Master Thesis in Theoretical High Energy Physics:

The Systematics of Radiative Corrections and A Proposed Approximative Evaluation Scheme

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Abstract

We propose an approximate scheme based on a saddle point approximation of propagators in higher order perturbation calculation. This scheme is applied to a general expression for one-loop scalar functions, and thereafter used to calculate the anomalous magnetic moment of the electron. The scheme is shown to be increasingly precise as the number of propagators increase, spanning a precision of 0.72 for the exact calculation for a three-point function, to 0.90 for a five-point function. The possible applications of this approximation, as well as it's extension to divergent diagrams and higher-loop calculations are discussed.

In addition, the general systematics of one-loop expressions are reviewed, as well as an introduction to regularization and renormalization.

Chapter 1

Introduction

1.1 General Introduction

As of today, through the scientific process of hypothesis and experimental verification, we have made the acquaintance of four fundamental forces of Nature. They are the strong and weak nuclear forces, the electromagnetic force and gravity. To the best of our understanding, these forces govern the behaviour of all constituents of the known universe, from the very small to the extremely large.

To account for characteristics of these forces, over the years, many different hypothesises have been formulated and tested against experiment. Some have been found to agree extremely well with results, and therefore been commonly accepted. However, these agreements have been observed in a particular domain of phase space, i.e. for some particular span of energy, mass, size, time etc. Several times, it turns out that when you test theories in new areas of phase space, they may very well offer results that are in variance with the well-tested theories, expressing a need for a more general theory. This theory is required to not only account for the new results, but also correctly predict all previous experiments.

During the past hundred years or so, by examining matter at smaller and smaller length scales, several discrepancies arose. These led to the birth of quantum mechanics. In parallel, by examining matter at larger and larger energies, Einstein's theories of Special and General Relativity were developed. Currently, no consistent manner of merging the two theories of General Relativity and Quantum Mechanics has been able to be experimentally verified. However, it has been possible to merge Special Relativity with Quantum Mechanics into a consistent framework. The framework that arises from merging these theories has been called Quantum Field Theory. There can be many forms of quantum field theories, since one may choose to include different terms in the Lagrangian of the theory, but whatever their form they follow certain principles. The particular breed of Quantum Field Theory that is currently by far the most successful and has accounted for all observed results so far is commonly referred to as the Standard Model. It consists of two separate theories, Quantum Chromodynamics (QCD) and the Electroweak theory. The former correctly accounts for the behaviour of the strong nuclear force, while the latter offers a unified description of the electromagnetic and weak nuclear force. As such, three of the four fundamental forces are accounted for. When testing with experiment, the fourth, gravity, is disregarded. Due to the fact that gravity, compared to the other forces, is extremely weak at the small scales of the experiments, this is usually a matter of small practical importance. However, it is from this perspective a fundamentally flawed theory and at some scale gravity (which currently is most precisely described by General Relativity) may no longer be disregarded, and a consistent theory that correctly describes all four forces will be required.

Leaving this at hand, the Standard Model has been tested to an astonishing precision. The most important method of testing theory with experiment has been through the means

of scattering processes in particle accelerators. In a scattering process, ideally the system is described by the so-called S-matrix. In Quantum Field Theory, very few systems can be exactly solved in this manner, and one therefore usually resorts to a perturbative approach, expanding in orders of the parameters of the theory. In this expansion, we think of the scattering process as a description where we know the incoming particles and the outgoing particles. For the process in between, we sum over all possible interactions (up until the given order of our perturbative calculation, roughly described by the 'number of reactions' that take place during the interaction).

When combined with a phase space integration, this offers a prediction that we can compare with experimental results in form of a cross-section (we can also calculate decays through decay widths in a similar manner).

As the precision of experiments increase, the need to include higher orders of perturbation theory increases. When calculating these higher-order perturbations, the computational complexity also increases accordingly. Due to a systematic treatment of this topic during the past 40 or so years, the first order corrections of perturbation theory can be considered to be completely worked out [6]. In this text, this progress is reviewed and the historically more relevant results are presented in some detail.

1.2 Novel results in this text

This text also presents an extension of work [1] done on evaluating higher order correlations through an approximative scheme. The scheme was originally devised for the N-point Veneziano model, and in this text, the process of applying the basic ideas of the scheme to QED calculations is commenced. The scope of this thesis covers applications of the scheme to convergent one-loop calculations. It also argues for the possibility to extend this scheme to higher-order divergent loop diagrams.

The application of the scheme to QED is discussed through several different calculational perspectives, which exemplifies the usefulness of the scheme in different settings. It does so by first comparing its precision in relation to the general one-loop integrals, comparing different types of diagrams. This is followed by a detailed discussion on the Anomalous Magnetic Moment, where different methods for carrying out the scheme are presented.

1.3 Outline

In chapter 2, the general framework of calculating first-order loops are reviewed. This starts with formulating a general one-loop expression in section 2.2 and its nomenclature in section 2.3. This is followed in section 2.4 by a general procedure, in which we show that general one-loop expression can be completely reduced to the evaluation of so-called scalar integrals. We continue with showing how to solve these scalar integrals in sections 2.5-2.9. The first steps of integration, Feynman parametrization, Wick rotation and momentum integration, are covered in section 2.5. The issue of dealing with divergent contributions is introduced in section 2.6 and applied to scalar integrals in sections 2.7 and 2.9, with an interlude of carrying out the integration over the previously introduced Feynman parameters in section 2.8.

In chapter 3, the saddle point approximation scheme is introduced. It begins with general considerations according to the scheme in section 3.2. As an introduction, we then apply the saddle point approximation to ϕ^4-theory. This is followed by an application of the saddle point approximation to scalar integrals in section 3.4. In section 3.5 we cover the details of saddle point approximation via the calculation of the anomalous magnetic moment. In section 3.6 we discuss the possibility to improve the accuracy of the approximation. This is followed by

a discussion on divergences in section 3.7 and higher order calculations in section 3.8. We follow this with a discussion on how a version of this method could be used to crudely estimate more sophisticated diagrams in section 3.9. We conclude by an outlook over possible future developments of the scheme in chapter 4.

During the discussion, we assume a basic proficiency in quantum field theory and a general knowledge of the Standard Model equivalent to having followed an introductory course in the topic.

Chapter 2

Perturbative Predictions

2.1 Introduction

The general procedure of perturbative Quantum Field Theory offers us a means for predicting the scattering experiments taking place in particle colliders. In general, the lowest order of the perturbative expansion is described by a tree level diagram, where no free internal momenta occur. As the accuracy of the experiments are increased, the need arises to take further terms of the expansion into consideration. It has turned out that some of these terms can be of considerable importance, for example the main channel aimed at the discovery of the Higgs boson, through two gluon fusion, has a correction at next to leading order (NLO) of about 50%.

These higher order terms are included by taking into account the possibility of additional internal particles that form and annihilate in the process, giving rise to free internal momenta to integrate over.

In this chapter, we will look closer at how to evaluate one-loop diagrams, starting with going through the general characteristics of a one-loop expression in section 2.2 and it's nomenclature in section 2.3. We will therafter review reductional schemes which puts any possible Feynman amplitude into the form of a linear combination of a small amount of integrals, so called scalar integrals, in section 2.4. This subcategory of integrations will be solved in section 2.5, through the stepwise application of the Schwinger Trick or Feynman parameters in subsection 2.5.2, Wick Rotation in subsection 2.5.3 and angular and radial integration in subsections 2.5.4 and 2.5.5. The solution will then be discussed further for tensorial integrals. In the sections to come, we will continue the discussion covering how to carry out the integration over Feynman parameters, in addition to introducing key concepts of regularization and renormalization.

2.2 General One-Loop Expression

There are many different possibilities for how a one-loop diagram may look like, as in figure 2.1

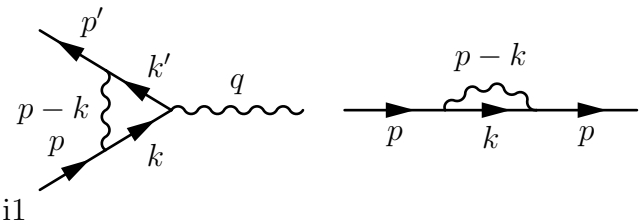

Figure 2.1: Schematic Feynman diagrams for the process by which the anomalous magnetic moment and the electron's self energy may be calculated.

When calculating a scattering amplitude, we do so by including all possible Feynman diagrams that contribute to the process up to a given order of the perturbation parameter. By using the Feynman rules to set up an algebraic expression, we can perform a calculation that gives us an prediction for the matrix element. In the case of the first order perturbation, we need to take expressions into account that include one internal momentum. These can be of many different shapes, some examples of which would include the Feynman diagrams involved in calculating the Anomalous Magnetic Moment and the self-energy of the electron, both of which are schematically shown in figure 2.1. There are truly many different first order corrections. For example, for a process $e\,e \to W\,W$, there are only three diagrams to consider at *tree-level* (that is, at the zeroth order of the perturbation), for first order corrections, we have 68 different diagrams to calculate. Therefore, it is not difficult to see the market for a procedure such as the one we are about to describe, which allows us to evaluate very many amplitudes while only having to carry out a few general calculations.

To this end, we want to write up a general one-loop tensorial integral for any N-point function, that is with any N number of internal propagators. This expression is illustrated in figure **??**, where the shorthand $p_{ij} = p_i - p_j$ is employed.

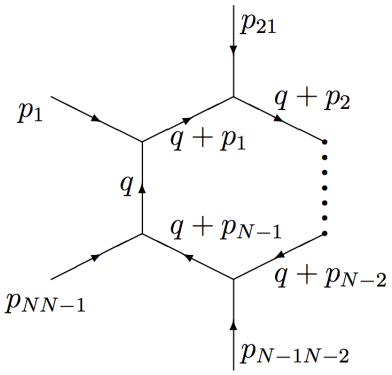

Figure 2.2: A general N-point integral (with q as the internal momentum variable), from [13]

This expression can in general be written in the form of the tensorial integral:

$$\begin{aligned} & T^n_{\mu_1 \ldots \mu_p}(p_1, ..., p_{n-1}, m_0, ..., m_{n-1}) \\ = \; & \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^d k \frac{\mathcal{N}(k_i, m_i, p_j)}{(k_0^2 - m_0^2 + i\epsilon)(k_1^2 - m_1^2 + i\epsilon)...(k_{n-1}^2 - m_{n-1}^2 + i\epsilon)} \end{aligned} \tag{2.1}$$

with

- n: number of internal lines in loops
- k_i: momentum of the i'th propagator, where $k_0 = k$ is the integration variable. Note that because of momentum conservation, both in the overall graph and at each vertex, there is only one independent internal momentum, which is the so-called loop momentum. This momentum may initially not be isolated in one propagator factor, but this can in this case always be achieved by a simple variable substitution.
- m_i: mass of the i'th propagator
- p_j: external momentum of the j'th particle
- $\mathcal{N}(k_i, m_i, p_j)$: function dependent on internal and external momenta, internal mass and the details of the diagram being considered that does not contain any divergences.

- d is the number of dimensions, in Minkowski space $d = 4$, which will be used as a variable in dimensional regularization to deal with divergent integrals (see below).
- $i\epsilon$ is an infinitesimal displacement into the complex plane, guaranteeing space-time causality. In this entire text the Feynman prescription is used (for conversion to other conventions, such as Advanced, Retarded and Dyson prescriptions, see the following note [26]). After integration it determines the correct imaginary parts of terms involving logarithms and dilogarithms.
- The parameter μ has mass dimension and therefore allows the dimensionality of the integral as a whole to be fixed for varying d.

2.3 Integral Nomenclature

Now for some remarks on nomenclature. Due to the rather large amount of different forms of the T^n-integral that occur in calculations, a typical naming scheme has been developed. The convention is to denote an integral by the n-th character of the alphabet, were n refers to the number of propagators in the denominator, i.e. an integral with three propagators is referred to as C and an integral with 15 propagators would be referred to with O. The tensorial quantities of internal loop moment in the numerator is referred to by it's Minkowski indices. A few examples that illustrate these conventions would be (suppressing the constant prefactor involving $\frac{(2\pi\mu)^{4-d}}{i\pi^2}$ throughout this section):

$$C^{\mu\nu} = \int d^d q \frac{q^\mu q^\nu}{P_0 P_1 P_2} \; ; \; B^{\mu\nu\rho} = \int d^d q \frac{q^\mu q^\nu q^\rho}{P_0 P_1} \tag{2.2}$$

where P_i is shorthand for the propagator from the previous section, $(k_i^2 - m_i^2 + i\epsilon)$. Within some reductional schemes, such as the Passarino-Veltman scheme that will be reviewed later, it happens that propagators are removed from integrals. In these cases the particular propagators indices are then superscripted to signify this. For example, for $C^{\mu\nu}$ from the previous example could have reductions such as:

$$B^{(1)\mu\nu} = \int d^d q \frac{q^\mu q^\nu}{P_0 P_2} \; ; \; A^{(1,2)\mu\nu} = \int d^d q \frac{q^\mu q^\nu}{P_0} \tag{2.3}$$

A final remark of nomenclature is that integrals that do not carry spacetime indices are referred to as *scalar integrals* and in the place of a spacetime index a zero is used, for example A_0,C_0 or O_0, while those integrals that still have spacetime indices are referred to as *tensorial integrals*, such as A^μ or $D^{\mu\nu\rho}$.

2.4 Passarino-Veltman Reduction

2.4.1 Introduction

One of the major techniques to make any general one-loop calculation manageable is by the means of a reduction scheme. The most commonly used, since its conception in 1979, is the Passarino-Veltman Scheme [11]. A systematic treatment of the method can be found in [13] and a selection of calculations employing the techniques is showcased in [14]. Through this section, we will first cover the basic idea of Passarino-Veltman Reduction by means of an illustrative example and then discuss how to generalize this method. We will conclude by discussing extensions and alternatives to the method.

2.4.2 Reduction of a tensorial integral

Passarino-Veltman Reduction aims to reduce any tensorial integral into a linear combination of scalar integrals. Thereby, any one-loop integral is reduced to a linear combination of a small subset of integrals. Since these can be worked out beforehand the scheme offers a way of calculating arbitrary one-loop integrals in a highly effective manner, which has been taken advantage of in several computer packages.

The notation for the general calculations can at times be quite lenghty, and it may be useful to illustrate this method by a worked out example. Therefore, let us go through the reduction scheme for C_μ. For this integral we have the expression (again suppressing the constant factor involving μ):

$$C_\mu\left(p_1, p_2, m_0, m_1, m_2\right) = \int d^d k \frac{k_\mu}{P_0 P_1 P_2} \tag{2.4}$$

where

- $P_0 = k^2 - m_0^2$
- $P_i = (k + p_i)^2 - m_i^2$
- p_i are external momenta, k_i the momenta of a propagator and m_i the corresponding mass.

Since the integral C_μ is a Lorentz vector, it may be written as a linear combination of the external momenta:

$$C_\mu = \bar{C}_1 p_{1\mu} + \bar{C}_2 p_{2\mu} \tag{2.5}$$

Where $\bar{C}_i$ are scalar coefficients. By solving for these coefficients, we can reduce the expression to combinations of scalar quantities with external momenta:

$$\begin{aligned} p_1^\mu C_\mu &= \bar{C}_1 p_1^2 + \bar{C}_2 p_1 \cdot p_2 \\ p_2^\mu C_\mu &= \bar{C}_1 p_1 \cdot p_2 + \bar{C}_2 p_2^2 \end{aligned} \tag{2.6}$$

The expression on the left hand side will contain an expression of the form $p_i \cdot k$, which we can rewrite:

$$2p_i \cdot k = k^2 + 2p_i \cdot k + p_i^2 - m_i^2 - (k^2 - m_0^2) - (p_i^2 - m_i^2 + m_0^2) = P_i - P_0 - (p_i^2 - m_i^2 + m_0^2) \tag{2.7}$$

Which yields

$$2p_i^\mu C_\mu = B_0^{(i)} - B_0^{(0)} - (p_i^2 - m_i^2 + m_0^2) C_0 \tag{2.8}$$

Where the parenthesized superscripts refer to the removed propagator. From this we obtain two linear equations with two unknowns, which we may solve for $\bar{C}_1$ and $\bar{C}_2$:

$$2 \underbrace{\begin{pmatrix} p_1^2 & p_1 \cdot p_2 \\ p_1 \cdot p_2 & p_2^2 \end{pmatrix}}_{\Delta_{ij}} \begin{pmatrix} \bar{C}_1 \\ \bar{C}_2 \end{pmatrix} = \begin{pmatrix} B_0^{(1)} - B_0^{(0)} - (p_1^2 - m_1^2 + m_0^2) C_0 \\ B_0^{(2)} - B_0^{(0)} - (p_2^2 - m_2^2 + m_0^2) C_0 \end{pmatrix} \tag{2.9}$$

Δ_{ij} is known as the *Gram matrix*. In the case that it can be inverted, there is a solution to the Passarino-Veltman reduction, given by:

$$\begin{pmatrix} \bar{C}_1 \\ \bar{C}_2 \end{pmatrix} = \frac{1}{2(p_1^2 p_2^2 - (p_1 \cdot p_2)^2)} \begin{pmatrix} p_2^2 & -p_1 \cdot p_2 \\ -p_1 \cdot p_2 & p_1^2 \end{pmatrix} \begin{pmatrix} B_0^{(1)} - B_0^{(0)} - (p_1^2 - m_1^2 + m_0^2) C_0 \\ B_0^{(2)} - B_0^{(0)} - (p_2^2 - m_2^2 + m_0^2) C_0 \end{pmatrix} \tag{2.10}$$

This solves the calculation of the coefficients of the expression, and we can therefore rewrite C_μ into a linear combination of scalar integrals, solving the issue at hand completely.

2.4.3 Extension for $n \leq 4$

This procedure can consistently be extended to include higher ranks of tensors and additional propagator terms (a general proof can be found in [13]). Let us begin by discussing the case when $n \leq 4$. For these expressions, Lorentz covariance preserves not only terms with external momenta p_i, but in addition terms proportional to the metric tensor $g_{\mu\nu}$. We can write this in form of a general expression:

$$T^n_{\mu_1 \ldots \mu_p}(p_1, ..., p_{n-1}, m_0, ..., m_{n-1}) = \sum_{i_1=0}^{n-1} ... \sum_{i_p=0}^{n-1} \bar{T}^n_{i_1 \ldots i_p} \times (p_{i_1} - p_{\mu_1}) \cdots (p_{i_p} - p_{\mu_p}) \tag{2.11}$$

Here the $\bar{T}^n$ are expansion coefficient functions that are totally symmetric in i_p. We have also introduced the artificial external momentum p_0 into the expression. This allows us to include the covariant terms for $g_{\mu\nu}$. If we remove all odd numbers of p_0's from the expression and replace products of even numbers of p_0's by a totally symmetric tensor constructed from $g_{\mu\nu}$, that is:

$$\begin{aligned} (p_0 - p_{\mu_1})(p_0 - p_{\mu_2}) &\rightarrow g_{\mu_1\mu_2} \\ (p_0 - p_{\mu_1})(p_0 - p_{\mu_2})(p_0 - p_{\mu_3})(p_0 - p_{\mu_4}) &\rightarrow g_{\mu_1\mu_2}g_{\mu_3\mu_4} + g_{\mu_1\mu_3}g_{\mu_2\mu_4} + g_{\mu_1\mu_4}g_{\mu_2\mu_3} \end{aligned} \tag{2.12}$$

we obtain an expression including all possible combinations of external momenta p_i, as well as all possible expressions involving the metric tensor $g_{\mu\nu}$.

To express the decompositions explicitly we first apply the shorthand notation $(p_A - p_B) = P_{AB}$.

The lowest order integrals are then given by:

$$
\begin{aligned}
B_\mu &= p_{1\mu} B_1 \\
B_{\mu\nu} &= g_{\mu\nu} B_{00} + p_{1\mu} p_{1\nu} B_{11}
\end{aligned}
$$

$$
\begin{aligned}
C_\mu &= p_{1\mu} C_1 + p_{2\mu} C_2 = \sum_{i=1}^{2} p_{i\mu} C_i \\
C_{\mu\nu} &= g_{\mu\nu} C_{00} + p_{1\mu} p_{1\nu} C_{11} + p_{2\mu} p_{2\nu} C_{22} + (p_{1\mu} p_{2\nu} + p_{2\mu} p_{1\nu}) C_{12} \\
&= g_{\mu\nu} C_{00} + \sum_{i,j=1}^{2} p_{i\mu} p_{j\nu} C_{ij} \\
C_{\mu\nu\rho} &= (g_{\mu\nu} p_{1\rho} + g_{\nu\rho} p_{1\mu} + g_{\rho\mu} p_{1\nu}) C_{001} + (g_{\mu\nu} p_{2\rho} + g_{\nu\rho} p_{2\mu} + g_{\rho\mu} p_{2\nu}) C_{002} \\
&+ p_{1\mu} p_{1\nu} p_{1\rho} C_{111} + + p_{2\mu} p_{2\nu} p_{2\rho} C_{222} \\
&+ (p_{1\mu} p_{1\nu} p_{2\rho} + p_{2\mu} p_{1\nu} p_{1\rho} + p_{1\mu} p_{2\nu} p_{1\rho}) C_{112} \\
&+ (p_{1\mu} p_{2\nu} p_{2\rho} + p_{2\mu} p_{1\nu} p_{2\rho} + p_{2\mu} p_{2\nu} p_{1\rho}) C_{122} \\
&= \sum_{i=1}^{2} (g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\rho\mu} p_{i\nu}) C_{00i} + \sum_{i,j,k=1}^{2} p_{i\mu} p_{j\nu} p_{k\rho} C_{ijk} \\
D_\mu &= \sum_{i=1}^{3} p_{i\mu} D_i \\
D_{\mu\nu} &= g_{\mu\nu} D_{00} + \sum_{i,j=1}^{3} p_{i\mu} p_{j\nu} D_{ij} \\
D_{\mu\nu\rho} &= \sum_{i=1}^{3} (g_{\mu\nu} p_{i\rho} + g_{\nu\rho} p_{i\mu} + g_{\rho\mu} p_{i\nu}) D_{00i} + \sum_{i,j,k=1}^{2} p_{i\mu} p_{j\nu} p_{k\rho} D_{ijk} \\
D_{\mu\nu\rho\sigma} &= (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}) D_{0000} \\
&+ \sum_{i,j=1}^{3} (g_{\mu\nu} p_{i\rho} p_{j\sigma} + g_{\nu\rho} p_{i\mu} p_{j\sigma} + g_{\mu\rho} p_{i\nu} p_{j\sigma} + g_{\mu\sigma} p_{i\nu} p_{j\rho} + g_{\nu\sigma} p_{i\mu} p_{j\rho} + g_{\rho\sigma} p_{i\mu} p_{j\nu}) D_{00ij} \\
&+ \sum_{i,j,k,l=1}^{3} p_{i\mu} p_{j\nu} p_{k\rho} p_{l\sigma}
\end{aligned}
\tag{2.13}
$$

These expressions can then be solved by applying principles similar to the example of the C_μ calculation. We will not cover the derivation of the general expression, but will refer the interested reader to [13].

