M3M6: Methods of Mathematical Physics (2017)

Mastery Sheet

Denote the Cauchy transform of f on $[0, \infty)$ by

$$\mathcal{C}f(z) = \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{f(x)}{x - z}\, \mathrm{d}x.$$

Problem 1.1 State how $\mathcal{C}f(z)$ can be reduced to an expression involving $\mathcal{C}g$ and $\mathcal{C}h$ where $g(x) = f(x^{3/2})$ and $h(x) = f(x^{3/2})/\sqrt{x}$. Show that your expression satisfies the criteria of Plemelj.

Problem 1.2 Use the previous part to derive a closed form expression for

$$\int_0^\infty \frac{\mathrm{e}^{-x^{2/3}}}{x - z}\, \mathrm{d}x$$

in terms of the incomplete Gamma function

$$\mathrm{Ei}(z) := \int_z^\infty \zeta^{a-1} \mathrm{e}^{-\zeta}\, \mathrm{d}\zeta,$$

where the path of integration can be chosen to be the union of two line segments: one segment from z to 1 and another segment from 1 to ∞.
