

Problem Sheet 3

This problem set considers the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + O(x^{n-1})$$

where $\alpha > -1$, which are orthogonal with respect to

$$\langle f, g \rangle_\alpha = \int_0^\infty f(x)g(x)x^\alpha e^{-x} dx$$

Problem 1.1 Show that the Rodrigues formula holds:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} [x^{\alpha+n} e^{-x}].$$

Problem 1.2 Show that the derivatives form a hierarchy: we have

$$\begin{aligned} \frac{dL_n^{(\alpha)}}{dx} &= -L_{n-1}^{(\alpha+1)}(x), \\ \frac{d}{dx} [x^{\alpha+1} e^{-x} L_n^{(\alpha+1)}(x)] &= (n+1)x^\alpha e^{-x} L_{n+1}^{(\alpha)}(x), \\ xL_n^{(\alpha+1)}(x) &= -(n+1)L_{n+1}^{(\alpha)}(x) + (n+\alpha+1)L_n^{(\alpha)}(x), \\ L_n^{(\alpha)}(x) &= L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x). \end{aligned}$$

Problem 1.3 Combine the results from Problem 1.2 to determine the three-term recurrence relationship and the top 5×5 block of the Jacobi operator.

Problem 2.1 Represent the ordinary differential operator

$$u'(x) - xu(x) \quad \text{for} \quad x \geq 0$$

as an operator on the coefficients of u in a weighted Laguerre expansion:

$$u(x) = \sum_{k=0}^{\infty} u_k e^{-x/2} L_k(x) = e^{-x/2} (L_0(x) \mid L_1(x) \mid \cdots) \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix},$$

where the range of the operator is specified in $e^{-x/2} (L_0^{(1)}(x) \mid L_1^{(1)}(x) \mid \cdots)$.

Problem 2.2 Show that the Laguerre polynomials are eigenfunctions of a Sturm–Liouville problem, that is, find $\lambda_n^{(\alpha)}$ so that

$$\frac{e^x}{x^\alpha} \frac{d}{dx} \left[x^{\alpha+1} e^{-x} \frac{dL_n^{(\alpha)}}{dx} \right] = \lambda_n^{(\alpha)} L_n^{(\alpha)}(x)$$

Re-express this as an ODE with polynomial coefficients.

This problem considers Cauchy and Logarithmic transforms of Laguerre polynomials. Recall from lectures that

$$\mathcal{C}_{[0,\infty)}[e^{-\diamond}](z) = -\frac{e^{-z}\text{Ei } z}{2\pi i}$$

for the exponential integral

$$\text{Ei } z = \int_{-\infty}^z \frac{e^{\zeta}}{\zeta} d\zeta.$$

Problem 3.1 What is

$$\mathcal{C}_{[0,\infty)}[\diamond e^{-\diamond} L_1^{(1)}(\diamond)](z) = \frac{1}{2\pi i} \int_0^\infty \frac{x^\alpha e^{-x} L_2^{(1)}(x)}{x-z} dx$$

in terms of $\text{Ei } z$?

Problem 3.2 What is

$$\frac{1}{\pi} \int_0^\infty e^{-x} L_2(x) \log |z-x| dx$$

in terms of the real and imaginary parts of $\text{Ei } z$?

Consider the incomplete Gamma function:

$$\Gamma(\alpha, z) = \int_z^\infty \zeta^{\alpha-1} e^{-\zeta} d\zeta,$$

where the contour of integration is two straight line segments from z to 1 to ∞ , hence this has a branch cut on $(-\infty, 0]$.

Problem 4.1 For $x < 0$, show that

$$\Gamma_+(\alpha, x) - e^{2i\pi\alpha} \Gamma_-(\alpha, x) = (1 - e^{2i\pi\alpha}) \Gamma(\alpha)$$

where $\Gamma(\alpha) = \Gamma(\alpha, 0) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is the Gamma function and

$$\Gamma_\pm(\alpha, x) = \lim_{\epsilon \rightarrow 0} \Gamma(\alpha, x \pm i\epsilon).$$

Problem 4.2 For $-1 < \alpha < 0$, express

$$\mathcal{C}_{[0,\infty)}[\diamond^\alpha e^{-\diamond}](z) = \frac{1}{2\pi i} \int_0^\infty \frac{x^\alpha e^{-x}}{x-z} dx$$

in terms of $\Gamma(-\alpha, -z)$ and $(-z)^\alpha e^z$ using Plemelj's lemma.