2.4.4 Extension for $n > 4$

Since only four Lorentz vectors are required to span four dimensional space, we can choose any set of four linearly independent Lorentz vectors $p_1, ..., p_4$ out of the external momenta at hand. It will be possible to write all other vector and tensor quantities as a linear combination of these vectors and therefore it is sufficent to decompose the expression into these vectors. This implies that we also leave out factors involving $g_{\mu\nu}$. This takes our general expression to the form:

$$
T^n_{\mu_1 \dots \mu_p}(p_1, ..., p_{n-1}, m_0, ..., m_{n-1}) = \sum_{i_1=1}^{4} ... \sum_{i_p=1}^{4} \bar{T}^n_{\mu_1 \dots \mu_p} \times (p_{i_1} - p_{\mu_1}) \cdots (p_{i_p} - p_{\mu_p}) \tag{2.14}
$$

Due to the fact that there are only four linearly independent Lorentz covariant vectors, the Gram determinant, which was inverted in the example to solve the system of equations, will always vanish for five-point functions or larger. Therefore, a calculational trick must be employed. In short, by expanding the determinant along the first column, one obtains a linear combination of tensor integrals with a smaller N. An extension of the reduction scheme from four to five point scalar integrals was first seen in [**?**] and a general discussion for N-point functions is covered in [13].

2.4.5 Application of Passarino-Veltman Reduction

As seen in the previous discussion, the Passarino-Veltman reduction in not only general in it's treatment, but performed in a systematic manner. Due to the possibility of running into quite lengthy expressions, it is therefore very well suited for implementation in automated algebraic computations. One well known implementation of this is the Mathematica [16] package *FeynCalc* [17]. It takes input either by hand or by output from *FeynArts* [19], a program which generates all diagrams for a given process. In addition, specific additional information can be provided, such as the kinematics involved in the process. An introduction to the package by one of the authors is available [20]

2.4.6 Limitations of the Passarino-Veltman Reduction

As discussed in section 2.4.4, the reduction is unsuccessful when the Gram determinant vanishes. This is possible also for $n \leq 4$, if the basis vectors of the decomposition are not linearly independent. In some cases it may be possible to aid this problem by a change of variables, but in some regions of phase space, such as the collinear region or at the threshold of a certain process, this approach may fail. The most straightforward way of extending the Passarino-Veltman reduction to these regions of phase space is to employ a similar approach to that at $n > 4$, expanding the Gram determinant into determinants of lower n. It may come to it that the reduced determinant is still vanishing, which signifies the existence of a leading Landau singularity, in which case one has to calculate the tensorial integral directly.

However, there have been numerous extensions to the Passarino-Veltman reduction which do seek to extend the method to all regions of phase space (except for these genuine singularities). The two main concerns with the Passarino-Veltman reduction for actual computer calculations are numerical stability and coverage of phase space. Regarding numerical stability, the decomposition may provide several large contributions from different scalar integrals that eventually come to cancel each other, leaving a difference that may be quite minute. This can lead to computational difficulties, as the precision in the calculations may be affected. van Oldenborgh and Vermaseren [18] have addressed this issue by employing a different tensorial basis, yielding expressions that simplify the contractions and offers a far improved numerical stability for very small, non-vanishing Gram determinants.

An alternative to the method described in the section 2.4.4 is to vary one of the parameters that lead to the vanishing Gram determinant away from its initial value and perform the Passarino-Veltman reduction. Thereafter, using l'Hôpital's Rule, one may take the limit where the parameter approaches it's initial value again. Here, it has been shown that the naïve approach reproduces the initial problems, however in [21], several additional relations between tensor integrals and scalar integrals have been derived that are only valid when the Gram determinant vanishes. These relations follow directly from the original reduction formulas given by Passarino and Veltman. By this method, a more transparent approach may be achieved in these situations.

2.4.7 Alternatives to the Passarino-Veltman Reduction

There exist a large selection of other schemes suitable for solving one-loop calculations, the development of which has been a priority to several research groups over the past years, due to the need for computationally effective methods to calculate higher order contributions to use in for example event generators. Some of these include

- Semi-Numerical Approach: which reduces the integrals to fewer tensor integrals and evaluates these numerically
- Numerical Approach: which computes the tensor integrals numerically
- Analytical Approach: a twistor-inspired approach which uses results from lower-loop, lower-point amplitudes and uses these at higher orders. Thereafter it calculates the scattering amplitudes by their poles and cuts. This approach has been implemented in calculations [22]
- OPP Reduction: Based on original results from in 1965 by Melrose and Källen, Ossola, Papadopoulos and Pittau developed a general scheme based on unitarity cuts to calculate higher order contributions. This approach has been inplemented in calculations [23]

There are two general concerns with using the Passarino-Veltman reduction at a large scale. The first is that inverting a Gram determinant (which can become rather large), can become computationally heavy to perform repeatedly. The second is that there are situations, as mentioned previously, where the Gram determinant vanishes due to the method (rather than reflecting the existance of a physical singularity). The last method of using unitarity cuts to practically build loop amplitudes out of physical tree amplitudes, recycling results from these, has been shown to be useful to aid both these issues. Two separate approaches, that both have shown different strengths, have been implemented in [22] and [23] respectively. However, to cover details of these methods are beyond the scope of this text.

2.5 The Standard Procedure for Calculating Scalar Integrals

2.5.1 Introduction

As seen in the previous section, it is clear that it is sufficient to explicitly calculate the scalar integrals and input in one's reduced expression. In this section we review the well-developed standard procedure for calculating said scalar integrals. The method is based on a series of steps:

- Applying the Schwinger Trick or Feynman parametrization
- Wick Rotation
- Change of variables to Spherical Coordinates
- Evaluation of Spherical Component
- Evaluation of Radial Component

Throughout this section, although we finally are looking for an expression in $d = 4$ dimensions, the calculations will be carried through in the general d-dimensional case. This is due to the fact that all divergences that arise in our expressions will be dealt with through dimensional regularization, which requires d-dimensional expressions and will be reviewed in section 2.6.

2.5.2 Schwinger Trick and Feynman parameters

For higher order loop calculations one of the most challenging aspects is carrying out the integration over the internal loop momenta, usually a spherically asymmetric, multi-dimensional integration to be carried out in a non-Eucledian metric, such as the Minkowski metric.

The first step to managing such an integration is usually to carry out a calculational trick to make the integral spherically symmetric. Secondly, by performing a Wick rotation (to be discussed in the next section) one obtains a spherically symmetric integral in Euclidean space. One can then reduce this to an integral along the radius of this higher-dimensional sphere times an integration over the solid angle, which greatly simplifies the integration.

There are two well-known procedures to 'spherify' the integral, i.e. to combine propagators, known as the Schwinger Trick [5] and the Feynman Parameters [4], both the namesakes of respective physicist. On a side note, apparently Schwinger remained bitter over the fact that the procedure was most commonly referred to as Feynman parametrization, since he came up with his version somewhat earlier.

The Schwinger Trick is based on the identity that:

$$\frac{1}{A} = \int_0^\infty dv e^{-Av} \tag{2.15}$$

for $A > 0$, where the integral is well-defined. We can apply this procedure to a product of propagators:

$$\prod_{i=1}^{N} \frac{1}{A_i} = \left(\prod_{i=1}^{N} \int_0^\infty dv_i \right) e^{-\sum\limits_{i=1}^{N} A_i v_i} \tag{2.16}$$

If we define $v = \sum v_i$ and $\alpha_i = v_i/v$ the following relation is true:

$$\prod_{i=1}^{N} dv_i = v^{N-1} dv \prod_{i=1}^{N} d\alpha_i \times \delta\left(1 - \sum_{i=1}^{N} \alpha_i\right) \tag{2.17}$$

which we can insert in our expression to obtain:

$$\prod_{i=1}^{N} \frac{1}{A_i} = \left(\prod_{i=1}^{N} \int_0^\infty d\alpha_i \right) \delta\left(1 - \sum_{i=1}^{N} \alpha_i\right) \int_0^\infty v^{N-1} dv \times e^{-v \sum\limits_{i=1}^{N} \alpha_i A_i} \tag{2.18}$$

If we now recognize the Euler Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt = (z-1)! \tag{2.19}$$

for which we have that:

$$\int_0^\infty t^{z-1} e^{-bt} dt = \frac{1}{b^z} \Gamma(z) \tag{2.20}$$

We find our final expression for The Schwinger Trick:

$$\prod_{i=1}^{N} \frac{1}{A_i} = \Gamma(N) \left(\prod_{i=1}^{N} \int_0^\infty d\alpha_i \right) \frac{\delta\left(1 - \sum\limits_{i=1}^{N} \alpha_i\right)}{\left(\sum\limits_{i=1}^{N} \alpha_i A_i \right)^N} \tag{2.21}$$

The Feynman parameters amounts to basically the same result, by initially noting that:

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{(xA + (1-x)B)^2} = \int_0^1 dx \int_0^1 dy \frac{\delta(1-x-y)}{(xA+yB)^2} \tag{2.22}$$

We can thereafter extend this derivation by repeatedly differentiating with respect to B:

$$\begin{aligned} \frac{1}{AB^2} &= \int_0^1 dx \int_0^1 dy \frac{2y \times \delta(1-x-y)}{(xA+yB)^3} \\ \frac{1}{AB^3} &= \int_0^1 dx \int_0^1 dy \frac{3y^2 \times \delta(1-x-y)}{(xA+yB)^4} \\ \frac{1}{AB^N} &= \int_0^1 dx \int_0^1 dy \frac{Ny^{N-1} \times \delta(1-x-y)}{(xA+yB)^{N+1}} \end{aligned} \tag{2.23}$$

This we may finally utilize to prove the general formula by induction. We will here refer the reader to [4], where the following is derived:

$$\prod_{i=1}^{N} \frac{1}{A_i} = (N-1)! \left(\prod_{i=1}^{N} \int_0^1 d\alpha_i \right) \frac{\delta\left(1 - \sum_{i=1}^{N} \alpha_i\right)}{\left(\sum_{i=1}^{N} \alpha_i A_i\right)^N} \tag{2.24}$$

where we have renamed the parameters to highlight the similarities with Schwinger's earlier calculational procedure.

Regardless of which calculational procedure is utilized, the procedure affects the integration in the following manner:

$$\begin{aligned} \int d^d k \frac{\mathcal{N}(k_i, m_i, k_j)}{(k_1^2 - m_1^2 + i\epsilon)(k_2^2 - m_2^2 + i\epsilon)...(k_n^2 - m_n^2 + i\epsilon)} = \\ (n-1)! \left(\prod_{i=1}^{n} \int_0^1 d\alpha_i \right) \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \int d^d k \frac{\mathcal{N}(k_i, m_i, p_j)}{(\alpha_1 A_1 + \alpha_2 A_2 + ... + \alpha_n A_n)^n} \end{aligned} \tag{2.25}$$

where $A_i = k_i^2 - m_i^2$. Due to the fact that the denominator only contains a second order polynomial in k we may complete the square by a change of variable $k \to q = k + \text{const}$, such that no linear factors of k remain. This leads to an expression:

$$(n-1)! \left(\prod_{i=1}^{n} \int_0^1 d\alpha_i \right) \delta\left(1 - \sum_{i=1}^{n} \alpha_i\right) \int d^d q \frac{\mathcal{N}(q, \alpha_i, m_i, k_j)}{(q^2 - M^2 + i\epsilon)^n} \tag{2.26}$$

In the denominator, only various powers of q^2 appear and thus the denominator will always be an even function.

The integral does not depend on any vectorial quantities and such if there are odd powers powers of q^μ in the numerator, these leave an odd integrand, causing the integral to vanish. Any even powers of q^μ reduce to $q^\mu q^\nu \propto g^{\mu\nu} q^2$. To see this, note that the integral will vanish by symmetry unless $\mu = \nu$. By Lorentz invariance, proportionality with $g^{\mu\nu}$ follows.

Our general expression thus reduces to a series of more manageable integrals:

$$\int d^d q \frac{q^{2m}}{(q^2 - M^2 + i\epsilon)^n} \tag{2.27}$$

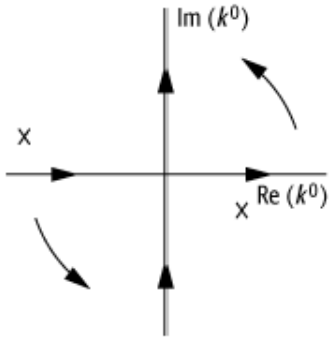

Figure 2.3: Wick rotation

2.5.3 Wick Rotation

The Minkowski metric $\{1; -1, -1, -1\}$ does not offer any simple means of integrating over solid angles over all four dimensions. A calculational procedure dubbed Wick rotation can however be used to turn the metric into the ordinary Euclidean metric, $\{1; 1, 1, 1\}$.

This Wick rotation is based on utilizing Cauchy's Integral Theorem on the 0-component of the internal momentum vector in the complex plane. As an example, consider the scalar integral for $m = 0$:

$$\int d^d q \frac{1}{(q^2 - M^2 + i\epsilon)^n} \tag{2.28}$$

For the integration over the 0-component of q in the complex plane we want to carry out a complex integration along a closed curve as in figure 2.3.

There are two poles whose location are determined by the $i\epsilon$-prescription of the propagators. We have been using the most common prescription, the Feynman prescription. The pole appears when $q^2 = M^2 - i\epsilon$, where the q_0 component corresponds to $q_0^2 = \vec{q}^2 + M^2 - i\epsilon$. Therefore, for the integration variable q_0 we have poles at

$$q_0 = \pm\sqrt{\vec{q}^2 + M^2 - i\epsilon} \approx \pm\sqrt{\vec{q}^2 + M^2}\left(1 - \frac{i\epsilon}{\vec{q}^2 + M^2}\right) \tag{2.29}$$

which corresponds to poles in the lower, right quadrature and upper, left quadrature.

If we integrate along a closed curve that follows the real and imaginary axes and connects these at infinity in the upper, right and lower, left quadrature we may close the curve without enclosing any poles. Naming these integrals by the subscripts R, I, UR and LL respectively, we notice that $I_R + I_I + I_{UR} + I_{LL} = 0$. For $n \geq 1$, which we require to have propagators, the integral vanishes for $|q_0| \to 0$, such that $I_{UR}, I_{LL} = 0$ and we get $I_R + I_I = 0$. Here the integration I_I extends as $\int_{+\,\mathrm{inf}}^{-\,\mathrm{inf}}$, while we would like to integrate $I_I' = \int_{-\,\mathrm{inf}}^{+\,\mathrm{inf}} = -\int_{+\,\mathrm{inf}}^{-\,\mathrm{inf}} = -I_I$. Thus $I_R = I_I'$. For convenience, we can define a new vector as follows:

$$q_0 = iq_{0,E}; \vec{q} = \vec{q_E} \tag{2.30}$$

We can then rewrite the $m = 0$ scalar integral into:

$$\lim_{\epsilon\to 0}\int d^d q \frac{1}{(q^2 - M^2 + i\epsilon)^n} = \int d^d q_E \frac{i}{(-1)^n}\frac{1}{(q_E^2 + M^2)^n} \tag{2.31}$$

where the i-factor comes from the integration measure $d^d q = id^d q_E$ and $(-1)^n$ comes from $q^2 = -q_E^2$.

The expression has now been turned into a spherically symmetric, even function of q in Euclidean space, and we can therefore carry out a change of variables to spherical coordinates:

$$\int d^d q = \int_0^\infty d|q||q|^{d-1} \int d\Omega^{d-1} \tag{2.32}$$

2.5.4 Angular integration in d dimensions

To calculate the solid angle contribution to the integration, we will evaluate one integral using two different methods and compare the results. Let us consider the integral:

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 ... \int_{-\infty}^{\infty} dx_d \exp[-(x_1^2 + x_2^2 + ... + x_d^2)] \tag{2.33}$$

$$\tag{2.34}$$

Since

$$\int_{\infty}^{\infty} dx \exp[-x^2] = \sqrt{\pi} \tag{2.35}$$

and the dimensions separate we get

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 ... \int_{-\infty}^{\infty} dx_d \exp[-(x_1^2 + x_2^2 + ... + x_d^2)] = \pi^{d/2} \tag{2.36}$$

We may also write the expression in spherical coordinates as hinted in the previous section by writing:

$$\int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{\infty} dx_2 ... \int_{-\infty}^{\infty} dx_d \exp[-(x_1^2 + x_2^2 + ... + x_d^2)] \tag{2.37}$$

$$=: \int d\Omega^{d-1} \int_0^{\infty} d|r||r|^{d-1} \exp[-r^2] \tag{2.38}$$

$$= \int d\Omega^{d-1} \int_0^{\infty} \frac{dt}{2\sqrt{t}} t^{\frac{d-1}{2}} \exp[-t] = \frac{\int d\Omega^{d-1}}{2} \int_0^{\infty} dt \ t^{\frac{d}{2}-1} \exp[-t] \tag{2.39}$$

$$= \frac{\int d\Omega^{d-1}}{2} \Gamma(d/2) \tag{2.40}$$

where the definition of the Gamma function has been used. We can thus see that:

$$\int d\Omega^{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)} \tag{2.41}$$

2.5.5 Radial integration

For the radial component, we are left with an integral of the form:

$$\int_0^\infty d|k| \frac{|k_E|^{d-1}}{(|k_E|^2 + m^2)^s} \tag{2.42}$$

We may evaluate this by a series of variable substitutions:

- $|k_E| \to m \cdot y$

$$\int_0^\infty d|k| \frac{|k_E|^{d-1}}{(|k_E|^2 + m^2)^s} = \frac{m^d}{m^{2s}} \int_0^\infty dy \frac{y^{d-1}}{(1+y^2)^s} \tag{2.43}$$

- $y = \sinh u$ where we use the following relations:

$$1 + y^2 = 1 + \sinh^2 u = \cosh^2 = (1 - \tanh^2 u)^{-1} \tag{2.44}$$

$$y = (1 + y^2 - 1)^{1/2} = \left(\frac{1}{1 - \tanh^2 u} - 1 \right)^{1/2} = \left(\frac{\tanh^2 u}{1 - \tanh^2 u} \right)^{1/2} \tag{2.45}$$

$$dy = \cosh u du = (1 - \tanh^2 u)^{-1/2} du \tag{2.46}$$

- $v = \tanh^2 u$:

$$1 + y^2 = (1 - v)^{-1} \tag{2.47}$$

$$y = \left(\frac{v}{1 - v} \right)^{1/2} \tag{2.48}$$

$$dv = 2 \tanh u \cosh^{-2} u du = 2 \tanh u (1 - \tanh^2 u) du \tag{2.49}$$

$$dy = (1 - \tanh^2 u)^{-1/2} du = (1 - v)^{-1/2} \frac{1}{2(v)^{1/2}(1 - v)} dv \tag{2.50}$$

Giving the integral:

$$\frac{m^d}{m^{2s}} \int_0^\infty dy \frac{y^{d-1}}{(1+y^2)^s} = \frac{m^{d-1}}{m^{2s}} \int_0^1 dv \frac{(1-v)^{-1/2}}{2(v)^{1/2}(1-v)} \times \left(\frac{v}{1-v} \right)^{(d-1)/2} \times (1-v)^s \tag{2.51}$$

$$= (m^2)^{\frac{d}{2}-s} \int_0^1 dv \frac{1}{2} \times v^{(d-1)/2 - 1/2} \times (1-v)^{s - 3/2 - (d-1)/2} \tag{2.52}$$

$$= (m^2)^{\frac{d}{2}-s} \int_0^1 dv \frac{1}{2} \times v^{\frac{d}{2}-1} \times (1-v)^{s - \frac{d}{2} - 1} \tag{2.53}$$

$$= \frac{(m^2)^{\frac{d}{2}-s}}{2} \frac{\Gamma(\frac{d}{2}) \Gamma(s - \frac{d}{2})}{\Gamma(s)} \tag{2.54}$$

Where we have used two definitions of the beta function:

$$\int_0^1 dt t^{x-1} (1-t)^{y-1} =: B(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2.55}$$

2.5.6 Angular and Radial Integration for Tensorial Integrals

Combining the angular and radial integration of the scalar integral gives:

$$\int d^dk \frac{1}{[k^2 - M^2]^n} = (-1)^n i\pi^{d/2} \frac{\Gamma(n - d/2)}{\Gamma(n)} (M^2)^{d/2-n} \tag{2.56}$$

In the language of the nomenclature for scalar integrals in one-loop calculations, this gives the contribution to the A_0 integral when $n = 1$, the B_0 integral when $n = 2$, and so on. These will then be integrated over their respective Feynman parameters to give a final result.

