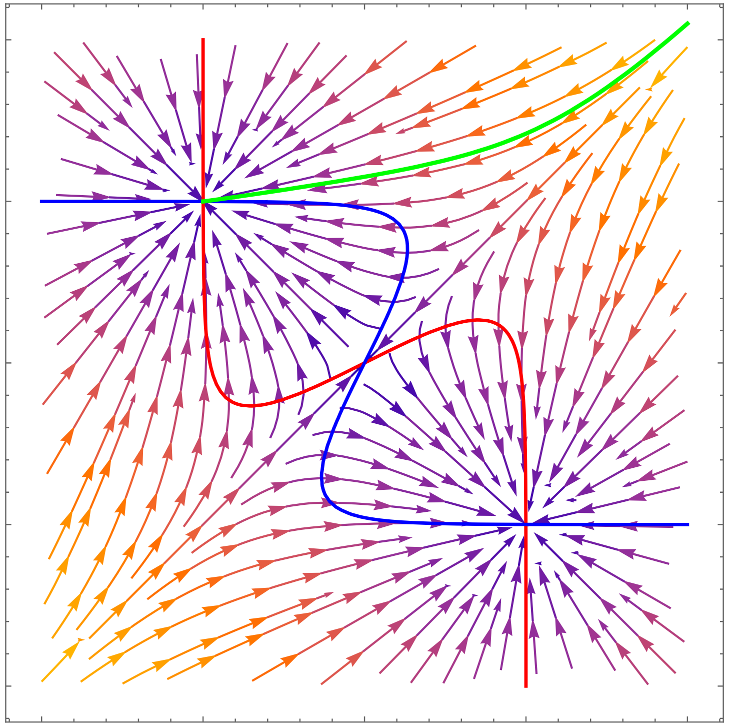
# Modelling decision-making with dynamical systems

**Konstanz summer school of collective behaviour – KSCB 2025**

**Marco Fele**



Adapted from notes and work by Alessio Franci and Anastasia Bizyaeva – all mistakes mine

# Topic summary

1. **Why dynamical systems are bad for modelling decision-making**

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2. **Introduce a baseline model of decision-making**

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5. Very brief introduction to symmetries in dynamical systems and their importance for decision-making

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3. Using bifurcations to describe decision-making
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5. Very brief introduction to symmetries in dynamical systems and their importance for decision-making
6. Multi-agent decision-making and the ring attractor

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2. **Introduce a baseline model of decision-making**
3. **Using bifurcations to describe decision-making**
4. **Separation of timescales**
5. **Very brief introduction to symmetries in dynamical systems and their importance for decision-making**
6. **Multi-agent decision-making and the ring attractor**

Focus on conceptual understanding rather than implementation
Answer questions
Code to replicate figures in GitHub repository

# What are dynamical systems

**We want to describe ant trails**

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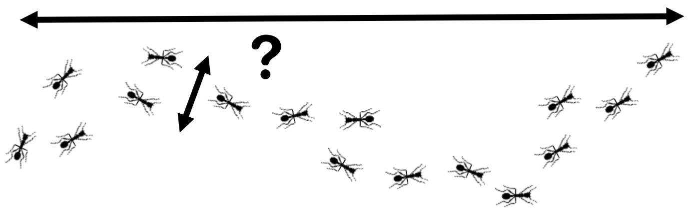
**Microscopic level**

# What are dynamical systems

We want to describe ant trails

Microscopic level

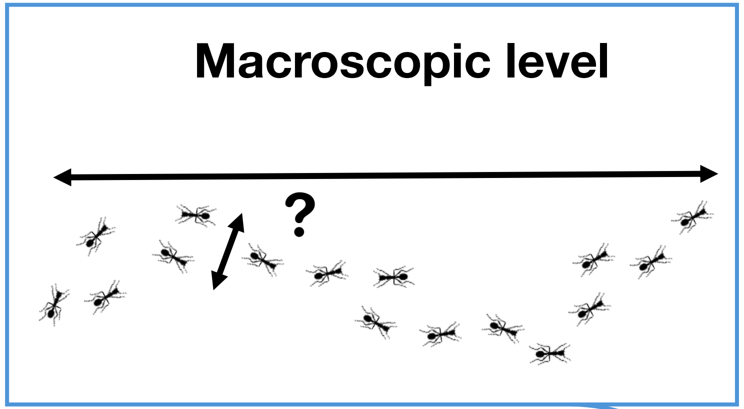

Macroscopic level



# What are dynamical systems

Dynamical system describe
the macroscopic level

Microscopic level


$$\dot{x} = f(x, p)$$

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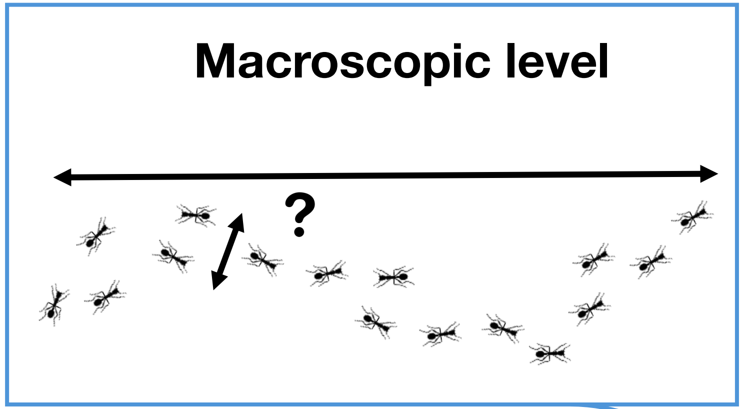
**Microscopic level**


**Macroscopic level**

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**State variables** $x \in \mathbb{R}^n$ **Parameters** $p \in \mathbb{R}^k$

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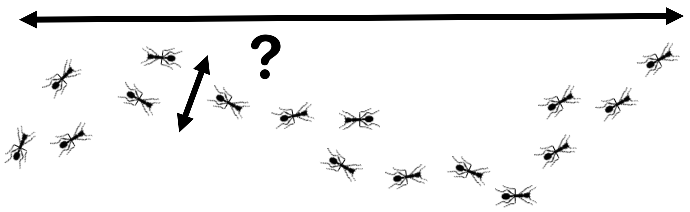


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***Can change***

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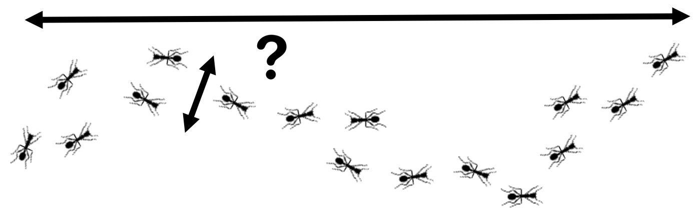


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**Important assumptions:**

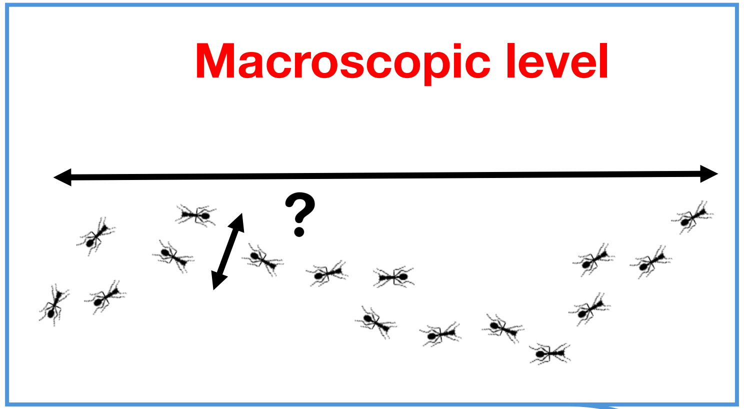
- **Continuous state variables**
- **Time evolution of state variables is deterministic**

# What are dynamical systems

**Problems**

**Microscopic level**


2.

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- **Time evolution of state variables is deterministic**

3.

# Part 1
# The problems with dynamical systems

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |

# 1. The problem with continuous state variables

**A colony of *Temnothorax* ants must choose a new nest**

A B

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**Tandem running**

Basari et al. 2014

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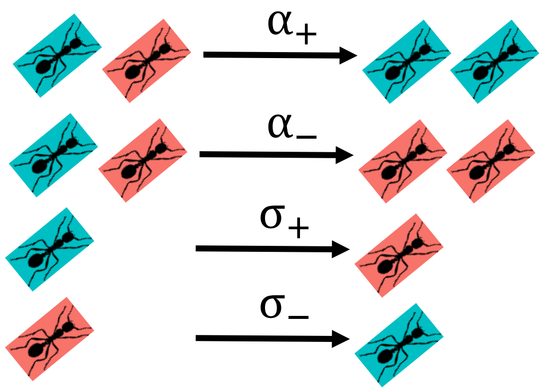
Voter model




Marshall et al. 2009

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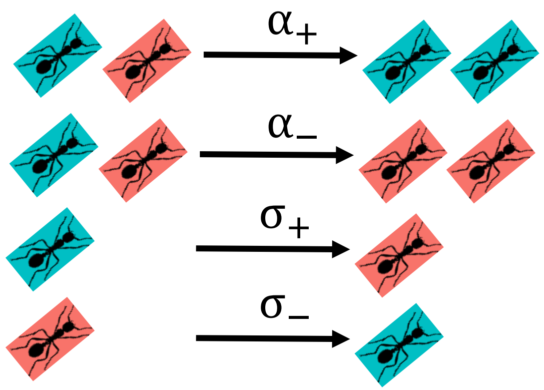
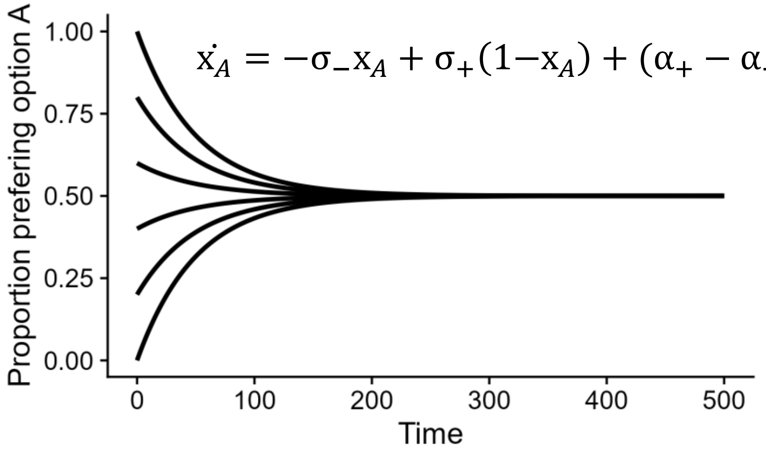



**Assuming infinite population size**




$$\dot{x}_A = -\sigma_- x_A + \sigma_+(1-x_A) + (\alpha_+ - \alpha_-)x_A(1-x_A)$$

Tomorrow - how to derive dynamical systems from microscopic rules

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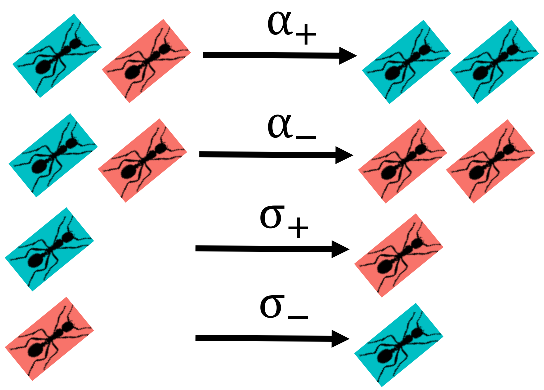
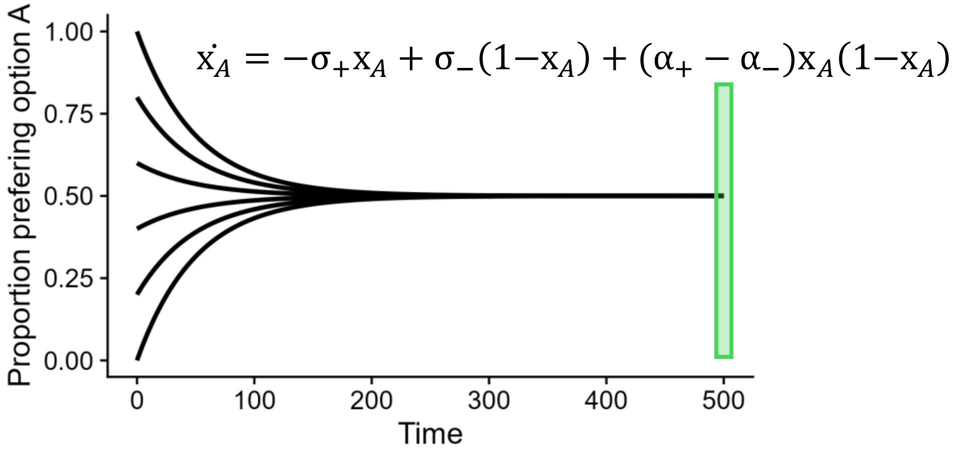


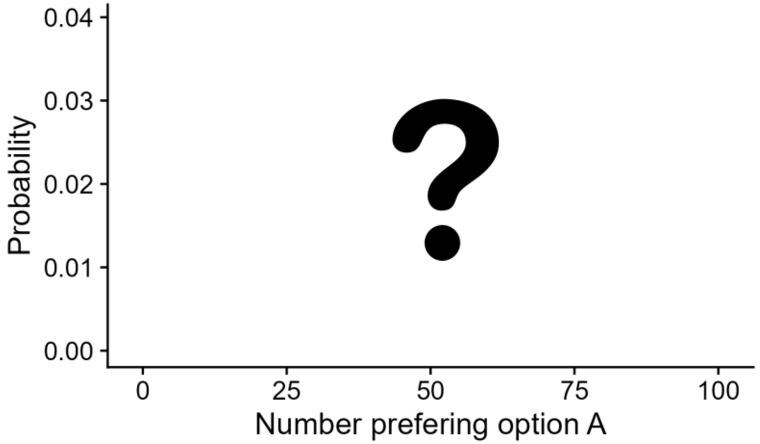




**At the end of the task, with 100 ants, what is the shape of the probability mass for the number of ants preferring option A:**

**a) Flat**

**b) Approximately gaussian (centered at 50)**

**c) Bimodal**



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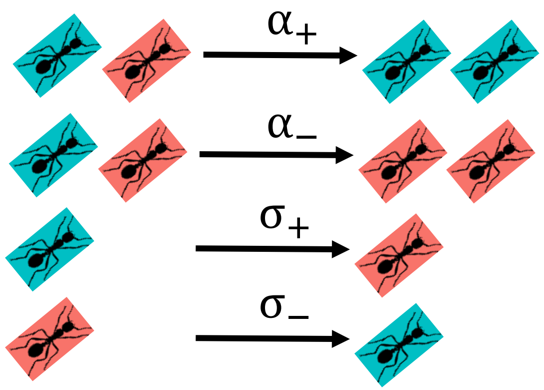


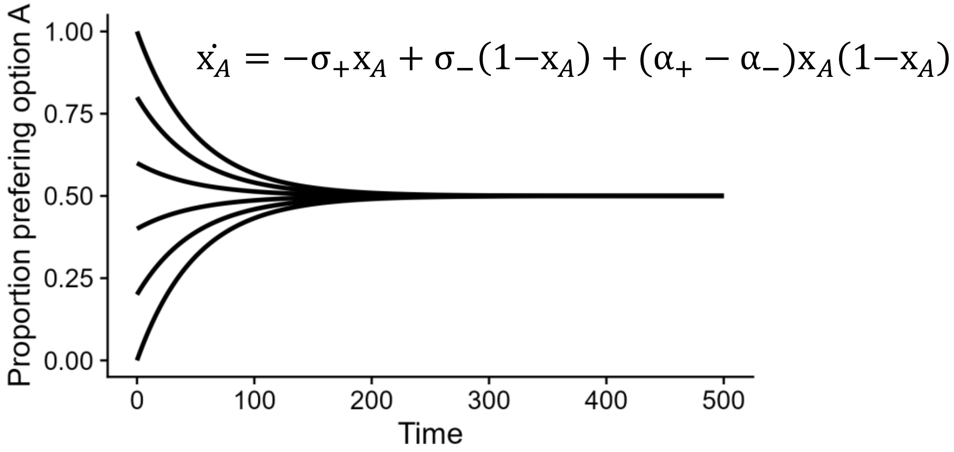


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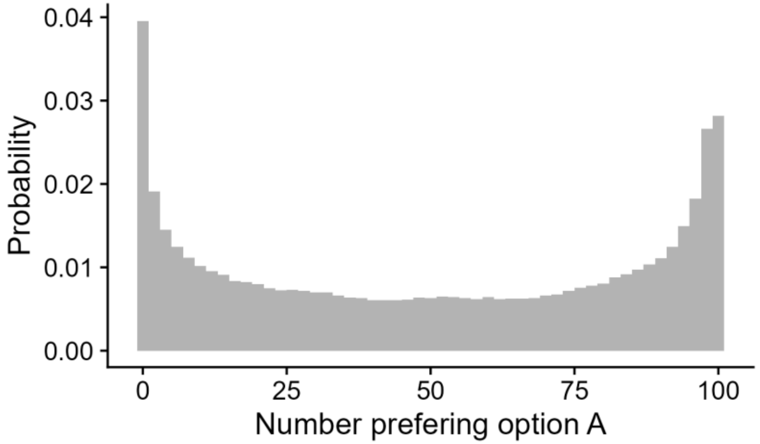

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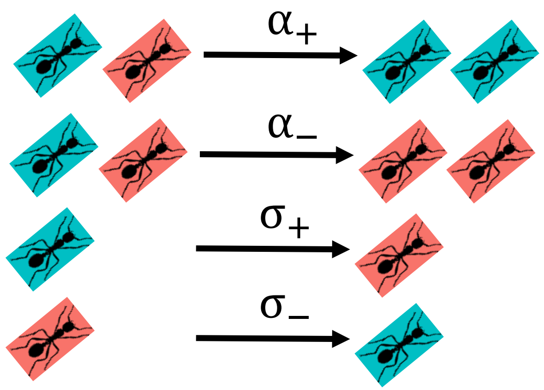
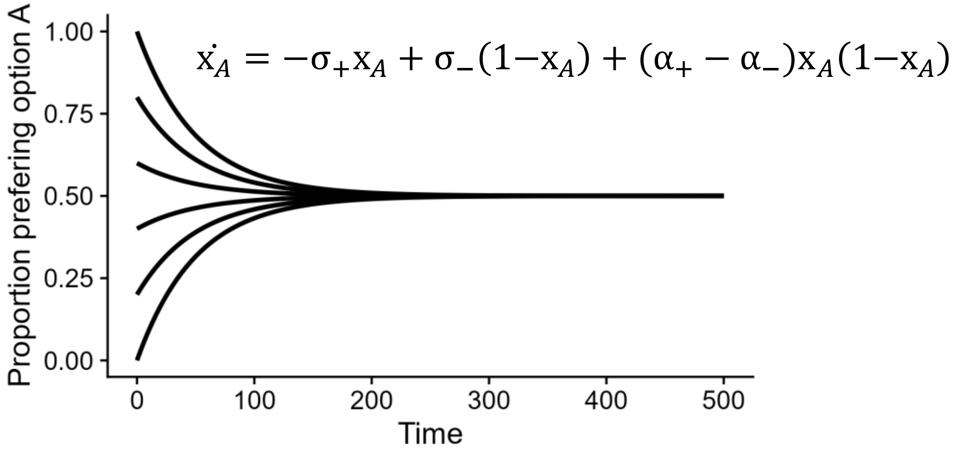
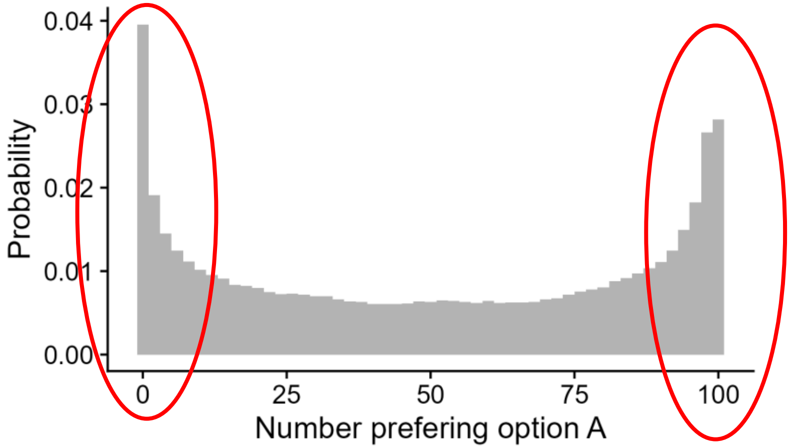



**Very relevant for collective decision-making: the colony mostly agrees!**



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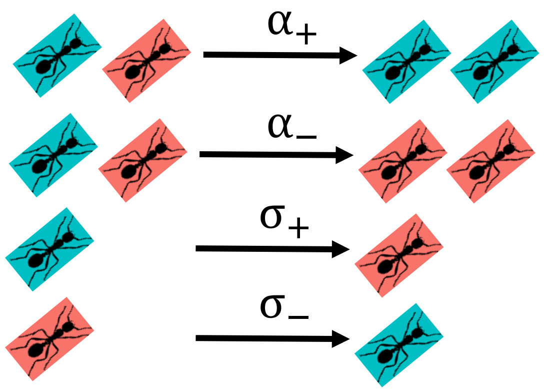
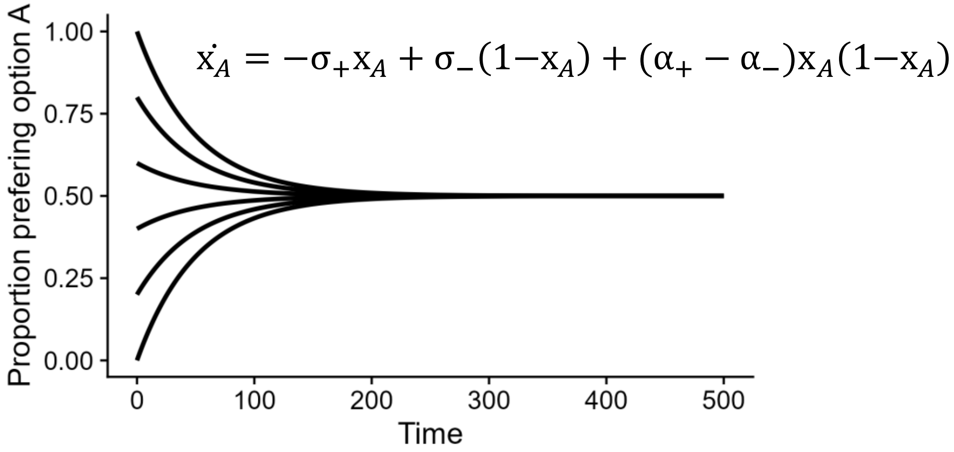
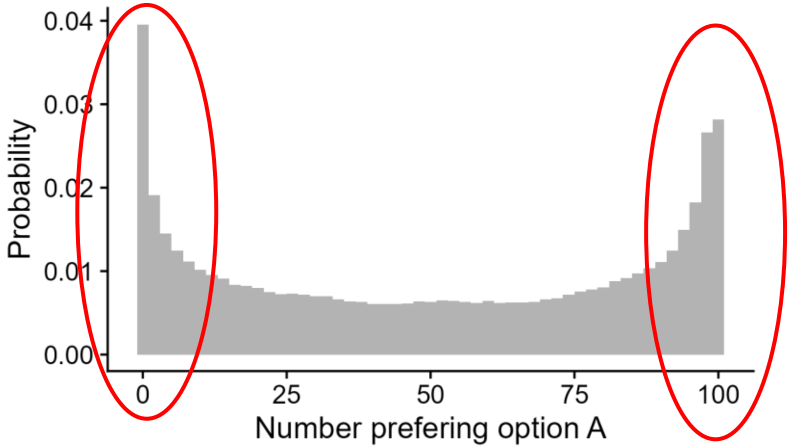


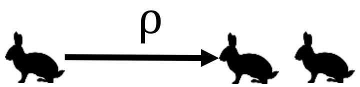

**Macroscopic models can miss dynamical outcomes**
**Noise induced effects**



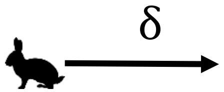
# 2. The problem with macroscopic descriptions

**Logistic growth model**

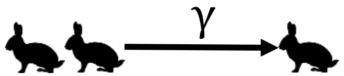
Clonal reproduction




Death




Density dependent death




$r$: population growth rate
$K$: carrying capacity

$$\dot{x} = r(1 - \frac{x}{K})$$

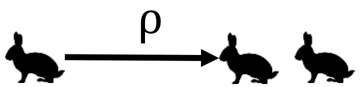
# 2. The problem with macroscopic descriptions

