# Modern Techniques for One-Loop Calculations

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**Abstract**

We review the techniques used for one-loop calculations with emphasis on practical applications. QED is used as an example but the methods can be used in any theory. The aim is to teach how to use modern techniques, like the symbolic package `FeynCalc` for `Mathematica` and the numerical package `LoopTools` for `Fortran` or `C++`, in one-loop calculations.

# Contents

# 1 Introduction

The techniques for doing one-loop calculations in Quantum Field Theory have been developed over the past 60 years and are now part of every textbook on this subject. However when we face a *real life* problem we get the impression that we have always to start from the first principles. This means that we have to introduce the Feynman parameters, dimensional regularization, Wick rotation, and so on, before we get a result.

There are however ways to make the calculation more automatic. These techniques use the Passarino-Veltman (PV) [1] decomposition, the `Mathematica` package `FeynCalc` [2] for symbolic computations and the `LoopTools` [3] package for numerical applications. This last package acts as a front end for the previous package `FF` [4,5] developed by van Oldenborgh for evaluation of the PV integrals. Although these techniques are by now quite standard they did not yet get into the textbooks. This is the gap that we want to fill in here.

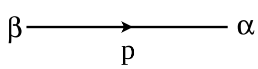
This text is organized as follows. In section 2 we review the renormalization program for QED using the usual technique of dimensional regularization. In section 3 we introduce the PV decomposition. As an example of its use we do again the renormalization of QED using this approach in section 4. In section 5.1 we compute the anomalous magnetic moment of the electron at one-loop and in section 5.2 we compute the radiative corrections to the Coulomb scattering taking special attention to the infrared (IR) divergences. In section 6 we use the process $\mu \to e\gamma$ in generic models to show the power of the techniques in real problems. Finally in the Appendix we collect many useful formulas for one-loop calculations.

# 2 Renormalization of QED at one-loop

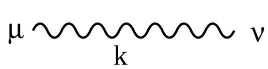
We will consider the theory described by the Lagrangian

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 + \overline{\psi} (i \not{\partial} + e \not{A} - m) \psi \ . \tag{2.1}$$

The free propagators are

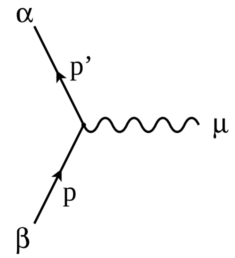



$$\left( \frac{i}{\not{p} - m + i\varepsilon} \right)_{\beta\alpha} \equiv S^0_{F\beta\alpha}(p) \tag{2.2}$$




$$\begin{aligned} & -i \left[ \frac{g_{\mu\nu}}{k^2 + i\varepsilon} + \frac{(\xi - 1)}{1} \frac{k_\mu k_\nu}{(k^2 + i\varepsilon)^2} \right] \\ = & -i \left\{ \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \frac{1}{k^2 + i\varepsilon} + \xi \frac{k_\mu k_\nu}{k^4} \right\} \\ \equiv & \ G^0_{F\mu\nu}(k) \end{aligned} \tag{2.3}$$

and the vertex




$$+ie(\gamma_\mu)_{\beta\alpha} \qquad e = |e| > 0 \tag{2.4}$$

We will now consider the one-loop corrections to the propagators and to the vertex. We will work in the Feynman gauge ($\xi = 1$).

## 2.1 Vacuum Polarization

In first order the contribution to the photon propagator is given by the diagram of Fig. 1 that we write in the form

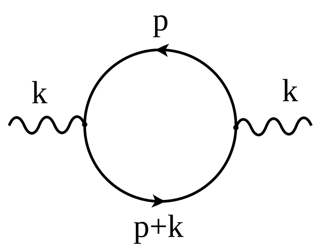



Figure 1:

$$G^{(1)}_{\mu\nu}(k) \equiv G^0_{\mu\mu'}\ i\,\Pi_{\mu'\nu'}(k) G^0_{\nu'\nu}(k) \tag{2.5}$$

where

$$\begin{aligned} i\,\Pi_{\mu\nu} &= -(+ie)^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left(\gamma_\mu \frac{i}{\not p - m + i\varepsilon} \gamma_\nu \frac{i}{\not p + \not k - m + i\varepsilon}\right) \\ &= -e^2 \int \frac{d^4p}{(2\pi)^4} \frac{\mathrm{Tr}[\gamma_\mu(\not p + m)\gamma_\nu(\not p + \not k + m)]}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \\ &= -4e^2 \int \frac{d^4p}{(2\pi)^4} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\nu - g_{\mu\nu}(p^2 + p\cdot k - m^2)}{(p^2 - m^2 + i\varepsilon)((p+k)^2 - m^2 + i\varepsilon)} \end{aligned} \tag{2.6}$$

Simple power counting indicates that this integral is quadratically divergent for large values of the internal loop momenta. In fact the divergence is milder, only logarithmic. The integral being divergent we have first to regularize it and then to define a renormalization procedure to cancel the infinities. For this purpose we will use the method of dimensional regularization. For a value of $d$ small enough the integral converges. If we define $\epsilon = 4 - d$, in the end we will have a divergent result in the limit $\epsilon \to 0$. We get therefore[1]

$$\begin{aligned}
i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{[2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2)]}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)} \\
&= -4e^2\,\mu^\epsilon \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{(p^2-m^2+i\varepsilon)((p+k)^2-m^2+i\varepsilon)}
\end{aligned} \tag{2.7}$$

where

$$N_{\mu\nu}(p,k) = 2p_\mu p_\nu + p_\mu k_\nu + p_\nu k_\mu - g_{\mu\nu}(p^2 + p\cdot k - m^2) \tag{2.8}$$

To evaluate this integral we first use the Feynman parameterization to rewrite the denominator as a single term. For that we use (see Appendix)

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax+b(1-x)]^2} \tag{2.9}$$

to get

$$\begin{aligned}
i\,\Pi_{\mu\nu}(k,\epsilon) &= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[x(p+k)^2 - xm^2 + (1-x)(p^2-m^2) + i\varepsilon]^2} \\
&= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[p^2 + k\cdot px + xk^2 - m^2 + i\varepsilon]^2} \\
&= -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p,k)}{[(p+kx)^2 + k^2x(1-x) - m^2 + i\varepsilon]^2}
\end{aligned} \tag{2.10}$$

For dimension $d$ sufficiently small this integral converges and we can change variables

$$p \to p - kx \tag{2.11}$$

We then get

$$i\,\Pi_{\mu\nu}(k,\epsilon) = -4e^2\,\mu^\epsilon \int_0^1 dx \int \frac{d^dp}{(2\pi)^d} \frac{N_{\mu\nu}(p-kx,k)}{[p^2 - C + i\epsilon]^2} \tag{2.12}$$

where

$$C = m^2 - k^2x(1-x) \tag{2.13}$$

$N_{\mu\nu}$ is a polynomial of second degree in the loop momenta as can be seen from Eq. (2.8). However as the denominator in Eq. (2.12) only depends on $p^2$ is it easy to show that

$$\begin{aligned}
&\int \frac{d^dp}{(2\pi)^d}\, \frac{p^\mu}{[p^2 - C + i\epsilon]^2} = 0 \\
&\int \frac{d^dp}{(2\pi)^d}\, \frac{p^\mu p^\nu}{[p^2 - C + i\epsilon]^2} = \frac{1}{d}\, g^{\mu\nu} \int \frac{d^dp}{(2\pi)^d}\, \frac{p^2}{[p^2 - C + i\epsilon]^2}
\end{aligned} \tag{2.14}$$

[1] Where $\mu$ is a parameter with dimensions of a mass that is introduced to ensure the correct dimensions of the coupling constant in dimension $d$, that is, $[e] = \frac{4-d}{2} = \frac{\epsilon}{2}$. We take then $e \to e\mu^{\frac{\epsilon}{2}}$. For more details see the Appendix.

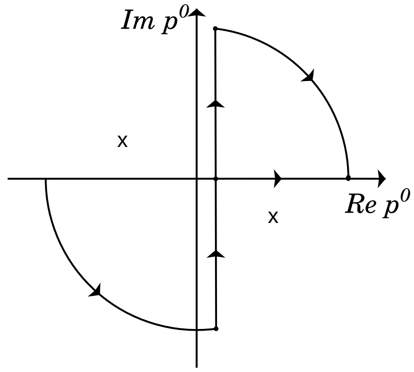



Figure 2:

This means that we only have to calculate integrals of the form

$$
\begin{aligned}
I_{r,m} &= \int \frac{d^d p}{(2\pi)^d} \, \frac{(p^2)^r}{[p^2 - C + i\epsilon]^m} \\
&= \int \frac{d^{d-1} p}{(2\pi)^d} \int dp^0 \frac{(p^2)^r}{[p^2 - C + i\epsilon]^m}
\end{aligned} \tag{2.15}
$$

To make this integration we will use integration in the plane of the complex variable $p^0$ as described in Fig. 2. The deformation of the contour corresponds to the so called Wick rotation,

$$
p^0 \to i p_E^0 \qquad ; \qquad \int_{-\infty}^{+\infty} \to i \int_{-\infty}^{+\infty} dp_E^0 \tag{2.16}
$$

and $p^2 = (p^0)^2 - |\vec{p}|^2 = -(p_E^0)^2 - |\vec{p}|^2 \equiv -p_E^2$, where $p_E = (p_E^0, \vec{p})$ is an euclidean vector, that is

$$
p_E^2 = (p_E^0)^2 + |\vec{p}|^2 \tag{2.17}
$$

We can then write (see the Appendix for more details),

$$
I_{r,m} = i(-1)^{r-m} \int \frac{d^d p_E}{(2\pi)^d} \, \frac{p_E^{2^r}}{[p_E^2 + C]^m} \tag{2.18}
$$

where we do not need the $i\epsilon$ anymore because the denominator is positive definite[2] ($C > 0$). To proceed with the evaluation of $I_{r,m}$ we write,

$$
\int d^d p_E = \int d\overline{p} \; \overline{p}^{\;d-1} \, d\Omega_{d-1} \tag{2.19}
$$

where $\overline{p} = \sqrt{(p_E^0)^2 + |\vec{p}|^2}$ is the length of of vector $p_E$ in the euclidean space with $d$ dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates. We can show (see Appendix) that

$$
\int d\Omega_{d-1} = 2 \, \frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{2.20}
$$

[2]The case when $C < 0$ is obtained by analytical continuation of the final result.

The $\overline{p}$ integral is done using the result,

$$\int_0^\infty dx\ \frac{x^p}{(x^n+a^n)^q} = \pi(-1)^{q-1}\, a^{p+1-nq}\ \frac{\Gamma(\frac{p+1}{n})}{n\sin(\pi\,\frac{p+1}{n})\,\Gamma(\frac{p+1}{2}-q+1)} \tag{2.21}$$

and we finally get

$$I_{r,m} = iC^{r-m+\frac{d}{2}}\ \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}}\ \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})}\ \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)} \tag{2.22}$$

Note that the integral representation of $I_{r,m}$, Eq. (2.15) is only valid for $d<2(m-r)$ to ensure the convergence of the integral when $\overline{p}\to\infty$. However the final form of Eq. (2.22) can be analytically continued for all the values of $d$ except for those where the function $\Gamma(m-r-d/2)$ has poles, which are (see section A.6),

$$m-r-\frac{d}{2}\neq 0,-1,-2,\dots \tag{2.23}$$

For the application to dimensional regularization it is convenient to write Eq. (2.22) after making the substitution $d=4-\epsilon$. We get

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2}\ \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}}\ C^{2+r-m}\ \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\ \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)} \tag{2.24}$$

that has poles for $m-r-2\leq 0$ (see section A.6).

We now go back to calculate $\Pi_{\mu\nu}$. First we notice that after the change of variables of Eq. (2.11) we get

$$N_{\mu\nu}(p-kx,k) = 2p_\mu p_\nu + 2x^2k_\mu k_\nu - 2xk_\mu k_\nu - g_{\mu\nu}\left(p^2+x^2k^2-xk^2-m^2\right) \tag{2.25}$$

and therefore

$$\begin{aligned}\mathcal{N}_{\mu\nu} &\equiv \mu^\epsilon\int\frac{d^dp}{(2\pi)^d}\ \frac{N_{\mu\nu}(p-kx,k)}{\left[p^2-C+i\epsilon\right]^2}\\ &= \left(\frac{2}{d}-1\right)g_{\mu\nu}\mu^\epsilon I_{1,2} + \Big[-2x(1-x)k_\mu k_\nu + x(1-x)k^2g_{\mu\nu} + g_{\mu\nu}m^2\Big]\mu^\epsilon I_{0,2}\end{aligned} \tag{2.26}$$

Using now Eq. (2.24) we can write

$$\begin{aligned}\mu^\epsilon I_{0,2} &= \frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}}\ \frac{\Gamma(\frac{\epsilon}{2})}{\Gamma(2)}\\ &= \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\end{aligned} \tag{2.27}$$

where we have used the expansion of the $\Gamma$ function, Eq. (A.47),

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon) \tag{2.28}$$

$\gamma$ being the Euler constant and we have defined, Eq. (A.50),

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \tag{2.29}$$

In a similar way

$$\begin{aligned}
\mu^\epsilon I_{1,2} &= -\frac{i}{16\pi^2}\left(\frac{4\pi\mu^2}{C}\right)^{\frac{\epsilon}{2}} C\, \frac{\Gamma(3-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\, \frac{\Gamma(-1+\frac{\epsilon}{2})}{\Gamma(2)} \\
&= \frac{i}{16\pi^2} C \left(1 + 2\Delta_\epsilon - 2\ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon) \qquad (2.30)
\end{aligned}$$

Due to the existence of a pole in $1/\epsilon$ in the previous equations we have to expand all quantities up to $\mathcal{O}(\epsilon)$. This means for instance, that

$$\frac{2}{d} - 1 = \frac{2}{4-\epsilon} - 1 = -\frac{1}{2} + \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2) \tag{2.31}$$

Substituting back into Eq. (2.26), and using Eq. (2.13), we obtain

$$\begin{aligned}
\mathcal{N}_{\mu\nu} &= g_{\mu\nu}\left[-\frac{1}{2} + \frac{1}{8}\epsilon + \mathcal{O}(\epsilon^2)\right]\left[\frac{i}{16\pi^2} C\left(1 + 2\Delta_\epsilon - 2\ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\right] \\
&\quad + \left[-2x(1-x)k_\mu k_\nu + x(1-x)k^2 g_{\mu\nu} + g_{\mu\nu} m^2\right]\left[\frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) + \mathcal{O}(\epsilon)\right] \\
&= -\frac{i}{16\pi^2} k_\mu k_\nu \left[\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) 2x(1-x)\right] \\
&\quad + \frac{i}{16\pi^2} g_{\mu\nu} k^2 \left[\Delta_\epsilon \Big(x(1-x) + x(1-x)\Big) + \ln\frac{C}{\mu^2}\Big(-x(1-x) - x(1-x)\Big)\right. \\
&\qquad\qquad \left. + x(1-x)\left(\frac{1}{2} - \frac{1}{2}\right)\right] \\
&\quad + \frac{i}{16\pi^2} g_{\mu\nu} m^2 \left[\Delta_\epsilon(-1+1) + \ln\frac{C}{\mu^2}(1-1) + (-\frac{1}{2} + \frac{1}{2})\right] \qquad (2.32)
\end{aligned}$$

and finally

$$\mathcal{N}_{\mu\nu} = \frac{i}{16\pi^2}\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\Big(g_{\mu\nu}k^2 - k_\mu k_\nu\Big)\, 2x(1-x) \tag{2.33}$$

Now using Eq. (2.7) we get

$$\begin{aligned}
\Pi_{\mu\nu} &= -4e^2 \frac{1}{16\pi^2}\Big(g_{\mu\nu}k^2 - k_\mu k_\nu\Big)\int_0^1 dx\; 2x(1-x)\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right) \\
&= -\Big(g_{\mu\nu}k^2 - k_\mu k_\nu\Big)\,\Pi(k^2,\epsilon) \qquad (2.34)
\end{aligned}$$

where

$$\Pi(k^2,\epsilon) \equiv \frac{2\alpha}{\pi}\int_0^1 dx\; x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2 - x(1-x)k^2}{\mu^2}\right] \tag{2.35}$$

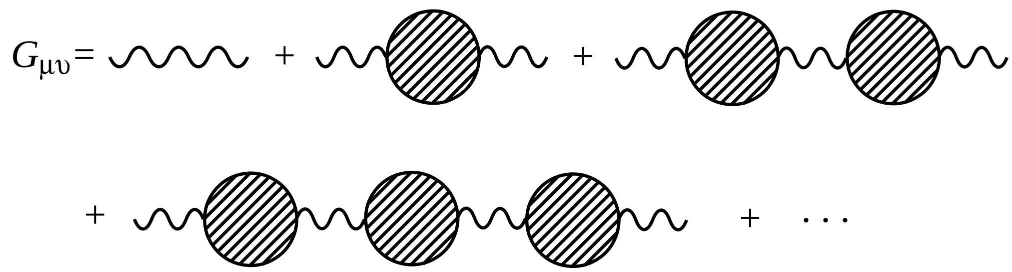



Figure 3:

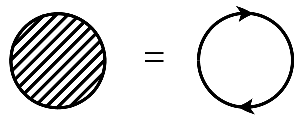
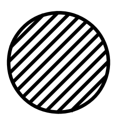



Figure 4:

This expression clearly diverges as $\epsilon \to 0$. Before we show how to renormalize it let us discuss the meaning of $\Pi_{\mu\nu}(k)$. The full photon propagator is given by the series represented in Fig. 3, where



$$\equiv i\,\Pi_{\mu\nu}(k) = \begin{array}{l}\text{sum of all one-particle irreducible} \\ (\textit{proper}) \text{ diagrams to } \textit{all} \text{ orders}\end{array} \tag{2.36}$$

In lowest order we have the contribution represented in Fig. 4, which is what we have just calculated. To continue it is convenient to rewrite the free propagator of the photon (in an arbitrary gauge $\xi$) in the following form

$$\begin{aligned} iG^0_{\mu\nu} &= \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} = P^T_{\mu\nu}\frac{1}{k^2} + \xi\frac{k_\mu k_\nu}{k^4} \\ &\equiv iG^{0T}_{\mu\nu} + iG^{0L}_{\mu\nu} \end{aligned} \tag{2.37}$$

where we have introduced the transversal projector $P^T_{\mu\nu}$ defined by

$$P^T_{\mu\nu} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{2.38}$$

obviously satisfying the relations,

$$\left\{ \begin{array}{l} k^\mu P^T_{\mu\nu} = 0 \\ \\ P^{T\,\nu}_{\mu} P^T_{\nu\rho} = P^T_{\mu\rho} \end{array} \right. \tag{2.39}$$

The full photon propagator can also in general be written separating its transversal an longitudinal parts

$$G_{\mu\nu} = G^T_{\mu\nu} + G^L_{\mu\nu} \tag{2.40}$$

where $G^T_{\mu\nu}$ satisfies

$$G^T_{\mu\nu} = P^T_{\mu\nu} G_{\mu\nu} \tag{2.41}$$

We have obtained, in first order, that the vacuum polarization tensor is transversal, that is

$$i\,\Pi_{\mu\nu}(k) = -ik^2 P^T_{\mu\nu}\,\Pi(k) \tag{2.42}$$

This result is in fact valid to all orders of perturbation theory, a result that can be shown using the Ward-Takahashi identities. This means that the longitudinal part of the photon propagator is not renormalized,

$$G^L_{\mu\nu} = G^{0L}_{\mu\nu} \tag{2.43}$$

For the transversal part we obtain

$$\begin{aligned} iG^T_{\mu\nu} &= P^T_{\mu\nu}\frac{1}{k^2} + P^T_{\mu\mu'}\frac{1}{k^2}(-i)k^2 P^{T\mu'\nu'}\,\Pi(k)(-i)P^T_{\nu'\nu}\frac{1}{k^2} \\ &\quad + P^T_{\mu\rho}\frac{1}{k^2}(-i)k^2 P^{T\rho\lambda}\;\Pi(k)(-i)P^T_{\lambda\tau}\frac{1}{k^2}(-i)k^2 P^{T\tau\sigma}\;\Pi(k)(-i)P^T_{\sigma\nu}\frac{1}{k^2} + \cdots \\ &= P^T_{\mu\nu}\frac{1}{k^2}\left[1 - \Pi(k) + \Pi^2(k^2) + \cdots\right] \end{aligned} \tag{2.44}$$

which gives, after summing the geometric series,

$$iG^T_{\mu\nu} = P^T_{\mu\nu}\frac{1}{k^2[1+\Pi(k)]} \tag{2.45}$$

All that we have done up to this point is formal because the function $\Pi(k)$ diverges. The most satisfying way to solve this problem is the following. The initial lagrangian from which we started has been obtained from the classical theory and nothing tell us that it should be exactly the same in quantum theory. In fact, has we have just seen, the normalization of the wave functions is changed when we calculate *one-loop* corrections, and the same happens to the physical parameters of the theory, the charge and the mass. Therefore we can think that the correct lagrangian is obtained by adding corrections to the classical lagrangian, order by order in perturbation theory, so that we keep the definitions of charge and mass as well as the normalization of the wave functions. The terms that we add to the lagrangian are called *counterterms*[3]. The total lagrangian is then,

[3]This interpretation in terms of quantum corrections makes sense. In fact we can show that an expansion in powers of the coupling constant can be interpreted as an expansion in $\hbar^L$, where $L$ is the number of the loops in the expansion term.

$$\mathcal{L}_{\text{total}} = \mathcal{L}(e, m, ...) + \Delta\mathcal{L} \tag{2.46}$$

Counterterms are defined from the normalization conditions that we impose on the fields are other parameters of the theory. In QED we have a our disposal the normalization of the electron and photon fields and of the two physical parameters, the electric charge and the electron mass. The normalization conditions are, to a large extent, arbitrary. It is however convenient to keep the expressions as close as possible to the free field case, that is, without radiative corrections. We define therefore the normalization of the photon field as,

$$\lim_{k\to 0} k^2 i G^{RT}_{\mu\nu} = 1 \cdot P^T_{\mu\nu} \tag{2.47}$$

where $G^{RT}_{\mu\nu}$ is the renormalized propagator (the transversal part) obtained from the lagrangian $\mathcal{L}_{\text{total}}$. The justification for this definition comes from the following argument. Consider the Coulomb scattering to all orders of perturbation theory. We have then the situation described in Fig. 5. Using the Ward-Takahashi identities one can show that the

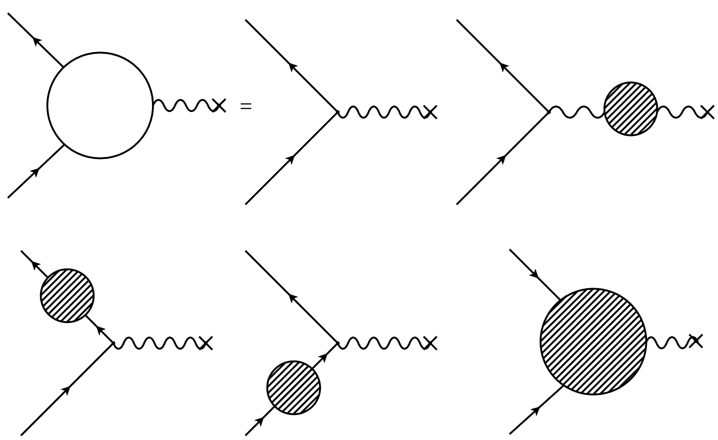



Figure 5:

last three diagrams cancel in the limit $q = p' - p \to 0$. Then the normalization condition, Eq. (2.47), means that we have the situation described in Fig. 6, that is, the experimental value of the electric charge is determined in the limit $q \to 0$ of the Coulomb scattering.