In principal, combined with a reduction scheme such as the Passarino-Veltman Reduction discussed previously, this is sufficient to completely calculate any one loop amplitude. In practice, it may at times be useful to calculate some tensorial integrals. Assuming that the integral has been made spherically symmetric by means of Feynman parametrization, these can be calculated with some slight modifications to the calculations carried out in these sections.

Firstly, note that for all *odd powers* of the internal loop momentum, k^μ, the integrand is odd and the integration vanishes. For *even powers* any factor $k^\mu k^\nu$ must be proportinal to the metric tensor $g^{\mu\nu}$, due to Lorentz invariance. By contracting both sides with $g_{\mu\nu}$, one may determine the the proportionality factor to be $k^\mu k^\nu \to \frac{1}{d} k^2 g^{\mu\nu}$. Thus our general numerator factor from eq [2.1], $\mathcal{N}$, is reduced to a linear combination of various integrals with $(k^2)^r$, where $r = 0, 1, 2, 3, ...$

From this point on, for even powers, we may merely note that for

$$|k_E|^{2r} |k_E|^{d-1} \tag{2.57}$$

which leads to an adjustment of the gamma functions in the denominator of the answer by:

$$\Gamma(d/2)\Gamma(n - d/2) \to m^{2r}\Gamma(d/2 + r)\Gamma(n - d/2 - r) = m^{2r} \left(\frac{d}{2}\right)^r \Gamma(d/2)\Gamma(n - d/2 - r) \tag{2.58}$$

Leading to the generalized integral:

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^{2r}}{[k^2 - m^2]^n} = \left(\frac{d}{2}\right)^r \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2 - r)}{\Gamma(n)} (M^2)^{d/2-n+r} \tag{2.59}$$

which for example, by $k^\mu k^\nu \to \frac{1}{d} k^2 g^{\mu\nu}$, gives the tensorial integral for $k^\mu k^\nu$:

$$\int \frac{d^dk}{(2\pi)^d} \frac{k^\mu k^\nu}{[k^2 - m^2]^n} = \frac{g^{\mu\nu}}{2} \frac{(-1)^{n-1} i}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)} (m^2)^{d/2-n+1} \tag{2.60}$$

Finally, we note that the Euler Gamma function, $\Gamma(x)$, diverges for integers $x \leq 0$. By inspection of the integrals in this section, we see that the factor $\Gamma(n - d/2)$ in the scalar integral and factors reminiscent of this in the tensorial integrals, may become ≤ 0 for small enough s, in the scalar case $n = 2$, the two-point one-loop integral, and for the k^2 case for $n = 3$, the three-point one-loop integral. These divergences will be the topic of the following section.

2.6 Dealing with Divergences

2.6.1 Introduction

As seen in the previous section, when calculating radiative corrections, the situation often arises when we encounter divergent integrals. We will therefore take a slight detour from the calculation of scalar integrals to delve into some depth of the background and solution for dealing with divergent integrals in particle physics.

The procedure for practical calculations with divergences is almost exclusively three-fold; the first step is to isolate divergences from finite terms in a *regularization procedure*. There is no unique way of doing this, but rather there are different schemes one can employ. Having done this, UV-divergences are dealt with through a *renormalization procedure*. Again, there is not a unique method of carrying out these calculations, but rather different schemes, with different merits. Finally, one can turn to the question of IR-divergences.

We start of by discussing the historical context in which they became relevant, and how to deal with UV-divergences by renormalization. We thereafter cover a discussion of infrared divergences, where we discuss the different approach these warrant. This is followed by a discussion of how regularization schemes are employed, and their use in separating divergent terms from finite contributions in the calculations. Finally we apply these concepts to the electron's self energy as an introductory example of their application. At this point, we are ready to return to the general discussion of solving scalar integrals, where we discuss how the divergent terms arise in their expression and explicitly give the results for one- to four-point functions and provide a list of all UV-divergent parts of any tensorial or scalar integral.

2.6.2 Occurence of renormalization procedures

The observation of internal divergences trace back to related issues in classical electrodynamics that were recognized already in the 19th century (however not explained satisfactory at that point). These issues have to do with the manner in which the intrinsic properties of a particle are connected to its environment. The most common example in this context is that of the electron's self energy. From the Maxwell equations, we find that the charge of the electron is tied to the electromagnetic field that is around it. When measuring for example the energy of an electron, one cannot achieve an arbitrary closeness to the electron, but will include some of the energy contained in the electromagnetic field surrounding the electron due to the inaccuracy of the testing equipment. The energy carried by the electron, the so-called self-energy, therefore includes the energy contained in the field around the electron. This surrounding energy scales inversely proportional to the distance by Coulumb's Law, meaning that it provides an infinite contribution to the electron's energy. However, when measured, the energy appears to be finite.

The energy of an electromagnetic field including a spherical source diverges in a linear fashion in the classical description. It turns out that it also diverges in the quantum mechanical regime. However, it diverges far slower in quantum mechanics than classically, merely diverging logarithmically, a result [7] found by Victor Weisskopf during his time as Wolfgang Pauli's assistant in Zurich in 1934, as opposed to its linear divergence in classical electromagnetic theory.

Historical background

The various issues in dealing with divergences were resolved through a series of insightful observations. In 1943 the Swiss Physicist Ernst Stuckelberg came up with what we nowadays call a renormalization program (however, his paper was rejected for publication by the Physical Review). Later, independently of this unpublished result, Hans Bethe came up with a similar scheme in 1947. Bethe's scheme originated in the experimental work [8] of Lamb and collaborators on the 'Lambshift' . The Lambshift shows level-splittings which are not predicted by the Dirac Equation. Later that year, Bethe came up with an answer that was, although approximate, sufficient to carry out what was to become renormalization [9]. In addition to offering a first theoretical calcuation of the Lambshift, it made it possible to deal with the divergences that occured. Basically, after attending a conference, Bethe completed the first non-relativistic computation of the Lambshift to the first order. However, he later found that

higher-order calculations led to infinities. Bethe attached these infinities to constants in the calculation, thereby rendering finite results.

In the upcoming two years, the birth of Quantum Electrodynamics lay at hand, spurred by calculations by Tomonaga, Schwinger and Feynman in a series of papers starting in 1948 for Tomonaga and Schwinger and 1949 for Feynman. They all proposed different methods for dealing with the calculations of quantum field theory, which at the time seemed contradictory.

Dyson, having studied the methods of Tomonaga, Schwinger and Feynman, wrote a series of papers, starting with [10] where he showed that the theories, although using quite different methods, were all equivalent.

Basic premise

The basic premise of the renormalization scheme, which has come to explain, among other things, the electron's self energy, is to replace every 'bare', diverging parameter, such as the electron's charge, by a renormalized finite parameter times an infinite factor Z_i, giving $e = Z_e e_0$, where e_0 is the bare charge and e is the renormalized charge. The Z-parameter is thereafter expanded in conjunction with the coupling constant, leading to expressions with various orders in Z. To lowest order, this expression is always taken to equal unity, thus preserving the non-divergent nature of first order calculations, while higher orders of Z have a divergent nature defined in such a manner that they cancel the infinities of higher order corrections of the amplitude.

This scheme allows us to absorb all divergences due to large energies, so called *ultraviolet* divergences, into the parameters of the theory. That the high energy behaviour of the fields is able to be absorbed for lower order calculations of the perturbation theory shows its inability to explain this region of phase space, but also illustrate that this information is not truly necessary for understanding the theory at its low energy approximation.

We will look closer at these principles in section 2.9.2.

2.6.3 Soft and collinear divergences

In the infrared region, there isn't an equivalent procedure that makes sense as renormalization. Not having this option at hand, what can be done is to consider the discrepancy between the actual physical situation being measured and the calculation.

The radiative corrections that we are treating involves taking the matrix element squared of several amplitudes with each other and a tree level diagram. In this process, it will be impossible to by experiment detect the differences between this process and the process including soft emission of particles, if the emission is of a lower energy than a cut-off energy given by the limitations of the experiment. Therefore, one includes *bremsstrahlung* diagrams, in the limit that the energy of the emitted particle goes to zero. When we include these physically indistinguishable amplitudes, the infrared divergences cancel out between them.

2.6.4 Regularization

Regularization is a method that assists in dealing with renormalization and the infrared and collinear divergences. Essentially, by regularizing your calculation, you are able to clarify and separate the divergences from the rest of your expression, making them easier to address.

The basic idea of regularization is to, in one or another manner, perturb a parameter of your expression by some α before calculation. While proceding with the calculation, the terms involving α are as thoroughly separated from the other terms as possible. After your calculation, you can let $\alpha \to 0$ and the divergences will be confined to those terms involving α.

Among the many different ways in which an expression may be perturbed, some schemes have proved highly successful over the years:

- **Cutoff regularization**: This more primitive regularization is achieved by simply introducing cutoffs for the integral limits. This leads to translation invariance only being conserved approximately, since integrals no longer run over the entire momentum space. One consequence of this is that one needs to work with modified Dirac delta functions.

  The typical regularization will give a resulting integral

  $$I = A(\Lambda) + B + C(\frac{1}{\Lambda}) \tag{2.61}$$

  We then want to take the physical limit of the calculation, i.e $\Lambda \to \infty$. This allows us to drop the C-factor directly. The A-factor, however, diverges, and to find a manner to deal with this is the topic of renormalization.

- **Pauli-Villars regularization**: This is a clear improvement over the naive cutoff regularization in that translation invariance is preserved. It regulates by introducing additional fields into the expression. The physical limit of the fields is achieved by sending the masses of the fields to infinity, causing their contribution to vanish. In a propagator term

  $$\frac{1}{p^2 - m^2} \to \lim_{\Lambda \to \infty} \frac{1}{p^2 - m^2} - \frac{1}{p^2 - \Lambda^2} \tag{2.62}$$

  this will typically leave an expression similar to the cutoff regularization, which, when taking the physical limit leaves us with one divergent and one finite term.

  The positive aspect of Pauli-Villars regularization is that it is quite intuitive in its procedure. However, in practical applications beyond the most straighforward calculations, it is often inconvenient. For example in a non-Abelian gauge theory, the self-interactions among the gauge bosons make calculations cumbersome. In addition, Pauli-Villars does not preserve gauge symmetry in theories such as electromagnetism, since, when introducing gauge fields with non-zero mass into a theory, this breaks the gauge symmetries of the theory. This gives the mass term that Pauli-Villars adds, with large Λ, more polarizations than a real photon. In QED calculations, the photon couples to a conserved current, so the longitudinal polarizations decouple, making it more manageable, but great care is still needed for these calculations.

- **Analytic regularisation**: The typical propagator $(p^2 + m^2)^{-1}$ is replaced by $(p^2 + m^2)^{-(1+\epsilon)}$. The physical limit is achieved found when letting $\epsilon \to 0$.

- **Dimensional regularization**: This method has, due to it being far more efficient in less trivial theories, become the most commonly practiced regularization scheme. This is also due to the fact that it respects far more symmetries than the previous schemes. In it, it is observed that divergences that arise in 4 dimensions, do not necessarily do so in other dimensionalities. Therefore, expressions are evaluated in d-dimensional space. After this, the dimensionality is shifted by $d = 4 - 2\varepsilon$, and one obtains results typically involving:

  $$I = A(\frac{1}{\varepsilon}) + B + C(\varepsilon) \tag{2.63}$$

  C-factors can be dropped as previously, while renormalization is needed to solve the situation for A-factors.

- **Lattice regularization** Lattice regularization looks at the issue from a slightly different perspective. In lattice regularization, you assume that the universe is not a continuum, but a discrete lattice. With a discrete lattice, integrals turns into sums. Therefore integration is no longer performed to infinity, but rather there is a natural cutoff, the lattice spacing a. Just as in other regularization schemes, the calculation is then taken to the physical limit; in this case that the lattice spacing $a \to 0$ (you then have to deal with a-dependent terms just as you would deal with for example Λ-dependent terms in Pauli-Villars)

  A unique feature of lattice regularization is that it is not necessarily perturbative, therefore you can use it to renormalize non-perturbative quantum field theory. In particular, this has been implemented in Lattice QCD calculations, where the strong force described by Quantum Chromodynamics is part of the expressions, since the scope of the perturbative regime of QCD is limited.

- **Zeta regularization:** Zeta regularization is based on an observation by Julian Schwinger that:

$$I(n,\Lambda) =: \int_0^\infty \approx 1 + 2^n + 3^n + ... + \Lambda^n = \sum_{i>0} i^{-(-n)} := \zeta(-n) \tag{2.64}$$

  While this turned out to give inconsistent results, an extension of these arguments by Hartle, J. Garcia, and E. Elizalde, was able to remove these inconsistencies. In brief, the ordinary Riemann zeta function is replaced by the generalized Riemann zeta function, also called the Hurwitz zeta function

$$\zeta(s,q) = \sum_{i>0} (q+i)^{-s} \tag{2.65}$$

  This is convergent for $s > 0$ and while q is not a negative integer or zero, however for these purposes, these are the interesting cases. Therefore, an analytic continuation to this area of parameter space can be done. This gives (for a thorough discussion on the topic, see [27]) a result which is a linear combination of (finite) zeta function values and a factor involving Bernoulli numbers. What is interesting to note about this is that it offers completely finite answers, thus providing both a regularization and renormalization scheme in one.

  On an interesting sidenote, this zeta regularization procedure also provides a means to make sens of such infinite series such as $1+1+1+1+....$ This is the value of the Riemann zeta function at $s = 0$. By performing the mentioned analytic continuation one in this case obtains

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s} \\
\to \zeta(s) &= 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\, \zeta(1-s) \\
\to \zeta(0) &= \frac{1}{\pi} \lim_{s\to 0} \sin\left(\frac{\pi s}{2}\right) \zeta(1-s) = \frac{1}{\pi} \lim_{s\to 0} \left(\frac{\pi s}{2} - \frac{\pi^3 s^3}{48} + ...\right) \left(-\frac{1}{s} + ...\right) = -\frac{1}{2}
\end{aligned} \tag{2.66}$$

  where the power series expansion of $\zeta(s)$ about $s = 1$ has the following expression due to the fact that the zeta function has a simple pole whose residue is 1.

2.6.5 Dimensional regularization

Introduction

One of these schemes, dimensional regularization, has proved highly effective in the calculation of higher order corrections. Introduced in 1972 by Veltman and 't Hooft (and by Bollini and Gambiagi), it is based on the observation that ultraviolet divergences converge for lower dimensions and infrared divergences converge for higher dimensionalities higher than 4. Therefore, it is reasonable to perform a sort of analytic continuation in the spacetime dimensions, which gives rise to the name of the procedure.

As hinted at in previous sections of this text, this principle has been the reason that we formally evaluated integrals in a general, d dimensional setting, rather than the 4 spacetime dimensions we are used to.

Dimensional regularization and Symmetries

For non-trivial calculations, is turns out to make calculations far more manageable than it's alternatives, such as Pauli-Villars regularization. However, it does have several other merits in addition to this fact.

One of these is that it preserves the Ward identities (or gauge invariance), which are essential to the proofs of the unitarity of the S-matrix, since it does not introduce any new massive fields.

Also, it preserves translational invariance. In for example cutoff regularizations, linear divergences do not preserve this symmetry, which leads to anomalies in these schemes. Translation invariance being one of the main definitions in d-dimensional integration, dimensional reduction does not suffer from such difficulties.

Dirac matrices in Dimensional Regularization

For d dimensions, contractions and traces of Dirac gamma matrices are slightly modified. To this note, start by contracting two metric tensors, we obtain $g_{\mu\nu}g^{\mu\nu} = d$. In turn, this modifies the contraction of gamma matrices:

$$\gamma_\mu\gamma^\mu = g_{\mu\nu}\gamma^\mu\gamma^\nu = \frac{1}{2}g_{\mu\nu}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu) = g_{\mu\nu}g^{\mu\nu} = d\mathbb{1} \tag{2.67}$$

For $d = 4 - 2\varepsilon$, we obtain for example:

$$\begin{aligned} \gamma_\mu\gamma^\nu\gamma^\mu &= -2(1-\varepsilon)\gamma^\nu, \\ \gamma_\mu\gamma^\nu\gamma^\rho\gamma^\mu &= 4g^{\nu\rho} - 2\varepsilon\gamma^\mu\gamma^\rho \end{aligned} \tag{2.68}$$

It is also worth noting that, for d dimensions, Tr $1 = 2^{d/2}$, but the trace is always associated with a fermion loop, and we can therefore adjust the fermion number in a continuous manner, cancelling the arising ε-dependence. We can therefore always use Tr $1 = 4$ in calculations.

Drawbacks of Dimensional Regularization

There are also setbacks in applying dimensional regularization. They include the inability to handle dimensionally dependent quantities such as the γ_5 matrix, the projection matrices $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ and the antisymmetric tensor $\epsilon_{\alpha\beta\gamma\delta}$.

In some cases these quantities plays such an important role that they are necessary to prove some of the abovementioned Ward identities. In these cases the method does fail to preserve these Ward identities due to its inability to generalize these quantities to d dimensions.

Let us discuss the first example, that of the Dirac matrix γ_5 in some more detail. In four dimensions, it is defined as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 \tag{2.69}$$

.

One solution that is often employed in these situations is to leave the γ_5 terms in four dimensions and split the other γ functions into one four dimensional part and one $d-4$ dimensional part, $\gamma_\mu = \hat{\gamma} + \tilde{\gamma}$, where the $\hat{\gamma}$ is the four dimensional term. By this definition, the anticommutation relation becomes

$$\{\gamma^\mu, \gamma_5\} = \begin{cases} 0 & \mu \in \{0,1,2,3\} \\ 2\tilde{\gamma^\mu}\gamma_5 & \text{otherwise.} \end{cases} . \tag{2.70}$$

The last line also implies that γ_5 commutes with $\tilde{\gamma}_\mu$ matrices, which can be seen as the γ_5 acting trivially in the $d-4$ dimensional subspace orthogonal to 'ordinary' spacetime.

Another solution [24], which preserves γ_5 anticommutation, is to redefine the Trace operation (which is involved in for example the definition of the trace operation using a projection onto four dimensional subspace). Instead one has to give up cyclicity for γ_5 odd traces. It is still, however, fully consistent to implement in regularization of UV-, collinear- (M) and IR-divergences. This scheme also allows for extension into any parity-violating theory consistently.

Regularization schemes

Related to this issue, several of the quantites typically involved in an amplitude expression, such as helicities of gauge bosons and massless fermions, usually introduce projection matrices. There exist different schemes for continuing the Dirac-algebra into d-dimensions:

- **Conventional dimensional regularisation (CDR):** All momenta and all polarisation vectors are continued into d dimensions.
- **'t Hooft-Veltman scheme (HV):** momenta and helicities of unobserved particles are continued into d- dimensions, while momenta and helicities of the observed particles are 4-dimensional.
- **Dimensional reduction (DR):** momenta and helicities of the observed particles are kept, and so are the internal polarisation vectors. Only unobserved particles momentum is continued into d-dimensions. On a side-note, this is the only one of these schemes which can respect supersymmetry. One then introduces extra particles, so-called ε-scalars, with the same mass and charge. The extra scalar fields introduced compensate for the degrees of freedom 'lost' in dimensional regularization. This is the practical way in which the vector fields keep their fourdimensionality mentioned.

At a one-loop level, the transition between different schemes has been worked out in detail [25].

Dimensionality considerations

In any quantum field theory, the action $S = \int d^dx\mathcal{L}$ is dimensionless (in units of $\hbar = 1$). In these units, the integral is of units $(\text{mass})^{-d}$, and thus the Lagrangian has units $(\text{mass})^d$. fermion fields enter as $\bar{\Psi}\ \not{\partial}\Psi$, leading to a mass dimension found by $d = 1 + 2D_F \rightarrow D_F = 3/2 - \varepsilon$, where D_F is the dimensionality of the fermion, and bosons enter as $(\partial\phi)^2$, leading to a mass dimension found by $d = 2 + D_B \rightarrow D_B = 1 - \varepsilon$, where D_B is the dimensionality of the boson.

If we now consider the interaction terms, we see that the coupling gets shifted away from its usual dimensionality in four dimensions (which is zero). For a renormalizable theory, it's

couplings are dimensionless in four dimensions. To remedy this, we extract the dimensionality of the coupling by $e_0 = \tilde{e_0}(\mu) - \mu^\epsilon$, where μ is a given mass scale and $\tilde{e_0}$ is a dimensionless quantity.

Since e_0^2 typically enters into the integrals we have considered, this given mass scale μ, is the constant that we included into our definitions for scalar integrals.

2.7 Regularization of Scalar Integrals

2.7.1 Introduction

We left our discussion on the general solution to scalar and tensorial integrals before integrating over the Feynman parameters. With this done, our solution of the scalar n-point functions would in principle be complete. The results we would obtain, as hinted earlier, would however be divergent. It is common practice to apply the means of dimensional regularization ahead of the integration over Feynman parameter space. We will therefore begin by discussing how the dimensional regularization to extract divergent terms is done in this section, and move on to the Feynman parameter integration in the following section. Let us now state some of the results from the previous section for ease and accessibility:

- The general integration over angular and radial components was given by

$$\int d^dk \frac{1}{[q^2 - M^2]^n} = (-1)^n i\pi^{d/2} \frac{\Gamma(n - d/2)}{\Gamma(n)} (M^2)^{d/2-s} \tag{2.71}$$

  This expression is for the shifted variables that arise from Feynman parametrization.

