M3M6: Methods of Mathematical Physics (2017)

Problem Sheet 1

Problem 1.1 Compute the residues of

1) $\mathrm{Res}_{z=a\mathrm{e}^{\mathrm{i}\pi/4}} \frac{z^3 \sin z}{z^4+a^4}$ where $a > 0$

2) $\mathrm{Res}_{z=1} \frac{z+1}{(z^2-1)^2}$ where $a \neq 0$

3) $\mathrm{Res}_{z=a} \frac{z^2 \mathrm{e}^z}{z^3-a^3}$ where $a \neq 0$

Problem 1.2 Use contour integration to find the values of

1) $\int_0^{2\pi} \frac{1}{5-4\cos\theta}\,\mathrm{d}\theta$

2) $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,\mathrm{d}\theta$

3) $\int_{-\infty}^{\infty} \frac{1}{(x^2+1)(x^2+4)}\,\mathrm{d}x$

4) $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,\mathrm{d}x$

5) $\int_{-\infty}^{\infty} \frac{1}{x+\mathrm{i}}\,\mathrm{d}x$

6) $\int_{-\infty}^{\infty} \frac{\sin 2x}{x^2+x+1}\,\mathrm{d}x$

7) $\int_{-\infty}^{\infty} \frac{\cos x}{x^2+4}\,\mathrm{d}x$

8) $\int_{-\infty}^{\infty} \frac{x \sin x}{x^2+1}\,\mathrm{d}x$

9) $\int_{-\infty}^{\infty} \frac{\cos ax-\cos bx}{x^2}\,\mathrm{d}x$ where $a, b > 0$

10) $\int_0^{2\pi} (\cos\theta)^n\,\mathrm{d}\theta$ where $n = 0, 1, 2, \ldots$ (Hint: consider even and odd n separately.)

Problem 2.1 By integrating around a rectangular contour with vertices at $\pm R$ and $\pi\mathrm{i} \pm R$ and letting $R \to \infty$, show that:

$$\int_0^{\infty} \operatorname{sech} x\,\mathrm{d}x = \frac{\pi}{2}$$

where $\operatorname{sech} x = \frac{2}{\mathrm{e}^{-x}+\mathrm{e}^{x}}$.

Problem 2.2 Show that the Fourier transform of $\operatorname{sech} x$ satisfies

$$\int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}kx} \operatorname{sech} x\,\mathrm{d}x = \pi \operatorname{sech} \frac{\pi k}{2}$$

Problem 3.1 Given $A \in \mathbb{R}^{n\times n}$ that is symmetric positive definite (that is, all eigenvalues of A are real and greater than zero) and $\boldsymbol{u}_0, \boldsymbol{v}_0 \in \mathbb{R}^n$, write a contour integral solution to the second-order linear constant coefficient ODE:

$$\boldsymbol{u}''(t) = A\boldsymbol{u}(t)$$
$$\boldsymbol{u}(0) = \boldsymbol{u}_0$$
$$\boldsymbol{u}'(0) = \boldsymbol{v}_0$$

Problem 3.2 Was the restriction to symmetric positive definite matrices necessary? Why or why not?

Problem 4.1 Suppose that $g(\theta)$ has absolutely summable Fourier coefficients, that is,

$$g(\theta) = \sum_{k=-\infty}^{\infty} g_k e^{\mathrm{i}k\theta} \qquad \text{where} \qquad \sum_{k=-\infty}^{\infty} |f_k| < \infty.$$

Show that the periodic trapezium rule satisfies

$$\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) = \cdots + g_{-2n} + g_{-n} + g_0 + g_n + g_{2n} + \cdots$$

where $\theta_j = \frac{2\pi j}{n}$.

Problem 4.2 Suppose that $g(\theta) = f(e^{i\theta})$ where $f(z)$ is holomorphic in an annulus $\{z : R^{-1} < |z| < R\}$. Prove that the periodic trapezium rule converges exponentially fast:

$$\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) \to \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,\mathrm{d}\theta.$$

Problem 4.3 Find an upper bound for the error

$$\left|\frac{1}{n}\sum_{j=0}^{n-1} g(\theta_j) - \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,\mathrm{d}\theta\right|$$

for $g(\theta) = \frac{1}{2-\cos\theta}$.
