_{\mathbb{T}}^- f(\zeta) = f(\zeta)$
- 3)  $\mathcal{C}_{\mathbb{T}} f(\infty) = 0$ .

**Problem 3.2** Show that it is the unique function  $\phi(z)$  satisfying

- 1)  $\phi(z)$  is analytic in  $\bar{\mathbb{C}} \setminus \mathbb{T}$
- 2)  $\phi^+(\zeta) - \phi^-(\zeta) = f(\zeta)$  where  $f$  is analytic in an annulus containing  $\mathbb{T}$ ,
- 3)  $\phi(\infty) = 0$ .

where

$$\phi^+(\zeta) = \lim_{\epsilon \rightarrow 0} \phi((1 - \epsilon)\zeta),$$

$$\phi^-(\zeta) = \lim_{\epsilon \rightarrow 0} \phi((1 + \epsilon)\zeta)$$

You can assume that  $\phi^\pm(\zeta)$  converges uniformly.

**Problem 3.3** What is  $\mathcal{C}_{\mathbb{T}}[\diamond^k](z)$ ? What about  $\operatorname{Re} \mathcal{C}_{\mathbb{T}}^-[\diamond^k](\zeta)$  and  $\operatorname{Im} \mathcal{C}_{\mathbb{T}}^-[\diamond^k](\zeta)$ ? Express your answers separately for negative and non-positive  $k$  and justify the answers using 3.1 and 3.2.

**Problem 3.4** Construct the solution to ideal fluid flow around a circle, that is, to find a function  $v(x, y)$  satisfying

- 1)  $v_{xx} + v_{yy} = 0$  for  $x^2 + y^2 > 1$ .
- 2)  $v(x, y) \sim y \cos \theta - x \sin \theta$
- 3)  $v(x, y) = 0$  for  $x^2 + y^2 = 1$ .

We investigate the solutions to  $\phi^+(x) - c\phi^-(x) = f(x)$  on  $[-1, 1]$ .

**Problem 4.1** Describe all solutions  $\kappa(z)$  to

- 1)  $\kappa(z)$  is analytic off  $[-1, 1]$ .
- 2)  $\kappa(\infty) = 0$ .
- 3)  $\kappa$  has weaker than pole singularities at  $\pm 1$ .
- 4)  $\kappa^+(x) - e^{i\theta}\kappa^-(x) = 0$  for  $-1 < x < 1$ .

Show that your answer satisfies all three properties.

**Problem 4.2** Construct all solutions  $\phi(z)$  to

- 1)  $\phi(z)$  is analytic off  $[-1, 1]$ .
- 2)  $\phi(\infty) = 0$ .
- 3)  $\phi$  has weaker than pole singularities at  $\pm 1$ .
- 4)  $\phi^+(x) - e^{i\theta}\phi^-(x) = f(x)$ .

in terms of a Cauchy transform involving  $f$ . You can assume that  $f$  is smooth (infinitely-differentiable on  $[-1, 1]$ ), and use the fact that

$$\mathcal{C}_{[-1,1]}[(1 - \diamond)^\alpha(1 + \diamond)^\beta f(\diamond)](z)$$

is bounded for all  $z$  if  $\alpha, \beta > 0$  when  $f$  is smooth.

**Problem 4.3** Let  $c \in \mathbb{C}$ . Repeat Problem 4.1 and Problem 4.2 for  $\phi^+(x) - c\phi^-(x) = f(x)$ .

We investigate the solutions to  $\phi^+(x) + \phi^-(x) = f(x)$  on two intervals  $(-1, -a) \cup (a, 1)$ , where  $0 < a < 1$ .

**Problem 5.1** Show that  $\kappa(z) = \frac{1}{\sqrt{z-1}\sqrt{z-a}\sqrt{z+a}\sqrt{z+1}}$  satisfies the following properties:

- 1)  $\kappa(z)$  is analytic off  $[-1, -a] \cup [1, a]$ .
- 2)  $\kappa^+(x) + \kappa^-(x) = 0$  for  $a < |x| < 1$ .
- 3)  $\kappa$  has weaker than pole singularities everywhere.
- 4)  $\kappa(\infty) = 0$ .

**Problem 5.2** Describe all solutions  $\psi(z)$  satisfying:

- 1)  $\psi(z)$  is analytic off  $[-1, -a] \cup [1, a]$ .
- 2)  $\psi^+(x) + \psi^-(x) = 0$  for  $a < |x| < 1$ .
- 3)  $\psi$  has weaker than pole singularities everywhere.
- 4)  $\psi(\infty) = 0$ .

**Problem 5.3** Assuming  $f(x)$  is smooth (infinitely differentiable on  $[-1, -a] \cup [a, 1]$ ), express in terms of a Cauchy transform involving  $f$  all solutions  $\phi(z)$  to the following:

- 1)  $\phi(z)$  is analytic off  $[-1, -a] \cup [1, a]$ .
  - 2)  $\phi^+(x) + \phi^-(x) = f(x)$  for  $a < |x| < 1$ .
  - 3)  $\phi$  has weaker than pole singularities everywhere.
  - 4)  $\phi(\infty) = 0$ .
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**Problem 6.1** Describe the limiting distribution of electric charges under the potential  $V(x) = x^4$ , that is, the limit of

$$\frac{d\lambda_k}{dt} = \sum_{\substack{j=1 \\ j \neq k}}^N \frac{1}{\lambda_k - \lambda_j} - V'(\lambda_k)$$

for  $k = 1, \dots, N$  as  $N \rightarrow \infty$ . (Hint: scale  $\lambda_k$  appropriately.)

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