- We used the Feynman parametrization to get our expression into the form

$$\begin{aligned} & \int d^dk \frac{1}{(k_1^2 - m_1^2 + i\epsilon)(k_2^2 - m_2^2 + i\epsilon)...(k_n^2 - m_n^2 + i\epsilon)} \\ = \; & (n-1)! \left(\prod_{i=1}^n \int_0^1 d\alpha_i \right) \delta \left(1 - \sum_{i=1}^n \alpha_i \right) \int d^dk \frac{1}{(\alpha_1 A_1 + \alpha_2 A_2 + ... + \alpha_n A_n)^n} \\ = \; & (n-1)! \left(\prod_{i=1}^n \int_0^1 d\alpha_i \right) \delta \left(1 - \sum_{i=1}^n \alpha_i \right) \int d^dq \frac{1}{(q^2 - M^2 + i\epsilon)^n} \end{aligned} \tag{2.72}$$

- We defined our general tensorial integral by the equation

$$\begin{aligned} & T^n_{\mu_1...\mu_p}(p_1, ..., p_{n-1}, m_0, ..., m_{n-1}) \\ = \; & \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk \frac{\mathcal{N}(q_i, m_i, p_j)}{(k_0^2 - m_0^2 + i\epsilon)(k_1^2 - m_1^2 + i\epsilon)...(k_{n-1}^2 - m_{n-1}^2 + i\epsilon)} \end{aligned} \tag{2.73}$$

- Note that the prefactor in the expression arises solely due to aesthetic purposes, and differs from the prefactors coming out of the Feynman rules by a factor $\frac{i\pi^2}{(2\pi)^4} \times \frac{1}{(4\pi\mu)^{4-d}}$.

- Since a reduction scheme such as the Passarino-Veltman reduction allows us to consistently reduce any tensorial integral to an integral which is a linear combination of different vector and tensor quantites times scalar coefficient functions, our problem is reduced to calculating coefficient functions of the form:

$$\bar{T}^n(p_1,...,p_{n-1},m_0,...,m_{n-1})$$
$$= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk \frac{1}{(k_0^2-m_0^2+i\epsilon)(k_1^2-m_1^2+i\epsilon)...(k_{n-1}^2-m_{n-1}^2+i\epsilon)}$$
$$= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\left[(n-1)!\left(\prod_{i=1}^{n}\int_0^1 d\alpha_i\right)\delta\left(1-\sum_{i=1}^{n}\alpha_i\right)\right]\left[(-1)^n i\pi^{d/2}\frac{\Gamma(n-d/2)}{\Gamma(n)}(M^2)^{d/2-n}\right] \tag{2.74}$$

where $M = M(m_i, p_{ij})$ is the effective mass term arising from the Feynman parametrization and q the shifted internal loop momentum.

2.7.2 Regularization

What we wish to do now is to carry out the dimensional regularization procedure and thereafter perform the integration over Feynman parameters.

To this end, let us begin by employing the standard dimensionality of dimensional regularization, that of applying the dimension $d = 4 - 2\varepsilon$. This allows us sort out our prefactors in the expression:

$$\frac{(2\pi\mu)^{4-d}}{i\pi^2}\times(-1)^n i\pi^{d/2} \to \frac{(2\pi\mu)^{2\varepsilon}}{i\pi^2}\times(-1)^n i\pi^{2-\varepsilon} = (-1)^n(4\pi\mu^2)^{\varepsilon} \tag{2.75}$$

Now, there are two factors which we need to look closer at, the the mass-terms and the Gamma factors.

Mass-terms Regularization

We begin by separating the mass-term in a dimensionality-independent term and a term containing the ε-terms:

$$\frac{M^2)^{d/2}}{(M^2)^s} \to \frac{(M^2)^{2-\varepsilon}}{(M^2)^s} = (M^2)^{2-s}\times(M^2)^{-\varepsilon} \tag{2.76}$$

In order to separate divergent and non-divergent terms, let us now expand the mass term. In practice, this expansion will be done together with the ε-dependent prefactor as

$$\left(\frac{M^2}{4\pi\mu^2}\right)^{-\varepsilon} = e^{\log\left(\frac{M^2}{4\pi\mu^2}\right)^{-\varepsilon}} = e^{-\varepsilon\log\left(\frac{M^2}{4\pi\mu^2}\right)} = 1-\varepsilon\log\left(\frac{M^2}{4\pi\mu^2}\right)+\mathcal{O}(\varepsilon^2) \tag{2.77}$$

Gamma Function Regularization

Depending on the value of n, that is, the number of propagators, the procedures for expansion varies slightly. However, they generally makes use of two properties of Gamma functions:

$$\Gamma(z+1) = z\Gamma(z) \to \Gamma(z) = \frac{\Gamma(z+1)}{z} \tag{2.78}$$

and the expansion of a Gamma function for small ε:

$$\Gamma(1\pm\varepsilon) = 1\mp\varepsilon\gamma_E+\mathcal{O}(\varepsilon^2) \tag{2.79}$$

where γ_E is the Euler-Mascharoni constant $\gamma_E = -\Gamma'(1) = \lim_{i\to\infty}\left(\sum_{k=1}^{i}\frac{1}{k}-\log i\right)\approx 0.57$.

In addition, a useful expansion to keep in mind will be:

$$\frac{1}{1-x} = 1 + x + \mathcal{O}(x^2) \tag{2.80}$$

By repeatedly using the first relation, it is possible to put the factor $\Gamma(n-2+\varepsilon)$ into $\Gamma(1+\varepsilon)$, thereby extracting the poles that contribute to divergences. Thereafter use the second relation to expand around this point. Terms of $\mathcal{O}(\varepsilon)$ or higher in our final expression are only relevant for two- or higher-loop diagrams.

- For $n = 3$, we trivially get the expression

$$\Gamma(3-2+\varepsilon) = \Gamma(1+\varepsilon) \approx 1 - \varepsilon\gamma_E + \mathcal{O}(\varepsilon^2) \tag{2.81}$$

- For $n = 2$, we get

$$\Gamma(2-2+\varepsilon) = \Gamma(\varepsilon) = \frac{\Gamma(1+\varepsilon)}{\varepsilon} \approx \frac{1}{\varepsilon}(1 - \varepsilon\gamma_E + \mathcal{O}(\varepsilon^2)) = \frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon) \tag{2.82}$$

- For $n = 1$, we get

$$\begin{aligned} \Gamma(1-2+\varepsilon) = \Gamma(-1+\varepsilon) = \frac{\Gamma(\varepsilon)}{\varepsilon-1} = \frac{\Gamma(1+\varepsilon)}{\varepsilon(\varepsilon-1)} &\approx \frac{1}{\varepsilon(\varepsilon-1)}(1 - \varepsilon\gamma_E + \mathcal{O}(\varepsilon^2)) \\ = \quad \frac{\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)}{(\varepsilon-1)} = -\frac{\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)}{(1-\varepsilon)} &= \left(\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon)\right)\left(1 + \varepsilon + \varepsilon^2 + \mathcal{O}(\varepsilon^3)\right) \\ = \quad \frac{1}{\varepsilon} - \gamma_E + 1 + \mathcal{O}(\varepsilon) & \end{aligned} \tag{2.83}$$

- For $n \geq 4$, the expression becomes:

$$\Gamma(n-2+\varepsilon) = \Gamma(1+\varepsilon)\prod_{i=1}^{n-3}[(n-2) - i + \varepsilon] \tag{2.84}$$

Results of regularization

Combining the results of the previous discussions, our general expression turns into:

$$\begin{aligned} &\frac{(2\pi\mu)^{4-d}}{i\pi^2}\left[(n-1)!\left(\prod_{i=1}^{n}\int_0^1 d\alpha_i\right)\delta\left(1 - \sum_{i=1}^{n}\alpha_i\right)\right]\left[(-1)^n i\pi^{d/2}\frac{\Gamma(n-d/2)}{\Gamma(n)}(M^2)^{d/2-s}\right] \\ &= (-1)^n(M^2)^{2-s}\left[\frac{\Gamma(n-d/2)}{\Gamma(n)}\right]\left[(n-1)!\left(\prod_{i=1}^{n}\int_0^1 d\alpha_i\right)\delta\left(1 - \sum_{i=1}^{n}\alpha_i\right)\right]\left[1 - \varepsilon\log\left(\frac{M^2}{4\pi\mu^2}\right) + \mathcal{O}(\varepsilon^2)\right] \end{aligned} \tag{2.85}$$

Here the factor involving Gamma functions is then to be calculated as outlined for various n-point functions in the previous section.

2.8 Feynman Parameter Integration

What is now left to do is to integrate over the relevant Feynman parameters of the problem. The only term that may depend on the Feynman parameters is the effective mass term M^2. As the number n of propagators grow with the number of legs, increasing the number of parameters to be integrated over, these integrals can become quite sophisticated, in particular due to the constraint of their sum equalling unity. In addition, through the momentum shift $k \to q$ performed, parameters start showing up in the numerator. There exist a consistent decomposition formula to put all integrals with Feynman parameters in the numerator into a sum of integrals without these numerator factors, greatly simplifying the calculation.

We will however consider the scalar integrals in this section, which does not include terms proportional to the loop momentum in the numerator, and therefore will not be shifted to include Feynman parameters in the numerator.

For these cases, it is often more straightforward to combine the terms from the Gamma function with the mass-term before performing the integral, something which will be illustrated for the scalar one- and two-point functions in the next sections.

2.8.1 Scalar One-point Function

For the scalar one-point function,

$$A_0(m) = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk \frac{1}{k^2-m^2} = \frac{(2\pi\mu)^{4-d}}{i\pi^2}\Gamma(1)\int_0^\infty dx \int d^dq \frac{\delta(1-x)}{x(q^2-M^2)} \tag{2.86}$$

the Feynman parametrization is trivially fulfilled (this may be seen by noting that $\frac{1}{P} = \Gamma(1)\int_0^\infty \frac{\delta(1-x)}{xP} = \frac{1}{P}\int_0^\infty \frac{\delta(1-x)}{x} = \frac{1}{P}$ or by simply not performing the Feynman parametrization). We therefore have $M \to m$ and $q \to k$-

Using equation 2.85 with the Gamma function for $n = 1$, we obtain:

$$\begin{aligned} A_0(m) &= (-1)^1(M^2)^{2-1}\left[\frac{1}{\varepsilon} - \gamma_E + 1 + \mathcal{O}(\varepsilon)\right]\left[1 - \varepsilon\log\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\varepsilon^2)\right] \\ &= -m^2\left[\frac{1}{\varepsilon} - \gamma_E + 1 - \log\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}(\varepsilon)\right] \\ &= m^2(\Delta - \log\frac{m^2}{\mu^2} + 1) + \mathcal{O}(\varepsilon) \end{aligned} \tag{2.87}$$

Where the UV-divergence is contained in the factor

$$\Delta = \frac{2}{4-d} - \gamma_E + \log 4\pi \tag{2.88}$$

with γ_E is Euler's constant.

2.8.2 Scalar Two-point Function

For the scalar two-point function,

$$\begin{aligned} B_0(p_{10}, m_0, m_1) &= \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk \frac{1}{(k^2-m_0^2)((k-p_{10})^2-m_1^2)} \\ &= \int_0^1 dx \int_0^1 dy \int d^dq \frac{\delta(1-x-y)}{(q^2-M^2)^2} \end{aligned} \tag{2.89}$$

Applying equation 2.85 with the Gamma functions for $n = 2$, we reach the result that

$$\begin{aligned} B_0(p_{10}, m_0, m_1) &= (-1)^2 (M^2)^{2-2} \left[(2-1)! \int_0^1 dx \int_0^1 dy \delta(1-x-y) \right] \times \\ &\times \left[\frac{1}{\varepsilon} - \gamma_E + \mathcal{O}(\varepsilon) \right] \left[1 - \varepsilon \log \left(\frac{M^2}{4\pi\mu^2} \right) + \mathcal{O}(\varepsilon^2) \right] \end{aligned} \tag{2.90}$$

Applying the same Δ that was defined in the previous section, we obtain the expression

$$B_0(p_{10}, m_0, m_1) = \Delta - \int_0^1 dx \log \frac{(p_{10}^2 x^2 - x(p_{10}^2 - m_0^2 + m_1^2) + m_1^2 - i\epsilon)}{\mu^2} + \mathcal{O}(\varepsilon) \tag{2.91}$$

This integral can be carried out as [12]:

$$\Delta + 2 - \log \frac{m_0 m_1}{\mu^2} + \frac{m_0^2 - m_1^2}{p_{10}^2} \log \frac{m_1}{m_0} - \frac{m_0 m_1}{p_{10}^2} \left(\frac{1}{r} - r \right) \log r + \mathcal{O}(\varepsilon) \tag{2.92}$$

where r and $\frac{1}{r}$ are determined by

$$x^2 + \frac{m_0^2 + m_1^2 - p_{10}^2 - i\epsilon}{m_0 m_1} x + 1 = (x + r) \left(x + \frac{1}{r} \right) \tag{2.93}$$

This result is valid for arbitrary physical parameters. This since the variable r never crosses the negative real axis even for complex physical mass-terms. For $r < 0$ the Feynman prescription gives $\Im r = \epsilon \left(r - \frac{1}{r} \right)$.

2.8.3 Scalar Three-point Function

The general result for the scalar three-point function has been shown to be possible to be put into the following form [12]:

$$\begin{aligned} & C_0(p_{10}, p_{20}, m_0, m_1, m_2) = \\ - & \int_0^1 dx \int_0^x dy (p_{21}^2 x^2 + p_{10}^2 y^2 + \left(p_{20}^2 - p_{10}^2 - p_{21}^2 \right) xy \\ + & \left(m_1^2 - m_2^2 - p_{21}^2 \right) x + \left(m_0^2 - m_1^2 + p_{21}^2 - p_{20}^2 \right) y + m_2^2 - i\epsilon \Big)^{-1} \\ = & \frac{1}{\alpha} \sum_{i=0}^{2} \left[\sum_{\sigma=\pm} \left(\mathrm{Li}_2 \left(\frac{y_{0i} - 1}{y_{i\sigma}} \right) - \mathrm{Li}_2 \left(\frac{y_{0i}}{y_{i\sigma}} \right) + \eta \left(1 - x_{i\sigma}, \frac{1}{y_{i\sigma}} \right) \log \frac{y_{0i} - 1}{y_{i\sigma}} - \eta \left(-x_{i\sigma}, \frac{1}{y_{i\sigma}} \right) \log \frac{y_{0i}}{y_{i\sigma}} \right) \right. \\ - & \left. \left[\eta(-x_{i+}, -x_{i-}) - \eta(y_{i+}, y_{i-}) - 2\pi i \theta(-p_{jk}^2) \theta(-\Im(y_{i+} y_{i-})) \right] \log \frac{1 - y_{i0}}{-y_{i0}} \right] \end{aligned} \tag{2.94}$$

with ($i, j, k = 0,1,2$ and cyclic)

$$
\begin{aligned}
y_{0i} &= \frac{1}{2\alpha p_{jk}^2}[p_{jk}^2(p_{jk}^2 - p_{ki}^2 - p_{ij}^2 + 2m_i^2 - m_j^2 - m_k^2 \\
&- (p_{ki}^2 - p_{ij}^2)(m_j^2 - m_k^2) + \alpha(p_{jk}^2 - m_j^2 + m_k^2)], \\
x_{i\pm} &= \frac{1}{2p_{jk}^2}[p_{jk}^2 - m_j^2 + m_k^2 \pm \alpha_i], \\
y_{i\pm} &= y_{0i} - x_{i\pm}, \\
\alpha &= \kappa(p_{10}^2, p_{21}^2, p_{20}^2), \\
\alpha_i &= \kappa(p_{jk}^2, m_j^2, m_k^2)(1 + i\epsilon p_{jk}^2)
\end{aligned} \tag{2.95}
$$

where κ is the Källén function

$$
\kappa(x, y, z) = \sqrt{x^2 + y^2 + z^2 - 2(xy + yz + zx)}. \tag{2.96}
$$

The dilogarithm or Spence function $\mathrm{Li}_2(x)$ is defined as

$$
\mathrm{Li}_2(x) = -\int_0^1 \frac{dt}{t} \log(1 - xt) \tag{2.97}
$$

where $|arg(1 - x)| < \pi$.

The η-function is a function which compensates for cut crossings on the Riemann-sheet of the logarithms and dilogarithms involved in the expression. The definition of of the function on the first Riemann-sheet is

$$
\log(ab) = \log(a) + \log(b) + \eta(a, b) \tag{2.98}
$$

where a, b are on the first Riemann-sheet. It is worth noting that the η-functions in the expression vanish when both α and the involved masses m_i are real and that this occurs in the case of α for on-shell decay and scattering processes.

2.8.4 Scalar Four-point Function

The scalar four-point function that was first solved in [12] can at it's most simple form be expressed in terms of 16 dilogarithms [13].

Although we will not cover the analytical form of the four-point function in this presentation, the same general procedure that has been explained are applied and the result is available for practical application in the Passarino-Veltman reduction.

2.8.5 First Order UV-divergent parts of Tensorial Integrals

Only a few of the coefficient functions in the Passarino-Veltman reduction are divergent and it is often useful to have access to a list of these explicitly. We will be stating these integrals in products of ε. In renormalizable theories up to terms of the order $\mathcal{O}\varepsilon$, the UV divergent coefficient integrals are [13]:

$$
\begin{aligned}
(\varepsilon)A_0(m) &= m^2, \\
(\varepsilon)B_0(p_1, m_0, m_1) &= 1, \\
(\varepsilon)B_1(p_1, m_0, m_1) &= -\frac{1}{2}, \\
(\varepsilon)B_{00}(p_1, m_0, m_1) &= -\frac{1}{12}(p_{10}^2 - 3m_0^2 - 3m_1^2), \\
(\varepsilon)B_{11}(p_1, m_0, m_1) &= \frac{1}{3}, \\
(\varepsilon)C_{00}(p_1, p_2, m_0, m_1, m_2) &= \frac{1}{4}, \\
(\varepsilon)C_{00i}(p_1, p_2, m_0, m_1, m_2) &= -\frac{1}{12}, \\
(\varepsilon)D_{0000}(p_1, p_2, p_3, m_0, m_1, m_2, m_3) &= \frac{1}{24}
\end{aligned} \tag{2.99}
$$

All other coefficient functions that were previously listed in (2.13) are UV-finite.

2.9 Applying Dimensional Regularization and Renormalization

2.9.1 Classical consideration

As mentioned previously, the classical electron's self energy diverges linearly for a point charge. In this section we aim to show that the means of dimensional regularization is not strictly speaking constrained to merely act within quantum field theory, but that it's princples also apply to for example classical electrodynamics. To this end, we will review a calculation of the potential due to an infinite line of charge, rather than the self energy, due to the fact that it diverges logarithmically, and therefore is nice to use as comparison with related quantum field theoretical results.

In this example, we will also get a hands-on comparison of a cutoff scheme, compared to dimensional regularization, in performing the regularization.

The Potential due to an Infinite Line of Charge

The potential is given by:

$$
V(r) = \frac{\lambda}{4\pi\varepsilon_0} \int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2 + r^2}} \tag{2.100}
$$

where r is the distance to from an arbitrary point of measurement to the line of charge along a perpendicular distance and x is the distance along the line. $\lambda = Q/x$ is the charge density.

Cutoff Regularization

If we want to carry out this integral naively, the result diverges. What we can do is to introduce an artificial regulator as:

$$
V(r) = \lim_{L\to\infty} \frac{\lambda}{4\pi\varepsilon_0} \int_{-L}^{L} \frac{dx}{\sqrt{x^2 + r^2}} = \lim_{L\to\infty} \frac{\lambda}{4\pi\varepsilon_0} \log\left(\frac{L + \sqrt{L^2 + r^2}}{-L + \sqrt{L^2 + r^2}}\right) \tag{2.101}
$$

We cannot let $\lim_{L\to\infty}$ without a divergent expression. However, for all physically observable quantities, all terms involving the regulator actually cancels, such as the electric field:

$$E = -\frac{\partial V(r)}{\partial r} = \lim_{L\to\infty} \frac{\lambda}{2\pi\varepsilon_0 r} \frac{L}{\sqrt{L^2 + r^2}} \to \frac{\lambda}{2\pi\varepsilon_0 r} \tag{2.102}$$

and the observed energy difference $\delta V = V(r_1) - V(r_2)$:

$$\delta V \to \frac{\lambda}{4\pi\varepsilon_0} \log\left(\frac{r_2^2}{r_1^2}\right) \tag{2.103}$$

However, the solution introduces the extra scale L, which results in a broken symmetry, translational invariance.

Dimensional regularization

To perform the same calculation with dimensional regularization, we must first obtain a d-dimensional expression for the infinite charge line.

