M3M6: Methods of Mathematical Physics (2018)

Problem Sheet 2

Problem 1.1 Use the Plemelj formulæ to calculate the following:

1) $\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{(1+t^2)(t-z)} \mathrm{d}t$ for $z \notin [-1,1]$.

2) $\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{1}{(t-z)(2+t)} \mathrm{d}t$ for $z \notin [-1,1]$.

3) $⨍_{-1}^{1} \frac{t}{(t-x)\sqrt{1-t^2}} \mathrm{d}t$ for $-1 < x < 1$

Problem 1.2 Find all solutions $\phi(z)$ analytic on $\bar{\mathbb{C}} \backslash [-1,1]$ with weaker than pole singularities satisfying the following, where $-1 < x < 1$:

1) $\phi_+(x) + \phi_-(x) = 1$ and $\phi(\infty) = 0$

2) $\phi_+(x) + \phi_-(x) = 0$ and $\phi(\infty) = 1$

3) $\phi_+(x) + \phi_-(x) = \sqrt{1-x^2}$ and $\phi(\infty) = 0$

4) $\phi_+(x) + \phi_-(x) = \frac{1}{x^2+1}$, $\phi(\infty) = 0$ and $\lim_{z\to\infty} z\phi(z) = 0$.

Problem 1.3 Use Plemelj formulæ to find all solutions $u(x)$ defined on $[-1,1]$ to the following, where $-1 < x < 1$:

1) $\frac{1}{\pi} ⨍_{-1}^{1} \frac{u(t)}{t-x} \mathrm{d}t = \frac{x}{\sqrt{1-x^2}}$.

2) $\frac{1}{\pi} ⨍_{-1}^{1} \frac{u(t)}{t-x} \mathrm{d}t = \frac{1}{2+x}$ where u is bounded at the right-endpoint.

In the following problems, use only the definitions

$$\log z = \int_1^z \frac{1}{\zeta} \mathrm{d}\zeta \qquad \text{for} \qquad z \notin (-\infty, 0]$$

$$\log_\pm x = \log(x \pm \mathrm{i}\epsilon) \qquad \text{for} \qquad x \in (-\infty, 0]$$

For example, do not use $\log z = \log|z| + \mathrm{i} \arg z$ as we need to prove it first! You can use the result from lectures that $\log z^{-1} = -\log z$ and $\log_\pm x = \log|x| \pm \mathrm{i}\pi$.

Problem 2.1 Show that $\log(ab) = \log a + \log b$ provided that the closed contour defined by the oriented line segments $\gamma = [1, 1/b] \cup [1/b, a] \cup [a, 1]$ does not surround the origin. Show that if γ surrounds the origin counter-clockwise then

$$\log(ab) = \log a + \log b + 2\pi\mathrm{i}$$

and if γ surrounds the origin clockwise then

$$\log(ab) = \log a + \log b - 2\pi\mathrm{i}$$

Use $\log_\pm x$ to express the equivalent formulæ for all cases where γ passes through the origin.

Problem 2.2 Show that $\overline{\log z} = \log \bar{z}$. Use this to show that $\log z = \log|z| + \mathrm{i} \arg z$. (Hint: deform along an arc for the imaginary part.)

Problem 2.3 Consider $\log_1 z$ defined off $[0,\infty)$ by

$$\log_1 z := \begin{cases} \log z & \text{if Im } z > 0 \\ \log_+ z & \text{if Im } z = 0 \text{ and } z < 0 \\ \log z + 2\pi\mathrm{i} & \text{if Im } z < 0 \end{cases}$$

show that $\log_1 z$ is analytic in $\mathbb{C}\backslash[0,\infty)$ and for $x > 0$

$$\lim_{\epsilon\to 0} \log_1(x - \mathrm{i}\epsilon) = \lim_{\epsilon\to 0} \log_1(x + \mathrm{i}\epsilon) + 2\pi\mathrm{i}.$$

(This is the analytic continuation of $\log z$ over its branch cut.)

Consider the Cauchy transform over the unit circle $\mathbb{T} = \{z : |z| = 1\}$:

$$\mathcal{C}_{\mathbb{T}} f(z) = \frac{1}{2\pi\mathrm{i}} \oint \frac{f(\zeta)}{\zeta - z} \mathrm{d}\zeta$$

Denote the limit from the left/right (inside/outside) as

$$\mathcal{C}^+_{\mathbb{T}} f(\zeta) = \lim_{\epsilon\to 0} \mathcal{C}_{\mathbb{T}} f((1-\epsilon)\zeta)$$
$$\mathcal{C}^-_{\mathbb{T}} f(\zeta) = \lim_{\epsilon\to 0} \mathcal{C}_{\mathbb{T}} f((1+\epsilon)\zeta)$$

Problem 3.1 Assuming that f is analytic in an annulus containing $\mathbb{T}$, show that $\mathcal{C}_{\mathbb{T}} f(z)$ satisfies the following (Plemelj formulæ on the circle):

1) $\mathcal{C}_{\mathbb{T}} f(z)$ is analytic in $\bar{\mathbb{C}}\backslash\mathbb{T}$
2) $\mathcal{C}^+_{\mathbb{T}} f(\zeta) - \mathcal{C}^-_{\mathbb{T}} f(\zeta) = f(\zeta)$
3) $\mathcal{C}_{\mathbb{T}} f(\infty) = 0$.