**By increasing the rate of clonal reproduction, the population will increase:**

a) **Faster**
b) **Same speed**
c) **Slower**

**Logistic growth model**

Clonal reproduction

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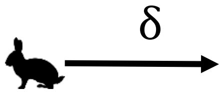


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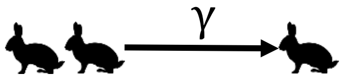


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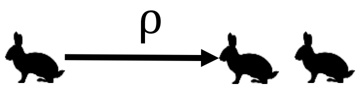


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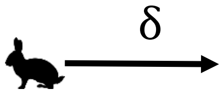


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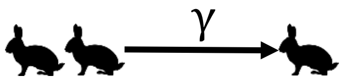


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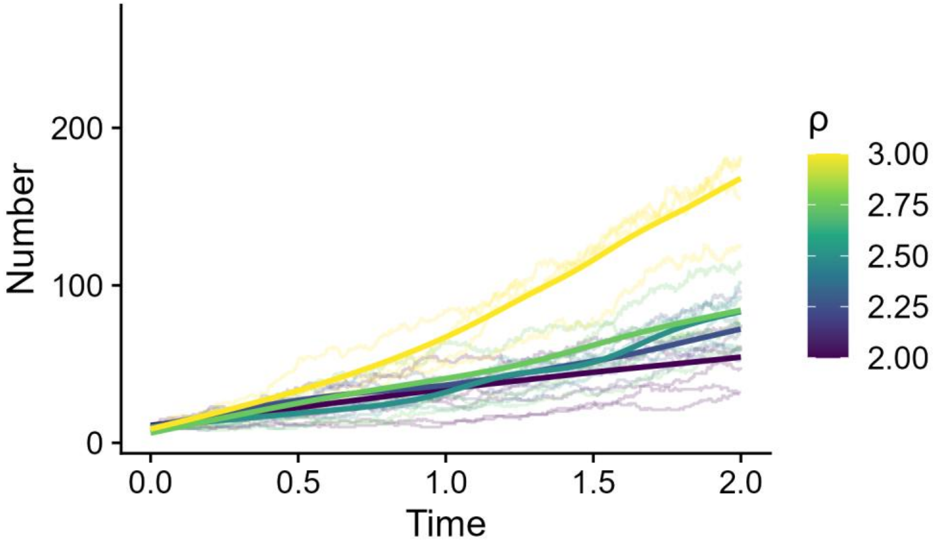

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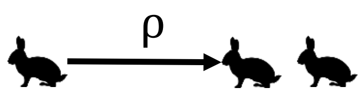


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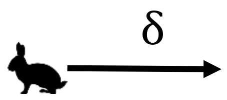


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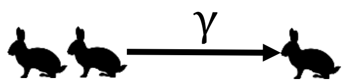


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$$\dot{x} = r(1 - \frac{x}{K})$$

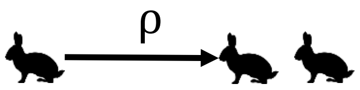
**By increasing the rate of clonal reproduction, the population will saturate:**

**a) At a lower number**
**b) At the same number**
**c) At a higher number**

# 2. The problem with macroscopic descriptions

### Logistic growth model

Clonal reproduction

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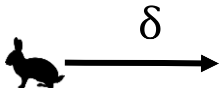


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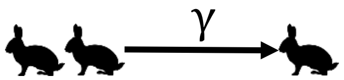


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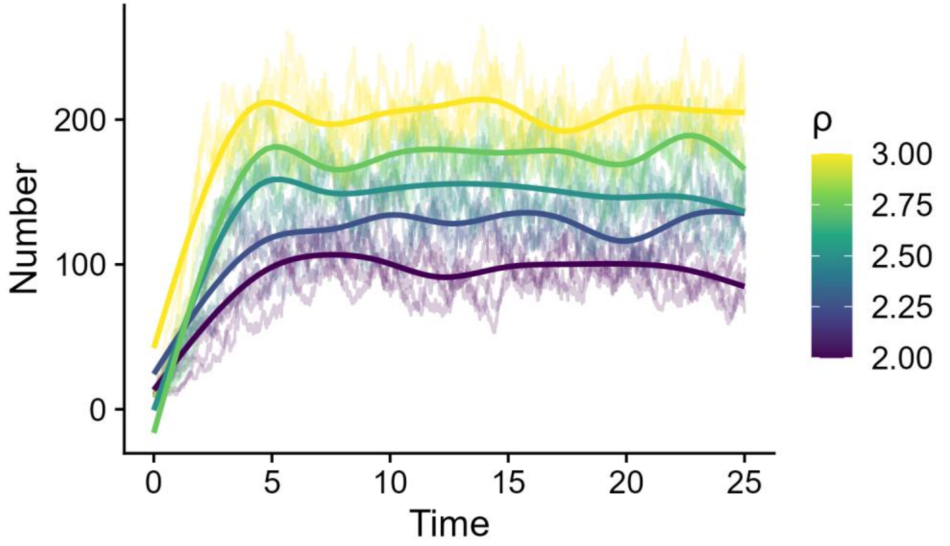

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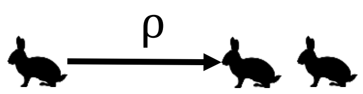


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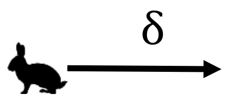


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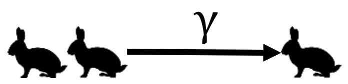


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$K$: carrying capacity

$$\dot{x} = r(1 - \frac{x}{K})$$

$$K = \frac{\gamma}{\rho - \delta}$$

$$r = \rho - \delta$$

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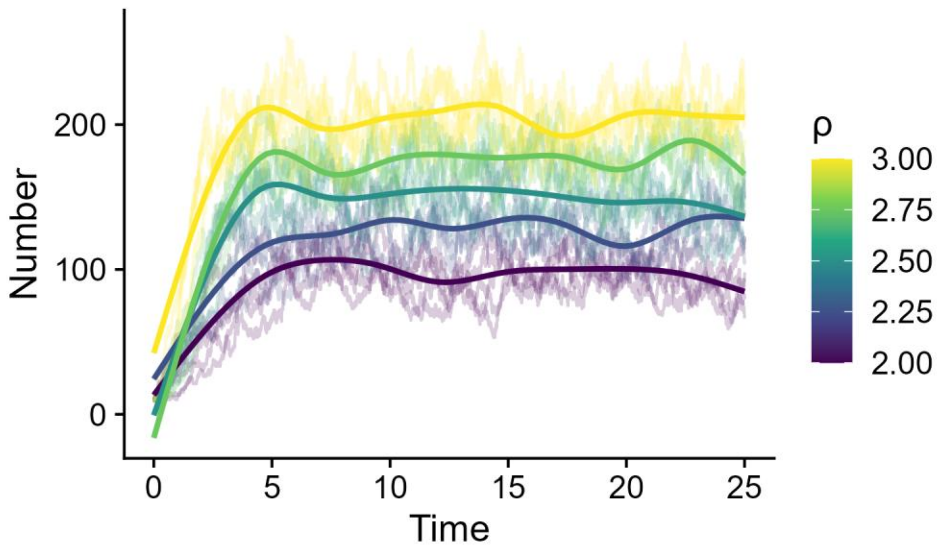


**Population growth rate and carrying capacity are NOT independent**

# 3. The problem with deterministic evolution of state variables

**Everything that is stochastic will be missed**

**Accuracy (probability of choosing best option) is stochastic!**

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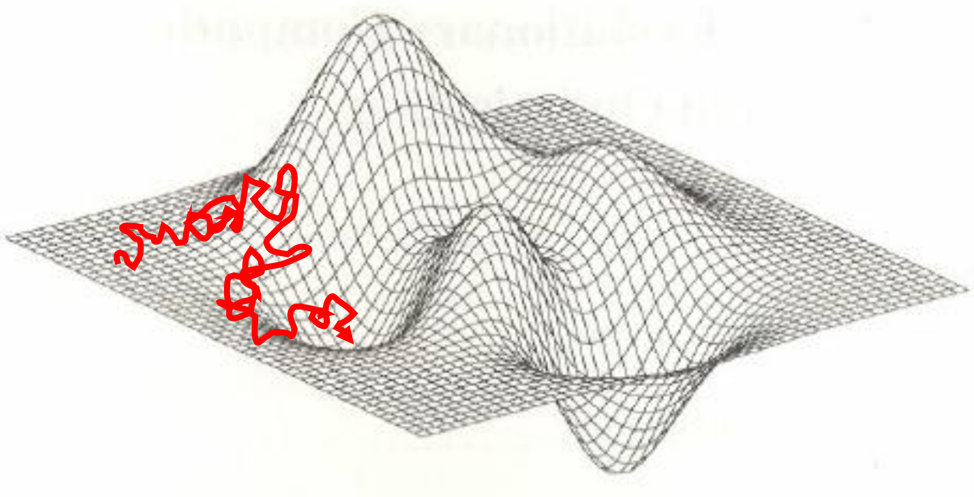


***Dynamical systems "describe the landscape" over which decisions "move" …***

# Part 2
# A baseline model of decision-making

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |

# Non-linear opinion dynamics model (NOD)

Decision variable

Input or bias

$$\dot{x} = -dx + \tanh(ux + b)$$

Leak

Attention

**One agent, two options**

# Non-linear opinion dynamics model (NOD)

Decision variable

Input or bias

$$\dot{x} = -dx + \tanh(ux + b)$$

Leak

Attention

**With $d>0$, does $x$ go to infinity:**
**a) Never**
**b) Depends on parameters**
**c) Always**

# Non-linear opinion dynamics model (NOD)

Linear Constant (for big |x|)

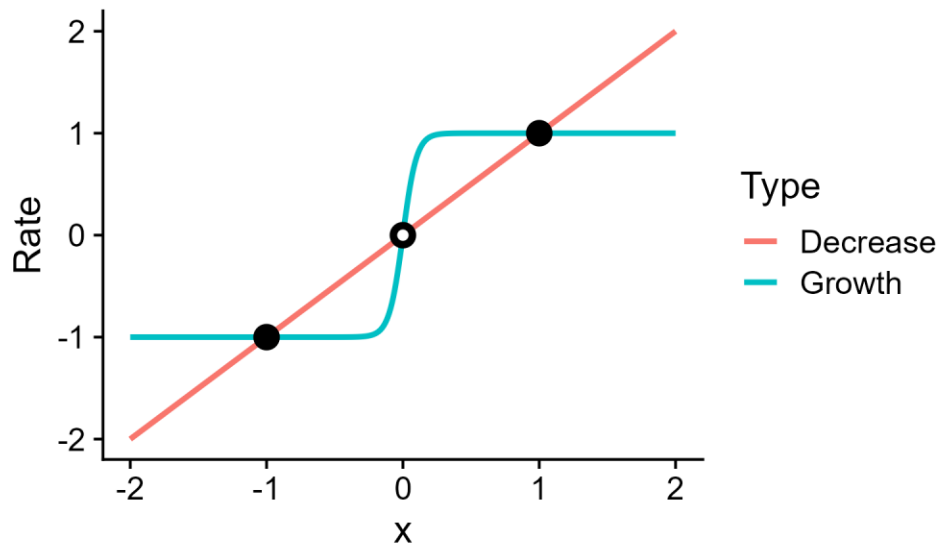
$$\dot{x} = \boxed{-dx} \boxed{+ \tanh(ux + b)}$$

With $d>0$, does $x$ go to infinity:

a) Never

b) Depends on parameters

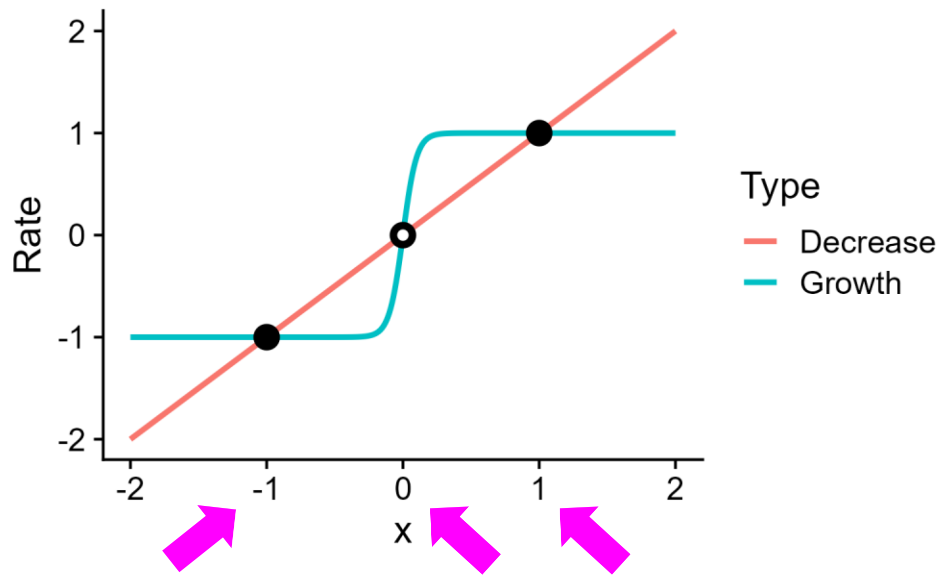
c) Always



# Non-linear opinion dynamics model (NOD)

$$\dot{x} = -dx + \tanh(ux + b)$$

**Equilibria** $\rightarrow f(\boldsymbol{x}^*, \boldsymbol{p}) = 0$



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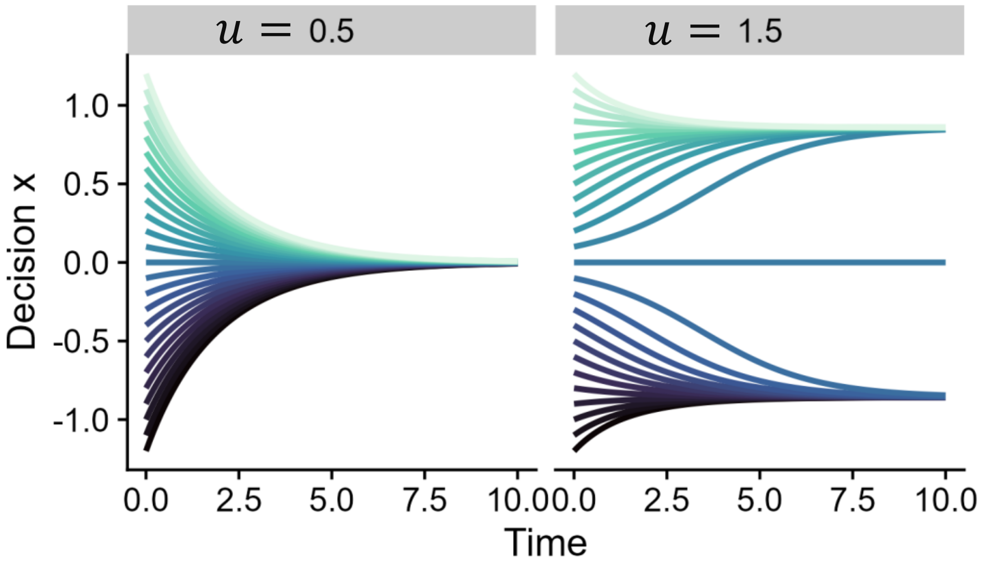
$$\dot{x} = -dx + \tanh(ux + b)$$

**Equilibria** $\rightarrow f(\boldsymbol{x}^*, \boldsymbol{p}) = 0$

**Qualitative analysis: explicit closed form solution for $\boldsymbol{x}^*$**

**Quantitative analysis: explicit closed form solution as function of time $\mathbf{x}(t) = g(\boldsymbol{p}, t)$**

**Time series**



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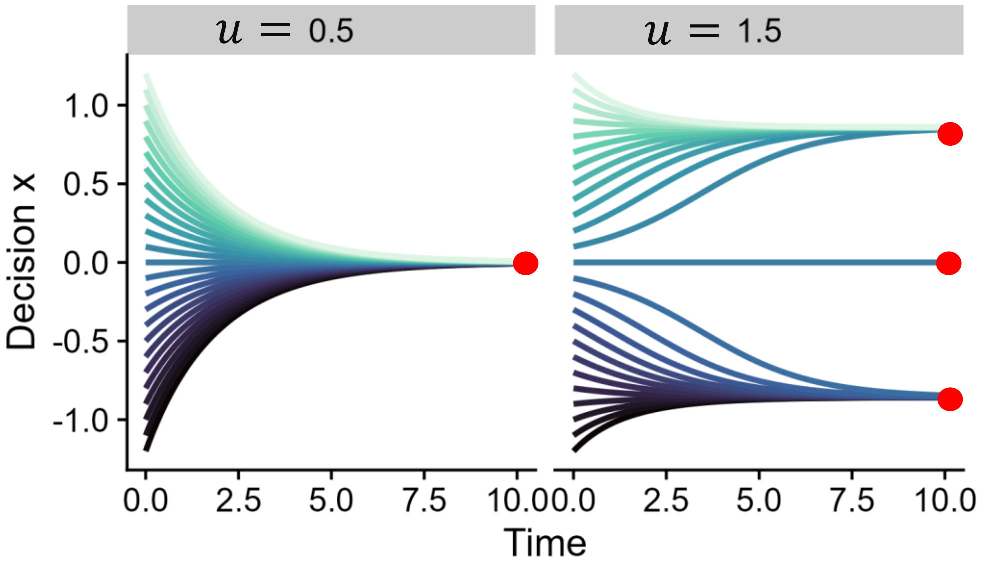
**Qualitative analysis: explicit closed form solution for $x^*$**

**Quantitative analysis: explicit closed form solution as function of time $\mathbf{x}(t) = g(\boldsymbol{p}, t)$**

**"Less information, more *informative*"**

**Time series**



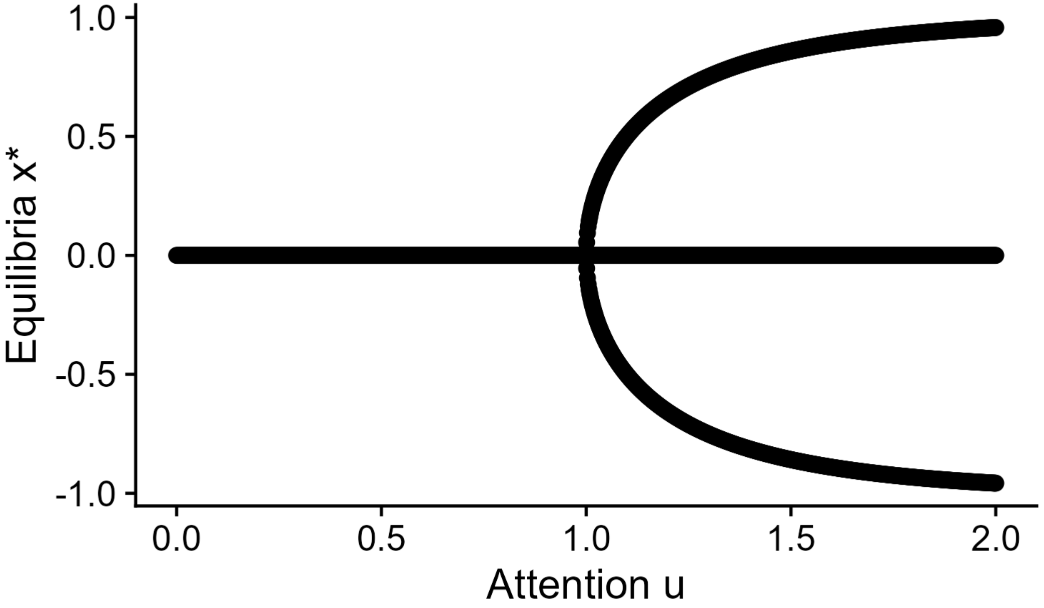
# Part 3
# Bifurcations

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |

# Bifurcation diagrams

**Equilibria instead of state variable**

$$\dot{x} = -x + \mathrm{Tanh}(ux)$$




**Parameter instead of time**

**Equilibrium branch**

# Bifurcation diagrams

**Equilibria instead of state variable**

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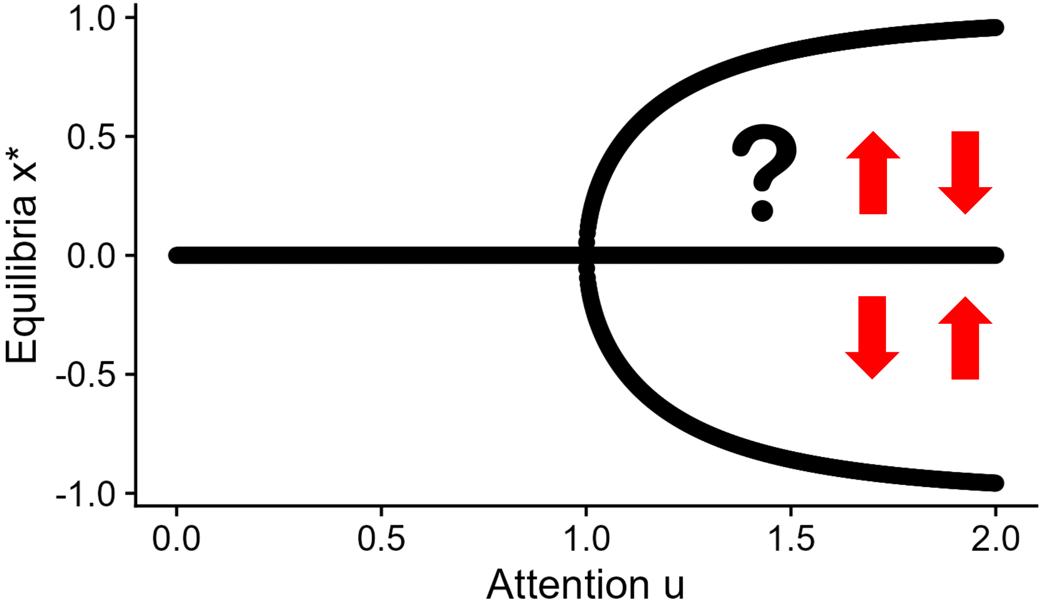


**Parameter instead of time**

**Stability of equilibria**

# Bifurcation diagrams

**Stability criterion:**

1. **derive the system in respect to the state variables (Jacobian matrix)**
2. **evaluate at the equilibrium**
3. **find max eigenvalue(s)** $\lambda_{max}$
4. **If** $\lambda_{max} < 0$ **stable,** $\lambda_{max} > 0$ **unstable**

$$J(\boldsymbol{x}^*, \boldsymbol{p}) = \left.\begin{pmatrix} \frac{\partial f_1(\boldsymbol{x}, \boldsymbol{p})}{\partial x_1} & \cdots & \frac{\partial f_1(\boldsymbol{x}, \boldsymbol{p})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\boldsymbol{x}, \boldsymbol{p})}{\partial x_1} & \cdots & \frac{\partial f_n(\boldsymbol{x}, \boldsymbol{p})}{\partial x_2} \end{pmatrix}\right|_{\boldsymbol{x}=\boldsymbol{x}^*}$$

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$$f(x,p) = -x + \tanh(10x)$$

$$f(x^*, p) = 0$$

$$J(x,p)$$

$$J(x,p)_{\,\boldsymbol{x}=\boldsymbol{x}^*}$$

**Calculate** $J(x,p)$

# Bifurcation diagrams

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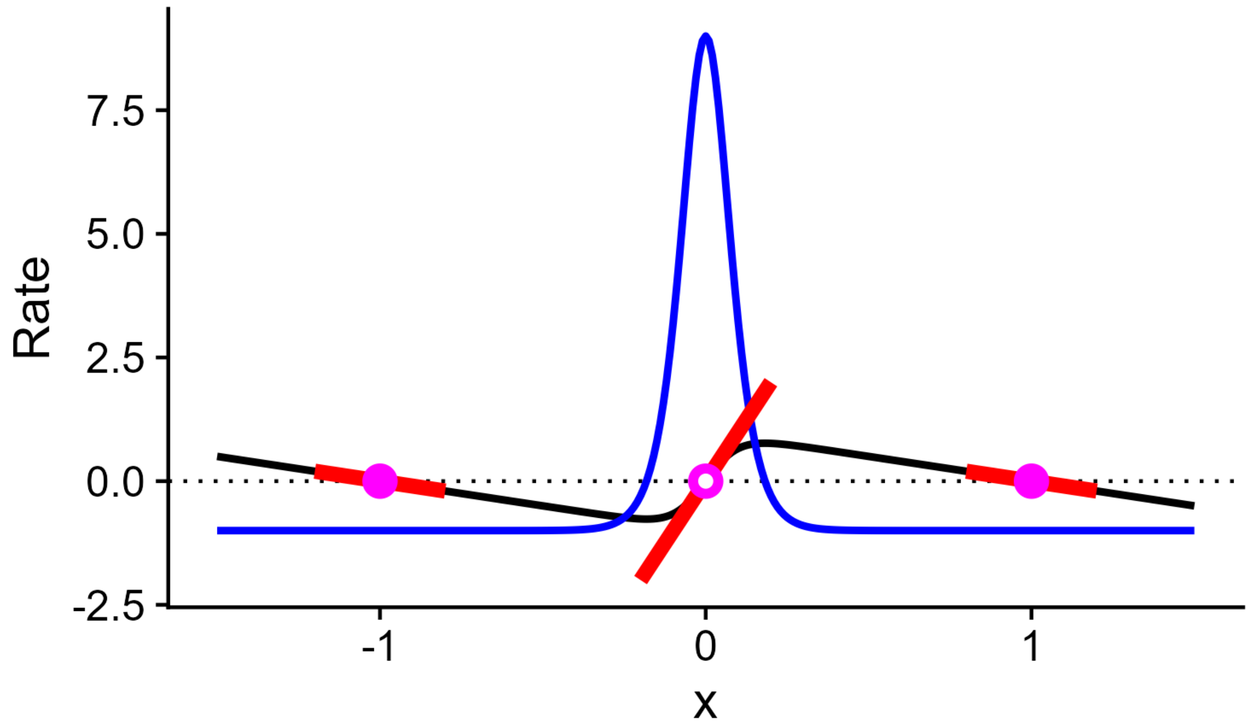
$J(x,p) = -1 + 10\mathrm{sech}(10x)^2$

