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Notes on SusyFit Project

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Abstract

We summarize the formulae used in our fitting codes for practical purposes.

Questions and comments

- **Questions on concepts:**
  1. **Should we employ the resummation formulae in Eqs. (57), (76) and (77)? Why aren't they used in some of recent papers?**
  2. **When should we use the coupling G_F instead of α? Which one is more appropriate when we don't use the resummation formulae? Why do the remainder terms in Eqs. (76) and (77) involve G_F?**
  3. **Can we use the approximate formulae for Δr^{α^2} and $\Delta\kappa^{\alpha^2}$ in Sec. D.3?**
- **Formulae which should be checked carefully:**
  1. **Check the second term in Eq. (83) and look for references to it.**
  2. **Check whether there is any missing piece for the imaginary parts of ρ_Z^f and κ_Z^f.**
  3. **Check the resummation formula for ρ_Z^b in Eq. (93), and look for reference.**
  4. **Write down the expression of the function V_2 in Eq. (113).**
  5. **Check the formulae for the W-boson width in Sec. 2.13.**
  6. **Check the formulas for the oblique parameters in Sec. 2.14.**
  7. **Rewrite Eq. (125) in terms of the variables defined in this write-up.**
  8. **Check inputs parameters in Table 1, especially recent studies on $\Delta\alpha_{\rm had}^{(5)}(M_Z^2)$.**
  9. **Check Eq. (160).**
  10. **Check the expression of the three-point PV function in Sec. B.3.**
  11. **Check the relation in Eq. (206).**
  12. **Check the function $F_1(x)$ in Eq. (258) for QCD corrections.**
  13. **What is the correction F in Eq. (264)? Look for reference.**
  14. **Check the expressions of the leading two-loop EW corrections in Sec. C.9, in which I refer to the formulae listed in the original papers. They should be checked carefully, since I might make mistakes when I copied them from the papers. Those formulae seem to be different from those adopted in the ZFITTER codes and also from those presented in [6]. Why?**
  15. **Check also all the other formulae carefully.**

Contents

1 Notations

1.1 Metric

The metric is defined as

$$g^{\mu\nu} = \mathrm{diag}\,(1,\ -1,\ -1,\ -1)\,, \tag{1}$$

which leads to the on-shell condition $p^2 = m^2$.

1.2 Top quark mass

M_t denotes the pole mass of the top quark.

1.3 SLHA convention

See Refs. [1, 2] for detail.

1.3.1 Superpotential

The superpotential is given by

$$W = \epsilon_{ab}\left[(Y_E)_{ij}H_1^a L_i^b \bar{E}_j + (Y_D)_{ij}H_1^a Q_i^b \bar{D}_j + (Y_U)_{ij}H_2^b Q_i^a \bar{U}_j - \mu H_1^a H_2^b\right], \tag{2}$$

where $\epsilon_{12} = \epsilon^{12} = 1$.

1.3.2 Higgs VEVs

The Higgs vacuum expectation values $\langle H_i^0\rangle = v_i/\sqrt{2}$ with $\tan\beta = v_2/v_1$ satisfy

$$v^2 = v_1^2 + v_2^2 \approx (246\ \text{GeV})^2, \tag{3}$$

which corresponds to $m_Z^2 = \frac{1}{4}(g'^2 + g^2)v^2$.

1.3.3 Soft breaking terms

The soft SUSY breaking terms are given in the sum of the trilinear scalar interaction potential V_3, the scalar bilinear interaction potential V_2, and the gaugino mass terms:

$$V_3 = \epsilon_{ab}\left[(T_E)_{ij}H_1^a \tilde{L}_{iL}^b \tilde{e}_{jR}^* + (T_D)_{ij}H_1^a \tilde{Q}_{iL}^b \tilde{d}_{jR}^* + (T_U)_{ij}H_2^b \tilde{Q}_{iL}^a \tilde{u}_{jR}^*\right] + \text{h.c.}, \tag{4}$$

$$\begin{aligned} V_2 = {} & m_{H_1}^2 H_{1a}^* H_1^a + m_{H_2}^2 H_{2a}^* H_2^a - (m_3^2 \epsilon_{ab} H_1^a H_2^b + \text{h.c.}) \\ & + \tilde{L}_{iLa}^* (m_{\tilde{L}}^2)_{ij} \tilde{L}_{jL}^a + \tilde{Q}_{iLa}^* (m_{\tilde{Q}}^2)_{ij} \tilde{Q}_{jL}^a \\ & + \tilde{e}_{iR} (m_{\tilde{e}}^2)_{ij} \tilde{e}_{jR}^* + \tilde{d}_{iR} (m_{\tilde{d}}^2)_{ij} \tilde{d}_{jR}^* + \tilde{u}_{iR} (m_{\tilde{u}}^2)_{ij} \tilde{u}_{jR}^*, \end{aligned} \tag{5}$$

$$\mathcal{L}_G = \frac{1}{2}\left(M_1 \tilde{b}\tilde{b} + M_2 \tilde{w}^A \tilde{w}^A + M_3 \tilde{g}^X \tilde{g}^X\right) + \text{h.c.}, \tag{6}$$

where $A = 1, 2, 3,\ X = 1, \cdots, 8$ and $A_{ij} \equiv T_{ij}/Y_{ij}$.

1.3.4 Neutralino mixing

1.3.5 Chargino mixing

1.3.6 Sfermion mixing

1.3.7 Higgs mixing

1.3.8 Tree-level mass matrices

2 EW precision observables in the SM

In this section, we present the formulae for the electroweak (EW) precision observables [3, 4] (see also [5]) in the SM. Many reviews on this subject can be found in the literature. For the formulae, we basically refer to Refs. [6, 7][1], in which the basis of the FORTRAN code ZFITTER [7, 8, 9, 10] are explained in detail[2]. New ingredients of our codes compared to ZFITTER are as follows:

[1] Note that $p^2 = -m^2$ in ZFITTER.

[2] Formulas presented in Ref. [7] contain a lot of errors, while the corresponding formulae in Ref. [6] seem to be correct in most cases.

- *complete* EW two-loop contributions not only for $\sin^2\theta^{\ell}_{\rm eff}$, but also for $\sin^2\theta^{q}_{\rm eff}$,
- two-loop approximate formulae for Δr^{α^2} and $\Delta\kappa^{\alpha^2}$,
- massless singlet contribution of $O(\alpha_s^4)$ terms in the radiator functions.

Practical introduction of the EW observables can also be found, e.g., in Refs. [11, 12, 13, 14].

For the $e^+e^- \to Z \to f\bar{f}$ processes, we have to consider the following types of radiative corrections:

- the self-energy corrections of the gauge bosons,
- vertex corrections,
- box diagrams, and
- QCD and QED corrections to the initial and final states.

The first one is independent of the flavour of the final-state fermions, while the second and third ones depend on the flavour. In general, the corrections from the box diagrams are tiny. Also the vertex corrections are sub-dominant, except for the case of $f = b$.

We employ the unitary gauge in the calculations of the radiative corrections, where the resultant EW observables are independent of the choice of the gauge. Loop functions with the superscript F, denoting finite parts, are defined at $\mu = M_W$ as an artifact of the definition of the finite parts [6, 7]: e.g.,

$$B_0(p^2; M_1, M_2) = \frac{1}{\bar{\varepsilon}} - \ln\frac{M_W^2}{\mu^2} + B_0^F(p^2; M_1, M_2)\,. \tag{7}$$

The finite part $B_0^F(p^2; M_1, M_2)$ corresponds to that in the $\overline{\rm MS}$ scheme with the renormalization scale $\mu = M_W$. For example, the quantity $\Delta\rho$ has a term with the above loop function at one-loop level as shown later: $\Delta\rho^F = CB_0^F(\cdots) + \cdots$, where C is a constant. In this case, $\Delta\rho^F$ should be understood as a finite quantity defined with the artificial scale $\mu = M_W$. We will use the notation $\Delta\rho^F|_{\mu=M_W}$ in some cases to stress the above facts, but not always. See Appendix B.4 for detail.

(In our codes, it's better to define a finite part at $\mu = M_Z$ from the beginning, for simplicity.)

2.1 On-mass-shell renormalization scheme

We use the on-mass-shell renormalization scheme [15, 16, 17, 18], adopted in the ZFITTER code [7, 8, 9, 10], where the weak mixing angle is defined in terms of the physical masses of the gauge bosons:

$$s_W^2 \equiv \sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}\,, \tag{8}$$

and $c_W^2 = 1 - s_W^2$. The Fermi constant G_μ in μ decay is taken as an input quantity instead of the W-boson mass, which has not been measured very precisely compared with G_μ. The relation between G_μ and M_W is given by

$$G_\mu = \frac{\pi\alpha(0)}{\sqrt{2}s_W^2 M_W^2}(1 + \Delta r)\,, \tag{9}$$

where Δr takes into account radiative corrections.

The input parameters for our SM analysis are as follows:

$$M_Z,\ \alpha_s(M_Z^2),\ G_\mu,\ \alpha(0),\ \Delta\alpha^{(5)}_{\rm had}(M_Z^2),\ m_h,\ \text{fermion masses.} \tag{10}$$

2.2 Renormalization

The bare parameters are expressed as the sums of their physical values and counter terms:

$$(M_W^{\rm bare})^2 = M_W^2 + \delta M_W^2 \,, \tag{11}$$

$$(M_Z^{\rm bare})^2 = M_Z^2 + \delta M_Z^2 \,, \tag{12}$$

$$(e^{\rm bare})^2 = e^2 + \delta e^2 \,, \tag{13}$$

$$(s_W^{\rm bare})^2 = 1 - \frac{(M_W^{\rm bare})^2}{(M_Z^{\rm bare})^2} = s_W^2 - c_W^2 \left(\frac{\delta M_W^2}{M_W^2} - \frac{\delta M_Z^2}{M_Z^2} \right) . \tag{14}$$

Let us consider radiative corrections to gauge-boson propagators. At one-loop level, a gauge-boson propagator is written as

$$\frac{-ig^{\mu\nu}}{q^2 - (M_V^{\rm bare})^2} (1 - \delta Z_V) + \frac{-ig^{\mu\rho}}{q^2 - (M_V^{\rm bare})^2} \, ig_{\rho\sigma} \left[\delta M_V^2 - \Sigma_{VV}(q^2) \right] \frac{-ig^{\sigma\nu}}{q^2 - (M_V^{\rm bare})^2} \,, \tag{15}$$

where only the transverse part of the self-energy, $\Sigma_{VV}^{\mu\nu} = g^{\mu\nu}\Sigma_{VV} + \cdots$, has been considered, since the longitudinal part coupled with fermion currents is proportional to light-fermion masses and hence negligible. Defining a finite self-energy function as $\hat{\Sigma}_{VV}(q^2) = \Sigma_{VV}(q^2) - \delta M_V^2 + [q^2 - (M_V^{\rm bare})^2]\delta Z_V$, the geometric series of the 1PI diagrams results in the propagator

$$\frac{-ig^{\mu\nu}}{q^2 - (M_V^{\rm bare})^2 + \hat{\Sigma}_{VV}(q^2)} \,, \tag{16}$$

Since the gauge-boson mass is defined as the pole of the real part of the propagator, the counter terms for the masses are given by[3]

$$\delta M_W^2 = {\rm Re}\, \hat{\Sigma}_{WW}(M_W^2) = \frac{\alpha(0)}{4\pi s_W^2} {\rm Re}\, \overline{\Sigma}_{WW}(M_W^2) \,, \tag{17}$$

$$\delta M_Z^2 = {\rm Re}\, \hat{\Sigma}_{ZZ}(M_Z^2) = \frac{\alpha(0)}{4\pi s_W^2 c_W^2} {\rm Re}\, \overline{\Sigma}_{ZZ}(M_Z^2) \,. \tag{18}$$

Rewriting the real part of $\hat{\Sigma}_{VV}(q^2)$ as

$${\rm Re}\, \hat{\Sigma}_{VV}(q^2) \equiv (q^2 - M_V^2) \hat{\Pi}_{VV}(q^2) \,, \tag{19}$$

we have a renormalized gauge-boson propagator,

$$\frac{1}{1 + \hat{\Pi}_{VV}(q^2)} \frac{-ig^{\mu\nu}}{q^2 - M_V^2 + i\, M_V \Gamma_V} \,, \tag{20}$$

where $M_V\Gamma_V \equiv {\rm Im}\hat{\Sigma}_{VV}(q^2)/(1 + \hat{\Pi}_{VV}(q^2))$. We introduce the scale-dependent coupling constant

$$g^{(\prime)2}(q^2) = \frac{g^{(\prime)2}}{1 + \hat{\Pi}_{VV}(q^2)} \,. \tag{21}$$

In the case of the photon, $\hat{\Pi}_{\gamma\gamma}(0) = 0$.

At last, the counter term for the electromagnetic coupling, defined at $q^2 = 0$, is given by[4]

$$\frac{\delta e^2}{e^2} = 2\, \frac{\delta e}{e} = \Pi_{\gamma\gamma}(0) + \frac{2 s_W}{c_W} \frac{\hat{\Sigma}_{\gamma Z}(0)}{M_Z^2} \approx \Pi_{\gamma\gamma}^{\rm fer}(0) = -\frac{\alpha(0)}{4\pi} \overline{\Pi}_{\gamma\gamma}^{\rm fer}(0) \,, \tag{22}$$

where $\hat{\Sigma}_{\gamma Z}(0) = 0$ for fermion loops.

[3] Our $\overline{\Sigma}_{VV}$ corresponds to Σ_{VV} used in ZFITTER.

[4] Our $\overline{\Pi}_{\gamma\gamma}$ corresponds to $\Pi_{\gamma\gamma}$ used in ZFITTER.

2.3 Electromagnetic coupling constant

The electromagnetic coupling constant at scale M_Z is given in terms of the fine-structure constant $\alpha(0)$ and $\Delta\alpha$, which contains the logarithmic corrections of the fermions:

$$\alpha(M_Z^2) = \alpha(0)\left[1 + \Delta\alpha(M_Z^2)\right] , \tag{23}$$

The quantity $\Delta\alpha$ is the sum of the leptonic correction $\Delta\alpha_{\rm lept}$ and the 5-flavour hadronic correction $\Delta\alpha_{\rm had}^{(5)}$ and the small top-quark correction $\Delta\alpha_{\rm top}$:

$$\Delta\alpha(M_Z^2) = \Delta\alpha_{\rm lept}(M_Z^2) + \Delta\alpha_{\rm had}^{(5)}(M_Z^2) + \Delta\alpha_{\rm top}(M_Z^2)\,, \tag{24}$$

where the leptonic and top-quark ones are given by[5]

$$\Delta\alpha_{\rm lept}(M_Z^2) = -\sum_{f=\text{leptons}}\Big[\mathrm{Re}\,\Pi^f_{\gamma\gamma}(M_Z^2) - \Pi^f_{\gamma\gamma}(0)\Big]\,, \tag{25}$$

$$\Delta\alpha_{\rm top}(M_Z^2) = -\left[\mathrm{Re}\,\Pi^t_{\gamma\gamma}(M_Z^2) - \Pi^t_{\gamma\gamma}(0)\right]. \tag{26}$$

Here, small imaginary parts are omitted. The quantity $\Delta\alpha_{\rm lept}$ is known up to three-loop order of $O(\alpha^3)$ [19], and $\Delta\alpha_{\rm top}$ is known up to three-loop order of $O(\alpha\alpha_s^2)$ [20, 21, 22, 23] (see Appendix C.5 for detail). The resummation of the quantity $\Delta\alpha$ yields the formula

$$\alpha(M_Z^2) = \frac{\alpha(0)}{1 - \Delta\alpha(M_Z^2)}\,. \tag{27}$$

Also we denote the sum of the leptonic and 5-flavour hadronic contributions as

$$\Delta\alpha^{\ell+5q}(M_Z^2) = \Delta\alpha_{\rm lept}(M_Z^2) + \Delta\alpha_{\rm had}^{(5)}(M_Z^2)\,. \tag{28}$$

2.4 Splitting of radiative corrections

In the subsequent sections, we will split a radiative correction into leading and remainder parts [7], such that

- the leading part can be resummed to all orders,
- the leading part is larger than the remainder one, and
- the two parts are separately gauge invariant in order to resum the leading part.

According to Ref. [7], the coupling constant for the leading part should be G_μ, instead of $\alpha(0)$. Namely, a leading correction δX_L^α, having the coupling $\alpha(0)$, should be replaced to $\delta X_L^{G_\mu}$, having the coupling G_μ:

$$\delta X_L^\alpha \;\to\; \delta X_L^{G_\mu} \equiv f\cdot\delta X_L^\alpha \tag{29}$$

with the conversion factor

$$f = \frac{\sqrt{2}G_\mu M_Z^2 s_W^2 c_W^2}{\pi\alpha(0)}\,. \tag{30}$$

[5]Note that $\mathrm{Re}\,\Pi^f_{\gamma\gamma}(M_Z^2) - \Pi^f_{\gamma\gamma}(0) = \mathrm{Re}\,\Pi^{f,F}_{\gamma\gamma}(M_Z^2) - \Pi^{f,F}_{\gamma\gamma}(0)$.

2.5 Effective neutral-current interactions

The four-fermi interaction through charged currents is described by the single effective coupling G_μ defined in Eq. (9):

$$\mathcal{L} = -\frac{G_\mu}{\sqrt{2}}\, J^\dagger_{\mathrm{cc}\,\mu} J^\mu_{\mathrm{cc}} \tag{31}$$

for the charged current $J^\mu_{\mathrm{cc}} = \bar{u}\gamma^\mu(1-\gamma_5)d + \cdots$. On the other hand, the interaction through neutral currents is given by the Lagrangian

$$\mathcal{L} = -\rho^{(0)}\, \frac{G_\mu}{\sqrt{2}}\, J^{(0)\dagger}_{\mathrm{nc}\,\mu} J^{(0)\mu}_{\mathrm{nc}} \tag{32}$$

at the tree level, where the neutral current is defined by $J^{(0)\mu}_{\mathrm{nc}} = \sum_f \bar{f}[(I^f_3 - 2Q_f s^2_W)\gamma^\mu - I^f_3\gamma^\mu\gamma_5]f$. We specify the tree-level variables by adding the superscript "(0)" hereafter. The ρ parameter, denoting the ratio of the neutral and charged current couplings [24], is defined at very low q^2:

$$\rho = \frac{G_N}{G_\mu}\bigg|_{q^2\approx 0} = \frac{1+\hat{\Pi}_{WW}(0)}{1+\hat{\Pi}_{ZZ}(0)} = 1 + \left(\frac{\hat{\Sigma}_{ZZ}(0)}{M^2_Z} - \frac{\hat{\Sigma}_{WW}(0)}{M^2_W}\right) \equiv 1 + \Delta\rho\,, \tag{33}$$

where $\mathrm{Re}\hat{\Sigma}_{VV}(0) = \hat{\Sigma}_{VV}(0)$. An alternative definition of the ρ parameter is

$$\rho_{\mathrm{mass}} \equiv \frac{M^2_W}{M^2_Z c^2_W} = 1\,, \tag{34}$$

which is not altered by the radiative corrections.

The effective interaction between the Z boson and the neutral current is written in terms of the effective $Zf\bar{f}$ couplings g^f_V and g^f_A, or ρ^f_Z and κ^f_Z:

$$\begin{aligned}\mathcal{L} &= -\left(\sqrt{2}G_\mu M^2_Z\right)^{1/2} Z_\mu \sum_f \bar{f}\left(g^f_V\gamma_\mu - g^f_A\gamma_\mu\gamma_5\right) f\,,\\ &= -\left(\sqrt{2}G_\mu M^2_Z\rho^f_Z\right)^{1/2} Z_\mu \sum_f \bar{f}\left[(I^f_3 - 2Q_f\kappa^f_Z s^2_W)\gamma^\mu - I^f_3\gamma^\mu\gamma_5\right] f\,,\end{aligned} \tag{35}$$

where the couplings satisfy the relations

$$g^f_V = \sqrt{\rho^f_Z}\, I^f_3(1-4|Q_f|\kappa^f_Z s^2_W) = \sqrt{\rho^f_Z}\,(I^f_3 - 2Q_f\kappa^f_Z s^2_W)\,, \tag{36}$$

$$g^f_A = \sqrt{\rho^f_Z}\, I^f_3\,. \tag{37}$$

At the tree level, $\rho^{f(0)}_Z = 1$ and $\kappa^{f(0)}_Z = 1$, and thus the tree-level couplings are defined as

$$v_f \equiv g^{f(0)}_V = I^f_3 - 2Q_f s^2_W = I^f_3(1-4|Q_f|s^2_W)\,, \tag{38}$$

$$a_f \equiv g^{f(0)}_A = I^f_3\,. \tag{39}$$

2.6 Radiative correction Δr

As shown in Eq. (9), the W-boson mass is given in terms of the quantity Δr:

$$s_W^2 M_W^2 = \frac{\pi\alpha(0)}{\sqrt{2}G_\mu}(1+\Delta r)\,. \tag{40}$$

Let us derive the quantity Δr. First of all, the effective coupling constant G_μ receives radiative corrections:

$$\begin{aligned}\frac{G_\mu}{\sqrt{2}} &= \frac{(e^{\rm bare})^2}{8(s_W^{\rm bare})^2(M_W^{\rm bare})^2}\left[1+\frac{\Sigma_{WW}(0)}{M_W^2}+(\text{vertex + box + WF rem. of external lines})\right],\\ &= \frac{e^2}{8s_W^2M_W^2}\left[1+\frac{\delta e^2}{e^2}+\frac{c_W^2}{s_W^2}\left(\frac{\delta M_W^2}{M_W^2}-\frac{\delta M_Z^2}{M_Z^2}\right)-\frac{\delta M_W^2}{M_W^2}+\frac{\Sigma_{WW}(0)}{M_W^2}\right.\\ &\qquad\left.+(\text{vertex + box + WF rem. of external lines})\right].\end{aligned} \tag{41}$$

The explicit one-loop expression for Δr at $\mu = M_W$, denoted by Δr^α, can be found in Eqs. (E.8) and (F.3) of Ref. [18] (see also Refs. [6, 7]):

$$\begin{aligned}\Delta r^\alpha = \Delta\alpha^{\ell+5q,\alpha}(M_Z^2)+\frac{\alpha(0)}{4\pi s_W^2}\Bigg[&-\frac{2}{3}s_W^2 - s_W^2\overline{\Pi}^{t,F}_{\gamma\gamma}(0)\big|_{\mu=M_W} - s_W^2\,{\rm Re}\Big[\overline{\Pi}^{\ell+5q,F}_{\gamma\gamma}(M_Z^2)\big|_{\mu=M_W}\Big]\\ &+\frac{c_W^2}{s_W^2}\Delta\overline{\rho}^F\big|_{\mu=M_W}+\Delta\overline{\rho}^F_W\big|_{\mu=M_W}+\frac{11}{2}-\frac{5}{8}c_W^2(1+c_W^2)+\frac{9c_W^2}{4s_W^2}\ln c_W^2\Bigg],\end{aligned} \tag{42}$$

where the quantities $\Delta\overline{\rho}^F\big|_{\mu=M_W}$ and $\Delta\overline{\rho}^F_W\big|_{\mu=M_W}$ are the combinations of the self-energies defined by[6]

$$\Delta\overline{\rho}^F\big|_{\mu=M_W} = \frac{1}{M_W^2}\left[{\rm Re}\,\overline{\Sigma}^F_{WW}(M_W^2)-{\rm Re}\,\overline{\Sigma}^F_{ZZ}(M_Z^2)\right], \tag{43}$$

$$\Delta\overline{\rho}^F_W\big|_{\mu=M_W} = \frac{1}{M_W^2}\left[\overline{\Sigma}^F_{WW}(0)-{\rm Re}\,\overline{\Sigma}^F_{WW}(M_W^2)\right]. \tag{44}$$

The one-loop expressions for the gauge-boson self-energies are given in Appendix C.2, while those for $\Delta\overline{\rho}^F$ and $\Delta\overline{\rho}^F_W$ are in Appendix C.6. The quantity $\Delta\overline{\rho}^F$, which is associated with the correction to the ρ parameter in Eq. (33), represents the size of isospin violation from the large quadratic corrections of the top-quark mass. Rescaling them to $\mu = M_Z$, Eq. (42) becomes [6, 7]

$$\begin{aligned}\Delta r^\alpha = \Delta\alpha^{\ell+5q,\alpha}(M_Z^2)+\frac{\alpha(0)}{4\pi s_W^2}\Bigg[&-\frac{2}{3}s_W^2 - s_W^2\overline{\Pi}^{t,F}_{\gamma\gamma}(0)\big|_{\mu=M_Z} - s_W^2\,{\rm Re}\Big[\overline{\Pi}^{\ell+5q,F}_{\gamma\gamma}(M_Z^2)\big|_{\mu=M_Z}\Big]+\frac{c_W^2}{s_W^2}\Delta\overline{\rho}^F\big|_{\mu=M_Z}\\ &+\Delta\overline{\rho}^F_W\big|_{\mu=M_Z}+\left(\frac{N_f^{\rm total}}{6}-\frac{1}{6}-7c_W^2\right)\ln c_W^2+\frac{11}{2}-\frac{5}{8}c_W^2(1+c_W^2)+\frac{9c_W^2}{4s_W^2}\ln c_W^2\Bigg],\end{aligned} \tag{45}$$

where $N_f^{\rm total} = \sum_f N_c^f = 24$ is the number of the fermions with $N_c^q = 3$ and $N_c^\ell = 1$. Here $\Delta\overline{\rho}^F|_{\mu=m_Z}$ is gauge invariant. The quantity Δr^α is split into the three parts:

$$\Delta r^\alpha = \Delta\alpha^{\ell+5q,\alpha}(M_Z^2)-\frac{c_W^2}{s_W^2}\,\Delta\rho^\alpha+\Delta r^\alpha_{\rm rem}\,, \tag{46}$$

[6] Our $\Delta\overline{\rho}^F_{(W)}$ corresponds to $\Delta\rho^F_{(W)}$ used in ZFITTER.

where the leading and small remainder contributions, $\Delta\rho^{G_\mu} = f\Delta\rho^\alpha$ and $\Delta r^\alpha_{\rm rem}$, are given by

$$\Delta\rho^{G_\mu} = -f\frac{\alpha(0)}{4\pi s_W^2}\Delta\overline{\rho}^F\big|_{\mu=M_Z}\,, \tag{47}$$

$$\begin{aligned}\Delta r^\alpha_{\rm rem} = \frac{\alpha(0)}{4\pi s_W^2}\bigg[-\frac{2}{3}s_W^2 - s_W^2\overline{\Pi}^{t,F}_{\gamma\gamma}(0)\big|_{\mu=M_Z} - s_W^2{\rm Re}\Big[\overline{\Pi}^{\ell+5q,F}_{\gamma\gamma}(M_Z^2)\big|_{\mu=M_Z}\Big] + \Delta\overline{\rho}^F_W\big|_{\mu=M_Z} \\ + \left(\frac{N_f^{\rm total}}{6} - \frac{1}{6} - 7c_W^2\right)\ln c_W^2 + \frac{11}{2} - \frac{5}{8}c_W^2(1+c_W^2) + \frac{9c_W^2}{4s_W^2}\ln c_W^2\bigg]\,. \end{aligned}\tag{48}$$

The leading term $\Delta\rho^{G_\mu}$ contains the large quadratic corrections of the top-quark mass.

Below we summarize the known higher-order contributions to Δr,

$$\Delta r = \Delta\alpha^{\ell+5q}(M_Z^2) - \frac{c_W^2}{s_W^2}\Delta\rho + \Delta r_{\rm rem}\,, \tag{49}$$

except for the corrections to $\Delta\alpha^{\ell+5q}(M_Z^2)$:

1. full one-loop EW corrections of $O(\alpha)$ [15, 16],
2. full two-loop QCD corrections of $O(\alpha\alpha_s)$ [25, 26, 27, 28, 29, 30, 31],
3. leading+subleading three-loop QCD corrections to $\Delta\rho$:
   - $O\left(G_\mu\alpha_s^2M_t^2\right)$ [32, 33],
   - $O\left(G_\mu\alpha_s^2M_t^2(1+(M_Z^2/M_t^2)^2+(M_Z^2/M_t^2)^4)\right)$ [34],
4. full two-loop EW corrections:
   - $O(G_\mu^2M_t^4)$ [35, 36, 37, 38] and $O(G_\mu^2M_t^2M_Z^2)$ [39, 40, 41],
   - full $O(\alpha^2)$ contribution in terms of one-dimensional integral representations [42, 43, 44, 45, 46, 47, 48],
   - an approximate fitting formula for M_W including the full $O(\alpha^2)$ contribution [49],
5. leading three-loop corrections of $O(G_\mu^2\alpha_sM_t^4)$ and $O(G_\mu^3M_t^6)$ to $\Delta\rho$ for $m_h = 0$ [50] and for $m_h \approx M_t$ and $m_h \gg M_t$ [51],
6. pure fermion-loop corrections up to four-loop order [52],
7. leading three-loop EW corrections of $O(G_\mu^3m_h^4)$ to $\Delta\rho$ in the large m_h limit [53, 54], and
8. leading four-loop QCD corrections of $O(G_\mu\alpha_s^3M_t^2)$ to $\Delta\rho$ from the singlet diagrams of the Z-boson self-energy [55] and the non-singlet diagrams [56, 57],

where the last three contributions are very small and negligible. The magnitude of each contribution is shown later in Eq. (334). In this work, we include the following corrections:

$$\Delta r = \Delta\alpha^{\ell+5q}(M_Z^2) + \Delta r^\alpha + \Delta r^{\alpha\alpha_s} + \Delta r^{\alpha\alpha_s^2} + \Delta r^{\alpha^2} + \Delta r^{G_\mu^2\alpha_sM_t^4} + \Delta r^{G_\mu^3M_t^6}\,, \tag{50}$$

where each term is split into the leading and remainder parts:

$$\Delta r^\alpha = -\frac{c_W^2}{s_W^2}\Delta\rho^{G_\mu} + \Delta r^\alpha_{\rm rem},$$

$$
\begin{aligned}
\Delta r^{\alpha\alpha_s} &= -\frac{c_W^2}{s_W^2}\,\Delta\rho^{G_\mu\alpha_s} + \Delta r_{\rm rem}^{\alpha\alpha_s}, & \Delta r^{\alpha\alpha_s^2} &= -\frac{c_W^2}{s_W^2}\,\Delta\rho^{G_\mu\alpha_s^2}, \\
\Delta r^{\alpha^2} &= -\frac{c_W^2}{s_W^2}\,\Delta\rho^{G_\mu^2} + \Delta r_{\rm rem}^{\alpha^2}, & & \\
\Delta r^{G_\mu^2\alpha_s M_t^4} &= -\frac{c_W^2}{s_W^2}\,\Delta\rho^{G_\mu^2\alpha_s M_t^4}, & \Delta r^{G_\mu^3 M_t^6} &= -\frac{c_W^2}{s_W^2}\,\Delta\rho^{G_\mu^3 M_t^6}.
\end{aligned} \tag{51}
$$

(G or α when not resummed???) The $O(\alpha)$ contributions are given in Eqs. (47) and (48). The leading QCD corrections of $O(\alpha\alpha_s)$ and $O(\alpha\alpha_s^2)$, where the latter includes subleading terms of expansions in M_Z^2/M_t^2 [32, 33, 34], are given by

$$
\Delta\rho^{G_\mu\alpha_s} = 3\,X_t\,\frac{\alpha_s(M_t^2)}{\pi}\,\delta_2^{\rm QCD}\,, \tag{52}
$$

$$
\Delta\rho^{G_\mu\alpha_s^2} = 3\,X_t\left(\frac{\alpha_s(M_t^2)}{\pi}\right)^2\delta_3^{\rm QCD} \tag{53}
$$

with

$$
X_t \equiv \frac{G_\mu M_t^2}{8\sqrt{2}\pi^2}\,, \tag{54}
$$

while the remainder contribution $\Delta r_{\rm rem}^{\alpha\alpha_s}$ are written as follows [27] (see also [6, 7]):

$$
\Delta r_{\rm rem}^{\alpha\alpha_s} = 2\,\Delta r^{ud} + \Delta r^{tb} + \frac{c_W^2}{s_W^2}\frac{1}{f}\,\Delta\rho^{G_\mu\alpha_s}\,. \tag{55}
$$

In the remainder contribution, Δr^{ud} and Δr^{tb} represent the corrections to the bosonic self-energies appearing in Δr from fermion loops of a light-quark doublet and the heavy-quark $t-b$ doublet, respectively. The explicit formulas for $\delta_2^{\rm QCD}$, $\delta_3^{\rm QCD}$, Δr^{ud} and Δr^{tb} are given in Appendix C.8, where the imaginary part of Δr^{tb} has been neglected in Eq. (55). The two-loop EW contribution Δr^{α^2} is given in Eqs. (339), whereas the formulae for the leading three-loop corrections of $O(G_\mu^2\alpha_s M_t^4)$ and $O(G_\mu^3 M_t^6)$ [50, 51] are given in Appendix C.10.

