PRACTICE EXAM

May – June 2018

Methods of Mathematical Physics (PRACTICE EXAM)

Time Allowed: 2 Hours for M3 paper; 2.5 Hours for M4/5 paper

This paper has *4 Questions (M3 version); 5 Questions (M4/5 version)*.

Candidates should start their solutions to each question in a new main answer book.

Supplementary books may only be used after the relevant main book(s) are full.

Statistical tables will not be provided.

- DO NOT OPEN THIS PAPER UNTIL THE INVIGILATOR TELLS YOU TO.
- Affix one of the labels provided to each answer book that you use, but DO NOT USE THE LABEL WITH YOUR NAME ON IT.
- Credit will be given for all questions attempted.
- Each question carries equal weight.
- Calculators may not be used.

1. (a) Use the Plemelj formulæ to calculate the following:

$$\frac{1}{2\pi \mathrm{i}} \int_{-1}^{1} \frac{\sqrt{1-t^2}}{(t^2+1)(t-z)} \mathrm{d}t \qquad \text{for} \qquad z \notin [-1,1].$$

Demonstrate that the proposed solution satisfies the requirements of the Plemelj formulæ.

(b) Find a closed form expression for all solutions $u(x)$ to the equation

$$\frac{1}{\pi} ⨍_{-1}^{1} \frac{u(t)}{t-x} \mathrm{d}t = \frac{1}{\sqrt{1-x^2}} \qquad \text{for} \qquad x \in [-1,1],$$

using the Hilbert inversion formula:

$$u(x) = -\frac{1}{\pi\sqrt{1-x^2}} ⨍_{-1}^{1} \frac{f(t)\sqrt{1-t^2}}{t-x} \mathrm{d}t - \frac{C}{\sqrt{1-x^2}}.$$

(c) Find a solution to the integral equation

$$\left(\int_{-2}^{-1} + \int_{1}^{2} \right) \log|t-x| u(t) \mathrm{d}t = C$$

where C is any non-zero constant. You do not need to give C explicitly, but need to show it is non-zero.

2. Let $u(x)$ solve the integral equation

$$u(x) + 4\int_0^\infty K(t-x)u(t)\mathrm{d}t = f(x) \qquad \text{for} \qquad x \geq 0$$

where

$$K(x) = \mathrm{e}^{-|x-t|} \qquad \text{and} \qquad f(x) = \mathrm{e}^{-x}.$$

We will use the notation

$$g_{\mathrm{L}}(x) := \begin{cases} g(x) & x < 0 \\ 0 & x \geq 0 \end{cases}, \qquad g_{\mathrm{R}}(x) := \begin{cases} 0 & x < 0 \\ g(x) & x \geq 0 \end{cases},$$

and the Fourier transform

$$\hat{f}(s) := \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-\mathrm{i}st}\mathrm{d}t.$$

(a) What are the regions of analyticity of $\hat{K}(s)$, and $\widehat{f_{\mathrm{R}}}(s)$? Assuming that $|u(x)| \leq \mathrm{e}^{-\delta x}$ for some $\delta > 0$, what is the region of analyticity of $\widehat{u_{\mathrm{R}}}(s)$ in terms of δ? Justify your answers without explicit calculation.

(b) For the integral equation above, set up a Riemann–Hilbert problem of the form

$$\Phi_+(s) - g(s)\Phi_-(s) = h(s) \qquad \text{for} \qquad s \in (-\infty, \infty)$$

where $\Phi(z)$ is analytic off the real axis,

$$\Phi_\pm(s) := \lim_{\epsilon \to 0^+} \Phi(s \pm \mathrm{i}\epsilon),$$

and

$$g(s) = \frac{s^2+9}{s^2+1}.$$

Explain precisely the definition of $\Phi(s)$, $g(s)$ and $h(s)$ in terms of the Fourier transforms of u and f.

(d) What is the solution to the homogeneous Riemann–Hilbert problem

$$\kappa_+(s) = g(s)\kappa_-(s) \qquad \text{for} \qquad s \in (-\infty, \infty)$$

where $\kappa(\infty) = 1$? Justify why this solution is unique.

(e) Determine $u(x)$.

3. Recall the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!}x^n + O(x^{n-1})$$

where $\alpha > -1$, which are orthogonal with respect to

$$\int_0^\infty f(x)g(x)x^\alpha \mathrm{e}^{-x}\mathrm{d}x.$$

Recall the Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha}\mathrm{e}^x}{n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}[x^{\alpha+n}\mathrm{e}^{-x}].$$

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}x}[x^{\alpha+1}\mathrm{e}^{-x}L_n^{(\alpha+1)}(x)] = (n+1)x^\alpha \mathrm{e}^{-x}L_{n+1}^{(\alpha)}(x)$$

and

$$xL_n^{(\alpha+1)}(x) = -(n+1)L_{n+1}^{(\alpha)}(x) + (n+\alpha+1)L_n^{(\alpha)}(x).$$

(b) For the Fourier transform

$$F_n(\omega) := \int_0^\infty L_n^{(0)}(x)\mathrm{e}^{-x}\mathrm{e}^{\mathrm{i}\omega x}\mathrm{d}x$$

use Part (a) to show that

$$F_n(\omega) = \left(\frac{\mathrm{i}\omega}{1-\mathrm{i}\omega}\right)^n \frac{1}{1-\mathrm{i}\omega}.$$

Hint: use integration by parts to determine a simple recurrence relationship for $F_n(\omega)$ in terms of $F_{n-1}(\omega)$.

(c) For what region of the complex plane in ω is the previous formula valid?

4. Recall the hypergeometric differential equation

$$z(1-z)\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} + [c-(a+b+1)z]\frac{\mathrm{d}u}{\mathrm{d}z} - abu = 0$$

which has the special solution ${}_2F_1\left({a,b \atop c}; z\right)$ that is analytic near zero satisfying ${}_2F_1\left({a,b \atop c}; 0\right) = 1$, assuming that c is not a negative integer.

(a) Show that

$$ {}_2F_1\left({a,a \atop a}; z\right) = (1-z)^{-a}$$

for a not equal to a negative integer.

(b) ${}_2F_1\left({a,b \atop c}; z\right)$ has a branch cut on $[1,\infty)$ on which it satisfies the subtractive jump

$$ {}_2F_1\left({a,b \atop c}; x\right)_+ - {}_2F_1\left({a,b \atop c}; x\right)_- = \frac{2\pi \mathrm{i}\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-a-b+1)}(x-1)^{c-a-b}{}_2F_1\left({c-a, c-b \atop c-a-b+1}; 1-x\right)$$

where

$$ {}_2F_1\left({a,b \atop c}; x\right)_\pm = \lim_{\epsilon\to 0^+} {}_2F_1\left({a,b \atop c}; x \pm \mathrm{i}\epsilon\right).$$

Use this fact alongside Part (a) to show that

$$\int_0^1 \frac{(1-t)^b}{t-z}\mathrm{d}t = -\frac{{}_2F_1\left({1,1 \atop 2+b}; z^{-1}\right)}{(1+b)z}$$

for $z \notin [0,1]$.