$J(x,p)_{x=x^*}$



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**Why?**

$f(x,p) = -x + \tanh(10x)$

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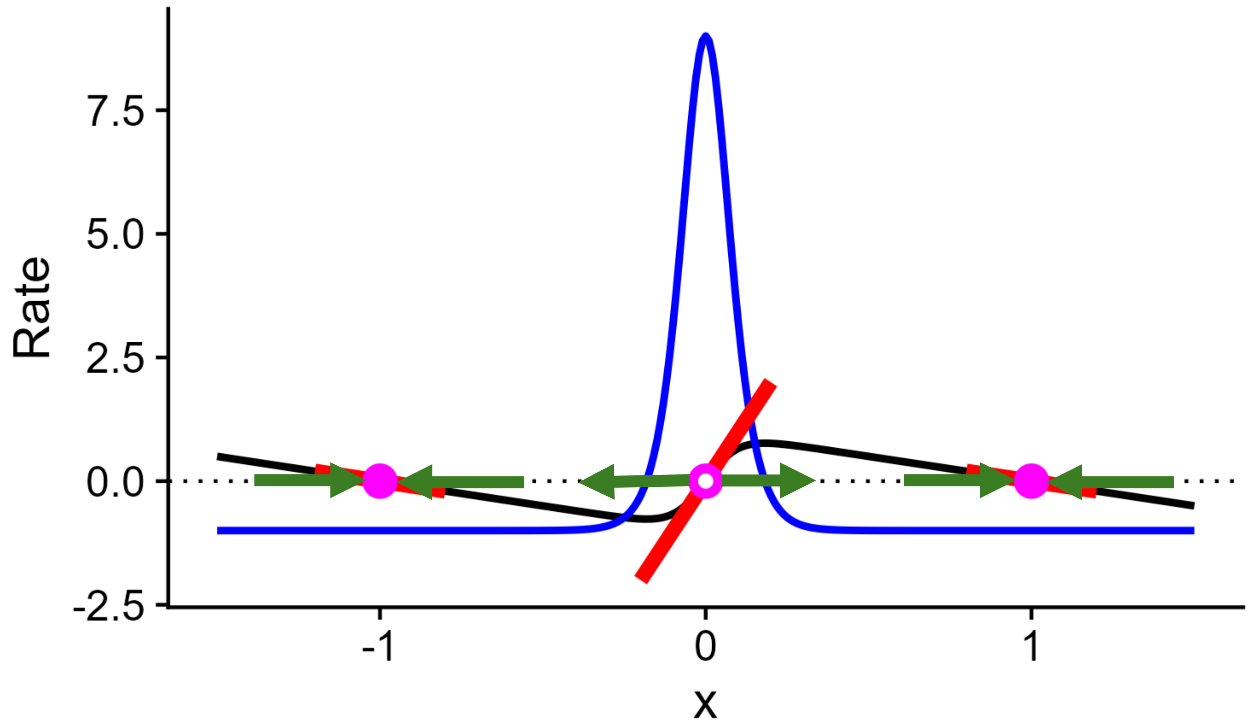

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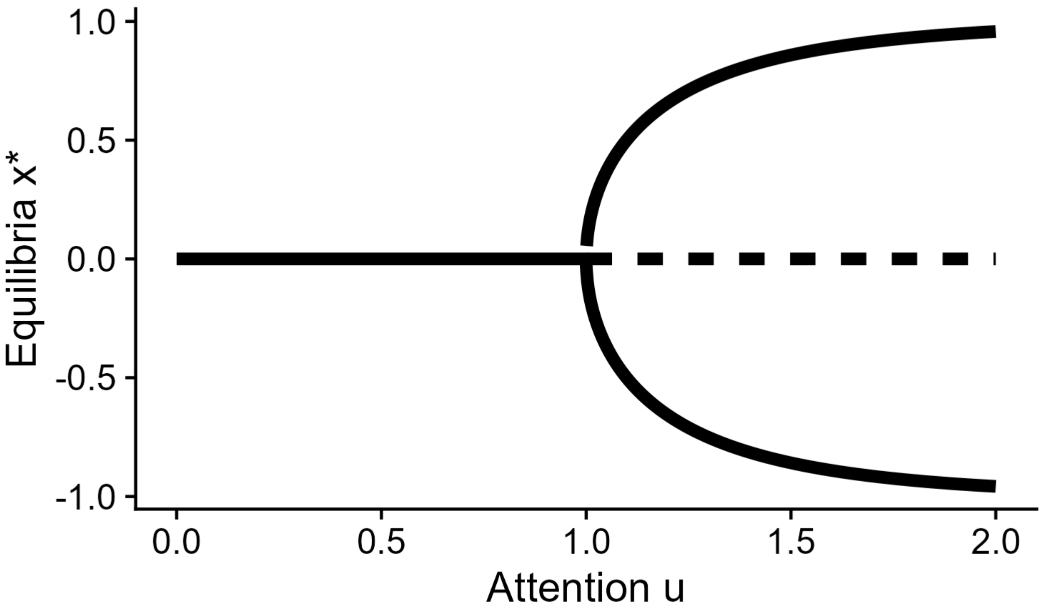
**Caution**

This criteria can fail when at least on eigenvalue is 0. Other tools can be used to define stability.
See end if curious

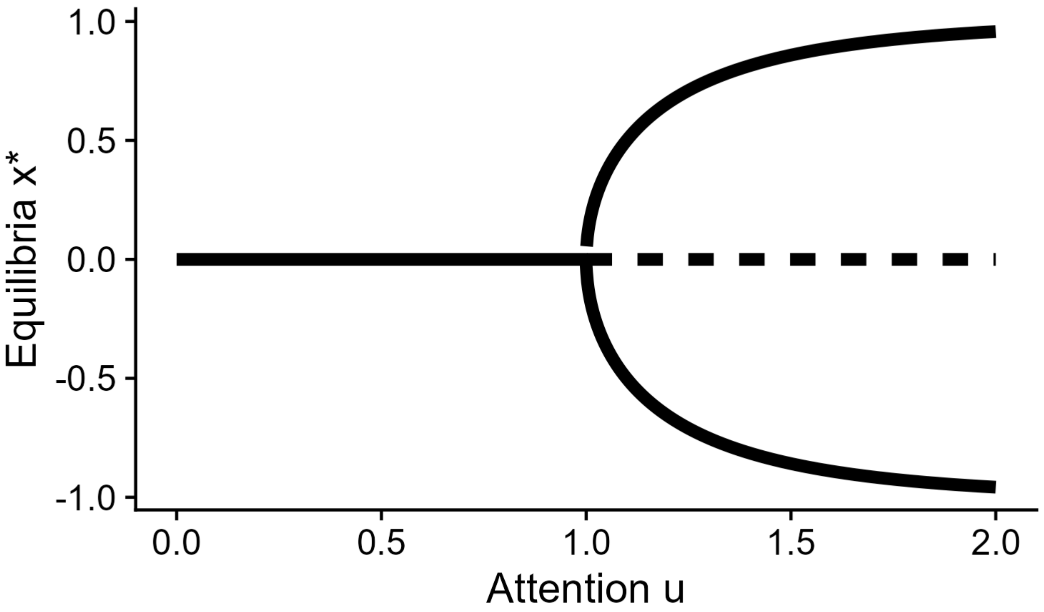
**Hyperbolic equilibria: linearization succeeds**
**Singular/critical points: linearization fails**

# Bifurcation diagrams



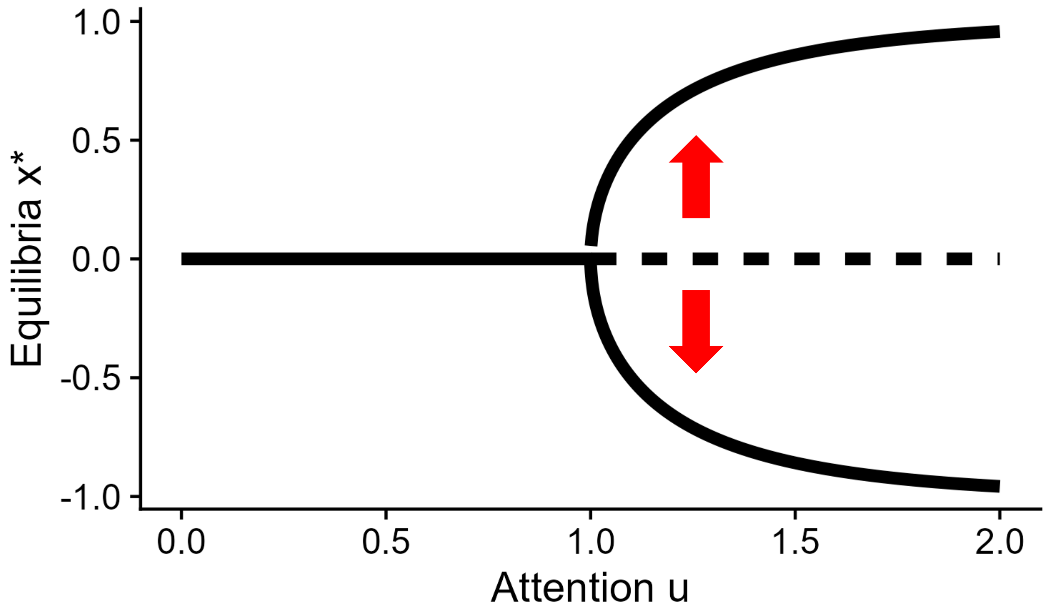
# Bifurcation diagrams

**What does this tell us about decision-making?**



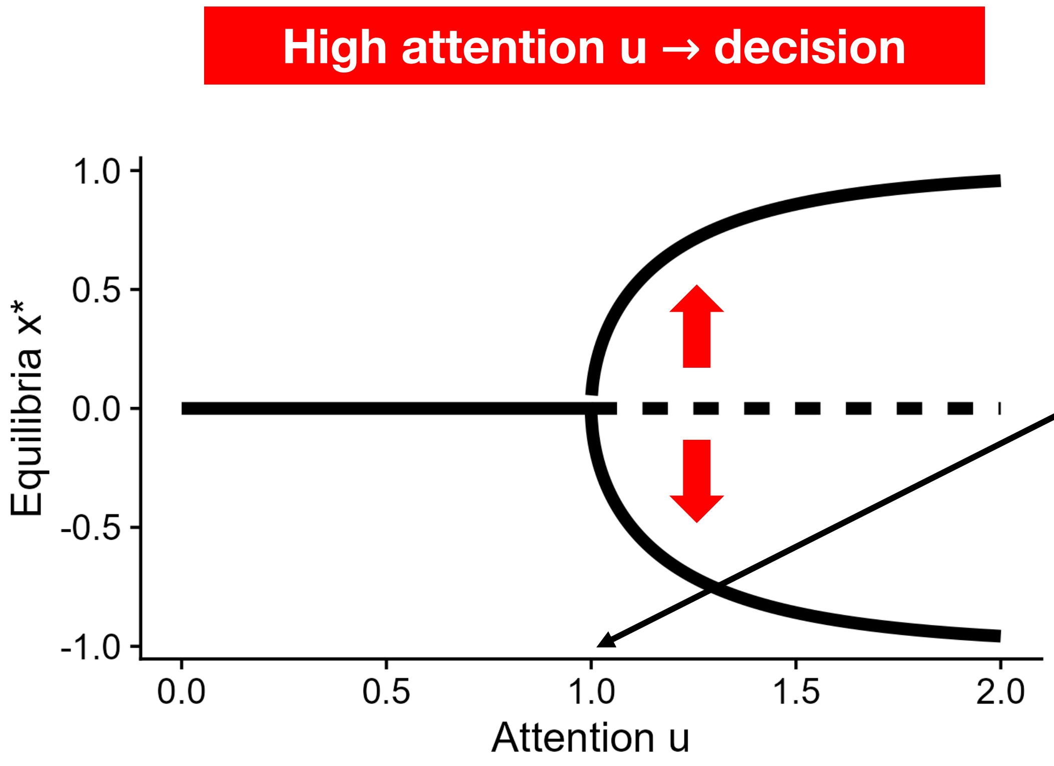
# Bifurcation diagrams

**High attention u → decision**




**In this case, multi-stability is associated with making decisions**

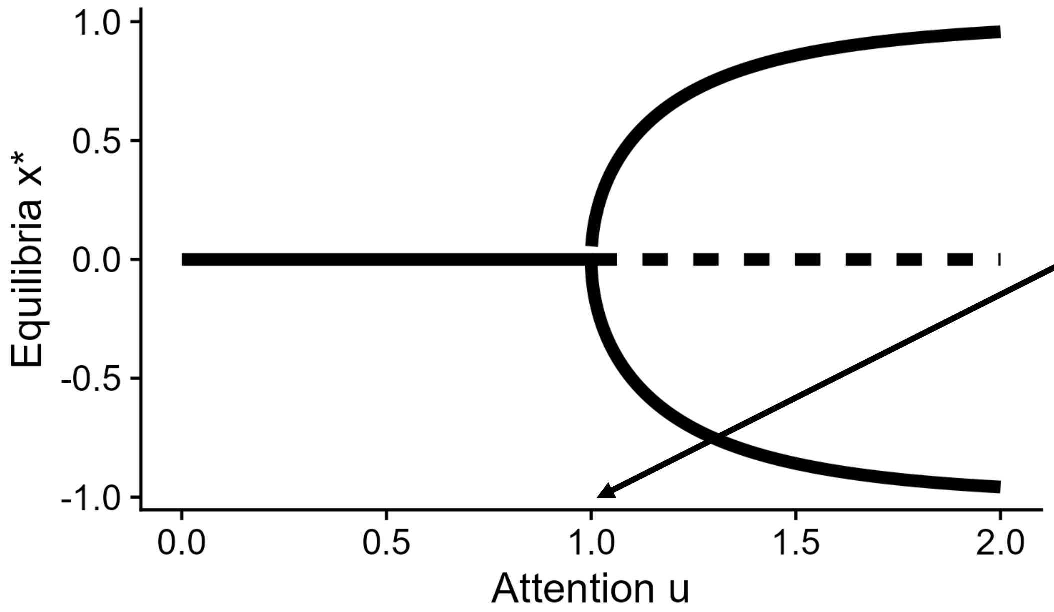
# Bifurcation diagrams




**Critical point $u^*$ $(Re(\lambda) = 0)$ divides different "regimes" (number/stability of equilibria) of decision-making**

**Could it be otherwise?**

# Bifurcation diagrams

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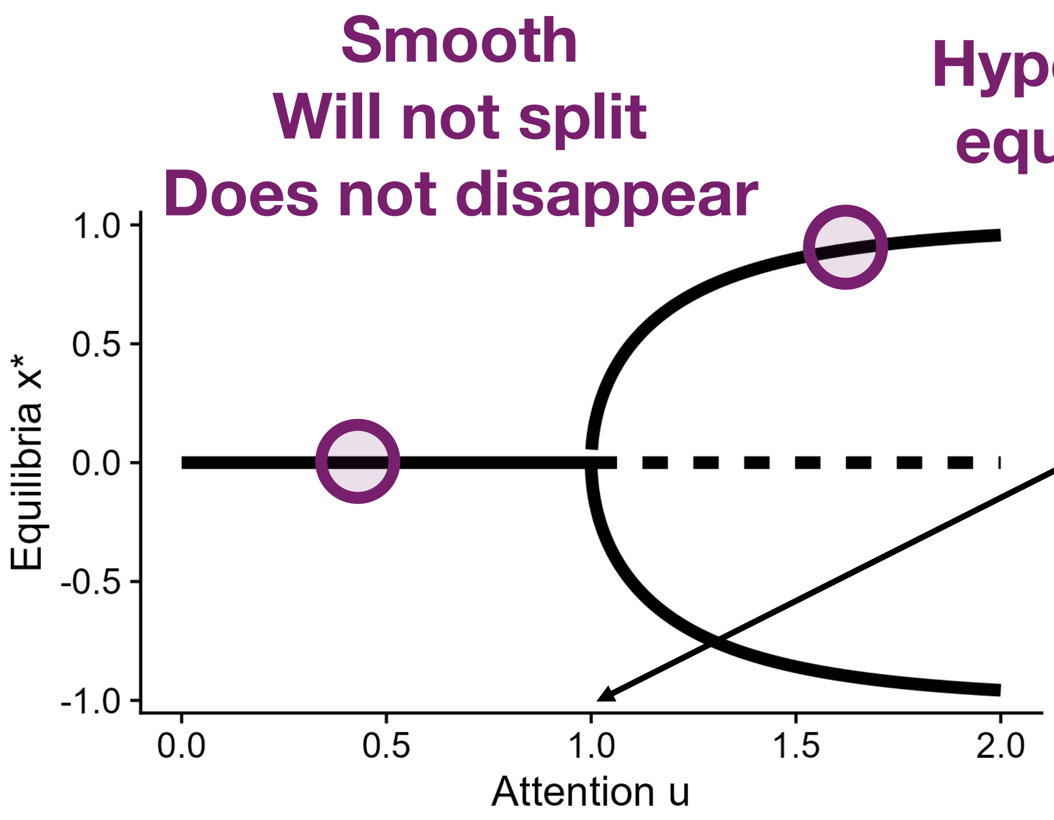
**Could it be otherwise?**

**No***

Implicit function theorem: if $f(\boldsymbol{x}^*, \boldsymbol{p}) = 0$ and $\boldsymbol{x}^*$ is not singular ($\boldsymbol{\lambda} \neq 0$), then locally there is a differentiable function $\chi(\boldsymbol{p}) = \boldsymbol{x}^*$

*Only considering *local bifurcations*; in *global bifurcations* (ex: homo/heteroclinic) the vector field changes without singular points

# Bifurcation diagrams

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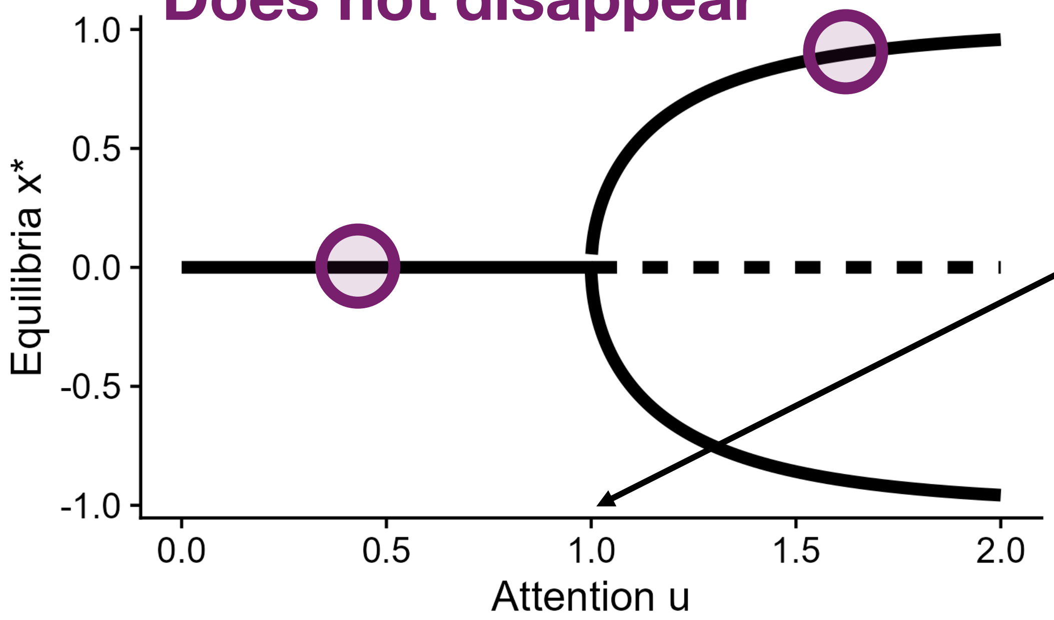
(NB bifurcations can occur even for non singular Jacobians, ex Hopf, but $Re(\lambda) = 0$ still holds)

# Bifurcation diagrams

**Smooth**
**Will not split**
**Does not disappear**

**Hyperbolic equilibria**

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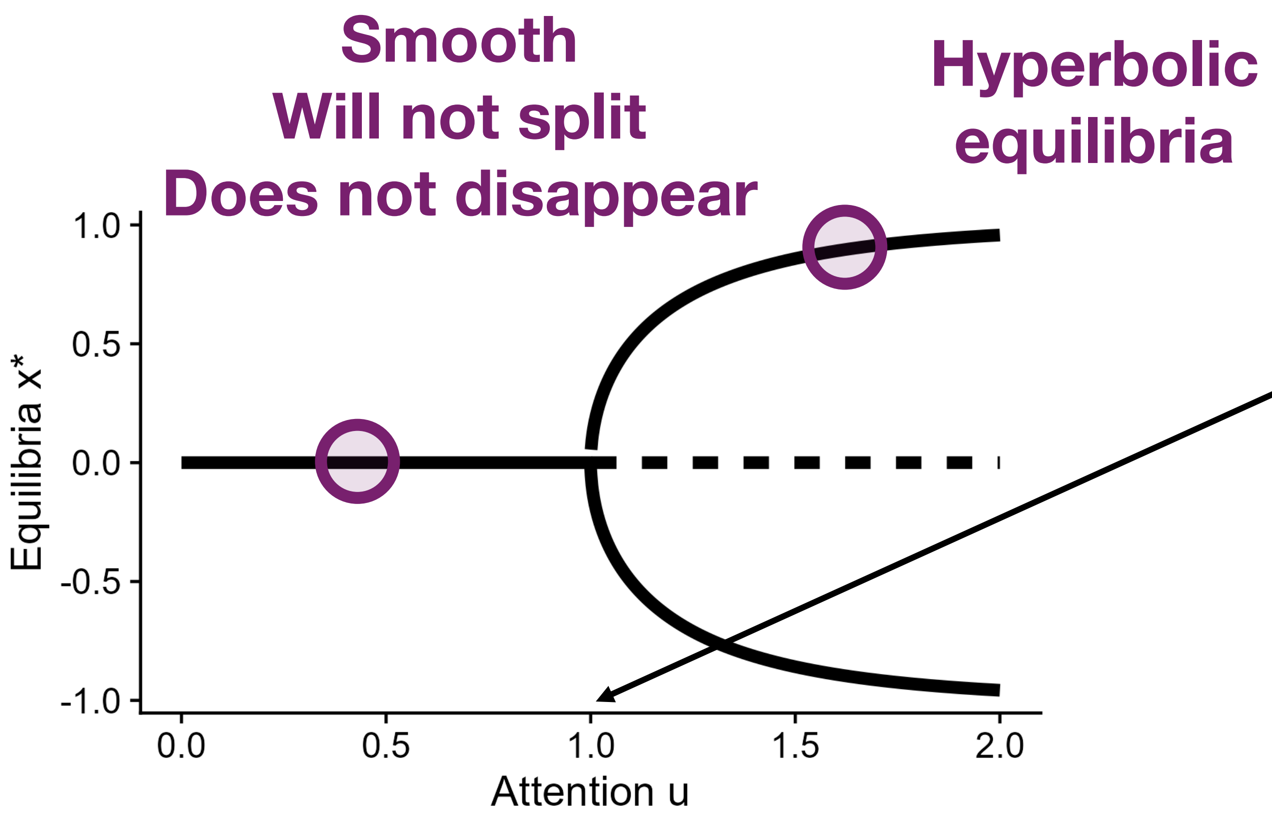
Numerical continuation techniques use this fact to draw bifurcation diagrams. Bifurcation/qualitative "counter-part" of time integration
*bifurcationKit* Julia - *matcont* Matlab - *dynamica* Mathematica ...

# Bifurcation diagrams




**Critical point $u^*$ ($\lambda = 0$) divides different "regimes" (number/stability of equilibria) of decision-making**

**Could it be otherwise?**

**No**

**In what/how many ways can equilibria branches change?**

# Elementary bifurcations

At the critical point, the bifurcation diagram will have specific shapes resembling a *normal form*: these are the elementary bifurcations.