Using relations from previous section, we end up with the expression:

$$\begin{aligned} V(r) &= \frac{\lambda}{4\pi\varepsilon_0} \int d^d x \frac{1}{\mu^{d-1}} \frac{1}{\sqrt{x^2 + r^2}} \\ &= \frac{\lambda}{4\pi\varepsilon_0} \frac{2\pi^{d/2}}{\Gamma(d/2)} \int dx \frac{x^{d-1}}{\mu^{d-1}} \frac{1}{\sqrt{x^2 + r^2}} \\ &= \frac{\lambda}{4\pi\varepsilon_0} \frac{\Gamma(\frac{1-d}{2})}{\left(\frac{r}{\mu}\sqrt{\pi}\right)^{1-d}} \end{aligned} \tag{2.104}$$

The problem is in $d = 1 - 2\varepsilon$, gives us

$$V(r) = \frac{\lambda}{4\pi\varepsilon_0} \left(\frac{\mu^2}{\pi r^2}\right)^{\varepsilon} \Gamma(\varepsilon) \tag{2.105}$$

Just as in the previous case, we cannot simply let $\varepsilon \to 0$. We can however extract the poles from the rest of the expression:

$$\begin{aligned} V(r) &= \frac{\lambda}{4\pi\varepsilon_0} \left(\frac{\mu^2}{\pi r^2}\right)^{\varepsilon} \Gamma(\varepsilon) \\ &= \frac{\lambda}{4\pi\varepsilon_0} \left(1 + \varepsilon \log\frac{\mu^2}{\pi r^2} + \mathcal{O}(\varepsilon^2)\right) \left(\frac{1}{\varepsilon} - \varepsilon\gamma_E + \mathcal{O}(\varepsilon)\right) \\ &= \frac{\lambda}{4\pi\varepsilon_0} \left(\frac{1}{\varepsilon} + \log\frac{\mu^2}{\pi r^2} - \varepsilon\gamma_E + \mathcal{O}(\varepsilon)\right) \end{aligned} \tag{2.106}$$

We can now note that for all physically observable quantities, our divergent quantities cancel, as in the observed energy difference $\delta V = V(r_1) - V(r_2)$:

$$\delta V \to \frac{\lambda}{4\pi\varepsilon_0} \log\left(\frac{r_2^2}{r_1^2}\right) \tag{2.107}$$

or the electric field:

$$E = -\frac{\partial V(r)}{\partial r} = \frac{\lambda}{4\pi\varepsilon_0} \frac{2}{r} = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{2.108}$$

Renormalization

In this manner, we wish to apply a renormalization scheme to the expression:

$$V(r) = \frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{\varepsilon} + \log\frac{\mu^2}{r^2} + \log\frac{1}{\pi} - \varepsilon\gamma_E + \mathcal{O}(\varepsilon)\right) \tag{2.109}$$

We could employ a Minimal Subtraction (MS) prescription:

$$V(r) = \frac{\lambda}{4\pi\varepsilon_0}\left(\quad + \log\frac{\mu^2}{r^2} + \log\frac{1}{\pi} - \varepsilon\gamma_E + \mathcal{O}(\varepsilon)\right) \tag{2.110}$$

As we have seen, some constant factors enter that will cancel each other for all physical observables. Therefore, we can even use a Modified Minimal Subtraction ($\overline{MS}$) presription:

$$V(r) = \frac{\lambda}{4\pi\varepsilon_0}\left(\quad + \log\frac{\mu^2}{r^2} + \qquad + \mathcal{O}(\varepsilon)\right) \tag{2.111}$$

After performing this prescription, we can safely let $\varepsilon \to 0$.

2.9.2 Renormalization in Quantum Field Theory

At this point we have calculated general results for the solution of scalar integrals, and described a reduction scheme that can reduce any tensorial quantity to a linear combination of scalar integrals. Some of these scalar integrals contain UV-divergences, which we have listed in equation 2.13. We are now to discuss how to handle these divergences.

The solution to this conundrum, in short, starts with the observation that there is no reason why the observables that we measure by experiment should have to be the parameters that occur in the Lagrangian of our theory. As such, it could very well be that the parameters of our Lagrangian, which give rise to our Feynman rules, which in turns ends up in our predictions, consist of a finite contribution and an infinite contribution, that may be separated in a consistent manner.

From a pedagogical perspective, we can approach this topic from two different angles; assuming so-called bare perturbation theory or renormalized perturbation theory. In bare perturbation theory, we start out with a Lagrangian containing 'bare' terms, that is terms that may have infinite contributions and do not necessarily correspond to physical quantities, carry through our calculations and in the end expand these bare quantities in terms of renormalized (finite) quantities.

If we, on the other hand, start out with the assumption that our parameters of the Lagrangian are separable into a renormalizable parameter and a infinite contribution and separate these from the start, it will lead to new terms in the Lagrangian, so-called counterterms. These will lead to the inclusion of new diagrams at each order (above tree-level) of our calculations.

If one compares the results of bare perturbation theory (having been renormalized) and renormalized perturbation theory, they provide the same results, although through slightly different intermediate calculational steps. Therefore, we are free to discuss either one, and we will in this text merely discuss renormalized perturbation theory.

Within the renormalization procedure of either formulation, there are however various ways in which to perform the renormalization, since there is an inherent ambiguity in which terms to include into the infinite terms that are to be removed by renormalization. The different schemes have their own particular strengths, and in this presentation we will almost exclusively stick to reviewing and comparing two such schemes, pole-mass renormalisation and minimal subtraction.

2.9.3 The Renormalized Lagrangian

Let us begin by stating the standard QED Lagrangian (adding 0 to the terms to clarify that we are at the moment talking about bare parameters):

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \bar{\psi}^0\left(i\not{\partial} - e_0\gamma^\mu A_\mu^0 - m_0\right)\psi^0 \tag{2.112}$$

.

For renormalization to be successful, the free parameters of the theory must be able to absorb every divergence within the theory's predictions. From the expression from the Lagrangian, there are two parameters that could potentially do this, m, and e, are both free parameters. However, as we have seen that there are quite a few divergences that arise from different one-loop corrections, the question arise whether there are more free parameters.

In fact, there is no a priori reason for the normalization of the wavefuction. So if we were to rescale the fields involved by

$$\psi^0 = \sqrt{Z_2}\psi^R \tag{2.113}$$
$$A_\mu^0 = \sqrt{Z_3}A_\mu^R \tag{2.114}$$

where R signifies the renormalized fields (that is to say with the usual normalization), we would obtain a modified Lagrangian

$$\mathcal{L} = -\frac{1}{4}Z_3F_{\mu\nu}^2 + iZ_2\bar{\psi}^R\not{\partial}\psi^R - e_0\gamma^\mu\bar{\psi}^RA_\mu^0\psi^R - Z_2m_0\bar{\psi}^R\psi^R \tag{2.115}$$

From this consideration it is clear that QED has one free parameter for each term of the Lagrangian. One usually defines the parameter

$$Z_1 = \frac{e_0}{e_R}Z_2\sqrt{Z_3} \tag{2.116}$$

which allows one to exchange e_0 for it:

$$\mathcal{L} = -\frac{1}{4}Z_3F_{\mu\nu}^2 + iZ_2\bar{\psi}^R\not{\partial}\psi^R - e_RZ_1\gamma^\mu\bar{\psi}^RA_\mu^R\psi^R - Z_2m_0\bar{\psi}^R\psi^R \tag{2.117}$$

To correspond with tree-level calculations, all of these parameters must have correspond to their physical, renormalized values, that is $Z_1 = Z_2 = Z_3 = 1$ and $m_0 = m_R$ (that $e_0 = e_R$ also follows). At next to leading order, all these constants have the potential of being infinite, and we can therefore expand them perturbatively:

$$Z_2 = 1 + \delta_2 \tag{2.118}$$
$$Z_3 = 1 + \delta_3 \tag{2.119}$$
$$Z_1 = 1 + \delta_1 = 1 + \delta_e + \delta_2 + \frac{1}{2}\delta_3 \tag{2.120}$$
$$Z_2m_0 = m_R + \delta_m \tag{2.121}$$

The Lagrangian becomes

$$\begin{aligned}\mathcal{L} = & -\frac{1}{4}F_{\mu\nu}^2 + i\bar{\psi}^R\not{\partial}\psi^R - e_R\gamma^\nu\bar{\psi}^RA_\mu^R\psi^R - m_R\bar{\psi}^R\psi^R \\ & -\frac{1}{4}\delta_3F_{\mu\nu}^2 + i\delta_2\bar{\psi}^R\not{\partial}\psi^R - e_R\delta_1\gamma^\mu\bar{\psi}^RA_\mu^R\psi^R - \delta_m\bar{\psi}^R\psi^R\end{aligned} \tag{2.122}$$

These so called counterterms will be able to compensate for divergent parts of calculations, in a clear and concise manner.

Another feature is that we can perform perturbation theory in e_R, rather than e_0, which is a nice feature, as an expansion in an infinite quantity is not well-defined.

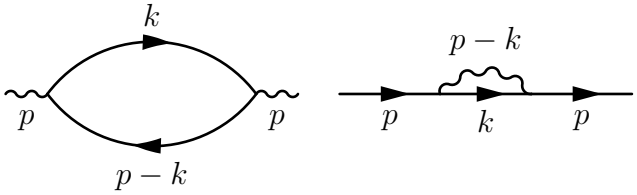

Figure 2.4: Schematic Feynman diagrams for the two non-vanishing first order contributions from two-point functions; the vacuum polarization and the electron's self energy may be calculated.

2.9.4 Cancellation of QED Divergences: Electron's self energy

To this end, we are now ready to apply our renormalization scheme to actual calculations.

There are two two-point functions of interest in QED perturbation theory:

In renormalized perturbation theory, the Feynman diagrams of the (amputated) fermion two-point function gives:

$$\langle 0|T\{\psi^R\psi^R\}|0\rangle = \quad + \quad = \frac{i}{\not k - m_R} + \frac{i}{\not k - m_R}[i\Sigma_2(\not k)]\frac{i}{\not k - m_R} \tag{2.123}$$

In addition, there is a counterterm:

$$= \frac{i}{\not k - m_R} i(\not k\delta_2 - \delta_m)\frac{i}{\not k - m_R} \tag{2.124}$$

We end up with a calculation containing:

$$\frac{i}{\not k - m_R} + \frac{i}{\not k - m_R} i[\Sigma_2(\not k) + \not k\delta_2 - \delta_m]\frac{i}{\not k - m_R} \tag{2.125}$$

We now want to evaluate Σ_2 and regularize the expression. To this end, we start by writing down its expression from the Feynman rules (suppressing the R subscript):

$$\begin{aligned} i\Sigma_2(\not q) &= \mu^{4-d}\int \frac{d^dk}{(2\pi)^4}(i\gamma^\nu e)\frac{i(\not k + m)}{k^2 - m^2 + i\epsilon}(i\gamma^\mu e)\frac{-ig_{\mu\nu}}{(p-k)^2 + i\epsilon} \\ &= \frac{ie^2}{(4\pi)^2}\frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk(-1)\gamma_\mu\frac{\not k + m}{P_0P_1}\gamma^\mu \\ &= \frac{ie^2}{(4\pi)^2}\int d^dk\frac{(2-d)\ \not k - dm}{P_0P_1} \\ &= \frac{ie^2}{(4\pi)^2}\left((2-d)\gamma^\mu B_\mu - dmB_0\right) \end{aligned} \tag{2.126}$$

where we have used the contraction idenities for gamma matrices and formulated our calculations in terms of a Passarino-Veltman reduction to show the usefulness of the formulae developed in the previous sections.

We now have one integral we can decompose, which is $B_\mu = p_\mu\bar{B}_1$. Let us repeat the calculation of a coefficient function, starting out with contracting the decomposition with the only Lorentz vector available, p^μ. The RHS can be written as $p_\mu p^\mu B_1 = p^2B_1$ and with the LHS we can calculate

$$p^\mu B_\mu = \int d^dk\frac{p\cdot k}{(k^2 - m^2 + i\epsilon(p-k)^2 + i\epsilon)} = \frac{p\cdot k}{P_0P_1} \tag{2.127}$$

Use that (suppressing $i\epsilon$)

$$p \cdot k = \frac{1}{2}\left(k^2 + 2p\cdot k + p^2 - k^2 - p^2 + m^2 - m^2\right) = \frac{1}{2}\left((k+p)^2 - (k^2-m^2) - (p^2-m^2)\right) \tag{2.128}$$

to exclude the first and second propagator respectively, allowing us to write the RHS as a linear combination of scalar integrals. The result becomes:

$$B_1(p^2, m^2, 0) = \frac{1}{2p^2}\left[A_0^{(1)}(m^2) - A_0^{(0)}(0) - (p^2+m^2)B_0(p^2, m^2, 0)\right] \tag{2.129}$$

where we have employed the standard notation for scalar integrals, with the first A-term has the photon propagator removed and the second A-term has the fermion propagator removed.

Divergences

If we now take a closer look at the self energy calculation, we notice that the products $(d-2)\gamma^\mu B_\mu$ and dmB_0 will contain factors relevant for evaluating divergences both in the integrals and the prefactors d and $(d-2)$. In this section we will therefore start by calculating the divergency B_μ from its Passarino-Veltman reduction, using the previous results from scalar integrals. We will follow this by taking the proper limit of the products just discussed. We are then finally at a position where we can discuss the renormalization of the diagram in question.

If we write down the linear combination that makes up the decomposition of B_μ, we find:

$$\begin{aligned} B_1(p^2, m^2, 0) &= \frac{1}{2p^2}\left[A_0^{(1)}(m^2) - A_0^{(0)}(0) - (p^2+m^2)B_0(p^2, m^2, 0)\right] \\ &\to \frac{1}{2p^2}\left[\left(m^2\Delta\right) - (p^2+m^2)\Delta + \text{ finite terms }\right] = -\frac{1}{2}\Delta \end{aligned} \tag{2.130}$$

This result is the same as we found in eq 2.13. However, since we are in a kinematical region where one of the masses is zero, some special care is needed to calculate the Passarino-Veltman reduction fully. We will circumvent this discussion by noting a simple observation regarding the B_1 coefficient function:

$$p_\mu \times B_1 = B_\mu = \int d^dk \frac{k_\mu}{P_0 P_1} \to \underbrace{\int d^dq \frac{q_\mu}{(q^2-m^2)^2}}_{\text{odd}\to 0} + \int d^dq \frac{-xp_\mu}{(q^2-m^2)^2} = p_\mu \times \int_0^1 dx(-x\tilde{B}_0) \tag{2.131}$$

where we have introduced the notation $\tilde{B}_0$ to signify the integrand of the B_0 integral before carrying out Feynman integration.

If we now return to our expression for Σ_2:

$$i\Sigma_2(\not{k}) = \frac{ie^2}{(4\pi)^2}\left((2-d)\gamma^\mu B_\mu - dmB_0\right) = \int_0^1 dx \frac{ie^2}{(4\pi)^2}\left(-(2-d)x\ \not{k} - dm\right)\tilde{B}_0 \tag{2.132}$$

If we apply $d = 4 - 2\varepsilon$

$$\begin{aligned} (2-d)\tilde{B}_0 &= (-2+2\varepsilon)\left(\Delta - \log\frac{(k_1^2x^2 - x\left(k_1^2+m^2\right) + m^2 - i\epsilon)}{\mu^2} + \mathcal{O}(\varepsilon)\right) = -2\tilde{B}_0 + 2 \\ (d)\tilde{B}_0 &= 4\tilde{B}_0 - 2 \end{aligned} \tag{2.133}$$

and our expression becomes:

$$\begin{aligned}
i\Sigma_2(\not{k}) &= \int_0^1 dx \frac{ie^2}{(4\pi)^2}\left(-(2-d)x\ \not{k} - dm\right)\tilde{B}_0 \\
&= \frac{ie^2}{(4\pi)^2}\int_0^1 dx\left(-(-2+2\varepsilon)x\ \not{k} - (4-2\varepsilon)m\right)\tilde{B}_0 \\
&= \frac{ie^2}{(4\pi)^2}\int_0^1 dx(2x\ \not{k} - 4m)\tilde{B}_0 - (2x\ \not{k} - 2m) \\
&= \frac{ie^2}{(4\pi)^2}\int_0^1 dx\left[(2x\ \not{k} - 4m)\left(\Delta - \log\frac{(k^2x^2 - x\left(k^2+m^2\right)+m^2-i\epsilon)}{\mu^2} + \mathcal{O}(\varepsilon)\right)\right. \\
&\qquad\qquad \left. -(2x\ \not{k} - 2m)\right] \\
&= \left[\frac{ie^2}{(4\pi)^2}(\not{k} - 4m)\Delta + (2m - \not{k})\right. \\
&\qquad\qquad \left. + \int_0^1 dx(2x\ \not{p} - 4m)\left(\log\frac{\mu^2}{(k^2x^2 - x\left(k^2+m^2\right)+m^2-i\epsilon)}\right)\right] + \mathcal{O}(\varepsilon)
\end{aligned} \tag{2.134}$$

We are now in the position of discussing how to renormalize our expression.

2.9.5 Renormalization conditions

As we now have calculated the terms of all the first-order contributions, what is left to do is to fix all the counterterms. This is done by defining the QED renormalization conditions. There is some freedom in the way we choose to fix these counter terms, and depending on how we choose the renormalization conditions, there are some differences in the interpretation of the renormalized theory. To this note, we will briefly review two different sets of normalization conditions, on-shell conditions and minimal subtraction.

2.9.6 Renormalization conditions of the electron self-energy diagram

For our discussion, let us define our entire first order correction as:

$$\Sigma(\not{k}) = \Sigma_2(\not{k}) + \delta_2\ \not{k} - \delta_m \tag{2.135}$$

Let us take a look at higher order contributions to the propagator:

$$\begin{aligned}
\frac{i}{\not{k} - m_R} &+ \frac{i}{\not{k} - m_R}[i\Sigma(\not{k})]\frac{i}{\not{k} - m_R} + ... \\
&= \frac{i}{\not{k} - m_R}\left(1 + \frac{-\Sigma(\not{k})}{\not{k} - m_R} + \left(\frac{-\Sigma(\not{k})}{\not{k} - m_R}\right)^2 + ...\right) \\
&= \frac{i}{\not{k} - m_R}\frac{1}{1 + \frac{\Sigma(\not{k})}{\not{k} - m_R}} = \frac{i}{\not{k} - m_R + \Sigma(\not{k})}
\end{aligned} \tag{2.136}$$

We see that, to all orders in the expansion, there is no need to introduce additional renormalization parameters.

Looking at the general expansion, we see that there seems to be a natural condition for fixing the renormalization parameters. The physical tree-level states have poles at their physical mass.

This relation is later used in, for example, the LSZ theorem, which makes a good argument for this position of the pole.

We can therefore define the renormalized mass by

$$\Sigma(m_R) = 0 \tag{2.137}$$

As we have two counterterms to consider, we have to define two requirements in order to make sure that the counterterms are fixed.

Another reasonable requirement in relation to the LSZ theorem is that the residue of the pole of the propagator is fixed to the same value as previously (in our convention to 1). This requirement give us the relationship:

$$1 = \lim_{\not{k}\to m_R} \frac{\not{k} - m_R}{\not{k} - m_R - \Sigma(\not{k})} = \lim_{\not{k}\to m_R} \frac{1}{1 - \frac{d}{d\not{k}}\Sigma(\not{k})} \tag{2.138}$$

where we have used l'Hopital's rule. Therefore

$$\left[\frac{d}{d\not{k}}\Sigma(\not{k})\right]_{\not{k}=m_R} \tag{2.139}$$

The additional counter terms in on-shell renormalization

In order to fix the remaining counter terms, there are two additional diagrams to consider. We will not cover the details of the calculations, but rather state the results which are discussed in depth in, for example, [4].

The other non-vanishing two-point function in QED is the photon propagator

$$\Pi^{\mu\nu}(p) = \langle 0|T\{A_R^\mu A_R^\nu\}|0\rangle \tag{2.140}$$

By the Feynman rules for QED, this propagator can, to first order of perturbations, include a vacuum polarization graph (or a photon self-energy if you will). Expanding the graphs to all orders in a procedure similar to that of our previous calculation, one obtains a modified propagator term:

$$\frac{(-i)g_{\mu\nu}}{p^2} \to \frac{(-i)g_{\mu\nu}}{p^2(1 - \Pi(p^2)} \tag{2.141}$$

where $\Pi(p^2)$ corresponds to the sum of the second order contributions from the renormalized fields and the counter terms (leaving out gauge dependent terms, that will drop out):

$$\Pi(p^2) = \Pi_2(p^2) + \delta_3 \tag{2.142}$$

This equation has a pole at $p = 0$, due to it's inverse proportionality to p^2. So we require that the value of the residue of the pole at $p = 0$ is kept at its tree-level expression as our renormalization condition. This gives us the condition:

$$\Pi(0) = 0 \tag{2.143}$$

implying

$$\delta_3 = -\Pi_2(0) \tag{2.144}$$

The entire Π-function converges from this consideration, being:

$$\Pi(p^2) = \Pi_2(p^2) - \Pi(0) \tag{2.145}$$

Having discussed all non-vanishing two-point functions, we may consider the simplest three-point function:

$$G_3 = \langle 0|T\{\bar{\psi}A_\mu\psi\}|0\rangle \tag{2.146}$$

At tree-level it is made up of the QED vertex $G_3 = ie\gamma^\mu$. To first order, the following loop is included:


$$= \Gamma_2^\mu = ie_R\left(F_1^{(2)}(q^2)\gamma^\mu + \frac{i\sigma^{\mu\nu}}{2m_R}q_\nu F_2^{(2)}(q^2)\right) \tag{2.147}$$

where q is the momentum of the incoming photon.

We also have a counter term contributing a factor $ie_R\delta_1\gamma^\mu$.

At tree-level, $F_1 = 1$ and $F_2 = 0$. The second formfactor, F_2, corresponds to the anomalous magnetic moment, and will be discussed at some depth later. The first form factor, F_1, gives rise to Coulomb's law at large distances, where the electric charge measured is $e_R F_1(0)$. To set this electric charge to equal e_R by definition, we find

$$F_1(0) = 1 + F_1^{(2)}(0) + \delta_1 \tag{2.148}$$

so we may identify

$$\delta_1 = -F_1^{(2)}(0) \tag{2.149}$$

We have thus fixed all four quantities based on physical considerations, and we now have a unique renormalization procedure, which, in short, preserves the electric charge and the position and value of the poles for fermions and photons.