In the ZFITTER codes, the leading contributions $\Delta\alpha$ and $\Delta\rho^\alpha$ in Δr are resummed separately [6, 7] (see Sec. 2.7). However, we do not undertake the resummation, and use the expanded form in Eq. (50), following the prescription in Ref. [49].

2.7 Resummation for Δr

The resummations of $\Delta\alpha$ and $\Delta\rho^\alpha$ in Eq. (46) are undertaken independently, such that the corrections $(\Delta\rho)^2$, $(\Delta\alpha\Delta\rho)$ and $(\Delta\alpha\Delta r_{\rm rem})$ at two-loop level and $(\Delta\alpha)^n$ to all orders can be correctly taken into account:

$$
1+\Delta r^\alpha \;\to\; \frac{1}{1-\Delta r^\alpha}\,, \tag{56}
$$

where

$$\frac{1}{1-\Delta r}=\begin{cases}\dfrac{1}{\left(1+\frac{c_W^2}{s_W^2}\,\Delta\rho^{(G)}\right)\left[1-\Delta\alpha^{\ell+5q}(M_Z^2)-\Delta r_{\rm rem}^{(\alpha)}\right]}\,,\\[2ex] \dfrac{1}{\left(1+\frac{c_W^2}{s_W^2}\,\Delta\rho^{(G)}\right)\left[1-\Delta\alpha^{\ell+5q}(M_Z^2)-\Delta r_{\rm rem}^{\alpha}\right]-\Delta r_{\rm rem}^{\alpha\alpha_s}-\Delta r_{\rm rem}^{\alpha^2}}\,,\\[2ex] \dfrac{1}{\left(1+\frac{c_W^2}{s_W^2}\,\Delta\rho^{(G)}\right)\left[1-\Delta\alpha^{\ell+5q}(M_Z^2)\right]-\left(1+\frac{c_W^2}{s_W^2}\,\Delta\rho^{G_\mu}\right)\Delta r_{\rm rem}^{\alpha}-\Delta r_{\rm rem}^{\alpha\alpha_s}-\Delta r_{\rm rem}^{\alpha^2}}\,,\end{cases}\tag{57}$$

in the so-called OMS-I, intermediate and OMS-II schemes [7], respectively. The superscripts (G) and (α) stands for the inclusion of the terms as

$$\Delta\rho^{(G)}=\Delta\rho^{G_\mu}+\Delta\rho^{G_\mu\alpha_s}+\Delta\rho^{G_\mu\alpha_s^2}+\Delta\rho^{G_\mu^2}+\Delta\rho^{G_\mu^2\alpha_s M_t^4}+\Delta\rho^{G_\mu^3 M_t^6},\tag{58}$$

$$\Delta r_{\rm rem}^{(\alpha)}=\Delta r_{\rm rem}^{\alpha}+\Delta r_{\rm rem}^{\alpha\alpha_s}+\Delta r_{\rm rem}^{\alpha^2}\,.\tag{59}$$

The W-boson mass is then given by

$$M_W^2=\frac{M_Z^2}{2}\left(1+\sqrt{1-\frac{4\pi\alpha}{\sqrt{2}G_\mu M_Z^2(1-\Delta r)}}\right).\tag{60}$$

According to Ref. [10], "**since typical resummation prescriptions for Δr (see Bardin et al., CERN95-03 and references therein) are problematic once the complete two-loop contributions are included, ...**"

2.8 W-boson mass

The W-boson mass is calculated from Eq. (40) as

$$M_W^2=\frac{M_Z^2}{2}\left(1+\sqrt{1-\frac{4\pi\alpha}{\sqrt{2}G_\mu M_Z^2}\,(1+\Delta r)}\right).\tag{61}$$

Note that the W-boson mass is computed from this relation iteratively, since the quantity Δr itself depends on M_W.

In Ref. [49], an approximate formula for the W-boson mass has been derived including full EW two-loop corrections of $O(\alpha^2)$ as well as leading $O(G_\mu^2\alpha_s M_t^4)$ and $O(G_\mu^3 M_t^6)$ corrections. See Appendix D.1 for detail. We will use the approximate formula, instead of the explicit formulae for the radiative corrections presented in the last subsections, when we compute observables in the SM. It is, however, noted that the approximate formula cannot be used when new physics contribution is taken into account, and the formula in Eq. (61) with the explicit formulae for the radiative corrections has to be employed.

2.9 Radiative corrections to ρ_Z^f and κ_Z^f

At one-loop level, the radiative corrections to ρ_Z^f and κ_Z^f are given in Refs. [58, 59] (see also [6, 7]):

$$\rho_Z^{f,\alpha}=1+\frac{\alpha(0)}{4\pi s_W^2}\left[-\frac{1}{c_W^2}{\rm Re}\big[\Sigma_{ZZ}^{\prime,F}(M_Z^2)\big|_{\mu=M_Z}\big]-\Delta\bar\rho_Z^F\big|_{\mu=M_Z}-\frac{11}{2}+\frac{5}{8}c_W^2(1+c_W^2)-\frac{9}{4}\frac{c_W^2}{s_W^2}\ln c_W^2+2u_f\right],$$

$$\equiv 1 + \Delta\rho^{\alpha} + \delta\rho^{f,\alpha}_{\rm rem}\,, \tag{62}$$

$$\kappa^{f,\alpha}_Z = 1 + \frac{\alpha(0)}{4\pi s^2_W}\left[-\frac{c^2_W}{s^2_W}\Delta\overline{\rho}^F\big|_{\mu=M_Z} - {\rm Re}\big[\overline{\Pi}^F_{Z\gamma}(M^2_Z)\big|_{\mu=M_Z}\big] + \frac{\delta^2_f}{4c^2_W}\mathcal{F}_Z(M^2_Z) - u_f\right],$$

$$\equiv 1 + \frac{c^2_W}{s^2_W}\Delta\rho^{\alpha} + \delta\kappa^{f,\alpha}_{\rm rem}\,, \tag{63}$$

(have to take real parts????) where the leading term $\Delta\rho^{\alpha}$ and the remainder terms $\delta\rho^{f,\alpha}_{\rm rem}$ and $\delta\kappa^{f,\alpha}_{\rm rem}$ are defined as

$$\Delta\rho^{\alpha} = -\frac{\alpha(0)}{4\pi s^2_W}\Delta\overline{\rho}^F\big|_{\mu=M_Z}\,, \tag{64}$$

$$\delta\rho^{f,\alpha}_{\rm rem} = \frac{\alpha(0)}{4\pi s^2_W}\left[-\frac{1}{c^2_W}{\rm Re}\big[\overline{\Sigma}'^{,F}_{ZZ}(M^2_Z)\big|_{\mu=M_Z}\big] - \Delta\overline{\rho}^F_W - \frac{11}{2} + \frac{5}{8}c^2_W(1+c^2_W) - \frac{9}{4}\frac{c^2_W}{s^2_W}\ln c^2_W + 2u_f\right], \tag{65}$$

$$\delta\kappa^{f,\alpha}_{\rm rem} = \frac{\alpha(0)}{4\pi s^2_W}\left[-{\rm Re}\big[\overline{\Pi}^F_{Z\gamma}(M^2_Z)\big|_{\mu=M_Z}\big] + \frac{\delta^2_f}{4c^2_W}\mathcal{F}_Z(M^2_Z) - u_f\right] \tag{66}$$

with

$$\Delta\overline{\rho}^F_Z = \frac{1}{M^2_W}\left[\overline{\Sigma}^F_{WW}(0) - {\rm Re}\,\overline{\Sigma}^F_{ZZ}(M^2_Z)\right] = \Delta\overline{\rho}^F + \Delta\overline{\rho}^F_W\,, \tag{67}$$

$$u_f = \frac{3v^2_f + a^2_f}{4c^2_W}\mathcal{F}_Z(M^2_Z) + \mathcal{F}_W(M^2_Z)\,, \tag{68}$$

$$\delta_f = v_f - a_f = -2Q_f s^2_W\,. \tag{69}$$

The expressions for the derivative of the Z-boson self-energy $\overline{\Sigma}'^{,F}_{ZZ}(M^2_Z)$ and those for the so-called unified form factors $\mathcal{F}_Z(M^2_Z)$ and $\mathcal{F}_W(M^2_Z)$, originating from radiative corrections to the $Zf\bar{f}$ vertex, are given in Appendix C.3 and C.7, respectively.

We include the higher-order corrections to $\Delta\rho$ as in Sec. 2.6, i.e.,

$$\Delta\rho = \Delta\rho^{G_\mu} + \Delta\rho^{G_\mu\alpha_s} + \Delta\rho^{G_\mu\alpha_s^2} + \Delta\rho^{G_\mu^2} + \Delta\rho^{G_\mu^2\alpha_s M_t^4} + \Delta\rho^{G_\mu^3 M_t^6}, \tag{70}$$

while those to $\delta\rho^f_{\rm rem}$ and $\delta\kappa^f_{\rm rem}$ are taken to be

$$\delta\rho^f_{\rm rem} = \delta\rho^{f,G_\mu}_{\rm rem} + \delta\rho^{f,G_\mu\alpha_s}_{\rm rem} + \delta\rho^{f,G_\mu^2}_{\rm rem}, \tag{71}$$

$$\delta\kappa^f_{\rm rem} = \delta\kappa^{f,G_\mu}_{\rm rem} + \delta\kappa^{f,G_\mu\alpha_s}_{\rm rem} + \delta\kappa^{f,G_\mu\alpha_s^2}_{\rm rem} + \delta\kappa^{f,G_\mu^2}_{\rm rem}. \tag{72}$$

The $O(\alpha)$ contributions $\delta\rho^{f,G_\mu}_{\rm rem}$ and $\delta\kappa^{f,G_\mu}_{\rm rem}$ are given in Eqs. (65) and (66), respectively, and the other contributions are given by

$$\delta\rho^{f,G_\mu\alpha_s}_{\rm rem} = f\left(2\,\Delta\rho^{ud} + \Delta\rho^{tb}\right) - \Delta\rho^{G_\mu\alpha_s}\,, \tag{73}$$

$$\delta\kappa^{f,G_\mu\alpha_s}_{\rm rem} = f\left(2\,\Delta\kappa^{ud} + \Delta\kappa^{tb}\right) - \frac{c^2_W}{s^2_W}\,\Delta\rho^{G_\mu\alpha_s}\,, \tag{74}$$

$$\delta\kappa^{f,G_\mu\alpha_s^2}_{\rm rem} = -3\,X_t\frac{c^2_W}{s^2_W}\left(\frac{\alpha_s(M^2_t)}{\pi}\right)^2\left(\delta^{\rm QCD}_3 + \delta^{\rm QCD}_{\kappa,3}\right), \tag{75}$$

where $\delta_3^{\rm QCD}$ also appears in Eq. (53), and $\Delta\rho^{ud}$, $\Delta\rho^{tb}$, $\Delta\kappa^{ud}$, $\Delta\kappa^{tb}$ and $\delta_{\kappa,3}^{\rm QCD}$ are given in Appendix C.8. The leading two-loop EW corrections of $O(G_\mu^2 M_t^4)$ [35, 36, 37, 38] and of $O(G_\mu^2 M_t^2 M_Z^2)$ [40, 41] are given in Appendix C.9.

The resummations of large corrections in the real parts of ρ_Z^f and κ_Z^f (i.e., $\Delta\rho^{tb}$, $\Delta\kappa^{tb}$ and $\delta_{\kappa,3}^{\rm QCD}$ in Eqs. (73), (74) and (75) must be taken to be their real parts) yield[7]

$$\rho_Z^f = \begin{cases} \dfrac{1+\delta\rho_{\rm rem}^{f,[G]}+\delta\rho_{\rm rem}^{f,G_\mu^2}}{1-\Delta\rho^{(G)}\left(1-\Delta\bar{r}_{\rm rem}^{G_\mu}\right)}\,, \\ \dfrac{1+\delta\rho_{\rm rem}^{f,[G]}}{1-\Delta\rho^{(G)}\left(1-\Delta\bar{r}_{\rm rem}^{G_\mu}\right)}+\delta\rho_{\rm rem}^{f,G_\mu^2}\,, \\ 1+\Delta\rho^{(G)}+\left(\Delta\rho^{G_\mu}\right)^2-\Delta\rho^{G_\mu}\Delta\bar{r}_{\rm rem}^{G_\mu}+\delta\rho_{\rm rem}^{f,[G]}\left(1+\Delta\rho^{G_\mu}\right)+\delta\rho_{\rm rem}^{f,G_\mu^2}\,, \end{cases} \tag{76}$$

$$\kappa_Z^f = \begin{cases} \left(1+\delta\kappa_{\rm rem}^{f,[G]}+\delta\kappa_{\rm rem}^{f,G_\mu^2}\right)\left[1+\dfrac{c_W^2}{s_W^2}\Delta\rho^{(G)}\left(1-\Delta\bar{r}_{\rm rem}^{G_\mu}\right)\right], \\ \left(1+\delta\kappa_{\rm rem}^{f,[G]}\right)\left[1+\dfrac{c_W^2}{s_W^2}\Delta\rho^{(G)}\left(1-\Delta\bar{r}_{\rm rem}^{G_\mu}\right)\right]+\delta\kappa_{\rm rem}^{f,G_\mu^2}\,, \\ 1+\dfrac{c_W^2}{s_W^2}\Delta\rho^{(G)}-\dfrac{c_W^2}{s_W^2}\Delta\rho^{G_\mu}\Delta\bar{r}_{\rm rem}^{G_\mu}+\delta\kappa_{\rm rem}^{f,[G]}\left(1+\dfrac{c_W^2}{s_W^2}\Delta\rho^{G_\mu}\right)+\delta\kappa_{\rm rem}^{f,G_\mu^2} \end{cases} \tag{77}$$

(Can we use the resummation formulae, when including $O(G_\mu^3 M_t^6)$ contribution?) in the OMS-I, intermediate and OMS-II schemes, respectively, where the superscripts (G) and $[G]$ stand for the inclusion of all relevant terms and that of one-loop EW corrections together with all known orders in α_s, $[G]=G_\mu+G_\mu\alpha_s+\cdots$:

$$\Delta\rho^{(G)} = \Delta\rho^{G_\mu}+\Delta\rho^{G_\mu\alpha_s}+\Delta\rho^{G_\mu\alpha_s^2}+\Delta\rho^{G_\mu^2}+\Delta\rho^{G_\mu^2\alpha_s M_t^4}+\Delta\rho^{G_\mu^3 M_t^6}, \tag{78}$$
$$\delta\rho_{\rm rem}^{f,[G]} = \delta\rho_{\rm rem}^{f,G_\mu}+\delta\rho_{\rm rem}^{f,G_\mu\alpha_s}, \tag{79}$$
$$\delta\kappa_{\rm rem}^{f,[G]} = \delta\kappa_{\rm rem}^{f,G_\mu}+\delta\kappa_{\rm rem}^{f,G_\mu\alpha_s}+\delta\kappa_{\rm rem}^{f,G_\mu\alpha_s^2}. \tag{80}$$

According to Refs. [6, 7], the one-loop remainder $\Delta\bar{r}_{\rm rem}^{G_\mu}$ is given by

$$\Delta\bar{r}_{\rm rem}^{G_\mu} = f\left[\Delta r_{\rm rem}^{\alpha}+\frac{\alpha(0)}{4\pi}\overline{\Pi}_{\gamma\gamma}^{t,F}(0)\big|_{\mu=M_Z}+\frac{\alpha(0)}{4\pi}{\rm Re}\left[\overline{\Pi}_{\gamma\gamma}^{\ell+5q,F}(M_Z^2)\big|_{\mu=M_Z}\right]\right], \tag{81}$$

where $\Delta r_{\rm rem}^{\alpha}$ is defined in Eq. (48).

For the imaginary parts of ρ_Z^f and κ_Z^f, we only keep the following contributions:

$${\rm Im}(\rho_Z^f) = {\rm Im}(\delta\rho_{\rm rem}^{f,\alpha})\,, \tag{82}$$
$${\rm Im}(\kappa_Z^f) = {\rm Im}(\delta\kappa_{\rm rem}^{f,\alpha})-\frac{\alpha(0)\alpha_s(M_Z^2)}{24\pi}\frac{(c_W^2-s_W^2)}{s_W^4}-\frac{1}{f}\,3\,X_t\left(\frac{\alpha_s(M_t^2)}{\pi}\right)^2{\rm Im}\left[\delta_{\kappa,3}^{\rm QCD}\right], \tag{83}$$

[7] In the ZFITTER paper [7], $\Delta r_{\rm rem}^{G_\mu\alpha_s}$ is added to $\Delta\bar{r}_{\rm rem}^{G_\mu}$, but it is not included in the ZFITTER code actually. See also Refs. [39, 40, 41].

where **the last term is absent in the ZFITTER codes.**

(Check the second term (taken from the ZFITTER codes))

(α or G_μ????)

Furthermore, κ_Z^f is modified by a sizable $O(\alpha^2)$ contribution originating from the product of the imaginary part of $\Delta\alpha$ and that of $\Pi_{Z\gamma}$ [6, 7][8]:

$$\begin{aligned}\mathrm{Re}(\kappa_Z^f) &= \mathrm{Re}(\kappa_Z^f)\big|_{\mathrm{Eq.}(77)} + \frac{1}{s_W^2}\left(\frac{\alpha(M_Z^2)}{4\pi}\right)^2 \mathrm{Im}\,\overline{\Pi}^{\mathrm{fer}}_{\gamma\gamma}(M_Z^2)\,\mathrm{Im}\,\overline{\Pi}^{\mathrm{fer}}_{Z\gamma}(M_Z^2)\\ &= \mathrm{Re}(\kappa_Z^f)\big|_{\mathrm{Eq.}(77)} + \frac{35\alpha^2(M_Z^2)}{18 s_W^2}\left(1-\frac{8}{3}\mathrm{Re}(\kappa_Z^f)s_W^2\right),\end{aligned} \tag{84}$$

where $\alpha(M_Z^2)$ is given in Eq. (27).

2.10 ρ_Z^b and κ_Z^b in $Z \to b\bar{b}$

In the SM, the large top-quark mass gives important corrections to the EW observables through the gauge-boson self-energies, i.e., $\Delta\rho$, and the $Zb\bar{b}$ vertex. The latter contribution, which affects ρ_Z^b and κ_Z^b for $Z \to b\bar{b}$, is parametrised by the quantity τ_b in the literature:

$$\tau_b = -2\,X_t\left[1-\frac{\pi}{3}\alpha_s(M_t^2) + X_t\,\tau^{(2)}\left(\frac{M_t^2}{M_Z^2}\right)\right], \tag{85}$$

where the $O(G_\mu\alpha_s M_t^2)$ term was calculated in Ref. [60, 61, 62, 63], and the $O(G_\mu^2 M_t^4)$ term can be found in Ref. [35, 36, 37, 38][9]. The explicit form of the function $\tau^{(2)}$ is given in Appendix C.11. It is noted that we have already included the virtual top-quark contribution of $O(G_\mu)$, which corresponds to the first term in Eq. (85), into Eqs. (65) and (66) through the unified form factors $\mathcal{F}_Z(M_Z^2)$ and $\mathcal{F}_W(M_Z^2)$ (see also Eqs. (243), (244) and (245)). Hence we have to subtract it from ρ_Z^b and κ_Z^b to avoid double counting:

$$\rho_Z^b \to \rho_Z^b + \frac{\alpha(0)}{4\pi s_W^2}\frac{M_t^2}{M_W^2}, \tag{86}$$

$$\kappa_Z^b \to \kappa_Z^b - \frac{\alpha(0)}{8\pi s_W^2}\frac{M_t^2}{M_W^2}. \tag{87}$$

The form factors ρ_Z^b and κ_Z^b are then obtained by taking into account τ_b as

$$\rho_Z^b \to \rho_Z^b(1+\tau_b)^2, \tag{88}$$

$$\kappa_Z^b \to \frac{\kappa_Z^b}{1+\tau_b}. \tag{89}$$

We use the above formulas for ρ_Z^b, but not for κ_Z^b. Instead, we use the approximate two-loop formula for κ_Z^b [64] presented in Appendix D.2. Let us explain the computation of ρ_Z^b more concretely. For consistency, we do not take into account the $O(G_\mu^2 M_t^2 M_Z^2)$ contribution to $\Delta\rho$ in ρ_Z^b:

$$\Delta\rho^{(G)} = \Delta\rho^{G_\mu} + \Delta\rho^{G_\mu\alpha_s} + \Delta\rho^{G_\mu\alpha_s^2} + \Delta\rho^{G_\mu^2 M_t^4}, \tag{90}$$

[8] In the ZFITTER codes, this factor is not taken into account for the $b\bar{b}$ channel.

[9] In the ZFITTER codes, the $O(G_\mu^2 M_t^2 M_Z^2)$ contribution to $\Delta\rho$ is neglected in the case of $Z \to b\bar{b}$, since the $O(G_\mu^2 M_t^2 M_Z^2)$ contribution to τ_b is not included.

where $\Delta\rho^{G_\mu^2 M_t^4} = 3X_t^2\rho^{(2)}$ is given in Appendix C.9. The remainder contribution $\delta\rho^f_{\rm rem}$ is the same as before, but must undertake the subtraction in Eq. (86):

$$\delta\rho^{b,[G]}_{\rm rem} = \left(\delta\rho^{b,G_\mu}_{\rm rem} + \frac{\alpha(0)}{4\pi s_W^2}\frac{M_t^2}{M_W^2}\right) + \delta\rho^{b,G_\mu\alpha_s}_{\rm rem}. \tag{91}$$

The leading contribution is then resummed and multiplied by the factor $(1+\tau_b)^2$ as

$$(\rho^b_Z)_L = \frac{1}{1-\Delta\rho^{(G)}}(1+\tau_b)^2. \tag{92}$$

Then, ρ^b_Z is given by **(Check!!!)**

$$\rho^b_Z = \begin{cases} \dfrac{1}{1-1/(\rho^b_Z)_L - \delta\rho^{b,[G]}_{\rm rem}}\,, \\[2ex] (\rho^b_Z)_L\left[1+(\rho^b_Z)_L\,\delta\rho^{b,[G]}_{\rm rem}\right], \\[2ex] (\rho^b_Z)_L + \delta\rho^{b,[G]}_{\rm rem}\,. \end{cases} \tag{93}$$

(taken from ZFITTER codes)

2.11 Effective weak mixing angle

We define the effective mixing angle for a given fermion f as

$$\sin^2\theta^f_{\rm eff} \equiv {\rm Re}(\kappa^f_Z)s_W^2 = \frac{1}{4|Q_f|}\left[1-{\rm Re}\left(\frac{g^f_V}{g^f_A}\right)\right]. \tag{94}$$

The radiative corrections to the weak mixing angle are given by two contributions: the corrections to M_W and those to $Zf\bar{f}$ vertices, the latter correction, entering into κ^f_Z, depends on the flavour of final-state fermions.

Recently, approximate formulae for the effective weak mixing angles have been presented in Ref. [65, 66, 64], including full EW two-loop corrections of $O(\alpha^2)$ as well as leading $O(G_\mu^2\alpha_s M_t^4)$ and $O(G_\mu^3 M_t^6)$ corrections, where the bosonic two-loop EW contribution is still missing only in the $b\bar{b}$ case. We can use those formulae to determine the EW form factor ${\rm Re}(\kappa^f_Z)$ through Eq. (94). See Appendix D.2 for detail.

On the other hand, the complete two-loop formulae for ρ^f_Z have been missing at the current moment. Therefore, we only include the partial two-loop contributions to ρ^f_Z as presented in the last subsections.

For the imaginary parts of ρ^f_Z and κ^f_Z, we use Eqs. (82) and (83) **(Check!!)**.

2.12 Z-pole observables

In this subsection, we present Z-pole observables, which depend on the radiative corrections through $|\rho^f_Z|$, ${\rm Re}\,(\kappa^f_Z)$ and ${\rm Im}\,(\kappa^f_Z)$ as well as Δr.

The asymmetry parameter $\mathcal{A}_f$ for a channel $Z \to f\bar{f}$ is defined in terms of the effective couplings:

$$\mathcal{A}_f = \frac{2\,\mathrm{Re}\left(g_V^f/g_A^f\right)}{1 + \left[\mathrm{Re}\left(g_V^f/g_A^f\right)\right]^2} \tag{95}$$

with

$$\mathrm{Re}\left(\frac{g_V^f}{g_A^f}\right) = 1 - 4|Q_f|\,\mathrm{Re}\left(\kappa_Z^f\right) s_W^2 \,. \tag{96}$$

Using the asymmetry parameter, the forward-backward asymmetry of a channel $f\bar{f}$, the longitudinal polarization of a final-state $\tau\bar{\tau}$, and the left-right asymmetry are given by

$$A_{\mathrm{FB}}^{0,f} = \frac{3}{4}\mathcal{A}_e\mathcal{A}_f \,. \tag{97}$$

$$P_\tau^{\mathrm{pol}} = \mathcal{A}_\tau \,. \tag{98}$$

$$A_{\mathrm{LR}}^0 = \mathcal{A}_e \,. \tag{99}$$

The partial width of Z decay into a lepton pair $\ell\bar{\ell}$, including contribution from final-state QED interactions, is given by

$$\Gamma_\ell = \Gamma_0\big|\rho_Z^f\big|\sqrt{1 - \frac{4m_\ell^2}{M_Z^2}}\left[\left(1 + \frac{2m_\ell^2}{M_Z^2}\right)\left(\left|\frac{g_V^\ell}{g_A^\ell}\right|^2 + 1\right) - \frac{6m_\ell^2}{M_Z^2}\right]\left(1 + \frac{3}{4}\frac{\alpha(M_Z^2)}{\pi}Q_\ell^2\right), \tag{100}$$

where $\Gamma_0 = G_\mu M_Z^3/(24\sqrt{2}\pi)$. In the case of the $Z \to q\bar{q}$ channels, final-state QCD interactions have to be taken into account in addition to the QED interactions:

$$\Gamma_q = N_c^q\,\Gamma_0\big|\rho_Z^q\big|\left[\left|\frac{g_V^q}{g_A^q}\right|^2 R_V^q(M_Z^2) + R_A^q(M_Z^2)\right] + \Delta_{\mathrm{EW/QCD}} \,, \tag{101}$$

where $R_V^q(s)$ and $R_A^q(s)$ are called the radiator factors defined in Appendix C.12 [67], and $\Delta_{\mathrm{EW/QCD}}$ denotes non-factorizable EW$\otimes$QCD corrections [7, 68, 69],

$$\Delta_{\mathrm{EW/QCD}} = \begin{cases} -0.113 \text{ MeV} & \text{for } q = u, c \,, \\ -0.160 \text{ MeV} & \text{for } q = d, s \,, \\ -0.040 \text{ MeV} & \text{for } q = b \,. \end{cases} \tag{102}$$

The total decay width of the Z boson, Γ_Z, is then given by the sum of all channels:

$$\Gamma_Z = \Gamma_h + \Gamma_e + \Gamma_\mu + \Gamma_\tau + \Gamma_{\mathrm{inv}} \tag{103}$$

with the hadronic width $\Gamma_h = \sum_q \Gamma_q$ and the invisible width $\Gamma_{\mathrm{inv}} = 3\,\Gamma_\nu$. The ratios of the widths are defined as

$$R_q^0 = \frac{\Gamma_q}{\Gamma_h} \,, \tag{104}$$

$$R_\ell^0 = \frac{\Gamma_h}{\Gamma_\ell} \,, \tag{105}$$

while the cross sections for $e^+e^- \to Z \to$ hadrons and $e^+e^- \to Z \to \ell\bar{\ell}$ at the Z pole are

$$\sigma_h^0 = \frac{12\pi}{M_Z^2}\frac{\Gamma_e\Gamma_h}{\Gamma_Z^2} \,, \tag{106}$$

$$\sigma_\ell^0 = \frac{12\pi}{M_Z^2}\frac{\Gamma_e\Gamma_\ell}{\Gamma_Z^2} \,. \tag{107}$$

2.13 W-boson width

In Ref. [70], the width of the W-boson decay into $f_i \bar{f}_j$ was calculated at one-loop level:

$$\Gamma_{ij}^W = N_c^f \, |V_{ij}|^2 \, \frac{G_\mu M_W^3}{6\sqrt{2}\,\pi} \, \rho_{ij}^W \tag{108}$$

where $V_{ij} = (V_{\mathrm{CKM}})_{ij}$ for quarks and $V_{ij} = \delta_{ij}$ for leptons, and the factor ρ_{ij}^W for EW radiative corrections is defined by

$$\rho_{ij}^W = 1 + \delta f_{ij}^W + \delta f_{ij}^{\mathrm{QED}} \tag{109}$$

with

$$\begin{aligned}
\delta f_{ij}^W = \frac{\alpha(0)}{4\pi s_W^2} \Bigg[& - \Delta\overline{\rho}_W^F - \mathrm{Re}\big[\overline{\Sigma}_{WW}^{\prime,F}(M_W^2)\big]\big|_{\mu=M_W} + \frac{5}{8} c_W^2 (1 + c_W^2) - \frac{11}{2} - \frac{9}{4}\frac{c_W^2}{s_W^2} \ln c_W^2 \\
& + \left(-1 + \frac{1}{2\,c_W^2} + \frac{2 s_W^4}{c_W^2} Q_i Q_j \right) \left(V_1(M_W^2, M_Z^2) + \frac{3}{2} \right) \\
& + 2\,c_W^2 \left(V_2(M_W^2, M_W^2, M_Z^2) + \frac{3}{2} \right) \Bigg] ,
\end{aligned} \tag{110}$$

$$\delta f_{ij}^{\mathrm{QED}} = \frac{\alpha(0)}{\pi} \left[\frac{85}{18} - \frac{\pi^2}{3} + \frac{3}{4} Q_i Q_j \right] . \tag{111}$$

The functions $V_1(M_W^2, M_Z^2)$ and $V_2(M_W^2, M_W^2, M_Z^2)$ are given in Ref. [18]:

$$\begin{aligned}
V_1(M_W^2, M_Z^2) &= \mathcal{F}_{Za}^0(M_W^2) - \frac{3}{2} , \\
&= -5 - \frac{2}{c_W^2} + \left(3 + \frac{2}{c_W^2} \right) \ln c_W^2 - \frac{2(1 + c_W^2)^2}{c_W^4} \left[\mathrm{Li}_2(1) - \mathrm{Li}_2(1 + c_W^2) \right] ,
\end{aligned} \tag{112}$$

$$V_2(M_W^2, M_W^2, M_Z^2) = \cdots \tag{113}$$

and the expression for $\overline{\Sigma}_{WW}^{\prime,F}(M_W^2)$ is presented in Appendix C.4. In the case of $W \to q_i \bar{q}_j$, the final-state QCD corrections have to be taken into account[10]:

$$\Gamma_{ij}^W \to \Gamma_{ij}^W \left(1 + \frac{\alpha_s(M_W^2)}{\pi} \right) . \tag{114}$$

See also Refs. [12, 71] for contributions from the finite fermion masses.