Problem 3.2 Show that it is the unique function $\phi(z)$ satisfying

1) $\phi(z)$ is analytic in $\bar{\mathbb{C}}\backslash\mathbb{T}$
2) $\phi^+(\zeta) - \phi^-(\zeta) = f(\zeta)$ where f is analytic in an annulus containing $\mathbb{T}$,
3) $\phi(\infty) = 0$.

where

$$\phi^+(\zeta) = \lim_{\epsilon\to 0} \phi((1-\epsilon)\zeta),$$
$$\phi^-(\zeta) = \lim_{\epsilon\to 0} \phi((1+\epsilon)\zeta)$$

You can assume that $\phi^\pm(\zeta)$ converges uniformly.

Problem 3.3 What is $\mathcal{C}_{\mathbb{T}}[\diamond^k](z)$? What about Re $\mathcal{C}^-_{\mathbb{T}}[\diamond^k](\zeta)$ and Im $\mathcal{C}^-_{\mathbb{T}}[\diamond^k](\zeta)$? Express your answers separately for negative and non-positive k and justify the answers using 3.1 and 3.2.

Problem 3.4 Construct the solution to ideal fluid flow around a circle, that is, to find a function $v(x, y)$ satisfying

1) $v_{xx} + v_{yy} = 0$ for $x^2 + y^2 > 1$.
2) $v(x, y) \sim y\cos\theta - x\sin\theta$
3) $v(x, y) = 0$ for $x^2 + y^2 = 1$.

We investigate the solutions to $\phi^+(x) - c\phi^-(x) = f(x)$ on $[-1, 1]$.

Problem 4.1 Describe all solutions $\kappa(z)$ to

1) $\kappa(z)$ is analytic off $[-1, 1]$.
2) $\kappa(\infty) = 0$.
3) κ has weaker than pole singularities at ± 1.
4) $\kappa^+(x) - \mathrm{e}^{\mathrm{i}\theta}\kappa^-(x) = 0$ for $-1 < x < 1$.

Show that your answer satisfies all three properties.

Problem 4.2 Construct all solutions $\phi(z)$ to

1) $\phi(z)$ is analytic off $[-1, 1]$.
2) $\phi(\infty) = 0$.
3) ϕ has weaker than pole singularities at ± 1.
4) $\phi^+(x) - \mathrm{e}^{\mathrm{i}\theta}\phi^-(x) = f(x)$.

in terms of a Cauchy transform involving f. You can assume that f is smooth (infinitely-differentiable on $[-1, 1]$), and use the fact that

$$\mathcal{C}_{[-1,1]}[(1-\diamond)^\alpha(1+\diamond)^\beta f(\diamond)](z)$$

is bounded for all z if $\alpha, \beta > 0$ when f is smooth.

Problem 4.3 Let $c \in \mathbb{C}$. Repeat Problem 4.1 and Problem 4.2 for $\phi^+(x) - c\phi^-(x) = f(x)$.

We investigate the solutions to $\phi^+(x) + \phi^-(x) = f(x)$ on two intervals $(-1, -a) \cup (a, 1)$, where $0 < a < 1$.

Problem 5.1 Show that $\kappa(z) = \frac{1}{\sqrt{z-1}\sqrt{z-a}\sqrt{z+a}\sqrt{z+1}}$ satisfies the following properties:

1) $\kappa(z)$ is analytic off $[-1, -a] \cup [1, a]$.
2) $\kappa^+(x) + \kappa^-(x) = 0$ for $a < |x| < 1$.
3) κ has weaker than pole singularities everywhere.
4) $\kappa(\infty) = 0$.

Problem 5.2 Describe all solutions $\psi(z)$ satisfying:

1) $\psi(z)$ is analytic off $[-1,-a] \cup [1,a]$.
2) $\psi^+(x) + \psi^-(x) = 0$ for $a < |x| < 1$.
3) ψ has weaker than pole singularities everywhere.
4) $\psi(\infty) = 0$.

Problem 5.3 Assuming $f(x)$ is smooth (infinitely differentiable on $[-1,-a] \cup [a,1]$), express in terms of a Cauchy transform involving f all solutions $\phi(z)$ to the following:

1) $\phi(z)$ is analytic off $[-1,-a] \cup [1,a]$.
2) $\phi^+(x) + \phi^-(x) = f(x)$ for $a < |x| < 1$.
3) ϕ has weaker than pole singularities everywhere.
4) $\phi(\infty) = 0$.

Problem 6.1 Describe the limiting distribution of electric charges under the potential $V(x) = x^4$, that is, the limit of

$$\frac{\mathrm{d}\lambda_k}{\mathrm{d}t} = \sum_{\substack{j=1 \\ j\neq k}}^{N} \frac{1}{\lambda_k - \lambda_j} - V'(\lambda_k)$$

for $k = 1, \ldots, N$ as $N \to \infty$. (Hint: scale λ_k appropriately.)