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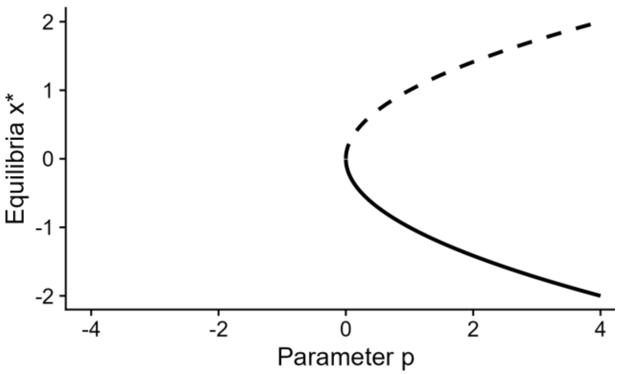
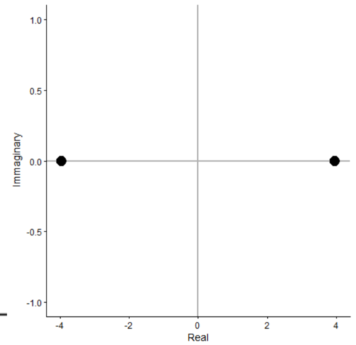
## Saddle node




$$\dot{x} = f(x,p) = \varepsilon_1 p + \varepsilon_2 x^2, \varepsilon_i \in \{-1,+1\}$$

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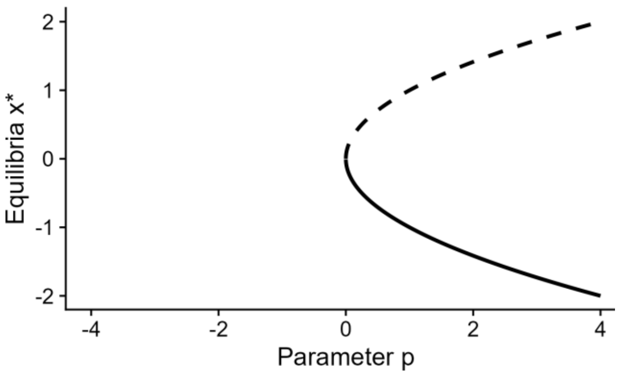


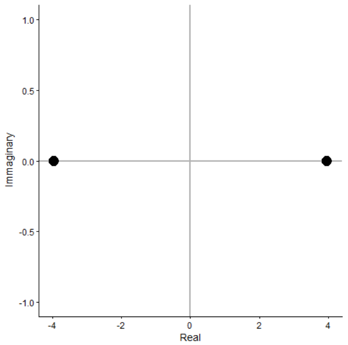


$$\dot{x} = f(x,p) = \varepsilon_1 p + \varepsilon_2 x^2, \varepsilon_i \in \{-1,+1\}$$

**For the shown graph:**

*a)* $\varepsilon_1 = -1, \varepsilon_2 = 1$
*b)* $\varepsilon_1 = 1, \varepsilon_2 = 1$
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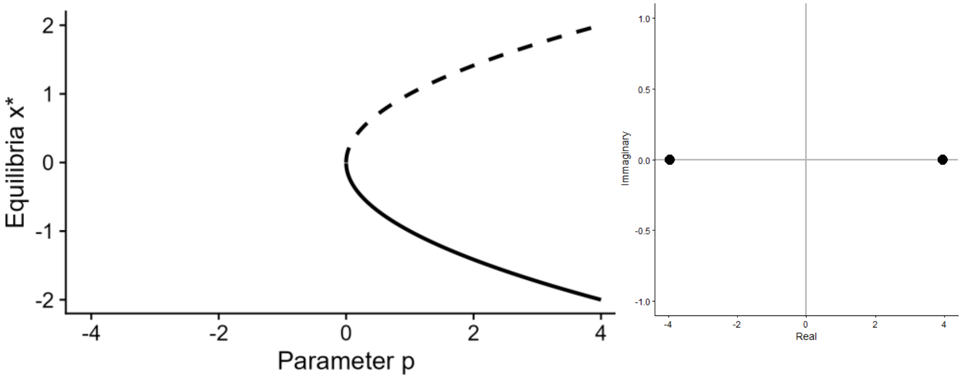


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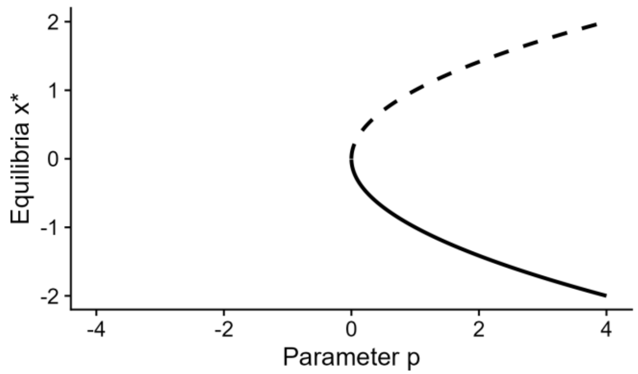


$$\dot{x} = f(x,p) = \varepsilon_1 p + \varepsilon_2 x^2, \varepsilon_i \in \{-1, +1\}$$

## Defining conditions

**a)** $\dfrac{\partial^2 f(x,p)}{\partial x^2} \neq 0$ **b)** $\dfrac{\partial f(x,p)}{\partial p} \neq 0$ **c)** $\dfrac{\partial f(x,p)}{\partial x} = 0$

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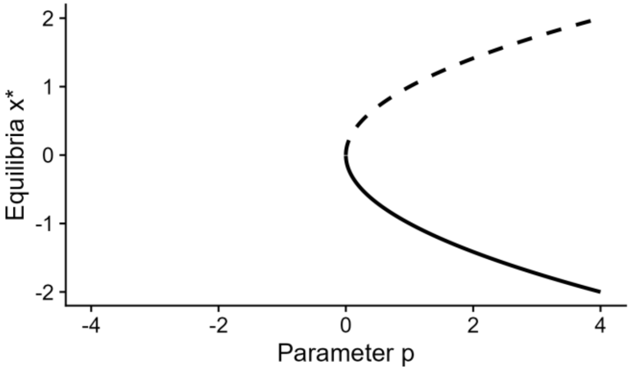


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**Defining conditions**

**Which defining condition is shared among all one-dimensional elementary bifurcations?**

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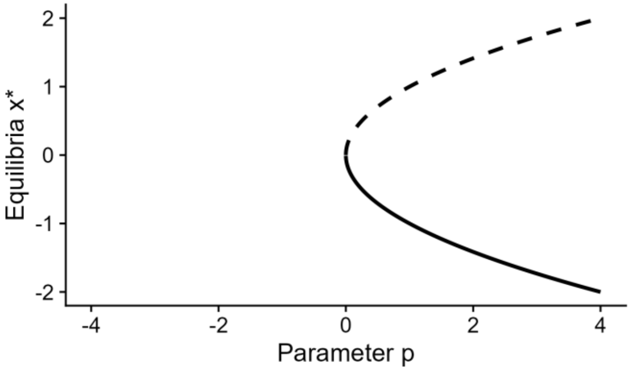


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**Condition for being a critical point**

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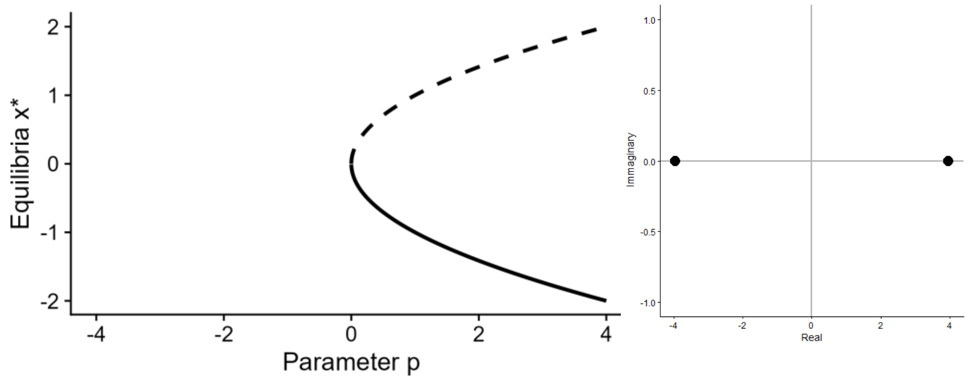
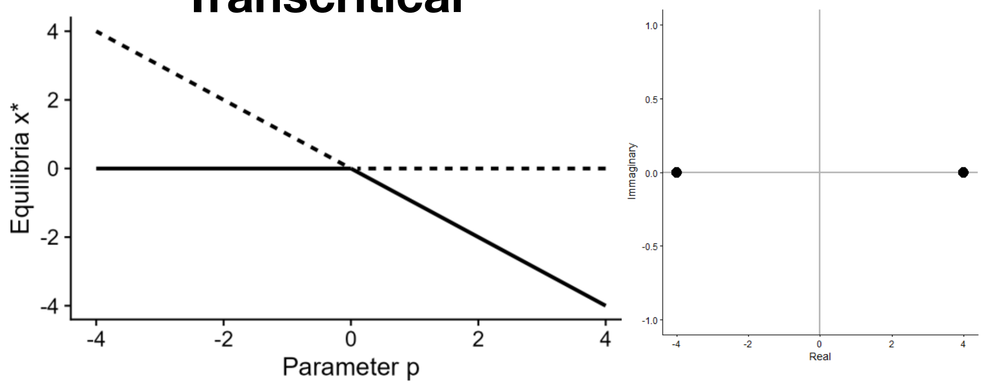



$$\dot{x} = f(x, p) = \varepsilon_1 p + \varepsilon_2 x^2, \varepsilon_i \in \{-1, +1\}$$

## Transcritical




$$\dot{x} = f(x, p) = px + \varepsilon x^2, \varepsilon \in \{-1, +1\}$$

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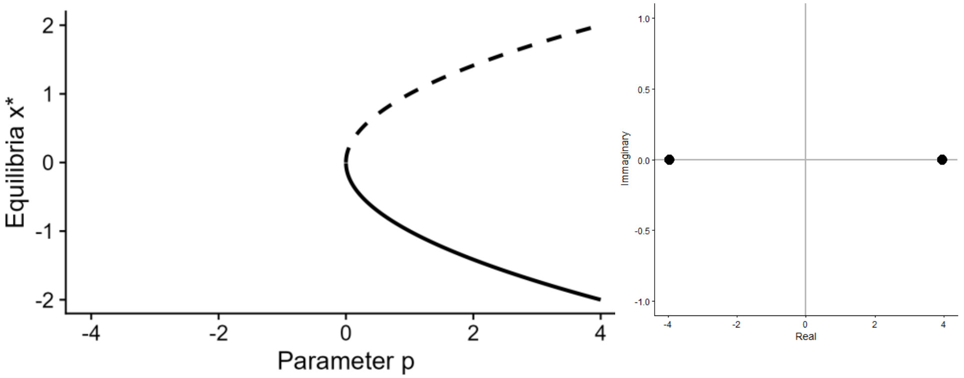


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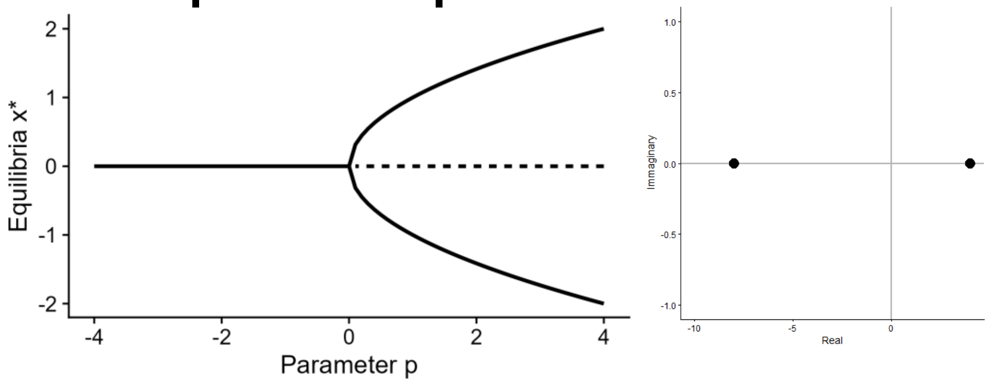
## Supercritical pitchfork

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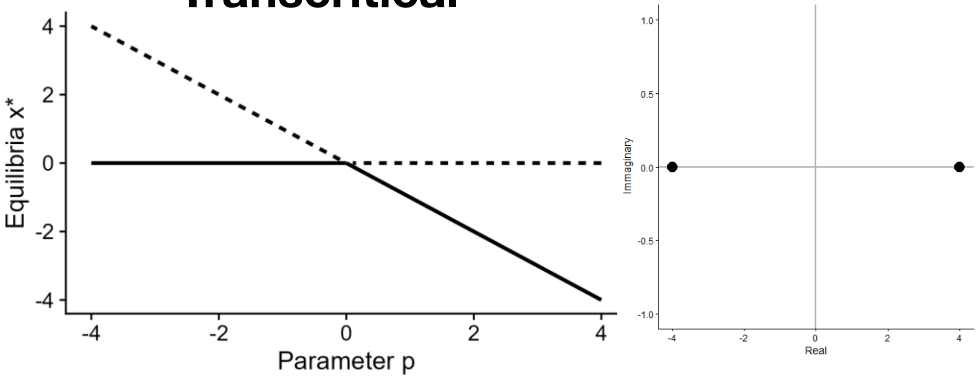


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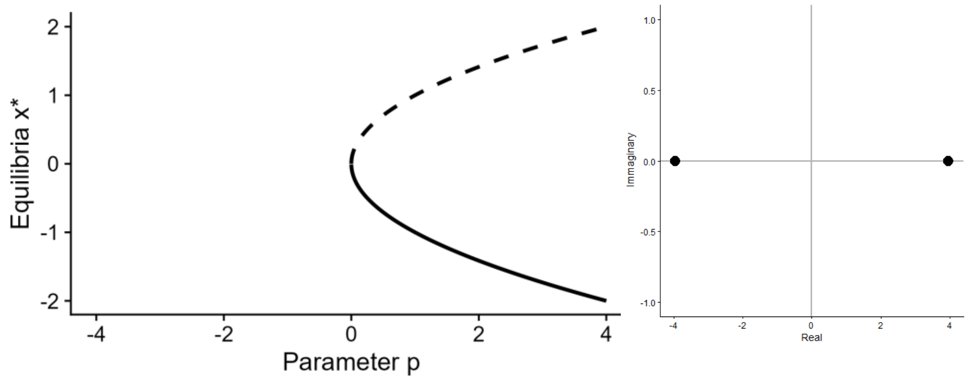


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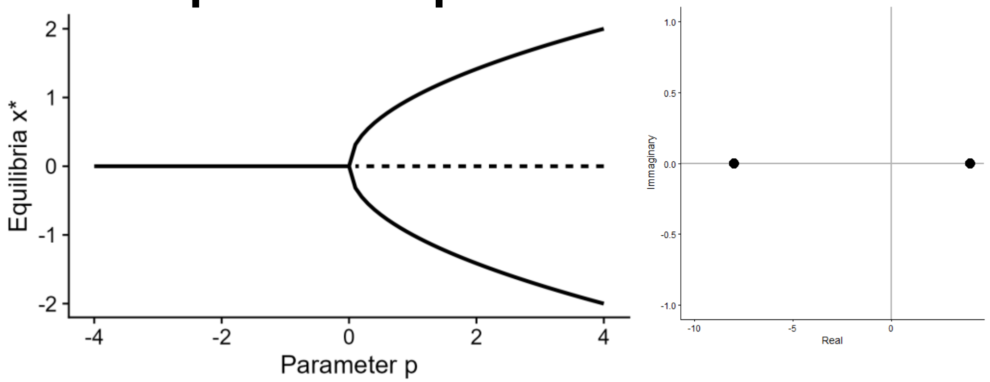


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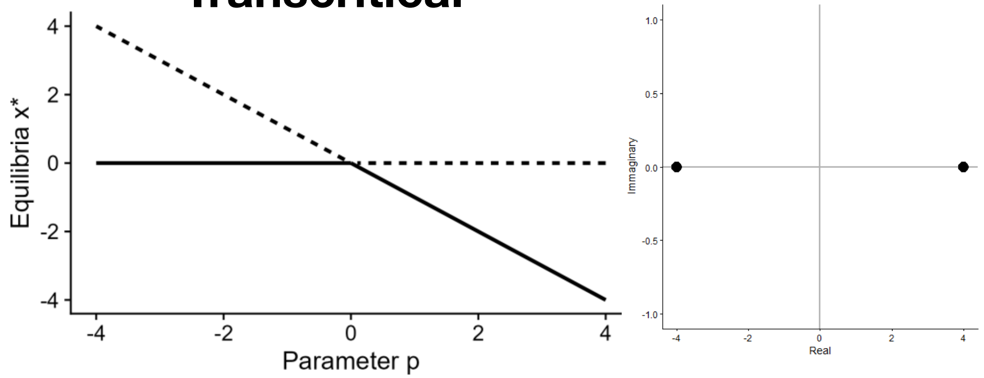


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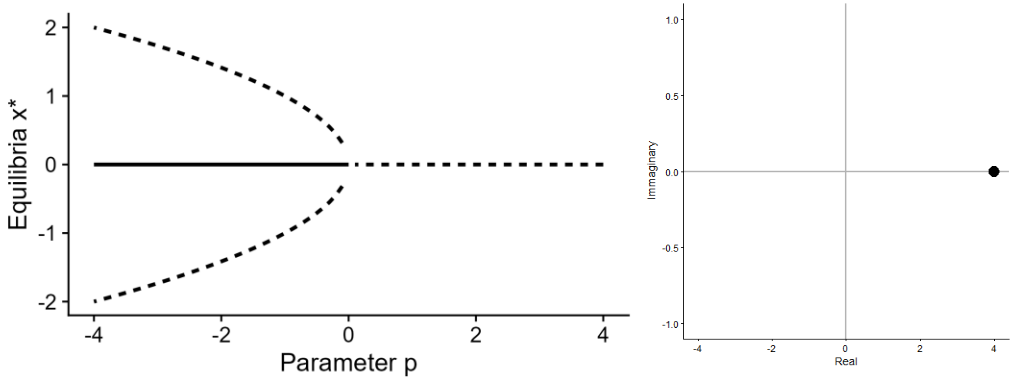
## Subcritical pitchfork




$$\dot{x} = f(x,p) = \varepsilon x p + x^3, \varepsilon \in \{-1,+1\}$$

# What we know

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**Elementary bifurcations can transform into each other**

**Only a finite way in which this can happen!**

# Universal unfolding theorem

There are a finite number of ways the bifurcation of $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{r} = 0)$ transforms into the bifurcation of $\dot{\boldsymbol{x}} = g(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{r} \neq 0)$. $\mathbf{r}$ are the *unfolding parameters* $\boldsymbol{r} \neq 0$ is called a *parametric perturbation*

## Universal unfolding of pitchfork

$$\dot{x} = g(x, p, a, b) = -x^3 + ax^2 + px + b$$

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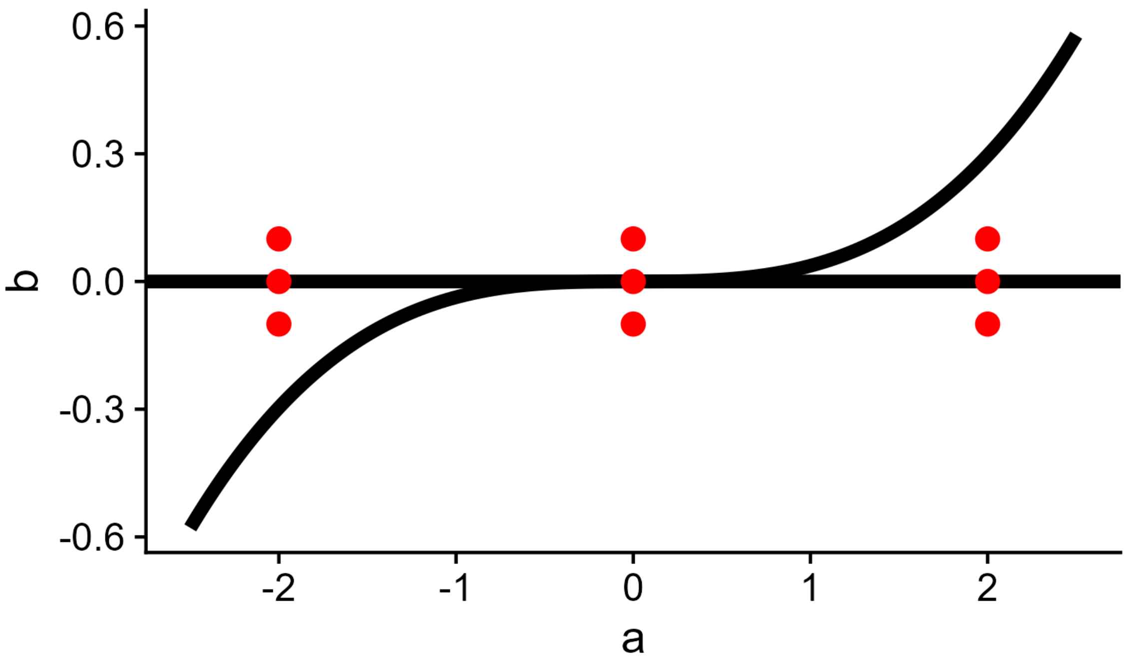
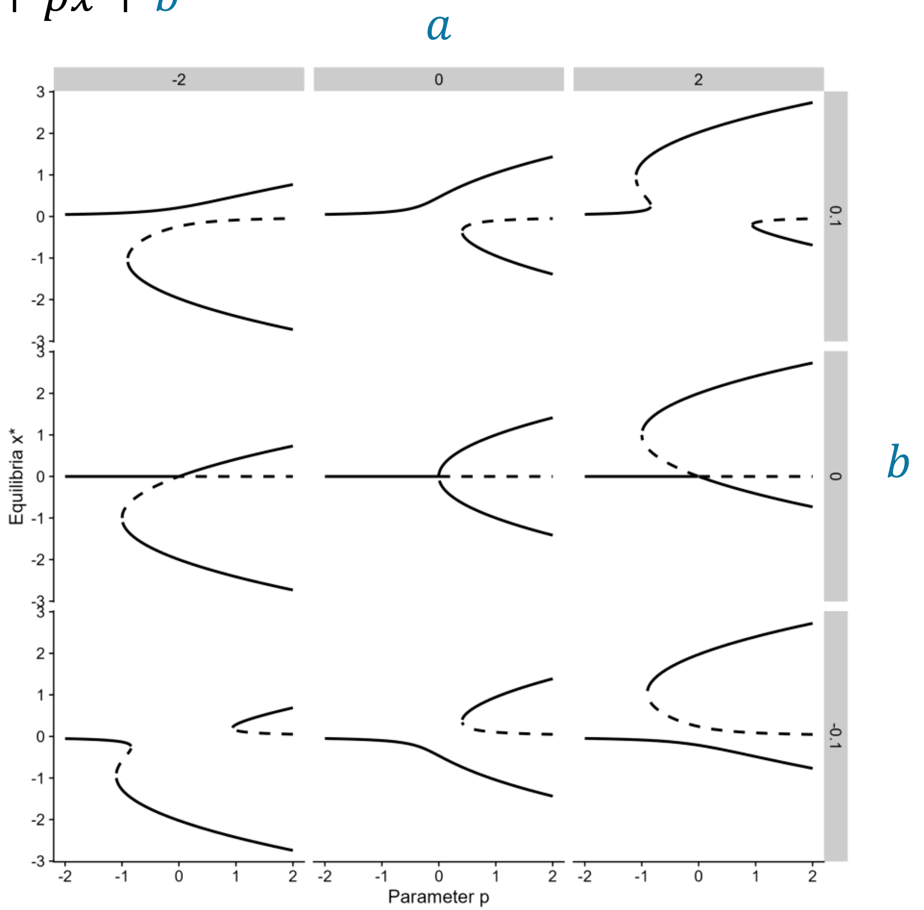
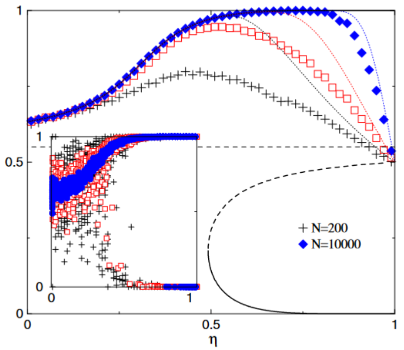