The counterterm lagrangian has to have the same form as the classical lagrangian to respect the symmetries of the theory. For the photon field it is traditional to write

$$\Delta\mathcal{L} = -\frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}\delta Z_3\, F_{\mu\nu}F^{\mu\nu} \tag{2.48}$$

corresponding to the following Feynman rule


$$-\,i\,\delta Z_3 k^2 \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \tag{2.49}$$

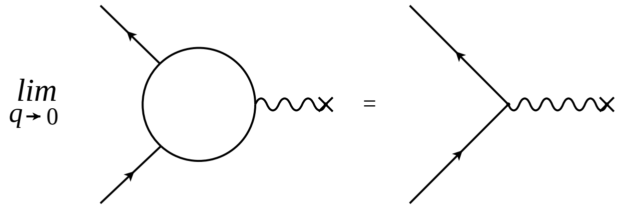



Figure 6:

We have then

$$
\begin{aligned}
i\Pi_{\mu\nu} &= i\Pi^{loop}_{\mu\nu} - i\,\delta Z_3 k^2 \left( g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \\
&= -i\ \left(\Pi(k,\epsilon) + \delta Z_3\right) P^T_{\mu\nu}
\end{aligned}
\tag{2.50}
$$

Therefore we should make the substitution

$$
\Pi(k,\epsilon) \to \Pi(k,\epsilon) + \delta Z_3 \tag{2.51}
$$

in the photon propagator. We obtain,

$$
iG^T_{\mu\nu} = P^T_{\mu\nu}\,\frac{1}{k^2}\,\frac{1}{1+\Pi(k,\epsilon)+\delta Z_3} \tag{2.52}
$$

The normalization condition, Eq. (2.47), implies

$$
\Pi(k,\epsilon) + \delta Z_3 = 0 \tag{2.53}
$$

from which one determines the constant $\delta Z_3$. We get

$$
\begin{aligned}
\delta Z_3 &= -\Pi(0,\epsilon) = -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right] \\
&= -\frac{\alpha}{3\pi}\left[\Delta_\epsilon - \ln\frac{m^2}{\mu^2}\right]
\end{aligned}
\tag{2.54}
$$

The renormalized photon propagator can then be written as [4]

$$
iG_{\mu\nu}(k) = \frac{P^T_{\mu\nu}}{k^2[1+\Pi(k,\epsilon)-\Pi(0,\epsilon)]} + i\,G^L_{\mu\nu} \tag{2.55}
$$

The *finite* radiative corrections are given through the function

$$
\Pi^R(k^2) \equiv \Pi(k^2,\epsilon) - \Pi(0,\epsilon)
$$

[4]Notice that the photon mass is not renormalized, that is the pole of the photon propagator still is at $k^2 = 0$.

$$= -\frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\ln\left[\frac{m^2 - x(1-x)k^2}{m^2}\right]$$

$$= -\frac{\alpha}{3\pi}\left\{\frac{1}{3} + 2\left(1+\frac{2m^2}{k^2}\right)\left[\left(\frac{4m^2}{k^2}-1\right)^{1/2}\cot^{-1}\left(\frac{4m^2}{k^2}-1\right)^{1/2} - 1\right]\right\} \tag{2.56}$$

where the last equation is valid for $k^2 < 4m^2$. For values $k^2 > 4m^2$ the result for $\Pi^R(k^2)$ can be obtained from Eq. (2.56) by analytical continuation. Using ($k^2 > 4m^2$)

$$\cot^{-1} iz = i\left(-\tanh^{-1} z + \frac{i\pi}{2}\right) \tag{2.57}$$

and

$$\left(\frac{4m^2}{k^2}-1\right)^{1/2} \to i\sqrt{1-\frac{4m^2}{k^2}} \tag{2.58}$$

we get

$$\Pi^R(k^2) \;=\; -\frac{\alpha}{3\pi}\left\{\frac{1}{3} + 2\left(1+\frac{2m^2}{k^2}\right)\left[-1+\sqrt{1-\frac{4m^2}{k^2}}\tanh^{-1}\left(1-\frac{4m^2}{k^2}\right)^{1/2}\right.\right. \tag{2.59}$$

$$\left.\left. -i\frac{\pi}{2}\sqrt{1-\frac{4m^2}{k^2}}\right]\right\} \tag{2.60}$$

The imaginary part of $\Pi^R$ is given by

$$Im\ \Pi^R(k^2) = \frac{\alpha}{3}\left(1+\frac{2m^2}{k^2}\right)\sqrt{1-\frac{4m^2}{k^2}}\,\theta\left(1-\frac{4m^2}{k^2}\right) \tag{2.61}$$

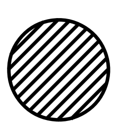
and it is related to the pair production that can occur [5] for $k^2 > 4m^2$.

## 2.2 Self-energy of the electron

The electron full propagator is given by the diagrammatic series of Fig. 7, which can be written as,

$$\begin{aligned} S(p) &= S^0(p) + S^0(p)\Big(-i\,\Sigma(p)\Big)S^0(p) + \cdots \\ &= S^0(p)\Big[1 - i\,\Sigma(p)S(p)\Big] \end{aligned} \tag{2.62}$$

where we have identified



$$\equiv -i\,\Sigma(p) \tag{2.63}$$

[5] For $k^2 > 4m^2$ there is the possibility of producing one pair $e^+e^-$. Therefore on top of a virtual process (vacuum polarization) there is a real process (pair production).

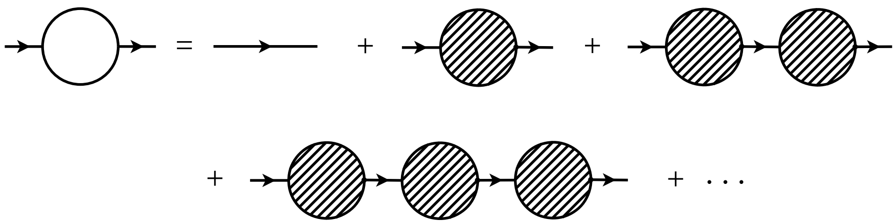


Figure 7:

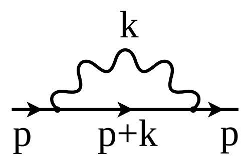



Figure 8:

Multiplying on the left with $S_0^{-1}(p)$ and on the right with $S^{-1}(p)$ we get

$$S_0^{-1}(p) = S^{-1}(p) - i\,\Sigma(p) \tag{2.64}$$

which we can rewrite as

$$S^{-1}(p) = S_0^{-1}(p) + i\,\Sigma(p) \tag{2.65}$$

Using the expression for the free field propagator,

$$S_0(p) = \frac{i}{\not{p} - m} \Longrightarrow S_0^{-1}(p) = -i(\not{p} - m) \tag{2.66}$$

we can then write

$$\begin{aligned} S^{-1}(p) &= S_0^{-1}(p) + i\Sigma(p) \\ &= -i\left[\not{p} - (m + \Sigma(p))\right] \end{aligned} \tag{2.67}$$

We conclude that it is enough to calculate $\Sigma(p)$ to all orders of perturbation theory to obtain the full electron propagator. The name *self-energy* given to $\Sigma(p)$ comes from the fact that, as can be seen in Eq. (2.67), it comes as an additional (momentum dependent) contribution to the mass.

In lowest order there is only the diagram of Fig. 8 contributing $\Sigma(p)$ and therefore we get,

$$-i\Sigma(p) = (+ie)^2 \int \frac{d^4k}{(2\pi)^4} (-i) \frac{g_{\mu\nu}}{k^2 - \lambda^2 + i\varepsilon} \gamma^\mu \frac{i}{\not{p} + \not{k} - m + i\varepsilon} \gamma^\nu \tag{2.68}$$

where we have chosen the Feynman gauge ($\xi = 1$) for the photon propagator and we have introduced a small mass for the photon $\lambda$, in order to control the infrared divergences

(IR) that will appear when $k^2 \to 0$ (see below). Using dimensional regularization and the results of the Dirac algebra in dimension $d$,

$$\begin{aligned}
\gamma_\mu(\not p+\not k)\gamma^\mu &= -(\not p+\not k)\gamma_\mu\gamma^\mu+2(\not p+\not k)=-(d-2)(\not p+\not k)\\
m\gamma_\mu\gamma^\mu &= m\,d
\end{aligned} \tag{2.69}$$

we get

$$\begin{aligned}
-i\,\Sigma(p) &= -\mu^\epsilon e^2\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-\lambda^2+\varepsilon}\gamma_\mu\frac{\not p+\not k+m}{(p+k)^2-m^2+i\varepsilon}\gamma^\mu\\
&= -\mu^\epsilon e^2\int\frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not p+\not k)+m\,d}{[k^2-\lambda^2+i\varepsilon]\,[(p+k)^2-m^2+i\varepsilon]}\\
&= -\mu^\epsilon e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not p+\not k)+m\,d}{[(k^2-\lambda^2)\,(1-x)+x(p+k)^2-xm^2+i\varepsilon]^2}\\
&= -\mu^\epsilon e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{-(d-2)(\not p+\not k)+m\,d}{[(k+px)^2+p^2x(1-x)-\lambda^2(1-x)-xm^2+i\varepsilon]^2}\\
&= -\mu^\epsilon e^2\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{-(d-2)\,[\not p(1-x)+\not k]+m\,d}{[k^2+p^2x(1-x)-\lambda^2(1-x)-xm^2+i\varepsilon]^2}\\
&= -\mu^\varepsilon e^2\int_0^1 dx\,\Big[-(d-2)\not p(1-x)+m\,d\Big]\,I_{0,2}
\end{aligned} \tag{2.70}$$

where[6]

$$I_{0,2}=\frac{i}{16\pi^2}\left[\Delta_\epsilon-\ln\left[-p^2x(1-x)+m^2x+\lambda^2(1-x)\right]\right] \tag{2.71}$$

The contribution from the *loop* in Fig. 8 to the electron self-energy $\Sigma(p)$ can then be written in the form,

$$\Sigma(p)^{loop}=A(p^2)+B(p^2)\,\not p \tag{2.72}$$

with

$$\begin{aligned}
A &= e^2\mu^\epsilon(4-\epsilon)m\frac{1}{16\pi^2}\int_0^1 dx\,\left[\Delta_\epsilon-\ln\left[-p^2x(1-x)+m^2x+\lambda^2(1-x)\right]\right]\\
B &= -e^2\mu^\epsilon(2-\epsilon)\frac{1}{16\pi^2}\int_0^1 dx\,(1-x)\Big[\Delta_\epsilon\\
&\qquad\qquad -\ln\left[-p^2x(1-x)+m^2x+\lambda^2(1-x)\right]\Big]
\end{aligned} \tag{2.73}$$

Using now the expansions

$$\begin{aligned}
\mu^\epsilon(4-\epsilon) &= 4\left[1+\epsilon\left(\ln\mu-\frac{1}{4}\right)+\mathcal{O}(\epsilon^2)\right]\\
\mu^\epsilon(4-\epsilon)\Delta_\epsilon &= 4\left[\Delta_\epsilon+2\left(\ln\mu-\frac{1}{4}\right)+\mathcal{O}(\epsilon)\right]
\end{aligned}$$

[6] The linear term in $k$ vanishes.

$$
\begin{aligned}
\mu^{\epsilon}(2-\epsilon) &= 2\left[1+\epsilon\left(\ln\mu-\frac{1}{2}\right)+\mathcal{O}(\epsilon^2)\right] \\
\mu^{\epsilon}(2-\epsilon)\Delta_\epsilon &= 2\left[\Delta_\epsilon+2\left(\ln\mu-\frac{1}{2}\right)+\mathcal{O}(\epsilon)\right]
\end{aligned}
\tag{2.74}
$$

we can finally write,

$$
A(p^2)=\frac{4\,e^2 m}{16\pi^2}\int_0^1 dx\,\left[\Delta_\epsilon-\frac{1}{2}-\ln\left[\frac{-p^2x(1-x)+m^2x+\lambda^2(1-x)}{\mu^2}\right]\right] \tag{2.75}
$$

e

$$
B(p^2)=-\frac{2\,e^2}{16\pi^2}\int_0^1 dx\,(1-x)\left[\Delta_\epsilon-1-\ln\left[\frac{-p^2x(1-x)+m^2x+\lambda^2(1-x)}{\mu^2}\right]\right] \tag{2.76}
$$

To continue with the renormalization program we have to introduce the counterterm lagrangian and define the normalization conditions. We have

$$
\Delta\mathcal{L}=i\,(Z_2-1)\,\overline{\psi}\gamma^\mu\partial_\mu\psi-(Z_2-1)\,m\,\overline{\psi}\psi+Z_2\delta m\,\overline{\psi}\psi+(Z_1-1)e\,\overline{\psi}\gamma^\mu\psi A_\mu \tag{2.77}
$$

and therefore we get for the self-energy

$$
-i\,\Sigma(p)=-i\,\Sigma^{loop}(p)+i\,\left(\not{p}-m\right)\delta Z_2+i\,\delta m \tag{2.78}
$$

Contrary to the case of the photon we see that we have two constants to determine. In the *on-shell* renormalization scheme that is normally used in QED the two constants are obtained by requiring that the pole of the propagator corresponds to the physical mass (hence the name of *on-shell* renormalization), and that the residue of the pole of the renormalized electron propagator has the same value as the free filed propagator. This implies,

$$
\begin{aligned}
&\Sigma(\not{p}=m)=0 \quad\rightarrow\quad \delta m=\Sigma^{loop}(\not{p}=m) \\
&\left.\frac{\partial\Sigma}{\partial\not{p}}\right|_{\not{p}=m}=0 \quad\rightarrow\quad \delta Z_2=\left.\frac{\partial\Sigma^{loop}}{\partial\not{p}}\right|_{\not{p}=m}
\end{aligned}
\tag{2.79}
$$

We then get for $\delta m$,

$$
\begin{aligned}
\delta m &= A(m^2)+m\,B(m^2) \\
&= \frac{2\,me^2}{16\pi^2}\int_0^1 dx\,\left\{\left[2\Delta_\epsilon-1-2\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\right. \\
&\qquad\left.-(1-x)\left[\Delta_\epsilon-1-\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right]\right\} \\
&= \frac{2\,m\,e^2}{16\pi^2}\left[\frac{3}{2}\Delta_\epsilon-\frac{1}{2}-\int_0^1 dx\,(1+x)\ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right] \\
&= \frac{3\alpha m}{4\pi}\left[\Delta_\epsilon-\frac{1}{3}-\frac{2}{3}\int_0^1 dx\,(1+x)\ln\left(\frac{m^2x^2}{\mu^2}\right)\right]
\end{aligned}
\tag{2.80}
$$

where in the last step in Eq. (2.80) we have taken the limit $\lambda \to 0$ because the integral does not diverge in that limit[7]. In a similar way we get for $\delta Z_2$,

$$\delta Z_2 = \left.\frac{\partial \Sigma^{loop}}{\partial \not{p}}\right|_{\not{p}=m} = \left.\frac{\partial A}{\partial \not{p}}\right|_{\not{p}=m} + B + m \left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m} \tag{2.81}$$

where

$$\begin{aligned}
\left.\frac{\partial A}{\partial \not{p}}\right|_{\not{p}=m} &= \frac{4\,e^2\,m^2}{16\pi^2} \int_0^1 dx \, \frac{2(1-1)x}{-m^2x(1-x) + m^2x + \lambda^2(1-x)} \\
&= \frac{2\,\alpha\, m^2}{\pi} \int_0^1 dx \, \frac{(1-x)x}{m^2x^2 + \lambda^2(1-x)} \\
B &= -\frac{\alpha}{2\pi} \int_0^1 dx\,(1-x) \left[\Delta_\epsilon - 1 - \ln\left(\frac{m^2x^2+\lambda^2(1-x)}{\mu^2}\right)\right] \\
m \left.\frac{\partial B}{\partial \not{p}}\right|_{\not{p}=m} &= -\frac{\alpha}{2\pi}\, m^2 \int_0^1 dx \, \frac{2x(1-x)^2}{m^2x^2+\lambda^2(1-x)}
\end{aligned} \tag{2.82}$$

Substituting Eq. (2.82) in Eq. (2.81) we get,

$$\begin{aligned}
\delta Z_2 &= -\frac{\alpha}{2\pi} \left[\frac{1}{2}\Delta_\epsilon - \frac{1}{2} - \int_0^1 dx\,(1-x)\ln\left(\frac{m^2x^2}{\mu^2}\right) - 2\int_0^1 dx \, \frac{(1+x)(1-x)xm^2}{m^2x^2+\lambda^2(1-x)}\right] \\
&= \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right]
\end{aligned} \tag{2.83}$$

where we have taken the $\lambda \to 0$ limit in all cases that was possible. It is clear that the final result in Eq. (2.83) diverges in that limit, therefore implying that $Z_2$ is IR divergent. This is not a problem for the theory because $\delta Z_2$ is not a physical parameter. We will see in section 5.2 that the IR diverges cancel for real processes. If we had taken a general gauge ($\xi \neq 1$) we will find out that $\delta m$ would not be changed but that $Z_2$ would show a gauge dependence. Again, in physical processes this should cancel in the end.

## 2.3 The Vertex

The diagram contributing to the QED vertex at one-loop is the one shown in Fig. 9. In the Feynman gauge ($\xi = 1$) this gives a contribution,

$$\begin{aligned}
ie\,\mu^{\epsilon/2}\Lambda_\mu^{loop}(p',p) &= (ie\,\mu^{\epsilon/2})^3 \int \frac{d^dk}{(2\pi)^d}(-i)\frac{g_{\rho\sigma}}{k^2-\lambda^2+i\varepsilon} \\
&\qquad \gamma^\sigma \frac{i[(\not{p}'+\not{k})+m]}{(p'+k)^2-m^2+i\varepsilon}\gamma_\mu \frac{i[(\not{p}+\not{k})+m]}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho
\end{aligned} \tag{2.84}$$

[7] $\delta m$ is not IR divergent.

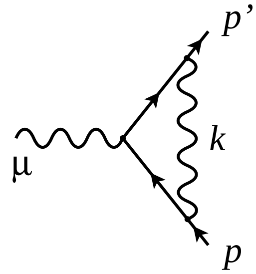



Figure 9:

where $\Lambda_\mu$ is related to the full vertex $\Gamma_\mu$ through the relation

$$
\begin{aligned}
i\Gamma_\mu &= ie\left(\gamma_\mu + \Lambda_\mu^{loop} + \gamma_\mu \delta Z_1\right) \\
&= ie\left(\gamma_\mu + \Lambda_\mu^R\right)
\end{aligned}
\tag{2.85}
$$

The integral that defines $\Lambda_\mu^{loop}(p', p)$ is divergent. As before we expect to solve this problem by regularizing the integral, introducing counterterms and normalization conditions. The counterterm has the same form as the vertex and is already included in Eq. (2.85). The normalization constant is determined by requiring that in the limit $q = p' - p \to 0$ the vertex reproduces the tree level vertex because this is what is consistent with the definition of the electric charge in the $q \to 0$ limit of the Coulomb scattering. Also this should be defined for on-shell electrons. We have therefore that the normalization condition gives,

$$
\overline{u}(p)\left(\Lambda_\mu^{loop} + \gamma_\mu \delta Z_1\right) u(p)\Big|_{\not p = m} = 0 \tag{2.86}
$$

If we are interested only in calculating $\delta Z_1$ and in showing that the divergences can be removed with the normalization condition then the problem is simpler. It can be done in two ways.

## 1<sup>st</sup> method

We use the fact that $\delta Z_1$ is to be calculated on-shell and for $p = p'$. Then

$$
i\Lambda_\mu^{loop}(p,p) = e^2 \mu^\epsilon \int \frac{dk^d}{(2\pi)^d} \frac{1}{k^2 - \lambda^2 + i\varepsilon} \gamma_\rho \frac{1}{\not p + \not k - m + i\varepsilon} \gamma_\mu \frac{1}{\not p + \not k - m + i\varepsilon} \gamma^\rho \tag{2.87}
$$

However we have

$$
\frac{1}{\not p + \not k - m + i\varepsilon} \gamma_\mu \frac{1}{\not p + \not k - m + i\varepsilon} = -\frac{\partial}{\partial p^\mu} \frac{1}{\not p + \not k - m + i\varepsilon} \tag{2.88}
$$

and therefore

$$i\Lambda^{loop}_\mu(p,p) = -e^2\mu^\epsilon \frac{\partial}{\partial p^\mu}\int \frac{dk^d}{(2\pi)^d}\frac{1}{k^2-\lambda^2+i\varepsilon}\gamma_\rho \frac{\not p+\not k+m\varepsilon}{(p+k)^2-m^2+i\varepsilon}\gamma^\rho \tag{2.89}$$

$$= -i\frac{\partial}{\partial p^\mu}\,\Sigma^{loop}(p) \tag{2.90}$$

We conclude then, that $\Lambda^{loop}_\mu(p,p)$ is related to the self-energy of the electron[8],

$$\Lambda^{loop}_\mu(p,p) = -\frac{\partial}{\partial p^\mu}\,\Sigma^{loop} \tag{2.91}$$

On-shell we have

$$\Lambda^{loop}_\mu(p,p)\Big|_{\not p=m} = -\left.\frac{\partial}{\partial p^\mu}\right|_{\not p=m} = -\delta Z_2\gamma_\mu \tag{2.92}$$

and the normalization condition, Eq. (2.86), gives

$$\delta Z_1 = \delta Z_2 \tag{2.93}$$

As we have already calculated $\delta Z_2$ in Eq. (2.83), then $\delta Z_1$ is determined.

## 2nd method

In this second method we do not rely in the Ward identity but just calculate the integrals for the vertex in Eq. (2.84). For the moment we do not put $p'=p$ but we will assume that the vertex form factors are to be evaluated for on-shell spinors. Then we have

$$\begin{aligned} i\,\overline{u}(p')\Lambda^{loop}_\mu u(p) &= e^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{\overline{u}(p)\gamma_\rho\left[\not p'+\not k+m)\right]\gamma_\mu\left[\not p+\not k+m)\right]\gamma^\rho u(p)}{D_0D_1D_2} \\ &= e^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\,\frac{\mathcal{N}_\mu}{D_0D_1D_2} \end{aligned} \tag{2.94}$$

where

$$\begin{aligned} \mathcal{N}_\mu &= \overline{u}(p)\Big[(-2+d)k^2\gamma_\mu + 4p\cdot p'\gamma_\mu + 4(p+p')\cdot k\,\gamma_\mu + 4m\,k_\mu \\ &\qquad -4\not k\,(p+p')_\mu + 2(2-d)\not k k_\mu\Big]\,u(p) \end{aligned} \tag{2.95}$$

$$D_0 = k^2-\lambda^2+i\epsilon \tag{2.96}$$

$$D_1 = (k+p')^2-m^2+i\epsilon \tag{2.97}$$

$$D_2 = (k+p)^2-m^2+i\epsilon \tag{2.98}$$

[8]This result is one of the forms of the Ward-Takahashi identity.

