M3M6: Methods of Mathematical Physics (2018)

Problem Sheet 3

This problem set considers the Laguerre polynomials

$$L_n^{(\alpha)}(x) = \frac{(-1)^n}{n!} x^n + O(x^{n-1})$$

where $\alpha > -1$, which are orthogonal with respect to

$$\langle f, g \rangle_\alpha = \int_0^\infty f(x) g(x) x^\alpha \mathrm{e}^{-x} \, \mathrm{d}x$$

We also use the notation $L_n(x) = L_n^{(0)}(x)$.

Problem 1.1 Show that the Rodrigues formula holds:

$$L_n^{(\alpha)}(x) = \frac{x^{-\alpha} \mathrm{e}^x}{n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} \left[x^{\alpha+n} \mathrm{e}^{-x} \right].$$

Problem 1.2 Show that the derivatives form a hierarchy: we have

$$\begin{aligned}
\frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} &= -L_{n-1}^{(\alpha+1)}(x), \\
\frac{\mathrm{d}}{\mathrm{d}x} \left[x^{\alpha+1} \mathrm{e}^{-x} L_n^{(\alpha+1)}(x) \right] &= (n+1) x^\alpha \mathrm{e}^{-x} L_{n+1}^{(\alpha)}(x), \\
x L_n^{(\alpha+1)}(x) &= -(n+1) L_{n+1}^{(\alpha)}(x) + (n+\alpha+1) L_n^{(\alpha)}(x), \\
L_n^{(\alpha)}(x) &= L_n^{(\alpha+1)}(x) - L_{n-1}^{(\alpha+1)}(x).
\end{aligned}$$

Problem 1.3 Combine the results from Problem 1.2 to determine the three-term recurrence relationship and the top 5×5 block of the Jacobi operator.

Problem 2.1 Represent the ordinary differential operator

$$u'(x) - x u(x) \qquad \text{for} \qquad x \geq 0$$

as an operator on the coefficients of u in a weighted Laguerre expansion:

$$u(x) = \sum_{k=0}^\infty u_k \mathrm{e}^{-x/2} L_k(x) = \mathrm{e}^{-x/2} \left(\, L_0(x) \mid L_1(x) \mid \cdots \right) \begin{pmatrix} u_0 \\ u_1 \\ \vdots \end{pmatrix},$$

where the range of the operator is specified in $\mathrm{e}^{-x/2} \left(\, L_0^{(1)}(x) \mid L_1^{(1)}(x) \mid \cdots \right)$.

Problem 2.2 Show that the Laguerre polynomials are eigenfunctions of a Sturm–Liouville problem, that is, find $\lambda_n^{(\alpha)}$ so that

$$\frac{\mathrm{e}^x}{x^\alpha} \frac{\mathrm{d}}{\mathrm{d}x} \left[x^{\alpha+1} \mathrm{e}^{-x} \frac{\mathrm{d}L_n^{(\alpha)}}{\mathrm{d}x} \right] = \lambda_n^{(\alpha)} L_n^{(\alpha)}(x)$$

Re-express this as an ODE with polynomial coefficients.

This problem considers Cauchy and Logarithmic transforms of Laguerre polynomials. Recall from lectures that

$$\mathcal{C}_{[0,\infty)}[\mathrm{e}^{-\diamond}](z) = -\frac{\mathrm{e}^{-z}\mathrm{Ei}\, z}{2\pi\mathrm{i}}$$

for the exponential integral

$$\mathrm{Ei}\, z = \int_{-\infty}^{z} \frac{\mathrm{e}^{\zeta}}{\zeta}\, \mathrm{d}\zeta.$$

Problem 3.1 What is

$$\mathcal{C}_{[0,\infty)}[\diamond \mathrm{e}^{-\diamond} L_1^{(1)}(\diamond)](z) := \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{x\mathrm{e}^{-x} L_1^{(1)}(x)}{x - z}\, \mathrm{d}x$$

in terms of $\mathrm{Ei}\, z$?

Problem 3.2 What is

$$\frac{1}{\pi} \int_0^\infty \mathrm{e}^{-x} L_2(x) \log|z - x|\, \mathrm{d}x$$

in terms of the real and imaginary parts of $\mathrm{Ei}\, z$?

Consider the incomplete Gamma function:

$$\Gamma(\alpha, z) = \int_z^\infty \zeta^{\alpha-1} \mathrm{e}^{-\zeta}\, \mathrm{d}\zeta,$$

where the contour of integration is two straight line segments from z to 1 to ∞, hence this has a branch cut on $(-\infty, 0]$.

Problem 4.1 For $x < 0$ and $\alpha > 0$, show that

$$\Gamma_+(\alpha, x) - \mathrm{e}^{2\mathrm{i}\pi\alpha}\Gamma_-(\alpha, x) = (1 - \mathrm{e}^{2\mathrm{i}\pi\alpha})\Gamma(\alpha)$$

where $\Gamma(\alpha) = \Gamma(\alpha, 0) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\, \mathrm{d}x$ is the Gamma function and

$$\Gamma_\pm(\alpha, x) = \lim_{\epsilon \to 0} \Gamma(\alpha, x \pm \mathrm{i}\epsilon).$$

Problem 4.2 For $-1 < \alpha < 0$, express

$$\mathcal{C}_{[0,\infty)}[\diamond^\alpha \mathrm{e}^{-\diamond}](z) = \frac{1}{2\pi\mathrm{i}} \int_0^\infty \frac{x^\alpha \mathrm{e}^{-x}}{x - z}\, \mathrm{d}x$$

in terms of $\Gamma(-\alpha, -z)$ and $(-z)^\alpha \mathrm{e}^z$ using Plemelj's lemma.