Journal of Statistical Mechanics: Theory and Experiment
An IOP and SISSA journal

# Phase coexistence in a forecasting game

Philippe Curty and Matteo Marsili



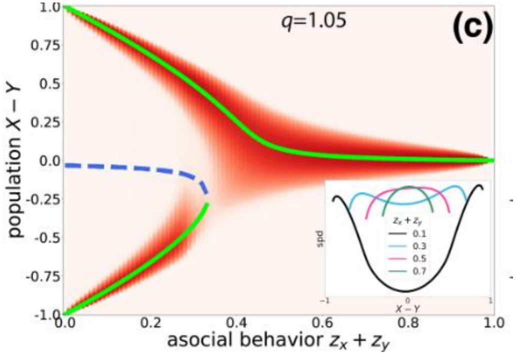

communications physics

ARTICLE

https://doi.org/10.1038/s42005-023-01345-3 OPEN

# Cross-inhibition leads to group consensus despite the presence of strongly opinionated minorities and asocial behaviour

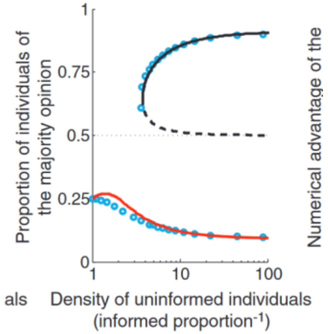
Andreagiovanni Reina[1,2]✉, Raina Zakir[1], Giulia De Masi[3,4] & Eliseo Ferrante[3,5]




REPORTS

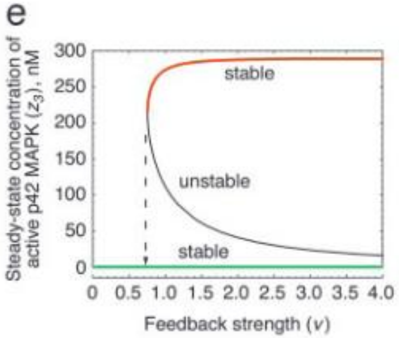
# Uninformed Individuals Promote Democratic Consensus in Animal Groups

Iain D. Couzin,[1]* Christos C. Ioannou,[1]† Güven Demirel,[2] Thilo Gross,[2]‡ Colin J. Torney,[1] Andrew Hartnett,[1] Larissa Conradt,[3]§ Simon A. Levin,[1] Naomi E. Leonard[4]




# Detection of multistability, bifurcations, and hysteresis in a large class of biological positive-feedback systems

David Angeli*, James E. Ferrell, Jr.†, and Eduardo D. Sontag‡§




RESEARCH ARTICLE | EVOLUTION |

# Weak individual preferences stabilize culture

Alberto Acerbi and Benoît de Courson Authors Info & Affiliations

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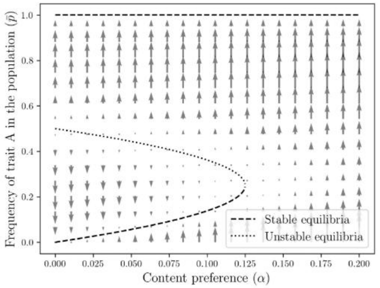
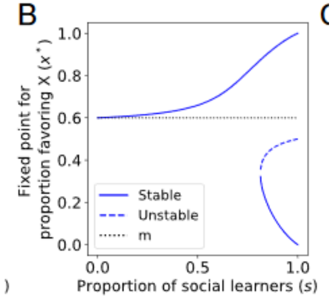
February 21, 2025 | 122 (8) e2412380122 | https://doi.org/10.1073/pnas.2412380122




# Dynamical system model predicts when social learners impair collective performance

Vicky Chuqiao Yang[a,1], Mirta Galesic[a,b,c], Harvey McGuinness[d], and Ani Harutyunyan[e]

[a]Santa Fe Institute, Santa Fe, NM 87501; [b]Complexity Science Hub Vienna, A-1080 Vienna, Austria; [c]Vermont Complex Systems Center, University of Vermont, Burlington, VT 05405; [d]Zanvyl Krieger School of Arts and Sciences, Johns Hopkins University, Baltimore, MD 21218; and [e]Sunwater Institute, North Bethesda, MD 20852

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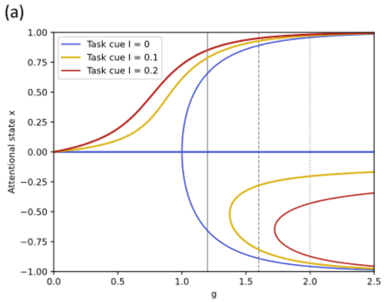
Review

# Examining cognitive flexibility and stability through the lens of dynamical systems

Sebastian Musslick[1,2,*] and Anastasia Bizyaeva[3,*]

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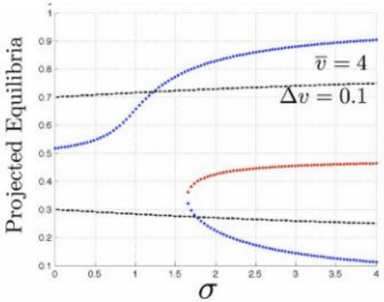
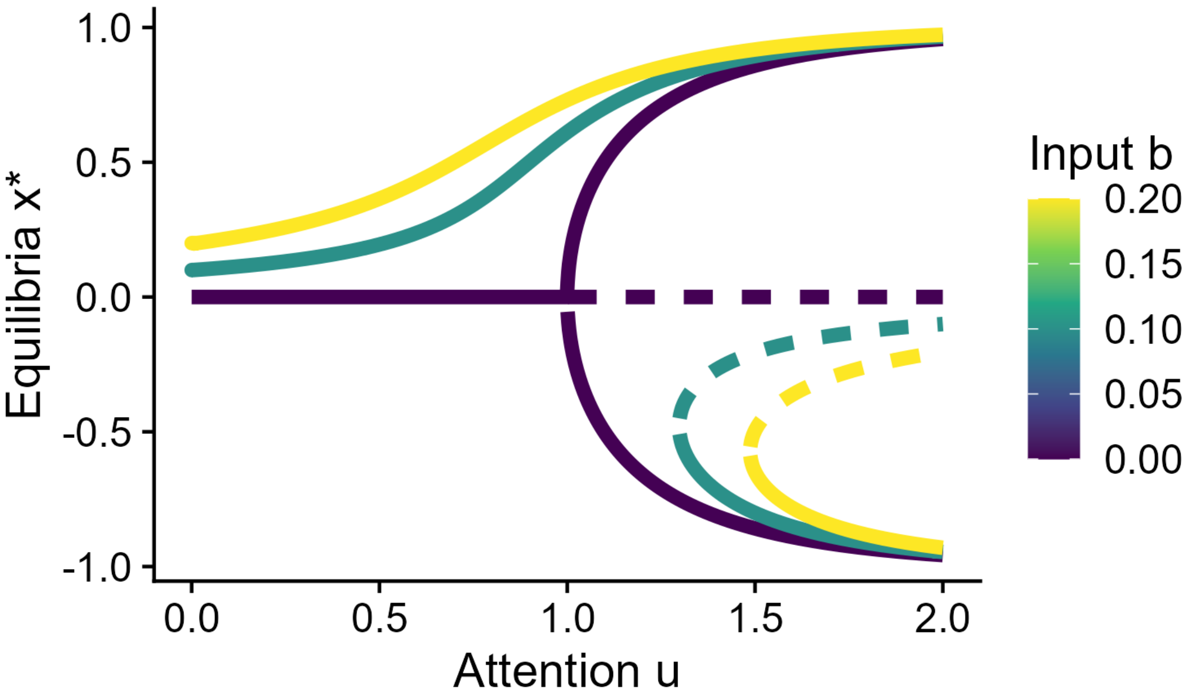
# A Mechanism for Value-Sensitive Decision-Making

Darren Pais[1], Patrick M. Hogan[2], Thomas Schlegel[3], Nigel R. Franks[3], Naomi E. Leonard[1], James A. R. Marshall[2]*



# Unfolding in the nonlinear opinion dynamics model

$$\dot{x} = -dx + \mathrm{Tanh}(ux + b)$$



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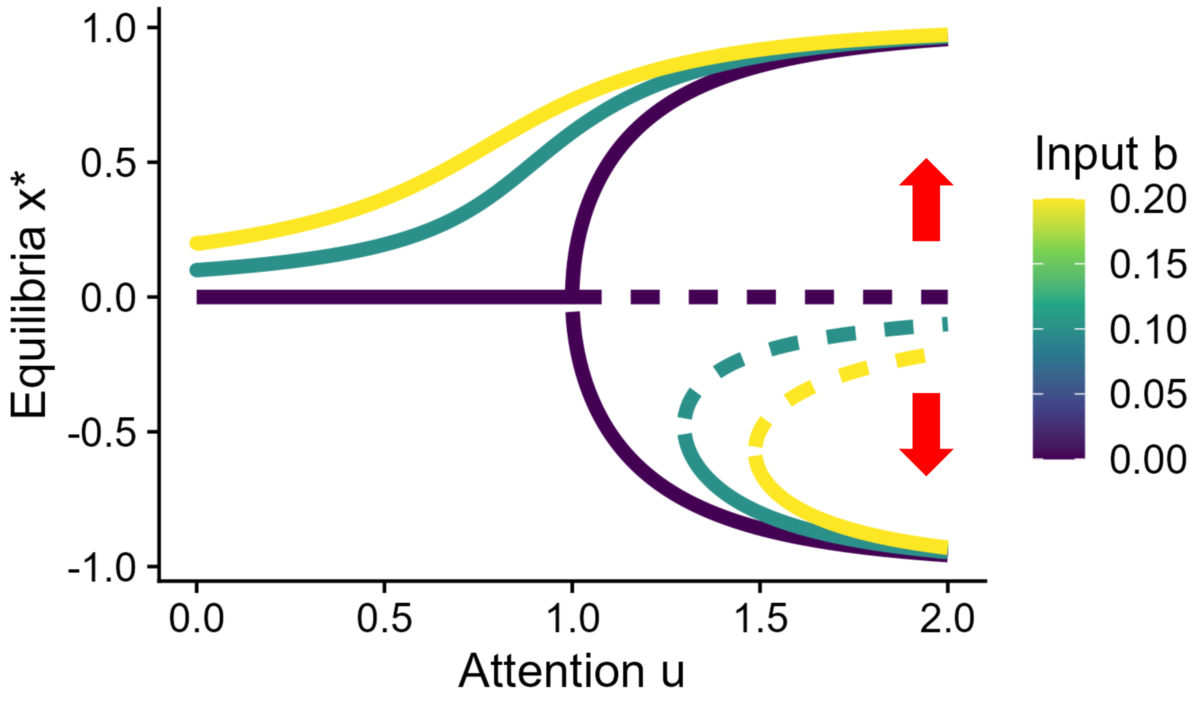
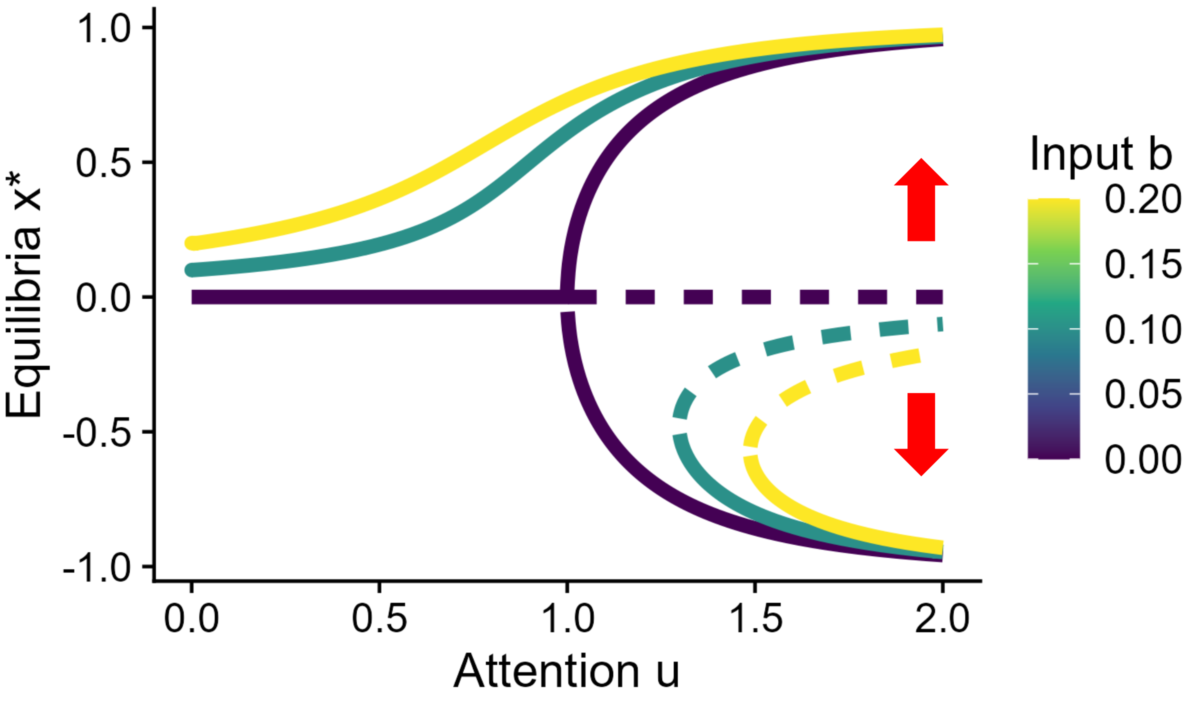



1. **Input increases monostable regime for best option**
2. **Non-linear effects can still amplify wrong information and lead to errors**
3. **Attention u (strength of positive feedback) creates a trade-off between making decisions and cascades to wrong decisions**

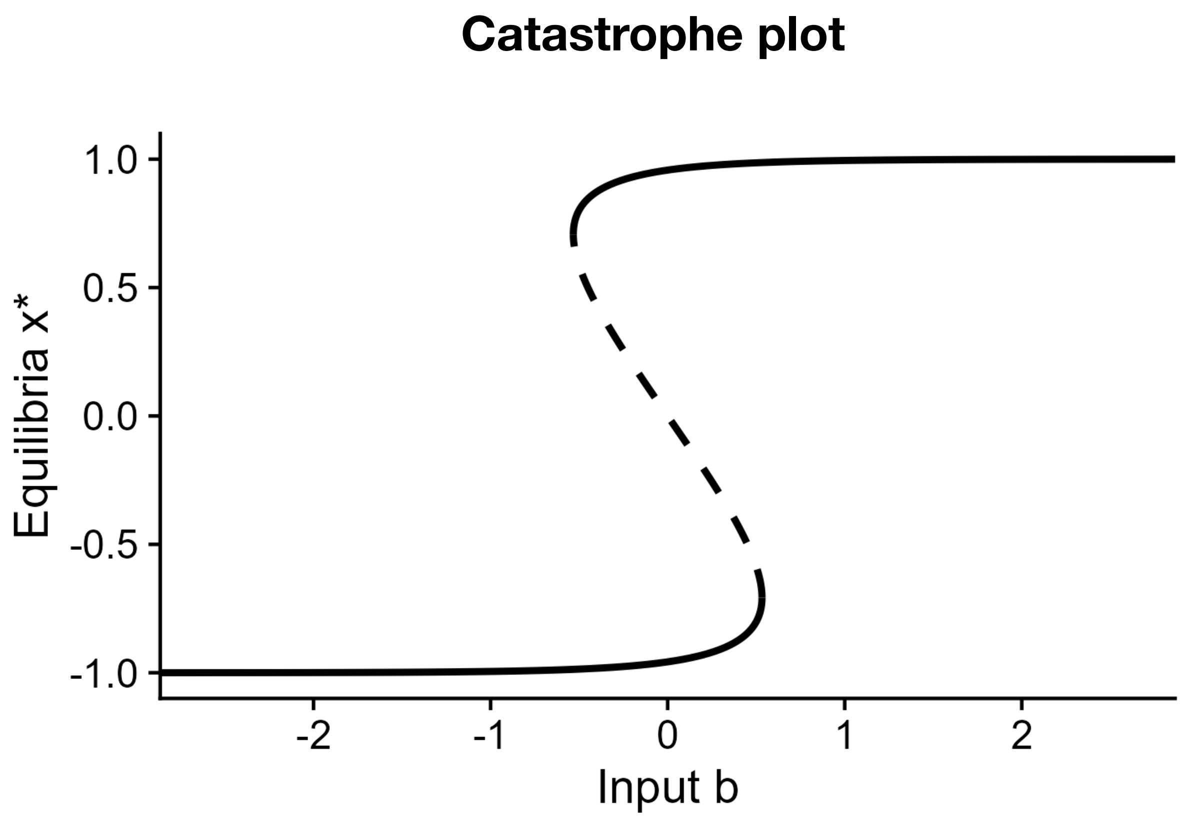
# Unfolding in the nonlinear opinion dynamics model

$$\dot{x} = -dx + \mathrm{Tanh}(ux + b)$$




**Draw the bifurcation diagram (u = 1.5) for input b as the bifurcating parameter**

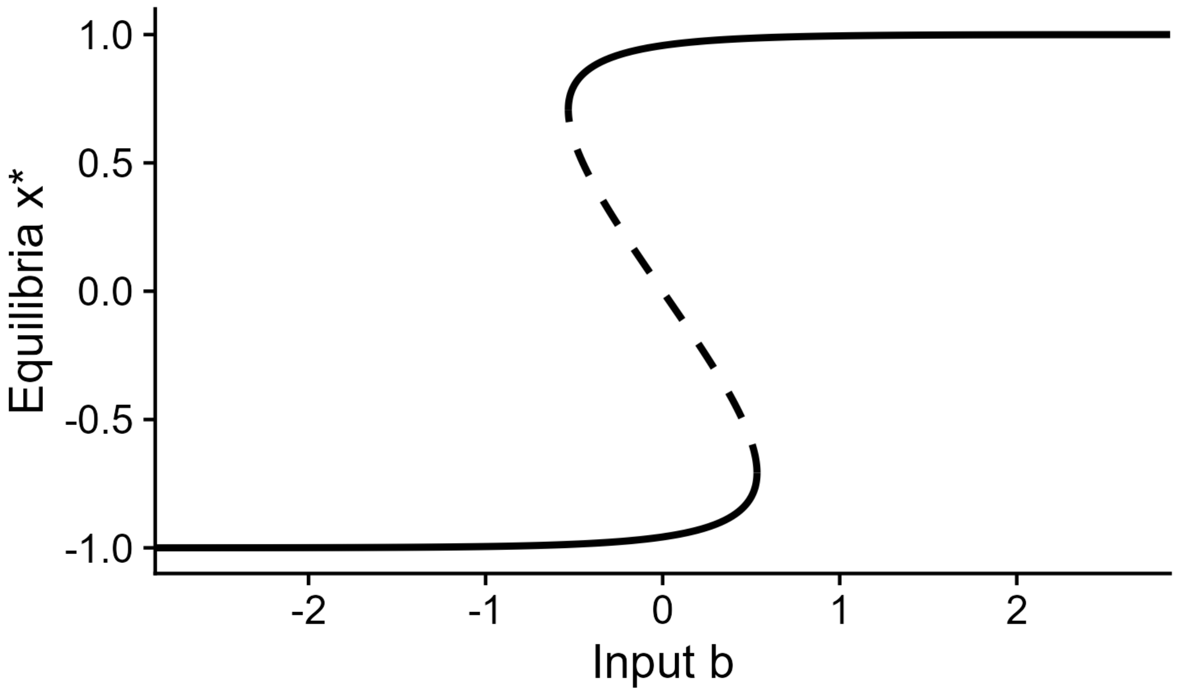
# Unfolding in the nonlinear opinion dynamics model




1. **Multi stability can lead to hysteresis/memory/path dependence**
2. **Hysteresis is bad for flexible decision-making**
3. **Small changes in inputs can lead to big changes in decision**

# Unfolding in the nonlinear opinion dynamics model

**Catastrophe plot**




**Problems**

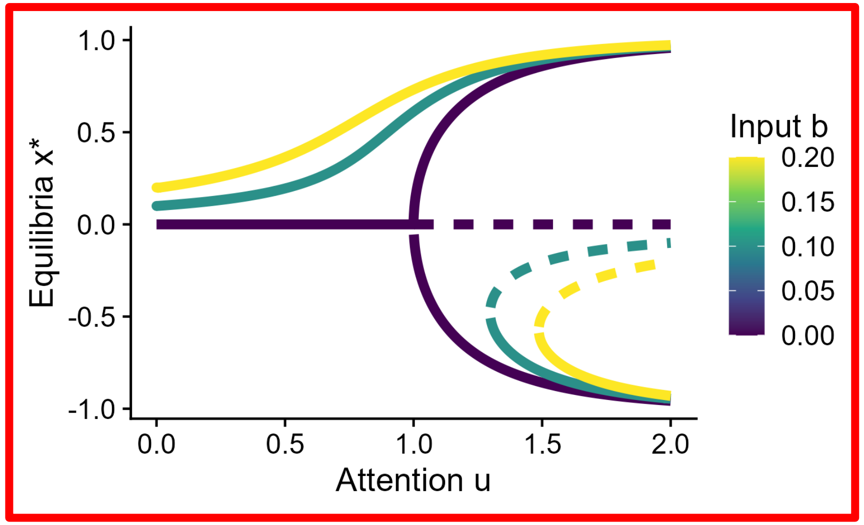
$$\dot{x} = f(x, p)$$

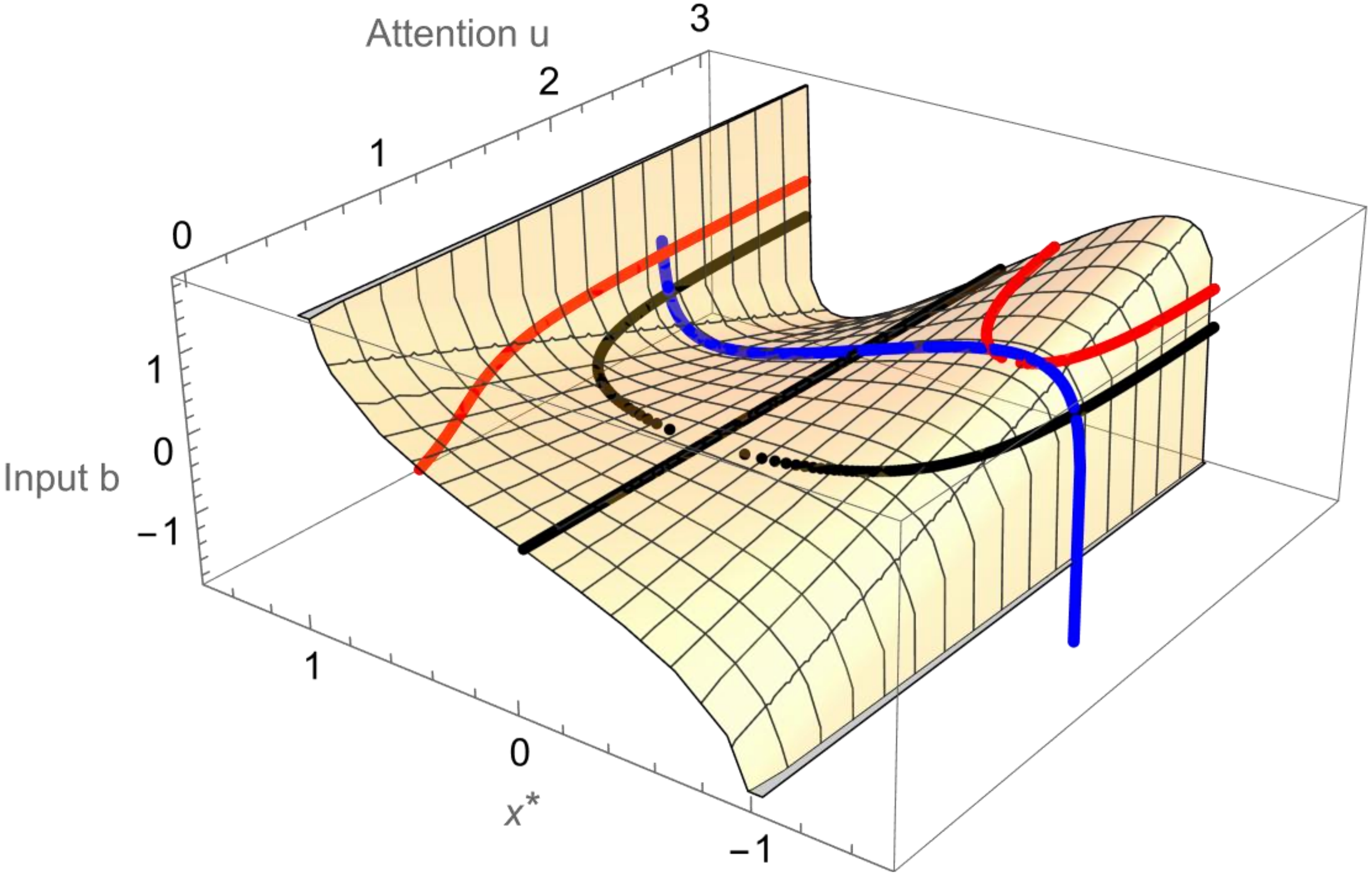
**State variables** $x \in \mathbb{R}^n$
***Can change***