Now using the results of section A.7.3 with

$$\begin{aligned} r_1^\mu &= p'^\mu \quad ; \quad r_2^\mu = p^\mu && (2.99)\\ P^\mu &= x_1 p'^\mu + x_2 p^\mu && (2.100)\\ C &= (x_1+x_2)^2 m^2 - x_1 x_2\, q^2 + (1-x_1-x_2)\lambda^2 && (2.101)\end{aligned}$$

where

$$q = p' - p \tag{2.102}$$

we get,

$$\begin{aligned} i\,\overline{u}(p')\Lambda_\mu^{loop} u(p) &= i\,\frac{\alpha}{4\pi}\Gamma(3)\int_0^1 dx_1 \int_0^{1-x_1} dx_2\, \frac{1}{2C}\\ &\quad \Bigg\{ \overline{u}(p')\gamma_\mu u(p)\Bigg[ -(-2+d)(x_1^2m^2 + x_2^2 m^2 + 2x_1x_2 p'\cdot p) - 4p'\cdot p\\ &\quad +4(p+p')\cdot(x_1p' + x_2 p) + \frac{(2-d)^2}{2}\, C\left(\Delta_\epsilon - \ln\frac{C}{\mu^2}\right)\Bigg]\\ &\quad +\overline{u}(p)u(p)m\,\Big[4(x_1p'+x_2p)_\mu - 4(p'+p)_\mu(x_1+x_2)\\ &\quad -2(2-d)(x_1+x_2)(x_1p'+x_2p)_\mu\Big]\Bigg\} && (2.103)\\ &= i\,\overline{u}(p)\Big[G(q^2)\,\gamma_\mu + H(q^2)\,(p+p')\Big]\,u(p) && (2.104)\end{aligned}$$

where we have defined[9],

$$\begin{aligned} G(q^2) \equiv \frac{\alpha}{4\pi}\Bigg[&\Delta_\epsilon - 2 - 2\int_0^1 dx_1\int_0^{1-x_1} dx_2\, \ln\frac{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}{\mu^2}\\ &+\int_0^1 dx_1\int_0^{1-x_1} dx_2\, \Bigg(\frac{-2(x_1+x_2)^2m^2 - x_1x_2q^2 - 4m^2 + 2q^2}{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}\\ &\qquad + \frac{2(x_1+x_2)(4m^2-q^2)}{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}\Bigg)\Bigg] && (2.105)\end{aligned}$$

$$H(q^2) \equiv \frac{\alpha}{4\pi}\left[\int_0^1 dx_1\int_0^{1-x_1} dx_2\, \frac{-2m\,(x_1+x_2) + 2m\,(x_1+x_2)^2}{(x_1+x_2)^2m^2 - x_1x_2q^2 + (1-x_1-x_2)\lambda^2}\right] \tag{2.106}$$

Now, using the definition of Eq. (2.85), we get for the renormalized vertex,

$$\overline{u}(p')\Lambda_\mu^R(p',p)u(p) = \overline{u}(p')\left[\Big(G(q^2) + \delta Z_1\Big)\,\gamma_\mu + H(q^2)\,(p+p')_\mu\right]u(p) \tag{2.107}$$

As $\delta Z_1$ is calculated in the limit of $q = p' - p \to 0$ it is convenient to use the Gordon identity to get rid of the $(p'+p)^\mu$ term. We have

$$\overline{u}(p')\,(p'+p)_\mu\, u(p) = \overline{u}(p')\left[2m\gamma_\mu - i\sigma_{\mu\nu}\, q^\nu\right]u(p) \tag{2.108}$$

[9]To obtain Eq. (2.106) one has to show that the symmetry of the integrals in $x_1 \leftrightarrow x_2$ implies that the coefficient of $p$ is equal to the coefficient of $p'$.

and therefore,

$$
\begin{aligned}
\overline{u}(p')\Lambda^R_\mu(p',p)u(p) &= \overline{u}(p')\left[\left(G(q^2)+2mH(q^2)+\delta Z_1\right)\gamma_\mu - i\,H(q^2)\,\sigma_{\mu\nu}\,q^\nu\right]u(p) \\
&= \gamma_\mu F_1(q^2)+\frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2) \qquad (2.109)
\end{aligned}
$$

where we have introduced the usual notation for the vertex form factors,

$$
F_1(q^2) \equiv G(q^2)+2mH(q^2)+\delta Z_1 \qquad (2.110)
$$

$$
F_2(q^2) \equiv -2mH(q^2) \qquad (2.111)
$$

The normalization condition of Eq. (2.86) implies $F_1(0)=0$, that is,

$$
\delta Z_1 = -G(0) - 2m\,H(0) \qquad (2.112)
$$

We have therefore to calculate $G(0)$ and $H(0)$. In this limit the integrals of Eqs. (2.105) and (2.106) are much simpler. We get (we change variables $x_1+x_2\to y$),

$$
\begin{aligned}
G(0) &= \frac{\alpha}{4\pi}\left[\Delta_\epsilon - 2 - 2\int_0^1 dx_1\int_{x_1}^1 dy\,\ln\frac{y^2m^2+(1-y)\lambda^2}{\mu^2}\right. \\
&\qquad \left. +\int_0^1 dx_1\int_{x_1}^1 dy\,\frac{-2y^2m^2-4m^2+8ym^2}{y^2m^2+(1-y)\lambda^2}\right] \qquad (2.113)
\end{aligned}
$$

$$
H(0) = \frac{\alpha}{4\pi}\int_0^1 dx_1\int_{x_1}^1 dy\,\frac{-2m\,y+2m\,y^2}{y^2m^2+(1-y)\lambda^2} \qquad (2.114)
$$

Now using

$$
\int_0^1 dx_1\int_{x_1}^1 dy\,\ln\frac{y^2m^2+(1-y)\lambda^2}{\mu^2} = \frac{1}{2}\left(\ln\frac{m^2}{\mu^2}-1\right) \qquad (2.115)
$$

$$
\int_0^1 dx_1\int_{x_1}^1 dy\,\frac{-2y^2m^2-4m^2+8ym^2}{y^2m^2+(1-y)\lambda^2} = 7+2\ln\frac{\lambda^2}{m^2} \qquad (2.116)
$$

$$
\int_0^1 dx_1\int_{x_1}^1 dy\,\frac{-2m\,y+2m\,y^2}{y^2m^2+(1-y)\lambda^2} = -\frac{1}{m} \qquad (2.117)
$$

(where we took the limit $\lambda\to 0$ if possible) we get,

$$
G(0) = \frac{\alpha}{4\pi}\left[\Delta_\epsilon + 6 - \ln\frac{m^2}{\mu^2} + 2\ln\frac{\lambda^2}{m^2}\right] \qquad (2.118)
$$

$$
H(0) = -\frac{\alpha}{4\pi}\frac{1}{m} \qquad (2.119)
$$

Substituting the previous expressions in Eq. (2.112) we get finally,

$$
\delta Z_1 = \frac{\alpha}{4\pi}\left[-\Delta_\epsilon - 4 + \ln\frac{m^2}{\mu^2} - 2\ln\frac{\lambda^2}{m^2}\right] \qquad (2.120)
$$

in agreement with Eq. (2.83) and Eq. (2.93). The general form of the form factors $F_i(q^2)$, for $q^2 \neq 0$, is quite complicated. We give here only the result for $q^2 < 0$ (in section 4.3 we will give a general formula for numerical evaluation of these functions),

$$\begin{aligned}
F_1(q^2) &= \frac{\alpha}{4\pi}\left\{\left(2\ln\frac{\lambda^2}{m^2}+4\right)(\theta\coth\theta-1)-\theta\,\tanh\frac{\theta}{2}-8\coth\theta\int_0^{\theta/2}\beta\tanh\beta d\beta\right\}\\
F_2(q^2) &= \frac{\alpha}{2\pi}\frac{\theta}{\sinh\theta}
\end{aligned} \tag{2.121}$$

where

$$\sinh^2\frac{\theta}{2} = -\frac{q^2}{4m^2}\cdot \tag{2.122}$$

In the limit of zero transfered momenta ($q = p' - p = 0$) we get

$$\begin{cases} F_1(0) = 0 \\ F_2(0) = \dfrac{\alpha}{2\pi} \end{cases} \tag{2.123}$$

a result that we will use in section 5.1 while discussing the anomalous magnetic moment of the electron.

# 3 Passarino-Veltman Integrals

In this section we present the Passarino-Veltman decomposition of the one-loop tensor integrals.

## 3.1 The general definition

The description of the previous sections works well if one just wants to calculate the divergent part of a diagram or to show the cancellation of divergences in a set of diagrams. In this last case one just uses the results of section A.8. If one actually wants to numerically calculate the integrals the task is normally quite complicated. Except for the *self-energy* type of diagrams, the integration over the Feynman parameters is normally quite difficult.

To overcome this problem a scheme was first proposed by Passarino and Veltman [1]. This scheme with the conventions of [6, 7] was latter implemented in the `Mathematica` package `FeynCalc` [2] and, for numerical evaluation, in the `LoopTools` package [3, 8]. The numerical evaluation follows the code developed earlier by van Oldenborgh [5]. We will now describe this scheme. We will write the generic one-loop tensor integral as

$$T_n^{\mu_1\cdots\mu_p} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\ \frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0D_1D_2\cdots D_{n-1}} \tag{3.1}$$

where we follow for the momenta the conventions of Fig. 10, with

$$D_i = (k + r_i)^2 - m_i^2 + i\epsilon \tag{3.2}$$

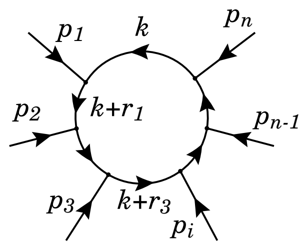



Figure 10:

and the momenta $r_i$ are related with the external momenta (all taken to be incoming) through the relations,

$$\begin{aligned} r_j &= \sum_{i=1}^{j} p_i \quad ; \quad j = 1, \ldots, n-1 \\ r_0 &= \sum_{i=1}^{n} p_i = 0 \end{aligned} \tag{3.3}$$

Notice that a factor of $i/16\pi^2$ is taken out. This is because, as we have seen in section 2, these integrals always give that pre-factor (see also A.3). So with our new convention that pre-factor **has** to included in the end. Factoring out the $i$ has also the convenience of dealing with real functions in many cases.[10] From all those integrals in Eq. (3.1) the scalar integrals are, has we have seen, of particular importance and deserve a special notation. It can be shown that there are only four independent such integrals, namely $(4 - d = \epsilon)$

$$A_0(m_0^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \, \frac{1}{k^2 - m_0^2} \tag{3.4}$$

$$B_0(r_{10}^2, m_0^2, m_1^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \prod_{i=0}^{1} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.5}$$

$$C_0(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.6}$$

$$D_0(r_{10}^2, r_{12}^2, r_{23}^2, r_{30}^2, r_{20}^2, r_{13}^2, m_0^2, \ldots, m_3^2) = \frac{(2\pi\mu)^\epsilon}{i\pi^2} \int d^dk \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.7}$$

where

$$r_{ij}^2 = (r_i - r_j)^2 \quad ; \quad \forall\, i, j = (0, n-1) \tag{3.8}$$

Remember that with our conventions $r_0 = 0$ so $r_{i0}^2 = r_i^2$. For simplicity, in all these expressions the $i\epsilon$ part of the denominator factors is suppressed. The general one-loop tensor integrals are not independent and therefore their decomposition is not unique. We

[10] The one loop functions are in general complex, but in some cases they can be real. These cases correspond to the situation where cutting the diagram does not corresponding to a kinematically allowed process.

follow the conventions of [2, 3] to write

$$B^{\mu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu} \prod_{i=0}^{1} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.9}$$

$$B^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu}k^{\nu} \prod_{i=0}^{1} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.10}$$

$$C^{\mu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu} \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.11}$$

$$C^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu}k^{\nu} \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.12}$$

$$C^{\mu\nu\rho} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu}k^{\nu}k^{\rho} \prod_{i=0}^{2} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.13}$$

$$D^{\mu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu} \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.14}$$

$$D^{\mu\nu} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu}k^{\nu} \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.15}$$

$$D^{\mu\nu\rho} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu}k^{\nu}k^{\rho} \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.16}$$

$$D^{\mu\nu\rho\sigma} \equiv \frac{(2\pi\mu)^{4-d}}{i\pi^2} \int d^dk\, k^{\mu}k^{\nu}k^{\rho}k^{\sigma} \prod_{i=0}^{3} \frac{1}{[(k+r_i)^2 - m_i^2]} \tag{3.17}$$

## 3.2 The tensor integrals decomposition

These integrals can be decomposed in terms of (reducible) functions in the following way:

$$B^{\mu} = r_1^{\mu}\, B_1 \tag{3.18}$$

$$B^{\mu\nu} = g^{\mu\nu}\, B_{00} + r_1^{\mu} r_1^{\nu}\, B_{11} \tag{3.19}$$

$$C^{\mu} = r_1^{\mu}\, C_1 + r_2^{\mu}\, C_2 \tag{3.20}$$

$$C^{\mu\nu} = g^{\mu\nu}\, C_{00} + \sum_{i=1}^{2} r_i^{\mu} r_j^{\nu}\, C_{ij} \tag{3.21}$$

$$C^{\mu\nu\rho} = \sum_{i=1}^{2} \left(g^{\mu\nu} r_i^{\rho} + g^{\nu\rho} r_i^{\mu} + g^{\rho\mu} r_i^{\nu}\right)\; C_{00i} + \sum_{i,j,k=1}^{2} r_i^{\mu} r_j^{\nu} r_k^{\rho}\, C_{ijk} \tag{3.22}$$

$$D^{\mu} = \sum_{i=1}^{3} r_i^{\mu}\, D_i \tag{3.23}$$

$$D^{\mu\nu} = g^{\mu\nu}\, D_{00} + \sum_{i=1}^{3} r_i^{\mu} r_j^{\nu}\, D_{ij} \tag{3.24}$$

$$D^{\mu\nu\rho} = \sum_{i=1}^{3} \left(g^{\mu\nu} r_i^{\rho} + g^{\nu\rho} r_i^{\mu} + g^{\rho\mu} r_i^{\nu}\right)\; D_{00i} + \sum_{i,j,k=1}^{2} r_i^{\mu} r_j^{\nu} r_k^{\rho}\, D_{ijk} \tag{3.25}$$

$$
\begin{aligned}
D^{\mu\nu\rho\sigma} \;=\;& (g_{\mu\nu}g_{\rho\sigma}+g_{\mu\rho}g_{\nu\sigma}+g_{\mu\sigma}g_{\nu\rho})\, D_{0000} + \sum_{i,j=1}^{3}\Big(g^{\mu\nu}r_i^\rho r_j^\sigma + g^{\nu\rho}r_i^\mu r_j^\sigma + g^{\mu\rho}r_i^\nu r_j^\sigma \\
& + g^{\mu\sigma}r_i^\nu r_j^\rho + g^{\nu\sigma}r_i^\mu r_j^\rho + g^{\rho\sigma}r_i^\mu r_j^\nu\Big)\, D_{00ij} + \sum_{i,j,k,l=1}^{3} r_i^\mu r_j^\nu r_k^\rho r_l^\sigma\, C_{ijkl}\,. \qquad (3.26)
\end{aligned}
$$

All coefficient functions have the same arguments as the corresponding scalar functions and are totally symmetric in their indices (not in their arguments). In the `FeynCalc` [2] package one generic notation is used,

$$
\mathrm{PaVe}\left[\mathrm{i},\mathrm{j},\ldots,\{\mathrm{r}_{10}^2,\mathrm{r}_{12}^2,\ldots\},\{\mathrm{m}_0^2,\mathrm{m}_1^2,\ldots\}\right]\ , \qquad (3.27)
$$

for instance

$$
B_{11}(r_{10}^2,m_0^2,m_1^2) = \mathrm{PaVe}\left[1,1,\{\mathrm{r}_{10}^2\},\{\mathrm{m}_0^2,\mathrm{m}_1^2\}\right]\ . \qquad (3.28)
$$

All these coefficient functions are not independent and can be reduced to the scalar functions. `FeynCalc` provides the command `PaVeReduce[...]` to accomplish that. This is very useful if one wants to check the cancellation of divergences or gauge invariance, where a number of diagrams have to cancel. We emphasize that the decomposition of the tensor integrals is not unique. Our choice is tied to the conventions of Fig. 10.

# 4 QED Renormalization with PV functions

In this section we will work out in detail a few examples of one-loop calculations using the `FeynCalc` package and the Passarino-Veltman scheme.

## 4.1 Vacuum Polarization in QED

We have done this example in section 2.1 using the techniques described in sections A.3, A.4 and A.5. Now we will use `FeynCalc`. The first step is to write the `Mathematica` program. We list it below:

```
(********************** Program VacPol.m ************************)

(* First input FeynCalc *)

<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)

dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
```

```
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(* Now write the numerator of the Feynman diagram. We define the
   constant

       C=alpha/(4 pi)
*)

num:= - C Tr[dm[mu] . (ds[q] + m) . dm[nu] .  (ds[q]+ds[k]+m)]
(* Tell FeynCalc to evaluate the integral in dimension D *)

SetOptions[OneLoop,Dimension->D]



(* Define the amplitude *)

amp:=num * FeynAmpDenominator[PropagatorDenominator[q+k,m], \
                              PropagatorDenominator[q,m]]

(* Calculate the result *)

res:=(-I / Pi^2) OneLoop[q,amp]
ans=Simplify[res]

(******************* End of Program VacPol.m *********************)
```

The output from `Mathematica` is:

```
                  2      2         2   2        2      2      2   2   2
Out[4]= (4  C  (k  + 6 m  B0[0, m , m ] - 3 (k  + 2 m ) B0[k , m , m ])
     2                                 2
   (k  g[mu, nu] - k[mu] k[nu])) / (9 k )
```

Now remembering that,

$$C = \frac{\alpha}{4\pi} \tag{4.1}$$

and

$$i\,\Pi_{\mu\nu}(k,\varepsilon) = -i\,k^2 P^T_{\mu\nu}\Pi(k,\varepsilon) \tag{4.2}$$

we get

$$\Pi(k,\varepsilon)=\frac{\alpha}{4\pi}\left[-\frac{4}{9}-\frac{8}{3}\frac{m^2}{k^2}\,B_0(0,m^2,m^2)+\frac{4}{3}\left(1+\frac{2m^2}{k^2}\right)\,B_0(k^2,m^2,m^2)\right] \tag{4.3}$$

To obtain the renormalized vacuum polarization one needs to know the value of $\Pi(0,\varepsilon)$. To do that one has to take the limit $k\to 0$ in Eq. (4.3). For that one uses the derivative of the $B_0$ function

$$B_0'(p^2,m_1^2,m_2^2)\equiv\frac{\partial}{\partial p^2}\,B_0(p^2,m_1^2,m_2^2) \tag{4.4}$$

to obtain

$$\Pi(0,\varepsilon)=\frac{\alpha}{4\pi}\left[-\frac{4}{9}+\frac{4}{3}B_0(0,m^2,m^2)+\frac{8}{3}m^2B_0'(0,m^2,m^2)\right] \tag{4.5}$$

Using

$$B_0'(0,m^2,m^2)=\frac{1}{6m^2} \tag{4.6}$$

we finally get

$$\Pi(0,\varepsilon)=-\delta Z_3=\frac{\alpha}{4\pi}\left[\frac{4}{3}\,B_0(0,m^2,m^2)\right] \tag{4.7}$$

and the final result for the renormalized vertex is:

$$\Pi^R(k)=\frac{\alpha}{3\pi}\left[-\frac{1}{3}+\left(1+\frac{2m^2}{k^2}\right)\Big(B_0(k^2,m^2,m^2)-B_0(0,m^2,m^2)\Big)\right] \tag{4.8}$$

If we want to compare with our earlier analytical results we need to know that

$$B_0(0,m^2,m^2)=\Delta_\varepsilon-\ln\frac{m^2}{\mu^2} \tag{4.9}$$

Then Eq. (4.8) reproduces the result of Eq. (2.54). The comparison between Eq. (4.8) and Eq. (2.56) can be done numerically using the package `LoopTools` [3].

## 4.2 Electron Self-Energy in QED

In this section we repeat the calculation of section 2.2 using the Passarino-Veltman scheme. We start with the `Mathematica` program,

```
(********************** Program SelfEnergy.m **********************)

(* First input FeynCalc *)

<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)
dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
```

```
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(* Tell FeynCalc to reduce the result to scalar functions *)

SetOptions[{B0,B1,B00,B11},BReduce->True]

(* Now write the numerator of the Feynman diagram. We define the
   constant

       C= - alpha/(4 pi)

  The minus sign comes from the photon propagator. The factor
  i/(16 pi^2) is already included in this definition.

*)

num:= C dm[mu] .  (ds[p]+ds[k]+m) . dm[mu]

(* Tell FeynCalc to evaluate the one-loop integral in dimension D *)

SetOptions[OneLoop,Dimension->D]

(* Define the amplitude *)

amp:= num \
FeynAmpDenominator[PropagatorDenominator[p+k,m], \
                   PropagatorDenominator[k]]

(* Calculate the result *)

res:=(-I / Pi^2) OneLoop[k,amp]
ans=-res;

(*
The minus sign in ans comes from the fact that -i \Sigma = diagram
*)

(* Calculate the functions A(p^2) and B(p^2) *)

A=Coefficient[ans,DiracSlash[p],0];
B=Coefficient[ans,DiracSlash[p],1];
```

```
(* Calculate deltm *)

delm=A + m B /. p->m

(* Calculate delZ2 *)

Ap2 = A /. ScalarProduct[p,p]->p2
Bp2 = B /. ScalarProduct[p,p]->p2

aux=2 m D[Ap2,p2] + Bp2 \
    + 2 m^2 D[Bp2,p2] /. D[B0[p2,0,m^2],p2]->DB0[p2,0,m^2]

aux2= aux /. p2->m^2

aux3= aux2 /. A0[m^2]->m^2 (B0[m^2,0,m^2] -1)

delZ2=Simplify[aux3]

(******** ********* End of Program SelfEnergy.m *******************)
```

The output from `Mathematica` is:

```
                             2      2
    A =  -(C (-2 m + 4 m B0[p , 0, m ]))


              2       2      2    2      2      2
          C (p  + A0[m ] - (m  + p ) B0[p , 0, m ])
    B = -(-----------------------------------------)
                                 2
                                p


            2       2       2     2      2
        C (m  - A0[m ] - 2 m  B0[m , 0, m ])
 delm = ------------------------------------
                          m


                  2      2       2      2      2
delZ2= C (-2 + B0[m , 0, m ] - 4 m  DB0[m , 0, m ])
```

We therefore get

$$\begin{aligned} A &= \frac{\alpha m}{\pi} \left[ -\frac{1}{2} + B_0(p^2, 0, m^2) \right] & (4.10) \\ B &= \frac{\alpha}{4\pi} \left[ 1 + \frac{1}{p^2} A_0(m^2) - \left( 1 + \frac{m^2}{p^2} \right) B_0(p^2, 0, m^2) \right] & (4.11) \\ \delta_m &= \frac{3\alpha m}{4\pi} \left[ -\frac{1}{3} + \frac{1}{3m^2} A_0(m^2) + \frac{2}{3} B_0(m^2, 0, m^2) \right] & (4.12) \end{aligned}$$

One can check that Eq. (4.12) is in agreement with Eq. (2.80). For that one needs the following relations,

$$\begin{aligned} A_0(m^2) &= m^2 \left( B_0(m^2, 0, m^2) - 1 \right) & (4.13) \\ B_0(m^2, 0, m^2) &= \Delta_\varepsilon + 2 - \ln \frac{m^2}{\mu^2} & (4.14) \\ \int_0^1 dx (1+x) \ln \frac{m^2 x^2}{\mu^2} &= -\frac{5}{2} + \frac{3}{2} \ln \frac{m^2}{\mu^2} & (4.15) \end{aligned}$$

For $\delta Z_2$ we get

$$\delta Z_2 = \frac{\alpha}{4\pi} \left[ 2 - B_0(m^2, 0, m^2) - 4m^2 B_0'(m^2, \lambda^2, m^2) \right] \tag{4.16}$$

This expression can be shown to be equal to Eq. (2.83) although this is not trivial. The reason is that $B_0'$ is IR divergent, hence the parameter $\lambda$ that controls the divergence. To show that the two expressions are equivalent we notice that in the limit $\lambda \to 0$ we have

$$\begin{aligned} \int_0^1 dx \, \frac{(1+x)(1-x)xm^2}{m^2x^2 + \lambda^2(1-x)} &= \int_0^1 dx \, \frac{(1-x)xm^2}{m^2x^2 + \lambda^2(1-x)} + \int_0^1 dx \, \frac{(1-x)x^2m^2}{m^2x^2 + \lambda^2(1-x)} \\ &= -m^2 B_0'(m^2, \lambda^2, m^2) + \frac{1}{2} & (4.17) \end{aligned}$$

where we have taken the $\lambda \to 0$ limit whenever possible and used Eq. (A.120). Also

$$\int_0^1 dx \, (1-x) \ln \left( \frac{m^2 x^2}{\mu^2} \right) = -\frac{3}{2} + \frac{1}{2} \ln \frac{m^2}{\mu^2} = \frac{1}{2} \Delta_\epsilon - \frac{1}{2} - \frac{1}{2} B_0(m^2, 0, m^2) \tag{4.18}$$

where in the last step we have used Eq. (A.118).