2.14 Oblique parameters

Simple parametrization of the radiative corrections to the EW observables, which are useful for analysis of new physics (NP) beyond the SM, are given in Refs. [72, 73] and [74]. The parameters S, T and U were introduced in the former, while the parameters ϵ_1, ϵ_2 and ϵ_3 were in the latter. The latter parametrization is given by

$$\epsilon_1 = \Delta\rho' \tag{115}$$

[10] In the ZFITTER code, $R_V^u(M_Z^2)$ is used for the final-state corrections. Probably, the contribution from Q_u in $R_V^u(M_Z^2)$ has been neglected, since it is tiny.

$$\epsilon_2 = c_0^2 \Delta\rho' + \frac{s_0^2}{c_0^2 - s_0^2} \Delta r_W - 2 s_0^2 \Delta\kappa' \,, \tag{116}$$

$$\epsilon_3 = c_0^2 \Delta\rho' + (c_0^2 - s_0^2) \Delta\kappa' \,, \tag{117}$$

where $\Delta\rho'$, Δ_W and $\Delta\kappa'$ are defined by

$$\sqrt{\rho_Z^e} = 1 + \frac{\Delta\rho'}{2} \,, \qquad \frac{1 - \Delta r_W}{1 - \Delta r} = \frac{\alpha(M_Z^2)}{\alpha(0)} \,, \qquad \sin^2\theta_{\rm eff}^e = (1 + \Delta\kappa') s_0^2 \,, \tag{118}$$

and s_0^2 and $c_0^2 \equiv 1 - s_0^2$ are given by

$$s_0^2 c_0^2 = \frac{\pi\alpha(M_Z^2)}{\sqrt{2}\, G_\mu M_Z^2} \,, \ \to \ s_0^2 = \frac{1}{2}\left(1 - \sqrt{1 - \frac{4\pi\alpha(M_Z)}{\sqrt{2}\, G_\mu M_Z^2}}\,\right) \,, \tag{119}$$

Moreover, the parameters S, T and U can be written in terms of ϵ_1, ϵ_2 and ϵ_3 as

$$S = \frac{4 s_0^2}{\alpha(0)}\, \epsilon_3 \,, \tag{120}$$

$$T = \frac{1}{\alpha(0)}\, \epsilon_1 \,, \tag{121}$$

$$U = -\frac{4 s_0^2}{\alpha(0)}\, \epsilon_2 \,. \tag{122}$$

2.15 Effective Lagrangian approach

NP may affect significantly one-loop vacuum polarization amplitudes for the gauge bosons. We consider the effective Lagrangian with the four vacuum polarization amplitudes[11]:

$$\mathcal{L}_{\rm vac-pol} = -\frac{1}{2} W_\mu^3\, \hat\Sigma_{33}(q^2) W^{3\mu} - \frac{1}{2} B_\mu\, \hat\Sigma_{00}(q^2) B^\mu - W_\mu^3\, \hat\Sigma_{30}(q^2) B^\mu - W_\mu^+ \hat\Sigma_{WW}(q^2) W^{-\mu}. \tag{123}$$

Assuming the NP scale is sensibly higher than the weak scale, the vacuum polarizations can be expanded in powers of q^2:

$$\hat\Sigma_{VV'} \simeq \hat\Sigma_{VV'}(0) + q^2\, \hat\Sigma'_{VV'}(0) + \frac{(q^2)^2}{2!}\, \hat\Sigma''_{VV'}(0) + \cdots \,, \tag{124}$$

where $VV' = \{33,\, 00,\, 30,\, WW\}$. There are $4 \times 3 = 12$ coefficients up to $O(q^4)$, where three of them are absorbed in the definitions of g, g' and v. Moreover, there exist two relations, $\hat\Sigma_{\gamma\gamma}(0) = \hat\Sigma_{Z\gamma}(0) = 0$ due to $U(1)_{\rm EM}$. Namely, we have 7 oblique parameters up to $O(q^4)$[12], parameterized as [75, 76]

$$\hat S = \frac{g}{g'}\, \hat\Sigma'_{30}(0) \,, \qquad \hat T = \frac{\hat\Sigma_{33}(0) - \hat\Sigma_{WW}(0)}{M_W^2} \,, \qquad \hat U = \hat\Sigma'_{WW}(0) - \hat\Sigma'_{33}(0) \,,$$

$$V = \frac{M_W^2}{2}\left[\hat\Sigma''_{33}(0) - \hat\Sigma''_{WW}(0)\right] , \quad X = \frac{M_W^2}{2}\, \hat\Sigma''_{30}(0) \,, \quad Y = \frac{M_W^2}{2}\, \hat\Sigma''_{00}(0) \,, \quad W = \frac{M_W^2}{2}\, \hat\Sigma''_{33}(0) \,, \tag{125}$$

where $\hat S$, $\hat T$ and $\hat U$ are related to S, T and U by $S = 4 s_W^2 \hat S/\alpha(0)$, $T = \hat T/\alpha(0)$ and $U = -4 s_W^2 \hat U/\alpha(0)$, respectively. The parameters $\hat T$, $\hat U$ and V break the custodial symmetry. Assuming $SU(2)_L$ invariance,

[11] The vacuum polarization amplitude $\hat\Sigma_{VV'}$ in this notes corresponds to $\Pi_{VV'}$ in Ref. [75].

[12] There exist $3 = 4 \times 2 - 5$ parameters up to $O(q^2)$.

the oblique parameters obey the hierarchies $\hat{S} \gg X$ and $\hat{T} \gg \hat{U} \gg V$, and thus the 4 parameters $\hat{S}$, $\hat{T}$, W and Y are enough to be considered for "universal" NP. Note that new observables ϵ_{ZZ}, $\epsilon_{Z\gamma}$ and $\epsilon_{\gamma\gamma}$ [75], associated with the differential cross sections for $e^+e^- \to f\bar{f}$ via the propagators of the Z-γ system, have to be considered in order to constrain Y and W as well as X, since the Z-pole observables discussed in the previous section correspond only to the 3 parameters $\hat{S}$, $\hat{T}$ and $\hat{U}$. The new observables are written in terms of the oblique parameters as

$$\epsilon_{ZZ} = c_W^2 W - 2 s_W c_W X + s_W^2 Y \,, \tag{126}$$

$$\epsilon_{Z\gamma} = (c_W^2 - s_W^2) X + s_W c_W (W - Y) \,, \tag{127}$$

$$\epsilon_{\gamma\gamma} = s_W^2 W + 2 s_W c_W X + c_W^2 Y \,. \tag{128}$$

3 Input data

Table 1: Inputs. **(Check!)**

Parameters	Input values
M_Z [GeV]	91.1875 ± 0.0021 [5]
$\alpha_s(M_Z^2)$	0.1184 ± 0.0007 [5]
G_μ [10^{-5} GeV^{-2}]	1.16637 ± 0.00001 [5]
$\alpha(0)$	$1/137.035999679(94)$ [5]
$\Delta\alpha^{(5)}_{\rm had}(M_Z^2)$	0.02758 ± 0.00035
m_h [GeV]	–
M_t [GeV]	173.3 ± 1.1 [Tevatron]

Table 2: Data. **(Check!)**

Observables	Data
Γ_Z [GeV]	2.4952 ± 0.0023 [5]
$A_{\rm FB}^{0,\ell}$	0.0171 ± 0.0010 [5]
$A_{\rm FB}^{0,c}$	0.0707 ± 0.0035 [5]
$A_{\rm FB}^{0,b}$	0.0992 ± 0.0016 [5]
$P_\tau^{\rm pol}$	0.1465 ± 0.0033
$\mathcal{A}_\ell$ (SLD)	0.1513 ± 0.0021
$\mathcal{A}_c$	0.670 ± 0.027 [5]
$\mathcal{A}_b$	0.923 ± 0.020 [5]
R_ℓ^0	20.767 ± 0.025 [5]
R_c^0	0.1721 ± 0.0030 [5]
R_b^0	0.21629 ± 0.00066 [5]
σ_h^0 [nb]	41.540 ± 0.037
$\sin^2\theta_{\rm eff}^{\rm lept}$	0.2324 ± 0.0012
M_W [GeV]	80.399 ± 0.023 [5]
Γ_W [GeV]	2.085 ± 0.042 [5]

4 SUSY contribution to the EW observables

4.1 W-boson mass

For vanishing complex phases and minimal flavour violation (MFV),

- the complete one-loop contribution [77, 78] (see also [79])
- two-loop QCD corrections of $O(\alpha\alpha_s)$ to $\Delta\rho$ [80, 81]
- two-loop EW corrections of $O(\alpha_t^2)$, $O(\alpha_t\alpha_b)$ and $O(\alpha_b^2)$ to $\Delta\rho$ [82, 83]

NMFV

- the leading one-loop contributions with non-minimal-flavour-violation (NMFV) effects [84]

With phases

- one-loop with phase to $\Delta\rho$ [85]

See also Ref. [86] for a summary of the SUSY contributions, and Ref. [87] for a global fit with flavour observables.

4.2 Effective weak mixing angle

4.3 Z-pole observables

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A Special functions

A.1 Logarithm function

$$\ln(-x \pm i\,\epsilon) \;=\; \ln x \pm i\,\pi\,, \qquad \text{for } x > 0\,. \tag{129}$$

A.2 Zeta functions

$$\zeta(2) = \frac{\pi^2}{6}\,, \tag{130}$$

A.3 Dilogarithm function

The dilogarithm function, also called the Spence function, is defined as the following integral form:

$$\mathrm{Li}_2(x) \;=\; -\int_0^x dt\,\frac{\ln(1-t)}{t}\,, \tag{131}$$

which sometimes appears in the integrals of loop calculations. We list some useful identities:

$$\mathrm{Li}_2(1) \;=\; \zeta(2)\,, \tag{132}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) \;=\; \zeta(2) - \ln(x)\ln(1-x)\,, \tag{133}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2\left(\frac{1}{x}\right) \;=\; -\zeta(2) - \frac{1}{2}\ln^2(-x)\,, \tag{134}$$

where the last two relations are valid for complex arguments.

A.4 Clausen function

$$\mathrm{Cl}_2(\phi) = \mathrm{Im}\big[\mathrm{Li}_2(e^{i\phi})\big]\,. \tag{135}$$

A.5 Some constants

The constants S_2, D_3, and B_4 are defined as

$$S_2 = \frac{4}{9\sqrt{3}}\,\mathrm{Cl}_2\left(\frac{\pi}{3}\right) \approx 0.260434137632162\,, \tag{136}$$

$$D_3 = 6\,\zeta(3) - \frac{15}{4}\,\zeta(4) - 6\left[\mathrm{Cl}_2\left(\frac{\pi}{3}\right)\right]^2 \approx -3.02700949398765\,, \tag{137}$$

$$B_4 = 16\,\mathrm{Li}_4\left(\frac{1}{2}\right) - 4\,\zeta(2)\ln^2 2 + \frac{2}{3}\ln^4 2 - \frac{13}{2}\,\zeta(4) \approx -1.76280008707377\,. \tag{138}$$

B Passarino-Veltman Functions

We define the general form of one-loop integrals as

$$T^N_{\mu_1\cdots\mu_P}(p_{10}^2, p_{21}^2\cdots, p_{0,N-1}^2; m_0, \cdots, m_{N-1}) \;=\; (-1)^N\,\frac{(2\pi\mu)^{4-d}}{i\,\pi^2}\int d^dk\,\frac{k_{\mu_1}\cdots k_{\mu_P}}{D_0 D_1\cdots D_{N-1}} \tag{139}$$

with $D_0 = k^2 - m_0^2 + i\,\varepsilon$ and $D_i = (k+p_i)^2 - m_i^2 + i\,\varepsilon$ for $i = 1, \cdots, N-1$. Here the factor $(-1)^N$ is added in order to employ the same definition of the Passarino-Veltman functions as in Refs. [6, 7], in which the sign of the metric is opposite to that adopted in this work. The one-point integral, two-point integrals, three-point integrals, etc., are denoted by the alphabet as $T^1 = A$, $T^2 = B$, $T^3 = C$. We also define external momenta as

$$p_{10} \;=\; p_1\,, \qquad p_{ij} \;=\; p_i - p_j\,, \qquad p_{0,N-1} \;=\; -p_{N-1}\,. \tag{140}$$

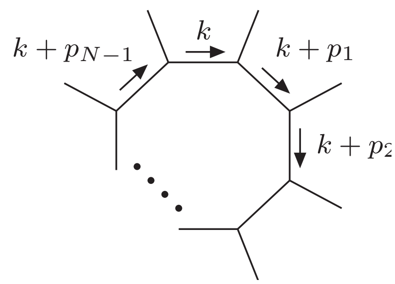

Figure 1: N-point one-loop diagram.

B.1 One-point function

The scalar one-point function is UV divergent:

$$\begin{aligned} A_0(m) \;&=\; -\frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,\frac{1}{k^2 - m^2 + i\varepsilon}\,, \\ &=\; -m^2\left(\frac{1}{\bar{\varepsilon}} - \ln\frac{m^2}{\mu^2} + 1\right), \end{aligned} \tag{141}$$

where $\bar{\varepsilon}$ is defined by

$$\frac{1}{\bar{\epsilon}} \;=\; \frac{2}{4-d} - \gamma_E + \ln(4\pi)\,. \tag{142}$$

In Refs. [6, 7], the factor $(2\pi)^{4-d}$ is not included in the definition of the loop functions, and thus $1/\bar{\epsilon} = 2/(4-d) - \gamma_E + \ln\pi$. Note that $A_0(0) = 0$ in the dimensional regularization.

B.2 Two-point function

According, *e.g.*, to Ref. [88], the scalar two-point function is given by

$$\begin{aligned} B_0(p^2;\, m_0, m_1) \;&=\; \frac{(2\pi\mu)^{4-d}}{i\pi^2}\int d^dk\,\frac{1}{(k^2 - m_0^2 + i\varepsilon)\left[(k+p)^2 - m_1^2 + i\varepsilon\right]}\,, \\ &=\; \frac{1}{\bar{\epsilon}} - \int_0^1 dx\,\ln\left(\frac{p^2x^2 - x(p^2 - m_0^2 + m_1^2) + m_1^2 - i\varepsilon}{\mu^2}\right), \\ &=\; \frac{1}{\bar{\epsilon}} - \ln\frac{m_0 m_1}{\mu^2} + \frac{m_0^2 - m_1^2}{2p^2}\ln\frac{m_1^2}{m_0^2} - R + 2 + i\pi\,\frac{\Lambda}{p^2}\,\theta\left(p^2 - (m_0+m_1)^2\right), \end{aligned} \tag{143}$$

where R is defined as

$$R = \begin{cases} -\dfrac{\Lambda}{p^2}\arctan\left(\dfrac{\Lambda}{p^2 - m_0^2 - m_1^2}\right) & \text{for } (m_0 - m_1)^2 < p^2 < (m_0 + m_1)^2, \\ \dfrac{\Lambda}{p^2}\ln\left|\dfrac{p^2 - m_0^2 - m_1^2 + \Lambda}{2m_0 m_1}\right| & \text{otherwise} \end{cases} \tag{144}$$

with $\Lambda = \sqrt{|(m_0^2 + m_1^2 - p^2)^2 - 4m_0^2 m_1^2|}$.

For other special cases, we have

$$\begin{aligned} B_0(p^2;\, 0, m) &= \frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2} - \left(1 - \frac{m^2}{p^2}\right)\ln\left(1 - \frac{p^2 + i\varepsilon}{m^2}\right) + 2\,, \\ &= \frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2} - \left(1 - \frac{m^2}{p^2}\right)\ln\left(1 - \frac{p^2}{m^2}\right)\theta(m^2 - p^2) \\ &\quad - \left(1 - \frac{m^2}{p^2}\right)\left[\ln\left(\frac{p^2}{m^2} - 1\right) - i\,\pi\right]\theta(p^2 - m^2) + 2\,, \end{aligned} \tag{145}$$

$$B_0(m^2;\, 0, m) = \frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2} + 2\,, \tag{146}$$

$$B_0(0;\, m_0, m_1) = \frac{1}{\bar{\epsilon}} - \frac{A_0(m_0) - A_0(m_1)}{m_0^2 - m_1^2} = \frac{1}{\bar{\epsilon}} - \frac{m_0^2}{m_0^2 - m_1^2}\ln\frac{m_0^2}{\mu^2} + \frac{m_1^2}{m_0^2 - m_1^2}\ln\frac{m_1^2}{\mu^2} + 1\,, \tag{147}$$

$$B_0(0;\, m, m) = \frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2}\,, \tag{148}$$

$$B_0(0;\, 0, m) = \frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2} + 1\,, \tag{149}$$

$$\begin{aligned} B_0(p^2;\, 0, 0) &= \frac{1}{\bar{\epsilon}} - \ln\frac{-p^2 - i\varepsilon}{\mu^2} + 2\,, \\ &= \frac{1}{\bar{\epsilon}} - \ln\frac{-p^2}{\mu^2}\theta(-p^2) - \left(\ln\frac{p^2}{\mu^2} - i\,\pi\right)\theta(p^2) + 2\,, \end{aligned} \tag{150}$$

In the case of $p^2 = 0$, $m_0^2 = 0$ and $m_1^2 = 0$, the integral, i.e $B_0(0; 0, 0)$, is also IR divergent. Noth that $\lim_{p^2 \to 0} p^2 B_0(p^2;\, m_0, m_1) \to \frac{1}{\bar{\epsilon}} + 0$ in the limit of $p^2 \to 0$.

The higher-rank two-point functions B_1 and B_{21} are given by

$$B_1(p^2;\, m_0, m_1) = -\frac{1}{2p^2}\left[A_0(m_0) - A_0(m_1) + \left(\Delta m_{01}^2 + p^2\right)B_0(p^2; m_0, m_1)\right], \tag{151}$$

$$\begin{aligned} B_{21}(p^2; m_0, m_1) = &-\frac{3(m_0^2 + m_1^2) - p^2}{18p^2} + \frac{\Delta m_{01}^2 + p^2}{3p^4}A_0(m_0) - \frac{\Delta m_{01}^2 + 2p^2}{3p^4}A_0(m_1) \\ &+ \frac{\bar{\Lambda}^2 + 3p^2 m_0^2}{3p^4}B_0(p^2; m_0, m_1) \end{aligned} \tag{152}$$

with $\Delta m_{01}^2 = m_0^2 - m_1^2$ and $\bar{\Lambda}^2 = (m_0^2 + m_1^2 - p^2)^2 - 4m_0^2 m_1^2$, and the function B_f is defined as a combination of them:

$$B_f(p^2; m_0, m_1) = 2\left[B_{21}(p^2; m_0, m_1) + B_1(p^2; m_0, m_1)\right]. \tag{153}$$

In the case of $p^2 = 0$, B_1 and B_{21} read

$$B_1(0;\, m_0, m_1) = -\frac{1}{2}\left[\frac{1}{\bar{\epsilon}} - \ln\frac{m_1^2}{\mu^2} - F_2\left(\frac{m_0^2}{\Delta m_{01}^2}\right)\right], \tag{154}$$

$$B_{21}(0;\,m_0,m_1) = \frac{1}{3}\left[\frac{1}{\bar{\epsilon}} - \ln\frac{m_1^2}{\mu^2} - F_3\left(\frac{m_0^2}{\Delta m_{01}^2}\right)\right] \tag{155}$$

with

$$F_n(x) = -\frac{1}{n} + xF_{n-1}(x)\,, \qquad F_0(x) = -\ln\left(1-\frac{1}{x}\right). \tag{156}$$

Moreover, we have

$$B_1(0;\,m,m) = -\frac{1}{2}\left(\frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2}\right), \tag{157}$$

$$B_{21}(0;\,m,m) = \frac{1}{3}\left(\frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2}\right), \tag{158}$$

$$B_1(0;\,0,m) = -\frac{1}{2}\left(\frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2} + \frac{1}{2}\right), \tag{159}$$

$$B_{21}(0;\,0,m) = \frac{1}{3}\left(\frac{1}{\bar{\epsilon}} - \ln\frac{m^2}{\mu^2} + \frac{1}{3}\right), ??? \tag{160}$$

and

$$\lim_{p^2\to 0} B_1(p^2;\,0,0) = 0\,. \tag{161}$$

We also define the derivatives of the two-point functions, which appear in the wave-function remornalizations,

$$B_{0p}(p^2;\,m_0,m_1) \equiv \frac{\partial}{\partial p^2}B_0(p^2;\,m_0.m_1)\,, \tag{162}$$

$$\begin{aligned}
B_{1p}(p^2;\,m_0,m_1) &\equiv \frac{\partial}{\partial p^2}B_1(p^2;\,m_0.m_1)\,,\\
&= \frac{1}{2p^4}\left[B_1(p^2;\,m_0,m_1) - p^2B_0(p^2;\,m_0,m_1) - p^4B_{0p}(p^2;\,m_0,m_1)\right]
\end{aligned} \tag{163}$$

$$\begin{aligned}
B_{21p}(p^2;\,m_0,m_1) &\equiv \frac{\partial}{\partial p^2}B_{21}(p^2;\,m_0.m_1)\,,\\
&= \frac{m_0^2+m_1^2}{6p^4} - \frac{2\Delta m_{01}^2+p^2}{3p^6}A_0(m_0) + \frac{2\left(\Delta m_{01}^2+p^2\right)}{3p^6}A_0(m_1)\\
&\quad + \frac{2m_1^2-m_0^2}{3p^4}B_0(p^2;m_0,m_1) + \frac{\bar{\Lambda}^2+3p^2m_0^2}{3p^4}B_{0p}(p^2;m_0,m_1)\,,
\end{aligned} \tag{164}$$

$$\begin{aligned}
B_{fp}(p^2;\,m_0,m_1) &\equiv \frac{\partial}{\partial p^2}B_f(p^2;\,m_0.m_1)\,,\\
&= 2\left[B_{21p}(p^2;m_0,m_1) + B_{1p}(p^2;m_0,m_1)\right].
\end{aligned} \tag{165}$$

The explicit expressions for B_{0p} are given as follows:

$$\begin{aligned}
B_{0p}(p^2;\,m_0,m_1) &= -\frac{m_0^2-m_1^2}{2p^4}\ln\frac{m_1^2}{m_0^2} - \frac{1}{p^2}(R'+1)\\
&\quad + i\,\pi\frac{1}{p^2}\left(\frac{p^2-m_0^2-m_1^2}{\Lambda} - \frac{\Lambda}{p^2}\right)\theta\left(p^2-(m_0+m_1)^2\right),
\end{aligned} \tag{166}$$

$$
\begin{aligned}
B_{0p}(p^2;\,0,m) &= -\frac{m^2}{p^4}\ln\left(1-\frac{p^2}{m^2}\right)\theta(m^2-p^2) \\
&\quad -\frac{m^2}{p^4}\left[\ln\left(\frac{p^2}{m^2}-1\right)-i\,\pi\right]\theta(p^2-m^2)+\frac{1}{m^2}\,, && (167) \\
B_{0p}(m^2;\,0,m) &= \frac{1}{2m^2}\left(\frac{1}{\epsilon_{\rm IR}}+\ln\frac{m^2}{\mu^2}-2\right), && (168) \\
B_{0p}(p^2;\,0,0) &= -\frac{1}{p^2}\,, && (169) \\
B_{0p}(0;\,m_0,m_1) &= \frac{m_0^2+m_1^2}{2(\Delta m_{01}^2)^2}+\frac{m_0^2 m_1^2}{(\Delta m_{01}^2)^3}\ln\frac{m_1^2}{m_0^2}\,, && (170) \\
B_{0p}(0;\,m,m) &= ??? && (171) \\
B_{0p}(0;\,0,m) &= \frac{1}{2m^2} && (172)
\end{aligned}
$$

with

$$
R' = \begin{cases}
\left(\dfrac{p^2-m_0^2-m_1^2}{\Lambda}+\dfrac{\Lambda}{p^2}\right)\arctan\left(\dfrac{\Lambda}{p^2-m_0^2-m_1^2}\right) & \text{for } (m_0-m_1)^2<p^2<(m_0+m_1)^2, \\
\left(\dfrac{p^2-m_0^2-m_1^2}{\Lambda}-\dfrac{\Lambda}{p^2}\right)\ln\left|\dfrac{p^2-m_0^2-m_1^2+\Lambda}{2m_0m_1}\right| & \text{otherwise}
\end{cases}
\tag{173}
$$

Note that B_{0p} is UV finite.