2.9.7 Minimal Subtraction

In this section we will look at an alternative set of renormalization conditions, known as *minimal subtraction*. It is not as physical as the on-shell conditions, but for complicated calculations it is far more useful.

The procedure of minimal subtraction (MS) is actually very straightforward. Although performed with slightly different conventions for the counterterms, the principle and the included diagrams are the same as for the on-shell conditions. The true difference lies in the fact that, rather than fixing the counter terms based on physical arguments, they are fixed to subtract *only* the actual terms involving divergences.

When using dimensional regularization, typically, as in for example the electron self energy diagram, minimal subtraction only removes the actually divergent factors $\frac{1}{\varepsilon}$, leaving the rest of the of the terms untouched. If we were to take a look at the electron self energy as an example, recall the solution:

$$\begin{aligned} i\Sigma_2(\not{k}) &= \frac{ie^2}{(4\pi)^2}(\not{k} - 4m)\Delta + (2m - \not{k}) \\ &\quad + \int_0^1 dx(2x\not{k} - 4m)\left(\log\frac{\mu^2}{(k^2x^2 - x(k^2+m^2) + m^2 - i\epsilon)}\right) + \mathcal{O}(\varepsilon) \end{aligned} \tag{2.150}$$

where we remember Δ to be

$$\Delta = \frac{1}{\varepsilon} - \gamma_E + 1 \tag{2.151}$$

In this case we would simply put $\not{p}\delta_2 = \not{p}/\varepsilon$ and similarly for δ_m. A common variation of this scheme is the *modified minimal subtraction* ($\overline{MS}$), where we have chosen to absorb convenient constants into counterterm, this most commonly being done by including the $-\gamma_E$-term.

With MS or $\overline{MS}$, there is no longer any reason why the pole should be at $\not{k} = m_R$ and its residue to equal 1. What we are really 'stuck' with is the definition of m_R as a term in the Lagrangian. Compared to the pole mass, we thus obtain an expression:

$$m_p = m_R(0) - \Sigma(m_P) \tag{2.152}$$

Inserting the expression for Σ, we can calculate a shift in the pole mass to first order in perturbation theory. These m_R masses set by minimal subtraction are often referred to as $\overline{MS}$ masses, and commonly used. They are even to prefer in some cases. For example, as quarks are not free, it does not make sense to talk about a free quark propagator, and the $\overline{MS}$ mass provides a good mass definition. It is interesting to note that for the difference between top quark mass's pole mass and $\overline{MS}$ mass is as much as much as a few GeV.

Chapter 3

saddle point Scheme to Evaluate Higher Order Contributions

3.1 Motivation

To evaluate higher order contributions to Feynman amplitudes is time-consuming and at times highly non-trivial. However, knowing their correct values is very valuable. With the advent of the Large Hadron Collider at CERN, many higher order calculations have become absolutely crucial for theoretical predictions to keep up with experimental precision. That this trend should continue seems probable, and there are currently great efforts being dedicated to calculating amplitudes to higher orders, both by going to higher-loops in internal momentum variables and higher numbers of external momentum variables. As this is both a laborious and time-consuming endeavour, it is our hope that the method that we have developed might offer some useful tools for calculating radiative corrections.

Our method aims at offering predictions to higher order contributions of order-of-magnitude precision or better. Thereby, the method may be able to contribute to practical calculation of perturbative predictions in many ways. Firstly, it may be used as a predictive tool, where one may do (relatively) expedient calculations of interesting higher-order calculations. Thereby, the magnitude of many different contributions could be foreseen, and thereby offering predictions of these calculations.

Secondly, it may offer a means of verification for higher order calculations, being a simple means of calculating an approximative value or distribution which one can compare to an exact calculation.

Thirdly, where there is a need for very fast higher order calculations, where one does not have the time to perform exact calculations, the method could in the future be implemented in order to speed up calculations that include multiple higher order contributions.

Fourthly, it may be so that some truly higher order contributions, that are currently not calculated due to their calculational complexity, could be approximated using a combination of this method with other useful packages.

Finally, by using the results of the theory in a crude manner, one might be able to become so proficient in calculating Feynman amplitudes to the correct order-of-magnitude that it takes about as little effort of calculating a Feynman diagram as it takes to draw it. In essence, radiative corrections could join the ranks of questions solvable as Fermi questions.

In this chapter we will begin by looking at the saddle point approximation from a general perspective, defining a standard procedure for both one and several dimensions. This section is followed by a discussion on the saddle point approximation in the context of solving scalar integrals, relevant due to the result from the previous chapter that all Feynman amplitudes may be reduced to scalar integrals. In the third section, we look closer at the example of

the anomalous magnetic moment, which offers an illustration of the accuracy of the method on a finite quantity. Finally, we discuss the matter of dealing with divergences and higher-loop calculations in the saddle point approximation, and its similarities and differences from calculating divergences with an exact method.

This work is an extension on a previous paper by H.B. Nielsen [1], where the same conceptual framework was applied to the N-point Veneziano model.

3.2 The saddle point approximation in general

A general, real function may be expanded as:

In one dimension:

$$\begin{aligned} f(x) &= f(a) + f'(a)(x-a) \\ &+ \frac{f''(a)}{2}(x-a)^2 + \mathcal{O}[(x-a)^3] \end{aligned} \quad (3.1)$$

where primes indicate partial integration with respect to x. We now wish to choose the value of a for which:

$$f'(x) = 0 \quad (3.2)$$

For this value, we denote the x value by $a = \hat{x}$. For an integral over $f(x)$, the dominant contribution comes from the region around the extremum and we are therefore interested in expanding our function around that position. In this case, our expansion reduces to:

$$\begin{aligned} f(x) &= f(a) + \frac{f''(\hat{x})}{2}(x-\hat{x})^2 + \mathcal{O}[(x-a)^3] \\ &= f(\hat{x})\left(1 + \frac{f''(\hat{x})}{2f(\hat{x})}(x-\hat{x})^2\right) \end{aligned} \quad (3.3)$$

For our purposes, we now wish to identify this, to leading order as:

$$\begin{aligned} & f(\hat{x})\left(1 + \frac{f''(\hat{x})}{2f(\hat{x})}(x-\hat{x})^2\right) \\ &= f(\hat{x}) \exp\left(\frac{f''(\hat{x})}{2f(\hat{x})}(x-\hat{x})^2\right) \end{aligned} \quad (3.4)$$

which, under integration, will behave as a 'Gaussian' or a standard normal distribution function with standard deviation $\sigma^2 = \frac{f(\hat{x})}{f''(\hat{x})}$.

In d-dimensions:
A d-dimensional function may be expanded as [2]

$$\begin{aligned} & f(\mathbf{x}) = f(\mathbf{a}) + \vec{\nabla} \cdot f(\vec{x}-\vec{a}) \\ & + \frac{1}{2}(\vec{x}-\vec{a})^T H[\mathbf{a}](\vec{x}-\vec{a}) + \mathcal{O}[(\vec{x}-\vec{a})^3] \end{aligned} \quad (3.5)$$

where $H[\mathbf{a}]$ is the Hessian matrix, the square matrix of second order partial derivatives.
By taking

$$\vec{\nabla} \cdot f(\vec{x}-\vec{a}) = 0 \quad (3.6)$$

we may find the extremum of the function, choose this point for our $\vec{a}$ in the previous expansion. For an integral over $f(\vec{x})$, the dominant contribution comes from the region around the extremum and we are therefore interested in expanding our function around that position. Here our expansion reduces to:

$$\begin{aligned} f(\vec{x}) &= f(\vec{a}) + \frac{1}{2}(\vec{x}-\vec{a})^T H[\vec{a}](\vec{x}-\vec{a}) + \mathcal{O}[(\vec{x}-\vec{a})^3] \\ &= f(\vec{a})\left[\mathbb{1} + \frac{1}{f(\vec{a})}(\vec{x}-\vec{a})^T H[\vec{a}](\vec{x}-\vec{a})\right] \\ &+ \mathcal{O}[(\vec{x}-\vec{a})^3] \end{aligned} \quad (3.7)$$

We now wish to identify this to leading order as

$$\begin{aligned} & f(\vec{a})\left[\mathbb{1} + \frac{1}{f(\vec{a})}(\vec{x}-\vec{a})^T H[\vec{a}](\vec{x}-\vec{a})\right] \\ &= f(\vec{a}) \times \exp\left[\mathbb{1} + \frac{1}{f(\vec{a})}(\vec{x}-\vec{a})^T H[\vec{a}](\vec{x}-\vec{a})\right] \end{aligned} \quad (3.8)$$

Where the exponential is defined by its expansion. Thus the expression will behave as a normal distribution in d-dimensions.

3.3 saddle point approximation in ϕ^4-theory

Let us, before turning to QED calculations, take a look at the saddle point approximation for ϕ^4-theory. We have the Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2}m^2|\phi|^2 - \frac{\lambda}{4!}|\phi|^4 \tag{3.9}$$

which gives the Feynman rules for:

- vertices:

$$-i\lambda \tag{3.10}$$

- and propagators:

$$\frac{i}{p^2 - m^2 + i\epsilon} \tag{3.11}$$

Let us now perform a saddle point approximation for each propagator factor individually (with their respective saddlepoints at zero):

$$\frac{i}{k^2 - m^2} \approx \frac{-i}{m^2} \exp\left(\frac{k^2}{m^2}\right) = \frac{-i}{m^2} \exp\left(\frac{-k_E^2}{m^2}\right) \tag{3.12}$$

where we in the last line have performed a Wick rotation of the variables.

Note that due to the form of the propagator, the dimensionalities now factorize:

$$\exp\left(\frac{-k_E^2}{m^2}\right) = \prod_{\mu=0,1,2,3} \exp\left(\frac{-k_{E,\mu}^2}{m^2}\right) \tag{3.13}$$

We will therefore,from now on, look at one particular spacetime dimension.

In order to extend this into a diagram setting, we introduce the notation of propagator momenta $k_i, k_j, k_k, ...$ For each vertex, four momentum conservation will apply:

$$k_{i,\mu}^E + k_{j,\mu}^E + k_{k,\mu}^E + ... = 0 \tag{3.14}$$

We can now find a common saddle-point, which will give the dominant contribution to the diagram. Since the propagators are all given by exponentials, it will suffice to find the extremal value for the exponential factor:

$$\frac{d}{dk_j} \sum_i = 0 \frac{-k_{E,i}^2}{m^2} \tag{3.15}$$

We may then shift the original propagator factors with the saddle-point factor, obtaining a common saddle point approximation for the diagram.

These conditions are in fact completely analogous to those of an electrical network. The primary condition, of the conservation of momenta, corresponds to Kirchhoff's Current Law:

$$\sum_i I_i = 0 \tag{3.16}$$

while the second corresponds to the task of minimizing the energy dissipation of a current. Since this procedure is relevant to the calculation of saddle points, let us briefly review it's results.

The energy of the entire circuit is given by:

$$\sum_i R_i I_i^2 \tag{3.17}$$

While respecting Kirchoff's Current Law in each vertex $[ijk]$ (denoted by the respective currents through the vertex), we want to minimize the energy dissipation:

$$\frac{\delta}{\delta I_k}\left(\sum_i R_i I_i^2 + \sum_{[ijk]} \lambda_{[ijk]}\left(I_i + I_j + I_k\right)\right) = 0 \tag{3.18}$$

Where the relevant Lagrange-multipliers have been inserted. For an example circuit consisting merely of one cord k between two vertices A, B, this results in

$$2R_k I_k + \lambda_A - \lambda_B = 0 \tag{3.19}$$

Identify $\lambda_A = 2V_A$ and $\lambda_B = 2V_B$, where we know V to be the voltage, and we thus obtain

$$R_k I_k = V_A - V_B \tag{3.20}$$

which we recognize to be Ohm's Law.

In summary, by identifying Kirchoff's Current Law and Ohm's Law, we see that the saddle-point approximation corresponds to, and can be solved by methods used in, an electrical network. This correspondance is given by:

$$\begin{aligned} -\frac{1}{m^2} &\leftrightarrow R_{i,\mu} \\ k_{i,\mu} &\leftrightarrow I_{i,\mu} \end{aligned} \tag{3.21}$$

for each spacetime dimension. Thus, for each dimension, the dominant contribution, for each diagram will flow through the diagram as an electrical network. We may thereafter shift the approximation according to the common saddlepoint.

If there are loops involved in the diagram, we can thereafter evaluate these rather easily, merely integrating the exponential factors for each spacetime dimension.

3.4 Evaluating scalar integrals with the saddle point approximation

For this section, our intention is to show the results of the saddle point approximation for scalar integrals with $m = 0$, that is, with a numerator = 1. We will use the familiar result for the angular integration, leading us to merely compare the terms of the radial integration. As given in section 2.5.5, eqn [2.54], the radial integration term is proportional to:

$$\int_0^\infty du \frac{u^{d-1}}{(1+u^2)^n} = \frac{\Gamma(d/2)\Gamma(-d/2+n)}{2\Gamma(n)} \tag{3.22}$$

Following the saddle point scheme, we equate the first derivative of the integrand to zero, finding the extremum to be:

$$u* = \sqrt{\frac{1-d}{-1+d-2n}} \tag{3.23}$$

If we then evaluate the saddle point approximation of this expression, we find it to be:

$$\int_{-\infty}^{\infty} f(u*)e^{\frac{f''(u*)}{2f(u*)}(u-u*)^2} \tag{3.24}$$

$$= 2^{\frac{1}{2}-n}\sqrt{\pi}\left(\frac{\sqrt{1-d}}{\sqrt{-1+d-2n}}\right)^{d-1}\left(\frac{n}{1-d+2n}\right)^{-n}\sqrt{\frac{n}{(-1+d-2n)^2}} \tag{3.25}$$

We are at this time interested in comparing convergent quantities, and therefore we look at $n \geq 3$ in $d = 4$ dimensions. In table 3.1 we have summarized some of these results for the various n-point functions of scalar integrals. We see that the saddle point approximation becomes a

n	Exact	saddle	Accuracy
3	1/4	$1/4 * \sqrt{\pi/6}$	0.723
4	0.0833	0.0711	0.853
5	0.0417	0.0378	0.906
8	0.0119	0.0115	0.964
15	0.002747	0.002741	0.998

Table 3.1: Comparison of the saddle point approximation to exact values of the radial integral

better and better method as n increases. A comparison of the difference between the exact and approximated results is shown in figure 3.1.

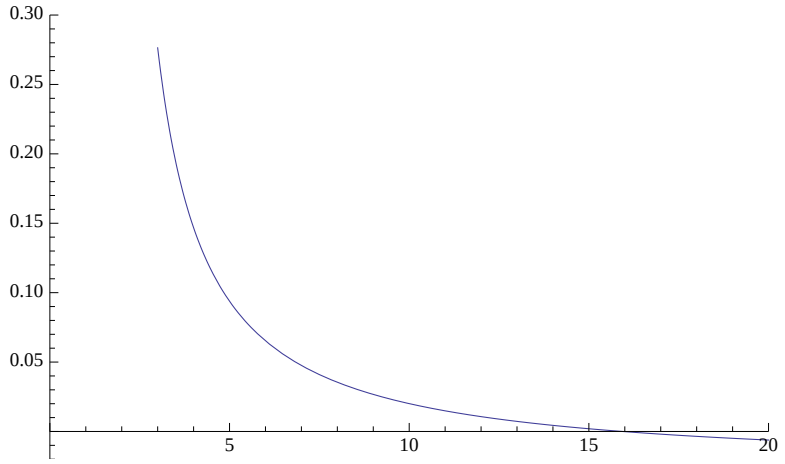

Figure 3.1: Difference between the exact and the approximative result for variable n with $m = 0$

From this comparison, it is clear that, at $n = 16$, the approximation grows to become slightly larger than the exact result (however, the scale of the difference is 10^{-6}). From numerical analysis, one can also see that this difference remains negative for $n > 16$, but decreases and for $n \to \infty$ the error becomes arbitrarily small.

To discuss the overall reason for this behaviour might be provide some insight into the potential of the method. The typical error function falls of faster when approaching infinity than a polynomial. However, as we increase the number of propagators, i.e as n increases, the exact expression tends towards infinity at a faster and faster rate. Therefore, the Gaussian approximation becomes more and more accurate. This is an argument in favour of implementing the saddle point approximation for more sophisticated diagrams, where higher order n-point functions are to be employed. To illustrate this result, we can compare the plots for the first non-diverging integral, $n = 3$, and that of $n = 15$ (see figure 3.2). Clearly, the relative rate at

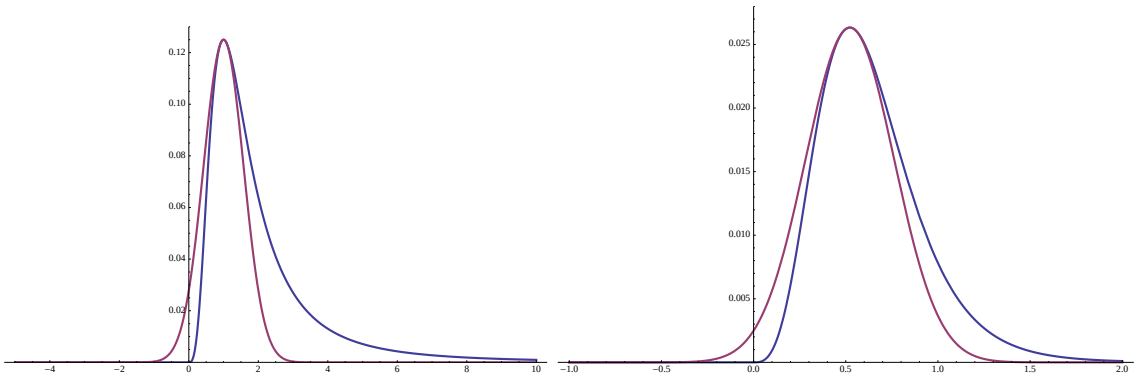

Figure 3.2: The scalar integral with $n = 3$, $m = 0$ and $n = y$, $m = 0$, respecively

which the integral approaches infinity differs slower and slower, which can be seen most clearly for the graphs towards positive infinity.

There are however cases when the naïve approximation is no longer a good approximation. This is when a divergence occurs in the integral, and the integral thus falls off significantly slower towards infinity. From figure 3.3, where the case of $n = 2$ is illustrated, it is clear that the current version of the saddle point approximation will need a clear manner to deal with divergences to accomodate such situations, a discussion which we will postpone until further along in the presentation.

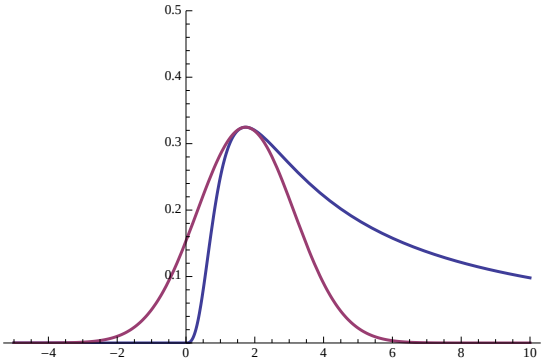

Figure 3.3: The diverging scalar integral $s = 2$ and $m = 0$

3.5 An example: the anomalous magnetic moment

3.5.1 Introduction

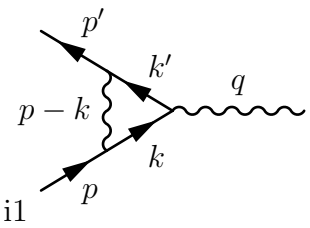

Figure 3.4: The Feynman diagram with contributions to the Anomalous Magnetic moment

The quantum field theory calculation of the anomalous magnetic moment stems back to

a calculation by Schwinger in 1948 [3], where we calculate the feynman amplitude for the Feynman diagram in figure 3.4 The Feynman rules for the diagram gives $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$ with [4]:

$$\begin{aligned}\delta\Gamma^\mu(p',p) = \int \frac{d^4k}{((2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')\left(-ie\gamma^\nu\right)\times \\ \frac{ik\!\!\!/'+m}{k'^2-m^2+i\epsilon}\gamma^m u\frac{i(k\!\!\!/+m)}{k^2-m^2+i\epsilon}\left(-ie\gamma^\rho\right)u(p) = \\ 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left(k\!\!\!/\gamma^\mu k\!\!\!/'+m^2\gamma^\mu-2m(k+k')^\mu\right)u(p)}{(k-p)^2+i\epsilon\left(k^2-m^2+i\epsilon\right)\left(k'^2-m^2+i\epsilon\right)}\end{aligned} \tag{3.26}$$

where we have utilized the anticommutation relation for gamma matrices repeatedly.

3.5.2 Standard result

Isolating the denominator for the moment, we may use Feynman parametrization:

$$\frac{1}{[(k-p)^2+i\epsilon][k'^2-m^2+i\epsilon][k'^2-m^2+i\epsilon]} = \int_0^1 dxdydz\delta(x+y+z-1)\frac{2}{D^3} \tag{3.27}$$

where the denominator term is described by:

$$D = x(k^2-m^2)+y(k'^2-m^2)+z(k-p)^2+(x+y+z)i\epsilon \tag{3.28}$$

By shifting k to $\ell =: k+yq-zp$ we can complete the square and after some algebra find:

$$D = \ell^2-\Delta+i\epsilon \tag{3.29}$$

where

$$\Delta =: -xyq^2+(1-z)^2m^2 \tag{3.30}$$

Where in a scattering process Δ is positive ($q<0$ in scattering) and it may be regarded as an effective mass term.