## 4.3 QED Vertex

In this section we repeat the calculation of section 2.3 for the QED vertex using the Passarino-Veltman scheme. The `Mathematica` program should by now be easy to understand. We just list it here,

```
(******************** Program QEDVertex.m *********************)

(* First input FeynCalc *)

<< FeynCalc.m

(* These are some shorthands for the FeynCalc notation *)
dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(* Tell FeynCalc to reduce the result to scalar functions *)

SetOptions[{B1,B00,B11},BReduce->True]

(* Now write the numerator of the Feynman diagram. We define the
   constant

       C=  alpha/(4 pi)

   The kinematics is: q = p1 -p2 and the internal momenta is k.

*)

num:=Spinor[p1,m] . dm[ro] . (ds[p1]-ds[k]+m) . ds[a] \
      . (ds[p2]-ds[k]+m) . dm[ro] . Spinor[p2,m]

SetOptions[OneLoop,Dimension->D]

amp:=C num \
FeynAmpDenominator[PropagatorDenominator[k,lbd], \
                   PropagatorDenominator[k-p1,m], \
                   PropagatorDenominator[k-p2,m]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m^2,ScalarProduct[p2,p2]->m^2, \
         ScalarProduct[p1,p2]->m^2-q2/2}
```

```
(* Define the divergent part of the relevant PV functions*)

div={B0[0,0,m^2]->Div,B0[0,m^2,m^2]->Div, \
     B0[m^2,0,m^2]->Div,B0[m^2,lbd^2,m^2]->Div,\
     B0[q2,m^2,m^2]->Div,B0[0,lbd^2,m^2]->Div}

res1:=(-I / Pi^2) OneLoop[k,amp]

res:=res1 /. onshell

auxV1:= res /.onshell
auxV2:= PaVeReduce[auxV1]
auxV3:= auxV2 /. div
divV:=Simplify[Div*Coefficient[auxV3,Div]]

(* Check  that the divergences do not cancel *)

testdiv:=Simplify[divV]

ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]
Set @@ {var, {ME[1],ME[2]}}

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 2 Standard Matrix
Elements. To have a simpler result we substitute these elements
by simpler expressions (ME[1],ME[2]).

 {StandardMatrixElement[u[p1, m1] . u[p2, m2]],

 StandardMatrixElement[u[p1, m1] . ga[mu] . u[p2, m2]]}

*)

ans2=Simplify[PaVeReduce[ans1]]

CE11=Coefficient[ans2,ScalarProduct[a,p1] ME[1]]
CE12=Coefficient[ans2,ScalarProduct[a,p2] ME[1]]
CE2=Coefficient[ans2, ME[2]]

ans3=CE11 (ScalarProduct[a,p1]+ScalarProduct[a,p2]) ME[1] + \
     CE2 ME[2]

test1:=Simplify[CE11-CE12]
test2:=Simplify[ans2-ans3]
```

```
(* ME[2] is \overline{u}(p')\gamma_{\mu}u(p) and ME[3] is
   \frac{i}{2m} \overline{u}(p')\sigma_{\mu\nu} q^{\u} u(p)
*)

ans4= ans3 /. {(ScalarProduct[a,p1]+ScalarProduct[a,p2]) \
              ME[1] -> 2 m ME[2] -2 m  ME[3]}

CGamma:=Coefficient[ans4,ME[2]]
CSigmaAux:=Coefficient[ans4,ME[3]]

test3:=Simplify[ans4-CGamma ME[2] -CSigmaAux ME[3]]

F2:=Simplify[CSigmaAux /. lbd->0]

delZ1aux= - CGamma /. q2->0

delZ1:= delZ1aux /. lbd->0

F1:=CGamma + delZ1 /. lbd->0

(**************** End of Program QEDVertex.m ******************)
```

From this program we can obtain first the value of $\delta Z_1$. We get

```
                          2             2   2           2      2
delZ1= C(1 - B0[0, 0, m ] + 2 B0[0, m , m ] - 2 B0[m , 0, m ] -

   2     2   2      2      2
4 m  C0[m , m , 0, m , 0, m ])
```

which can be written as

$$\delta Z_1 = \frac{\alpha}{2\pi}\Big[1 - B_0(0,0,m^2) + 2B_0(0,m^2,m^2) - 2B_0(m^2,m^2) -4m^2 C_0(m^2,m^2,0,m^2,\lambda^2,m^2)\Big] \tag{4.19}$$

where we have introduced a small mass for the photon in the function $C_0(m^2,m^2,0,m^2,\lambda^2,m^2)$ because it is IR divergent when $\lambda \to 0$ (see Eq. (A.134)). Using the results of Eqs. (A.117), (A.118), (A.119) and Eq. (A.134) we can show the important result

$$\delta Z_1 = \delta Z_2 \tag{4.20}$$

where $\delta Z_2$ was defined in Eq. (4.16). After performing the renormalization the coefficient $F_1(k^2)$ is finite and given by

```
                                        2                 2   2
               q2         q2 B0[0, 0, m ]   2 q2 B0[0, m , m ]
 F1 =   C (-(---------) - --------------- + ------------------ -
                     2                2                    2
            q2 - 4 m          q2 - 4 m             q2 - 4 m

    2        2   2                   2   2        2         2   2
  8 m  B0[0, m , m ]   3 q2 B0[q2, m , m ]   8 m  B0[q2, m , m ]
  ------------------ - ------------------- + ------------------- +
               2                     2                      2
      q2 - 4 m              q2 - 4 m              q2 - 4 m

          2       2            2      2   2       2       2
  2 q2 B0[m , 0, m ]   4 q2 m  C0[m , m , 0, m , 0, m ]
  ------------------ - -------------------------------- +
               2                                 2
      q2 - 4 m                          q2 - 4 m

     4     2   2      2       2         2     2   2       2       2
  16 m  C0[m , m , 0, m , 0, m ]   2 q2  C0[m , m , q2, m , 0, m ]
  ------------------------------ - ------------------------------- +
                  2                                  2
          q2 - 4 m                           q2 - 4 m

        2     2   2       2       2        4     2   2       2       2
  12 q2 m  C0[m , m , q2, m , 0, m ]   16 m  C0[m , m , q2, m , 0, m ]
  ---------------------------------- - ------------------------------- )
                     2                                  2
            q2 - 4 m                           q2 - 4 m

In[5]:= F1 /. q2->0

Out[5]= 0
```

while the coefficient $F_2(q^2)$ does not need renormalization and it is given by,

```
           2                 2             2   2           2      2
        4 m  (1 + B0[0, 0, m ] + B0[q2, m , m ] - 2 B0[m , 0, m ])
 F2= C ---------------------------------------------------------
                                        2
                               q2 - 4 m
```

and for $F_2(0)$ we get

```
                             2             2   2            2       2
F2[0]= -(C (1 + B0[0, 0, m ] + B0[0, m , m ] - 2 B0[m , 0, m ]))
```

Using the results of the Appendix we can show that,

$$F_2(0) = \frac{\alpha}{2\pi} \tag{4.21}$$

a well known result, first obtained by Schwinger even before the renormalization program was fully understood ($F_2(q^2)$ is finite).

# 5 Finite contributions from RC to physical processes

In the previous sections we have discussed the formalism of the renormalization of QED using a large number of techniques. As we have shown, in the end all quantities are finite. The question that might arise is, if we can compare the results with the experiment, the final test in physics. In this section we will show that this is indeed possible (and necessary) in two physical situations.

## 5.1 Anomalous electron magnetic moment

We will show here, for the case of the electron anomalous moment, how the finite part of the radiative corrections can be compared with experiment, given credibility to the renormalization program. In fact we will just consider the first order, while to compare with the present experimental limit one has to go to fourth order in QED and to include also the weak and QCD corrections. The electron magnetic moment is given by

$$\vec{\mu} = \frac{e}{2m} g \frac{\vec{\sigma}}{2} \tag{5.1}$$

where $e = -|e|$ for the electron. One of the biggest achievements of the Dirac equation was precisely to predict the value $g = 2$. Experimentally we know that $g$ is close to, but not exactly, 2. It is usual to define this difference as the anomalous magnetic moment. More precisely,

$$g = 2(1 + a) \tag{5.2}$$

or

$$a = \frac{g}{2} - 1 \tag{5.3}$$

Our task is to calculate $a$ from the radiative corrections that we have computed in the previous sections. To do that let us start to show how a value $a \neq 0$ will appear in non relativistic quantum mechanics. Schrödinger's equation for a charged particle in an exterior field is,

$$i\frac{\partial \varphi}{\partial t} = \left[ \frac{(\vec{p} - e\vec{A})^2}{2m} + e\phi - \frac{e}{2m}(1 + a)\vec{\sigma} \cdot \vec{B} \right] \varphi \tag{5.4}$$

Now we consider that the external field is a magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$. Then keeping only terms first order in $e$ we get

$$
\begin{aligned}
H &= \frac{p^2}{2m} - e\frac{\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}}{2m} - \frac{e}{2m}(1+a)\vec{\sigma}\cdot\vec{\nabla}\times\vec{A} \\
&\equiv H_0 + H_{int}
\end{aligned} \tag{5.5}
$$

With this interaction Hamiltonian we calculate the transition amplitude between two electron states of momenta $p$ and $p'$. We get

$$
\begin{aligned}
\langle p'|\, H_{int}\, |p\rangle &= -\frac{e}{2m}\int \frac{d^3x}{(2\pi)^3}\chi^\dagger e^{-i\vec{p}'\cdot\vec{x}}[\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}+(1+a)\vec{\sigma}\times\vec{\nabla}\cdot\vec{A}]e^{i\vec{p}\cdot x}\chi \\
&= -\frac{e}{2m}\int \frac{d^3x}{(2\pi)^3}\chi^\dagger[(\vec{p}'+\vec{p})\cdot\vec{A}+i(1+a)\sigma^i\epsilon^{ijk}q^jA^k]e^{-i\vec{q}\cdot x}\chi \\
&= -\frac{e}{2m}\chi^\dagger[(p'+p)^k + i(1+a)\sigma^i\epsilon^{ijk}q^j]A^k(q)\chi
\end{aligned} \tag{5.6}
$$

This is the result that we want to compare with the non relativistic limit of the renormalized vertex. The amplitude is given by,

$$
\begin{aligned}
A &= e\overline{u}(p')(\gamma_\mu + \Lambda^R_\mu)u(p)A^\mu(q) \\
&= e\overline{u}(p')\left[\gamma_\mu(1+F_1(q^2)) + \frac{i}{2m}\sigma_{\mu\nu}q^\nu F_2(q^2)\right]u(p)A^\mu(q) \\
&= \frac{e}{2m}\overline{u}(p')\Big\{(p'+p)_\mu\left[1+F_1(q^2)\right] + i\sigma_{\mu\nu}q^\nu\left[1+F_1(q^2)+F_2(q^2)\right]\Big\}u(p)A^\mu(q)
\end{aligned} \tag{5.7}
$$

where we have used Gordon's identity. For an external magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}$ and in the limit $q^2 \to 0$ this expression reduces to

$$
\begin{aligned}
A &= \frac{e}{2m}\overline{u}(p')\left\{(p'+p)_k[1+F_1(0)] + i\sigma_{kj}q^j[1+F_1(0)+F_2(0)]\right\}u(p)A^k(q) \\
&= \frac{e}{2m}\overline{u}(p')\left[-(p'+p)^k + i\Sigma^i\epsilon^{kij}q^j\left(1+\frac{\alpha}{2\pi}\right)\right]u(p)A^k(q)
\end{aligned} \tag{5.8}
$$

where we have used the results of Eq. (2.123),

$$
\begin{cases}
F_1(0) = 0 \\
F_2(0) = \dfrac{\alpha}{2\pi}
\end{cases} \tag{5.9}
$$

Using the explicit form for the spinors $u$

$$
u(p) = \begin{pmatrix} \chi \\ \dfrac{\vec{\sigma}\cdot(\vec{p}-e\vec{A})}{2m}\,\chi \end{pmatrix} \tag{5.10}
$$

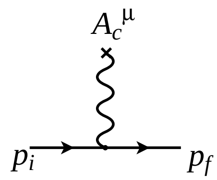



Figure 11:

we can write in the non relativistic limit,

$$A = -\frac{e}{2m}\chi^\dagger \left[ (p'+p)^k + i\left(1+\frac{\alpha}{2\pi}\right)\sigma^i\epsilon^{ijk}q^j \right]\chi A^k \tag{5.11}$$

which after comparing with Eq. (5.6) leads to

$$a^e_{th} = \frac{\alpha}{2\pi} \tag{5.12}$$

This result obtained for the first time by Schwinger and experimentally confirmed, was very important in the acceptance of the renormalization program of Feynman, Dyson and Schwinger for QED.

## 5.2 Cancellation of IR divergences in Coulomb scattering

In this section we will show how the IR divergences cancel in physical processes. We will take as an example the Coulomb scattering from a fixed nucleus. This is better done if we start from first principles. Coulomb scattering corresponds to the diagram of Fig. 11, which gives the following matrix element for the $S$ matrix,

$$S_{fi} = iZe^2(2\pi)\delta(E_i - E_f)\frac{1}{|\vec{q}|^2}\ \overline{u}(p_f)\gamma^0 u(p_i) \tag{5.13}$$

We will now study the radiative corrections to this result in lowest order in perturbation theory. Due to the IR divergences it is convenient to introduce a mass $\lambda$ for the photon. For a classical field, as we are considering, this means a screening. If we take,

$$A^0_c(x) = Ze\frac{e^{-\lambda|\vec{x}|}}{4\pi|\vec{x}|} \tag{5.14}$$

the Fourier transform will be,

$$A^0_c(q) = Ze\frac{1}{|\vec{q}|^2+\lambda^2} \tag{5.15}$$

that shows that the screening is equivalent to a mass for the photon. With these modifications we have,

$$S_{fi} = iZe^2(2\pi)\delta(E_f - E_i)\ \frac{i}{|\vec{q}|^2+\lambda^2}\ \overline{u}(p_f)\gamma^0 u(p_i) \tag{5.16}$$

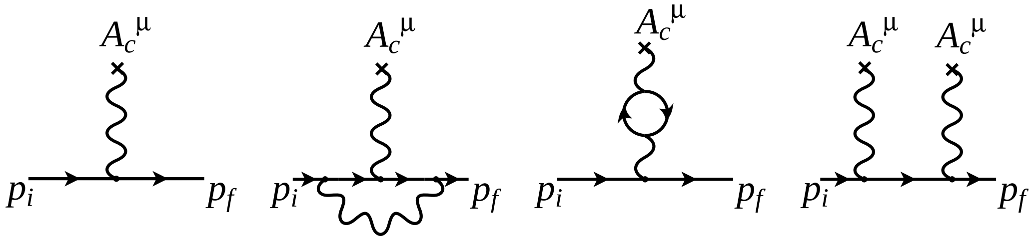



Figure 12:

We are interested in calculating the corrections up to order $e^3$ in the amplitude. To this contribute the diagrams of Fig. 12. Diagram 1 is of order $e^2$ while diagrams $2, 3, 4$ are of order $e^4$. Therefore the interference between 1 and $(2+3+4)$ is of order $\alpha^3$ and should be added to the result of the bremsstrahlung in a Coulomb field. The contribution from $1+2+3$ can be easily obtained by noticing that

$$eA_c^\mu\gamma_\mu \rightarrow eA_c^\mu(\gamma_\mu + \Lambda_\mu^R + \Pi_{\mu\nu}^R G^{\nu\rho}\gamma_\rho) \tag{5.17}$$

where $\Lambda_\mu^R$ e $\Pi_{\mu\nu}^R$ have been calculated before. We get

$$\begin{aligned} S_{fi}^{(1+2+3)} &= iZe^2(2\pi)\delta(E_i - E_f)\frac{1}{|\vec{q}|^2+\lambda^2}\overline{u}(p_f)\gamma^0\left\{1+\frac{\alpha}{\pi}\left[-\frac{1}{2}\varphi\tanh\varphi\right.\right. \\ &\quad \left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi - 1) - 2\coth 2\varphi\int_0^\varphi \beta\tanh\beta d\beta \\ &\quad \left.\left. + \left(1-\frac{\coth^2\varphi}{\beta}\right)(\varphi\coth\varphi - 1) + \frac{1}{9}\right] - \frac{\not{q}}{2m}\frac{\alpha}{\pi}\frac{\varphi}{\sinh 2\varphi}\right\}u(p_i) \end{aligned} \tag{5.18}$$

where

$$\frac{|\vec{q}|^2}{4m} = \sinh^2\varphi\,. \tag{5.19}$$

Finally the fourth diagram gives

$$\begin{aligned} S_{fi}^{(4)} &= (iZe)^2(e)^2\int\frac{d^4k}{(2\pi)^4}\overline{u}(p_f)\left[\frac{2\pi\delta(E_f-k^0)}{(p_f-k)^2-\lambda^2}\gamma^0\frac{i}{\not{k}-m+i\varepsilon}\gamma^0\frac{2\pi\delta(k^0-E_i)}{(k-p_i)^2-\lambda^2}\right] \\ &= -2i\frac{Z^2\alpha^2}{\pi}2\pi\delta(E_f-E_i)\ \overline{u}(p_f)\left[m(I_1-I_2)+\gamma^0E_i(I_1+I_2)\right]u(p_i) \end{aligned} \tag{5.20}$$

with

$$I_1 = \int d^3k\frac{1}{[(\vec{p}_f-\vec{k})^2+\lambda^2][(\vec{p}_i-\vec{k})^2+\lambda^2][(\vec{p})^2-(\vec{k})^2+i\varepsilon]} \tag{5.21}$$

and

$$\frac{1}{2}(\vec{p}_i+\vec{p}_f)I_2 \equiv \int d^3k \frac{\vec{k}}{[(\vec{p}_f-\vec{k})^2+\lambda^2][(\vec{p}_i-\vec{k})^2+\lambda^2][(\vec{p})^2-(\vec{k})^2+i\varepsilon]}\,. \tag{5.22}$$

In the limit $\lambda \to 0$ it can be shown that

$$I_1 = \frac{\pi^2}{2ip^3\sin^2\theta/2}\ln\left(\frac{2p\sin(\theta/2)}{\lambda}\right) \tag{5.23}$$

$$I_2 = \frac{\pi^2}{2p^3\cos^2\theta/2}\left\{\frac{\pi}{2}\left[1-\frac{1}{\sin\theta/2}\right]-i\left[\frac{1}{\sin^2\theta/2}\ln\left(\frac{2p\sin\theta/2}{\lambda}\right)+\ln\frac{\lambda}{2p}\right]\right\} \tag{5.24}$$

With these expressions we get for the cross section

$$\frac{d\sigma}{d\Omega} = \frac{Z^2\alpha^2}{|\vec{q}|^2}\frac{1}{2}\sum_{pol}|\overline{u}(p_f)\Gamma u(p_i)|^2 \tag{5.25}$$

where

$$\Gamma = \gamma^0(1+A)+\gamma^0\frac{\not{q}}{2m}B+C \tag{5.26}$$

and

$$\begin{aligned} A &= \frac{\alpha}{\pi}\left[\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth 2\varphi-1)-2\coth 2\varphi\int_0^{\varphi}d\beta\beta\tanh\beta-\frac{\varphi}{2}\tanh\varphi\right. \\ &\quad \left.+\left(1-\frac{1}{3}\coth^2\varphi\right)(\varphi\coth\varphi-1)+\frac{1}{9}\right]-\frac{Z\alpha}{2\pi^2}|\vec{q}|^2E(I_1+I_2) \end{aligned} \tag{5.27}$$

$$B = -\frac{\alpha}{\pi}\frac{\varphi}{\sinh 2\varphi} \tag{5.28}$$

$$C = -\frac{Z\alpha}{2\pi^2}m|\vec{q}|^2(I_1-I_2) \tag{5.29}$$

Therefore

$$\begin{aligned} \frac{1}{4}\sum_{pol}|\overline{u}(p_f)pu(p_i)|^2 &= \frac{1}{4}\mathrm{Tr}[\Gamma(\not{p}_i+m)\overline{\Gamma}(\not{p}_f+m)] \\ &= 2E^2(1-\beta^2\sin^2\theta/2)+2E^2 2B\beta^2\sin^2\frac{\theta}{2} \\ &\quad +2E^2 2ReA\left(1-\beta^2\sin^2\frac{\theta}{2}\right)+2ReC(2mE)+O(\alpha^2) \end{aligned} \tag{5.30}$$

Notice that $A, B$ e $C$ are of order $\alpha$. Therefore the final result is, up to order $\alpha^3$:

$$
\begin{aligned}
\left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \left\{1 + \frac{2\alpha}{\pi}\left[\left(1+\ln\frac{\lambda}{m}\right)(2\varphi\coth\varphi - 1) - \frac{\varphi}{2}\tanh\varphi\right.\right. \\
&\quad -2\coth 2\varphi \int_0^{\varphi} d\beta\beta\tanh\beta + \left(-\frac{\coth^2\varphi}{3}\right)(\varphi\coth\varphi - 1) + \frac{1}{9} \\
&\quad \left.\left. -\frac{\varphi}{\sinh 2\varphi}\frac{B^2\sin^2\theta/2}{1-\beta^2\sin^2\theta/2}\right] + Z\alpha\pi\frac{\beta\sin\frac{\theta}{2}[1-\sin\theta/2]}{1-\beta^2\sin^2\theta/2}\right\} \qquad (5.31)
\end{aligned}
$$

As we had said before the result is IR in the limit $\lambda \to 0$. This divergence is not physical and can be removed in the following way. The detectors have an energy threshold, below which they can not detect. Therefore in the limit $\omega \to 0$ bremsstrahlung in a Coulomb field and Coulomb scattering can not be distinguished. This means that we have to add both results. If we consider an energy interval $\Delta E$ with $\lambda \leq \Delta E \leq E$ we get

$$
\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \int_{\omega\leq\Delta E} \frac{d^3k}{2\omega(2\pi)^3} e^2 \left[\frac{2p_i\cdot p_f}{k_i\cdot p_i k\cdot p_f} - \frac{m^2}{(k\cdot p\cdot)^2} - \frac{m^2}{(k\cdot p_f)^2}\right] \qquad (5.32)
$$

Giving a mass to the photon (that is $\omega = (|\vec{k}|^2 + \lambda^2)^{1/2}$) the integral can be done with the result,

$$
\begin{aligned}
\left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} = \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \frac{2\alpha}{\pi} &\left\{(2\varphi\coth 2\varphi - 1)\ln\frac{2\Delta E}{\lambda} + \frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}\right. \\
&\left. -\frac{1}{2}\cosh 2\varphi \frac{1-\beta^2}{\beta\sin\theta/2}\int_{\cos\theta/2}^{1} d\xi \frac{1}{(1-\beta^2\xi^2)[\xi-\cos^2\theta/2]^{1/2}} \ln\frac{1+\beta\xi}{1-\beta\xi}\right\}
\end{aligned}
\qquad (5.33)
$$

The inclusive cross section can now be written as

$$
\begin{aligned}
\frac{d\sigma}{d\Omega}(\Delta E) &= \left(\frac{d\sigma}{d\Omega}\right)_{\text{elastic}} + \left[\frac{d\sigma}{d\Omega}(\Delta E)\right]_{BR} \\
&= \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} (1 - \delta_R + \delta_B) \qquad (5.34)
\end{aligned}
$$

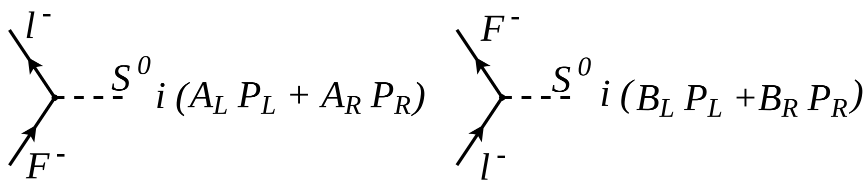
where $\delta_R$ and $\delta_B$ are complicated expressions that depend on the resolution of the detector $\Delta E$ but do not depend on $\lambda$ that can be finally put to zero. One can show that in QED all the IR divergences can be treated in a similar way. One should note that the final effect of the bremsstrahlung is finite and can be important.

