M3M6: Methods of Mathematical Physics (2017)

Problem Sheet 4

This set of problems investigates the analyticity properties of the half-Fourier transform. Recall the definitions

$$u_{\mathrm{R}}(x) = \begin{cases} u(x) & x \geq 0 \\ 0 & \text{otherwise} \end{cases},$$

$$u_{\mathrm{L}}(x) = \begin{cases} u(x) & x < 0 \\ 0 & \text{otherwise} \end{cases},$$

The Fourier transform

$$\hat{u}(s) = \int_{-\infty}^{\infty} u(x) \mathrm{e}^{-\mathrm{i}xs} \, \mathrm{d}x,$$

and the inverse Fourier transform

$$u(x) = \frac{1}{2\pi} \int_{-\infty+\mathrm{i}\gamma}^{\infty+\mathrm{i}\gamma} \hat{u}(s) \mathrm{e}^{\mathrm{i}sx} \, \mathrm{d}s$$

where the choice of γ is dictated by the analyticity of $\hat{u}(z)$.

Problem 1.1 Consider $f(x) = x$. Without computing it, in what strip, if any, is $\hat{f}(z)$ analytic? For what choice of γ, if any, does the inverse Fourier transform recover x?

Problem 1.2 Consider $f(x) = \frac{1}{1+\mathrm{e}^x}$. Without computing it, in what strip, if any, is $\hat{f}(z)$ analytic? For what choice of γ, if any, does the inverse Fourier transform recover x?

Problem 1.3 Consider $f(x) = \mathrm{e}^{2x}$.Without computing it, in what strip, if any, is $\widehat{f_{\mathrm{R}}}(z)$ analytic? For what choice of γ, if any, does the inverse Fourier transform recover x?

Problem 1.4 Consider $f(x) = x$. Without computing it, in what strip, if any, is $\widehat{f_{\mathrm{L}}}(z)$ analytic? For what choice of γ, if any, does the inverse Fourier transform recover x?

Problem 1.5 Calculate the Fourier transforms in the above problems and confirm your statements.

Problem 1.6 What is the Fourier transform of $\delta(x)$, i.e., the Dirac delta function satisfying

$$\int_{-\infty}^{\infty} \delta(x) g(x) \, \mathrm{d}x = g(0)$$

for smooth test functions g. Where is it analytic?

This set of problems considers extensions of the Wiener–Hopf method to functions that do not decay, degenerate integral equations, and to integro-differential equations. Please be precise on which contour the Riemann–Hilbert problem is solved on and the inverse Fourier transforms taken.

Problem 2.1 The function $u(x)$ is bounded by a polynomial for all $x \geq 0$, including as $x \to \infty$, and satisfies the integral equation for $x \geq 0$,

$$u(x) + \frac{3}{2} \int_0^{\infty} \mathrm{e}^{-|x-t|} u(t) \, \mathrm{d}t = 1 + \alpha x$$

where α is a positive constant. Find $u(x)$ for $x \geq 0$. Hint: set up a Riemann–Hilbert problem on the contour $\mathbb{R} + \mathrm{i}\gamma$ where $-1 < \gamma < 0$ is arbitrary.

Problem 2.2 The function $u(x)$ is bounded by a polynomial for all $x \geq 0$, including as $x \to \infty$, and satisfies the integral equation for $x \geq 0$,

$$\int_0^\infty \mathrm{e}^{-\alpha|x-t|} u(t)\,\mathrm{d}t = 1 + \alpha x$$

where α is a positive constant. Find $u(x)$ for $x \geq 0$. Hint: If you proceed naïvely, we arrive at a Riemann–Hilbert problem of the form

$$\Phi_+(s) - g(s)\Phi_-(s) = f(s) \qquad \text{and} \qquad \lim_{z\to\infty}(\infty) = 0$$

but where $g(\infty) = 0$ instead of $g(\infty) = 1$. This is not in canonical form, but maybe this example is special. Try writing $\Phi(z) = \kappa(z)Y(z)$ as before but allowing different asymptotic behaviour in κ and Y in the different half planes in a way that they cancel out so that $\lim_{z\to\infty}\Phi(z) = 0$:

$$\kappa(z) = \begin{cases} O(z^{-1}) & \text{Im } z > 0 \\ O(z) & \text{Im } z < 0 \end{cases}$$

$$Y(z) = \begin{cases} O(1) & \text{Im } z > 0 \\ O(z^{-2}) & \text{Im } z < 0 \end{cases}.$$

Problem 2.3 Consider the integral equation

$$u(t) - \lambda \int_0^\infty \mathrm{e}^{-|x-t|} u(t)\,\mathrm{d}t = x$$

where $0 < \lambda < \frac{1}{2}$. Show that, for $x \geq 0$,

$$u(x) = A(\sinh\gamma x + \gamma\cosh\gamma x) + \frac{1}{\gamma^2}\left[x + \left(\gamma - \frac{1}{\gamma}\right)\sinh\gamma x\right]$$

where $\gamma^2 = 1 - 2\lambda$ and A is an arbitrary constant.