In the limit of $p^2\to 0$, the Taylor expantions of the B_0 function are given by

$$
\begin{aligned}
\lim_{p^2\to 0} B_0(p^2;\,m_0,m_1) &= \frac{1}{\bar\epsilon}-\frac{A_0(m_0)-A_0(m_1)}{m_0^2-m_1^2}+p^2 B_{0p}(0;\,m_0,m_1)\,, && (174) \\
\lim_{p^2\to 0} B_0(p^2;\,m,m) &= \frac{1}{\bar\epsilon}-\ln\frac{m^2}{\mu^2}+\frac{p^2}{6m^2}\,, && (175) \\
\lim_{p^2\to 0} B_0(p^2;\,0,m) &= \frac{1}{\bar\epsilon}-\ln\frac{m^2}{\mu^2}+1+\frac{p^2}{2m^2}\,. && (176)
\end{aligned}
$$

B.3 Three-point function

In this study, we only need the following function with $p_1^2=p_2^2=0$ and $p^2\equiv(p_1+p_2)^2$:

$$
\begin{aligned}
C_0(p^2;\,m_0,m_1,m_0) &\equiv -\frac{1}{i\pi^2}\int d^4k\,\frac{1}{(k^2-m_1^2+i\varepsilon)\left[(k+p_1)^2-m_0^2+i\varepsilon\right]\left[(k-p_2)^2-m_0^2+i\varepsilon\right]}\,, \\
&= -\frac{1}{p^2}\Bigg[\mathrm{Li}_2\left(\frac{x_0}{x_0-x_1}\right)-\mathrm{Li}_2\left(\frac{x_0-1}{x_0-x_1}\right)+\mathrm{Li}_2\left(\frac{x_0}{x_0-x_2}\right) \\
&\qquad -\mathrm{Li}_2\left(\frac{x_0-1}{x_0-x_2}\right)-\mathrm{Li}_2\left(\frac{x_0}{x_0-x_3}\right)+\mathrm{Li}_2\left(\frac{x_0-1}{x_0-x_3}\right)\Bigg] && (177)
\end{aligned}
$$

with

$$
x_0 = 1-\frac{m_0^2-m_1^2}{p^2}\,,\quad x_{1,2}=\frac{1}{2}\left(1\pm\sqrt{1-\frac{4m_0^2}{p^2}}\right),\quad x_3=\frac{m_0^2}{m_0^2-m_1^2}\,, \tag{178}
$$

(**this formula should be checked!!**) where all masses squared are understood to have infinitesimal imaginary part as $m_i^2 \to m_i^2 - i\varepsilon$. For some special cases, we have

$$C_0(p^2;\, 0, m^2, 0) = -\frac{1}{p^2}\left[\mathrm{Li}_2(1) - \mathrm{Li}_2\left(1 + \frac{p^2 + i\varepsilon}{m^2}\right)\right], \tag{179}$$

$$C_0(p^2;\, m^2, 0, m^2) = -\frac{1}{p^2}\ln^2\frac{\sqrt{1 - 4m^2/p^2} + 1}{\sqrt{1 - 4m^2/p^2} - 1}\,. \tag{180}$$

B.4 Splitting of pole and finite parts

In Refs. [6, 7], the loop functions A_0 and B_0 are split into pole and finite parts at an artificial scale $\mu = M_W$, where the former contains the singular term and the μ-dependent term, while the latter is the rest:

$$A_0(M) = -M^2\left(\frac{1}{\bar{\epsilon}} - \ln\frac{M_W^2}{\mu^2}\right) + A_0^F(M)\,, \tag{181}$$

$$B_0(p^2; M_1, M_2) = \frac{1}{\bar{\epsilon}} - \ln\frac{M_W^2}{\mu^2} + B_0^F(p^2; M_1, M_2) \tag{182}$$

with the finite term

$$A_0^F(M) = M^2\left(\ln\frac{M^2}{M_W^2} - 1\right), \tag{183}$$

and so on. Note that the above definition of finite contribution corresponds to the $\overline{\mathrm{MS}}$ scheme with $\mu = M_W$: e.g.,

$$B_0^F(p^2; M_1, M_2) = B_0(p^2; M_1, M_2)\big|_{\overline{\mathrm{MS}},\, \mu = M_W}\,, \tag{184}$$

and we have the relation

$$B_0^F(p^2; M_1, M_2)\big|_{\mu = M_W} = B_0^F(p^2; M_1, M_2)\big|_{\mu = M_Z} + \ln c_W^2\,, \tag{185}$$

$$B_f^F(p^2; 0, 0)\big|_{\mu = M_W} = B_f^F(p^2; 0, 0)\big|_{\mu = M_Z} - \frac{1}{3}\ln c_W^2\,. \tag{186}$$

C Formulae for radiative corrections in the SM

C.1 Notations

- the color factor: $N_c^f = 3$ for quarks and $N_c^f = 1$ for leptons.
- the total number of the fermions: $N_f^{\mathrm{total}} = \sum_f N_c^f = 24$.
- the weak mixing angle:

$$s_W^2 = \sin^2\theta_W = 1 - \frac{M_W^2}{M_Z^2}\,, \qquad c_W^2 = 1 - s_W^2\,. \tag{187}$$

- the tree-level couplings:

$$
\begin{aligned}
v_f &= I_3^f - 2Q_f s_W^2 = I_3^f (1 - 4|Q_f| s_W^2)\,, && (188)\\
a_f &= I_3^f\,, && (189)\\
\sigma_f^a &= |v_f + a_f| = 1 - 2|Q_f| s_W^2\,, && (190)\\
\delta_f &= v_f - a_f = -2Q_f s_W^2\,. && (191)
\end{aligned}
$$

- some shorthand notations:

$$
\begin{aligned}
&R_W = \frac{M_W^2}{s}\,, \qquad R_Z = \frac{M_Z^2}{s}\,, \qquad r_W = \frac{m_h^2}{M_W^2}\,, \qquad r_Z = \frac{m_h^2}{M_Z^2}\,,\\
&w_t = \frac{M_t^2}{M_W^2}\,, \qquad z_t = \frac{M_t^2}{M_Z^2}\,, \qquad r_{4t}^Z = \frac{M_Z^2 + i\varepsilon}{4M_t^2}\,, \qquad r_{4t}^s = \frac{s + i\varepsilon}{4M_t^2}\,, \qquad x_t^W = \frac{M_W^2 + i\varepsilon}{M_t^2}\,. \qquad (192)
\end{aligned}
$$

C.2 Self-energies of the gauge bosons of $O(\alpha)$

The self-energy functions of the gauge bosons (see Appendix A.3 of Ref. [7] for their definitions) are the sums of bosonic and fermionic contributions:

$$
\begin{aligned}
\overline{\Sigma}_{WW}(s) &= \overline{\Sigma}^{\rm bos}_{WW}(s) + \overline{\Sigma}^{\rm fer}_{WW}(s)\,, && (193)\\
\overline{\Sigma}_{ZZ}(s) &= \overline{\Sigma}^{\rm bos}_{ZZ}(s) + \overline{\Sigma}^{\rm fer}_{ZZ}(s)\,, && (194)\\
\overline{\Pi}_{\gamma\gamma}(s) &= \overline{\Pi}^{\rm bos}_{\gamma\gamma}(s) + \overline{\Pi}^{\rm fer}_{\gamma\gamma}(s)\,, && (195)\\
\overline{\Pi}_{Z\gamma}(s) &= \overline{\Pi}^{\rm bos}_{Z\gamma}(s) + \overline{\Pi}^{\rm fer}_{Z\gamma}(s)\,. && (196)
\end{aligned}
$$

In the unitary gauge, the bosonic parts are given by

$$
\begin{aligned}
\overline{\Sigma}^{\rm bos}_{WW}(s) = M_W^2 \Biggl\{ \Biggl[& \left(\frac{1}{12 c_W^4} + \frac{2}{3}\frac{1}{c_W^2} - \frac{3}{2} + \frac{2}{3} c_W^2 + \frac{1}{12} c_W^4 \right) R_W + \frac{2}{3} \left(\frac{1}{c_W^2} - 4 - 4c_W^2 + c_W^4 \right)\\
& - \left(\frac{3}{2} + \frac{8}{3} c_W^2 + \frac{3}{2} c_W^4 \right) \frac{1}{R_W} + \frac{2}{3} c_W^2 \left(1 + c_W^2\right) \frac{1}{R_W^2} + \frac{1}{12} c_W^4 \frac{1}{R_W^3} \Biggr] B_0(s; M_Z, M_W)\\
& - \frac{s_W^2}{6} \left(-5R_W + 17 + 17\frac{1}{R_W} - 5\frac{1}{R_W^2} \right) B_0(s; 0, M_W)\\
& - \frac{1}{12} \left[-\left(1 - r_W\right)^2 R_W - 10 + 2r_W - \frac{1}{R_W} \right] B_0(s; m_h, M_W)\\
& - \frac{1}{12} \left[\left(\frac{1}{c_W^2} - 2 + c_W^2 - c_W^4 + r_W \right) R_W + 24 - 2c_W^2 + c_W^4 + \frac{-10 + c_W^2 + c_W^4}{R_W} - \frac{c_W^4}{R_W^2} \right] \frac{A_0(M_W)}{M_W^2}\\
& - \frac{1}{12} \left[-\left(\frac{1}{c_W^2} + 9 - 9c_W^2 - c_W^4 \right) R_W + 1 + 14c_W^2 + 9c_W^4 + \frac{c_W^2}{R_W}\left(1 - 9c_W^2\right) - \frac{c_W^4}{R_W^2} \right] \frac{A_0(M_Z)}{M_W^2}\\
& + \frac{1}{12} \left(\frac{m_h^2 - M_W^2}{s} - 2 \right) \frac{A_0(m_h)}{M_W^2} - \frac{1}{6} \left(\frac{1}{c_W^2} + 22 + c_W^2 + c_W^4 + r_W \right)\\
& + \frac{1}{9} \left[\left(6 + 3c_W^2 + \frac{7}{2} c_W^4 \right) \frac{1}{R_W} - \left(1 + \frac{3}{2} c_W^2 + \frac{5}{2} c_W^4 \right) \frac{1}{R_W^2} + \frac{c_W^4}{2}\frac{1}{R_W^3} \right] \Biggr\}\,, \qquad (197)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Sigma}^{\text{bos}}_{WW}(0) &= \lim_{p^2\to 0} \overline{\Sigma}^{\text{bos}}_{WW}(p^2)\,,\\
&= M_W^2 \Bigg[\frac{2}{3}\left(\frac{1}{c_W^2} - 4 - 4c_W^2 + c_W^4\right) B_0(0; M_Z, M_W) \\
&\quad + \left(\frac{1}{12 c_W^4} + \frac{2}{3}\frac{1}{c_W^2} - \frac{3}{2} + \frac{2}{3}c_W^2 + \frac{1}{12}c_W^4\right) M_W^2 B_{0p}(0; M_Z, M_W) \\
&\quad - \frac{17 s_W^2}{6} B_0(0; 0, M_W) + \frac{5 s_W^2}{12} \\
&\quad - \frac{1}{12}(-10 + 2r_W) B_0(0; m_h, M_W) + \frac{1}{12}(1 - r_W^2) M_W^2 B_{0p}(0; m_h, M_W) \\
&\quad - \frac{1}{12}(24 - 2c_W^2 + c_W^4)\frac{A_0(M_W)}{M_W^2} - \frac{1}{6}\frac{A_0(m_h)}{M_W^2} \\
&\quad - \frac{1}{12}(1 + 14 c_W^2 + 9 c_W^4)\frac{A_0(M_Z)}{M_W^2} - \frac{1}{6}\left(\frac{1}{c_W^2} + 22 + c_W^2 + c_W^4 + r_W\right)\Bigg]\,, \qquad (198)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Sigma}^{\text{bos}}_{ZZ}(s) &= M_W^2 \Bigg\{ - c_W^4 \left(4 + \frac{17}{3}\frac{1}{R_W} - \frac{4}{3}\frac{1}{R_W^2} - \frac{1}{12}\frac{1}{R_W^3}\right) B_0(s; M_W, M_W) \\
&\quad + \frac{1}{12}\left[\left(\frac{1}{c_W^4} - \frac{2}{c_W^2} r_W + r_W^2\right) R_W + \frac{10}{c_W^2} - 2r_W + \frac{1}{R_W}\right] B_0(s; m_h, M_Z) \\
&\quad - c_W^2 \left(4 - \frac{4}{3}\frac{1}{R_W} - \frac{1}{6}\frac{1}{R_W^2}\right)\frac{A_0(M_W)}{M_Z^2} \\
&\quad + \frac{1}{12}\left(\frac{M_Z^2 - m_h^2}{s} + 1\right)\frac{A_0(M_Z) - A_0(m_h)}{c_W^2 M_Z^2} - \frac{1}{12}\frac{A_0(m_h)}{c_W^2 M_Z^2} \\
&\quad - \left[\frac{1}{6 c_W^2} + 4c_W^4 + \frac{1}{6} r_W - \left(\frac{1}{18} + \frac{4}{3}c_W^4\right)\frac{1}{R_W} + \frac{5}{9}c_W^4 \frac{1}{R_W^2} - \frac{1}{18}c_W^4 \frac{1}{R_W^3}\right]\Bigg\}\,, \qquad (199)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\Pi}^{\text{bos}}_{\gamma\gamma}(s) &= R_W \Bigg\{ \left[4 + \frac{17}{3}\frac{1}{R_W} - \frac{4}{3}\frac{1}{R_W^2} - \frac{1}{12}\frac{1}{R_W^3}\right] B_0(s; M_W, M_W) \\
&\quad + \left(4 - \frac{4}{3}\frac{1}{R_W} - \frac{1}{6}\frac{1}{R_W^2}\right)\left[\frac{A_0(M_W)}{M_W^2} + 1\right] - \frac{1}{18}\frac{1}{R_W^3} + \frac{13}{18}\frac{1}{R_W^2}\Bigg\}\,, \qquad (200)
\end{aligned}
$$

$$
\overline{\Pi}^{\text{bos}}_{Z\gamma}(s) = c_W^2\, \overline{\Pi}^{\text{bos}}_{\gamma\gamma}(s)\,, \qquad (201)
$$

while the fermionic parts are given by

$$
\overline{\Sigma}^{\text{fer}}_{WW}(s) = \sum_{\{f,f'\}=\text{doublets}} N_c^f \left[-s\, B_f(s; m_{f'}, m_f) + m_f^2\, B_1(s; m_{f'}, m_f) + m_{f'}^2\, B_1(s; m_f, m_{f'})\right]\,, \qquad (202)
$$

$$
\overline{\Sigma}^{\text{fer}}_{ZZ}(s) = \sum_f N_c^f \left[-(v_f^2 + a_f^2)\, s\, B_f(s; m_f, m_f) - 2a_f^2 m_f^2\, B_0(s; m_f, m_f)\right]\,, \qquad (203)
$$

$$
\overline{\Pi}^{\text{fer}}_{\gamma\gamma}(s) = 4 \sum_f N_c^f Q_f^2 B_f(s; m_f, m_f)\,, \qquad (204)
$$

$$
\overline{\Pi}^{\text{fer}}_{Z\gamma}(s) = \sum_f N_c^f \left(|Q_f| - 4 s_W^2 Q_f^2\right) B_f(s; m_f, m_f)\,. \qquad (205)
$$

where the sum in the first line runs over the 6 fermion doublets in the SM. We denote each contribution from a fermion f as $\overline{\Sigma}^{f}_{WW}(s)$, $\overline{\Sigma}^{f}_{ZZ}(s)$, $\overline{\Pi}^{f}_{\gamma\gamma}(s)$ and $\overline{\Pi}^{f}_{Z\gamma}(s)$. In ZFITTER, the self-energy for the Z boson is corrected by $O(\alpha^2)$ contribtuion from the Z-γ mixing (Dyson resummation):

$$\overline{\Sigma}^{\rm fer}_{ZZ}(M_Z^2) \to \overline{\Sigma}^{\rm fer}_{ZZ}(M_Z^2) - \frac{\alpha(0)}{4\pi} M_W^2 \left[\overline{\Pi}^{\rm fer}_{Z\gamma}(M_Z^2)\right]^2 . \tag{206}$$

(This relation is taken from ZFITTER codes. Should be checked carefully! See also Sec. 6.11 of [6].) The finite part of the fermionic self-energies for $\mu = M_W$, $\overline{\Sigma}^{{\rm fer},F}_{WW}(s)|_{\mu=M_W}$, $\overline{\Sigma}^{{\rm fer},F}_{ZZ}(s)|_{\mu=M_W}$, $\overline{\Pi}^{{\rm fer},F}_{\gamma\gamma}(s)|_{\mu=M_W}$ and $\overline{\Pi}^{{\rm fer},F}_{Z\gamma}(s)|_{\mu=M_W}$, are obtained by replacing the B functions to the corresponding their finite parts: $B_{0,1,f} \to B^F_{0,1,f}$. For example, the finite contributions for the last twos defined at $\mu = M_W$ are rescaled to those at $\mu = M_Z$ with the relation

$$\overline{\Pi}^{{\rm fer},F}_{\gamma\gamma}(s)\big|_{\mu=M_W} = \overline{\Pi}^{{\rm fer},F}_{\gamma\gamma}(s)\big|_{\mu=M_Z} - \frac{4}{3}\sum_f N_c^f Q_f^2 \ln c_W^2 \,, \tag{207}$$

$$\overline{\Pi}^{{\rm fer},F}_{Z\gamma}(s)\big|_{\mu=M_W} = \overline{\Pi}^{{\rm fer},F}_{Z\gamma}(s)\big|_{\mu=M_Z} - \frac{1}{3}\sum_f N_c^f (|Q_f| - 4 s_W^2 Q_f^2) \ln c_W^2 \,. \tag{208}$$

Moreover, we have

$$\overline{\Pi}^{{\rm bos},F}_{Z\gamma}(s)\big|_{\mu=M_W} = \overline{\Pi}^{{\rm bos},F}_{Z\gamma}(s)\big|_{\mu=M_Z} + c_W^2 R_W \left[4 + \frac{17}{3}\frac{1}{R_W} - \frac{4}{3}\frac{1}{R_W^2} - \frac{1}{12}\frac{1}{R_W^3}\right] \ln c_W^2 \,. \tag{209}$$

C.3 Z-boson wave-function renormalization of $O(\alpha)$

The wave-function renormalization of the Z boson, $Z_0^\mu = \mathcal{Z}_Z^{1/2} Z_R^\mu$, satisfies

$$\mathcal{Z}_Z - 1 = -\frac{\alpha(0)}{4\pi s_W^2 c_W^2} \frac{\partial\, \overline{\Sigma}_{ZZ}(p^2)}{\partial p^2}\bigg|_{p^2=M_Z^2} \equiv -\frac{\alpha(0)}{4\pi s_W^2 c_W^2} \overline{\Sigma}'_{ZZ}(M_Z^2) \,. \tag{210}$$

where the on-shell derivative is split into bosonic and fermionic parts:

$$\frac{1}{c_W^2}\overline{\Sigma}'_{ZZ}(M_Z^2) = \frac{1}{c_W^2}\overline{\Sigma}'^{,{\rm bos}}_{ZZ}(M_Z^2) + \frac{1}{c_W^2}\overline{\Sigma}'^{,{\rm fer}}_{ZZ}(M_Z^2) \,. \tag{211}$$

The bosonic contribution is given by[13]

$$\begin{aligned}
\frac{1}{c_W^2}\overline{\Sigma}'^{,{\rm bos}}_{ZZ}(M_Z^2) = &\left(\frac{1}{4c_W^2} + \frac{8}{3} - \frac{17}{3}c_W^2\right) B_0(M_Z^2; M_W, M_W) \\
&+ \left(\frac{1}{12 c_W^2} + \frac{4}{3} - \frac{17}{3}c_W^2 - 4c_W^4\right) M_Z^2\, B_{0p}(M_Z^2; M_W, M_W) \\
&+ \frac{r_W}{6}\left(1 - \frac{r_Z}{2}\right) B_0(M_Z^2; m_h, M_Z) + \left(1 - \frac{r_Z}{3} + \frac{r_Z^2}{12}\right)\frac{M_Z^2}{c_W^2}\, B_{0p}(M_Z^2; m_h, M_Z)
\end{aligned}$$

[13] Useful relations

$$M_Z^2 \frac{\partial}{\partial s} R_W\Big|_{s=M_Z^2} = -c_W^2 \,, \quad M_Z^2 \frac{\partial}{\partial s}\frac{1}{R_W}\Big|_{s=M_Z^2} = \frac{1}{c_W^2} \,, \quad M_Z^2 \frac{\partial}{\partial s}\frac{1}{R_W^2}\Big|_{s=M_Z^2} = \frac{2}{c_W^4} \,, \quad M_Z^2 \frac{\partial}{\partial s}\frac{1}{R_W^3}\Big|_{s=M_Z^2} = \frac{3}{c_W^6} \,. \tag{212}$$

$$
\begin{aligned}
&+\frac{1}{3c_W^2}\left(1+4c_W^2\right)\frac{A_0(M_W)}{M_Z^2}-\frac{(1-r_Z)}{12c_W^2}\frac{[A_0(M_Z)-A_0(m_h)]}{M_Z^2}\\
&+\frac{2}{9c_W^2}-\frac{10}{9}+\frac{4}{3}c_W^2\,.
\end{aligned}
\tag{213}
$$

On the other hand, the fermionic contribution reads as

$$
\begin{aligned}
\frac{1}{c_W^2}\overline{\Sigma}'^{,\mathrm{fer}}_{ZZ}(M_Z^2)=-\frac{1}{c_W^2}\sum_f N_c^f\Big\{&(v_f^2+a_f^2)\left[B_f(M_Z^2;m_f,m_f)+M_Z^2\,B_{fp}(M_Z^2;m_f,m_f)\right]\\
&+2a_f^2m_f^2\,B_{0p}(M_Z^2;m_f,m_f)\Big\}\,.
\end{aligned}
\tag{214}
$$

Note that we have the following relations for the finite parts:

$$
\frac{1}{c_W^2}\overline{\Sigma}'^{\mathrm{bos},F}_{ZZ}(M_Z^2)\Big|_{\mu=M_W}=\frac{1}{c_W^2}\overline{\Sigma}'^{,\mathrm{bos},F}_{ZZ}(M_Z^2)\Big|_{\mu=M_Z}-\left(-\frac{1}{3\,c_W^2}-\frac{7}{3}+7\,c_W^2\right)\ln c_W^2\,,
\tag{215}
$$

$$
\frac{1}{c_W^2}\overline{\Sigma}'^{,\mathrm{fer},F}_{ZZ}(M_Z^2)\Big|_{\mu=M_W}=\frac{1}{c_W^2}\overline{\Sigma}'^{,\mathrm{fer},F}_{ZZ}(M_Z^2)\Big|_{\mu=M_Z}-\left[\frac{N_f^{\mathrm{total}}}{6}\left(\frac{1}{c_W^2}-2\right)-\frac{4\,s_W^4}{3\,c_W^2}\sum_f N_c^fQ_f^2\right]\ln c_W^2\,.
\tag{216}
$$

C.4 W-boson wave-function renormalization of $O(\alpha)$

The wave-function renormalization of the W boson, $W_0^\mu=\mathcal{Z}_W^{1/2}W_R^\mu$, satisfies

$$
\mathcal{Z}_W-1=-\frac{\alpha(0)}{4\pi s_W^2}\frac{\partial\,\overline{\Sigma}_{WW}(p^2)}{\partial p^2}\Bigg|_{p^2=M_W^2}\equiv-\frac{\alpha(0)}{4\pi s_W^2}\overline{\Sigma}'_{WW}(M_W^2)\,.
\tag{217}
$$

The on-shell derivative is split into bosonic and fermionic parts:

$$
\overline{\Sigma}'_{WW}(M_W^2)=\overline{\Sigma}'^{,\mathrm{bos}}_{WW}(M_W^2)+\overline{\Sigma}'^{,\mathrm{fer}}_{WW}(M_W^2)
\tag{218}
$$

with each contribution being[14]

$$
\begin{aligned}
\overline{\Sigma}'^{,\mathrm{bos}}_{WW}(M_W^2)=&-\left(\frac{1}{12c_W^4}+\frac{2}{3c_W^2}+2c_W^2\right)B_0(M_W^2;M_Z,M_W)\\
&+\left(\frac{1}{12c_W^4}+\frac{4}{3c_W^2}-\frac{17}{3}-4c_W^2\right)M_W^2B_{0p}(M_W^2;M_Z,M_W)\\
&-2s_W^2B_0(M_W^2;0,M_W)-4s_W^2M_W^2B_{0p}(M_W^2;0,M_W)\\
&+\frac{r_W}{6}\left(1-\frac{r_W}{2}\right)B_0(M_W^2;m_h,M_W)+\left(1-\frac{r_W}{3}+\frac{r_W^2}{12}\right)M_W^2B_{0p}(M_W^2;m_h,M_W)\\
&+\frac{1}{12}\left(\frac{1}{c_W^2}+8+r_W\right)\frac{A_0(M_W)}{M_W^2}-\frac{1}{12}\left(\frac{1}{c_W^2}+9-8c_W^2-12c_W^4\right)\frac{A_0(M_Z)}{M_W^2}
\end{aligned}
$$

[14] Useful relations

$$
M_W^2\frac{\partial}{\partial s}R_W\Big|_{s=M_W^2}=-1\,,\quad M_W^2\frac{\partial}{\partial s}\frac{1}{R_W}\Big|_{s=M_W^2}=1\,,\quad M_W^2\frac{\partial}{\partial s}\frac{1}{R_W^2}\Big|_{s=M_W^2}=2\,,\quad M_W^2\frac{\partial}{\partial s}\frac{1}{R_W^3}\Big|_{s=M_W^2}=3\,,
\tag{219}
$$

$$-\frac{(r_W-1)}{12}\frac{A_0(m_h)}{M_W^2}+\frac{4}{9}\,, \tag{220}$$

$$\overline{\Sigma}'^{,\mathrm{fer}}_{WW}(M_W^2) = -\sum_{\{f,f'\}=\mathrm{doublets}} N_c^f\,\big[B_f(M_W^2;m_{f'},m_f)+M_W^2\,B_{fp}(M_W^2;m_{f'},m_f)$$
$$-m_f^2\,B_{1p}(M_W^2;m_{f'},m_f)-m_{f'}^2\,B_{1p}(M_W^2;m_f,m_{f'})\big]\,. \tag{221}$$

C.5 Corrections to electromagnetic coupling

The leptonic contribution $\Delta\alpha_{\mathrm{lept}}$ is known up to three-loop order of $O(\alpha^3)$ in the limit of $q^2 \gg m_\ell^2$ [19]:

$$\Delta\alpha_{\mathrm{lept}}(M_Z^2) = -\frac{\alpha(0)}{4\pi}\sum_{M_1}\mathrm{Re}\left[\Pi^{(0)}+\frac{\alpha(0)}{\pi}\Pi^{(1)}+\left(\frac{\alpha(0)}{\pi}\right)^2\left(\Pi_A^{(2)}+\sum_{M_2}\Pi_l^{(2)}+\Pi_F^{(2)}+\sum_{M_2}\Pi_h^{(2)}\right)\right] \tag{222}$$

with

$$\Pi^{(0)} = \frac{20}{9}-\frac{4}{3}\ln\left(-\frac{M_Z^2}{M_1^2}\right)+8\frac{M_1^2}{M_Z^2}\,, \tag{223}$$

$$\Pi^{(1)} = \frac{5}{6}-4\,\zeta(3)-\ln\left(-\frac{M_Z^2}{M_1^2}\right)-12\frac{M_1^2}{M_Z^2}\ln\left(-\frac{M_Z^2}{M_1^2}\right)\,, \tag{224}$$

$$\Pi_A^{(2)} = -\frac{121}{48}+(-5+8\ln 2)\,\zeta(2)-\frac{99}{16}\,\zeta(3)+10\,\zeta(5)+\frac{1}{8}\ln\left(-\frac{M_Z^2}{M_1^2}\right)\,, \tag{225}$$

$$\Pi_l^{(2)} = -\frac{116}{27}+\frac{4}{3}\,\zeta(2)+\frac{38}{9}\,\zeta(3)+\frac{14}{9}\ln\left(-\frac{M_Z^2}{M_1^2}\right)+\left(\frac{5}{18}-\frac{4}{3}\,\zeta(3)\right)\ln\left(-\frac{M_Z^2}{M_2^2}\right)$$
$$+\frac{1}{6}\ln^2\left(-\frac{M_Z^2}{M_1^2}\right)-\frac{1}{3}\ln\left(-\frac{M_Z^2}{M_1^2}\right)\ln\left(-\frac{M_Z^2}{M_2^2}\right)\,, \tag{226}$$

$$\Pi_F^{(2)} = -\frac{307}{216}-\frac{8}{3}\,\zeta(2)+\frac{545}{144}\,\zeta(3)+\left(\frac{11}{6}-\frac{4}{3}\,\zeta(3)\right)\ln\left(-\frac{M_Z^2}{M_1^2}\right)-\frac{1}{6}\ln^2\left(-\frac{M_Z^2}{M_1^2}\right)\,, \tag{227}$$

$$\Pi_h^{(2)} = -\frac{37}{6}+\frac{38}{9}\,\zeta(3)+\left(\frac{11}{6}-\frac{4}{3}\,\zeta(3)\right)\ln\left(-\frac{M_Z^2}{M_2^2}\right)-\frac{1}{6}\ln^2\left(-\frac{M_Z^2}{M_2^2}\right)\,, \tag{228}$$

where M_1 and M_2 are summed over m_e, m_μ and m_τ. According to Ref. [19], "*Special care has to be taken for those contributions where two masses are involved: In the case of the electron* $\Pi_l^{(2)}$ *is not present and* $\Pi_h^{(2)}$ *has to be evaluated with* $M_2 = m_\mu$ *and* $M_2 = m_\tau$*. For the muon* $\Pi_l^{(2)}$ *is used with* $M_2 = m_e$ *and* $\Pi_h^{(2)}$ *with* $M_2 = m_\tau$*. For the contribution from the tau lepton* $\Pi_l^{(2)}$ *has to be evaluated with* $M_2 = m_e$ *and* $M_2 = m_\mu$*.*" The one-, two- and three-loop contributions are estimated as $\Delta\alpha_{\mathrm{lept}}(M_Z^2)|_{1-\mathrm{loop}} \approx 314.19\times 10^{-4}$, $\Delta\alpha_{\mathrm{lept}}(M_Z^2)|_{2-\mathrm{loop}} \approx 0.78\times 10^{-4}$ and $\Delta\alpha_{\mathrm{lept}}(M_Z^2)|_{3-\mathrm{loop}} \approx 0.01\times 10^{-4}$, respectively.

On the other hand, the top-quark contribution $\Delta\alpha_{\mathrm{top}}$ is known up to three-loop order of $O(\alpha\alpha_s^2)$ [20, 21, 22, 23]:

$$\Delta\alpha_{\mathrm{top}}(M_Z^2) = -\frac{4}{45}\frac{\alpha(0)}{\pi}\frac{M_Z^2}{M_t^2}\Bigg\{1+5.062\,\frac{\alpha_s(\mu^2)}{\pi}+\left(28.220+9.702\,\ln\frac{\mu^2}{M_t^2}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2$$
$$+\frac{M_Z^2}{M_t^2}\left[0.1071+0.8315\,\frac{\alpha_s(\mu^2)}{\pi}+\left(6.924+1.594\,\ln\frac{\mu^2}{M_t^2}\right)\left(\frac{\alpha_s(\mu^2)}{\pi}\right)^2\right]\Bigg\}\,, \tag{229}$$

where μ is an arbitrary renormalization scale, taken to be $\mu = M_Z$, and $\Delta\alpha_{\rm top}(M_Z^2) \approx -0.7 \times 10^{-4}$.

C.6 $\Delta\overline{\rho}$ and $\Delta\overline{\rho}_W$ of $O(\alpha)$

Formulas presented in this subsections are not necessary for our fitting codes. Instead, we can use the formulas for the self-energies themselves. However, the formulas below may be useful for tests.

