M3M6: Methods of Mathematical Physics (2018)

Problem Sheet 1

Problem 1.1 Compute the residues of

1) $\operatorname{Res}_{z=ae^{\mathrm{i}\pi/4}} \frac{z^3 \sin z}{z^4+a^4}$ where a > 0

2) $\operatorname{Res}_{z=1} \frac{z+1}{(z^2-1)^2}$

3) $\operatorname{Res}_{z=a} \frac{z^2 e^z}{z^3 - a^3}$ where $a \neq 0$

Problem 1.2 Use contour integration to find the values of

 $1) \qquad \int_0^{2\pi} \frac{1}{5 - 4\cos\theta} \,\mathrm{d}\theta$

2) $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos \theta} d\theta$

3) $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)} \, \mathrm{d}x$

4) $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9} \, \mathrm{d}x$

5) $\int_{-\infty}^{\infty} \frac{1}{x+i} \, \mathrm{d}x$

 $6) \qquad \int_{-\infty}^{\infty} \frac{\sin 2x}{x^2 + x + 1} \, \mathrm{d}x$

7) $\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + 4} \, \mathrm{d}x$

 $8) \qquad \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 1} \, \mathrm{d}x$

9) $\int_{-\infty}^{\infty} \frac{\cos ax - \cos bx}{x^2} dx \quad \text{where} \quad a, b > 0$

10) $\int_0^{2\pi} (\cos \theta)^n d\theta$ where n = 0, 1, 2, ... (Hint: consider even and odd n separately.)

Problem 2.1 By integrating around a rectangular contour with vertices at $\pm R$ and $\pi i \pm R$ and letting $R \to \infty$, show that:

$$\int_0^\infty \operatorname{sech} x \, \mathrm{d}x = \frac{\pi}{2}$$

where sech $x = \frac{2}{e^{-x} + e^x}$.

Problem 2.2 Show that the Fourier transform of sech x satisfies

$$\int_{-\infty}^{\infty} e^{ikx} \operatorname{sech} x \, \mathrm{d}x = \pi \operatorname{sech} \frac{\pi k}{2}$$

Let

$$A = \begin{pmatrix} 1 & 2 & -1 \\ -2 & 2 & 1 \\ 0 & 1 & 4 \end{pmatrix}$$

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Problem 3.1 Use Gershgorin's theorem to bound the eigenvalues of A.

Problem 3.2 Recall that the eigenvalues of A and A^{\top} are the same. Use this fact to find a bound on the eigenvalues based on the absolute column sums.

Problem 3.3 Design a circular contour surrounding the spectrum of A.

Problem 4.1 Given $A \in \mathbb{R}^{n \times n}$ that is symmetric positive definite (that is, all eigenvalues of A are real and greater than zero) and $\mathbf{u}_0, \mathbf{v}_0 \in \mathbb{R}^n$, write a contour integral solution to the second-order linear constant coefficient ODE:

$$\boldsymbol{u}''(t) = A\boldsymbol{u}(t)$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0$$

$$\boldsymbol{u}'(0) = \boldsymbol{v}_0$$

Problem 4.2 Was the restriction to symmetric positive definite matrices necessary? Why or why not?

Problem 5.1 Suppose that $g(\theta)$ has absolutely summable Fourier coefficients, that is,

$$g(\theta) = \sum_{k=-\infty}^{\infty} g_k e^{ik\theta}$$
 where $\sum_{k=-\infty}^{\infty} |f_k| < \infty$.

Show that the periodic trapezium rule satisfies

$$\frac{1}{n}\sum_{j=0}^{n-1}g(\theta_j) = \dots + g_{-2n} + g_{-n} + g_0 + g_n + g_{2n} + \dots$$

where $\theta_j = \frac{2\pi j}{n}$. Hint: use the geometric series to simplify $\sum_{j=0}^{n-1} e^{ik\theta_j}$.

Problem 5.2 Suppose that $g(\theta) = f(e^{i\theta})$ where f(z) is holomorphic in an annulus $\{z : R^{-1} < |z| < R\}$. Prove that the periodic trapezium rule converges exponentially fast:

$$\frac{1}{n} \sum_{j=0}^{n-1} g(\theta_j) \to \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta.$$

Problem 5.3 Find an upper bound for the error

$$\left| \frac{1}{n} \sum_{j=0}^{n-1} g(\theta_j) - \frac{1}{2\pi} \int_0^{2\pi} g(\theta) d\theta \right|$$

for $g(\theta) = \frac{1}{2 - \cos \theta}$.