We can now reformulate the numerator in our new coordinates and utilize anticommutation relations and the Dirac equation to put it into the useful form:

$$\begin{aligned}&\bar{u}(p')\left[k\!\!\!/\gamma^\mu k\!\!\!/'+m^2\gamma^\mu-2m(k+k')^\mu\right]u(p) \\ &\to\bar{u}(p')\left[\gamma^\mu\cdot\left(-(1/2)\ell^2+(1-x)(1-y)q^2+(1-2z-z^2)m^2\right)+\right. \\ &\qquad\left.+(p'^\mu+p^\mu)\cdot mz(z-1)+q^\mu\cdot m(z-2)(x-y)\right]u(p)\end{aligned} \tag{3.31}$$

The q^μ-term vanishes according to the Ward identity (for a derivation of this, see for example [4]). This may also be seen by noting that the term with q^μ is odd under interchange of x and y while the denominator does not contain terms of this nature and is thus even under interchange of x and y. As such, this term vanishes.

In addition, we may now rewrite our expression using the so-called Gordon identity (for a derivation of this, see for example [4]), which replaces $(p+p')^\mu \to i\sigma^{\mu\nu}q_\nu$ by

$$\bar{u}(p')\gamma^\mu u(p) = \bar{u}(p')\left[\frac{p'^\mu+p^\mu}{2m}+\frac{i\sigma^{\mu\nu}q_\nu}{2m}\right]u(p) \tag{3.32}$$

These two results will allow us to rewrite our expression from the general form (motivated by Lorentz invariance) into the two well-known formfactors:

$$\begin{aligned}\Gamma(p',p) &= \gamma^\mu \cdot A + (p'^\mu + p^\mu)\cdot B + (p'^\mu - p^\mu)\cdot C \\ &\to \gamma^\mu \cdot F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\cdot F_2(q^2)\end{aligned} \tag{3.33}$$

leaving us with the expressions:

$$\begin{aligned}\bar{u}(p')\Gamma(p',p)u(p) = 4ie^2\int\frac{d^4l}{(2\pi)^4}\int_0^1 dxdydz\delta(x+y+z-1) \\ \times\bar{u}(p')\left[\gamma^\mu\cdot\frac{(-\frac{1}{2}\ell^2+(1-x)(1-y)q^2+(1-4z+z^2)m^2)}{(\ell^2-m^2)^3}\right. \\ \left. + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\frac{2m^2z(1-z)}{(\ell^2-m^2)^3}\right]\end{aligned} \tag{3.34}$$

where we have used the identities:

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu}{D^3} = 0 \tag{3.35}$$

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^\mu\ell^\nu}{D^3} = \int\frac{d^4\ell}{(2\pi)^4}\frac{\frac{1}{4}g^{\mu\nu}\ell^2}{D^3} \tag{3.36}$$

which were motivated in section 2.5.2. The first identity is due to that the numerator is odd and the denominator even, while the other is due to Lorentz invariance and the factor 1/4 is given by contracting both sides of the equation.

The expression is now reduced to performing two integrals:

$$\begin{aligned}\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{(\ell^2-\Delta)^3} \\ \int\frac{d^4\ell}{(2\pi)^4}\frac{\ell^2}{(\ell^2-\Delta)^3}\end{aligned} \tag{3.37}$$

These are standard scalar integrals, whose integration has been reviewed in section 2.5. It is worthwhile to note that, although F_1 contains both infrared and ultraviolet divergences, which must be handled with renormalization and regularization schemes (discussed in general principles in the previous section, the anomalous magnetic moment, $F_2[0]$, is completely convergent.

By stating the result from previous chapter for this particular case,

$$\int\frac{d^4\ell}{(2\pi)^4}\frac{1}{[\ell^2-\Delta]^3} = \frac{-i}{(4\pi)^2}\frac{1}{2}\frac{1}{\Delta} \tag{3.38}$$

We can write down the result for the anomalous magnetic moment:

$$\begin{aligned}F_2[q^2=0] &= \frac{\alpha}{2\pi}\int_0^1\delta(x+y+z-1)\frac{2m^2z(1-z)}{m^2(1-z)^2} \\ &= \frac{\alpha}{\pi}\int_0^1 dz\int_0^{1-z}dy\frac{z}{1-z} = \frac{\alpha}{2\pi}\end{aligned} \tag{3.39}$$

where α is the finestructure constant, $e^2/4\pi$, in suitable units and we have obtained the form factor F_2 to first order in α.

3.5.3 Passarino-Veltman Reduction

The same result as has been shown here, can also be reached by Passarino-Veltman reduction [21]. However, although the anomalous magnetic moment in itself is a convergent quantity, the internal cancellation of divergences in this calculation is extensive and the calculation occurs in the region of a vanishing Gram determinant. In short, the reduction is made to:

$$F_2 \propto \left(1 + B_0(0, m^2, m^2) - B_0\left(-m^2, 0, m^2\right)\right) \tag{3.40}$$

By using the relation that $B_0(-m^2, 0, m^2) = B_0(0, m^2, m^2) + 2$, the scalar integrals drops out and one is left with the value of the anomalous magnetic moment without carrying out an integration.

Although this is a nice result, this also means that we cannot apply a standard Passarino-Veltman reduction on our integral, but that we must begin by manipulating our expression to put it into a suitable form. This could also in principle be amended by using a reduction scheme in a different basis. Although relevant to the question of generalizing saddle point approximations, this will not be developed further in this text.

3.5.4 A saddle point approximation for the Anomalous Magnetic Moment with Standard Calculational Techniques

In the previous section, as a means to illustrate the usefulness of the saddle point approximations, the evaluation of the scalar integral C_0 by these techniques can yield an approximate value for the anomalous magnetic moment.

The scalar integral we want to calculate is:

$$\int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^3} = \frac{1}{4\Delta} \tag{3.41}$$

and an alternative means of achieving the saddle point approximation is given here to confirm its results.

Start by the standard means of evaluating the integral:

$$\begin{aligned} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[\ell^2 - \Delta]^3} &= \frac{-i}{(2\pi)^4} \int d^4\ell_E \frac{1}{[\ell_E^2 + \Delta]^3} \\ &= \frac{-i}{(2\pi)^4} \int d\Omega_4 \int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^3} \\ &= \frac{-i}{8\pi^2} \int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^3} \end{aligned} \tag{3.42}$$

We now wish to calculate this integral using a saddle point approximation. We can rewrite the integral to

$$\int_0^\infty d\ell_E \frac{\ell_E^3}{[\ell_E^2 + \Delta]^3} = \int_0^\infty d\ell_E \left[\frac{\ell_E}{\ell_E^2 + \Delta}\right]^3 \tag{3.43}$$

Looking closer at the integrand:

$$\frac{\ell_E}{\ell_E^2 + \sqrt{\Delta}^2} = \frac{1}{\sqrt{\Delta}} \frac{1}{\frac{\ell_E}{\sqrt{\Delta}} + \frac{\sqrt{\Delta}}{\ell_E}} \tag{3.44}$$

Taking the ratio $\ell_E/\sqrt{\Delta} =: 1+x$ where x is small as we are close to being on shell the expression becomes:

$$\frac{1}{\sqrt{\Delta}}\frac{1}{1+x+\frac{1}{1+x}} = \frac{1}{\sqrt{\Delta}}\frac{1}{1+x+1-x+x^2} = \frac{1}{2\sqrt{\Delta}}\frac{1}{1+\frac{1}{2}x^2} \tag{3.45}$$

From this point we use the saddle point approximation:

$$\frac{1}{2\sqrt{\Delta}}\frac{1}{1+\frac{1}{2}x^2} = \frac{1}{2\sqrt{\Delta}}\exp\left[-\frac{1}{2\Delta}(\ell_E-\sqrt{\Delta})^2\right] \tag{3.46}$$

Inserting the integrand into our previous expression we obtain

$$\frac{1}{8\Delta^{3/2}}\int_{-\infty}^{\infty} d\ell_E \exp\left[-\frac{3}{2\Delta}(\ell_E-\sqrt{\Delta})^2\right] = \frac{1}{4\Delta}\sqrt{\frac{\pi}{6}} \tag{3.47}$$

In this case the expression thus differs from the exact value by $\sqrt{\frac{\pi}{6}} = 0.72$, the same factor as was derived in the previous section on scalar integrals and which depends on the dimensionality of space and the number of propagators n. From this point the calculation is identical to the standard method of calculation, thus the saddle point method gives a bare result of $0.72 * \frac{\alpha}{2\pi}$ for the value of the Anomalous Magnetic Moment.

3.5.5 The saddle point approximation with a slowly varying numerator

Setting Up the Feynman Rules

Following the calculation in Peskin and Schroeder [4] up to $\Gamma^\mu = \gamma^\mu + \delta\Gamma^\mu$ with

$$\begin{aligned}\delta\Gamma^\mu(p',p) = \int \frac{d^4k}{((2\pi)^4}\frac{-ig_{\nu\rho}}{(k-p)^2+i\epsilon}\bar{u}(p')\left(-ie\gamma^\nu\right)\times \\ \frac{ik\!\!\!/'+m}{k'^2-m^2+i\epsilon}\gamma^\mu\frac{i(k\!\!\!/+m)}{k^2-m^2+i\epsilon}\left(-ie\gamma^\rho\right)u(p) = \\ 2ie^2\int\frac{d^4k}{(2\pi)^4}\frac{\bar{u}(p')\left(k\!\!\!/\gamma^\mu k\!\!\!/'+m^2\gamma^\mu-2m(k+k')^\mu\right)u(p)}{(k-p)^2+i\epsilon\left(k^2-m^2+i\epsilon\right)\left(k'^2-m^2+i\epsilon\right)}\end{aligned} \tag{3.48}$$

Linearity of internal and external momentum

Since there is only one scalar product involved in the integrand, we may choose to work in the rest frame of $p = (p^0;0,0,0)$. This allows us to separate the k^0-integration. We may then adjust the integration measure accordingly as $\int d^4k = \int\limits_{-\infty}^{\infty} dk_0 \int d^3k_\perp$. The variables are then given in the following sense:

$$|k_\parallel| =: \frac{k\cdot p}{\sqrt{p^2}} \tag{3.49}$$

$$|k_\perp| =: \sqrt{k_\parallel^2 - k^2} \tag{3.50}$$

$$k_\perp \cdot p = 0 \tag{3.51}$$

$$k_\perp \cdot k_\parallel = 0 \tag{3.52}$$

Further, we may split up $k_\parallel$ into two parts, by grouping all parts proportional to p and gather the remainder in a nonlinear term, $k_\parallel = k_0 + k_{\text{non-lin}}$. We will later determine this proportionality constant at the saddle point, and we will therefore name it by the relation that $k_0 = \alpha p_0$, where the name is chosen due to it's relation to p_0, the only non-vanishing component of p in our reference frame and α in general is a function. In short, for arbitrarily small excess terms $k_{\text{non-lin}}$, we therefore have the relation $k^2 = \alpha^2 p_0^2 - k_\perp^2$.

Numerator Algebra

While handling the numerator, we will repeatedly use the anticommutation relations, $\{\gamma^\mu, \gamma^\nu\} = 2$, as well as the Dirac equation, $\not{p}u(p) = mu(p)$ and $\bar{u}(p')\not{p}' = \bar{u}(p')m$.

During the calculation we will also make the approximation that the numerator varies slowly in comparison to the denominator, and therefore use $k^\mu = \alpha p_0^\mu$ for these terms, where we from now on assume that α is a constant (which will be determined at the saddle point).

In addition, we will do well to remember the definitions $p' = p+q$ and $k' = k+q$, as well as the relation from the previous paragraph, $k^\mu = \alpha p^\mu$. We will also suppress the $u(p)$ and $\bar{u}(p')$ terms.

$$\begin{aligned}
\not{k}\gamma^\mu\not{k}' + m^2\gamma^\mu - 2m(k+k')^\mu &= \\
= \not{k}\gamma^\mu\not{k} + \not{k}\gamma^\mu\not{q} + m^2\gamma^\mu - 2m(k+k')^\mu &\\
= \alpha m(2k^\mu - \gamma^\mu\not{k}) + \alpha m\gamma^\mu\not{q} - \not{q}\gamma^\mu\not{q} + m^2\gamma^\mu - 2m(k+k')^\mu &\\
= 2\alpha m k^\mu - \alpha^2 m^2\gamma^\mu + \alpha m(2q^\mu - \not{q}\gamma^\mu) &\\
+(q^2+m^2)\gamma^\mu - 2m(k+k')^\mu &\\
= 2\alpha m k^\mu - \alpha^2 m^2\gamma^\mu + \alpha m(2q^\mu + \not{p}\gamma^\mu - \not{p}'\gamma^\mu) &\\
+(q^2+m^2)\gamma^\mu - 2m(k+k')^\mu &\\
= 2\alpha m k^\mu - \alpha^2 m^2\gamma^\mu + \alpha m(2q^\mu + 2p^\mu - 2m\gamma^\mu) &\\
+(q^2+m^2)\gamma^\mu - 2m(k+k')^\mu &
\end{aligned} \tag{3.53}$$

The ambition is to rewrite the expression above in the form $\gamma\cdot[A] + (p+p')^\mu\cdot[B] + q^\mu\cdot[C]$. Anticipating this, the expression can be rewritten in the following form:

$$\begin{aligned}
\gamma^\mu\cdot(-\alpha^2m^2 + q^2 + m^2 - 2m\alpha) + &\\
k^\mu\cdot(2\alpha m) + p'^\mu(2\alpha m) - 2m(k+k)^\mu &\\
= \gamma^\mu\cdot(-\alpha^2m^2 + q^2 + m^2 - 2m\alpha) + &\\
p^\mu\cdot(2\alpha^2 m) + p^\mu(2\alpha m) + q^\mu(2\alpha m) - 2m\alpha(p+p')^\mu &\\
= \gamma^\mu\cdot(-\alpha^2m^2 + q^2 + m^2 - 2m\alpha) + &\\
p^\mu(2\alpha^2 + 2\alpha m - 4\alpha m) + q^\mu(2\alpha m - 2\alpha m) &\\
= \gamma^\mu\cdot(-\alpha^2m^2 + q^2 + m^2 - 2m\alpha) + &\\
p^\mu(2\alpha m(\alpha - 1)) &\\
= \gamma^\mu\cdot(-\alpha^2m^2 + q^2 + m^2 - 2m\alpha) + &\\
(p+p')^\mu(\alpha m(\alpha-1)) - q^\mu(\alpha m(\alpha-1)) &
\end{aligned} \tag{3.54}$$

The Gordon identity, $2m\gamma^\mu = (p+p')^\mu + i\sigma^{\mu\nu}q_\nu$ may now be used to put the expression in the standard form for evaluating the anomalous magnetic moment, $\gamma\cdot[F_1] + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\cdot[F_2] + q^\mu\cdot[C]$ where $F_2[0]$ is the anomalous magnetic moment. Therefore:

$$(p+p')^\mu\cdot m\alpha(1-\alpha) = \frac{i\sigma^{\mu\nu}q_\nu}{2m}\cdot 2m^2\alpha(1-\alpha) + 2m\gamma^\mu\cdot m\alpha(1-\alpha) \tag{3.55}$$

From this, we may easily read the value for the second form factor, F_2, as being the expression $2m^2\alpha(1-\alpha)$ times the denominator.

Wick Rotation

Putting together the expression for the formfactor, it is given by:

$$\begin{aligned} F_2(q \to 0) &= \lim_{q\to 0} 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2m^2\alpha(1-\alpha)}{(k-p)^2(k^2-m^2)(k'^2-m^2)} \\ &= 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2m^2\alpha(1-\alpha)}{(k-p)^2(k^2-m^2)^2} \end{aligned} \tag{3.56}$$

In this treatment we assume that the factors in the denominator vary appreciably much faster than those in the numerator. Thus, we keep the value of the numerator constant at the value of the saddle-point (which is yet to be calculated).

In order to make the expression more manageable, we want to perform a Wick rotation. In our situation, the Wick rotation will be slightly modified, since our expression involves factors that are usually excluded by the process of Feynman parametrization, which we will not employ in this calculation.

We are interested in performing a Wick rotation, not through the origin, but through the line $k_0 = -p_0/3$. When analyzing the various poles of the expression, we notice that one pole is inside the the area covered by the complex integration over k_0. Thus, we have a modified expression:

$$\begin{aligned} 2ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{2m^2\alpha(1-\alpha)}{(k-p)^2(k^2-m^2)^2} &= 2ie^2 \times \frac{i}{-1} \int \frac{d^4k_E}{(2\pi)^4} \frac{2m^2\alpha(1-\alpha)}{(k_E^0+ip)^2(k_E^2+m^2)^2} \\ &+ 2ie^2 \times 2m^2 \int \frac{d^3\vec{k}}{(2\pi)^4} 2\pi i \ \mathrm{Res}(k_0') \end{aligned} \tag{3.57}$$

where k_0' is the location of the pole, which we will now calculate:

The interesting pole that may interfere with the Wick rotation should come from the photon propagator, since the electron propagator terms are of the usual form of the Feynman $i\epsilon$-prescription centered at the origin. However, the Feynman $i\epsilon$-prescription for the photon propagator is centered at $k_0 = p = p_0$ (where we, as previously noted, chosen a frame $p = p_0$). So the poles for the photon propagator emerge at:

$$\begin{aligned} 0 &= (k-p)^2 + i\epsilon = k^2 - 2k\cdot p_0 + p_0^2 \\ &= k_0^2 - k_\perp^2 - 2k\cdot p_0 + p_0^2 = k_0^2 - k_\perp^2 - 2k_0\cdot p_0 + p_0^2 \\ &= (k_0 - p_0)^2 - k_\perp^2 + i\epsilon \end{aligned} \tag{3.58}$$

Thus we can calculate:

$$\begin{aligned} (k_0 - p_0) &= \pm\sqrt{k_\perp^2 - i\epsilon} \\ |k_0| &= |p_0| \pm |k_\perp| \sqrt{1 - \frac{i\epsilon}{|k_\perp|^2}} \\ &\approx |p_0| \pm |k_\perp| \left(1 - \frac{i\epsilon}{2|k_\perp|^2}\right) \\ &=: |p_0| \pm (|k_\perp| - i\epsilon') \end{aligned} \tag{3.59}$$

From inspection, the pole that is within the curve integral is $|p_0| - |k_0| + i\epsilon'$. If we separate our expression as:

$$\frac{\alpha(1-\alpha)}{(|k_0| - |p_0| - |k_\perp|)(|k_0| - |p_0| + |k_\perp|)(|k_0|^2 - |k_\perp|^2 - m^2)^2} \tag{3.60}$$

We see that the simple pole of the residue is seen more clearly and we may calculate it as (supressing $i\epsilon$):

$$\begin{aligned}
\text{Res}\ (|p_0|-|k_\perp|) &= \left[\frac{\alpha(1-\alpha)}{(|k_0|-|p_0|-|k_\perp|)(|k_0|^2-|k_\perp|^2-m^2)^2}\right]_{|k_0|=|p_0|-|k_\perp|} \\
&= \left[\frac{\alpha(1-\alpha)}{(-2|k_\perp|)(|p_0|^2-2|p_0||k_\perp|+|k_\perp|^2-|k_\perp|^2-m^2)^2}\right]_{|k_0|=|p_0|-|k_\perp|} \\
&= \left[\frac{\alpha(1-\alpha)}{(-2|k_\perp|)(-2|p_0||k_\perp|)^2}\right]_{|k_0|=|p_0|-|k_\perp|} \\
&= \left[\frac{1}{(-2|k_\perp|)}\times\frac{1}{4|p_0|^2}\times\frac{|k_0|}{|p_0|}\left(1-\frac{|k_0|}{|p_0|}\right)\right]_{|k_0|=|p_0|-|k_\perp|} \\
&= \frac{1}{(-2|k_\perp|)}\times\frac{1}{4|p_0|^2}\times\frac{|p_0|-|k_\perp|}{|p_0|}\left(\frac{|p_0|-(|p_0|-|k_\perp|)}{|p_0|}\right) \\
&= \frac{1}{-8|p_0|^4}\frac{1}{|k_\perp|^2}(|p_0|-|k_\perp|) \qquad (3.61)
\end{aligned}$$

We may now insert this residue into the expression (3.57), allowing us to carry out the remaining three integrations:

$$\begin{aligned}
2\pi i &\int\frac{d^3k_\perp}{(2\pi)^4}\frac{1}{-8|p_0|^4}\frac{1}{|k_\perp|^2}(|p_0|-|k_\perp|) \\
&= i\frac{2\pi}{(2\pi)^4}\int d\Omega\int d|k_\perp||k_\perp|^2\times \\
&\times \frac{1}{-8|p_0|^4}\frac{1}{|k_\perp|^2}(|p_0|-|k_\perp|) \\
&= \frac{i}{4}\frac{(2\pi)^2}{(2\pi)^4}\int d|k_\perp|\frac{1}{-|p_0|^4}(|p_0|-|k_\perp|) \qquad (3.62)
\end{aligned}$$

From this expression, we can see that the pole contains terms linear and quadratic in $|k_\perp|$. However, for means of our Wick rotation, the pole will only be situated within the bounds of the integral's path while $\alpha \geq -1/3$. Therefore, we may simply perform the integral with these amended bounds. We will therefore rewrite our expression in terms of α, for a moment treating α as a function as a means of calculation:

$$\begin{aligned}
&\frac{i}{4}\frac{(2\pi)^2}{(2\pi)^4}\int d|k_\perp|\frac{1}{-|p_0|^4}(|p_0|-|k_\perp|) = \left\{\begin{array}{r} |k_\perp| = |p_0|-|k_0| \\ |k_0| = |p_0|-|k_\perp| \\ d|k_0| = -d|k_\perp| \end{array}\right\} \\
&= \frac{i}{4(2\pi)^2}\frac{1}{|p_0|^4}\int d|k_0||k_0| = \left\{\begin{array}{r} \alpha = |k_0|/|p_0| \\ d\alpha = d|k_0|/|p_0| \end{array}\right\} \\
&= \frac{i}{16\pi^2}\frac{1}{|p_0|^4}\int |p_0|d\alpha|p_0|\alpha \\
&= \frac{i}{16\pi^2}\frac{1}{|p_0|^2}\int_{\alpha=1}^{-1/3} d\alpha\alpha = \frac{i}{16\pi^2}\frac{1}{|p_0|^2}\left[\frac{\alpha^2}{2}\right]_{\alpha=1}^{-1/3} = \frac{i}{16\pi^2}\frac{1}{|p_0|^2}\left[\frac{-8}{18}\right] \qquad (3.63)
\end{aligned}$$

From (3.57), the residue calculation gives a contribution of:

$$2ie^2\times 2m^2\times\frac{i}{16\pi^2}\frac{1}{|p_0|^2}\left[\frac{-8}{18}\right] = \frac{8}{9}\alpha_{AMM} \qquad (3.64)$$

where α_{AMM} is the finestructure constant in natural units, and $\frac{\alpha_{AMM}}{2\pi}$ is the exact value of the anomalous magnetic moment.

saddle point approximation over $k_\perp$

For the denominator, we may carry out a saddle-point approximation for the terms transverse to the external momentum as follows:

$$\begin{aligned}
\frac{1}{(k-p)^2(k^2-m^2)^2} =& \frac{1}{(k_0+k_\perp-p)^2(k_0^2-k_\perp^2-m^2)^2} && (3.65)\\
=& \frac{1}{[(k_0-p)^2-k_\perp^2](k_0^2-k_\perp^2-m^2)^2} && (3.66)\\
\approx& \frac{1}{(k_0-p)^2(k_0^2-m^2)^2}\times && (3.67)\\
& \exp\left(-k_\perp^2\left(-\frac{1}{(\alpha-1)^2p^2}-\frac{2}{\alpha^2p^2-m^2}\right)\right) && (3.68)\\
=& \frac{1}{(k_0-p)^2(k_0^2-m^2)^2}\exp\left(\frac{-k_\perp^2}{p^2}\cdot C\right) && (3.69)
\end{aligned}$$

where in the first equality $k=\alpha p+k_T$ is used, in the second equality the orthogonality of p and k_T is used, in the third equality a saddle-point approximation is made and in the fourth equality the constant C is defined.