# 6 Modern techniques in a real problem: $\mu \to e\gamma$

In the previous sections we have redone most of the QED standard textbook examples using the PV decomposition and automatic tools. Here we want to present a more complex example, the calculation of the partial width $\mu \to e\gamma$ in an arbitrary theory where the charged leptons couple to scalars and fermions, charged or neutral. This has been done in Ref. [9] for fermions and bosons of arbitrary charge $Q_F$ and $Q_B$, but for simplicity I will consider here separately the cases of neutral and charged scalars.

## 6.1 Neutral scalar charged fermion loop

We will consider a theory with the following interactions,




where $F^-$ is a fermion with mass $m_F$ and $S^0$ a neutral scalar with mass $m_S$. In fact $B_{L,R}$ are not independent of $A_{L,R}$ but it is easier for our programming to consider them completely general. The Feynman rule for the coupling of the photon with the lepton is $-i\, e\, Q_\ell\, \gamma^\mu$ where $e$ is the positron charge (for an electron $Q_\ell = -1$). $\ell_i^-$ can be any of the leptons but we will omit all indices in the program, the lepton being identified by its mass and from the assumed kinematics

$$\ell_2(p_2) \to \ell_1(p_1) + \gamma(k) \tag{6.1}$$

The diagrams contributing to the process are given in Fig. 13,

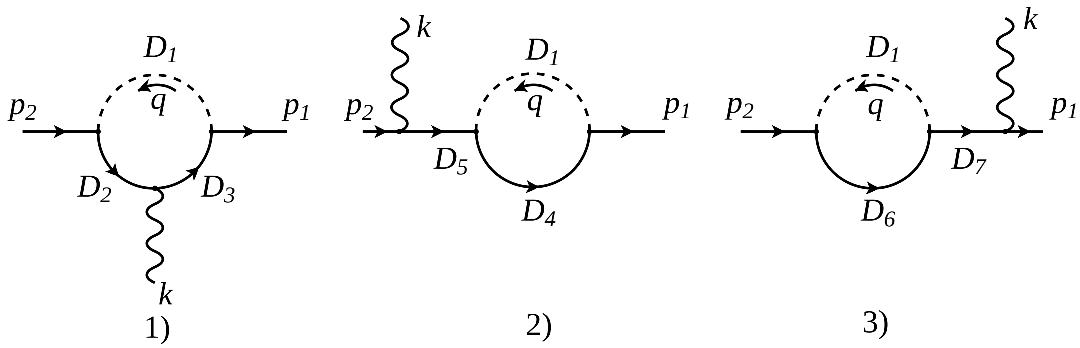



Figure 13:

where

$$D_1 = q^2 - m_S^2 \quad ; \quad D_2 = (p_2+q)^2 - m_F^2 \quad ; \quad D_3 = (q+p2-k)^2 - m_F^2 \tag{6.2}$$

$$D_4 = D_3 \quad ; \quad D_6 = D_2 \quad ; \quad D_5 = (p_2-k)^2 - m_2^2 = -2p_2\cdot k \tag{6.3}$$

$$D_7 = (p_1+k)^2 - m_1^2 = 2p_1\cdot k = -D_5 \tag{6.4}$$

The amplitudes are

$$
\begin{aligned}
iM_1 &= \frac{e\,Q_\ell}{D_1 D_2 D_3}\overline{u}(p_1)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 - \not{k} + m_F)\,\gamma^\mu\;(\not{q} + \not{p}_2 + m_F) \\
&\qquad (B_L P_L + B_R P_R)\,u(p_2)\;\varepsilon_\mu(k) \qquad (6.5)\\
iM_2 &= \frac{e\,Q_\ell}{D_1 D_4 D_5}\overline{u}(p_1)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 - \not{k} + m_F)\,(B_L P_L + B_R P_R) \\
&\qquad (\not{p}_2 - \not{k}_2 + m_2)\,\gamma^\mu u(p_2)\;\varepsilon_\mu(k) \qquad (6.6)\\
iM_3 &= \frac{e\,Q_\ell}{D_1 D_6 D_7}\overline{u}(p_1)\gamma^\mu\;(\not{p}_1 + \not{k} + m_F)\,(A_L P_L + A_R P_R)\,(\not{q} + \not{p}_2 + m_1) \\
&\qquad (B_L P_L + B_R P_R)\,u(p_2)\;\varepsilon_\mu(k) \qquad (6.7)
\end{aligned}
$$

On-shell the amplitude will take the form (we have $p_1 \cdot k = p_2 \cdot k$)

$$
\begin{aligned}
iM &= 2p_2 \cdot \varepsilon(k)\left[\;C_L \overline{u}(p_1) P_L u(p_2) + C_R \overline{u}(p_1) P_R u(p_2)\right] \\
&\qquad + D_L u(p_1) \not{\varepsilon} P_L u(p_2) + D_R u(p_1) \not{\varepsilon} P_R u(p_2) \qquad (6.8)
\end{aligned}
$$

If we write the amplitude as

$$M = M_\mu\;\varepsilon^\mu(k) \tag{6.9}$$

then gauge invariance implies

$$M_\mu k^\mu = 0 \tag{6.10}$$

Imposing this condition on Eq. (6.8) we get the relations

$$
\begin{aligned}
D_L &= -m_2 C_R - m_1 C_L \qquad (6.11)\\
D_R &= -m_1 C_R - m_2 C_L \qquad (6.12)
\end{aligned}
$$

Assuming these relations the amplitude can be written as

$$
\begin{aligned}
iM = C_L &\left[2p_2 \cdot \varepsilon(k) \overline{u}(p_1) P_L u(p_2) - m_1 \overline{u}(p_1) \not{\varepsilon}(k) P_L u(p_2) - m_2 \overline{u}(p_1) \not{\varepsilon}(k) P_R u(p_2)\right] \\
+ C_R &\left[2p_2 \cdot \varepsilon(k) \overline{u}(p_1) P_R u(p_2) - m_2 \overline{u}(p_1) \not{\varepsilon}(k) P_L u(p_2) - m_1 \overline{u}(p_1) \not{\varepsilon}(k) P_R u(p_2)\right] \qquad (6.13)
\end{aligned}
$$

and the decay width will be

$$\Gamma = \frac{1}{16\pi m_2^3}\;\left(m_2^2 - m_1^2\right)^3 \left(|C_L|^2 + |C_R|^2\right) \tag{6.14}$$

As the coefficient of $p_2 \cdot \varepsilon(k)$ only comes from the 3-point function (amplitude $M_1$) this justifies the usual procedure of just calculating that coefficient and forgetting about the self-energies (amplitudes $M_2$ and $M_3$). However these amplitudes are crucial for the cancellation of divergences and for gauge invariance. Now we will show the power of the automatic `FeynCalc` [2] program and calculate both the coefficients $C_{L,R}$ and $D_{L,R}$, showing the cancellation of the divergences and that the relations, Eqs. (6.11) and (6.12) needed for gauge invariance are satisfied. We start by writing the `mathematica` program:

```
(*********************** Program mueg-ns.m *************************)
(*
This program calculates the COMPLETE (both the 3 point amplitude and
the two self energy type on each external line) amplitudes for
\mu -> e \gamma when the fermion line in the loop is charged and the
neutral line is a scalar. The \mu has momentum p2 and mass m2, the
electron (p1,m1) and the photon momentum k. The momentum in the loop
is q.

The assumed vertices are,

1) Electron-Scalar-Fermion:

  Spinor[p1,m1] (AL P_L + AR P_R) Spinor [pf,mf]

2) Fermion-Scalar-Muon:

   Spinor[pf,mf] (BL P_L + BR P_R) Spinor [p2,m2]
*)

dm[mu_]:=DiracMatrix[mu,Dimension->D]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(*
SetOptions[{B0,B1,B00,B11},BReduce->True]
 *)

gA:= AL DiracMatrix[7] + AR DiracMatrix[6]
gB:= BL DiracMatrix[7] + BR DiracMatrix[6]
```

```
num1:=Spinor[p1,m1] . gA . (ds[q]+ds[p2]-ds[k]+mf) . ds[Polarization[k]]\
      . (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]


num2:=Spinor[p1,m1] . gA . (ds[q]+ds[p1]+mf) . gB . (ds[p1]+m2) . \
       ds[Polarization[k]] . Spinor[p2,m2]


num3:=Spinor[p1,m1] . ds[Polarization[k]] . (ds[p2]+m1) . gA . \
      (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]



SetOptions[OneLoop,Dimension->D]

amp1:=num1 \
FeynAmpDenominator[PropagatorDenominator[q+p2-k,mf], \
                   PropagatorDenominator[q+p2,mf], \
                   PropagatorDenominator[q,ms]]
amp2:=num2 \
FeynAmpDenominator[PropagatorDenominator[q+p1,mf], \
                   PropagatorDenominator[p2-k,m2],  \
                   PropagatorDenominator[q,ms]]
amp3:=num3 \
FeynAmpDenominator[PropagatorDenominator[p1+k,m1], \
                   PropagatorDenominator[q+p2,mf],  \
                   PropagatorDenominator[q,ms]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m1^2,ScalarProduct[p2,p2]->m2^2, \
        ScalarProduct[k,k]->0,ScalarProduct[p1,k]->(m2^2-m1^2)/2,\
        ScalarProduct[p2,k]->(m2^2-m1^2)/2, \
        ScalarProduct[p2,Polarization[k]]->p2epk, \
        ScalarProduct[p1,Polarization[k]]->p2epk}

(* Define the divergent part of the relevant PV functions*)

div={B0[m1^2,mf^2,ms^2]->Div,B0[m2^2,mf^2,ms^2]->Div, \
  B0[0,mf^2,ms^2]->Div,B0[0,mf^2,mf^2]->Div,B0[0,ms^2,ms^2]->Div}

res1:=(-I / Pi^2) OneLoop[q,amp1]
res2:=(-I / Pi^2) OneLoop[q,amp2]
res3:=(-I / Pi^2) OneLoop[q,amp3]
```

```
res:=res1+res2+res3 /. onshell

auxT1:= res1 /.onshell
auxT2:= PaVeReduce[auxT1]
auxT3:= auxT2 /. div
divT:=Simplify[Div*Coefficient[auxT3,Div]]

auxS1:= res2 + res3 /.onshell
auxS2:= PaVeReduce[auxS1]
auxS3:= auxS2 /. div
divS:=Simplify[Div*Coefficient[auxS3,Div]]

(* Check cancellation of divergences

   testdiv should be zero because divT=-divS

*)

testdiv:=Simplify[divT + divS]

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 8 Standard Matrix Elements.
To have a simpler result we substitute these elements by simpler
expressions (ME[1],...ME[8]). But they are not all independent. The
final result can just be written in terms of 4 Matrix Elements.

{StandardMatrixElement[p2epk u[p1,m1] . ga[6] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . ga[7] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . gs[k] . ga[6] . u[p2,m2]],

StandardMatrixElement[p2epk u[p1,m1] . gs[k] . ga[7] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[ep[k]] . ga[6] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[ep[k]] . ga[7] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[k] . gs[ep[k]] . ga[6] . u[p2,m2]],

StandardMatrixElement[u[p1,m1] . gs[k] . gs[ep[k]] . ga[7] . u[p2,m2]]}
*)
```

```
ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]
Set @@ {var, {ME[1],ME[2],ME[3],ME[4],ME[5],ME[6],ME[7],ME[8]}}
identities={ME[3]->-m1 ME[1] + m2 ME[2], ME[4]->-m1 ME[2] + m2 ME[1],
            ME[7]->-m1 ME[5] - m2 ME[6] + 2  ME[1],
            ME[8]->-m1 ME[6] - m2 ME[5] + 2  ME[2]}

ans2 =ans1 /. identities ;
ans=Simplify[ans2];

CR=Coefficient[ans,ME[1]]/2;
CL=Coefficient[ans,ME[2]]/2;
DR=Coefficient[ans,ME[5]];
DL=Coefficient[ans,ME[6]];

(* Test to see if we did not forget any term *)

test1:=Simplify[ans-2 CR*ME[1]-2 CL*ME[2]-DR*ME[5]-DL*ME[6]]


(* Test that the divergences cancel term by term *)

auxCL=PaVeReduce[CL] /. div ;
testdivCL:=Simplify[Coefficient[auxCL,Div]]

auxCR=PaVeReduce[CR] /. div ;
testdivCR:=Simplify[Coefficient[auxCR,Div]]

auxDL=PaVeReduce[DL] /. div ;
testdivDL:=Simplify[Coefficient[auxDL,Div]]

auxDR=PaVeReduce[DR] /. div ;
testdivDR:=Simplify[Coefficient[auxDR,Div]]

(* Test the gauge invariance relations *)
testGI1:=Simplify[PaVeReduce[(m2^2-m1^2)*CR - DR*m1 + DL*m2]]

testGI2:=Simplify[PaVeReduce[(m2^2-m1^2)*CL + DR*m2 - DL*m1]]

(********************** End Program mueg-ns.m  ************************)
```

We first do the tests. The output of `mathematica` is

```
(**********************  Mathematica output   ************************)
In[3]:= << FeynCalc.m

FeynCalc4.1.0.3b Type ?FeynCalc for help or visit
http://www.feyncalc.org

In[4]:= << mueg-ns.m

In[5]:= test1

Out[5]= 0

In[6]:= testdiv

Out[6]= 0

In[7]:= testdivCL

Out[7]= 0

In[8]:= testdivCR

Out[8]= 0

In[9]:= testdivDL

Out[9]= 0

In[10]:= testdivDR

Out[10]= 0

In[11]:= testGI1

Out[11]= 0

In[12]:= testGI2

Out[12]= 0
(******************  End of Mathematica output   ********************)
```

Now we obtain the results for $C_L$

```
(**********************  Mathematica output   ************************)
In[13]:= CL
                              2    2    2    2    2
Out[13]= (-4 AL BL mf C0[0, m2 , m1 , mf , mf , ms ] +

                             2    2      2    2    2
    4 AL BR m2 PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                             2    2      2    2    2
    4 AL BL mf PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                                2    2      2    2    2
    4 AR BL m1 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                2    2      2    2    2
    4 AL BR m2 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                2    2      2    2    2
    4 AL BR m2 PaVe[2, 2, {0, m1 , m2 }, {mf , mf , ms }]) / 4
```

and for $C_R$

```
In[15]:= CR
                              2    2    2    2    2
Out[15]= (-4 AR BR mf C0[0, m2 , m1 , mf , mf , ms ] +

                             2    2      2    2    2
    4 AR BL m2 PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                             2    2      2    2    2
    4 AR BR mf PaVe[2, {0, m1 , m2 }, {mf , mf , ms }] -

                                2    2      2    2    2
    4 AL BR m1 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                2    2      2    2    2
    4 AR BL m2 PaVe[1, 2, {0, m1 , m2 }, {mf , mf , ms }] +

                                2    2      2    2    2
    4 AR BL m2 PaVe[2, 2, {0, m1 , m2 }, {mf , mf , ms }]) / 4
(****************** End of  Mathematica output  *********************)
```

The expressions for $D_{L,R}$ are quite complicated. They are not normally calculated because they can be related to $C_{L,R}$ by gauge invariance. However the power of this automatic program can be illustrated by asking for these functions. As they are very long we calculate them by pieces. We just calculate $D_L$ because one can easily check that $D_R = D_L(L \leftrightarrow R)$.

```
(**********************  Mathematica output   ************************)

In[12]:= Coefficient[PaVeReduce[DL],AL BL]

                   2    2    2                2    2    2
         m1 mf B0[m1 , mf , ms ]   m1 mf B0[m2 , mf , ms ]
Out[12]= ----------------------- - ----------------------- +
                  2     2                   2     2
                m1  - m2                  m1  - m2

            2    2       2    2    2
  m1 mf C0[m1 , m2 , 0, mf , ms , mf ]

In[13]:= Coefficient[PaVeReduce[DL],AL BR]

            2     2          2    2
         (mf  - ms ) B0[0, mf , ms ]
Out[13]= -------------------------- -
                   2 m1 m2

     2           2         2       2    2    2
  (m1  m2 - m2 mf  + m2 ms ) B0[m1 , mf , ms ]
  -------------------------------------------- +
                         2     2
                2 m1 (m1  - m2 )

        2        2         2       2    2    2
  (m1 m2  - m1 mf  + m1 ms ) B0[m2 , mf , ms ]
  --------------------------------------------
                         2     2
                2 m2 (m1  - m2 )

In[14]:= Coefficient[PaVeReduce[DL],AR BL]

                      2          2       2    2    2
         1   (-2 m1 mf  + 2 m1 ms ) B0[m1 , mf , ms ]
Out[14]= - - --------------------------------------- +
         2                        2     2
                         2 m1 (m1  - m2 )

        2           2       2    2    2
  (-2 m2 mf  + 2 m2 ms ) B0[m2 , mf , ms ]
  ---------------------------------------
                       2     2
              2 m2 (m1  - m2 )
```

```
     2       2    2        2    2    2
  + mf  C0[m1 , m2 , 0, mf , ms , mf ]

In[15]:= Coefficient[PaVeReduce[DL],AR BR]

                    2    2    2                2    2    2
         m2 mf B0[m1 , mf , ms ]   m2 mf B0[m2 , mf , ms ]
Out[15]= ----------------------- - -----------------------
                  2     2                    2     2
                m1  - m2                   m1  - m2

              2    2        2    2    2
  + m2 mf C0[m1 , m2 , 0, mf , ms , mf ]

(****************** End of  Mathematica output  *********************)
```

From these expressions one can immediately verify that the divergences cancel in $D_{L,R}$ and that they are not present in $C_{L,R}$. To finish this section we just rewrite the $C_{L,R}$ in our usual notation. We get

$$\begin{aligned}
C_L \;&=\; \frac{e\,Q_\ell}{16\pi^2}\Big[A_L B_L m_F\left(-C_0(0,m_2^2,m_1^2,m_F^2,m_F^2,m_S^2)-C_2(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\right)\\
&\qquad +A_L B_R m_2\left(C_2(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)+C_{12}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\right.\\
&\qquad\qquad \left.+C_{22}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\right)\\
&\qquad + A_R B_L m_1\, C_{12}(0,m_1^2,m_2^2,m_F^2,m_F^2,m_S^2)\Big] \qquad (6.15)\\
C_R \;&=\; C_L(L\leftrightarrow R) \qquad (6.16)
\end{aligned}$$

These equations are in agreement with Eqs. (32-34) and Eqs. (38-39) of Ref. [9], although some work has to be done in order to verify that[11]. This has to do with the fact that the PV decomposition functions are not independent (see the Appendix for further details on this point). We can however use the power of `FeynCalc` to verify this. We list below a simple program to accomplish that.

```
(*******************  Program lavoura-ns.m  *************************)
(*
This program tests the results of my program mueg-ns.m against the
results obtained by L. Lavoura (hepph/0302221).
*)

(* First load FeynCalc.m and mueg-ns.m *)
```

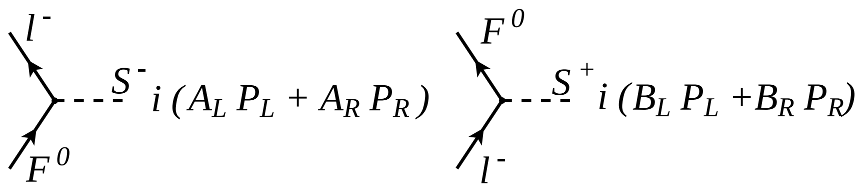
[11] An important difference between our conventions and those of Ref. [9] is that $p_1$ and $p_2$ (and obviously $m_1$ and $m_2$) are interchanged.

```
<< FeynCalc.m
<< mueg-ns.m

(*
Now write Lavoura integrals in the notation of FeynCalc. Be careful
with the order of the entries.
*)

c1:=PaVe[1,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
c2:=PaVe[2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

d1:=PaVe[1,1,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]
d2:=PaVe[2,2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

f:=PaVe[1,2,{m2^2,0,m1^2},{ms^2,mf^2,mf^2}]

(* Write Eqs. (32)-(34) of hepph/0302221 in our notation *)

k1:=PaVeReduce[m2*(c1+d1+f)]
k2:=PaVeReduce[m1*(c2+d2+f)]
k3:=PaVeReduce[mf*(c1+c2)]

(*
Now test the results. For this we should use the equivalences:

\rho    -> AL BR
\lambda -> AR BL
\xi     -> AR BR
\nu     -> AL BL

*)

testCLALBR:=Simplify[PaVeReduce[Coefficient[CL, AL BR]-k1]]
testCLARBL:=Simplify[PaVeReduce[Coefficient[CL, AR BL]-k2]]
testCLALBL:=Simplify[PaVeReduce[Coefficient[CL, AL BL]-k3]]

testCRALBR:=Simplify[PaVeReduce[Coefficient[CR, AL BR]-k2]]
testCRARBL:=Simplify[PaVeReduce[Coefficient[CR, AR BL]-k1]]
testCRARBR:=Simplify[PaVeReduce[Coefficient[CR, AR BR]-k3]]

(***************** End of Program lavoura-ns.m  *********************)
```

One can easily check that the output of the six tests is zero, showing the equivalence between our results. And all this is done in a few seconds.