Problem 2.4 A bounded, smooth, function $u(x)$ satisfies the integro-differential equation

$$u''(x) - \frac{72}{5}\int_0^\infty \mathrm{e}^{-5|x-t|} u(t)\,\mathrm{d}t = 1 \qquad \text{for} \qquad x \geq 0$$

with $u(0) = 0$.

1) Rewrite the integral equation on the half line in the form:

$$u''_{\mathrm{R}}(x) - \frac{72}{5}\int_{-\infty}^\infty \mathrm{e}^{-5|x-t|} u(t)\,\mathrm{d}t = 1_{\mathrm{R}}(x) + \alpha\delta(x) + p_{\mathrm{L}}(x)$$

for $\alpha = u'(0)$ and a to-be-specified $p(x)$. Here δ is the Dirac delta function, that is,

$$\int_{-\infty}^\infty f(x)\delta(x)\,\mathrm{d}x = f(0).$$

2) Use integration by parts to determine that

$$\widehat{u''_{\mathrm{R}}}(s) = -u'(0) - s^2\hat{u}_{\mathrm{R}}(s).$$

What is $\hat{\delta}(s)$? Use these to translate the equation to Fourier space on a contour $s \in \mathbb{R} + \mathrm{i}\gamma$. What choices of γ are suitable?

3) Define $\Phi(z)$ in terms of $\widehat{p_{\rm L}}(z)$ and $\widehat{u_{\rm R}}(z)$ so that it satisfies the following (non-standard) RH problem

$$\Phi_+(s) - \frac{(s^2+9)(s^2+16)}{s^2+25}\Phi_-(s) = \alpha + \frac{1}{\mathrm{i}s}$$
$$\lim_{\substack{z\to\infty \\ \mathrm{Im}\, z>\gamma}} \Phi(z) = \alpha$$
$$\lim_{\substack{z\to\infty \\ \mathrm{Im}\, z<\gamma}} \Phi(z) = 0.$$

4) Solve the Riemann–Hilbert problem for Φ. Hint: write $\Phi(z) = \kappa(z)Y(z)$ where

$$\kappa(z) = \begin{cases} O(z) & \mathrm{Im}\, z > \gamma \\ O(z^{-1}) & \mathrm{Im}\, z < \gamma \end{cases},$$
$$Y(z) = O(z^{-1}).$$

Hint: $Y(z)$ does not depend on α in the lower-half plane.

5) Recover $u(x)$ by taking the inverse Fourier transform of $\Phi_-(s)$.

This problem set considers analytic properties of solutions to the Hypergeometric equation

$$z(1-z)\frac{\mathrm{d}^2u}{\mathrm{d}z^2} + [c-(a+b+1)z]\frac{\mathrm{d}u}{\mathrm{d}z} - abu = 0$$

using the special solution that is analytic near zero satisfying $u(0)=1$, which we saw had the Taylor series"

$${}_2F_1\left(\begin{matrix} a,b \\ c \end{matrix}; z\right) = \sum_{k=0} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

for the Pochhammer symbol

$$(a)_n := a(a+1)(a+2)\cdots(a+n-1)$$

Problem 3.1 Use the indicial equation to argue that there is also a solution analytic in a neighbourhood of $z=1$, that is, there is a solution with a convergent Taylor series

$$u(z) = \sum_{k=0}^{\infty} u_k(z-1)^k$$

Integrating the ODE on straight lines, what is the domain of analyticity? What is the form of singularity near $z=1$ for the second solution?

Problem 3.2 What are the Taylor coefficients u_k of this solution analytic near $z=1$.

Problem 3.3 Show that $u(z)$ can be expressed as

$$u(z) = {}_2F_1\left(\begin{matrix} \tilde{a},\tilde{b} \\ \tilde{c} \end{matrix}; 1-z\right)$$

for to-be-specified parameters $\tilde{a},\tilde{b},\tilde{c}$. What is the formula for the second solution in terms of ${}_2F_1$?

Problem 3.4 What is the equation number corresponding to Problem 3.3 on the DLMF website (`http://dlmf.nist.gov`)?