Since the integration is merely over the completely separable subspace perpendicular to the external momenta p, it is internally spherically symmetric and we may rewrite the expression in Euclidean, three-dimensional, spherical coordinates:

$$\begin{aligned}
& \int d^3k_\perp \exp\left(\frac{-|k_\perp|^2}{p^2}\cdot C\right)\\
=& \int_0^{4\pi} d\Omega \int_0^{\infty} d|k_\perp||k_\perp|^2 \exp\left(\frac{-|k_\perp|^2}{p^2}\cdot C\right)\\
=& -\frac{4\pi}{C}\frac{d}{d\frac{1}{p^2}}\int_0^{\infty} d|k_\perp| \exp\left(\frac{-|k_\perp|^2}{p^2}\cdot C\right)\\
=& -\frac{4\pi}{C}\frac{d}{d\frac{1}{p^2}}\left(\frac{1}{2}\sqrt{\frac{\pi p^2}{C}}\right)\\
=& -\frac{4\pi}{2C}\times -\frac{C}{2\pi}\left(\sqrt{\frac{\pi p^2}{C}}\right)^3 = \left(\sqrt{\frac{\pi p^2}{C}}\right)^3 && (3.70)
\end{aligned}$$

saddle point approximation over k_0

We are now interested in performing the saddle-point approximation for the k_0-subspace. Since this subspace is not centered around zero, as it was in the previous case, we will go through the various steps of the saddle-point approximation explicitly for clarity. If we take the derivative of

$$\frac{1}{(k_0-p_0)^2(k_0^2-m^2)^2} \qquad (3.71)$$

We find it to be

$$\frac{-6k_0^2 + 2m^2 + 4k_0p_0}{(k_0^2 - m^2)^3(k_0 - p_0)^3} \tag{3.72}$$

This has two extrema, those of $k_0 = p_0$ and $k_0 = -p_0/3$. During this treatment we will only calculate an on-shell situation where q=0. In addition, as discussed earlier in this section, we are calculating in a frame where $p = p_0$, and looking at a situation where k_0 is appreciably larger than $k_\perp$. We can therefore always use $k^2 = p^2$, which in our approximation gives the relation $k_0^2 = p_0^2$. Therefore, the extremum at $k_0 = p_0$ will be excluded, since at this value, the expression diverges.

We now define the function $H(k)$ for notational simplicity.

$$\begin{aligned}
F_2[0] &= 2ie^2 \times 2\left(\sqrt{\frac{\pi p^2}{C}}\right)^3 \int \frac{dk_0}{(2\pi)^4} \frac{2m^2\alpha(1-\alpha)}{(k_0 - p_0)^2(k_0^2 - m^2)^2} \\
&= 2ie^2 \times 2\left(\sqrt{\frac{\pi p^2}{C}}\right)^3 \times \frac{-8m^2}{9} \times \frac{1}{(2\pi)^4} \int dk_0 \frac{1}{(k_0 - p_0)^2(k_0^2 - m^2)^2} \\
&= 2ie^2 \times 2\left(\sqrt{\frac{\pi p^2}{C}}\right)^3 \times \frac{-8m^2}{9} \times \frac{1}{(2\pi)^4} \times i \int dk_{E0} \frac{1}{(ik_{E0} - p_0)^2((ik_{E0})^2 - m^2)^2} \\
&= 2ie^2 \times 2\left(\sqrt{\frac{\pi p^2}{C}}\right)^3 \times \frac{-8m^2}{9} \times \frac{1}{(2\pi)^4} \times -i \int dk_{E0} \frac{1}{(k_{E0} + ip_0)^2((k_{E0})^2 + m^2)^2} \\
&=: 2ie^2 \times 2\left(\sqrt{\frac{\pi p^2}{C}}\right)^3 \times \frac{-8m^2}{9} \times \frac{1}{(2\pi)^4} \times -i \int dk_{E0} H(k_{E0})
\end{aligned} \tag{3.73}$$

Let us now perform our saddle point approximation for the expression $H(k_{E0})$.

$$\begin{aligned}
&\int dk_{E0} H(k_{E0}) \rightarrow \\
&\int dk_{E0} H(ip/3) \times \exp\left(\frac{-H''(ip/3)(k_{E0} - (ip/3))^2}{2H(ip/3)}\right) = \frac{0.68i}{p^4\sqrt{p^2}}
\end{aligned} \tag{3.74}$$

We will also need to calculate C:

$$C = \left[-\frac{1}{(\alpha-1)^2} - \frac{2}{\alpha^2 - 1}\right]_{\alpha=-1/3} = -\frac{1}{16/9} - \frac{2}{8/9} = \frac{-27}{16} \tag{3.75}$$

We can now calculate the contribution to the anomalous magnetic moment:

$$2ie^2 \times \left(\sqrt{-\frac{16\pi p^2}{27}}\right)^3 \times \frac{-8m^2}{9} \times \frac{1}{(2\pi)^4} \times -i \times \frac{0.68i}{p^4\sqrt{p^2}} = -0.16 \times \frac{\alpha_{AMM}}{2\pi} \tag{3.76}$$

where α_{AMM} is the finestructure constant in natural units, and $\frac{\alpha_{AMM}}{2\pi}$ is the exact value of the anomalous magnetic moment.

The combined result

Based on equation 3.57, we may now add the two calculational results 3.64 and **??**, to obtain the common value:

$$\left(\frac{8}{9} - 0.16\right) \times \frac{\alpha_{AMM}}{2\pi} = 0.73 \times \frac{\alpha_{AMM}}{2\pi} \tag{3.77}$$

which is very close to $\sqrt{\pi/6}$

How does the saddle point approximations compare in separable vector space and in spherified space?

If we compare the results of the four-dimensional spherically symmetrical integral (which was calculated with the standard method of Feynman parameters), with that discussed in this section (which utilizes the separability of the parallel and orthogonal vector space of the external momentum), we see that the results are fairly similar. This is not a priori clear, as the methods employ different approximations, in particular of the numerator factors involved in the expression.

A merit of this calculational method is that it is more clear how the contributions are adding up in the computation, offering some insight into the process at hand. Since the calculation is performed with the physical quantities at hand, although analytically continued into complex space by the $i\epsilon$-prescription, it offers a greatly increased oversight over the process at hand.

That it is a saddle point approximation also shows that there is a typical saddle-point value of the momenta of the process, with which we gain some insight over what is happening in the process during the calculation.

3.6 Correction factors to the saddle point approximation

Having looked at how the saddle point approximation behaves, it is clear that it generally is strictly smaller than the propagator terms, due to the faster convergence of the exponential function. Due to this argument, it is quite likely that one could obtain a better agreement on a whole if one were to use a saddle point approximation which corrects for this.

In principle, one could then have the approximation overestimating the propagator terms near the saddle point, crossing the exact answer at a later time. To this end, we could fit the two occurences of parameters m_e and m_d:

$$\frac{\exp(-\frac{k^2}{m_e^2})}{m_d^2} \tag{3.78}$$

This fit could either be from theoretical arguments, such as that the integrand from the approximation should equal the exact integrand at a typical value, or from an actual fitting to the exact answer.

The issue with the first approach is that it is not clear at the time which 'typical value' is suitable for the applying the correction factor, and this is a question open for further investigation. The problem with the second approach is that it assumes knowledge of the exact solution to the problem. It may be so that the fit for one calculation agrees with other calculations, and that would allow one to 'train' the saddle point approximation on already known diagrams, and then use the 'trained' saddle point approximation on new diagrams. Whether this approach would be successful is also still an open question.

3.7 Divergences and the saddle point approximation

As we know, not all diagrams yields such simple, convergent predictions as the anomalous magnetic moment at first glance. In fact, the F_1-contribution of the same diagram, the correction to the electric charge, contains both infrared and ultraviolet divergences. As discussed in the previous chapter, there are many means of regularizing these integrals. Before regularizing, simply by comparing the diagrams of the scalar integrals with their saddle point approximation, as is done in figure 3.3, it is clear that the method is no longer particularly effective.

At this point, there is no clear choice among the various well-established regularization schemes that has particular merit, but rather the approach of an on-shell renormalization scheme could perhaps be effective. Rather than evaluating each of these diagrams by itself, one could evaluate the saddle point approximation of their difference, circumventing any possible divergences to come. As an example, the $F_1(q^2)$-factor from the AMM example uses the approach of:

$$\delta F_1(q^2) \to \delta F_1(q^2) - \delta F_1(0) \tag{3.79}$$

A scheme such as dimensional regularization with minimal subtraction runs into problems in that it lets removes poles *after* evaluating scalar and tensorial integrals in d-dimensions. However, these integrals, including divergent terms, are included into the saddle point approximation, leading to unsuccessful results.

Similar issues arise when introducing fictious photon masses as in a Pauli-Villars regularization and therefore the most straightforward, albeit lengthy, way of dealing with divergences would therefore be that of evaluating sums of Feynman diagrams, where divergences are cancelled internally.

This is a topic which lends itself to further investigation, and a clear prediction is not yet established. One possible procedure, which one is invited to look closer at, due to the nature of the approximation, is to use the well-known calculational trick of absorbing terms into the exponential function by partial integration. In princple, this could allow one to evaluate the derivatives of the Feynman diagram one is interested in calculating. Finding this derivative term to be convergent, one could thereafter integrate to obtain the amplitude of interest, having applied the saddle point approximation for the convergent case.

3.8 Higher Order Contributions

The applicability to the standard method is useful in several respects; it provides a useful point of comparison for other calculational methods and it allows one to apply the method to a very general procedure, through Passarino-Veltman reduction. However, obviously, the results of one-loop calculations with few external particles is not a relevant problem where the saddle point approximation may provide real utility, since these more elementary integrals are already worked out exactly and implemented in fast programs (examples of which could be CutTools [23], FeynCalc [17] and XLOOPS [28]) it is a good indication of the usefulness of the method.

The truly interesting feature of this calculational procedure is that it is quite straightforward to extend to higher orders of perturbation theory. This is the area were this calculational procedure would be useful to implement. For example, from the one loop to the two loop case, the integration measure changes as:

$$\int d^d k = \int_{-\infty}^{\infty} dk_0 \int d^{d-1}k_\perp = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})} \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} d|k_\perp| \times |k_\perp|^{d-2}$$

$$\rightarrow \int d^d k \int d^d l = \int_{-\infty}^{\infty} dk_0 \int d^{d-1}k_\perp \int_{-\infty}^{\infty} dl_0 \int d^{d-1}l_\perp$$

$$= \int_{-\infty}^{\infty} dk_0 \int_0^{\infty} d|k_\perp||k_\perp|^{d-2} \int_{-\infty}^{\infty} dl_0 \int_0^{\infty} dl_\perp |l_\perp|^{d-2} \int_0^{\pi} d\theta sin^{d-3}\theta \tag{3.80}$$

And as such the expression has five integrals to be performed, instead of the previous two (one, however, is quite trivial; the integral over θ, the angle between the external momenta). Therefore, this method of calculation is quite possible to extend beyond results that are easily accessible at present, quite possibly even beyond two loop calculations, with a manageable increase in computational complexity.

3.9 Crude Estimations of diagrams

I would also be very interesting to use the method to make crude rules for how one can use somewhat intuitive decent approximations to calculate very big diagrams almost by just counting loops, propagators, vertices and crudely approximating determinants as almost diagonal. The nicest and perhaps most accurate approach would be to simply evaluate the integrand of the Feynman diagram at the 'typical point in the integration region', and then in addition evaluate or estimate the width of the region over which the integrand is close to the value in this 'typical value region'. This suggestive proceedure is of course in reality to be interpreted as a saddle point-like method, that has some hope of providing order-of-magnitude predictions.

Chapter 4

Discussion and Outlook

We have, through the previous chapters, seen how the systematic treatment of one-loop calculations has offered renormalizable, well-defined predictions for QED. As such, the problem of evaluating first order corrections to perturbative quantum field theory can be considered possible to solve completely. These methods have also been shown to be rather non-trivial at some kinematical regions, where non-inherent divergences may arise due to the method of calculation. It has also been mentioned that these, and other modern approaches, have been implemented in computer packages, that makes these results easily available.

That it would be interesting to have a simple and accessible method to evaluate even more complex Feynman diagrams, albeit with an approximative method, seems to be a warranted statement. It has many possible applications and is also of interest from an intuitive point of view regarding the calculation of radiative corrections. At present, the method is clearly not able to handle all kinds of diagrams that one would be interested in evaluating, but the initial results presented in this text shows promise. Hopefully, it will be possible to consistently extend this mode of calculation to areas where the answers are not as easily exactly calculable.

There are still many open questions regarding the theory, and to round off this presentation, we summarize the three main questions at the moment:

- How can we consistently deal with approximating divergent integrals?
- How can we use an *ad hoc* (although warranted) correction factor to improve the accuracy of the method?
- How can we extend the scheme to apply for higher order contributions?

Bibliography

[1] 'An Almost Physical Interpretation of the Integrand of the N-point Veneziano Model', Holger Bech Nielsen, preprint, 1970

[2] http://dtc.pima.edu/~hacker/vectorcalc/classnotes/vector-calc-chapter3.10-multi-dim-taylor-series.pdf

[3] J. Schwinger, Phys. Rev. 73, 416L (1948)

[4] Michael E. Peskin and Daniel V. Schroeder, An Introduction to Quantum Field Theory, Addison-Wesley, Reading, 1995.

[5] Shapiro, J. 'Schwinger trick and Feynman Parameters', http://www.physics.rutgers.edu/grad/615/lects/schwingertrick.pdf

[6] G. 't Hooft and M. Veltman, Scalar One Loop Integrals, Nucl.Phys. B153 (1979) 365-401.
A. Denner, U. Nierste, and R. Scharf, A Compact expression for the scalar one loop four point function, Nucl. Phys. B367 (1991) 637-656.
R. K. Ellis and G. Zanderighi, Scalar one-loop integrals for QCD, JHEP 02 (2008) 002, [0712.1851].
R. K. Ellis and G. Zanderighi, QCDLoop. http://qcdloop.fnal.gov.
A. Denner and S. Dittmaier, Scalar one-loop 4-point integrals, Nucl. Phys. B844 (2011) 199-242, [1005.2076].
A. van Hameren, OneLoop: for the evaluation of one-loop scalar functions, 1007.4716.

[7] Weisskopf, V. 1934, Z. Phys. **89**,27;**90**,817.

[8] Lamb, W. E. And Retherford, R. C. (1947). Fine Structures of the Hydrogen Atom by a Microwave Method, Phys. Rev. 72, 241-243.

[9] Bethe, H.A. (1947). The Electromagnetic Shift of Energy Levels, Phys. Rev. 72, 339-341

[10] Dyson, F. J. The Radiation Theories of Tomonaga, Schroedinger and Feynman, Phys. Rev. 75, 486-502, 1949

[11] G. Passarino and M. J. G. Veltman, 'One Loop Corrections for e+ e- Annihilation Into mu+ mu- in the Weinberg Model', Nucl. Phys. B160 (1979) 151.

[12] G. 't Hooft and M. Veltman, Scalar One Loop Integrals, Nucl.Phys. B153 (1979) 365-401.

[13] A. Denner, Techniques for Calculation of Electroweak Radiative Corrections at the One Loop Level and Results for W Physics at LEP-200, Fortschr. Phys. 41 (1993) 307-420, arXiv:0709.1075 [hep-ph].

[14] D. Ross, 'Modern methods in perturbative QCD', http://www.hep.phys.soton.ac.uk/hepwww/staff/D.Ross/mhv/

[15] Z. Bern, L. J. Dixon, and D. A. Kosower, 'Dimensionally Regulated Pentagon Integrals,' Nucl. Phys. B412 (1994) 751-816, arXiv:hep-ph/9306240

[16] Wolfram Research, Inc., Mathematica, Version 8.0, Champaign, IL (2011).

[17] R. Mertig, M. Bohm , and A. Denner, Comp.Phys.Comm. 64 (1991) 345.

[18] Selection of relevant articles are available in chapter 3 of the review article
Ellis, Kunszt, Melnikov and Zanderighi 'One-loop calculations in quantum field theory: from Feynman diagrams to unitarity cuts', 2011, [arXiv:1105.4319]

[19] J.Kublbeck, M.Bohm and A. Denner, Comp.Phys.Comm. 60 (1990) 165

[20] Guide to FeynCalc 1.0. Rolf Mertig. Physikalisches Institut. Am Hubland. Universität Würzburg. 8700 Würzburg, Germany. 1992, www.feyncalc.org/oldfc/FCGuide3.ps.gz

[21] Devaraj, G. and Stuart, R., 'Reduction of one loop tensor form-factors to scalar integrals: A General scheme', Nucl.Phys. B519 (483-513) 1998 DOI:10.1016/S0550-3213(98)00035-2 [arXiv:9704.308]

[22] Berger, C.F. and Bern, Z. and Dixon, Lance J. and Febres Cordero, F. and Forde, D. and others 'One-Loop Calculations with BlackHat' Nucl.Phys.Proc.Suppl.183 (313-319) 2008 DOI:10.1016/j.nuclphysbps.2008.09.123 [arXiv:0807.3705] (and references therein)

[23] Ossola, Giovanni and Papadopoulos, Costas G. and Pittau, Roberto 'CutTools: A Program implementing the OPP reduction method to compute one-loop amplitudes' JHEP 0803 (042) 2008 DOI:10.1088/1126-6708/2008/03/042 [arXiv:0711.3596] (and references therein)

[24] Korner, J. G.; Kreimer, D.; Schilcher, K, 'A practicable γ 5-scheme in dimensional regularization' Zeitschrift fur Physik C Particles and Fields, Volume 54, Issue 3, pp.503-512 DOI: 10.1007/BF01559471

[25] Kunszt, Signer and Trocsanyi, One Loop Helicity Amplitudes For All $2 \to 2$ Processes In QCD And N=1 Supersymmetric Yang-Mills Theory, Nucl. Phys. B 411 (1994) 397 [arXiv:hep-ph/9305239]

Catani, Seymour and Trocsanyi, Regularization scheme independence and unitarity in QCD cross sections, Phys. Rev. D 55 (1997) 6819 [arXiv:hep-ph/9610553].

A. Signer and D. Stockinger, Nucl. Phys. B 808 (2009) 88 [arXiv:0807.4424 [hep-ph]].

[26] Espriu, D. 'Propagators and Green functions'
http://www.ecm.ub.es/˜ espriu/teaching/classes/fae/LECT5.pdf

[27] E. Elizalde 'Zeta regularization techniques with Applications.' World Scientific (1994)

[28] Franzkowski, J. et al. 'Automatic calculation of massive two-loop self-energies with XLOOPS'. Nuclear Instruments and Methods in Physics Research A 389 (1-2): 333 (1997) [arXiv:9611.378] DOI:10.1016/S0168-9002(97)00121-6. Brucher, L. et al. 'Automatic Feynman diagram calculation with xloops: A Short overview' (2000) arXiv:hep-ph/0002028 [hep-ph].