## 6.2 Charged scalar neutral fermion loop

We consider now the case of the scalar being charged and the scalar neutral. The general case of both charged [9] can also be easily implemented, but for simplicity we do not consider it here. The couplings are now




and the diagrams contributing to the process are given in Fig. 14, where all the denomi-

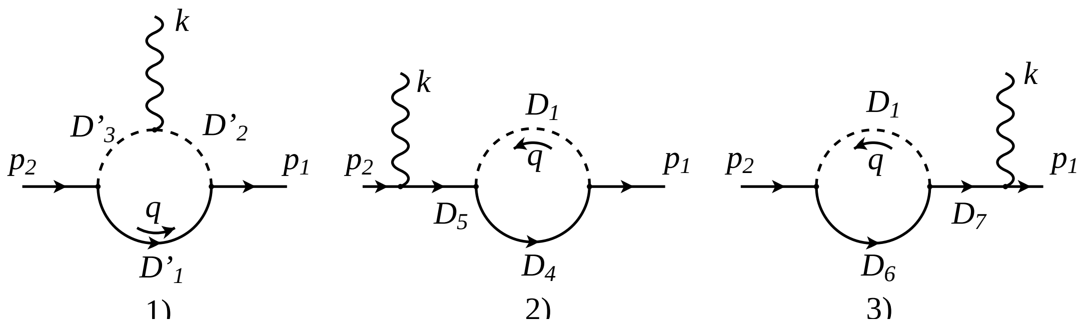



Figure 14:

nators are as in Eqs. (6.2)- (6.4) except that

$$D_1' = q^2 - m_F^2 \quad ; \quad D_2' = (q-p_1)^2 - m_S^2 \quad ; \quad D_3' = (q-p_1-k)^2 - m_S^2 \tag{6.17}$$

Also the coupling of the photon to the charged scalar is, in our notation,

$$-ie\, Q_\ell\,(-2q+p_1+p_2)^\mu \tag{6.18}$$

The procedure is very similar to the neutral scalar case and we just present here the `mathematica` program and the final result. All the checks of finiteness and gauge invariance can be done as before.

```
(*********************** Program mueg-cs.m *************************)
(*
This program calculates the COMPLETE (both the 3 point amplitude and
the two self energy type on each external line) amplitudes for
\mu -> e \gamma when the fermion line in the loop is neutral and the
charged line is a scalar. The \mu has momentum p2 and mass m2, the
electron (p1,m1) and the photon momentum k. The momentum in the loop
is q.
```

```
The assumed vertices are,

1) Electron-Scalar-Fermion:

  Spinor[p1,m1] (AL P_L + AR P_R) Spinor [pf,mf]

2) Fermion-Scalar-Muon:

   Spinor[pf,mf] (BL P_L + BR P_R) Spinor [p2,m2]
*)

dm[mu_]:=DiracMatrix[mu,Dimension->4]
dm[5]:=DiracMatrix[5]
ds[p_]:=DiracSlash[p]
mt[mu_,nu_]:=MetricTensor[mu,nu]
fv[p_,mu_]:=FourVector[p,mu]
epsilon[a_,b_,c_,d_]:=LeviCivita[a,b,c,d]
id[n_]:=IdentityMatrix[n]
sp[p_,q_]:=ScalarProduct[p,q]
li[mu_]:=LorentzIndex[mu]
L:=dm[7]
R:=dm[6]

(*
SetOptions[{B0,B1,B00,B11},BReduce->True]
 *)

gA:= AL DiracMatrix[7] + AR DiracMatrix[6]
gB:= BL DiracMatrix[7] + BR DiracMatrix[6]

num1:= Spinor[p1,m1] . gA . (ds[q]+mf) . gB . Spinor[p2,m2]  \
PolarizationVector[k,mu] ( - 2 fv[q,mu] + fv[p1,mu] + fv[p2,mu] )

num11:=DiracSimplify[num1];

num2:=Spinor[p1,m1] . gA . (ds[q]+ds[p1]+mf) . gB . (ds[p1]+m2) . \
       ds[Polarization[k]] . Spinor[p2,m2]


num3:=Spinor[p1,m1] . ds[Polarization[k]] . (ds[p2]+m1) . gA . \
      (ds[q]+ds[p2]+mf) . gB . Spinor[p2,m2]



SetOptions[OneLoop,Dimension->D]
```

```
amp1:=num1 \
FeynAmpDenominator[PropagatorDenominator[q,mf],\
                   PropagatorDenominator[q-p1,ms],\
                   PropagatorDenominator[q-p1-k,ms]]

amp2:=num2 \
FeynAmpDenominator[PropagatorDenominator[q+p1,mf], \
                   PropagatorDenominator[p2-k,m2],  \
                   PropagatorDenominator[q,ms]]
amp3:=num3 \
FeynAmpDenominator[PropagatorDenominator[p1+k,m1], \
                   PropagatorDenominator[q+p2,mf],  \
                   PropagatorDenominator[q,ms]]

(* Define the on-shell kinematics *)

onshell={ScalarProduct[p1,p1]->m1^2,ScalarProduct[p2,p2]->m2^2, \
         ScalarProduct[k,k]->0,ScalarProduct[p1,k]->(m2^2-m1^2)/2, \
         ScalarProduct[p2,k]->(m2^2-m1^2)/2, \
         ScalarProduct[p2,Polarization[k]]->p2epk, \
         ScalarProduct[p1,Polarization[k]]->p2epk}

(* Define the divergent part of the relevant PV functions*)

div={B0[m1^2,mf^2,ms^2]->Div,B0[m2^2,mf^2,ms^2]->Div, \
     B0[0,mf^2,ms^2]->Div,B0[0,mf^2,mf^2]->Div,B0[0,ms^2,ms^2]->Div}

res1:=(-I / Pi^2) OneLoop[q,amp1]
res2:=(-I / Pi^2) OneLoop[q,amp2]
res3:=(-I / Pi^2) OneLoop[q,amp3]

res:=res1+res2+res3 /. onshell


auxT1:= res1 /.onshell
auxT2:= PaVeReduce[auxT1]
auxT3:= auxT2 /. div
divT:=Simplify[Div*Coefficient[auxT3,Div]]

auxS1:= res2 + res3 /.onshell
auxS2:= PaVeReduce[auxS1]
auxS3:= auxS2 /. div
divS:=Simplify[Div*Coefficient[auxS3,Div]]
```

```
(* Check cancellation of divergences

   testdiv should be zero because divT=-divS

*)

testdiv:=Simplify[divT + divS]

(* Extract the different Matrix Elements

Mathematica writes the result in terms of 6 Standard Matrix Elements.
To have a simpler result we substitute these elements by simpler
expressions (ME[1],...ME[6]). Not all are independent.

{StandardMatrixElement[p2epk u[p1, m1] . ga[6] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . ga[7] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . gs[k] . ga[6] . u[p2, m2]],

StandardMatrixElement[p2epk u[p1, m1] . gs[k] . ga[7] . u[p2, m2]],

StandardMatrixElement[u[p1, m1] . gs[ep[k]] . ga[6] . u[p2, m2]],

StandardMatrixElement[u[p1, m1] . gs[ep[k]] . ga[7] . u[p2, m2]]}
*)

ans1=res;
var=Select[Variables[ans1],(Head[#]===StandardMatrixElement)&]

Set @@ {var, {ME[1],ME[2],ME[3],ME[4],ME[5],ME[6]}}
identities={ME[3]->-m1 ME[1] + m2 ME[2],ME[4]->-m1 ME[2] + m2 ME[1]}

ans2 =ans1 /. identities ;
ans=Simplify[ans2];


CR=Coefficient[ans,ME[1]]/2;
CL=Coefficient[ans,ME[2]]/2;
DR=Coefficient[ans,ME[5]];
DL=Coefficient[ans,ME[6]];

(* Test to see if we did not forget any term *)

test1:=Simplify[ans-2*CR*ME[1]-2*CL*ME[2]-DR*ME[5]-DL*ME[6]]
```

```
(* Test that the divergences cancel term by term *)

auxCL:=PaVeReduce[CL] /. div ;
testdivCL:=Simplify[Coefficient[auxCL,Div]]

auxCR:=PaVeReduce[CR] /. div ;
testdivCR:=Simplify[Coefficient[auxCR,Div]]

auxDL:=PaVeReduce[DL] /. div ;
testdivDL:=Simplify[Coefficient[auxDL,Div]]

auxDR:=PaVeReduce[DR] /. div ;
testdivDR:=Simplify[Coefficient[auxDR,Div]]

(* Test the gauge invariance relations *)
testGI1:=PaVeReduce[(m2^2-m1^2)*CR - DR*m1 + DL*m2]

testGI2:=PaVeReduce[(m2^2-m1^2)*CL + DR*m2 - DL*m1]

(********************* End Program mueg-cs.m  **********************)
```

Note that although these programs look large, in fact they are very simple. Most of it are comments and tests. The output of this program gives,

```
(********************  Mathematica output   ***********************)
In[3]:= CL

                             2    2    2    2    2
Out[3]= (-2 AR BL m1 C0[0, m1 , m2 , ms , ms , mf ] -

                          2        2       2    2    2
    2 AR BL m1 PaVe[1, {m1 , 0, m2 }, {mf , ms , ms }] -

                          2    2           2    2    2
    4 AR BL m1 PaVe[1, {m1 , m2 , 0}, {ms , mf , ms }] -

                          2    2           2    2    2
    2 AL BL mf PaVe[1, {m1 , m2 , 0}, {ms , mf , ms }] -

                          2        2       2    2    2
    2 AL BR m2 PaVe[2, {m1 , 0, m2 }, {mf , ms , ms }] -

                          2    2           2    2    2
    2 AR BL m1 PaVe[2, {m1 , m2 , 0}, {ms , mf , ms }] +
```

```
                             2    2          2    2    2
   2 AL BR m2 PaVe[2, {m1 , m2 , 0}, {ms , mf , ms }] -

                                2    2          2    2    2
   2 AR BL m1 PaVe[1, 1, {m1 , m2 , 0}, {ms , mf , ms }] -

                                2    2          2    2    2
   2 AR BL m1 PaVe[1, 2, {m1 , m2 , 0}, {ms , mf , ms }] +

                                2    2          2    2    2
   2 AL BR m2 PaVe[1, 2, {m1 , m2 , 0}, {ms , mf , ms }]) / 2

(****************** End of  Mathematica output  *******************)
```

To finish this section we just rewrite the $C_{L,R}$ in our usual notation. We get

$$
\begin{aligned}
C_L \;=\; & \frac{e\,Q_\ell}{16\pi^2}\Big[A_L B_L m_F\left(-C_1(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\right) \\
& \quad + A_L B_R m_2\Big(-C_2(m_1^2,0,m_2^2,m_F^2,m_S^2,m_S^2)+C_2(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2) \\
& \qquad\qquad + C_{12}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\Big) \\
& \quad + A_R B_L m_1\Big(-C_0(0,m_1^2,m_2^2,m_S^2,m_S^2,m_F^2)-C_1(m_1^2,0,m_2^2,m_F^2,m_S^2,m_S^2) \\
& \qquad\qquad -2C_1(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)-C_2(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2) \\
& \qquad\qquad -C_{11}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)-C_{12}(m_1^2,m_2^2,0,m_S^2,m_F^2,m_S^2)\Big)\Big] \\
C_R \;=\; & C_L(L\leftrightarrow R) \qquad (6.19)
\end{aligned}
$$

It is left as an exercise to write a `mathematica` program that proves that these equations are in agreement with Eqs. (35-37) and Eqs. (38-39) of Ref. [9].

# A Useful techniques and formulas for the renormalization

## A.1 Parameter $\mu$

The reason for the $\mu$ parameter introduced in section 2.1 is the following. In dimension $d = 4 - \epsilon$, the fields $A_\mu$ and $\psi$ have dimensions given by the kinetic terms in the action,

$$\int d^d x \left[ -\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \overline{\psi} \gamma \cdot \partial \psi \right] \tag{A.1}$$

We have therefore

$$\begin{array}{lll} 0 = -d + 2 + 2[A_\mu] & \Rightarrow [A_\mu] & = \frac{1}{2}(d-2) = 1 - \frac{\epsilon}{2} \\ 0 = -d + 1 + 2[\psi] & \Rightarrow [\psi] & = \frac{1}{2}(d-1) = \frac{3}{2} - \frac{\epsilon}{2} \end{array} \tag{A.2}$$

Using these dimensions in the interaction term

$$S_I = \int d^d x \; e \overline{\psi} \gamma_\mu \psi A^\mu \tag{A.3}$$

we get

$$\begin{aligned} [S_I] &= -d + [e] + 2[\psi] + [A] \\ &= -4 + \epsilon + [e] + 3 - \epsilon + 1 - \frac{\epsilon}{2} \\ &= [e] - \frac{\epsilon}{2} \end{aligned} \tag{A.4}$$

Therefore, if we want the action to be dimensionless (remember that we use the system where $\hbar = c = 1$), we have to set

$$[e] = \frac{\epsilon}{2} \tag{A.5}$$

We see then that in dimensions $d \neq 4$ the coupling constant has dimensions. As it is more convenient to work with a dimensionless coupling constant we introduce a parameter $\mu$ with dimensions of a mass and in $d \neq 4$ we will make the substitution

$$e \rightarrow e \mu^{\frac{\epsilon}{2}} \qquad (\epsilon = 4 - d) \tag{A.6}$$

while keeping $e$ dimensionless.

## A.2 Feynman parameterization

The most general form for a *1–loop* é [12]

$$\hat{T}_n^{\mu_1 \cdots \mu_p} \equiv \int \frac{d^d k}{(2\pi)^d} \frac{k^{\mu_1} \cdots k^{\mu_p}}{D_0 D_1 \cdots D_{n-1}} \tag{A.7}$$

where

$$D_i = (k + r_i)^2 - m_i^2 + i\epsilon \tag{A.8}$$

and the momenta $r_i$ are related with the external momenta (all taken to be incoming) through the relations,

$$\begin{aligned} r_j &= \sum_{i=1}^{j} p_i \quad ; \quad j = 1, \ldots, n-1 \\ r_0 &= \sum_{i=1}^{n} p_i = 0 \end{aligned} \tag{A.9}$$

as indicated in Fig. (15). In these expressions there appear in the denominators products

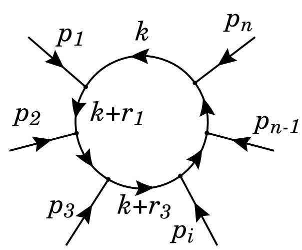



Figure 15:

of the denominators of the propagators of the particles in the loop. It is convenient to combine these products in just one common denominator. This is achieved by a technique due to Feynman. Let us exemplify with two denominators.

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax + b(1-x)]^2} \tag{A.10}$$

The proof is trivial. In fact

$$\int dx \, \frac{1}{[ax + b(1-x)]^2} = \frac{x}{b\,[(a-b)x + b]} \tag{A.11}$$

and therefore Eq. (A.10) immediately follows. Taking successive derivatives with respect to $a$ and $b$ we get

$$\frac{1}{a^p b^q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 dx \, \frac{x^{p-1}(1-x)^{q-1}}{[ax + b(1-x)]^{p+q}} \tag{A.12}$$

[12] We introduce here the notation $\hat{T}$ to distinguish from a more standard notation that will be explained in subsection 3.

and using induction we obtain a general formula

$$\frac{1}{a_1 a_2 \cdots a_n} = \Gamma(n) \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots$$
$$\int_0^{1-x_1-\cdots-x_{n-1}} \frac{dx_{n-1}}{[a_1 x_1 + a_2 x_2 + \cdots + a_n(1 - x_1 - \cdots - x_{n-1})]^n} \qquad \text{(A.13)}$$

Before closing the section let us give an example that will be useful in the self-energy case. Consider the situation with the kinematics described in Fig. (16).

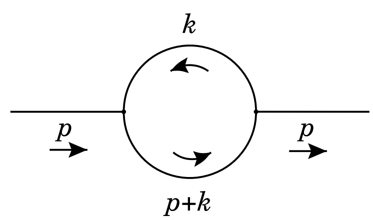



Figure 16:

We get

$$\begin{aligned}
I &= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k+p)^2 - m_1^2 + i\epsilon]\,[k^2 - m_2^2 + i\epsilon]} \\
&= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 + 2p \cdot k\, x + p^2\, x - m_1^2\, x - m_2^2\, (1-x) + i\epsilon]^2} \\
&= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[k^2 + 2P \cdot k - M^2 + i\epsilon]^2} \\
&= \int_0^1 dx \int \frac{d^d k}{(2\pi)^d} \frac{1}{[(k+P)^2 - P^2 - M^2 + i\epsilon]^2}
\end{aligned} \qquad \text{(A.14)}$$

where in the last line we have completed the square in the term with the loop momenta $k$. The quantities $P$ and $M^2$ are, in this case, defined by

$$P = xp \qquad \text{(A.15)}$$

and

$$M^2 = -x\, p^2 + m_1^2\, x + m_2^2\, (1-x) \qquad \text{(A.16)}$$

They depend on the masses, external momenta and Feynman parameters, but not in the loop momenta. Now changing variables $k \to k - P$ we get rid of the linear terms in $k$ and finally obtain

$$I = \int_0^1 dx \int \frac{d^d x}{(2\pi)^d} \frac{1}{[k^2 - C + i\epsilon]^2} \qquad \text{(A.17)}$$

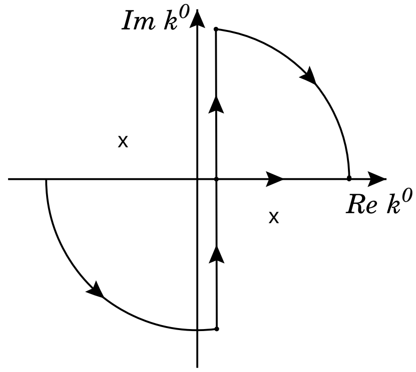



Figure 17:

where $C$ is independent of the loop momenta $k$ and it is given by

$$C = P^2 + M^2 \tag{A.18}$$

Notice that the $i\epsilon$ factors will add correctly and can all be put as in Eq. (A.17).

## A.3 Wick Rotation

From the example of the last section we can conclude that all the scalar integrals can be reduced to the form

$$I_{r,m} = \int \frac{d^d x}{(2\pi)^d} \, \frac{k^{2^r}}{[k^2 - C + i\epsilon]^m} \tag{A.19}$$

It is also easy to realize that also all the tensor integrals can be obtained from the scalar integrals. For instance

$$\begin{aligned} &\int \frac{d^d x}{(2\pi)^d} \, \frac{k^\mu}{[k^2 - C + i\epsilon]^m} = 0 \\ &\int \frac{d^d x}{(2\pi)^d} \, \frac{k^\mu k^\nu}{[k^2 - C + i\epsilon]^m} = \frac{1}{d} \, g^{\mu\nu} \int d^d x \, \frac{k^2}{[k^2 - C + i\epsilon]^m} \end{aligned} \tag{A.20}$$

and so on. Therefore the integrals $I_{r,m}$ are the important quantities to evaluate. We will consider that $C > 0$. The case $C < 0$ can be done by analytical continuation of the final formula for $C > 0$.
To evaluate the integral $I_{r,m}$ we will use integration in the complex plane of the variable $k^0$ as described in Fig. 17. We can then write

$$I_{r,m} = \int \frac{d^{d-1}k}{(2\pi)^d} \int dk^0 \, \frac{k^{2^r}}{\left[k_0^2 - |\vec{k}|^2 - C + i\epsilon\right]^m} \tag{A.21}$$

The function under the integral has poles for

$$k^0 = \pm \left( \sqrt{|\vec{k}|^2 + C} - i\epsilon \right) \tag{A.22}$$

has shown in Fig. 17. Using the properties of functions of complex variables (Cauchy theorem) we can deform the contour, changing the integration from the real to the imaginary axis plus the two arcs at infinity. This can be done because in deforming the contour we do not cross any pole. Notice the importance of the $i\epsilon$ prescription to be able to do this. The contribution from the arcs at infinity vanishes in dimension sufficiently low for the integral to converge, as we assume in dimensional regularization. We have then changed the integration along the real axis into an integration along the imaginary axis in the plane of the complex variable $k^0$. If we write

$$k^0 = i k_E^0 \qquad \text{com} \qquad \int_{-\infty}^{+\infty} dk^0 \to i \int_{-\infty}^{+\infty} dk_E^0 \tag{A.23}$$

and $k^2 = (k^0)^2 - |\vec{k}|^2 = -(k_E^0)^2 - |\vec{k}|^2 \equiv -k_E^2$, where $k_E = (k_E^0, \vec{k})$ is an euclidean vector. By this we mean that is we calculate the scalar product using the euclidean metric diag$(+,+,+,+)$,

$$k_E^2 = (k_E^0)^2 + |\vec{k}|^2 \tag{A.24}$$

We can them write

$$I_{r,m} = i(-1)^{r-m} \int \frac{d^d k_E}{(2\pi)^d} \, \frac{k_E^{2r}}{[k_E^2 + C]^m} \tag{A.25}$$

where we do not need the $i\epsilon$ because the denominator is strictly positive ($C > 0$). This procedure is known as *Wick Rotation*. We note that the Feynman prescription for the propagators that originated the $i\epsilon$ rule for the denominators is crucial for the Wick rotation to be possible.