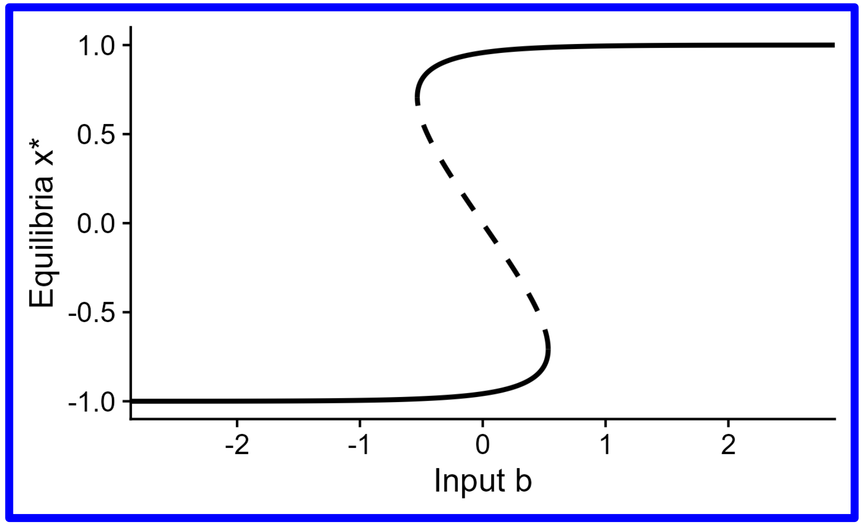
**Parameters** $p \in \mathbb{R}^k$
***Don't change*** **4.**

1. **Multi stability can lead to hysteresis/memory/path dependence**
2. **Hysteresis is bad for flexible decision-making**
3. **Small changes in inputs can lead to big changes in decision**
4. **Dynamical systems treat inputs as parameters (they must change slowly compared to the decision's dynamics)**

# Unfolding in the nonlinear opinion dynamics model



# Part 4
# Separation of timescales

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |

# State dependent attention

Decision variable

Input or bias

$$\dot{x} = -dx + \tanh(ux + b)$$

Leak

Attention

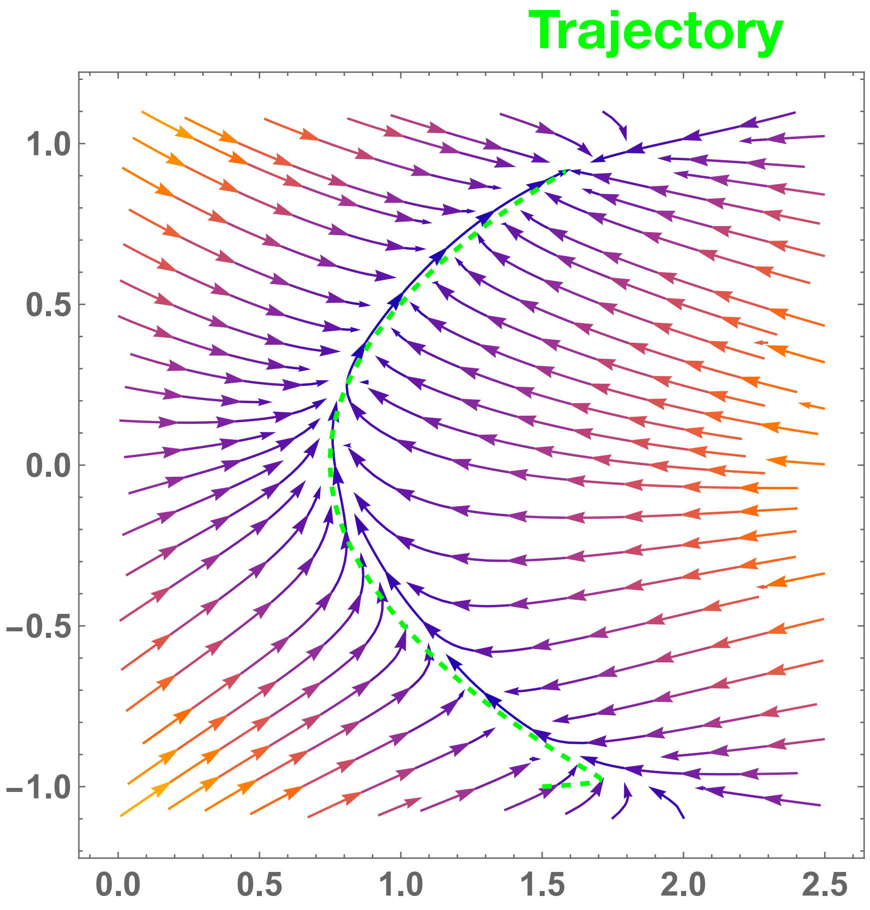
Characteristic timescale

$$\tau\dot{u} = -\alpha u + u_0 + kx^2$$

Gain

Baseline attention

# Phase plane



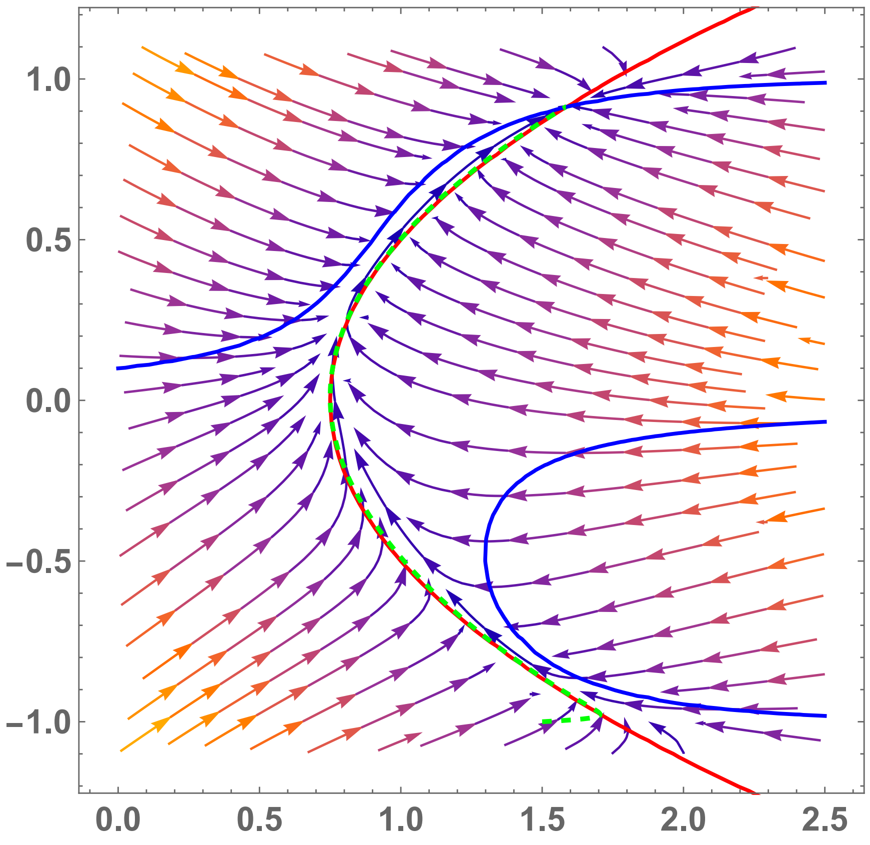

**Decision variable**
**x**

**Attention**
**u**

# Nullclines

**Decision variable**
**x**

$$\dot{x} = f(x, u, p) = 0$$



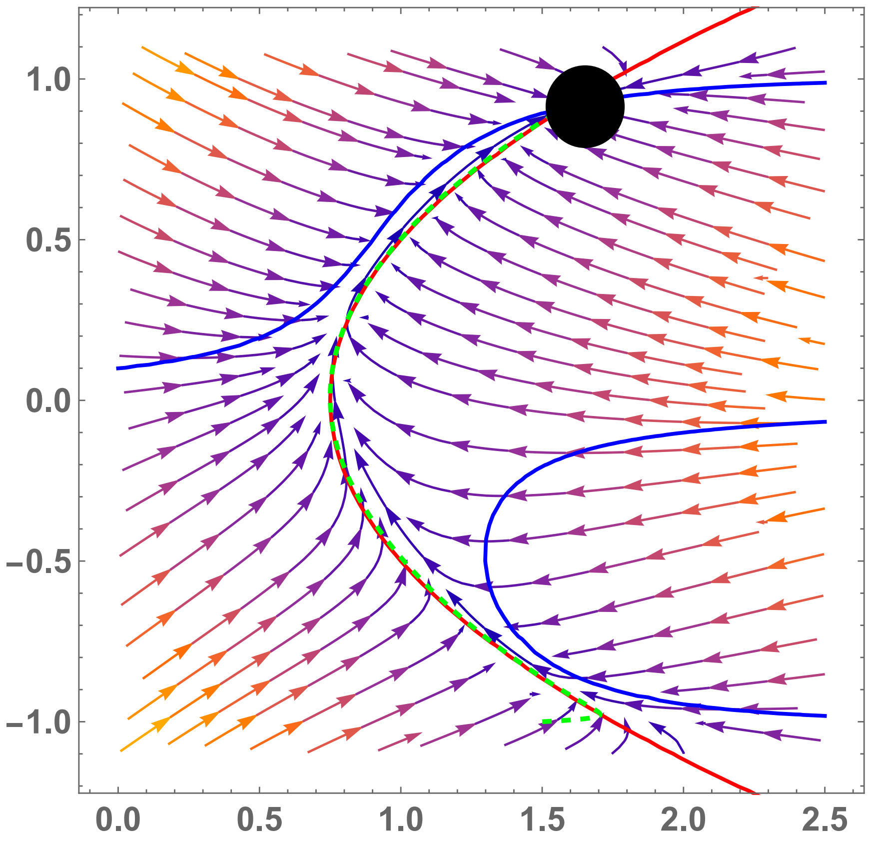

**Attention**
**u**

$$\tau\dot{u} = g(x, u, p) = 0$$

# Nullclines

**Decision variable**
**x**

$\dot{x} = f(x, u, p) = 0$




**Attention**
**u**

$\tau\dot{u} = g(x, u, p) = 0$

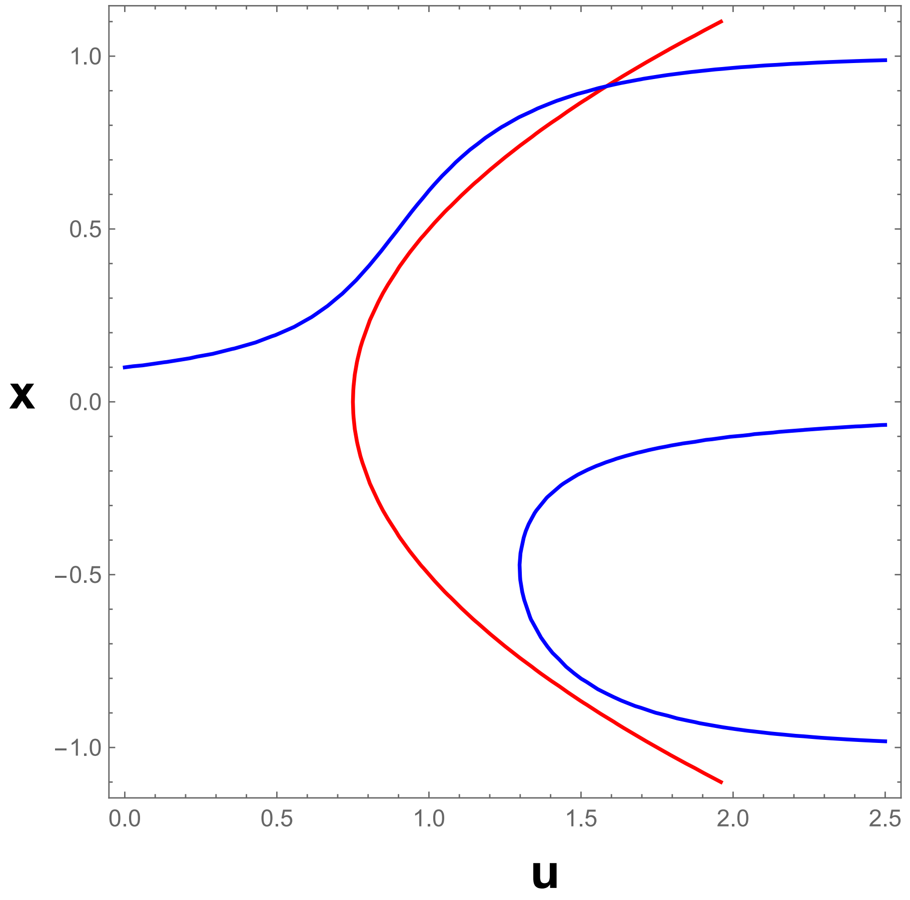
**Global equilibria are where nullclines cross**

# Separation of timescales

$1)\ \dot{x} = f(x, u, p)$

$2)\ \tau\dot{u} = g(x, u, p)$

**τ is small**



# Separation of timescales

$1)\ \dot{x} = f(x, u, p)$

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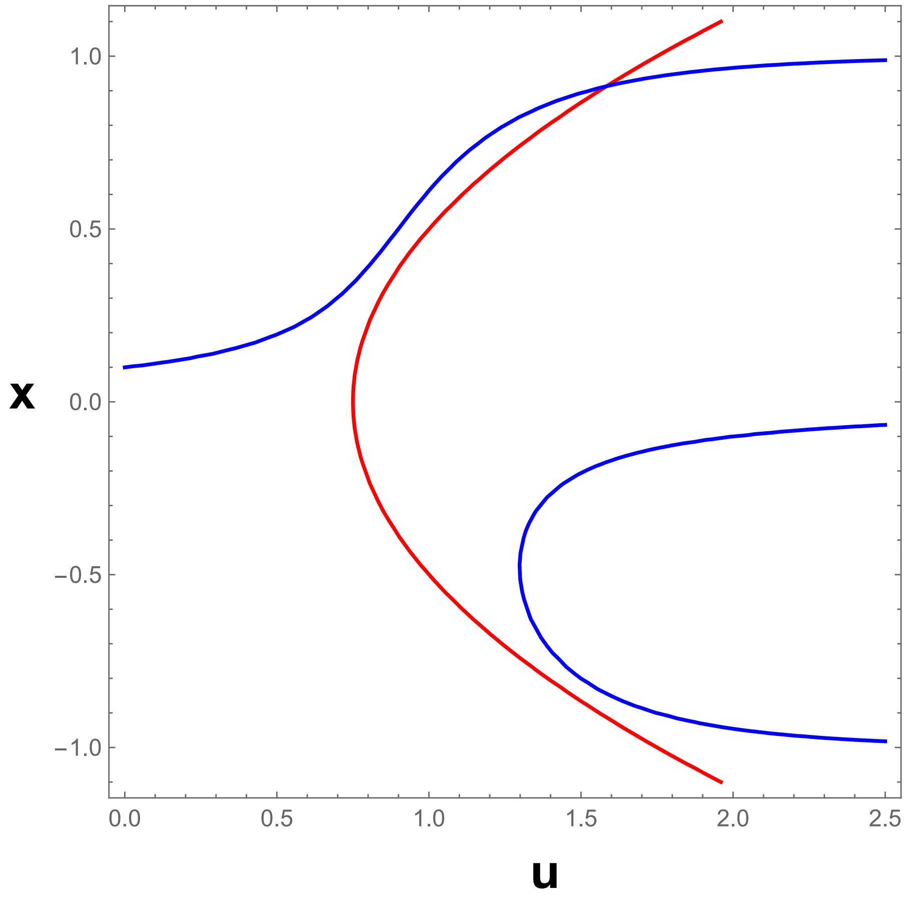
$\tau$ **is small**

**Which dynamic will be faster?**

**1)**

**2)**



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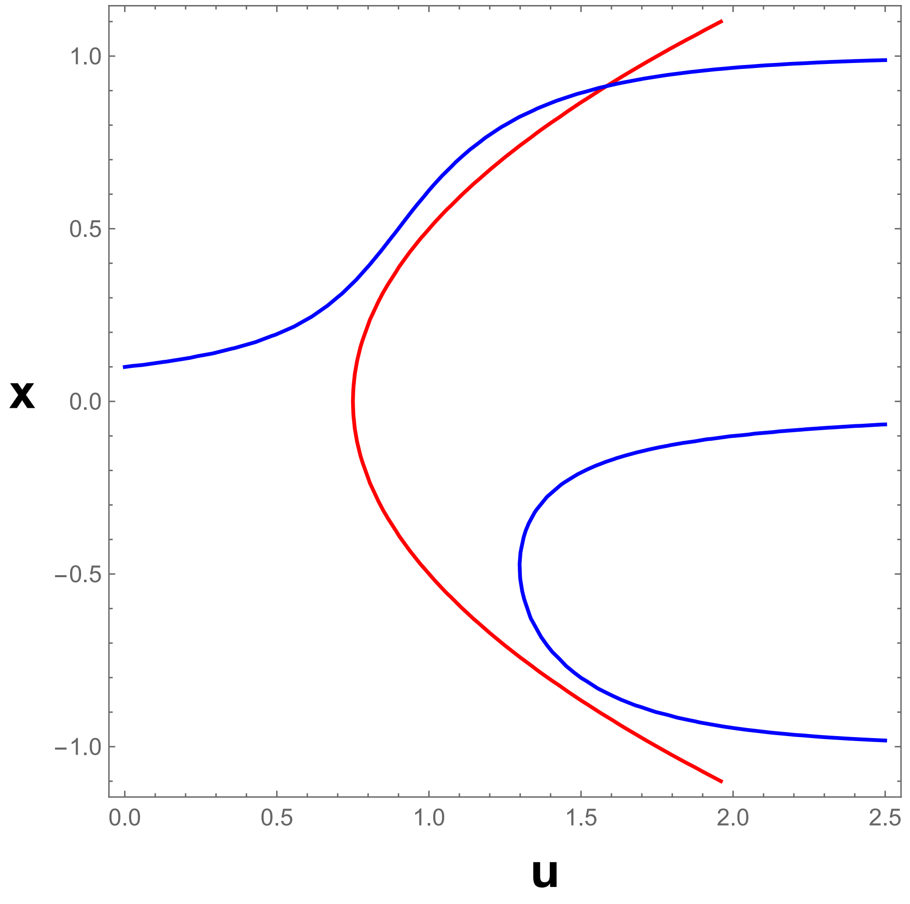


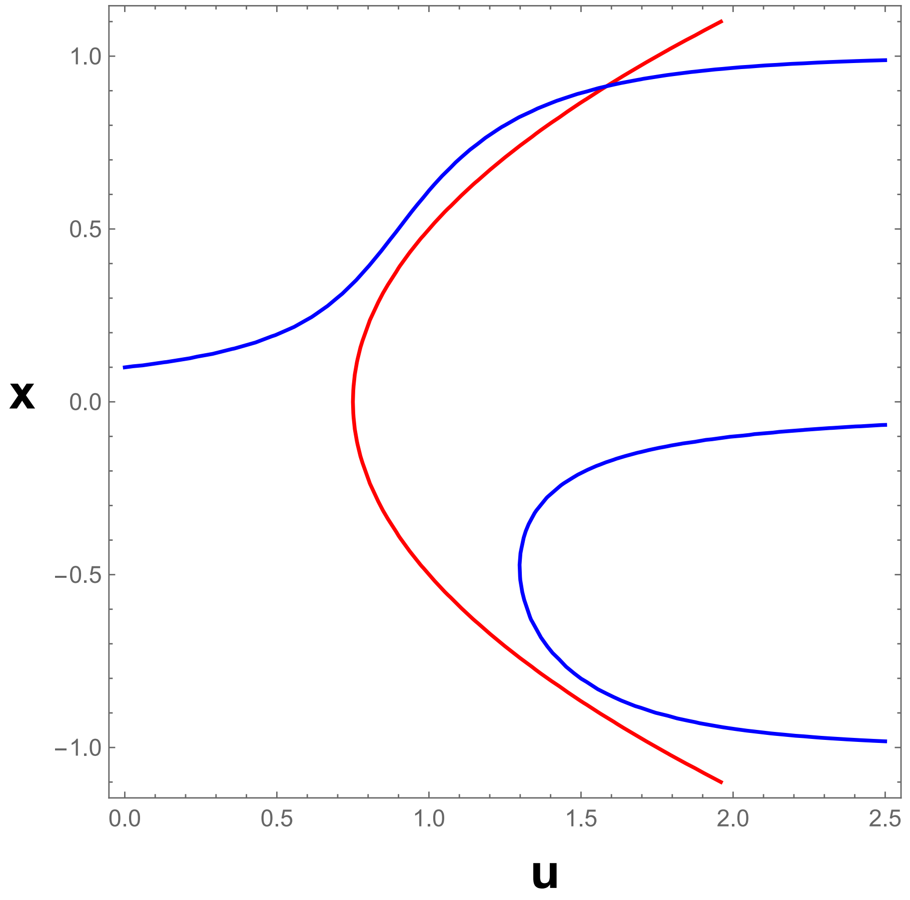
# Separation of timescales

$$1)\ \dot{x} = f(x, u, p)$$

$$2)\ \tau\dot{u} = g(x, u, p)$$

**$\tau$ is small**

**Decompose the dynamics along the systems' dimensions**



# Separation of timescales

1) $\dot{x} = f(x, u, p)$
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**The fast dynamics occurs along attention u**

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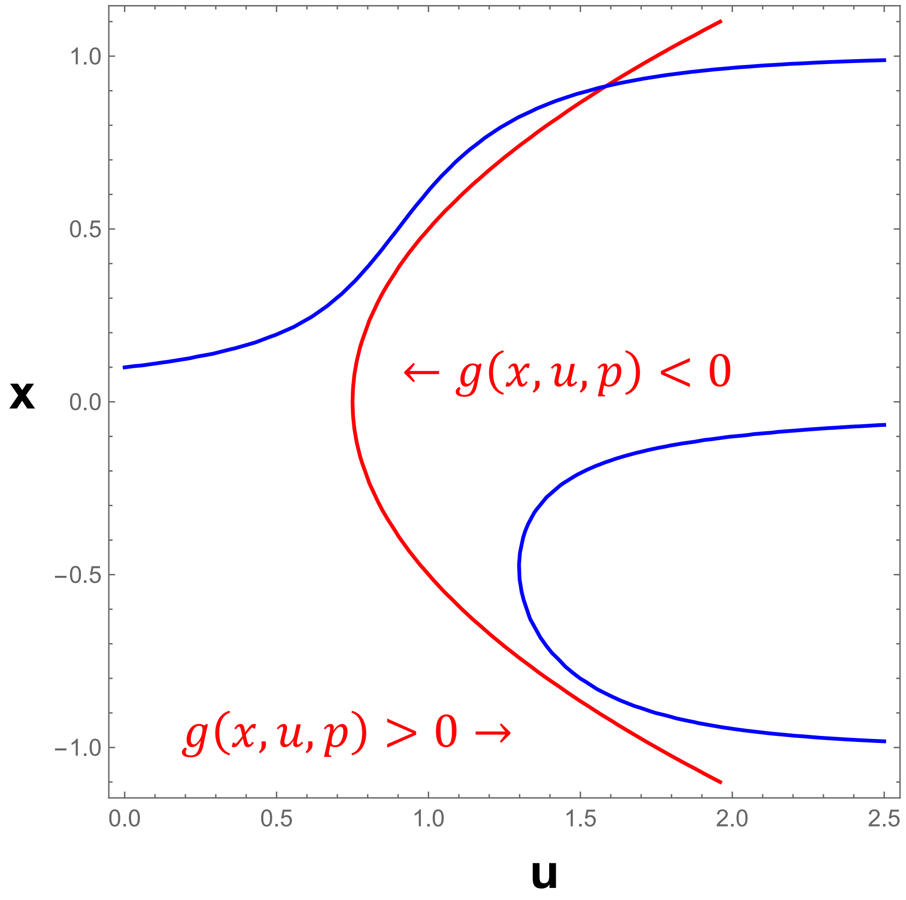

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The fast dynamics occurs along attention u