The finite parts of $\Delta\overline{\rho}$ and $\Delta\overline{\rho}_W$ are given by

$$\Delta\overline{\rho}^F\big|_{\mu=M_W} = \Delta\overline{\rho}^{{\rm bos},F}\big|_{\mu=M_W} + \Delta\overline{\rho}^{{\rm fer},F}\big|_{\mu=M_W} \tag{230}$$

$$\Delta\overline{\rho}_W^F\big|_{\mu=M_W} = \Delta\overline{\rho}_W^{{\rm bos},F}\big|_{\mu=M_W} + \Delta\overline{\rho}_W^{{\rm fer},F}\big|_{\mu=M_W} \tag{231}$$

with their bosonic and fermionic parts

$$\begin{aligned}
\Delta\overline{\rho}^{{\rm bos},F}\big|_{\mu=M_W} &= \frac{1}{M_W^2}\left[{\rm Re}\,\overline{\Sigma}_{WW}^{{\rm bos},F}(M_W^2) - {\rm Re}\,\overline{\Sigma}_{ZZ}^{{\rm bos},F}(M_Z^2)\right], \\
&= -\left(\frac{1}{12c_W^2} + \frac{4}{3} - \frac{17}{3}c_W^2 - 4c_W^4\right){\rm Re}\left[B_0^F(M_Z^2; M_W, M_W) - \frac{1}{c_W^2}B_0^F(M_W^2; M_Z, M_W)\right] \\
&\quad + \left(1 - \frac{1}{3}r_W + \frac{1}{12}r_W^2\right){\rm Re}\,B_0^F(M_W^2; m_h, M_W) \\
&\quad - \left(1 - \frac{1}{3}r_Z + \frac{1}{12}r_Z^2\right)\frac{1}{c_W^2}{\rm Re}\,B_0^F(M_Z^2; m_h, M_Z) \\
&\quad + \frac{1}{12}s_W^2 r_W^2(\ln r_W - 1) - \left(\frac{1}{12c_W^4} + \frac{1}{2c_W^2} - 2 + \frac{1}{12}r_W\right)\ln c_W^2 \\
&\quad - \frac{1}{12c_W^4} - \frac{19}{36c_W^2} - \frac{133}{18} + 8c_W^2\,,
\end{aligned} \tag{232}$$

$$\Delta\overline{\rho}^{{\rm fer},F}\big|_{\mu=M_W} = \frac{1}{M_W^2}\left[{\rm Re}\,\overline{\Sigma}_{WW}^{{\rm fer},F}(M_W^2) - {\rm Re}\,\overline{\Sigma}_{ZZ}^{{\rm fer},F}(M_Z^2)\right], \tag{233}$$

$$\begin{aligned}
\Delta\overline{\rho}_W^{{\rm bos},F}\big|_{\mu=M_W} &= \frac{1}{M_W^2}\left[\overline{\Sigma}_{WW}^{{\rm bos},F}(0) - {\rm Re}\,\overline{\Sigma}_{WW}^{{\rm bos},F}(M_W^2)\right], \\
&= -\left(\frac{1}{12c_W^4} + \frac{4}{3c_W^2} - \frac{17}{3} - 4c_W^2\right){\rm Re}\,B_0^F(M_W^2; M_Z, M_W) \\
&\quad - \left(1 - \frac{1}{3}r_W + \frac{1}{12}r_W^2\right){\rm Re}\,B_0^F(M_W^2; m_h, M_W) \\
&\quad + \left(\frac{3}{4(1-r_W)} + \frac{1}{4} - \frac{1}{12}r_W\right)r_W\ln r_W + \left(\frac{1}{12c_W^4} + \frac{17}{12c_W^2} - \frac{3}{s_W^2} + \frac{1}{4}\right)\ln c_W^2 \\
&\quad + \frac{1}{12c_W^4} + \frac{11}{8c_W^2} + \frac{139}{36} - \frac{177}{24}c_W^2 + \frac{5}{8}c_W^4 - \frac{1}{12}r_W\left(\frac{7}{2} - r_W\right),
\end{aligned} \tag{234}$$

$$\Delta\overline{\rho}_W^{{\rm fer},F}\big|_{\mu=M_W} = \frac{1}{M_W^2}\left[\overline{\Sigma}_{WW}^{{\rm fer},F}(0) - {\rm Re}\,\overline{\Sigma}_{WW}^{{\rm fer},F}(M_W^2)\right]. \tag{235}$$

The finite contributions defined at $\mu = M_W$ are rescaled to those at $\mu = M_Z$ with the relations

$$\Delta\overline{\rho}^F\big|_{\mu=M_W} = \Delta\overline{\rho}^F\big|_{\mu=M_Z} + \frac{s_W^2}{c_W^2}\left(\frac{N_f^{\rm total}}{6} - \frac{4}{3}s_W^2\sum_f N_c^f Q_f^2 - \frac{1}{6} - 7c_W^2\right)\ln c_W^2\,, \tag{236}$$

$$\Delta\overline{\rho}_W^F\big|_{\mu=M_W} = \Delta\overline{\rho}_W^F\big|_{\mu=M_Z}\,. \tag{237}$$

C.7 Unified form factors for $Zf\bar{f}$ vertices of $O(\alpha)$

The so-called unified form factors $\mathcal{F}_Z$ and $\mathcal{F}_W$, which appear in the radiative corrections to the $Zf\bar{f}$ vertices with the virtual Z boson and with the virtual W boson(s), respectively, are defined as[15]

$$\mathcal{F}_Z(s) \approx \mathcal{F}^0_{Za}(s)\,, \tag{238}$$

$$\mathcal{F}_W(s) \approx c_W^2 \mathcal{F}^0_{Wn}(s) - \frac{1}{2}\sigma^a_{f'}\mathcal{F}^0_{Wa}(s) - \frac{1}{2}\overline{\mathcal{F}}^0_{Wa}(s)\,, \tag{239}$$

where $\sigma^a_{f'} = |v_{f'} + a_{f'}| = 1 - 2|Q_{f'}|s_W^2 = 2c_W^2 - 1 + 2|Q_f|s_W^2$ with f' being the partner of f in the $SU(2)_L$ doublet, the superscript "0" denotes the chiral limit, and the subscripts "a" and "n" stand for contributions from abelian and non-abelian diagrams, respectively. The form factor are given in the limit of vanishing masses for the external fermions by

$$\mathcal{F}^0_{Va}(s) = 2(R_V+1)^2 s\, C_0(s;0,(\widetilde{M}_V^2)^{1/2},0) - (2R_V+3)\ln\left(-\frac{\widetilde{M}_V^2}{s}\right) - 2R_V - \frac{7}{2}\,, \tag{240}$$

$$\overline{\mathcal{F}}^0_{Wa}(s) = 0\,, \tag{241}$$

$$\begin{aligned}\mathcal{F}^0_{Wn}(s) &= -2(R_W+2)M_W^2 C_0(s;M_W,0,M_W)\\ &\quad - \left(2R_W + \frac{7}{3} - \frac{3}{2R_W} - \frac{1}{12R_W^2}\right) B_0^F(s;M_W,M_W) + 2R_W + \frac{9}{2} - \frac{11}{18R_W} + \frac{1}{18R_W^2}\,,\end{aligned} \tag{242}$$

where $\widetilde{M}_Z^2 \equiv M_V^2 - i\,M_V\Gamma_V \approx M_V^2 - i\epsilon$, and $\mathcal{F}^0_{Va}(s)$ develops imaginary contribution. In the case of the $b\bar{b}$ channel, the contributions from the large top-quark mass must be taken into account in addition to the contributions in the chiral limit above:

$$\begin{aligned}\mathcal{F}^t_{Wa}(s) &= 2(R_W+1)^2 s\left[C_0(s;M_t,M_W,M_t) - C_0(s;0,(\widetilde{M}_W^2)^{1/2},0)\right]\\ &\quad + (2R_W+3)\left[-B_0^F(s;M_t,M_t) + \ln\left(-\frac{\widetilde{M}_W^2}{s}\right) + 2\right]\\ &\quad - r_t\Bigg\{\left(3R_W + 2 - r_t - r_t^2 R_W\right) M_W^2 C_0(s;M_t,M_W,M_t)\\ &\qquad + \left(R_W + \frac{1}{2} + r_t R_W\right)\left[1 - B_0^F(s;M_t,M_t)\right]\\ &\qquad - \left(2R_W + \frac{1}{2} - \frac{2}{r_t-1} + \frac{3}{2}\frac{1}{(r_t-1)^2} + r_t R_W\right)\ln r_t + \frac{3}{2}\frac{1}{r_t-1} + \frac{3}{4}\Bigg\}\,,\end{aligned} \tag{243}$$

$$\begin{aligned}\overline{\mathcal{F}}^t_{Wa}(s) &= -r_t\Bigg\{\left[R_W + 2 - r_t(2-r_t)R_W\right] M_W^2 C_0(s;M_t,M_W,M_t)\\ &\qquad - \left(\frac{1}{2} - R_W + r_t R_W\right)\left[-B_0^F(s;M_t,M_t) + 1\right] + r_t R_W \ln r_t\Bigg\}\,,\end{aligned} \tag{244}$$

$$\begin{aligned}\mathcal{F}^t_{Wn}(s) &= -2(R_W+2)M_W^2\left[C_0(s;M_W,M_t,M_W) - C_0(s;M_W,0,M_W)\right]\\ &\quad + r_t\Bigg\{\left[3R_W + \frac{5}{2} - \frac{2}{R_W} - r_t\left(2 - \frac{1}{2R_W}\right) + r_t^2\left(\frac{1}{2} - R_W\right)\right] M_W^2 C_0(s;M_W,M_t,M_W)\end{aligned}$$

[15] The formulae for the unified form factors in Appendix of Ref. [7] contain a couple of errors. Instead, we refer to those in Ref. [6] and so on.

$$+\left[R_W+1-\frac{1}{4R_W}-r_t\left(\frac{1}{2}-R_W\right)\right]\left[B_0^F(s;M_W,M_W)-1\right]$$
$$+\left[2R_W+\frac{1}{2}-\frac{2}{r_t-1}+\frac{3}{2}\frac{1}{(r_t-1)^2}-r_t\left(\frac{1}{2}-R_W\right)\right]\ln r_t-\frac{3}{2}\frac{1}{r_t-1}+\frac{1}{4}\Bigg\}, \quad (245)$$

and

$$\mathcal{F}_W(s)=c_W^2\left(\mathcal{F}^0_{Wn}(s)+\mathcal{F}^t_{Wn}(s)\right)-\frac{1}{2}\sigma^a_{f'}\left(\mathcal{F}^0_{Wa}(s)+\mathcal{F}^t_{Wa}(s)\right)-\frac{1}{2}\left(\overline{\mathcal{F}}^0_{Wa}(s)+\overline{\mathcal{F}}^t_{Wa}(s)\right), \quad (246)$$

where we have assumed $V_{tb}=1$.

C.8 QCD corrections of $O(\alpha\alpha_s)$ and $O(\alpha\alpha_s^2)$

The analytic formulas for the $O(G_\mu\alpha_s M_t^2)$ and $O\left(G_\mu\alpha_s^2 M_t^2(1+(M_Z^2/M_t^2)^2+(M_Z^2/M_t^2)^4)\right)$ contributions to the leading term $\delta\hat{\rho}$ (see Eqs. (52) and (53)) are given as follows[16] [32, 33, 34] (see also [6]):

$$\delta_2^{\rm QCD}=-\frac{2}{3}\left[1+2\,\zeta(2)\right], \quad (247)$$
$$\delta_3^{\rm QCD}=\frac{157}{648}-\frac{3313}{162}\,\zeta(2)-\frac{308}{27}\,\zeta(3)+\frac{143}{18}\,\zeta(4)-\frac{4}{3}\,\zeta(2)\,\ln 2+\frac{441}{8}\,S_2-\frac{B_4}{9}-\frac{D_3}{18}$$
$$-\left[\frac{1}{18}-\frac{13}{9}\,\zeta(2)+\frac{4}{9}\,\zeta(3)\right]n_f$$
$$+\frac{M_Z^2}{M_t^2}\Big[-17.224+0.08829\,l_z+0.4722\,l_Z^2+\left(22.6367+1.2527\,l_Z-0.8519\,l_Z^2\right)s_W^2\Big]$$
$$+\left(\frac{M_Z^2}{M_t^2}\right)^2\Big[-7.7781-0.07226\,l_z+0.004938\,l_Z^2$$
$$+\left(21.497+0.05794\,l_Z-0.006584\,l_Z^2\right)s_W^2-21.0799\,s_W^4\Big], \quad (248)$$

where $l_z\equiv\ln(M_Z^2/M_t^2)$ and $n_f=6$, and the constants S_2, D_3 and B_4 are given in Appendix A.5. Moreover, the contribution to κ_Z^f of $O\left(G_\mu\alpha_s^2 M_t^2(1+(M_Z^2/M_t^2)^2+(M_Z^2/M_t^2)^4)\right)$ is given by [34]

$$\delta^{\rm QCD}_{\kappa,3}=-\delta_3^{\rm QCD}$$
$$+\frac{M_Z^2}{M_t^2}\Big[\left(22.6367+1.2527\,l_Z-0.8519\,l_Z^2\right)s_W^2+\left(-11.3184-0.6263\,l_z+0.4259\,l_Z^2\right)s_W^2\Big]$$
$$+\left(\frac{M_Z^2}{M_t^2}\right)^2\Big[\left(21.497+0.05794\,l_Z-0.006584\,l_Z^2\right)s_W^2$$
$$+\left(-16.0186-0.02897\,l_Z+0.003292\,l_Z^2\right)s_W^2-21.0799\,s_W^4+10.54\,s_W^4\Big]$$
$$+i\,\frac{M_Z^2}{M_t^2}\Big[\left(-1.968+2.676\,l_Z\right)s_W^2+\left(2.6235-3.5682\,l_z\right)s_W^4\Big]$$
$$+i\left(\frac{M_Z^2}{M_t^2}\right)^2\Big[\left(-0.09102+0.02069\,l_Z\right)s_W^2+\left(0.1214-0.02758\,l_z\right)s_W^4\Big] \quad (249)$$

[16] Subleading terms of an expansion in M_Z^2/M_t^2 are not included in $\delta_2^{\rm QCD}$, since they are included in the remainder contribution $\Delta^{\alpha\alpha_s}_{\rm rem}$ via r^{tb}.

Below we present the full $O(\alpha\alpha_s)$ contributions, which involve the leading contribution associated with $\delta_2^{\rm QCD}$. The $O(\alpha\alpha_s)$ QCD corrections to the bosonic self-energies appearing in $\Delta r_{\rm rem}$ through a light-quark doublet and the heavy-quark $t-b$ doublet, denoted by Δr^{ud} and Δr^{tb} in Eq. (55), respectively, are given in Ref. [27] (see also [6, 7]):

$$\Delta r^{ud} = -\frac{\alpha(0)\alpha_s(M_Z^2)}{\pi^2}\frac{c_W^2-s_W^2}{4s_W^4}\ln c_W^2\,, \tag{250}$$

$$\begin{aligned}\Delta r^{tb} = \frac{\alpha(0)\alpha_s(M_t^2)}{\pi^2}\Bigg\{ & Q_t^2 V_1'(0) + \frac{c_W^2}{s_W^4}\frac{w_t}{4}\left[\zeta(2)+\frac{1}{2}\right] - \frac{z_t}{4s_W^4}{\rm Re}\left[(\sigma_t^a)^2 V_1(r_{4t}^Z) + A_1(r_{4t}^Z) - A_1(0)\right] \\ & + \frac{c_W^2-s_W^2}{s_W^4}\,w_t\,{\rm Re}\left[F_1(x_t^W) - F_1(0)\right] - \frac{\sigma_b^a}{8s_W^4}\ln z_t\Bigg\}\,,\end{aligned} \tag{251}$$

while the corresponding corrections to $\delta\rho_{\rm rem}^f$ and $\delta\kappa_{\rm rem}^f$ in Eqs. (73) and (74) are given by

$$\Delta\rho^{ud} = -\frac{\alpha(0)\alpha_s(M_Z^2)}{\pi^2}\frac{1}{16s_W^2c_W^2}\left[2+(\sigma_u^a)^2+(\sigma_d^a)^2\right]\frac{s}{s-M_Z^2}\ln R_Z\,, \tag{252}$$

$$\Delta\kappa^{ud} = \frac{\alpha(0)\alpha_s(M_Z^2)}{\pi^2}\frac{1}{4s_W^4}\left\{c_W^2\ln c_W^2 + s_W^2\left[1-4s_W^2(Q_u^2+Q_d^2)\right]\ln R_Z\right\}, \tag{253}$$

$$\begin{aligned}\Delta\rho^{tb} = \frac{\alpha(0)\alpha_s(M_t^2)}{\pi^2}\frac{1}{4s_W^2c_W^2}\Bigg\{ & z_t\left[(\sigma_t^a)^2V_1(r_{4t}^Z) + A_1(r_{4t}^Z)\right] \\ & - \frac{M_t^2}{s-M_Z^2}\left\{(\sigma_t^a)^2\left[V_1(r_{4t}^s) - V_1(r_{4t}^Z)\right] + A_1(r_{4t}^s) - A_1(r_{4t}^Z)\right\} \\ & - \frac{1}{4}\left[1+(\sigma_b^a)^2\right]\frac{M_t^2}{s-M_Z^2}\ln R_Z - 2z_t\left[\frac{23}{8} - \zeta(2) - 3\,\zeta(3)\right]\Bigg\},\end{aligned} \tag{254}$$

$$\begin{aligned}\Delta\kappa^{tb} = \frac{\alpha(0)\alpha_s(M_t^2)}{\pi^2}\Bigg\{ & \frac{c_W^2}{4s_W^4}w_t\left[(\sigma_t^a)^2V_1(r_{4t}^Z) + A_1(r_{4t}^Z)\right] - \frac{c_W^2}{s_W^4}\,w_t\,F_1(x_t^w) + \frac{\sigma_t^a}{s_W^2}\,|Q_t|\,w_t\,V_1(r_{4t}^s) \\ & - \frac{1}{16s_W^4}\left[-(1+\sigma_b^a)\ln z_t - 4\sigma_b^a\,|Q_b|\,s_W^2\ln R_Z\right]\Bigg\}.\end{aligned} \tag{255}$$

In the above equations, the two-loop functions V_1, A_1 and F_1 are defined as

$$\begin{aligned}V_1(r) = {} & 4\left(r-\frac{1}{4r}\right)\left\{2\,{\rm Li}_3(r_-^2) - {\rm Li}_3(r_-^4) + \frac{8f}{3}\left[{\rm Li}_2(r_-^2) - {\rm Li}_2(r_-^4)\right] + 4f^2\left(-f+\frac{g}{3}+\frac{2h}{3}\right)\right\} \\ & + \sqrt{1-\frac{1}{r}}\left\{\frac{8}{3}\left(r+\frac{1}{2}\right)\left[{\rm Li}_2(r_-^2) - {\rm Li}_2(r_-^4) + f\,(-3f+2g+4h)\right] - 2f\left(r+\frac{3}{2}\right)\right\} \\ & - 8f^2\left(r-\frac{1}{6}-\frac{7}{48r}\right) + \frac{13}{6} + \frac{\zeta(3)}{r}\,,\end{aligned} \tag{256}$$

$$\begin{aligned}A_1(r) = {} & 4\left(r-\frac{3}{2}+\frac{1}{2r}\right)\left\{2\,{\rm Li}_3(r_-^2) - {\rm Li}_3(r_-^4) + \frac{8f}{3}\left[{\rm Li}_2(r_-^2) - {\rm Li}_2(r_-^4)\right] + 4f^2\left(-f+\frac{g}{3}+\frac{2h}{3}\right)\right\} \\ & + \sqrt{1-\frac{1}{r}}\left\{\frac{8}{3}\,(r-1)\left[{\rm Li}_2(r_-^2) - {\rm Li}_2(r_-^4) + f\,(-3f+2g+4h)\right] - 2f\left(r-3+\frac{1}{4r}\right)\right\}\end{aligned}$$

$$
\begin{aligned}
&\quad - 8f^2\left(r - \frac{11}{12} + \frac{5}{48r} + \frac{1}{32r^2}\right) + \frac{13}{6} - 3\,\zeta(2) + \left[-2\,\zeta(3) + \frac{1}{4}\right]\frac{1}{r}\,, &(257)\\
F_1(x) &= \left(x - \frac{3}{2} + \frac{1}{2x^2}\right)\left[\mathrm{Li}_3\left(\frac{1}{1-x}\right) + \frac{2b}{3}\,\mathrm{Li}_2\left(\frac{1}{1-x}\right) - \frac{b^2}{6}(a-b)\right]\\
&\quad + \frac{1}{3}\left(x + \frac{1}{2} - \frac{1}{2x}\right)\left[\mathrm{Li}_2\left(\frac{1}{1-x}\right) - ab\right]\\
&\quad + \frac{b^2}{3}\left(x - \frac{1}{8} - \frac{1}{x} + \frac{5}{8x^2}\right) - \frac{b}{4}\left(x - \frac{5}{2} + \frac{2}{3x} + \frac{5}{6x^2}\right) + \frac{13}{12} - \frac{3}{4}\,\zeta(2) - \frac{5}{24x} - \frac{\zeta(3)}{2x^2} &(258)
\end{aligned}
$$

with

$$
\begin{aligned}
&r_\pm \equiv \sqrt{1-r} \pm \sqrt{-r}\,, \qquad f \equiv \ln(r_+)\,, \qquad g \equiv \ln(r_+ - r_-)\,, \qquad h \equiv \ln(r_+ + r_-)\,,\\
&a \equiv \ln(-x)\,, \qquad b \equiv \ln(1-x)\,. &(259)
\end{aligned}
$$

(Here the expression for F_1 is taken from [6] and the ZFITTER codes, which is a bit different from that in [27]. Why??) For some special cases, we have

$$V_1(0) = 0\,, \tag{260}$$

$$V_1'(0) = 4\,\zeta(3) - \frac{5}{6}\,, \tag{261}$$

$$A_1(0) = 3\left[\frac{7}{4} - \zeta(2) - 2\,\zeta(3)\right]\,, \tag{262}$$

$$F_1(0) = \frac{23}{16} - \frac{1}{2}\,\zeta(2) - \frac{3}{2}\,\zeta(3)\,. \tag{263}$$

In the codes of ZFITTER, Δr^{ud} is multiplied by the following factor to take into account higher-order corrections **(Why????????)**:

$$F \equiv 1 + 1.409\frac{\alpha_s(M_Z^2)}{\pi} - 12.805\left(\frac{\alpha_s(M_Z^2)}{\pi}\right)^2\,, \tag{264}$$

i.e., $\Delta r^{ud} \to \Delta r^{ud} F$, where the R-ratio is given by $R(M_Z^2) = \frac{11}{3}[1 + \frac{\alpha_s(M_Z^2)}{\pi}F]$ for $n_f = 5$ [89].

C.9 Two-loop EW corrections of $O(G_\mu^2 M_t^4)$ and $O(G_\mu^2 M_t^2 M_Z^2)$

Leading two-loop EW contributions of $O(G_\mu^2 M_t^4)$ in expansions in the ratio M_Z/M_t were calculated in Refs. [35, 36, 37, 38], and next-to-leading power contributions of $O(G_\mu^2 M_t^2 M_Z^2)$ were also calculated in the limits of $m_h \gg M_t$ and $M_t \gg m_h$ in Refs. [39, 40, 41], where the latter contributions could be comparable in size to the former ones.

First we present the formula for the $O(G_\mu^2 M_t^4)$ contribution to $\Delta\rho$, written as $\Delta\rho^{G_\mu^2 M_t^4} = 3X_t^2\rho^{(2)}$, which is used in the computation of ρ_Z^b (see Sec. 2.10). According to Refs. [37, 38], the function $\rho^{(2)}$ is given by a simple form,

$$
\begin{aligned}
\rho^{(2)} &= 25 - 4a + \frac{1}{2}(a^2 - 12a - 12)\ln a + \frac{a-2}{2a}\,\pi^2 + \frac{1}{2}(a-4)\sqrt{a}\,g(a)\\
&\quad - \frac{3}{a}(a-1)^2(a-2)\,f(a,0) + 3(a^2 - 6a + 10)\,f(a,1)\,, &(265)
\end{aligned}
$$

where the functions $g(a)$, $f(a,0)$ and $f(a,1)$ are given by

$$g(a) = \begin{cases} \sqrt{4-a}\,(\pi-\varphi)\,, & 0 \le a \le 4\,, \\ \sqrt{a-4}\,\ln(-\xi)\,, & a \ge 4\,, \end{cases} \tag{266}$$

$$f(a,0) = \mathrm{Li}_2(1-a) = \mathrm{Li}_2(1) - \mathrm{Li}_2(a) - \ln(a)\ln(1-a)\,, \tag{267}$$

$$f(a,1) = \begin{cases} -\dfrac{2}{\sqrt{y-1}}\,\mathrm{Cl}_2(\varphi)\,, & 0 \le a \le 4\,, \\ -\dfrac{1}{\sqrt{1-y}}\left[\mathrm{Li}_2(1) + 2\,\mathrm{Li}_2(\xi)\right], & a \ge 4 \end{cases} \tag{268}$$

with $y \equiv 4/a$, $\xi \equiv (\sqrt{1-y}-1)/(\sqrt{1-y}+1)$, $\varphi \equiv 2\arcsin(\sqrt{a/4})$, and $\xi \equiv e^{i\varphi}$ for $0 \le a \le 4$. For some special cases, we have $f(4,1) = -4\ln 2$, $f(1,0) = 0$ and $f(0,0) = \pi^2/6$.

In Ref. [39], the $\overline{\mathrm{MS}}$ scheme was employed to calculate the two-loop contributions of $O(G_\mu^2 M_t^2 M_Z^2)$ in order to undertake resummations correctly. In subsequent papers [40, 41], the resultant two-loop contributions were rewritten in terms of parameters in the on-shell scheme. Below we present the two-loop formulas, in which quantities having the subscript "add" represent additional contributions due to the rewrite.

$$\Delta\rho^{G_\mu^2} = 3X_t^2\left(\Delta\hat{\rho}^{(2)}\big|_{[39]} + 4\,zt\,c_W^2\,\Delta\bar{\rho}^{(2)}_{\mathrm{add}}\big|_{[40]}\right), \tag{269}$$

$$\Delta r^{\alpha^2}_{\mathrm{rem}} = 3\left(\frac{\alpha(0)}{4\pi s_W^2}\right)^2 \frac{M_t^2}{M_W^2}\left[\Delta\hat{r}_W^{(2)}\big|_{[39]} + s_W^2\left(\frac{\delta e}{e}\right)^{(2)}\Big|_{[41]} + \frac{1}{4}\,\bar{f}^{(2)}_{\mathrm{add}}\big|_{[40]}\right], \tag{270}$$

$$\delta\rho^{f,G_\mu^2}_{\mathrm{rem}}(s) = 3X_t^2\left[16\,zt\,c_W^2\,\Delta\hat{\eta}^{(2)}\big|_{[41]} + 4\,zt\,c_W^2\,\Delta\bar{\eta}^{(2)}_{\mathrm{add}}\big|_{[41]}\right], \tag{271}$$

$$\delta\kappa^{f,G_\mu^2}_{\mathrm{rem}}(s) = 3X_t^2\left[16\,zt\,c_W^2\,\Delta\hat{k}^{(2)}\big|_{[41]} + 4\,zt\,c_W^2\,\Delta\bar{k}^{(2)}_{\mathrm{add}}\big|_{[41]}\right], \tag{272}$$

(Check carefully!!!) where we define ht (used below) and zt as

$$ht \equiv \left(\frac{m_h}{M_t}\right)^2, \qquad zt \equiv \left(\frac{M_Z}{M_t}\right)^2. \tag{273}$$

For a light Higgs $m_h \ll M_t$ ($m_h < M_t/2$ in ZFITTER), $\Delta\hat{\rho}^{(2)}$ is given by

$$\begin{aligned}
\Delta\hat{\rho}^{(2)} = {} & 19 - \frac{53\,ht}{3} + \frac{3\,ht^{3/2}\pi}{2} + \frac{8\,ht^2}{9\,zt} - \frac{5\,ht^2}{9\,c_W^2 zt} + \left(\frac{845}{27} - \frac{1}{3\,c_W^2} + \frac{427\,c_W^2}{27} - \frac{122\,c_W^4}{9}\right) zt \\
& + \frac{\pi^2}{27}\left(54\,ht - 54 - 119\,zt + 44\,c_W^2 zt\right) + \frac{4}{27}\sqrt{ht}\,\pi\left(-27 + 34\,zt - 116\,c_W^2 zt + 64\,c_W^4 zt\right) \\
& + \left(\frac{32\,ht}{9} - \frac{8\,ht^2}{9\,zt} - \frac{32\,zt}{3}\right) B_0^F(M_Z^2;\, m_h, M_Z)\big|_{\mu=M_t} \\
& + \left(1 + 20\,c_W^2 - 24\,c_W^4\right)\frac{zt}{3\,c_W^2} B_0^F(M_W^2;\, M_W, M_Z)\big|_{\mu=M_t} \\
& - \frac{2}{3}\left(1 + 18\,c_W^2 - 16\,c_W^4\right) zt\, B_0^F(M_Z^2;\, M_W, M_W)\big|_{\mu=M_t} \\
& - \frac{5}{9}\left(4\,ht - \frac{ht^2}{c_W^2 zt} - 12\,c_W^2 zt\right) B_0^F(M_W^2;\, m_h, M_W)\big|_{\mu=M_t} \\
& - \frac{1}{9}\left(5\,ht + 3\,zt + 32\,c_W^2 zt + 48\,c_W^4 zt\right)\ln c_W^2 + \frac{ht}{9c_W^2 zt}\left(5\,ht - 8\,c_W^2 ht - 18\,c_W^2 zt\right)\ln ht
\end{aligned}$$

$$+\frac{8}{9}\left(4-26\,c_W^2-5\,c_W^4\right)zt\,\ln\frac{M_t^2}{\mu^2}+\left(\frac{ht}{3}-\frac{11\,zt}{9}+\frac{zt}{3\,c_W^2}-\frac{16\,c_W^2\,zt}{9}-\frac{16\,c_W^4\,zt}{3}\right)\ln zt\,,\quad(274)$$

while, in the case of a heavy Higgs ($m_h \geq 2M_t$ in ZFITTER),

$$\begin{aligned}
\Delta\hat{\rho}^{(2)} = {} & 25-4\,ht+\left(\frac{1}{2}-\frac{1}{ht}\right)\pi^2+\frac{(ht-4)\sqrt{ht}\,g(ht)}{2}+\left(-6-6\,ht+\frac{ht^2}{2}\right)\ln ht \\
& +\left(\frac{6}{ht}-15+12\,ht-3\,ht^2\right)\mathrm{Li}_2\left(1-ht\right)+\frac{3}{2}\left(-10+6\,ht-ht^2\right)\phi\left(\frac{ht}{4}\right) \\
& +zt\left\{\frac{1}{54c_W^2\,(ht-4)\,ht}\Big[-1776\,c_W^4+\left(72-6250\,c_W^2-3056\,c_W^4+3696\,c_W^6\right)ht\right. \\
& \qquad +\left(-18+1283\,c_W^2+1371\,c_W^4-1436\,c_W^6\right)ht^2+\left(68\,c_W^2-124\,c_W^4+128\,c_W^6\right)ht^3\Big] \\
& \quad +\frac{\left(6\,c_W^2\,ht-37\,c_W^2-119\,ht^2+56\,c_W^2\,ht^2\right)\pi^2}{27\,ht^2} \\
& \quad +\left(\frac{32\,c_W^4}{3}-\frac{2}{3}-12\,c_W^2\right)B_0^F(M_Z^2;\,M_W,M_W)\big|_{\mu=M_t} \\
& \quad +\left(\frac{20}{3}+\frac{1}{3\,c_W^2}-8\,c_W^2\right)B_0^F(M_W^2;\,M_W,M_Z)\big|_{\mu=M_t} \\
& \quad +\frac{\left(17-58\,c_W^2+32\,c_W^4\right)(4-ht)\sqrt{ht}\,g(ht)}{27} \\
& \quad -\frac{40s_W^2\,(4-ht)\,\Lambda(ht)}{3\,ht}+\frac{2\,c_W^2\left(37-6\,ht-12\,ht^2-22\,ht^3+9\,ht^4\right)\mathrm{Li}_2(1-ht)}{9\,ht^2} \\
& \quad -\frac{\left(1+14\,c_W^2+16\,c_W^4\right)\ln c_W^2}{3} \\
& \quad +\Big[11520-15072\,c_W^2-(7170-8928\,c_W^2-768\,c_W^4)ht \\
& \qquad +(3411-7062\,c_W^2+3264\,c_W^4)ht^2-(1259-3547\,c_W^2+2144\,c_W^4)ht^3 \\
& \qquad +(238-758\,c_W^2+448\,c_W^4)ht^4-(17-58\,c_W^2+32\,c_W^4)ht^5\Big]\frac{\ln ht}{27\,(ht-4)^2\,ht} \\
& \quad +\frac{8}{9}\left(4-26\,c_W^2-5\,c_W^4\right)\ln\frac{M_t^2}{\mu^2}+\frac{\left(3+5\,c_W^2-26\,c_W^4-48\,c_W^6\right)\ln zt}{9\,c_W^2} \\
& \quad +\Big[3840s_W^2-(4310-4224\,c_W^2-256\,c_W^4)ht+(1706-1312\,c_W^2-320\,c_W^4)ht^2 \\
& \qquad -(315+476\,c_W^2-64\,c_W^4)ht^3+(24+454\,c_W^2)ht^4-112\,c_W^2\,ht^5 \\
& \qquad \left.+9\,c_W^2\,ht^6\Big]\frac{\phi\left(\frac{ht}{4}\right)}{9\,(ht-4)^2\,ht^2}\right\},
\end{aligned}\qquad(275)$$

where

$$g(x)=\begin{cases}\sqrt{4-x}\left(\pi-2\arcsin\sqrt{x/4}\right) & 0<x\leq 4\,,\\[2ex] 2\sqrt{x/4-1}\,\ln\left(\dfrac{1-\sqrt{1-4/x}}{1+\sqrt{1-4/x}}\right) & x>4\,,\end{cases}\qquad(276)$$

$$
\Lambda(x) = \begin{cases} -\dfrac{1}{2\sqrt{x}}\, g(x) + \dfrac{\pi}{2}\,\sqrt{4/x-1} & 0 < x \le 4 \\ -\dfrac{1}{2\sqrt{x}}\, g(x) & x > 4\,, \end{cases} \tag{277}
$$

$$
\phi(x) = \begin{cases} 4\sqrt{\dfrac{x}{1-x}}\,\mathrm{Cl}_2\left(2\arcsin\sqrt{x}\right) & 0 < x \le 1\,, \\ \dfrac{1}{\lambda}\left[-4\,\mathrm{Li}_2\left(\dfrac{1-\lambda}{2}\right) + 2\ln^2\left(\dfrac{1-\lambda}{2}\right) - \ln^2(4x) + \dfrac{\pi^2}{3}\right] & x > 1\,, \end{cases} \tag{278}
$$

with $\lambda = \sqrt{1-1/x}$. An interpolation function for the region between the above twos is given by

$$
\begin{aligned}
\Delta\hat{\rho}^{(2)} &= -15.642 + 0.036382\,M_t + ht^{1/4}(2.301 - 0.01343\,M_t) + ht^{1/2}(0.01809\,M_t - 9.953) \\
&\quad + ht\,(5.687 - 0.01568\,M_t) + ht^{3/2}(0.005369\,M_t - 1.647) + ht^2(0.1852 - 0.000646\,M_t)\,.
\end{aligned} \tag{279}
$$