## A.4 Scalar integrals in dimensional regularization

We have seen in the last section that the scalar integrals to be calculated with dimensional regularization had the general form of Eq. (A.25). We are now going to find a general formula for $I_{r,m}$. We begin by writing

$$\int d^d k_E = \int d\overline{k} \; \overline{k}^{\,d-1} \, d\Omega_{d-1} \tag{A.26}$$

where $\overline{k} = \sqrt{(k_E^0)^2 + |\vec{k}|^2}$ is the length of the vector $k_E$ in the euclidean space in $d$ dimensions and $d\Omega_{d-1}$ is the solid angle that generalizes spherical coordinates in that euclidean space. The angles are defined by

$$k_E = \overline{k}(\cos\theta_1, \sin\theta_1\cos\theta_2, \sin\theta_1\sin\theta_2, \sin\theta_1\sin\theta_2\cos\theta_3, \ldots, \sin\theta_1\cdots\sin\theta_{d-1}) \tag{A.27}$$

We can then write

$$\int d\Omega_{d-1} = \int_0^{\pi} \sin\theta_1^{d-2} \, d\theta_1 \cdots \int_0^{2\pi} d\theta_{d-1} \tag{A.28}$$

Using now

$$\int_0^{\pi} \sin\theta^m \, d\theta = \sqrt{\pi} \, \frac{\Gamma(\frac{m+1}{2})}{\Gamma(\frac{m+2}{2})} \tag{A.29}$$

where $\Gamma(z)$ is the gamma function (see section A.6) we get

$$\int d\Omega_{d-1} = 2\,\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})} \tag{A.30}$$

The integration in $\overline{k}$ is done using the result

$$\int_0^\infty dx\ \frac{x^p}{(x^n+a^n)^q} = \pi(-1)^{q-1}\,a^{p+1-nq}\ \frac{\Gamma(\frac{p+1}{n})}{n\sin(\pi\,\frac{p+1}{n})\,\Gamma(\frac{p+1}{2}-q+1)} \tag{A.31}$$

and we finally get

$$I_{r,m} = iC^{r-m+\frac{d}{2}}\ \frac{(-1)^{r-m}}{(4\pi)^{\frac{d}{2}}}\ \frac{\Gamma(r+\frac{d}{2})}{\Gamma(\frac{d}{2})}\ \frac{\Gamma(m-r-\frac{d}{2})}{\Gamma(m)} \tag{A.32}$$

Before ending the section we note that the integral representation for $I_{r,m}$, Eq. (A.19), is valid only for $d<2(m-r)$ to ensure convergence when $\overline{k}\to\infty$. However the final form in Eq. (A.32) can be analytically continued for all values of $d$ except for those where the function $\Gamma(m-r-d/2)$ has poles, that is for (see section A.6),

$$m-r-\frac{d}{2}\neq 0,-1,-2,\ldots \tag{A.33}$$

For the application in dimensional regularization it is convenient to rewrite Eq. (A.32) using the relation $d=4-\epsilon$. we get

$$I_{r,m} = i\frac{(-1)^{r-m}}{(4\pi)^2}\ \left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}}\ C^{2+r-m}\ \frac{\Gamma(2+r-\frac{\epsilon}{2})}{\Gamma(2-\frac{\epsilon}{2})}\ \frac{\Gamma(m-r-2+\frac{\epsilon}{2})}{\Gamma(m)} \tag{A.34}$$

## A.5 Tensor integrals in dimensional regularization

We are frequently faced with the task of evaluating the tensor integrals of the form of Eq. (A.7),

$$\hat{T}_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d}\ \frac{k^{\mu_1}\cdots k^{\mu_p}}{D_0D_1\cdots D_{n-1}} \tag{A.35}$$

The first step is to reduce to one common denominator using the Feynman parameterization technique. The result is,

$$\begin{aligned}
\hat{T}_n^{\mu_1\cdots\mu_p} &= \Gamma(n)\int_0^1 dx_1\cdots\int_0^{1-x_1-\cdots-x_{n-1}} dx_{n-1}\int\frac{d^dk}{(2\pi)^d}\ \frac{k^{\mu_1}\cdots k^{\mu_p}}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^n}\\
&= \Gamma(n)\int_0^1 dx_1\cdots\int_0^{1-x_1-\cdots-x_{n-1}} dx_{n-1}\,I_n^{\mu_1\cdots\mu_p}
\end{aligned} \tag{A.36}$$

where we have defined

$$I_n^{\mu_1\cdots\mu_p} \equiv \int \frac{d^dk}{(2\pi)^d}\ \frac{k^{\mu_1}\cdots k^{\mu_p}}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^n} \tag{A.37}$$

that we call, from now on, the tensor integral. In principle all these integrals can be written in terms of scalar integrals. It is however convenient to have a general formula for them. This formula can be obtained by noticing that

$$\frac{\partial}{\partial P^\mu}\,\frac{1}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^n}=-n\,\frac{2k_\mu}{\left[k^2+2k\cdot P-M^2+i\epsilon\right]^{n+1}} \tag{A.38}$$

Using the last equation one can write the final result

$$I_n^{\mu_1\cdots\mu_p}=\frac{i}{16\pi^2}\,\frac{(4\pi)^{\epsilon/2}}{\Gamma(n)}\int_0^\infty\frac{dt}{(2t)^p}\;t^{n-3+\epsilon/2}\;\frac{\partial}{\partial P_{\mu_1}}\cdots\frac{\partial}{\partial P_{\mu_p}}\,e^{-t\,C} \tag{A.39}$$

where $C=P^2+M^2$. After doing the derivatives the remaining integrals can be done using the properties of the $\Gamma$ function (see section A.6). Notice that $P$, $M^2$ and therefore also $C$ depend not only in the Feynman parameters but also in the exterior momenta. The advantage of having a general formula is that it can be programmed [10] and all the integrals can then be obtained automatically.

## A.6 $\Gamma$ function and useful relations

The $\Gamma$ function is defined by the integral

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}dt \tag{A.40}$$

or equivalently

$$\int_0^\infty t^{z-1}e^{-\mu t}dt=\mu^{-z}\Gamma(z) \tag{A.41}$$

The function $\Gamma(z)$ has the following important properties

$$\begin{aligned}\Gamma(z+1)&=z\Gamma(z)\\ \Gamma(n+1)&=n!\end{aligned} \tag{A.42}$$

Another related function is the logarithmic derivative of the $\Gamma$ function, with the properties,

$$\psi(z)=\frac{d}{dz}\ln\Gamma(z) \tag{A.43}$$

$$\psi(1)=-\gamma \tag{A.44}$$

$$\psi(z+1)=\psi(z)+\frac{1}{z} \tag{A.45}$$

where $\gamma$ is the Euler constant. The function $\Gamma(z)$ has poles for $z=0,-1,-2,\cdots$. Near the pole $z=-m$ we have

$$\Gamma(z) = \frac{(-1)^m}{m!}\frac{1}{m+z} + \frac{(-1)^m}{m!}\psi(m+1) + O(z+m) \tag{A.46}$$

From this we conclude that when $\epsilon \to 0$

$$\Gamma\left(\frac{\epsilon}{2}\right) = \frac{2}{\epsilon} + \psi(1) + O(\epsilon) \qquad \Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{2}{\epsilon} + \psi(n+1) + 1\right] \tag{A.47}$$

and

$$\Gamma(1+\epsilon) = 1 - \gamma\epsilon + \left(\gamma^2 + \frac{\pi^2}{6}\right)\frac{\epsilon^2}{2!} + \cdots \quad , \quad \epsilon \to 0 \tag{A.48}$$

Using these results we can expand our integrals in powers of $\epsilon$ and separate the divergent and finite parts. For instance for the one of the integrals of the self-energy,

$$\begin{aligned} I_{0,2} &= \frac{i}{(4\pi)^2}\left(\frac{4\pi}{C}\right)^{\frac{\epsilon}{2}}\frac{2\,\Gamma(1+\frac{\epsilon}{2})}{\epsilon} \\ &= \frac{i}{16\pi^2}\left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln C + O(\epsilon)\right] \\ &= \frac{i}{16\pi^2}\left[\Delta_\epsilon - \ln C + O(\epsilon)\right] \end{aligned} \tag{A.49}$$

where we have introduced the notation

$$\Delta_\epsilon = \frac{2}{\epsilon} - \gamma + \ln 4\pi \tag{A.50}$$

for a combination that will appear in all expressions.

## A.7 Explicit formulas for the *1–loop* integrals

Although we have presented in the previous sections the general formulas for all the integrals that appear in *1–loop*, Eqs. (A.34) and (A.39), in practice it is convenient to have expressions for the most important cases with the expansion on the $\epsilon$ already done. The results presented below were generated with the `Mathematica` package `OneLoop` [10] from the general expressions. In these results the integration on the Feynman parameters has still to be done. This is in general a difficult problem and we have presented in section 3 an alternative way of expressing these integrals, more convenient for a numerical evaluation.

### A.7.1 *Tadpole* integrals

With the definitions of Eqs. (A.34) and (A.39) we get

$$I_{0,1} = \frac{i}{16\pi^2}C(1 + \Delta_\epsilon - \ln C) \tag{A.51}$$

$$I_1^\mu = 0 \tag{A.52}$$

$$I_1^{\mu\nu} = \frac{i}{16\pi^2}\frac{1}{8}C^2\, g^{\mu\nu}(3 + 2\Delta_\epsilon - 2\ln C) \tag{A.53}$$

where for the *tadpole* integrals

$$P = 0 \quad ; \quad C = m^2 \tag{A.54}$$

because there are no Feynman parameters and there is only one mass. In this case the above results are final.

### A.7.2 *Self–Energy* integrals

For the integrals with two denominators we get,

$$I_{0,2} = \frac{i}{16\pi^2} \left(\Delta_\epsilon - \ln C\right) \tag{A.55}$$

$$I_2^\mu = \frac{i}{16\pi^2} (-\Delta_\epsilon + \ln C) P^\mu \tag{A.56}$$

$$I_2^{\mu\nu} = \frac{i}{16\pi^2} \frac{1}{2} \left[ C g^{\mu\nu} (1 + \Delta_\epsilon - \ln C) + 2(\Delta_\epsilon - \ln C) P^\mu P^\nu \right] \tag{A.57}$$

$$I_2^{\mu\nu\alpha} = \frac{i}{16\pi^2} \frac{1}{2} \Big[ - C g^{\mu\nu} (1 + \Delta_\epsilon - \ln C) P^\alpha - C g^{\nu\alpha} (1 + \Delta_\epsilon - \ln C) P^\mu \tag{A.58}$$

$$- C g^{\mu\alpha} (1 + \Delta_\epsilon - \ln C) P^\nu - (2\Delta_\epsilon P^\alpha P^\mu - 2 \ln C P^\alpha P^\mu) P^\nu \Big] \tag{A.59}$$

where, with the notation and conventions of Fig. (15), we have

$$P^\mu = x\, r_1^\mu \quad ; \quad C = x^2\, r_1^2 + (1-x)\, m_0^2 + x\, m_1^2 - x\, r_1^2 \tag{A.60}$$

### A.7.3 *Triangle* integrals

For the integrals with three denominators we get,

$$I_{0,3} = \frac{i}{16\pi^2} \frac{-1}{2C} \tag{A.61}$$

$$I_3^\mu = \frac{i}{16\pi^2} \frac{1}{2C}\, P^\mu \tag{A.62}$$

$$I_3^{\mu\nu} = \frac{i}{16\pi^2} \frac{1}{4C} \left[ C g^{\mu\nu} (\Delta_\epsilon - \ln C) - 2 P^\mu P^\nu \right] \tag{A.63}$$

$$I_3^{\mu\nu\alpha} = \frac{i}{16\pi^2} \frac{1}{4C} \Big[ C g^{\mu\nu} (-\Delta_\epsilon + \ln C) P^\alpha + C g^{\nu\alpha} (-\Delta_\epsilon + \ln C) P^\mu \tag{A.64}$$

$$+ C g^{\mu\alpha} (-\Delta_\epsilon + \ln C) P^\nu + 2 P^\alpha P^\mu P^\nu \Big] \tag{A.65}$$

$$I_3^{\mu\nu\alpha\beta} = \frac{i}{16\pi^2} \frac{1}{8C} \Big[ C^2 \left(1 + \Delta_\epsilon - \ln C\right) \left( g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} + g^{\alpha\beta} g^{\mu\nu} \right) \tag{A.66}$$

$$+2C \left(\Delta_\epsilon - \ln C\right) \Big( g^{\mu\nu} P^\alpha P^\beta + g^{\nu\beta} P^\alpha P^\mu + g^{\nu\alpha} P^\beta P^\mu + g^{\mu\alpha} P^\beta P^\nu \tag{A.67}$$

$$+g^{\mu\beta} P^\alpha P^\nu + g^{\alpha\beta} P^\mu P^\nu \Big) - 4 P^\alpha P^\beta P^\mu P^\nu \Big] \tag{A.68}$$

where

$$P^\mu = x_1\, r_1^\mu + x_2\, r_2^\mu \tag{A.69}$$

$$C = x_1^2\, r_1^2 + x_2^2\, r_2^2 + 2 x_1\, x_2\, r_1 \cdot r_2 + x_1\, m_1^2 + x_2\, m_2^2 \tag{A.70}$$

$$+(1 - x_1 - x_2)\, m_0^2 - x_1\, r_1^2 - x_2\, r_2^2 \tag{A.71}$$

### A.7.4 *Box* integrals

$$I_{0,4} = \frac{i}{16\pi^2} \frac{1}{6C^2} \tag{A.72}$$

$$I_4^\mu = \frac{i}{16\pi^2} \frac{-1}{6C^2}\; P^\mu \tag{A.73}$$

$$I_4^{\mu\nu} = \frac{i}{16\pi^2} \frac{-1}{12C^2} \Big[ C g^{\mu\nu} - 2 P^\mu P^\nu \Big] \tag{A.74}$$

$$I_4^{\mu\nu\alpha} = \frac{i}{16\pi^2} \frac{1}{12C^2} \Big[ C \left( g^{\mu\nu} P^\alpha + g^{\nu\alpha} P^\mu + g^{\mu\alpha} P^\nu \right) - 2 P^\alpha P^\mu P^\nu \Big] \tag{A.75}$$

$$I_4^{\mu\nu\alpha\beta} = \frac{i}{16\pi^2} \frac{1}{12C^2} \Big[ C^2 \left( \Delta_\epsilon - \ln C \right) \Big( g^{\mu\alpha} g^{\nu\beta} + g^{\mu\beta} g^{\nu\alpha} + g^{\alpha\beta} g^{\mu\nu} \Big) \tag{A.76}$$

$$-2C \Big( g^{\mu\nu} P^\alpha P^\beta + g^{\nu\beta} P^\alpha P^\mu + g^{\nu\alpha} P^\beta P^\mu + g^{\mu\alpha} P^\beta P^\nu \tag{A.77}$$

$$+\, g^{\mu\beta} P^\alpha P^\nu + g^{\alpha\beta} P^\mu P^\nu \Big) + 4 P^\alpha P^\beta P^\mu P^\nu \Big] \tag{A.78}$$

where

$$P^\mu = x_1\, r_1^\mu + x_2\, r_2^\mu + x_3\, r_3^\mu \tag{A.79}$$

$$C = x_1^2\, r_1^2 + x_2^2\, r_2^2 + x_3^2\, r_3^2 + 2 x_1\, x_2\, r_1 \cdot r_2 + 2 x_1\, x_3\, r_1 \cdot r_3 + 2 x_2\, x_3\, r_2 \cdot r_3 \tag{A.80}$$

$$+x_1\, m_1^2 + x_2\, m_2^2 + x_3\, m_3^2 + (1 - x_1 - x_2 - x_3)\, m_0^2 \tag{A.81}$$

$$-x_1\, r_1^2 - x_2\, r_2^2 - x_3\, r_3^2 \tag{A.82}$$

## A.8 Divergent part of *1–loop* integrals

When we want to study the renormalization of a given theory it is often convenient to have expressions for the divergent part of the one-loop integrals, with the integration on the Feynman parameters already done. We present here the results for the most important cases. These divergent parts were calculated with the help of the package `OneLoop` [10].

### A.8.1 *Tadpole* integrals

$$\text{Div}\left[I_{0,1}\right] = \frac{i}{16\pi^2}\,\Delta_\epsilon\, m^2 \tag{A.83}$$

$$\text{Div}\left[I_1^\mu\right] = 0 \tag{A.84}$$

$$\text{Div}\left[I_1^{\mu\nu}\right] = \frac{i}{16\pi^2}\,\frac{1}{4}\,\Delta_\epsilon\, m^4\, g^{\mu\nu} \tag{A.85}$$

### A.8.2 *Self–Energy* integrals

$$\text{Div}\left[I_{0,2}\right] = \frac{i}{16\pi^2}\,\Delta_\epsilon \tag{A.86}$$

$$\text{Div}\left[I_2^\mu\right] = \frac{i}{16\pi^2}\left(-\frac{1}{2}\right)\Delta_\epsilon\, r_1^\mu \tag{A.87}$$

$$\text{Div}\left[I_2^{\mu\nu}\right] = \frac{i}{16\pi^2}\,\frac{1}{12}\,\Delta_\epsilon\left[(3m_1^2+3m_0^2-r_1^2)g^{\mu\nu}+4r_1^\mu r_1^\nu\right] \tag{A.88}$$

$$\text{Div}\left[I_2^{\mu\nu\alpha}\right] = \frac{i}{16\pi^2}\left(-\frac{1}{24}\right)\Delta_\epsilon\left[(4m_1^2+2m_2^2-r_1^2)\left(g^{\mu\nu}r_1^\alpha+g^{\nu\alpha}r_1^\mu+g^{\mu\alpha}r_1^\nu\right)\right. \tag{A.89}$$

$$\left.+\,6\,r_1^\alpha r_1^\mu r_1^\nu\right] \tag{A.90}$$

### A.8.3 *Triangle* integrals

$$\text{Div}\left[I_{0,3}\right]=0 \tag{A.91}$$

$$\text{Div}\left[I_3^\mu\right]=0 \tag{A.92}$$

$$\text{Div}\left[I_3^{\mu\nu}\right]=\frac{i}{16\pi^2}\,\frac{1}{4}\,\Delta_\epsilon\, g^{\mu\nu} \tag{A.93}$$

$$\text{Div}\left[I_3^{\mu\nu\alpha}\right]=\frac{i}{16\pi^2}\left(-\frac{1}{12}\right)\Delta_\epsilon\left[g^{\mu\nu}(r_1^\alpha+r_2^\alpha)+g^{\nu\alpha}(r_1^\mu+r_2^\mu)+g^{\mu\alpha}(r_1^\nu+r_2^\nu)\right] \tag{A.94}$$

$$\text{Div}\left[I_3^{\mu\nu\alpha\beta}\right]=\frac{i}{16\pi^2}\,\frac{1}{48}\,\Delta_\epsilon\left[(2m_1^2+2m_2^2+2m_3^2)\left(g^{\mu\alpha}g^{\nu\beta}+g^{\alpha\beta}g^{\mu\nu}+g^{\mu\beta}g^{\nu\alpha}\right)\right. \tag{A.95}$$

$$+g^{\alpha\beta}\left[2r_1^\mu r_1^\nu+r_1^\mu r_2^\nu+(r_1\leftrightarrow r_2)\right]+g^{\mu\beta}\left[2r_1^\alpha r_1^\nu+r_1^\alpha r_2^\nu+(r_1\leftrightarrow r_2)\right] \tag{A.96}$$

$$+g^{\nu\beta}\left[2r_1^\alpha r_1^\mu+r_1^\alpha r_2^\mu+(r_1\leftrightarrow r_2)\right]+g^{\mu\nu}\left[2r_1^\alpha r_1^\beta+r_1^\alpha r_2^\beta+(r_1\leftrightarrow r_2)\right] \tag{A.97}$$

$$+g^{\mu\alpha}\left[2r_1^\beta r_1^\nu+r_1^\beta r_2^\nu+(r_1\leftrightarrow r_2)\right]+g^{\nu\alpha}\left[2r_1^\beta r_1^\mu+r_1^\beta r_2^\mu+(r_1\leftrightarrow r_2)\right] \tag{A.98}$$

$$\left.+\left(-r_1^2+r_1\cdot r_2-r_2^2\right)\left(g^{\mu\alpha}g^{\nu\beta}+g^{\alpha\beta}g^{\mu\nu}+g^{\mu\beta}g^{\nu\alpha}\right)\right] \tag{A.99}$$

### A.8.4 *Box* integrals

$$\mathrm{Div}\left[I_{0,4}\right] = \mathrm{Div}\left[I_4^{\mu}\right] = \mathrm{Div}\left[I_4^{\mu\nu}\right] = \mathrm{Div}\left[I_4^{\mu\nu\alpha}\right] = 0 \tag{A.100}$$

$$\mathrm{Div}\left[I_4^{\mu\nu\alpha\beta}\right] = \frac{i}{16\pi^2}\frac{1}{24}\Delta_\epsilon\left[g^{\mu\nu}g^{\alpha\beta} + g^{\mu\beta}g^{\alpha\nu} + g^{\mu\alpha}g^{\nu\beta}\right] \tag{A.101}$$

## A.9 Useful results for PV integrals

Although the PV approach is intended primarily to be used numerically there are situations where one wants to have explicit results. These can be useful to check cancellation of divergences or because in some simple cases the integrals can be done analytically. We note that as our conventions for the momenta are the same in sections 3 and A.7 one can read immediately the integral representation of the PV in terms of the Feynman parameters just by comparing both expressions, not forgetting to take out the $i/(16\pi^2)$ factor. For instance, from Eq. (3.21) for $C^{\mu\nu}$ and Eq. (A.64) for $I_3^{\mu\nu}$ we get

$$C_{12}(r_1^2, r_{12}^2, r_2^2, m_0^2, m_1^2, m_2^2) = -\Gamma(3)\frac{2}{4}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1 x_2}{\mathrm{C}} \tag{A.102}$$

with

$$\begin{aligned} C &= x_1^2\, r_1^2 + x_2^2\, r_2^2 + x_1\, x_2\,(r_1^2 + r_2^2 - r_{12}^2) + x_1\, m_1^2 + x_2\, m_2^2 \\ &\quad +(1 - x_1 - x_2)\, m_0^2 - x_1\, r_1^2 - x_2\, r_2^2 \end{aligned} \tag{A.103}$$

### A.9.1 Divergent part of the PV integrals

The package `LoopTools` provides ways to numerically check for the cancellation of divergences. However it is useful to know the divergent part of the Passarino-Veltman integrals. Only a small number of these integrals are divergent. They are

$$\mathrm{Div}\left[\mathrm{A_0(m_0^2)}\right] = \Delta_\epsilon\, m_0^2 \tag{A.104}$$

$$\mathrm{Div}\left[\mathrm{B_0(r_{10}^2, m_0^2, m_1^2)}\right] = \Delta_\epsilon \tag{A.105}$$

$$\mathrm{Div}\left[\mathrm{B_1(r_{10}^2, m_0^2, m_1^2)}\right] = -\frac{1}{2}\,\Delta_\epsilon \tag{A.106}$$

$$\mathrm{Div}\left[\mathrm{B_{00}(r_{10}^2, m_0^2, m_1^2)}\right] = \frac{1}{12}\,\Delta_\epsilon\,\left(3m_0^2 + 3m_1^2 - r_{10}^2\right) \tag{A.107}$$

$$\mathrm{Div}\left[\mathrm{B_{11}(r_{10}^2, m_0^2, m_1^2)}\right] = \frac{1}{3}\,\Delta_\epsilon \tag{A.108}$$

$$\mathrm{Div}\left[\mathrm{C_{00}(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2)}\right] = \frac{1}{4}\,\Delta_\epsilon \tag{A.109}$$

$$\mathrm{Div}\left[\mathrm{C_{001}(r_{10}^2, r_{12}^2, r_{20}^2, m_0^2, m_1^2, m_2^2)}\right] = -\frac{1}{12}\,\Delta_\epsilon \tag{A.110}$$

$$\text{Div}\left[\text{C}_{002}(\text{r}_{10}^2,\text{r}_{12}^2,\text{r}_{20}^2,\text{m}_0^2,\text{m}_1^2,\text{m}_2^2)\right] = -\frac{1}{12}\Delta_\epsilon \tag{A.111}$$

$$\text{Div}\left[\text{D}_{0000}(\text{r}_{10}^2,\ldots,\text{m}_0^2,\ldots)\right] = \frac{1}{24}\Delta_\epsilon \tag{A.112}$$

These results were obtained with the package LoopTools, after reducing to the scalar integrals with the command `PaVeReduce`, but they can be verified by comparing with our results of section A.8, after factoring out the $i/(16\pi^2)$.