The slow dynamics occurs along the decision variable x

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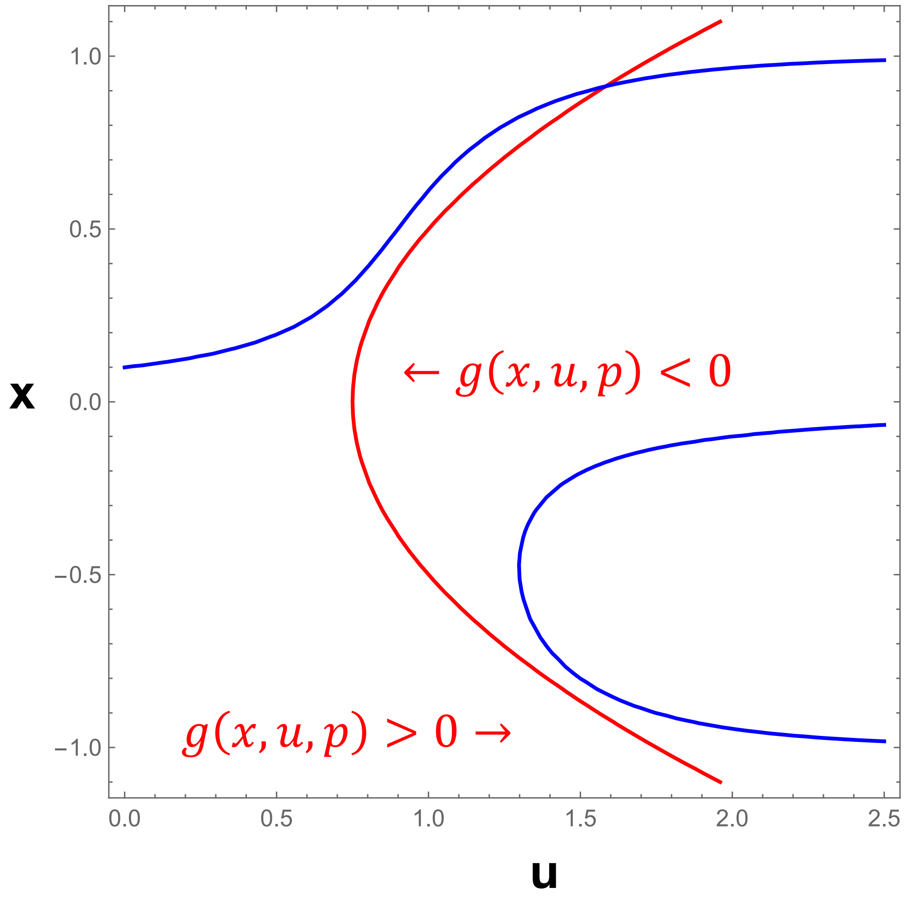

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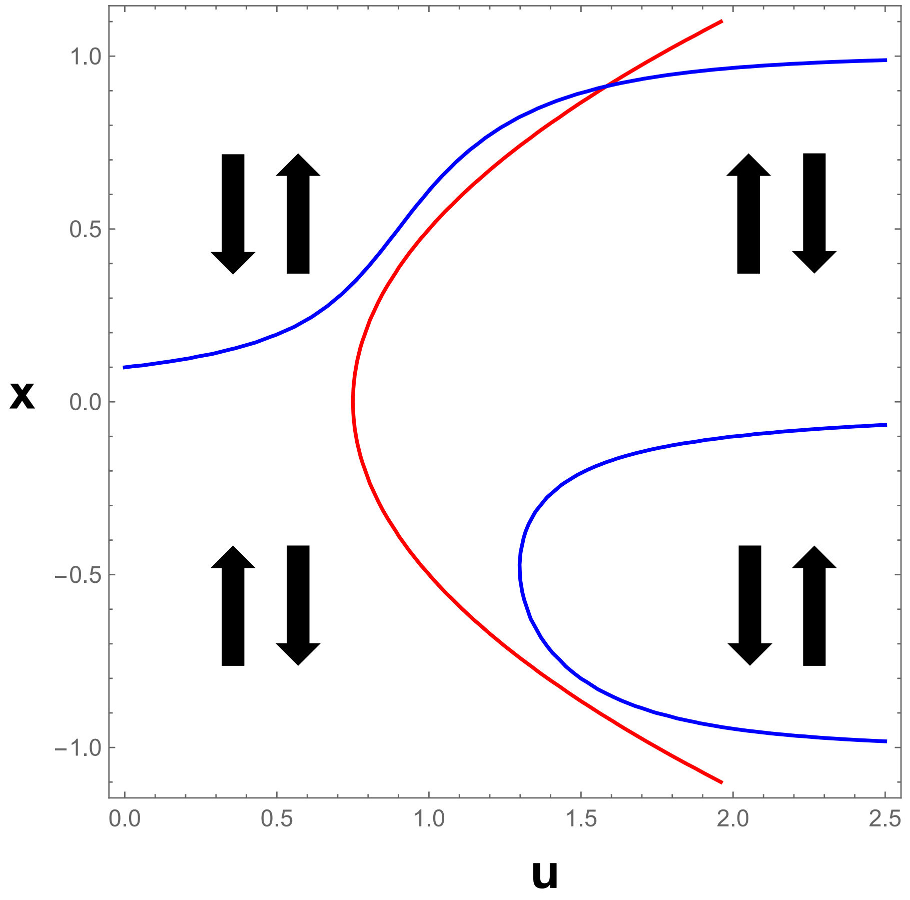
**Which arrows describe the slow dynamics?**
**1) left**
**2) right**



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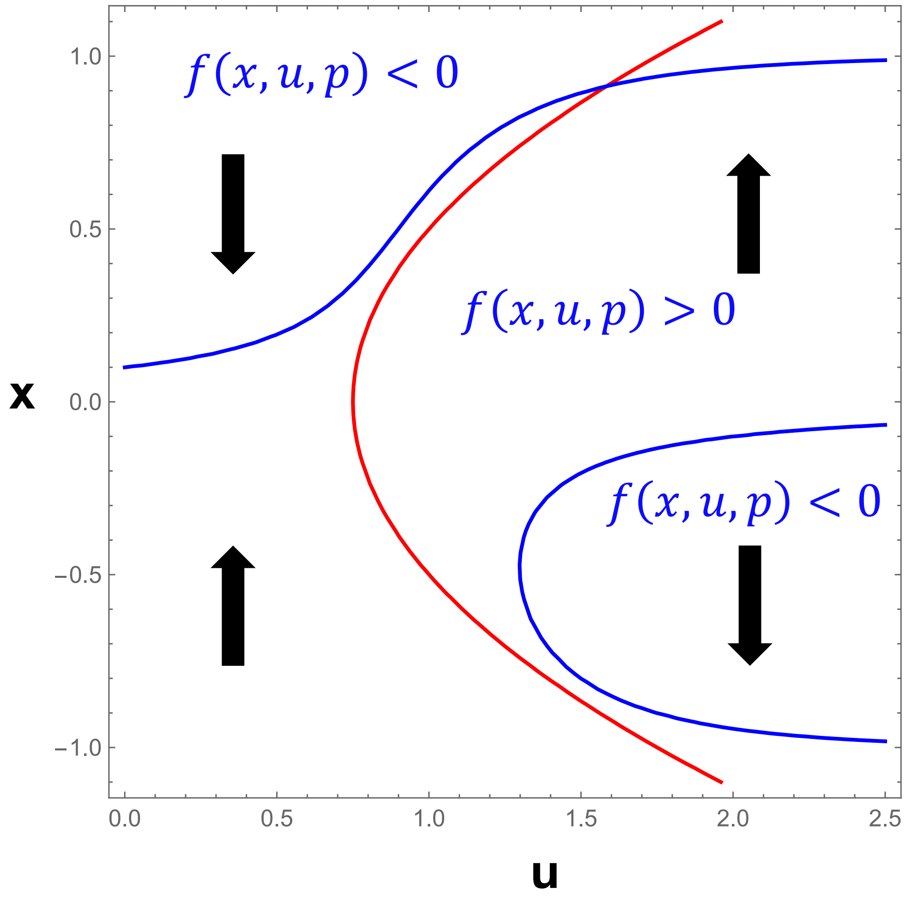
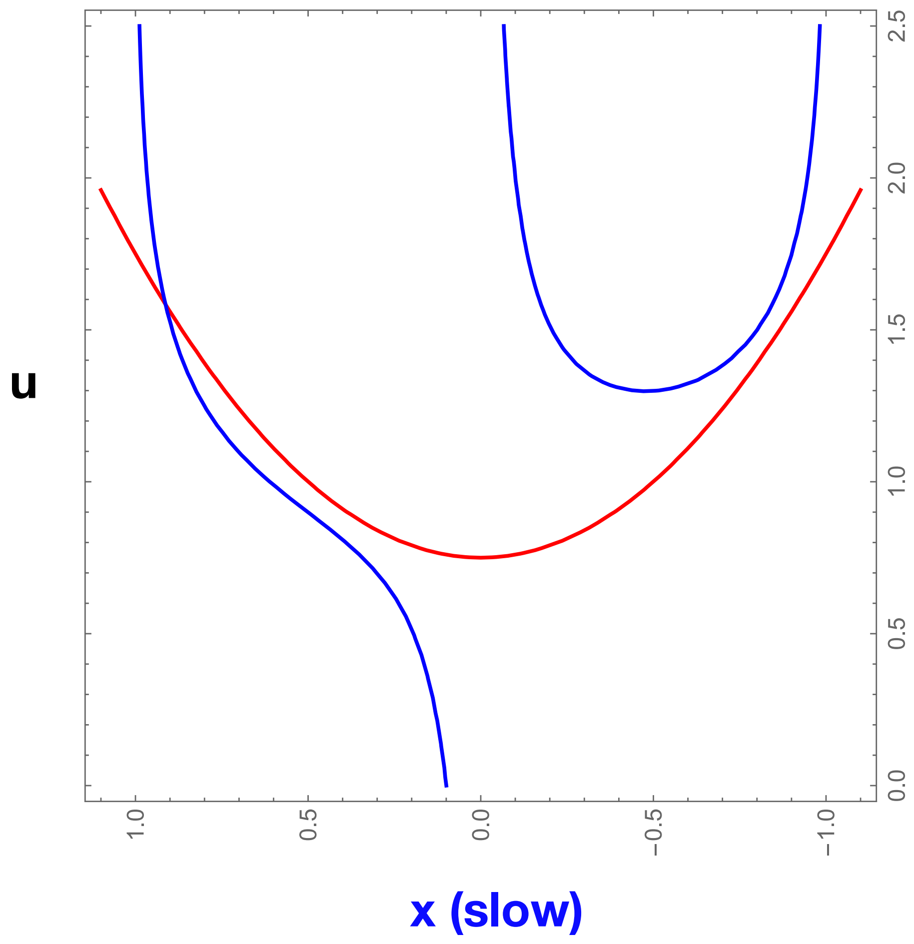


# Separation of timescales

$$1)\ \dot{x} = f(x, u, p)$$
$$2)\ \tau\dot{u} = g(x, u, p)$$

**You can consider the slow variable a parameter, along which there is an equilibrium branch for the fast variable**



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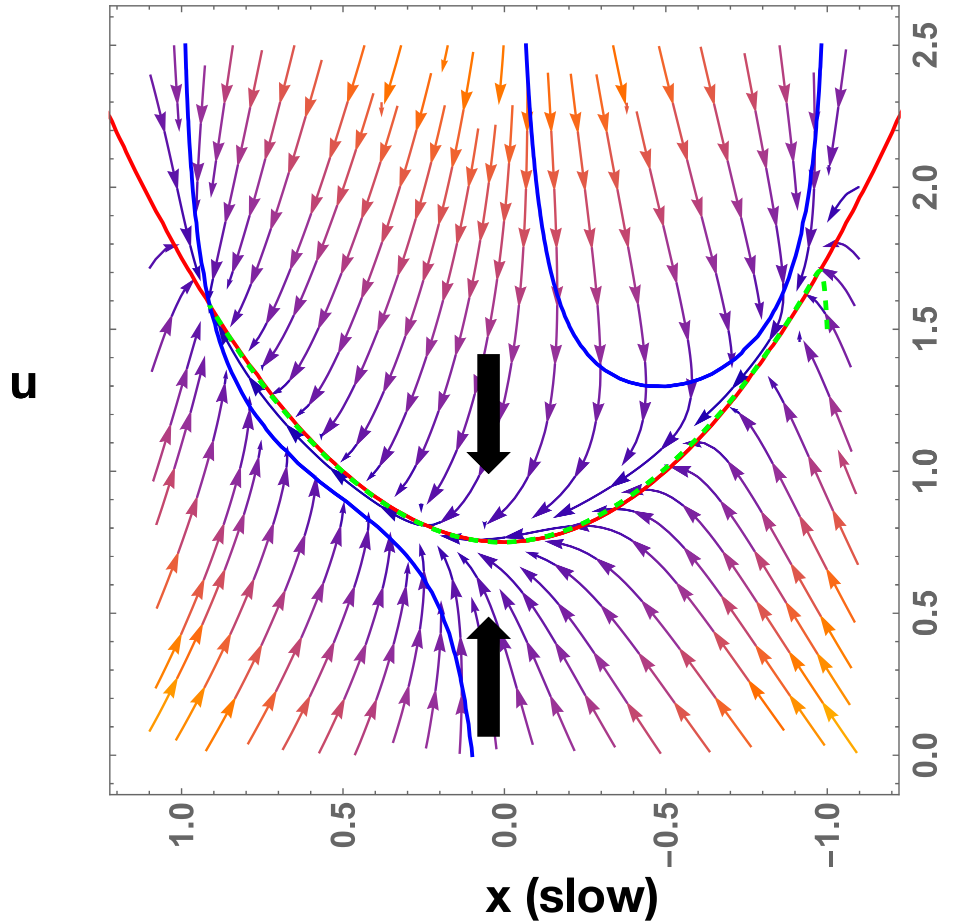



**You can consider the slow variable a parameter, along which there is an equilibrium branch for the fast variable**

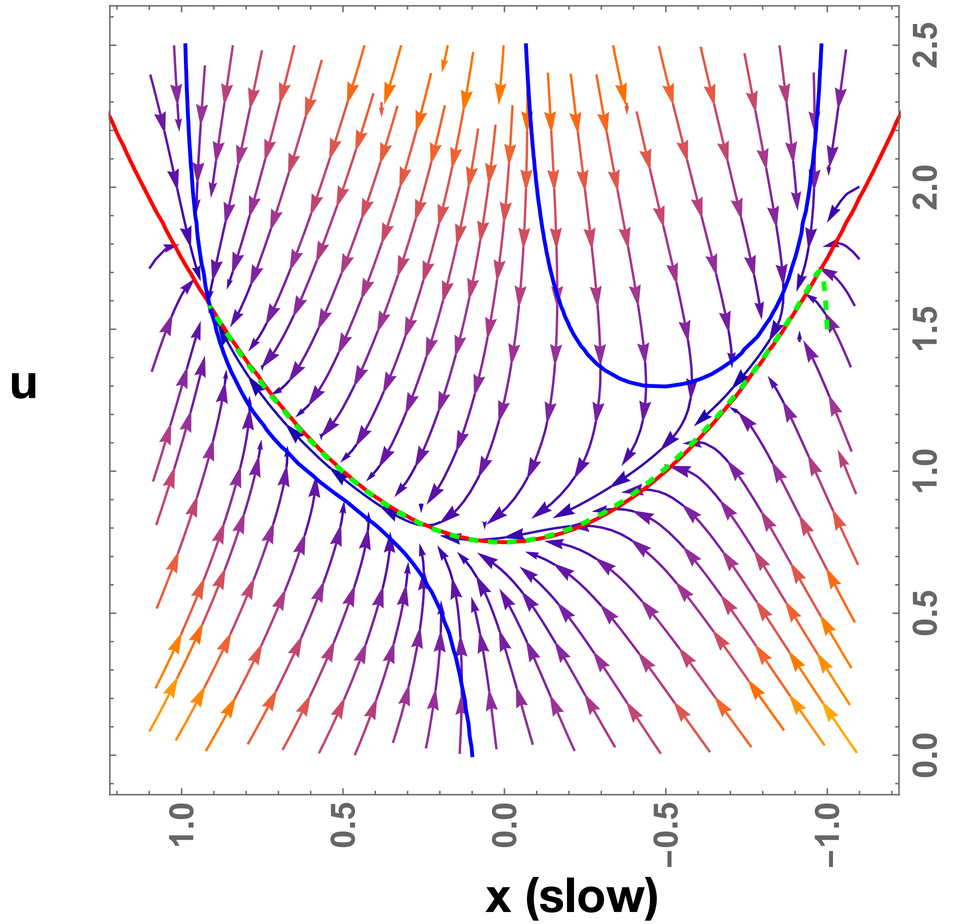
**If the state was not on the fast nullcline, the fast dynamics would take the state back to the fast nullcline**

# Separation of timescales

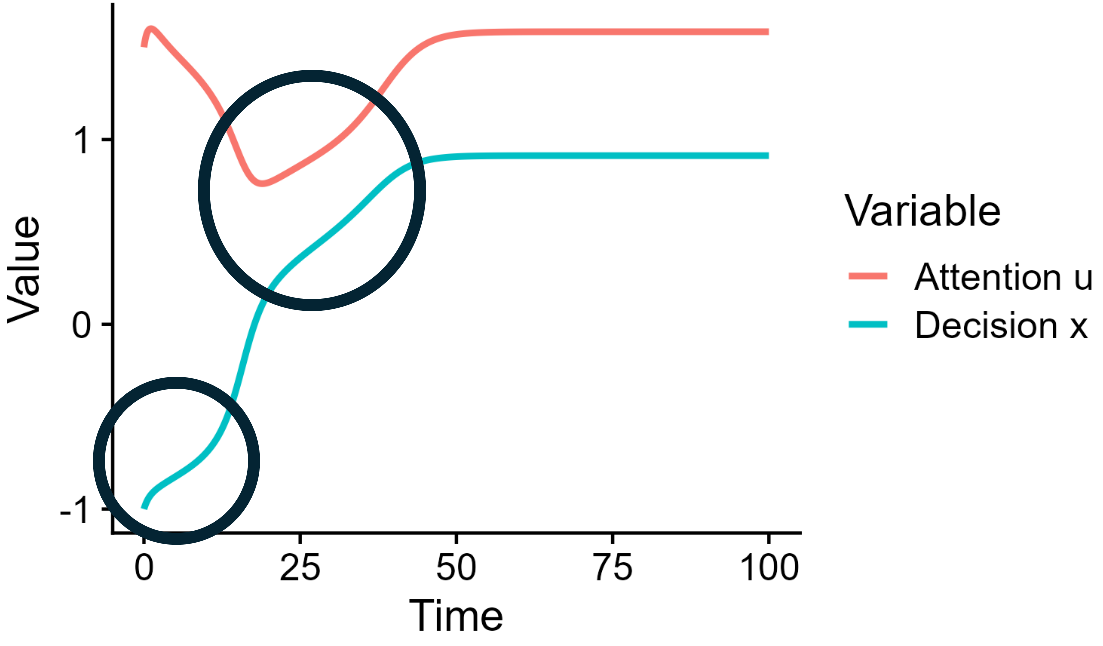
$$1)\ \dot{x} = f(x,u,p)$$
$$2)\ \tau\dot{u} = g(x,u,p)$$

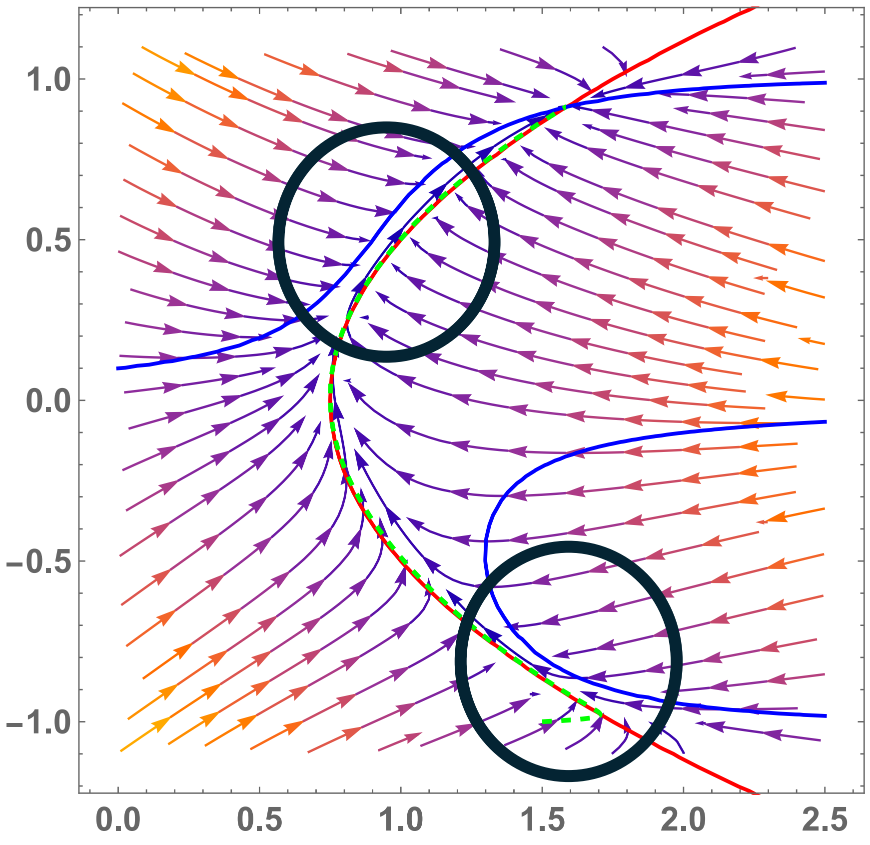
**The dynamics mainly occur along the fast nullcline(s) in the direction indicated by the slow variable**

**If the state was not on the fast nullcline, the fast dynamics would take the state back to the fast nullcline**



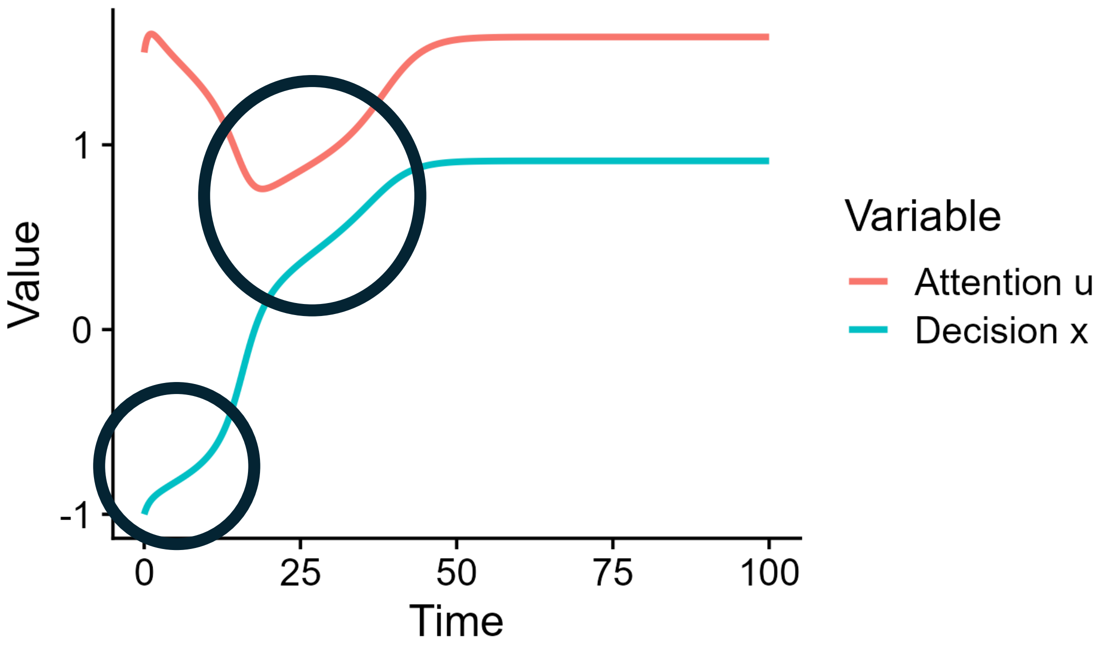
# Ghost equilibria




**When passing near previous global equilibria the dynamics slow down**

# Ghost equilibria

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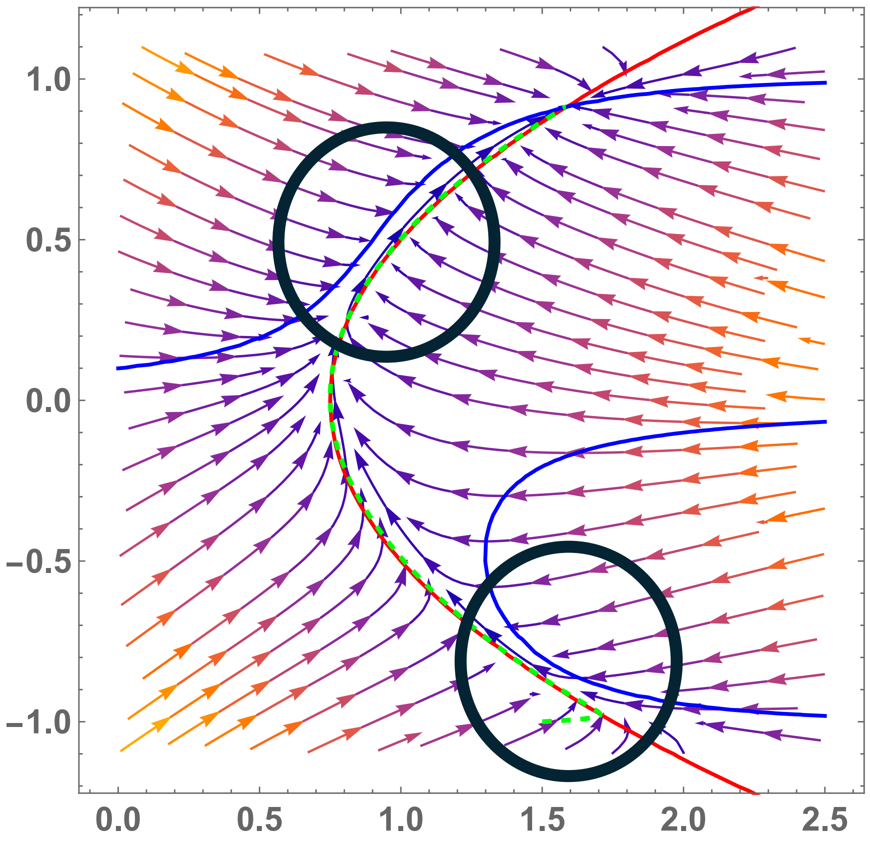