The additional contribution $\Delta\bar{\rho}^{(2)}_{\mathrm{add}}$ is given by

$$
\begin{aligned}
\Delta\bar{\rho}^{(2)}_{\mathrm{add}} &= \frac{542}{27} - \frac{2}{3\,c_W^2} - \frac{800\,c_W^2}{27} + \frac{1}{3}\left(1 + 26\,c_W^2 + 24\,c_W^4\right) B_0^F(M_z^2;\, M_w, M_w)\big|_{\mu=M_t} \\
&\quad + 4\,c_W^2 B_0(M_w^2;\, 0, M_w)\big|_{\mu=M_t} - \left(\frac{11}{3} + \frac{1}{3\,c_W^2} + 4\,c_W^2\right) B_0^F(M_w^2;\, M_w, M_z)\big|_{\mu=M_t} \\
&\quad - \left(\frac{2}{3} + \frac{4\,c_W^2}{3} - 8\,c_W^4\right)\ln c_W^2 + \left(\frac{1}{c_W^2} - \frac{38}{3} + \frac{34\,c_W^2}{3}\right)\ln\frac{M_t^2}{\mu^2} \\
&\quad + \frac{2\left(3 - 62\,c_W^2 + 74\,c_W^4 + 36\,c_W^6\right)\ln zt}{9\,c_W^2}\,.
\end{aligned} \tag{280}
$$

For a light Higgs $m_h \ll M_t$ ($m_h/M_t < 0.3$ in the ZFITTER codes),

$$
\begin{aligned}
\Delta\hat{r}_W^{(2)} &= -\frac{13}{144} - \frac{1}{48\,c_W^4} - \frac{41}{96\,c_W^2} + \frac{61\,c_W^2}{72} + \frac{7-16\,c_W^2}{27}\,\pi\sqrt{ht} - \frac{\pi^2}{36} - \frac{5\,ht^2}{144\,c_W^4\,zt^2} + \frac{35\,ht}{288\,c_W^2\,zt} \\
&\quad + \frac{5}{12}\left(1 + \frac{ht^2}{12\,c_W^4\,zt^2} - \frac{ht}{3\,c_W^2\,zt}\right) B_0^F(M_W^2;\, m_h, M_W)\big|_{\mu=M_t} \\
&\quad + \frac{1 + 20\,c_W^2 - 24\,c_W^4}{48\,c_W^4} B_0^F(M_W^2;\, M_W, M_Z)\big|_{\mu=M_t} \\
&\quad - \frac{\left(5\,s_W^2\,ht^2 + 3\,ht\,zt + 48\,c_W^2\,ht\,zt - 60\,c_W^4\,ht\,zt - 3\,c_W^2\,zt^2 - 8\,c_W^4\,zt^2 + 20\,c_W^6\,zt^2\right)\ln c_W^2}{144\,c_W^2\,s_W^2\,zt\left(ht - c_W^2\,zt\right)} \\
&\quad + \frac{5\,ht\left(ht^2 - 4\,c_W^2\,ht\,zt + 12\,c_W^4\,zt^2\right)\ln ht}{144\,c_W^4\,zt^2\left(ht - c_W^2\,zt\right)} + \left(\frac{17}{36} - \frac{13\,c_W^2}{18}\right)\ln\frac{M_t^2}{\mu^2} \\
&\quad - \frac{\left[5\,c_W^2\,ht^2 - 3\,ht\,zt - 60\,c_W^2\,ht\,zt + 60\,c_W^4\,ht\,zt + \left(3\,c_W^2 + 60\,c_W^4 - 20\,c_W^6\right)zt^2\right]\ln zt}{144\,c_W^4\,zt\left(ht - c_W^2\,zt\right)}\,,
\end{aligned} \tag{281}
$$

while, for $m_h \gg M_Z$ ($m_h/M_t \ge 0.3$ in the ZFITTER codes),

$$
\Delta\hat{r}_W^{(2)} = -\frac{121}{288} - \frac{1}{48\,c_W^4} - \frac{41}{96\,c_W^2} + \frac{77\,c_W^2}{12} + \frac{19}{72\,ht} + \left(\frac{41}{216} - \frac{4\,c_W^2}{27}\right)ht - \frac{(19 + 21\,ht)\,\pi^2}{432\,ht^2}
$$

$$
\begin{aligned}
&-\left(\frac{1}{2}-\frac{1}{48\,c_W^4}-\frac{5}{12\,c_W^2}\right)B_0^F(M_W^2;\,M_W,M_Z)\big|_{\mu=M_t}+\frac{16\,c_W^2-7}{216}(ht-4)\sqrt{ht}\,g(ht)\\
&-\left(\frac{1}{12}-\frac{1}{3\,ht}\right)\Lambda(ht)+\frac{\left(19+21ht-12ht^2-31\,ht^3+9\,ht^4\right)}{72\,ht^2}\,\mathrm{Li}_2(1-ht)\\
&-\frac{\left(1+21\,c_W^2-25\,c_W^4\right)\ln c_W^2}{48\,c_W^2\,s_W^2}+\left(\frac{17}{36}-\frac{13\,c_W^2}{18}\right)\ln\frac{M_t^2}{\mu^2}+\frac{\left(1+20\,c_W^2-25\,c_W^4\right)\ln zt}{48\,c_W^4}\\
&+\frac{372+(96\,c_W^2-213)ht+(432\,c_W^2-318)ht^2+(97-160\,c_W^2)ht^3-(7-16\,c_W^2)ht^4}{216(ht-4)ht}\ \ln ht\\
&+\frac{96-(384-64\,c_W^2)ht-(2+64\,c_W^2)ht^2+231\,ht^3-85\,ht^4+9\,ht^5}{144\,(ht-4)ht^2}\ \phi\left(\frac{ht}{4}\right). \qquad (282)
\end{aligned}
$$

For $m_h \ll M_t$ ($m_h/M_t < 0.3$ in the ZFITTER codes), ,

$$
\left(\frac{\delta e}{e}\right)^{(2)}=\frac{61}{72}-\frac{16\sqrt{ht}\,\pi}{27}-\frac{13}{18}\,\ln\frac{M_t^2}{\mu^2}\,, \qquad (283)
$$

while, for $m_h \gg M_Z$ ($m_h/M_t \geq 0.3$ in the ZFITTER codes), ,

$$
\begin{aligned}
\left(\frac{\delta e}{e}\right)^{(2)}=&\ \frac{231-32\,ht}{216}-\frac{2}{27}\,(4-ht)\sqrt{ht}\,g(ht)+\frac{2\left(6+27\,ht-10\,ht^2+ht^3\right)}{27\,(ht-4)}\ln ht\\
&-\frac{13}{18}\ln\frac{M_t^2}{\mu^2}-\frac{4\,(ht-1)}{9\,(ht-4)\,ht}\,\phi\left(\frac{ht}{4}\right). \qquad (284)
\end{aligned}
$$

The additional contribution $\bar{f}^{(2)}_{\mathrm{add}}$ is given by

$$
\begin{aligned}
\bar{f}^{(2)}_{\mathrm{add}}=&\ \frac{10}{3}+\frac{1}{3\,c_W^2}+4\,c_W^2B_0^F(M_W^2;\,0,M_W)\big|_{\mu=M_t}-\left(\frac{11}{3}+\frac{1}{3\,c_W^2}+4\,c_W^2\right)B_0^F(M_W^2;\,M_W,M_Z)\big|_{\mu=M_t}\\
&+\frac{\left(11-8\,c_W^2\right)\ln c_W^2}{6\,s_W^2}-\left(\frac{11}{3}+\frac{1}{3\,c_W^2}\right)\ln zt\,. \qquad (285)
\end{aligned}
$$

For $m_h \ll M_t$ ($m_h/M_t < 0.57$ in the ZFITTER codes), ,

$$
\begin{aligned}
\Delta\hat{\eta}^{(2)}=&\ \frac{ht^3-6\,ht^2\,zt+11\,ht\,zt^2}{9\,c_W^2\,(ht-4\,zt)\,zt^2}+\frac{49-289\,c_W^2-349\,c_W^4+292\,c_W^6}{216\,c_W^2\left(1-4\,c_W^2\right)}+\frac{1+18\,c_W^2-16\,c_W^4}{12\left(1-4\,c_W^2\right)}\ln c_W^2\\
&-\frac{17-40\,c_W^2+32\,c_W^4}{54\,c_W^2}(\sqrt{ht}\,\pi-2)+\frac{11\,ht^2\,zt-2\,ht^3-24\,ht\,zt^2+24\,zt^3}{18\,c_W^2\,(ht-4\,zt)\,zt^2}\ln ht\\
&+\frac{1-4\,c_W^2+44\,c_W^4-32\,c_W^6}{24\,c_W^2(1-4\,c_W^2)}B_0^F(M_Z^2;\,M_W,M_W)\big|_{\mu=M_t}\\
&+\frac{13\,ht^2\,zt-2\,ht^3-32\,ht\,zt^2+36\,zt^3}{18\,c_W^2\,(ht-4\,zt)\,zt^2}B_0^F(M_Z^2;\,m_h,M_Z)\big|_{\mu=M_t}-\frac{17-34\,c_W^2+26\,c_W^4}{36\,c_W^2}\ln\frac{M_t^2}{\mu^2}\\
&+\left(\frac{ht\,(2ht-5zt)}{18\,c_W^2\,zt\,(ht-4\,zt)}+\frac{10-39\,c_W^2-70\,c_W^4+48\,c_W^6}{36\,c_W^2\left(4\,c_W^2-1\right)}\right)\ln zt\,, \qquad (286)
\end{aligned}
$$

while, for $m_h \gg M_Z$ ($m_h/M_t \geq 0.57$ in the ZFITTER codes),

$$
\Delta\hat{\eta}^{(2)}=\frac{\left(-17+40\,c_W^2-32\,c_W^4\right)ht}{216\,c_W^2}+\frac{5}{144\,c_W^2\,(ht-4)}+\frac{707-4720\,c_W^2+5900\,c_W^4-3696\,c_W^6}{864\,c_W^2\left(1-4\,c_W^2\right)}
$$

$$+\left(\frac{10}{27}-\frac{17}{108\,c_W^2}-\frac{8\,c_W^2}{27}\right)\left(1-\frac{ht}{4}\right)\sqrt{ht}\,g\,(ht)+\frac{1+18\,c_W^2-16\,c_W^4}{12\left(1-4\,c_W^2\right)}\ln c_W^2+\frac{4-ht}{12\,c_W^2 ht}\Lambda\,(ht)$$
$$+\frac{2-7\,c_W^2-70\,c_W^4+48\,c_W^6}{36\,c_W^2\left(4\,c_W^2-1\right)}\ln zt+\frac{1-4\,c_W^2+44\,c_W^4-32\,c_W^6}{24\,c_W^2\left(1-4\,c_W^2\right)}B_0^F(M_Z^2;\,M_W,M_W)\big|_{\mu=M_t}$$
$$-\frac{17-34\,c_W^2+26\,c_W^4}{36\,c_W^2}\ln\frac{M_t^2}{\mu^2}$$
$$+\left[\frac{(4\,c_W^2-5)\left(6+27\,ht-10\,ht^2+ht^3\right)}{54\,(ht-4)}\right.$$
$$\left.-\frac{1152+606\,ht+1467\,ht^2-1097\,ht^3+238\,ht^4-17\,ht^5}{432\,c_W^2\,(ht-4)^2\,ht}\right]\ln ht$$
$$+\left[\frac{\left(5-4\,c_W^2\right)(ht-1)}{9\,(ht-4)\,ht}-\frac{384+10\,ht-238\,ht^2+63\,ht^3-3\,ht^4}{144\,c_W^2\,(ht-4)^2\,ht^2}\right]\phi\left(\frac{ht}{4}\right). \tag{287}$$

The additional contribution $\Delta\bar{\eta}^{(2)}_{\rm add}$ is given by

$$\Delta\bar{\eta}^{(2)}_{\rm add}=16\pi^2\Delta\bar{\eta}^{(1)}_f+V_{\rm add}-\frac{197-1378\,c_W^2+1064\,c_W^4}{27\left(1-4\,c_W^2\right)}$$
$$-\frac{\left(1+16\,c_W^2-20\,c_W^4+48\,c_W^6\right)}{3(1-4\,c_W^2)}B_0^F(M_Z^2;\,M_W,M_W)\big|_{\mu=M_t}-\frac{2\,c_W^2\left(1+26\,c_W^2+24\,c_W^4\right)}{3(1-4c_W^2)}\ln c_W^2$$
$$+\left(\frac{41}{3}-\frac{46\,c_W^2}{3}\right)\ln\frac{M_t^2}{\mu^2}+\frac{2\left(50-283\,c_W^2+242\,c_W^4-72\,c_W^6\right)}{9\left(1-4\,c_W^2\right)}\ln zt\,. \tag{288}$$

In this formula, $V_{\rm add}$ contains the shifts of the one-loop vertices,

$$V_{\rm add}=8\,c_W^2\ln\frac{M_W^2}{\mu^2}+3\left(I_f^3Q_f-4s_W^2Q_f^2\right)\mathcal{F}_V(1)-16\,c_W^2\mathcal{G}_V\left(c_W^{-2}\right)$$
$$+\left[1-4\,c_W^2-2(1-2\,c_W^2)I_f^3Q_f\right]\mathcal{F}_V\left(c_W^{-2}\right), \tag{289}$$

and $\Delta\bar{\eta}^{(1)}_f$ is given by

$$\Delta\bar{\eta}^{(1)}_f=\frac{1}{16\pi^2}\left[-\frac{1}{c_W^2}\overline{\Sigma}'^{,F}_{ZZ}(M_Z^2)\Big|_{\mu=M_Z^2}-4\,c_W^2\ln c_W^2+\mathcal{V}_{fi}(M_Z^2)\right] \tag{290}$$

with

$$\mathcal{V}_{fi}(q^2)=\left[1-s_W^2(2-I_a^3Q_a^2)\right]\mathcal{F}\left(\frac{q^2}{M_W^2}\right)+8\,c_W^2\,\mathcal{G}\left(\frac{q^2}{M_W^2}\right)$$
$$-\frac{1-3s_W^2I_a^3Q_a+6s_W^2Q_aQ_a}{2c_W^2}\mathcal{F}\left(\frac{q^2}{M_Z^2}\right), \tag{291}$$

where a repeated a-index denotes summation over the initial and final fermions, i.e. $I_a^3Q_a=I_i^3Q_i+I_f^3Q_f$ [90]. The functions $\mathcal{F}(x)$ and $\mathcal{G}(x)$ correspond to $f(x)$ and $g(x)$, respectively, in Ref. [90]:

$$\mathcal{F}(x)=i\,\pi\left[\frac{2}{x}+3-2\left(1+\frac{1}{x}\right)^2\ln(1+x)\right]+\frac{2}{x}+\frac{7}{2}-\left(3+\frac{2}{x}\right)\ln x$$

$$
+\left(1+\frac{1}{x}\right)^2\left[2\mathrm{Li}_2\left(\frac{1}{1+x}\right)-\frac{\pi^2}{3}+\ln^2(1+x)\right], \tag{292}
$$

$$
\mathcal{G}(x)=\left(\sqrt{\frac{4-x}{x}}\,\arctan\sqrt{\frac{x}{4-x}}-1\right)\left(\frac{1}{x}+\frac{1}{2}\right)+\frac{9}{8}+\frac{1}{2x}-\left(1+\frac{1}{2x}\right)\frac{4}{x}\left(\arctan\sqrt{\frac{x}{4-x}}\right)^2. \tag{293}
$$

For $m_h \ll M_t$,

$$
\begin{aligned}
\Delta\hat{k}^{(2)} &= \frac{-175+366\,s_W^2}{432}+\left(\frac{3}{8}-\frac{s_W^2}{3}\right)B_0^F(M_Z^2;\,M_W,M_W)\big|_{\mu=M_t}-\frac{c_W^2}{6}\ln c_W^2 \\
&\quad -\frac{2\pi}{27}\sqrt{ht}\,(8\,s_W^2-3)-\left(\frac{1}{4}+\frac{2}{9}\,s_W^2\right)\ln\frac{M_t^2}{\mu^2}+\frac{(3\,s_W^2-2)}{18}\ln zt\,,
\end{aligned} \tag{294}
$$

while, for $m_h \gg M_Z$,

$$
\begin{aligned}
\Delta\hat{k}^{(2)} &= \frac{-211+24\,ht+462\,s_W^2-64\,ht\,s_W^2}{432}+\left(\frac{3}{8}-\frac{s_W^2}{3}\right)B_0^F(M_Z^2;\,M_W,M_W)\big|_{\mu=M_t}-\frac{c_W^2}{6}\ln c_W^2 \\
&\quad +\frac{(ht-4)\sqrt{ht}\,(8\,s_W^2-3)\,g(ht)}{108}-\frac{\left(6+27\,ht-10\,ht^2+ht^3\right)\left(3-8\,s_W^2\right)}{108\,(ht-4)}\ln ht \\
&\quad -\left(\frac{1}{4}+\frac{2}{9}\,s_W^2\right)\ln\frac{M_t^2}{\mu^2}+\frac{\left(3\,s_W^2-2\right)}{18}\ln zt+\frac{(ht-1)\left(8\,s_W^2-3\right)}{18\,(4-ht)\,ht}\,\phi\left(\frac{ht}{4}\right).
\end{aligned} \tag{295}
$$

The additional contribution $\Delta\bar{k}^{(2)}_{\rm add}$ is given by

$$
\begin{aligned}
\Delta\bar{k}^{(2)}_{\rm add} &= \frac{-238\,c_W^2}{27}+8\,c_W^4-2\,c_W^2\sqrt{4\,c_W^2-1}\,(3+4\,c_W^2)\arctan\left(\frac{1}{\sqrt{4\,c_W^2-1}}\right)-\frac{16}{9}\,c_W^2\ln zt \\
&\quad +\frac{1-6\,I_f^3\,Q_f+8\,Q_f^2-8\,c_W^4Q_f^2}{4\,c_W^2}\,f_V(1)+4\,c_W^2g_V\left(c_W^{-2}\right)-7\,c_W^2\ln c_W^2-\frac{17}{3}\,c_W^2\ln\frac{\mu^2}{M_z^2} \\
&\quad +c_W^2\,(1-Q_fI_f^3)f_V\left(c_W^{-2}\right)-\frac{80}{9}\,i\pi\,.
\end{aligned} \tag{296}
$$

C.10 Three-loop corrections of $O(G_\mu^2\alpha_s M_t^4)$ and $O(G_\mu^3 M_t^6)$

Analytical results for the leading three-loop corrections to $\Delta\rho$ of $O(G_\mu^2\alpha_s M_t^4)$ and $O(G_\mu^3 M_t^6)$ are available for $m_h = 0$ in Ref. [50] and for $m_h \approx M_t$ and $m_h \gg M_t$ in Ref. [51][17]. For $m_h = 0$, the corrections are given by

$$
\begin{aligned}
\Delta\rho^{G_\mu^2\alpha_s M_t^4} &= 4\,X_t^2\,\frac{\alpha_s(M_t^2)}{\pi}\left(\frac{185}{3}+\frac{729}{4}\,S_2-48\,\zeta(2)\ln 2-\frac{151}{6}\,\zeta(2)+29\,\zeta(3)-24\,\zeta(4)+12\,B_4\right), \\
&\approx 2.939\,X_t^2\,\frac{\alpha_s(M_t^2)}{\pi}\,,
\end{aligned} \tag{297}
$$

$$
\Delta\rho^{G_\mu^3 M_t^6} = 3\,X_t^3\Big[\big(68+729\,S_2+36\,D_3+96\,\zeta(2)\ln 2+6\,\zeta(2)-612\,\zeta(3)+324\,\zeta(4)-72\,B_4\big)
$$

[17] $\Delta\rho$ in [50, 51] corresponds to $-\delta\hat{\rho}$ here.

$$
+\,3\bigg(-\frac{6572}{15}-\frac{4374}{5}\,S_2+\frac{1472}{15}\,\zeta(2)+440\,\zeta(3)\bigg)\bigg]\,,
$$
$$
\approx 249.74\,X_t^3\,, \tag{298}
$$

where the constants S_2, D_3 and B_4 are given in Appendix A.5. For $m_h \approx M_t$ (valid for $0 \lesssim m_h \lesssim 2.5M_t$), they are given by

$$
\Delta\rho^{G_\mu^2\alpha_s M_t^4} = X_t^2\,\frac{\alpha_s(M_t^2)}{\pi}\left(157.295+112.00\,\delta-24.73\,\delta^2+7.39\,\delta^3-3.52\,\delta^4+2.06\,\delta^5\right), \tag{299}
$$
$$
\Delta\rho^{G_\mu^3 M_t^6} = X_t^3\left(95.92-111.98\,\delta+8.099\,\delta^2+9.36\,\delta^3+7.27\,\delta^4-15.60\,\delta^5\right) \tag{300}
$$

with $m_h \equiv M_t(1+\delta)$, while the results in the limit $m_h \gg M_t$ (valid for $m_h \gtrsim 2.5M_t$) are

$$
\begin{aligned}
\Delta\rho^{G_\mu^2\alpha_s M_t^4} = X_t^2\,\frac{\alpha_s(M_t^2)}{\pi}\Big[&79.73-47.77\,\ln Y+42.07\,\ln^2 Y+9.00\,\ln^3 Y\\
&+Y\left(225.16-179.74\,\ln Y+70.22\,\ln^2 Y-19.22\,\ln^3 Y\right)\\
&+Y^2\left(-76.07+25.33\,\ln Y-9.17\,\ln^2 Y-5.57\,\ln^3 Y\right)\\
&+Y^3\left(-10.10-24.69\,\ln Y-0.30\,\ln^2 Y-5.46\,\ln^3 Y\right)\\
&+Y^4\left(-4.52-32.85\,\ln Y+0.72\,\ln^2 Y-5.25\,\ln^3 Y\right)\\
&+Y^5\left(-2.55-36.61\,\ln Y+1.06\,\ln^2 Y-5.14\,\ln^3 Y\right)\Big]\,,
\end{aligned} \tag{301}
$$
$$
\begin{aligned}
\Delta\rho^{G_\mu^3 M_t^6} = X_t^3\Big[&Y^{-1}\left(-3.17-83.25\,\ln Y\right)\\
&-189.93-231.48\,\ln Y-142.06\,\ln^2 Y+2.75\,\ln^3 Y\\
&+Y\left(-332.34+77.71\,\ln Y-68.67\,\ln^2 Y+51.79\,\ln^3 Y\right)\\
&+Y^2\left(227.55-510.55\,\ln Y+87.77\,\ln^2 Y+6.41\,\ln^3 Y\right)\\
&+Y^3\left(-58.40-329.18\,\ln Y+20.42\,\ln^2 Y+14.54\,\ln^3 Y\right)\\
&+Y^4\left(-36.14-381.88\,\ln Y+18.63\,\ln^2 Y+15.04\,\ln^3 Y\right)\\
&+Y^5\left(-39.08-416.36\,\ln Y+13.76\,\ln^2 Y+17.19\,\ln^3 Y\right)\Big]
\end{aligned} \tag{302}
$$

with $Y \equiv 4M_t^2/m_h^2$.

C.11 Corrections to the $Zb\bar{b}$ vertex of $O(G_\mu^2 M_t^4)$

In Refs. [35, 36], Barbieri *et al.* calculated the function $\tau^{(2)}(M_t^2/M_Z^2)$ in Eq. (85) with asymptotic expansions in terms of M_t^2/M_Z^2:

$$
\begin{aligned}
\tau^{(2)}(r) = \frac{1}{144}\Big[&311+24\pi^2+282\,\ln r+90\,\ln^2 r-4\,r\left(40+6\pi^2+15\,\ln r+18\,\ln^2 r\right)\\
&+\frac{3}{100}\,r^2\left(24209-6000\pi^2-45420\,\ln r-1800\,\ln^2 r\right)\Big]+O(r^3)
\end{aligned} \tag{303}
$$

for $r \equiv (M_t/m_h)^2 \ll 1$, and

$$
\tau^{(2)}(r) = \frac{27-\pi^2}{3}-\frac{4\pi}{\sqrt{r}}+O\left(\frac{\ln r}{r}\right) \tag{304}
$$

for $r \gg 1$. In Refs. [37, 38], Fleischer *et al.* reported the full analytical expression of $\tau_b^{(2)}(a)$ for $a \equiv (m_h/M_t)^2$ without the expansions:

$$\tau^{(2)}(1/a) = 9 - \frac{13}{4}a - 2a^2 - \frac{a}{4}(19+6a)\ln a - \frac{a^2}{4}(7-6a)\ln^2 a - \left(\frac{1}{4}+\frac{7}{2}a^2-3a^2\right)\frac{\pi^2}{6} + \left(\frac{a}{2}-2\right)\sqrt{a}\,g(a) + (a-1)^2\left(4a-\frac{7}{4}\right)f(a,0) - \left(a^3-\frac{33}{4}a^2+18a-7\right)f(a,1)\,, \quad (305)$$

where the functions $g(a)$, $f(a,0)$ and $f(a,1)$ are given in Eqs. (266), (267) and (268).