### A.9.2 Explicit expression for $A_0$

This integral is trivial. There is no Feynman parameter and the integral can be read from Eq. (A.51). We get, after factoring out the $i/(16\pi^2)$,

$$A_0(m^2) = m^2\left(\Delta_\epsilon + 1 - \ln\frac{m^2}{\mu^2}\right) \tag{A.113}$$

### A.9.3 Explicit expressions for the $B$ functions

**Function $B_0$**

The general form of the integral $B_0(p^2, m_1^2, m_2^2)$ can be read from Eq. (A.55). We obtain

$$B_0(p^2, m_0^2, m_1^2) = \Delta_\epsilon - \int_0^1 dx \ln\left[\frac{-x(1-x)p^2 + xm_1^2 + (1-x)m_0^2}{\mu^2}\right] \tag{A.114}$$

From this expression one can easily get the following results,

$$B_0(0, m_0^2, m_1^2) = \Delta_\epsilon + 1 - \frac{m_0^2\ln m_0^2 - m_1^2\ln m_1^2}{m_0^2 - m_1^2} \tag{A.115}$$

$$B_0(0, m_0^2, m_1^2) = \frac{A_0(m_0^2) - A_0(m_1^2)}{m_0^2 - m_1^2} \tag{A.116}$$

$$B_0(0, m^2, m^2) = \Delta_\epsilon - \ln\frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} - 1 \tag{A.117}$$

$$B_0(m^2, 0, m^2) = \Delta_\epsilon + 2 - \ln\frac{m^2}{\mu^2} = \frac{A_0(m^2)}{m^2} + 1 \tag{A.118}$$

$$B_0(0, 0, m^2) = \Delta_\epsilon + 1 - \ln\frac{m^2}{\mu^2} \tag{A.119}$$

**Function $B_0'$**

The derivative of the $B_0$ function with respect to $p^2$ appears many times. From Eq. (A.114) one can derive an integral representation,

$$B_0'(p^2, m_0^2, m_1^2) = -\int_0^1 dx\, \frac{x(1-x)}{-p^2 x(1-x) + x m_1^2 + (1-x) m_0^2} \tag{A.120}$$

An important particular case corresponds to $B_0'(m^2, m_0^2, m^2)$ that appears in the self-energy of the electron. In this case $m$ is the electron mass and $m_0 = \lambda$ is the photon mass that one has to introduce to regularize the IR divergent integral. The integral in this case reduces to

$$\begin{aligned} B_0'(m^2, \lambda^2, m^2) &= -\int_0^1 dx\, \frac{x(1-x)}{m^2 x^2 + (1-x)\lambda^2} \\ &= \frac{1}{m^2} + \frac{1}{2m^2} \ln \frac{\lambda^2}{m^2} \end{aligned} \tag{A.121}$$

It is clear that in the limit $\lambda \to 0$ this integral diverges.

**Function $B_1$**

The explicit expression can be read from Eq. (A.56). We have

$$B_1(p^2, m_0^2, m_1^2) = -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx x \ln\left[\frac{-x(1-x)p^2 + x m_1^2 + (1-x) m_0^2}{\mu^2}\right] \tag{A.122}$$

For $p^2 = 0$ this integral can be easily evaluated to give

$$B_1(0, m_0^2, m_1^2) = -\frac{1}{2}\Delta_\epsilon + \frac{1}{2} \ln\left(\frac{m_0^2}{\mu^2}\right) + \frac{-3 + 4t - t^2 - 4t\ln t + 2t^2 \ln t}{4(-1+t)^2} \tag{A.123}$$

where we defined

$$t = \frac{m_1^2}{m_0^2} \tag{A.124}$$

From Eq. (A.123) one can shown that even for $p^2 = 0$ $B_1$ is **not** a symmetric function of the masses,

$$B_1(p^2, m_0^2, m_1^2) \neq B_1(p^2, m_1^2, m_0^2) \tag{A.125}$$

As this might appear strange let us show with one example how the coefficient functions are tied to our conventions about the order of the momenta and Feynman parameters. Let us consider the contribution to the self-energy of a fermion of mass $m_f$ of the exchange of a scalar with mass $m_s$. We can consider the two choices in Fig. 18,

Now with the first choice (diagram on the left of Fig. 18) we have

$$\begin{aligned} -i\Sigma_1 &= \frac{i}{16\pi^2}\left[(\not p + m_f) B_0(p^2, m_s^2, m_f^2) + \not p B_1(p^2, m_s^2, m_f^2)\right] \\ &= \frac{i}{16\pi^2}\left[\not p \left(B_0(p^2, m_s^2, m_f^2) + B_1(p^2, m_s^2, m_f^2)\right) + m_f B_0(p^2, m_s^2, m_f^2)\right] \end{aligned} \tag{A.126}$$

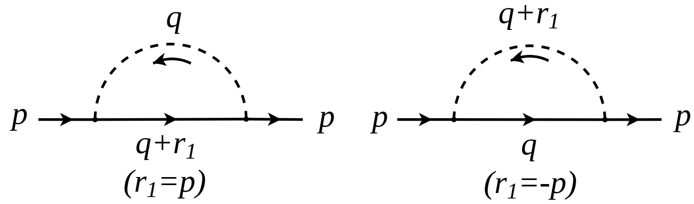



Figure 18:

while with the second choice we have

$$-i\Sigma_2 = \frac{i}{16\pi^2}\left[-\not{p}B_1(p^2,m_f^2,m_s^2)+m_f B_0(p^2,m_f^2,m_s^2)\right] \tag{A.127}$$

How can these two expressions be equal? The reason has precisely to do with the non symmetry of $B_1$ with respect to the mass entries. In fact from Eq. (A.122) we have

$$\begin{aligned}
B_1(p^2,m_0^2,m_1^2) &= -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx x \ln\left[\frac{-x(1-x)p^2+xm_1^2+(1-x)m_0^2}{\mu^2}\right] \\
&= -\frac{1}{2}\Delta_\epsilon + \int_0^1 dx(1-x) \ln\left[\frac{-x(1-x)p^2+(1-x)m_1^2+xm_0^2}{\mu^2}\right] \\
&= -\frac{1}{2}\Delta_\epsilon + \Big(\Delta_\epsilon - B_0(p^2,m_1^2,m_0^2)\Big) - \Big(\frac{1}{2}\Delta_\epsilon + B_1(p^2,m_1^2,m_0^2)\Big) \\
&= -\Big(B_0(p^2,m_1^2,m_0^2)+B_1(p^2,m_1^2,m_0^2)\Big)
\end{aligned} \tag{A.128}$$

where we have changed variables ($x \to 1-x$) in the integral and used the definitions of $B_0$ and $B_1$. We have then, remembering that $B_0(p^2,m_s^2,m_f^2)=B_0(p^2,m_f^2,m_s^2)$,

$$B_1(p^2,m_f^2,m_s^2) = -\Big(B_0(p^2,m_s^2,m_f^2)+B_1(p^2,m_s^2,m_f^2)\Big) \tag{A.129}$$

and therefore Eqs. (A.126) and (A.127) are equivalent.

### A.9.4 Explicit expressions for the $C$ functions

In Eq. (A.102) we have already given the general form of $C_{12}$. The other functions are very similar. In the following we just present the results for the particular case of $p^2=0$. This case is important in many situations where it is a good approximation to neglect the external momenta in comparison with the masses of the particles in the loop. We also warn the reader that the coefficient functions $C_i, C_{ij}$ obtained from `LoopTools` are not well defined in this limit. Hence there is some utility in given them here.

**Function $C_0$**

$$C_0(0,0,0,m_0^2,m_1^2,m_2^2) = -\Gamma(3)\frac{1}{2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1m_1^2+x_2m_2^2+(1-x_1-x_2)m_0^2}$$

$$
\begin{aligned}
&= -\frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{1}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\
&= -\frac{1}{m_0^2}\frac{-t_1\ln t_1 + t_1 t_2 \ln t_1 + t_2 \ln t_2 - t_1 t_2 \ln t_2}{(-1+t_1)(t_1-t_2)(-1+t_2)} \qquad \text{(A.130)}
\end{aligned}
$$

where

$$
t_1 = \frac{m_1^2}{m_0^2} \quad ; \quad t_2 = \frac{m_2^2}{m_0^2} \qquad \text{(A.131)}
$$

Using the properties of the logarithms one can show that in this limit $C_0$ is a symmetric function of the masses. This expression is further simplified when two of the masses are equal, as it happens in the $\mu \to e\gamma$ problem. Then $t = t_1 = t_2$,

$$
C_0(0,0,0,m_0^2,m_1^2,m_1^2) = -\frac{1}{m_0^2}\frac{-1+t-\ln t}{(-1+t)^2} \qquad \text{(A.132)}
$$

in agreement with Eq.(20) of [9]. In the case of equal masses for all the loop particles we have

$$
C_0(0,0,0,m_0^2,m_0^2,m_0^2) = -\frac{1}{2m_0^2} \qquad \text{(A.133)}
$$

Before we close this section on $C_0$ there is another particular case when it is useful to have an explicit case for it. This in the case when it is IR divergent as in the QED vertex. The functions needed is $C_0(m^2, m^2, 0, m^2, \lambda^2, m^2)$. Using the definition we have

$$
\begin{aligned}
C_0(m^2,m^2,0,m^2,\lambda^2,m^2) &= -\int_0^1 dx_1 \int_0^{1-x1} dx_2 \frac{1}{m^2(1-2x_1+x_1^2)+x_1\lambda^2} \\
&= -\int_0^1 dx_1 \frac{1-x_1}{m^2(1-x_1)^2+(1-x_1)\lambda^2} \\
&= -\int_0^1 dx \frac{x}{m^2x^2+x\lambda^2} \\
&= \frac{1}{2m^2}\ln\frac{\lambda^2}{m^2} = B_0'(m^2,\lambda^2,m^2) - \frac{1}{m^2} \qquad \text{(A.134)}
\end{aligned}
$$

**Function $C_{00}$**

$$
\begin{aligned}
C_{00}(0,0,0,m_0^2,m_1^2,m_2^2) &= \Gamma(3)\frac{1}{4}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \left[\Delta_\epsilon - \ln\left(\frac{C}{\mu^2}\right)\right] \\
&= \frac{1}{4}\Delta_\epsilon - \frac{1}{2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \ln\left[\frac{x_1 m_1^2 + x_2 m_2^2 + (1-x_1-x_2)m_0^2}{\mu^2}\right] \\
&= \frac{1}{4}\left(\Delta_\epsilon - \ln\frac{m_0^2}{\mu^2}\right) + \frac{3}{8} - \frac{t_1^2}{4(t_1-1)(t_1-t_2)}\ln t_1 \\
&\quad -\frac{t_2^2}{4(t_2-1)(t_1-t_2)}\ln t_2 \qquad \text{(A.135)}
\end{aligned}
$$

where, as before

$$t_1 = \frac{m_1^2}{m_0^2} \quad ; \quad t_2 = \frac{m_2^2}{m_0^2} \tag{A.136}$$

Using the properties of the logarithms one can show that in this limit $C_{00}$ is a symmetric function of the masses. This expression is further simplified when two of the masses are equal. Then $t = t_1 = t_2$,

$$\begin{aligned} C_{00}(0,0,0,m_0^2,m_1^2,m_1^2) &= \frac{1}{4}\left(\Delta_\epsilon - \ln\frac{m_0^2}{\mu^2}\right) - \frac{-3+4t-t^2-4t\,\ln t+2t^2\,\ln t}{8(t-1)^2} \\ &= -\frac{1}{2}B_1(0,m_0^2,m_1^2) \end{aligned} \tag{A.137}$$

**Functions $C_i$ and $C_{ij}$**

We recall that the definition of the coefficient functions is not unique, it is tied to a particular convention for assigning the loop momenta and Feynman parameters, as shown in Fig. 15. For the particular case of the $C$ functions we show our conventions in Fig. 19.

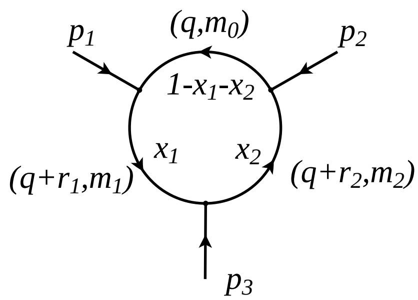



Figure 19:

With the same techniques we obtain,

$$\begin{aligned} C_1(0,0,0,m_0^2,m_1^2,m_2^2) &= \frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_1}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\ &= -\frac{1}{m_0^2}\left[\frac{t_1}{2(-1+t_1)(t_1-t_2)} - \frac{t_1(t_1-2t_2+t_1t_2)}{2(-1+t_1)^2(t_1-t_2)^2}\ln t_1 \right. \\ &\qquad \left. + \frac{{t_2}^2 - 2t_1{t_2}^2 + {t_1}^2{t_2}^2}{2(-1+t_1)^2(t_1-t_2)^2(-1+t_2)}\ln t_2\right] \end{aligned} \tag{A.138}$$

$$\begin{aligned} C_2(0,0,0,m_0^2,m_1^2,m_2^2) &= \frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_2}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \\ &= -\frac{1}{m_0^2}\left[-\frac{t_2}{2(t_1-t_2)(-1+t_2)} + \frac{\ln t_1}{2(-1+t_1)(-1+t_2)^2}\right. \end{aligned}$$

$$+\frac{2t_1t_2 - 2{t_1}^2t_2 - {t_2}^2 + {t_1}^2{t_2}^2}{2(-1+t_1)(t_1-t_2)^2(-1+t_2)^2}\ln\left(\frac{t_1}{t_2}\right)\Bigg] \quad \text{(A.139)}$$

$$C_{ij}(0,0,0,m_0^2,m_1^2,m_2^2) = -\frac{1}{m_0^2}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \frac{x_i x_j}{x_1 t_1 + x_2 t_2 + (1-x_1-x_2)} \quad \text{(A.140)}$$

where we have not written explicitly the $C_{ij}$ for $i,j=1,2$ because they are rather lengthy. However a simple `Fortran` program can be developed [10] to calculate all the three point functions in the zero external limit case. This is useful because in this case some of the functions from `LoopTools` will fail. Notice that the $C_i$ and $C_{ij}$ functions are not symmetric in their arguments. This a consequence of their non-uniqueness, they are tied to a particular convention. This is very important when ones compares with other results. However using their definition one can get some relations. For instance we can show

$$\begin{aligned} C_1(0,0,0,m_0^2,m_1^2,m_2^2) &= C_1(0,0,0,m_2^2,m_1^2,m_0^2) && \text{(A.141)} \\ C_2(0,0,0,m_0^2,m_1^2,m_2^2) &= -C_0(0,0,0,m_2^2,m_1^2,m_0^2) - C_1(0,0,0,m_2^2,m_1^2,m_0^2) \\ &\quad -C_2(0,0,0,m_2^2,m_1^2,m_0^2) && \text{(A.142)} \end{aligned}$$

In the limit $m_1 = m_2$ we get the simple expressions,

$$\begin{aligned} C_1(0,0,0,m_0^2,m_1^2,m_1^2) &= C_2(0,0,0,m_0^2,m_1^2,m_1^2) \\ &= -\frac{1}{m_0^2}\frac{3-4t+t^2+2\ln t}{4(-1+t)^3} && \text{(A.143)} \\ C_{11}(0,0,0,m_0^2,m_1^2,m_1^2) &= C_{22}(0,0,0,m_0^2,m_1^2,m_1^2) = 2\ C_{12}(0,0,0,m_0^2,m_1^2,m_1^2) \\ &= -\frac{1}{m_0^2}\frac{-11+18t-9t^2+2t^3-6\ln t}{18(-1+t)^4} && \text{(A.144)} \end{aligned}$$

in agreement with Eqs. (21-22) of [9]. The case of masses equal gives

$$\begin{aligned} C_1(0,0,0,m_0^2,m_0^2,m_0^2) &= C_2(0,0,0,m_0^2,m_0^2,m_0^2) = \frac{1}{6m_0^2} && \text{(A.145)} \\ C_{11}(0,0,0,m_0^2,m_0^2,m_0^2) &= C_{22}(0,0,0,m_0^2,m_0^2,m_0^2) = -\frac{1}{12m_0^2} && \text{(A.146)} \\ C_{12}(0,0,0,m_0^2,m_0^2,m_0^2) &= -\frac{1}{24m_0^2} && \text{(A.147)} \end{aligned}$$

### A.9.5 The package PVzem

As we said before, in many situations it is a good approximation to neglect the external momenta. In this case, the loop functions are easier to evaluate and one approach is for each problem to evaluate them. However our approach here is more in the direction of automatically evaluating the one-loop amplitudes. If one does that with the use of FeynCalc, has we have been doing, then the result is given in terms of standard functions that can be numerically evaluated with the package LoopTools. However this package has problems with this limit. This is because this limit is unphysical. Let us illustrate this point calculating the functions $C_1(m^2, 0, 0, m_S^2, m_F^2, m_F^2)$ and $C_2(m^2, 0, 0, m_S^2, m_F^2, m_F^2)$ for $m_B = 100$ GeV, $m_F = 80$ GeV and $m_2$ ranging from $10^{-6}$ to 100 GeV. To better illustrate our point we show two plots with different scales on the axis.

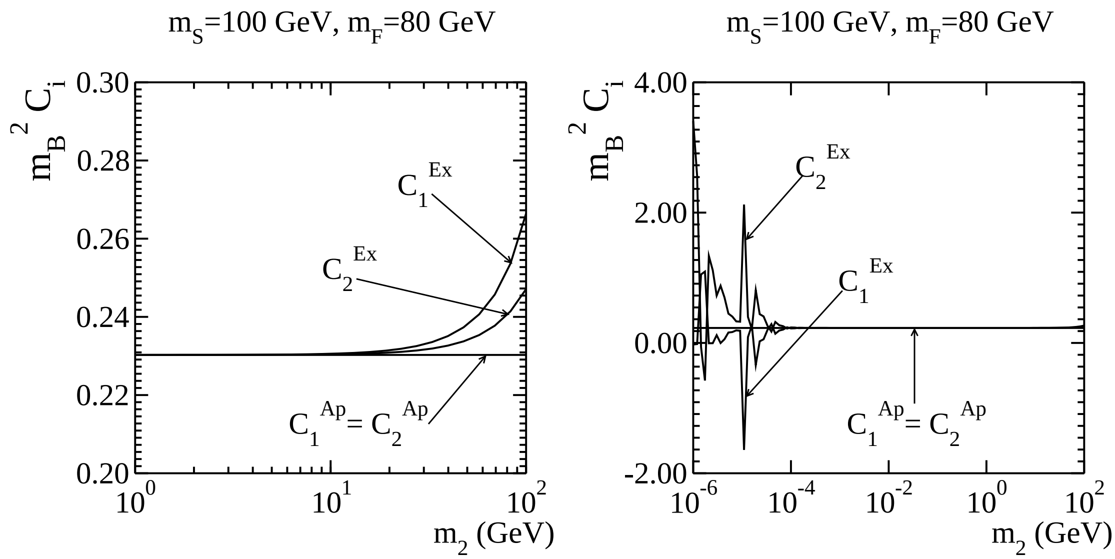



Figure 20:

In these plots, $C_i^{\rm Ex}$ are the exact $C_i$ functions calculated with LoopTools and $C_i^{\rm Ap}$ are the $C_i$ calculated in the zero momenta limit. We can see that only for external momenta (in this case corresponding to the mass $m_2$) close enough to the masses of the particles in the loop, the exact result deviates from the approximate one. However for very small values of the external momenta, LoopTools has numerical problems as shown in the right panel of Fig. 20. To overcome this problem I have developed a Fortran package that evaluates all the $C$ functions in the zero external momenta limit. There are no restrictions on the masses being equal or different and the conventions are the same as in FeynCalc and LoopTools, for instance,

$$c12zem(m02, m12, m22) = c0i(cc12, 0, 0, 0, m02, m12, m22) \tag{A.148}$$

where $c0i(cc12, \cdots)$ is the LoopTools notation and $c12zem(\cdots)$ is the notation of my package, called PVzem. It can be obtained from the address indicated in Ref. [10]. The approximate functions shown in Fig. 20 were calculated using that package. We include here the Fortran code used to produce that figure.

```
*************************************************************
*                                                           *
*               Program LoopToolsExample                    *
*                                                           *
*   This program calculates the values used in the plots    *
```

```
*   of Figure 20. For the exact results the LoopTools       *
*   package was used. The package PVzem was used for the    *
*   approximate results.                                    *
*                                                           *
*              Version of  16/03/2003                       *
*                                                           *
*     Author: Jorge C. Romao                                *
*     e-mail: jorge.romao@ist.utl.pt                        *
*************************************************************

      program LoopToolsExample
      implicit none

*
*  LoopTools has to be used with FORTRAN programs with the
*  extension .F in order to have the header file "looptools.h"
*  preprocessed. This file includes all the definitions used
*  by LoopTools.
*
*  Functions c1zem and c2zem are provided by the package PVzem.
*

#include "looptools.h"

      integer i
      real*8 m2,mF2,mS2,m
      real*8 lgmmin,lgmmax,lgm,step
      real*8 rc1,rc2
      real*8 c1zem,c2zem

      mS2=100.d0**2
      mF2=80.d0**2

*
*  Initialize LoopTools. See the LoopTools manual for further
*  details. There you can also learn how to set the scale MU
*  and how to handle the UR and IR divergences.
*

      call ffini

      lgmmax=log10(100.d0)
      lgmmin=log10(1.d-6)
      step=(lgmmax-lgmmin)/100.d0
      lgm=lgmmin-step
```

```
      open(10,file='plot.dat',status='unknown')

      do i=1,101
         lgm=lgm+step
         m=10.d0**lgm
         m2=m**2
*
*  In LoopTools the c0i(...) are complex functions. For the
*  kinematics chosen here they are real, so we take the real
*  part for comparison.
*

      rc1=dble(c0i(cc1,m2,0.d0,0.d0,mS2,mF2,mF2))
      rc2=dble(c0i(cc2,m2,0.d0,0.d0,mS2,mF2,mF2))
      write(10,100)m,rc1*mS2,rc2*mS2,c1zem(mS2,mF2,mF2)*mS2,
     &            c2zem(mS2,mF2,mF2)*mS2
      enddo

 100  format(5(e22.14))

      end
************** End of Program LoopToolsExample.F ************
```

When the above program is compiled, the location of the header file `looptools.h` must be known by the compiler. This is best achieved by using a `Makefile`. We give below, as an example, the one that was used with the above program. Depending on the installation details of `LoopTools` the paths might be different.

```
FC          = g77
LT          = /usr/local/lib/LoopTools/LT2/i386-linux
FFLAGS      =  -c -O -I$(LT)/include
LDFLAGS     =
LINKER      = $(FC)

LIB         = -L$(LT)/lib
LIBS        = -looptools

.f.o:
        $(FC) $(FFLAGS) $*.F

files   = LoopToolsExample.o PVzem.o

all:    $(files)
        $(LINKER) $(LDFLAGS) -o Example $(files) $(LIB) $(LIBS)
```

### A.9.6 Explicit expressions for the $D$ functions

**Function $D_0$**

The various $D$ functions can be calculated in a similar way. However they are rather lengthy and have to handled numerically [10]. Here we just give $D_0$ for the equal masses case.

$$
\begin{aligned}
D_0(0,\cdots,0,m^2,m^2,m^2,m^2) &= \Gamma(4)\frac{1}{6}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \, \frac{1}{(m^2)^2} \\
&= \frac{1}{m^4}\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \\
&= \frac{1}{6m^4} \qquad \text{(A.149)}
\end{aligned}
$$

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