C.12 Final state corrections to the widths for $Z \to q\bar{q}$

For the $q\bar{q}$ channels, the final-state QCD corrections have to be taken into account as well as the QED corrections. The radiator factors $R_V^q(s)$ and $R_A^q(s)$ are then given as follows [67] (see also [7, 91] and Sec. 12.4 of [6]):

$$\begin{aligned}
R_V^q(s) = {} & 1 + \frac{3Q_q^2}{4}\frac{\alpha(s)}{\pi} + \frac{\alpha_s(s)}{\pi} - \frac{Q_q^2}{4}\frac{\alpha(s)}{\pi}\frac{\alpha_s(s)}{\pi} \\
& + \left[C_{02} + C_2^t\left(\frac{s}{M_t^2}\right)\right]\left(\frac{\alpha_s(s)}{\pi}\right)^2 + C_{03}\left(\frac{\alpha_s(s)}{\pi}\right)^3 + C_{04}\left(\frac{\alpha_s(s)}{\pi}\right)^4 \\
& + \frac{m_c^2(s)+m_b^2(s)}{s}C_{23}\left(\frac{\alpha_s(s)}{\pi}\right)^3 + \frac{m_q^2(s)}{s}\left[C_{21}^V\frac{\alpha_s(s)}{\pi} + C_{22}^V\left(\frac{\alpha_s(s)}{\pi}\right)^2 + C_{23}^V\left(\frac{\alpha_s(s)}{\pi}\right)^3\right] \\
& + \frac{m_c^4(s)}{s^2}\left(C_{42} - \ln\frac{m_c^2(s)}{s}\right)\left(\frac{\alpha_s(s)}{\pi}\right)^2 + \frac{m_b^4(s)}{s^2}\left(C_{42} - \ln\frac{m_b^2(s)}{s}\right)\left(\frac{\alpha_s(s)}{\pi}\right)^2 \\
& + \frac{m_q^4(s)}{s^2}\left[C_{41}^V\frac{\alpha_s(s)}{\pi} + \left(C_{42}^V + C_{42}^{V,L}\ln\frac{m_q^2(s)}{s}\right)\left(\frac{\alpha_s(s)}{\pi}\right)^2\right] + 12\frac{m_q'^4(s)}{s^2}\left(\frac{\alpha_s(s)}{\pi}\right)^2 \\
& - \frac{m_q^6(s)}{s^3}\left[8 + \frac{16}{27}\left(155 + 6\ln\frac{m_q^2(s)}{s}\right)\frac{\alpha_s(s)}{\pi}\right], \qquad (306)
\end{aligned}$$

$$\begin{aligned}
R_A^q(s) = {} & 1 + \frac{3Q_q^2}{4}\frac{\alpha(s)}{\pi} + \frac{\alpha_s(s)}{\pi} - \frac{Q_q^2}{4}\frac{\alpha(s)}{\pi}\frac{\alpha_s(s)}{\pi} + \left[C_{02} + C_2^t\left(\frac{s}{M_t^2}\right) - \left(2I_q^{(3)}\right)\mathcal{I}^{(2)}\left(\frac{s}{M_t^2}\right)\right]\left(\frac{\alpha_s(s)}{\pi}\right)^2 \\
& + \left[C_{03} - \left(2I_q^{(3)}\right)\mathcal{I}^{(3)}\left(\frac{s}{M_t^2}\right)\right]\left(\frac{\alpha_s(s)}{\pi}\right)^3 + C_{04}\left(\frac{\alpha_s(s)}{\pi}\right)^4 + \frac{m_c^2(s)+m_b^2(s)}{s}C_{23}\left(\frac{\alpha_s(s)}{\pi}\right)^3 \\
& + \frac{m_q^2(s)}{s}\left[C_{20}^A + C_{21}^A\frac{\alpha_s(s)}{\pi} + C_{22}^A\left(\frac{\alpha_s(s)}{\pi}\right)^2 + 6\left(3+\ln\frac{M_t^2}{s}\right)\left(\frac{\alpha_s(s)}{\pi}\right)^2 + C_{23}^A\left(\frac{\alpha_s(s)}{\pi}\right)^3\right] \\
& - 10\frac{m_q^2(s)}{M_t^2}\left(\frac{8}{81} + \frac{1}{54}\ln\frac{M_t^2}{s}\right)\left(\frac{\alpha_s(s)}{\pi}\right)^2 \\
& + \frac{m_c^4(s)}{s^2}\left(C_{42} - \ln\frac{m_c^2(s)}{s}\right)\left(\frac{\alpha_s(s)}{\pi}\right)^2 + \frac{m_b^4(s)}{s^2}\left(C_{42} - \ln\frac{m_b^2(s)}{s}\right)\left(\frac{\alpha_s(s)}{\pi}\right)^2 \\
& + \frac{m_q^4(s)}{s^2}\left[C_{40}^A + C_{41}^A\frac{\alpha_s(s)}{\pi} + \left(C_{42}^A + C_{42}^{A,L}\ln\frac{m_q^2(s)}{s}\right)\left(\frac{\alpha_s(s)}{\pi}\right)^2\right] - 12\frac{m_q'^4(s)}{s^2}\left(\frac{\alpha_s(s)}{\pi}\right)^2, \qquad (307)
\end{aligned}$$

where the $\overline{\mathrm{MS}}$ mass $m_q(s)$ is taken to be non-zero only for the charm and bottom quarks, i.e., $m_q(s) = 0$ for $q = u, d, s$, and the primed mass $m_q'(s)$ is taken to be $m_c'(s) = m_b(s)$ and $m_b'(s) = m_c(s)$. The top-quark mass M_t is taken to be the pole mass. The formulae presented here are valid when no cuts are applied in the final state. The coefficients are given as follows:

- massless non-singlet corrections of $O(\alpha_s^2)$ [92, 93, 94], $O(\alpha_s^3)$ [89] and $O(\alpha_s^4)$ [95]:

$$C_{02} = \frac{365}{24} - 11\zeta(3) + \left(-\frac{11}{12} + \frac{2}{3}\zeta(3)\right) n_f\,, \tag{308}$$

$$\begin{aligned} C_{03} &= \frac{87029}{288} - \frac{121}{8}\zeta(2) - \frac{1103}{4}\zeta(3) + \frac{275}{6}\zeta(5) \\ &\quad + \left(-\frac{7847}{216} + \frac{11}{6}\zeta(2) + \frac{262}{9}\zeta(3) - \frac{25}{9}\zeta(5)\right) n_f + \left(\frac{151}{162} - \frac{1}{18}\zeta(2) - \frac{19}{27}\zeta(3)\right) n_f^2\,, \end{aligned} \tag{309}$$

$$C_{04} = -156.61 + 18.77 n_f - 0.7974 n_f^2 + 0.0215 n_f^3\,; \tag{310}$$

- quadratic massive correction [67]:

$$C_{23} = -80 + 60\zeta(3) + \left(\frac{32}{9} - \frac{8}{3}\zeta(3)\right) n_f\,, \tag{311}$$

$$C_{21}^V = 12\,, \tag{312}$$

$$C_{22}^V = \frac{253}{2} - \frac{13}{3} n_f\,, \tag{313}$$

$$\begin{aligned} C_{23}^V &= 2522 - \frac{855}{2}\zeta(2) + \frac{310}{3}\zeta(3) - \frac{5225}{6}\zeta(5) \\ &\quad + \left(-\frac{4942}{27} + 34\zeta(2) - \frac{394}{27}\zeta(3) + \frac{1045}{27}\zeta(5)\right) n_f + \left(\frac{125}{54} - \frac{2}{3}\zeta(2)\right) n_f^2\,, \end{aligned} \tag{314}$$

$$C_{20}^A = -6\,, \tag{315}$$

$$C_{21}^A = -22\,, \tag{316}$$

$$C_{22}^A = -\frac{8221}{24} + 57\zeta(2) + 117\zeta(3) + \left(\frac{151}{12} - 2\zeta(2) - 4\zeta(3)\right) n_f\,, \tag{317}$$

$$\begin{aligned} C_{23}^A &= -\frac{4544045}{864} + 1340\zeta(2) + \frac{118915}{36}\zeta(3) - 127\zeta(5) \\ &\quad + \left(\frac{71621}{162} - \frac{209}{2}\zeta(2) - 216\zeta(3) + 5\zeta(4) + 55\zeta(5)\right) n_f \\ &\quad + \left(-\frac{13171}{1944} + \frac{16}{9}\zeta(2) + \frac{26}{9}\zeta(3)\right) n_f^2\,; \end{aligned} \tag{318}$$

- quartic massive corrections [67]:

$$C_{42} = \frac{13}{3} - 4\zeta(3)\,, \tag{319}$$

$$C_{40}^V = -6\,, \tag{320}$$

$$C_{41}^V = -22\,, \tag{321}$$

$$C_{42}^V = -\frac{3029}{12} + 162\zeta(2) + 112\zeta(3) + \left(\frac{143}{18} - 4\zeta(2) - \frac{8}{3}\zeta(3)\right) n_f\,, \tag{322}$$

$$C_{42}^{V,L} = -\frac{11}{2} + \frac{1}{3} n_f\,, \tag{323}$$

$$C_{40}^{A} = 6\,, \tag{324}$$

$$C_{41}^{A} = 10\,, \tag{325}$$

$$C_{42}^{A} = \frac{3389}{12} - 162\zeta(2) - 220\zeta(3) + \left(-\frac{41}{6} + 4\zeta(2) + \frac{16}{3}\zeta(3)\right) n_f\,, \tag{326}$$

$$C_{42}^{A,L} = \frac{77}{2} - \frac{7}{3} n_f\,, \tag{327}$$

- power suppressed top-mass correction [67]:

$$C_2^t(x) = x\left(\frac{44}{675} - \frac{2}{135}\ln x\right)\,, \tag{328}$$

- singlet axial-vector corrections [67]:

$$\mathcal{I}^{(2)}(x) = -\frac{37}{12} + \ln x + \frac{7}{81}x + 0.0132x^2\,, \tag{329}$$

$$\mathcal{I}^{(3)}(x) = -\frac{5075}{216} + \frac{23}{6}\zeta(2) + \zeta(3) + \frac{67}{18}\ln x + \frac{23}{12}\ln^2 x\,, \tag{330}$$

where $n_f = 5$ in the above coefficients. The $O(\alpha_s^4)$ terms with the coefficient C_{04} in the massless singlet contribution were not included in ZFITTER [7], while it was included in Gfitter [91].

D Approximate two-loop formulae in the SM

D.1 W-boson mass

A simple parametrisation of the complete two-loop result for M_W was suggested in Ref. [49]:

$$\begin{aligned} M_W = M_W^0 &- c_1\,\mathrm{dH} - c_2\,\mathrm{dH}^2 + c_3\,\mathrm{dH}^4 + c_4\,(\mathrm{dh}-1) - c_5\,\mathrm{d}\alpha + c_6\,\mathrm{dt} - c_7\,\mathrm{dt}^2 \\ &- c_8\,\mathrm{dH}\,\mathrm{dt} + c_9\,\mathrm{dh}\,\mathrm{dt} - c_{10}\,\mathrm{d}\alpha_s + c_{11}\,\mathrm{dZ}\,, \end{aligned} \tag{331}$$

where

$$\begin{aligned} &\mathrm{dH} = \ln\left(\frac{m_h}{100\,\mathrm{GeV}}\right)\,, \quad \mathrm{dh} = \left(\frac{m_h}{100\,\mathrm{GeV}}\right)^2\,, \quad \mathrm{dt} = \left(\frac{M_t}{174.3\,\mathrm{GeV}}\right)^2 - 1\,, \\ &\mathrm{dZ} = \frac{M_Z}{91.1875\,\mathrm{GeV}} - 1\,, \quad \mathrm{d}\alpha = \frac{\Delta\alpha}{0.05907} - 1\,, \quad \mathrm{d}\alpha_s = \frac{\alpha_s(M_Z)}{0.119} - 1\,, \end{aligned} \tag{332}$$

and the values of the coefficients are given by

$$\begin{aligned} &M_W^0 = 80.3799\,(80.3800)\,, && c_1 = 0.05429\,(0.05253)\,, && c_2 = 0.008939\,(0.010345)\,, \\ &c_3 = 0.0000890\,(0.001021)\,, && c_4 = 0.000161\,(-0.000070)\,, && c_5 = 1.070\,(1.077)\,, \\ &c_6 = 0.5256\,(0.5270)\,, && c_7 = 0.0678\,(0.0698)\,, && c_8 = 0.00179\,(0.004055)\,, \\ &c_9 = 0.0000659\,(0.000110)\,, && c_{10} = 0.0737\,(0.0716)\,, && c_{11} = 114.9\,(115.0) \end{aligned} \tag{333}$$

in units of GeV. This parametrisation approximates the full result for M_W to better than 0.5 (0.2) MeV over the range of $10\,\mathrm{GeV} \le m_h \le 1\,\mathrm{TeV}$ ($100\,\mathrm{GeV} \le m_h \le 1\,\mathrm{TeV}$), if all other inputs vary within their combined 2σ region around their central values, reported in 2003. In the above formula, the following corrections to Δr are taken into account:

$$
\begin{aligned}
\Delta r &= \Delta r^{\alpha} + \Delta r^{\alpha\alpha_s} + \Delta r^{\alpha\alpha_s^2} + \Delta r^{\alpha^2} + \Delta r^{G_\mu^2 \alpha_s M_t^4} + \Delta r^{G_\mu^3 M_t^6}\,, \\
&\approx \Big[283.41|_{\alpha} + 35.89|_{\alpha\alpha_s} + 7.23|_{\alpha\alpha_s^2} + (28.56|_{\alpha^2,\mathrm{ferm}} + 0.64|_{\alpha^2,\mathrm{bos}}) \\
&\qquad - 1.27|_{G_\mu^2 \alpha_s M_t^4} - 0.16|_{G_\mu^3 M_t^6} \Big] \times 10^{-4}\,,
\end{aligned} \tag{334}
$$

where the numerical values are estimated with $m_h = 100$ GeV, etc. [49]. The remaining theoretical uncertainty coming from missing higher-order corrections was estimated to be 4 MeV for $m_h \lesssim 300$ GeV [49], while the current experimental uncertainty is about 20 MeV [4, 5].

The resummations of $\Delta\alpha$ and $\Delta\rho$ are not used in Ref. [49], since the the higher-order corrections involve powers of $\Delta\alpha$ and $\Delta\rho$ beyond two-loop order. Namely, the W-boson mass is related to Δr as

$$
M_W^2 = \frac{M_Z^2}{2}\left(1 + \sqrt{1 - \frac{4\pi\alpha(0)}{\sqrt{2} G_\mu M_Z^2}(1+\Delta r)}\,\right). \tag{335}
$$

D.2 Effective weak mixing angle

A simple formula for the effective weak mixing angle for a final state $f\bar{f}$ is given by [65, 66, 64]

$$
\begin{aligned}
\sin^2\theta_{\mathrm{eff}}^{f} = s_0 &+ d_1 L_H + d_2 L_H^2 + d_3 L_H^4 + d_4(\Delta_H^2 - 1) + d_5 \Delta_\alpha \\
&+ d_6 \Delta_t + d_7 \Delta_t^2 + d_8 \Delta_t(\Delta_H - 1) + d_9 \Delta_{\alpha_s} + d_{10}\Delta_Z\,,
\end{aligned} \tag{336}
$$

where

$$
\begin{aligned}
&L_H = \ln\left(\frac{m_h}{100\,\mathrm{GeV}}\right), \qquad \Delta_H = \frac{m_h}{100\,\mathrm{GeV}}\,, \qquad \Delta_\alpha = \frac{\Delta\alpha}{0.05907} - 1\,, \\
&\Delta_t = \left(\frac{M_t}{178.0\,\mathrm{GeV}}\right)^2 - 1\,, \quad \Delta_{\alpha_s} = \frac{\alpha_s(M_Z)}{0.117} - 1\,, \quad \Delta_Z = \frac{M_Z}{91.1876\,\mathrm{GeV}} - 1\,,
\end{aligned} \tag{337}
$$

and the coefficients are listed in Table 3. This parametrisation reproduces the exact result with maximal deviations of 4.5×10^{-6} (4.3×10^{-6}) for the Higgs mass $10\,\mathrm{GeV} \le m_h \le 1\,\mathrm{TeV}$ in the case of $f = e, \mu, \tau, \nu_{e,\mu,\tau}, u, d, s, c$ ($f = b$), if all other inputs vary within their combined 2σ region around their central values, reported in 2003 (or 2004?). As in the case of M_W in Sec. D.1, the EW/QCD one-loop and two-loop corrections, and the leading three-loop corrections of $O(G_\mu^2 \alpha_s M_t^4)$ and $O(G_\mu^3 M_t^6)$ are incorporated in the above result. Namely, the quantity $\Delta\kappa$ for $\kappa_Z^{\mathrm{lept}} = 1 + \Delta\kappa$ is written as

$$
\begin{aligned}
\Delta\kappa &= \Delta\kappa^{\alpha} + \Delta\kappa^{\alpha\alpha_s} + \Delta\kappa^{\alpha\alpha_s^2} + \Delta\kappa^{\alpha^2} + \Delta\kappa^{G_\mu^2 \alpha_s M_t^4} + \Delta\kappa^{G_\mu^3 M_t^6}\,, \\
&\approx \Big[413.33|_{\alpha} - 35.58|_{\alpha\alpha_s} - 7.25|_{\alpha\alpha_s^2} + (1.07|_{\alpha^2,\mathrm{ferm}} - 0.74|_{\alpha^2,\mathrm{bos}}) \\
&\qquad + 1.15|_{G_\mu^2 \alpha_s M_t^4} + 0.14|_{G_\mu^3 M_t^6} \Big] \times 10^{-4}\,,
\end{aligned} \tag{338}
$$

where the numerical values are estimated with $m_h = 100$ GeV, etc. [66]. The theoretical uncertainty from missing higher-order corrections is estimated to be 4.7×10^{-5} for the leptonic channels [65, 66], while the current experimental uncertainty is of $O(10^{-4})$ [4, 5].

Note that the bosonic two-loop contribution of $O(\alpha^2)$ is not included in the case of the $b\bar{b}$ mode. For the $Zb\bar{b}$ vertex, leading four-loop QCD corrections to $\Delta\rho$ from top-bottom loops [56, 57] are also taken into account.

Table 3: Coefficients of $\sin^2\theta^f_{\rm eff}$ for different final states $f\bar{f}$. **Why is the sign of d_{10} positive only for $f = b$?**

f	e,μ,τ	$\nu_{e,\mu,\tau}$	u,c	d,s	b
s_0	0.2312527	0.2308772	0.2311395	0.2310286	0.2327580
d_1 $[10^{-4}]$	4.729	4.713	4.726	4.720	4.749
d_2 $[10^{-5}]$	2.07	2.05	2.07	2.06	2.03
d_3 $[10^{-6}]$	3.85	3.85	3.85	3.85	3.94
d_4 $[10^{-6}]$	-1.85	-1.85	-1.85	-1.85	-1.84
d_5 $[10^{-2}]$	2.07	2.06	2.07	2.07	2.08
d_6 $[10^{-3}]$	-2.851	-2.850	-2.853	-2.848	-0.993
d_7 $[10^{-4}]$	1.82	1.82	1.83	1.81	0.708
d_8 $[10^{-6}]$	-9.74	-9.71	-9.73	-9.73	-7.61
d_9 $[10^{-4}]$	3.98	3.96	3.98	3.97	4.03
d_{10} $[10^{-1}]$	-6.55	-6.54	-6.55	-6.55	6.61

D.3 Δr^{α^2} and $\Delta\kappa^{\alpha^2}$

The approximate formulae for Δr and $\Delta\kappa$ $(=\kappa_Z^{\rm lept}-1)$ at the pure EW two-loop level are also presented in Ref. [66]:

$$\Delta r^{\alpha^2} = (\Delta\alpha)^2 + 2\Delta\alpha\,\Delta r^{\alpha} + \Delta r^{\alpha^2}_{\rm rem}\,, \tag{339}$$

$$\begin{aligned}\Delta r^{\alpha^2}_{\rm rem} &= r_0 + r_1\,L_H + r_2\,L_H^2 + r_3\,L_H^4 + r_4\,(\Delta_H^2-1) + r_5\,\Delta_t \\ &\quad + r_6\,\Delta_t^2 + r_7\,\Delta_t L_H + r_8\,\Delta_W + r_9\,\Delta_W\Delta_t + r_{10}\,\Delta_Z\,,\end{aligned} \tag{340}$$

$$\Delta\kappa^{\alpha^2} = \Delta\alpha\,\Delta\kappa^{\alpha} + \Delta\kappa^{\alpha^2}_{\rm rem}\,, \tag{341}$$

$$\begin{aligned}\Delta\kappa^{\alpha^2}_{\rm rem} &= k_0 + k_1\,L_H + k_2\,L_H^2 + k_3\,L_H^4 + k_4\,(\Delta_H^2-1) + k_5\,\Delta_t \\ &\quad + k_6\,\Delta_t^2 + k_7\,\Delta_t L_H + k_8\,\Delta_W + k_9\,\Delta_W\Delta_t + k_{10}\,\Delta_Z\,,\end{aligned} \tag{342}$$

where L_H, Δ_H, Δ_t and Δ_Z are given in Eq. (337), and

$$\Delta_W = \frac{M_W}{80.404\,\text{GeV}} - 1\,. \tag{343}$$

The coefficients are given by

$$\begin{aligned} &r_0 = 0.003354\,, && r_1 = -2.09\times10^{-4}\,, && r_2 = 2.54\times10^{-5}\,, && r_3 = -7.85\times10^{-6}\,, \\ &r_4 = -2.33\times10^{-6}\,, && r_5 = 7.83\times10^{-3}\,, && r_6 = 3.38\times10^{-3}\,, && r_7 = -9.89\times10^{-6}\,, \\ &r_8 = 0.0939\,, && r_9 = 0.204\,, && r_{10} = -0.103\,, \end{aligned} \tag{344}$$

and

$$\begin{aligned} &k_0 = -0.002711\,, && k_1 = -3.12\times10^{-5}\,, && k_2 = -4.12\times10^{-5}\,, && k_3 = 5.28\times10^{-6}\,, \\ &k_4 = 3.75\times10^{-6}\,, && k_5 = -5.16\times10^{-3}\,, && k_6 = -2.06\times10^{-3}\,, && k_7 = -2.32\times10^{-4}\,, \\ &k_8 = -0.0647\,, && k_9 = -0.129\,, && k_{10} = 0.0712\,. \end{aligned} \tag{345}$$

Moreover, the two-loop fermionic EW corrections to κ_Z^b for the $b\bar{b}$ channel can be approximated as [64]

$$\Delta\kappa_{b\bar{b}}^{\alpha^2,\text{ferm}} = \Delta\alpha\,\Delta\kappa_{b\bar{b}}^{\alpha} + \Delta\kappa_{b\bar{b},\text{rem}}^{\alpha^2,\text{ferm}}\,, \tag{346}$$

$$\begin{aligned}\Delta\kappa_{b\bar{b},\text{rem}}^{\alpha^2,\text{ferm}} &= \bar{k}_0 + \bar{k}_1\,L_H + \bar{k}_2\,L_H^2 + \bar{k}_3\,L_H^4 + \bar{k}_4\,(\Delta_H^2 - 1) + \bar{k}_5\,\Delta_t \\ &\quad + \bar{k}_6\,\Delta_t^2 + \bar{k}_7\,\Delta_t L_H + \bar{k}_8\,\Delta_W + \bar{k}_9\,\Delta_W\Delta_t + \bar{k}_{10}\,\Delta_Z\end{aligned} \tag{347}$$

with

$$\begin{array}{llll} \bar{k}_0 = -0.002666\,, & \bar{k}_1 = -5.92\times10^{-5}\,, & \bar{k}_2 = -3.29\times10^{-6}\,, & \bar{k}_3 = 3.49\times10^{-6}\,, \\ \bar{k}_4 = 2.83\times10^{-6}\,, & \bar{k}_5 = -5.34\times10^{-3}\,, & \bar{k}_6 = -2.10\times10^{-3}\,, & \bar{k}_7 = -2.19\times10^{-4}\,, \\ \bar{k}_8 = -0.0631\,, & \bar{k}_9 = -0.126\,, & \bar{k}_{10} = 0.0647\,. & \end{array} \tag{348}$$

E Our Codes

Brief usage of the codes for EW physics:

1. Create a StandardModel object and set parameters:

```
StandardModel* myModel;
myModel = new StandardModel();

std::map<std::string, double> Parameters;
Parameters["GF"] = 1.16637E-5;
Parameters["mneutrino_1"] = 0.0;
....
....
myModel->Init(Parameters);
```

2. Create an EW object, and define flags for resummation scheme and higher-order corrections:

```
EW myEW(*myModel);

const EW::schemes_EW schemeMw = EW::OMSI;
const EW::schemes_EW schemeRhoZ = EW::OMSI;
const EW::schemes_EW schemeKappaZ = EW::OMSI;
const bool flag_order[EWSM::orders_EW_size] = {true,true,true,true,true,true};
```

3. Computes the radiative corrections to M_W, ρ_Z^f and κ_Z^f with EWSM or ZFitter class:

```
myEW.ComputeEWSM(schemeMw, schemeRhoZ, schemeKappaZ, flag_order);
```

   or

```
myEW.ComputeZFitter(schemeMw, schemeRhoZ, schemeKappaZ, flag_order);
```

4. Computes observables, e.g.,

```
Mw myMw(myEW);
cout << "Mw = " << myMw.getThValue() << endl;

sin2thetaEff mySin2thetaEff(myEW);
cout << "sin2thetaEff = " << mySin2thetaEff.getThValue() << endl;
```

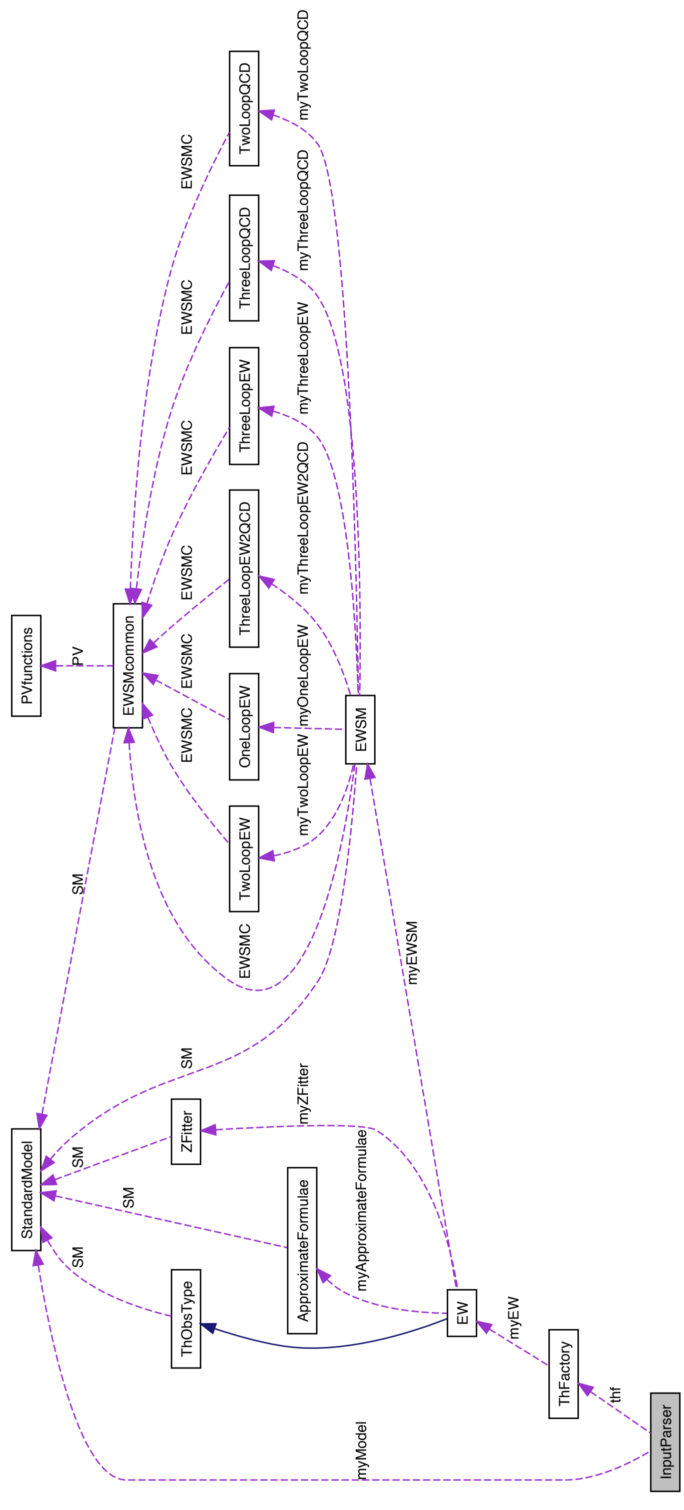

Figure 2: Class hierarchy of our codes for EW physics.

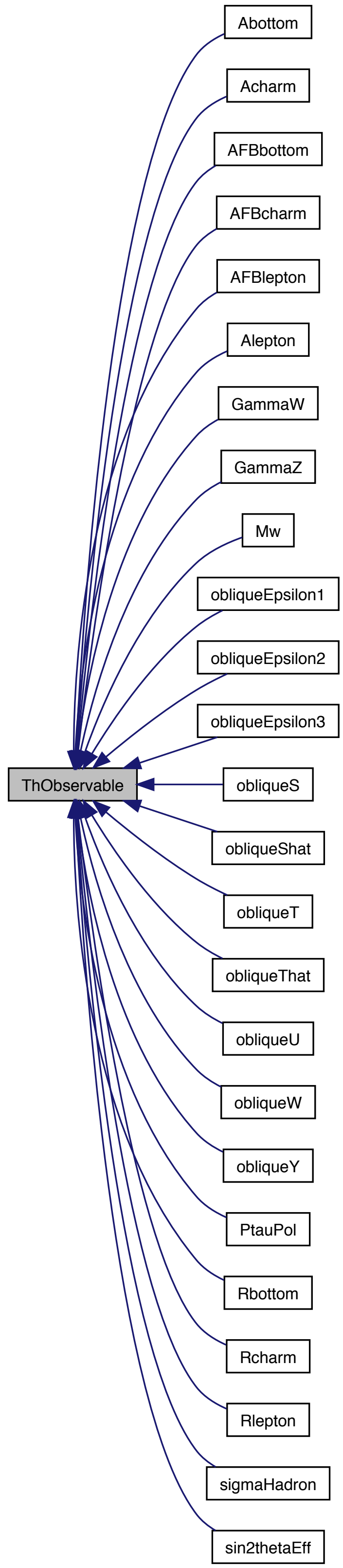

Figure 3: Classes for precision observables in EW project.

F Other public codes

F.1 ZFITTER

ZFITTER [7, 9, 10] is a FORTRAN program for the evaluation of EW precision observables including important radiative corrections. The current version, v6.43 (as of May 18, 2011), is available online at [8]. We summarize some important files of ZFITTER below:

- `zfusr6_43.f` — subroutines/functions for users:

  `ZUINIT()`: sets flags and cuts.

  `ZUWEAK()`: calls `ZDIZET()` and `ROKANC()`.

  `ZDIZET()`: calls `DIZET()`.

- `dizet6_43.f` — codes for one-loop corrections to Δr, ρ_Z^f and κ_Z^f and those for QCD corrections to $\delta\hat{\rho}(= -\Delta\rho)$, (see Fig. 4); computes M_W, s_W^2, Γ_f and $\sin^2\theta_{\text{eff}}^f$ with the two-loop corrections provided by `bkqcdl5_14.f` and `m2tcor5_11.f`:

  `DIZET()`: calls `CONST1()`, `QCDCOF()`, `SETCON()` and `ZWRATE()`.

  `ROKANC()`: computes the effective couplings for $e^+e^- \to f\bar{f}$, which will be stored into `ALLCH[0-11]` in the common block `EWFORM`.

  `XFOTF3()`: computes $\Delta\alpha * 4\pi/\alpha(0)$, i.e.,

$$\alpha(s) = \frac{\alpha(0)}{1 - \frac{\alpha(0)}{4\pi}\,\mathtt{DREAL}(\mathtt{XFOTF3}())}\,. \tag{349}$$

  `CONST1()`: sets the coupling constants and the fermion masses.

  `QCDCOF()`: computes $R^q_{V,A}(s)$.

  `SETCON()`: calls `CONST2()` and `SEARCH()`, and computes M_W iteratively.

  `CONST2()`: computes self-energies.

  `SEARCH()`: computes Δr.

  `ZWRATE()`: calls `VERTZW()` and `FOKAPP()` (`ROKAPP()` for $b\bar{b}$), and computes $\sin^2\theta_{\text{eff}}^f$ and Γ_f.

  `VERTZW()`: computes $\mathrm{Re}[\mathcal{F}^0_{Za}]$, $\mathrm{Re}[\mathcal{F}^0_{Wa}]$ and $\mathcal{F}^0_{Wn}$, and calls `VTBANA()`.

  `VTBANA()`: computes $\mathcal{F}^t_{Wa}$, $\overline{\mathcal{F}}^t_{Wa}$ and $\mathcal{F}^t_{Wn}$.

  `FOKAPP()`: calls `XV1B()`, `AFMT3()` and `TBQCDR()`, and computes ρ_Z^f and κ_Z^f for $f \neq b$.

  `ROKAPP()`: calls `AFMT3()` and `FBARBB()`, and computes ρ_Z^b and κ_Z^b.

  `XV1B()`: computes $\mathrm{Im}[\mathcal{F}^0_{Za}]$ and $\mathrm{Im}[\mathcal{F}^0_{Wa}]$.

  `AFMT3()`: computes the leading QCD corrections to $\delta\hat{\rho}$ of $O(G_\mu\alpha_s M_t^2)$ and $O\left(G_\mu\alpha_s^2 M_t^2(1 + (M_Z^2/M_t^2)^2 + (M_Z^2/M_t^2)^4)\right)$ based on [32, 34]:

  `TBQCDR()`: computes $O\left(G_\mu\alpha_s^2 M_t^2((M_Z^2/M_t^2)^2 + (M_Z^2/M_t^2)^4)\right)$ contribution to $\delta\kappa_{\text{rem}}$.

  `FBARB()`: computes $\rho^{(2)}$ of $O(G_\mu^2 M_t^4)$.

  `FBARBB()`: computes Barbieri's $\tau^{(2)}$ of $O(G_\mu^2 M_t^4)$.

- `bkqcdl5_14.f` — codes for QCD corrections of $O(\alpha\alpha_s)$ based on [27]:

  `DRMQCD()`: computes Δr^{tb}.

  `XRMQCD()`: computes $\Delta\rho^{tb}$.

Table 4: Inputs and outputs of DIZET().

Inputs					
M_Z	AMZ	M_t	AMT	m_h	AMH
$\alpha_s(M_Z^2)$	ALSTR	$\Delta\alpha_{\rm had}^{(5)}$	DAL5H		
Outputs					
M_W	AMW	$\alpha(M_Z^2)$	ALQED	$\alpha_s(M_t^2)$	ALSTRT
Δr	DR	$\Delta r_{\rm rem}^{\alpha}$	DR	s_W^2	SW2
$\sin^2\theta_{\rm eff}^f$	ZPAR(5-14)	$R_{V,A}^q(M_Z^2)$	ZPAR(17-28)	Γ_f	PARTZ(0-9)
Γ_h	PARTZ(10)	Γ_Z	PARTZ(11)		

`XKMQCD()`: computes $\Delta\kappa^{tb}$.

`CV1()`: computes $V_1(r)$.

`CA1()`: computes $A_1(r)$.

`CF1()`: computes $F_1(x)$.

- `m2tcor5_11.f` — codes for two-loop EW corrections of $O(G_\mu^2 M_t^4)$ and $O(G_\mu^2 M_t^2 M_Z^2)$ based on [39, 40, 41]:

  `GDEGNL()`: computes $\delta\hat{\rho}^{G^2}$, $\Delta r_{\rm rem}^{\alpha^2}$, $\delta\rho_{\rm rem}^{f,G^2}$ and $\delta\kappa_{\rm rem}^{f,G^2}$.

The input and output parameters of `DIZET()` are summarized in Table 4, and output variables in the common block `CDZRKZ` are listed in Table 11.

A couple of constants are fixed inside the ZFITTER codes: G_μ and $\alpha(0)$ are set in subroutines `CONST1()` and `EWINIT()` in `zfbib6_40.f` (and other places?), and all lepton and quark masses (u,d,s,c,b) are set in `CONST1()`. The two important flags that should be changed from their default values are

- `AMT4=6`: The two-loop formulas for M_W and $\sin^2\theta_{\rm eff}^{\rm lept}$ in Secs. D.1 and D.2 are employed.
- `ALEM=2`: $\Delta\alpha_{\rm had}^{(5)}$ is supplied by the user as input.

The two-loop approximate formula for $\sin^2\theta_{\rm eff}^q$ has not been included in ZFITTER, and the two-loop EW contribution for κ_Z^q is taken to be the same as that for the leptonic modes, which is obtained from the approximate formula for $\sin^2\theta_{\rm eff}^{\rm lept}$.

$$\frac{2p^2}{M_W^2}\left[B_0(p^2;m_0,m_1)\big|_{\mu=M_W}+\ln\frac{m_0m_1}{M_W^2}-\frac{m_0^2-m_1^2}{2p^2}\ln\frac{m_1^2}{m_0^2}-2\right]=\frac{1}{M_W^2}L(-p^2;m_0^2,m_1^2)\Big|_{[17]}. \quad (350)$$

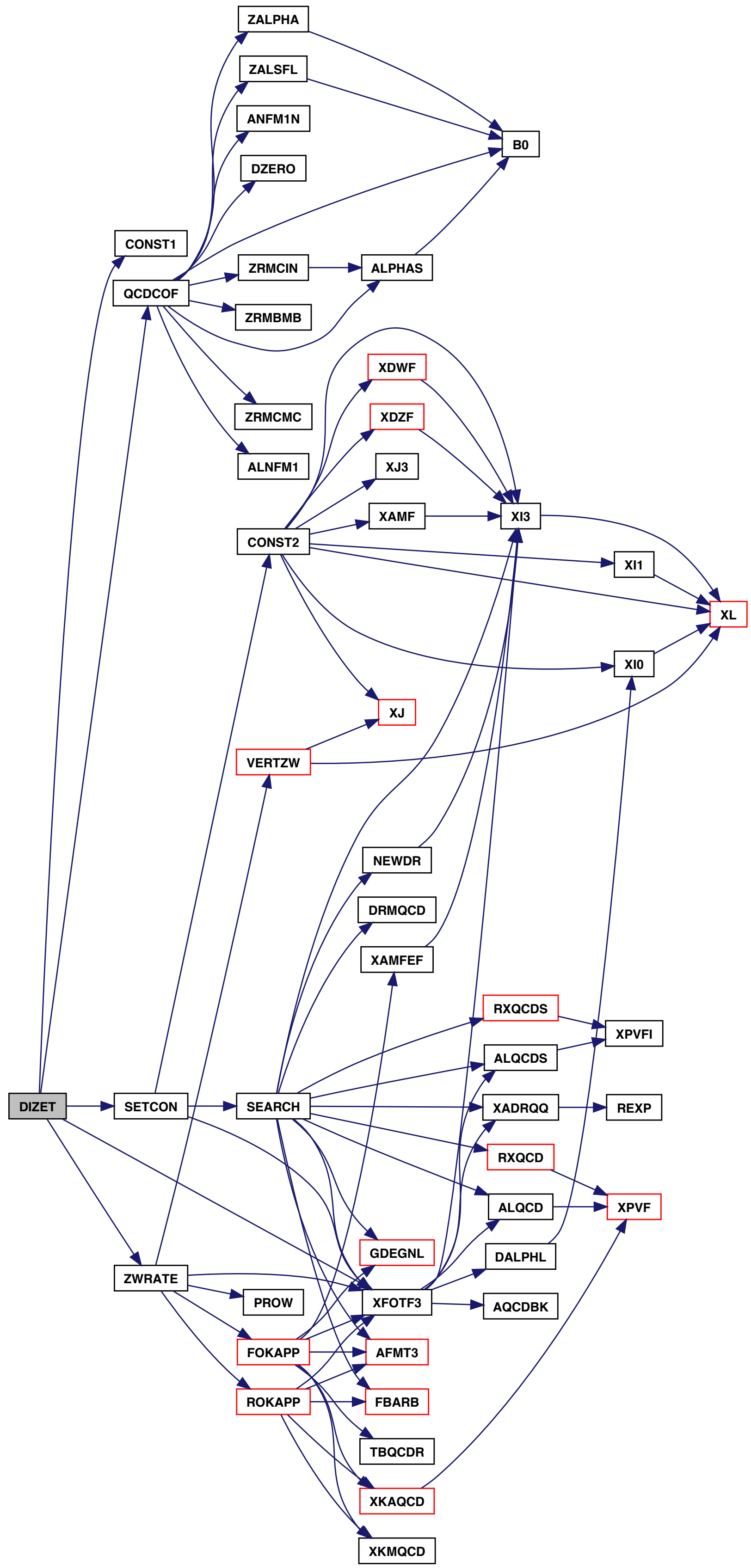

Figure 4: Subroutines and functions called from DIZET.

Table 5: Variables calculated in `CONST2()`, where the column "check" shows the results of numerical comparisons between ZFITTER and our codes.

	codes	check
$2\left[B_0(M_W^2;m_h,M_W)\vert_{\mu=M_W}+\cdots\right]=L(-M_W^2;m_h^2,M_W^2)/M_W^2$	XL1	**OK**
$2\left[B_0(M_W^2;M_Z,M_W)\vert_{\mu=M_W}+\cdots\right]=L(-M_W^2;M_W^2,M_Z^2)/M_W^2$	XL2	**OK**
$\frac{2}{c_W^2}\left[B_0(M_Z^2;m_h,M_Z)\vert_{\mu=M_W}+\cdots\right]=L(-M_Z^2;m_h^2,M_Z^2)/M_W^2$	XL3	**OK**
$\frac{2}{c_W^2}\left[B_0(M_Z^2;M_W,M_W)\vert_{\mu=M_W}-2\right]=L(-M_z^2;M_W^2,M_W^2)/M_W^2$	XL4	**OK**
$\frac{1}{M_W^2}\overline{\Sigma}_{WW}^{\mathrm{bos},F}(0)\vert_{\mu=M_Z}$	W0	**OK**
$\frac{1}{M_W^2}\overline{\Sigma}_{WW}^{\mathrm{fer},F}(0)\vert_{\mu=M_Z}$	W0F	**OK**
$\frac{1}{M_W^2}\overline{\Sigma}_{WW}^{\mathrm{bos},F}(M_W^2)\vert_{\mu=M_Z}$	XWM1	**OK**
$\frac{1}{M_W^2}\overline{\Sigma}_{WW}^{\mathrm{fer},F}(M_W^2)\vert_{\mu=M_Z}$	XWM1F	**OK**
$\frac{1}{M_W^2}\overline{\Sigma}_{ZZ}^{\mathrm{bos},F}(M_Z^2)\vert_{\mu=M_Z}$	XZM1	**OK**
$\frac{1}{M_W^2}\overline{\Sigma}_{ZZ}^{\mathrm{fer},F}(M_Z^2)\vert_{\mu=M_Z}$	XZM1F	**OK**
$-\overline{\Pi}_{Z\gamma}^{\mathrm{bos},F}(M_Z^2)\vert_{\mu=M_Z}$	XAMM1	**OK**
$-\overline{\Pi}_{Z\gamma}^{\mathrm{fer},F}(M_Z^2)\vert_{\mu=M_Z}$	XAMM1F	**OK**
$\overline{\Pi}_{\gamma\gamma}^{\mathrm{fer},F}(0)\vert_{\mu=M_Z}$	$-\frac{4}{3}$(SL2+SQ2)	**OK**
$\frac{1}{M_W^2}\overline{\Sigma}_{WW}^{\mathrm{bos},F}(0)\vert_{\mu=M_Z}$	W0$+\cdots$	
$\frac{1}{M_W^2}\overline{\Sigma}_{WW}^{\mathrm{fer},F}(0)\vert_{\mu=M_Z}$	W0F$+\cdots$	
$\frac{1}{M_W^2}\overline{\Sigma}_{WW}^{\mathrm{bos},F}(M_W^2)\vert_{\mu=M_Z}$	XWM1$+\cdots$	
$\frac{1}{M_W^2}\overline{\Sigma}_{WW}^{\mathrm{fer},F}(M_W^2)\vert_{\mu=M_Z}$	XWM1F$+\cdots$	
$\frac{1}{M_W^2}\overline{\Sigma}_{ZZ}^{\mathrm{bos},F}(M_Z^2)\vert_{\mu=M_Z}$	XZM1$+\cdots$	
$\frac{1}{M_W^2}\overline{\Sigma}_{ZZ}^{\mathrm{fer},F}(M_Z^2)\vert_{\mu=M_Z}$	XZM1F$+\cdots$	
$-\overline{\Pi}_{Z\gamma}^{\mathrm{bos},F}(M_Z^2)\vert_{\mu=M_Z}$	XAMM1$+\cdots$	
$-\overline{\Pi}_{Z\gamma}^{\mathrm{fer},F}(M_Z^2)\vert_{\mu=M_Z}$	XAMM1F$+\cdots$	
$\overline{\Pi}_{\gamma\gamma}^{\mathrm{fer},F}(0)\vert_{\mu=M_Z}$	$-\frac{4}{3}$(SL2+SQ2)$+\cdots$	
$\overline{\Sigma}_{WW}^{\prime,\mathrm{bos},F}(M_W^2)\vert_{\mu=M_W}$	XWFM1	
$\overline{\Sigma}_{WW}^{\prime,\mathrm{fer},F}(M_W^2)\vert_{\mu=M_W}$	XWFM1F	
$\frac{1}{c_W^2}\overline{\Sigma}_{ZZ}^{\prime,\mathrm{bos},F}(M_Z^2)\vert_{\mu=M_W}$	XZFM1	
$\frac{1}{c_W^2}\overline{\Sigma}_{ZZ}^{\prime,\mathrm{fer},F}(M_Z^2)\vert_{\mu=M_W}$	XZFM1F	

Table 6: Passarino-Veltman functions

	codes	check
$B_f(s;\, m_1, m_2)\vert_{\mu=M}$	`2*XI3`$(M^2, -s, M_1^2, M_2^2)$	
$C_0(s;\, M_t, M_W, M_t)$	`XS3T`	
$C_0(s;\, 0, M_W, 0)$	`XS3T0`	
$C_0(s;\, M_W, M_t, M_W)$	`XS3W`	
$C_0(s;\, 0, M_t, 0)$	`XS3W0`	
These functions are computed with subroutine `S3WANA()`.		

Table 7: Variables used in `SEARCH()`, where the column "check" shows the results of numerical comparisons between ZFITTER and our codes.

	codes	check
s_W^2	`R1`	**OK**
c_W^2	`R`	**OK**
$\Delta\bar{\rho}^{\rm bos}\vert_{\mu=M_W}$	$s_W^2 *$ `XWZ1R1`	**OK**
$\Delta\bar{\rho}^{\rm fer}\vert_{\mu=M_W}$	$s_W^2 *$ `XDWZ1F`	**OK**
$\Delta\bar{\rho}_W^{\rm bos}\vert_{\mu=M_W}$	`W0-DREAL(XWM1)`	**OK**
$\Delta\bar{\rho}_W^{\rm fer}\vert_{\mu=M_W}$	`W0F-DREAL(XWM1F)`	**OK**
$\frac{\alpha(0)}{4\pi s_W^2}\frac{c_W^2}{s_W^2}\left(\Delta\bar{\rho}^F\vert_{\mu=M_Z} - \Delta\bar{\rho}^F\vert_{\mu=M_W}\right)$	`SCALE`	
$V_1(r)$	`CV1()`	
$A_1(r)$	`CA1()`	
$F_1(x)$	`CF1()`	
Δr^{ud}	`CLQQCD`	
Δr^{tb}	`XTBQCD`	
$\frac{\alpha_s(M_t^2)}{\pi}\,\delta_2^{\rm QCD} + \left(\frac{\alpha_s(M_t^2)}{\pi}\right)^2 \delta_3^{\rm QCD}$	`TBQCD0`	
$-\frac{c_W^2}{s_W^2}\frac{1}{f}\,\Delta\rho^{G\alpha_s}$	`TBQCDL`	
$\Delta\alpha^{\ell+5q}$	`DALFA`	
$\Delta\rho^G$	$-f\left(\frac{\alpha(0)}{4\pi}{\rm Re}[\texttt{XWZ1R1} + \texttt{XDWZ1F}] + \frac{s_W^2}{c_W^2}\texttt{SCALE}\right)$	
$\Delta\rho^{G\alpha_s} + \Delta\rho^{G\alpha_s^2}$	$\frac{3\sqrt{2}\,G_\mu M_t^2}{16\pi^2} *$ `TBQCD0`	
$\Delta\rho^{G^2}$	`DRHOD`	
$\Delta\rho^{(G)} = \Delta\rho^G + \Delta\rho^{G\alpha_s} + \Delta\rho^{G\alpha_s^2} + \Delta\rho^{G^2}$	`DROBAR`	
$\Delta r_{\rm rem}^{\alpha}$	`DRREMN`	
$\Delta r_{\rm rem}^{\alpha\alpha_s}$	${\rm Re}[\texttt{XTBQCD}] - \texttt{TBQCDL} + 2*\texttt{CLQQCD}$	
$\Delta r_{\rm rem}^{\alpha} + \Delta r_{\rm rem}^{\alpha\alpha_s}$	`DRREM`	
$\Delta r_{\rm rem}^{\alpha^2}$	`DRDREM`	
Δr	`DR`	

Table 8: Variables used in `FOKAPP()`, where the column "check" shows the results of numerical comparisons between ZFITTER and our codes.

	codes	check
$\mathrm{Re}[\mathcal{F}^0_{Za}(M_Z^2)]$	`V1ZZ`	
$\mathrm{Im}[\mathcal{F}^0_{Za}(M_Z^2)]$	`V1ZIM`	
$\mathrm{Re}[\mathcal{F}^0_{Wa}(M_Z^2)]$	`V1ZW`	
$\mathrm{Im}[\mathcal{F}^0_{Wa}(M_Z^2)]$	`V1WIM`	
$\mathcal{F}^0_{Wn}(M_Z^2)$	`V2ZWW`	
$c_W^2\mathcal{F}^t_{Wn}(s) - \frac{1}{2}\sigma^a_{f'}\mathcal{F}^t_{Wa}(s) - \frac{1}{2}\overline{\mathcal{F}}^t_{Wa}(s)$	`VTB`	
f	`RENORM`	
$\Delta\rho^{\alpha\alpha_s}$	`TBQCDL`	
$-\Delta\rho^{\alpha}$	`CORRHO`	
$-\frac{c_W^2}{s_W^2}\Delta\rho^{\alpha}$	`CORKAP`	
$\Delta\bar{r}^{\alpha}_{\mathrm{rem}}$	`DRREMD`	
$\lvert 1+\Delta\rho^{\alpha}+\delta\rho^{f,\alpha}_{\mathrm{rem}}\rvert$	`RO1`	
$\mathrm{Re}[\Delta\rho^{tb}] + 2\Delta\rho^{ud}$	`ROQCD`	
$\mathrm{Re}[1+\frac{c_W^2}{s_W^2}\Delta\rho^{\alpha}+\delta\kappa^{f,\alpha}_{\mathrm{rem}}]$	`AK1`	
$\mathrm{Re}[\Delta\kappa^{tb}] + 2\Delta\kappa^{ud}$	`AKQCD`	
$\Delta\rho^{(G)} = \Delta\rho^{G} + \Delta\rho^{G\alpha_s} + \Delta\rho^{G\alpha_s^2} + \Delta\rho^{G^2}$	`DROBAR`	
$\Delta\rho^{G}$	`DROBRO`	
$\Delta\bar{r}^{[G]}_{\mathrm{rem}}$	`RENORM*DRREMD`	
$\delta\rho^{f,[G]}_{\mathrm{rem}}$	`RENORM*[(RO1+ROQCD)-(1-CORRHO+TBQCDL)]`	
$\delta\rho^{f,G^2}_{\mathrm{rem}}$	`DROREM`	
$\delta\kappa^{f,[G]}_{\mathrm{rem}}$	`RENORM*[(AK1+AKQCD)-(1-CORKAP+R/SW2*TBQCDL)+DKTB3R]`	
$\delta\kappa^{f,G\alpha_s^2}_{\mathrm{rem}}$	`DKTB3R`	
$\delta\kappa^{f,G^2}_{\mathrm{rem}}$	`DKDREM`	
$\frac{35\alpha^2(M_Z^2)}{18}\,(1-\frac{8}{3}\,\mathrm{Re}(\kappa_Z^f)s_W^2)$	`ADDIM`	
$\mathrm{Re}[\rho_Z^f]$	`ROFAC`	
$\mathrm{Im}[\rho_Z^f]$	`AR1IM`	
$\mathrm{Re}[\kappa_Z^f]$	`AKFAC+ADDIM/`s_W^2	
$\mathrm{Im}[\kappa_Z^f]$	`AK1IM`	

Table 9: Variables used in `ROKAPP()`, where the column "check" shows the results of numerical comparisons between ZFITTER and our codes.

	codes	check
$2u_b$	`UFF`	
$\mathrm{Re}[\Delta\rho^{\alpha}+\delta\rho^{b,\alpha}_{\mathrm{rem}}]$	`RO1`	
$\mathrm{Re}[\frac{c_W^2}{s_W^2}\Delta\rho^{\alpha}+\delta\kappa^{b,\alpha}_{\mathrm{rem}}]$	`AK1`	
$\mathrm{Re}[\Delta\rho^{tb}]+2\Delta\rho^{ud}$	`ROQCD`	
$\mathrm{Re}[\Delta\kappa^{tb}]+2\Delta\kappa^{ud}$	`AKQCD`	
$\rho^{(2)}(M_t^2/m_h^2)$	`AMT4C`	
$\tau^{(2)}(M_t^2/m_h^2)$	`AMT4B`	
X_t	`TOPX2`	
τ_b	`TAUBB1+TAUBB2`	
$\frac{\alpha(0)}{8\pi s_W^2}\frac{M_t^2}{M_W^2}$	`CORBB`	
$\frac{c_W^2}{s_W^4}\Delta\overline{\rho}^F$	`DWZ1AL`	
f	`RENORM`	
$-\frac{c_W^2}{s_W^2}\Delta\rho^{\alpha}$	`CORKAP`	
$\Delta\rho^{\alpha}_L=\frac{1}{f}3X_t$	`DRHOT`	
$\Delta\rho_L^{G_\mu}+\Delta\rho_L^{G_\mu^2}$	`DRHOT4`	
$\Delta\rho_L^{G\alpha_s}+\Delta\rho_L^{G\alpha_s^2}$	`TBQCD0`	
$\Delta\rho_L^{\alpha\alpha_s}$	`TBQCDL`	
$(1+\tau_b)^2/(1-\Delta\rho_L^{G_\mu}-\Delta\rho_L^{G_\mu^2}-\Delta\rho_L^{G\alpha_s}-\Delta\rho_L^{G\alpha_s^2})$	`ROFACL`	
	`AKFACL`	
$\delta\rho^{b,\alpha}_{\mathrm{rem}}+\delta\rho^{b,\alpha\alpha_s}_{\mathrm{rem}}$	`ROFACR`	
	`AKFACR`	
$\mathrm{Re}[\rho_Z^b]$	`ROFACI`	
$\mathrm{Re}[\kappa_Z^b]$	`AKFACI=AKFACL+AKFACR`	

Table 10: Variables used in `ZWRATE()`, where the column "check" shows the results of numerical comparisons between ZFITTER and our codes.

	codes	check
s_W^2	`SW2M`	
corrected $\mathrm{Re}[\rho_Z^b]$	`ROFABI`	
corrected $\mathrm{Re}[\kappa_Z^b]$	`AKFABI`	
$\mathrm{Re}[\rho_Z^f]$	`ROFACI`	
$\mathrm{Im}[\rho_Z^f]$	`AR1IM`	
$\mathrm{Re}[\kappa_Z^f]$	`AKFACI`	
$\mathrm{Im}[\kappa_Z^f]$	`AK1IM`	
$\mathrm{Re}[g_V^f/g_A^f]$	`1-4*AKFACI*SW2M*`$\|Q_f\|$	
$\mathrm{Im}[g_V^f/g_A^f]$	`-4*AK1IM*SW2M*`$\|Q_f\|$	
$\sin^2\theta_{\mathrm{eff}}^f = \mathrm{Re}[\kappa_Z^f]s_W^2$	`AKFACI*SW2M`	
$R_V^f(M_Z^2)$	`RQCDV`	
$R_A^f(M_Z^2)$	`RQCDA`	
Γ_f	`GAM1I*`N_c	
Γ_b	`BAM1I*`N_c	
Γ_h	`GAM1H`	
Γ_Z	`GAM1T`	

Table 11: Common block `CDZRKZ`.

	codes	check
$\mathrm{Re}[\rho_Z^f]$	`AROTFZ[0-11]`	
$\mathrm{Im}[\rho_Z^f]$	`AIROFZ[0-11]`	
$\mathrm{Re}[\kappa_Z^f]$	`ARKAFZ[0-11]`	
$\mathrm{Im}[\kappa_Z^f]$	`AIKAFZ[0-11]`	
$\mathrm{Re}[g_V^f/g_A^f]$	`ARVEFZ[0-11]`	
$\mathrm{Im}[g_V^f/g_A^f]$	`AIVEFZ[0-11]`	
$\sin^2\theta_{\mathrm{eff}}^f = \mathrm{Re}[\kappa_Z^f]s_W^2$	`ARSEFZ[0-11]`	

F.2 Gfitter

See Ref. [91] and their web page [96] for detail.

F.3 FeynHiggs

FeynHiggs (v2.8.0 as of May 12, 2011; available online at [97]) [98, 99, 100, 101] is a FORTRAN code, which computes the masses and couplings of the Higgs sector in the MSSM at the two-loop level, including one-loop non-minimal-flavour-violation (NMFV) effects and the effects of the complex phases. The self-energy corrections of the Higgs bosons, $\hat{\Sigma}_{hh,hH,Hh,HH,hA,Ah,HA,AH,AA,H^+H^-}$, are evaluated with the following corrections:

- full one-loop corrections in the MSSM with complex parameters [101],
- leading $O(\alpha_t\alpha_s)$ two-loop corrections in the MSSM with complex parameters [102], and
- leading NMFV one-loop corrections in the MSSM with real parameters [84].

In addition, the following contributions are also included in the self-energies $\hat{\Sigma}_{hh,hH,Hh,HH}$:

- $O(\alpha_t^2)$ and $O(\alpha_b\alpha_s, \alpha_t\alpha_b, \alpha_b^2)$ two-loop corrections in the MSSM with real parameters [103, 104, 105].

The bottom Yukawa coupling Y_b receives potentially large corrections for large $\tan\beta$, which requires a resummation of the $O(\alpha_s^n \tan^n\beta)$ and $O(\alpha_t^n \tan^n\beta)$ terms. FeynHiggs uses the Δ_b's of Refs. [104] and [106], which have to be used with care to be consistent with the two-loop corrections.

FeynHiggs also provides the production cross sections and the total and partial decay widths of the Higgs bosons as well as some of EW-precision and flavour observables, such as

- $\Delta\rho^{\rm MSSM}$ with $O(\alpha\alpha_s)$ two-loop corrections [80, 81], including one-loop NMFV effects [84],
- M_W and $\sin^2\theta_{\rm eff}$ via SM two-loop formulae [49, 65, 66, 64] in Secs. D.1 and D.2 and $\Delta\rho^{\rm MSSM}$, where Δ_α in Eqs. (332) and (337) are set to be zero,
- $B(b\to s\gamma)$ at one-loop level (no mass insertion), including NMFV effects [107],
- ΔM_s at one-loop level (no mass insertion), including NMFV effects [107],
- $B(B_s\to\mu^+\mu^-)$ (dummy for v2.8.0; part of future extensions),
- $(g_\mu - 2)$ with some two-loop MSSM corrections,

  * two-loop leading QED logs in hep-ph/9803384,

  * two-loop corrections in hep-ph/0312264,
- the EDMs of the electron, neutron and mercury,

  * one-loop gluino, neutralino and chargino contributions in hep-ph/97084563 (see Eqs. (12), (13), (19), (22), (27-29) and (36-39)),

  * leading two-loop contributions in hep-ph/0311314 and hep-ph/9811202.

In FeynHiggs, M_W and $\sin^2\theta_{\rm eff}$ are computed by simply adding the following shifts to their SM contributions:

$$\delta M_W^{\rm MSSM} \approx \left(\frac{M_W}{2}\frac{c_W^2}{c_W^2 - s_W^2}\right)_{\rm SM} \Delta\rho^{\rm MSSM}\,, \tag{351}$$

$$\delta \sin^2\theta_{\rm eff}^{\rm MSSM} \approx -\left(\frac{c_W^2 s_W^2}{c_W^2 - s_W^2}\right)_{\rm SM} \Delta\rho^{\rm MSSM}\,. \tag{352}$$

The input parameters of FeynHiggs, which can be read from a text file in the SLHA format, are as follows:

SM inputs:

$\alpha^{-1}(M_Z)$, $\alpha_s(M_Z)$, G_μ,
m_s(2 GeV), $m_c(m_c)$, $m_b(m_b)$, M_t, M_W, M_Z,
λ, A, $\bar\rho$, $\bar\eta$

MFV MSSM inputs:

scale factor = (RG scale)/M_t,
$\tan\beta$,
m_{A^0} or m_{H^+},
$m^2_{\tilde e_L}$, $m^2_{\tilde\mu_L}$, $m^2_{\tilde\tau_L}$, $m^2_{\tilde e_R}$, $m^2_{\tilde\mu_R}$, $m^2_{\tilde\tau_R}$, $m^2_{\tilde q_{1L}}$, $m^2_{\tilde q_{2L}}$, $m^2_{\tilde q_{3L}}$, $m^2_{\tilde u_R}$, $m^2_{\tilde c_R}$, $m^2_{\tilde t_R}$, $m^2_{\tilde d_R}$, $m^2_{\tilde s_R}$, $m^2_{\tilde b_R}$,
μ,
A_e, A_μ, A_τ, A_u, A_c, A_t, A_d, A_s, A_b,
M_1, M_2, M_3,
Q_τ, Q_b, Q_t (the scales at which the 3rd-generation sfermion masses are given)

NMFV MSSM inputs:

the off-diagonal entries of $m^2_{\tilde Q}$, $m^2_{\tilde u}$, $m^2_{\tilde d}$, A_U and A_D

The subroutines which are essential for our study are as follows:

Important subroutines:

- FHSetFlags() — sets 9 flags
- FHSetSMPara() and FHSetPara() — set the SM and MFV MSSM inputs, respectively, where M_t is set in the latter
- FHSetNMFV() — sets the off-diagonal entries of the $m^2_{\tilde Q}$, $m^2_{\tilde U}$, $m^2_{\tilde D}$, A_U and A_D
- FHSetSLHA() — sets the parameters from SLHA data, used with SLHARead()
- FHHiggsCorr() — computes the Higgs masses
- FHCouplings() — computes the couplings, decay widths and branching ratios
- FHHiggsProd() — computes the production cross sections
- FHConstraints() – computes the EW observables

- FHFlavour() – computes the flavour observables

The subroutines FHSetFlags and FHSetSMPara/FHSetPara/FHSetNMFV (or FHSetSLHA) must be called before the other routines.

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