**Dynamics always slow down near the critical point (critical slowing down)**

**Even if the equilibrium disappeared! (the further away, the smaller the effect)**

# Part 5
# Symmetries in dynamical systems

| | | | | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 | 6 |

# Why do we care about symmetries in dynamical systems?

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## 1) Model development

- **All decision-making systems should exhibit some symmetry. This models decision-making in absence of evidence.**

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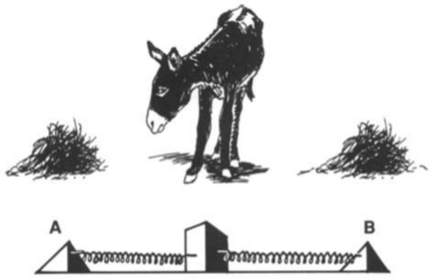
- Problem of symmetry-breaking: problem of deadlock (individual) and consensus formation (collective)

Buridan’s ass



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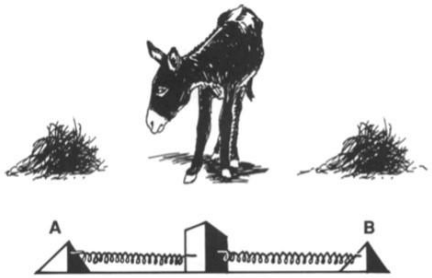
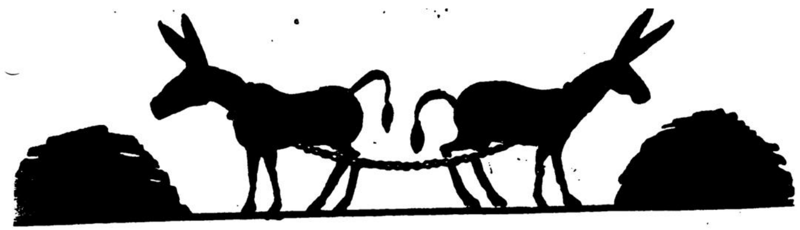



**Consensus formation**



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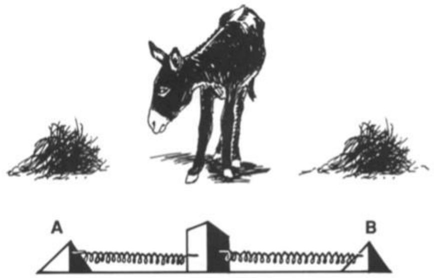
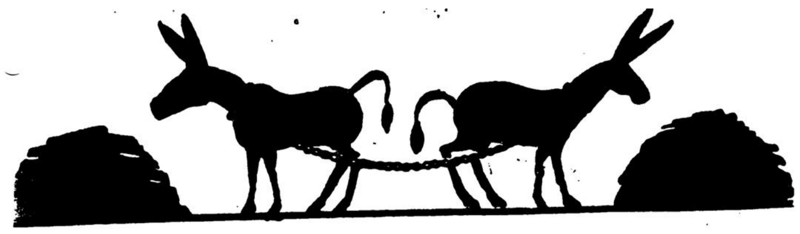



Consensus formation

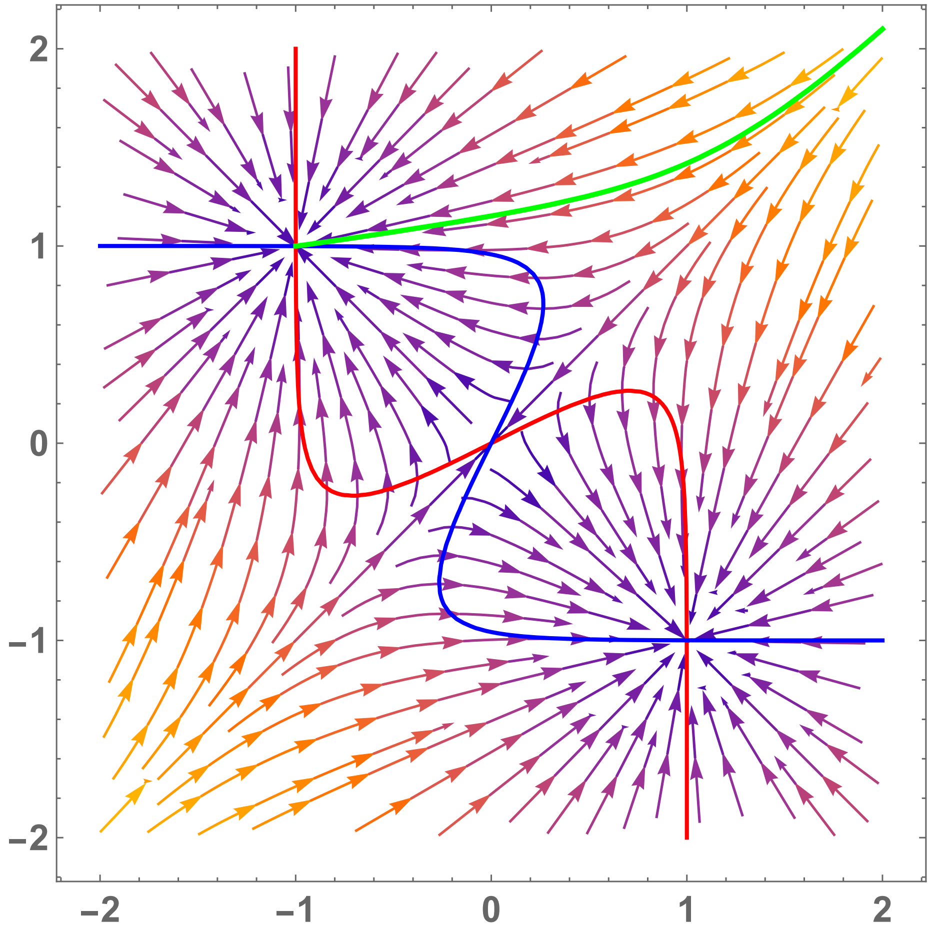


## 2) Model analysis

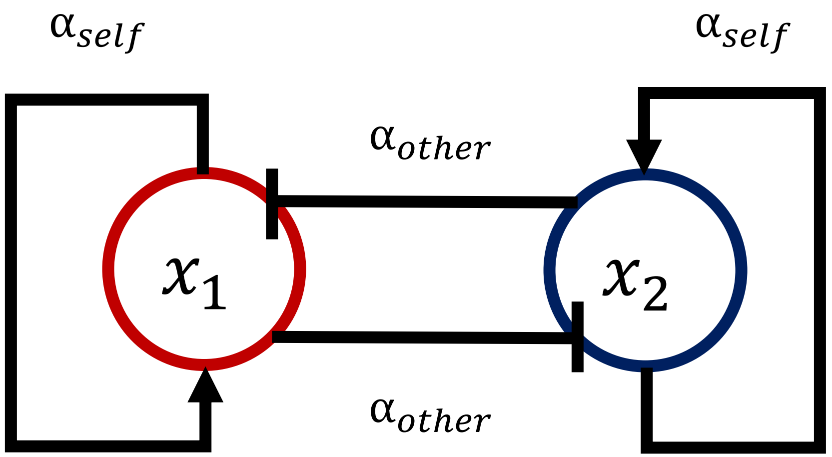
A model with symmetry can be reduced and simplified.
Example: mapping between ring attractor (many dimensions) and drift-diffusion (one dimensional)

# Symmetries in dynamical systems

Equivariance: given a system $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{p})$, if for every solution $\boldsymbol{x}(t)$, $\gamma\boldsymbol{x}(t)$ is also a solution, the system is symmetric with respect to $\gamma$ (for our purposes $\gamma$ is a matrix) (formal definition at the end).




$$\dot{x_1} = -\delta x_1 + \tanh(u(\alpha_{self} x_1 + \alpha_{other} x_2)) \qquad \alpha_{self} > 0$$
$$\dot{x_2} = -\delta x_2 + \tanh(u(\alpha_{self} x_2 + \alpha_{other} x_1)) \qquad \alpha_{other} < 0$$



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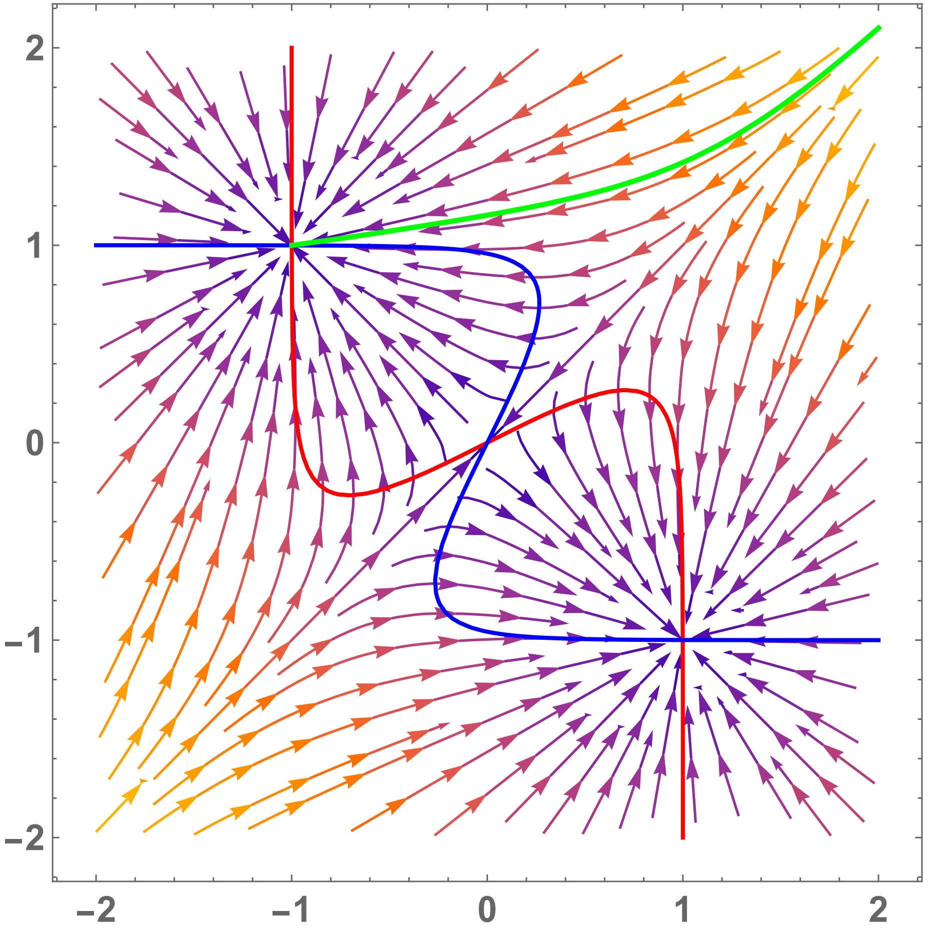


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**Two ways to establish symmetry:**

1) **Verify the criteria** $\gamma f(\boldsymbol{x}, \boldsymbol{p}) = f(\gamma \boldsymbol{x}, \boldsymbol{p})$ Proof: $\dot{x} = f(x), y = \gamma x, \dot{y} = f(\gamma x), \gamma \dot{x} = \gamma f(x) \rightarrow \gamma f(x) = f(\gamma x)$
2) **Just look at the vector field**

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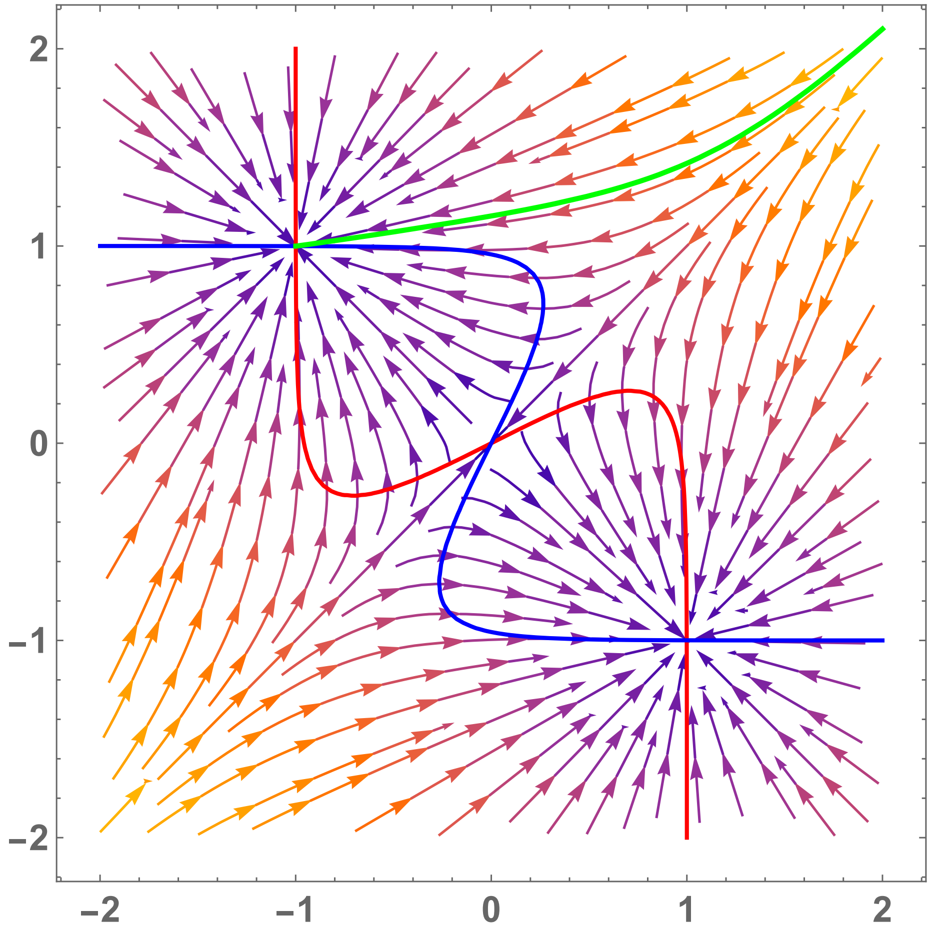


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2) **Just look at the vector field**

**Under what transformation(s) is the system equivariant:**

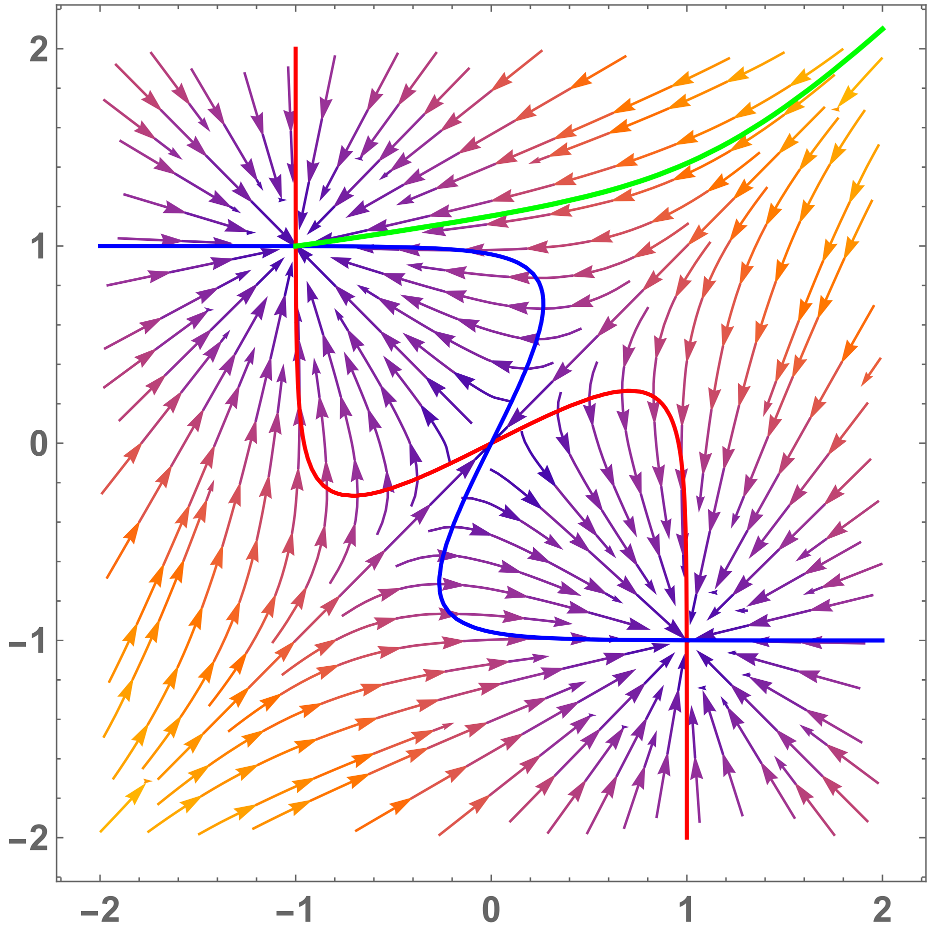
**a)** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **b)** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

**c)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **d)** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

**e)** $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **f)** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

# Symmetries in dynamical systems

Equivariance: given a system $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{p})$, if for every solution $\boldsymbol{x}(t)$, $\gamma\boldsymbol{x}(t)$ is also a solution, the system is symmetric with respect to $\gamma$ (for our purposes $\gamma$ is a matrix) (formal definition at the end).

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$$\dot{x_2} = -\delta x_2 + \tanh(u(\alpha_{self} x_2 + \alpha_{other} x_1)) \qquad \alpha_{other} < 0$$

**Two ways to establish symmetry:**

1) **Verify the criteria** $\gamma f(\boldsymbol{x}, \boldsymbol{p}) = f(\gamma\boldsymbol{x}, \boldsymbol{p})$ Proof: $\dot{x} = f(x), y = \gamma x, \dot{y} = f(\gamma x), \gamma\dot{x} = \gamma f(x) \rightarrow \gamma f(x) = f(\gamma x)$
2) **Just look at the vector field**

**Under what transformation(s) is the system equivariant:**

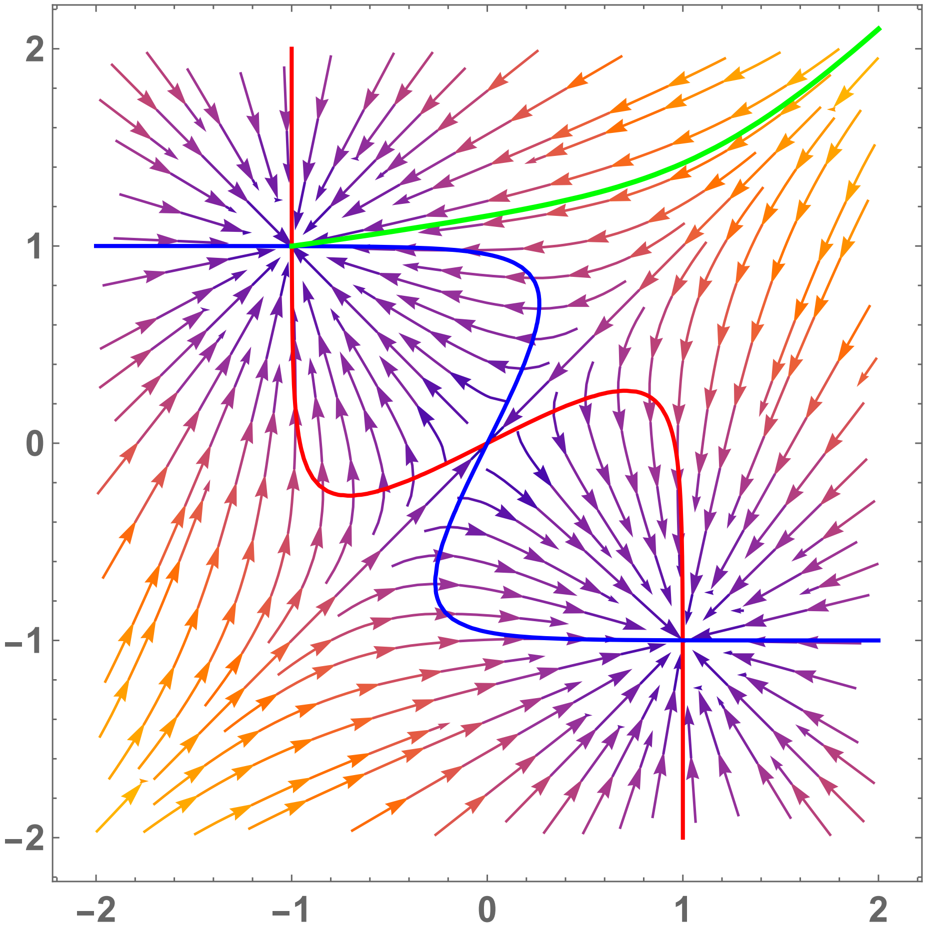
**a)** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **b)** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

**c)** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **d)** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

**e)** $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **f)** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

# Symmetries in dynamical systems

Equivariance: given a system $\dot{\boldsymbol{x}} = f(\boldsymbol{x}, \boldsymbol{p})$, if for every solution $\boldsymbol{x}(t)$, $\gamma\boldsymbol{x}(t)$ is also a solution, the system is symmetric with respect to γ (for our purposes γ is a matrix) (formal definition at the end).

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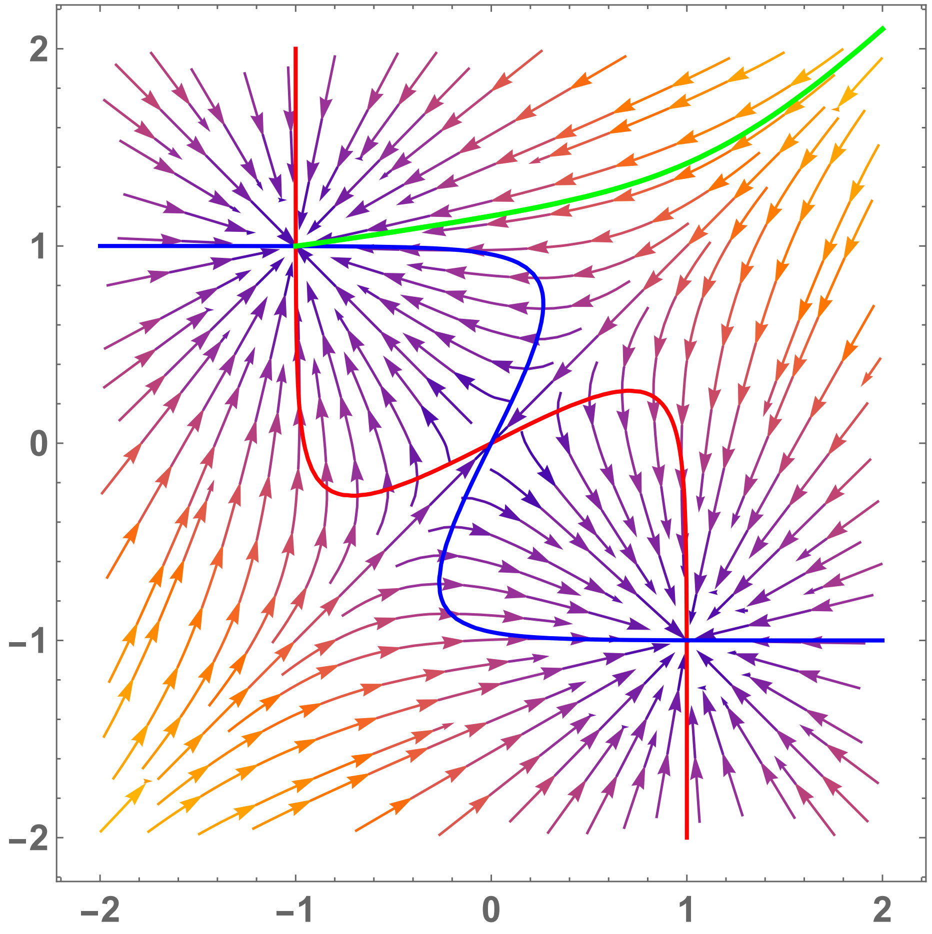


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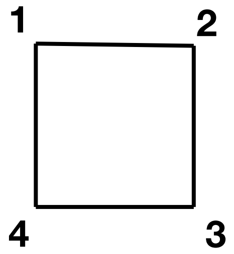
**Is the shape symmetric under $S_4$?**

**a) No**

**b) Yes**



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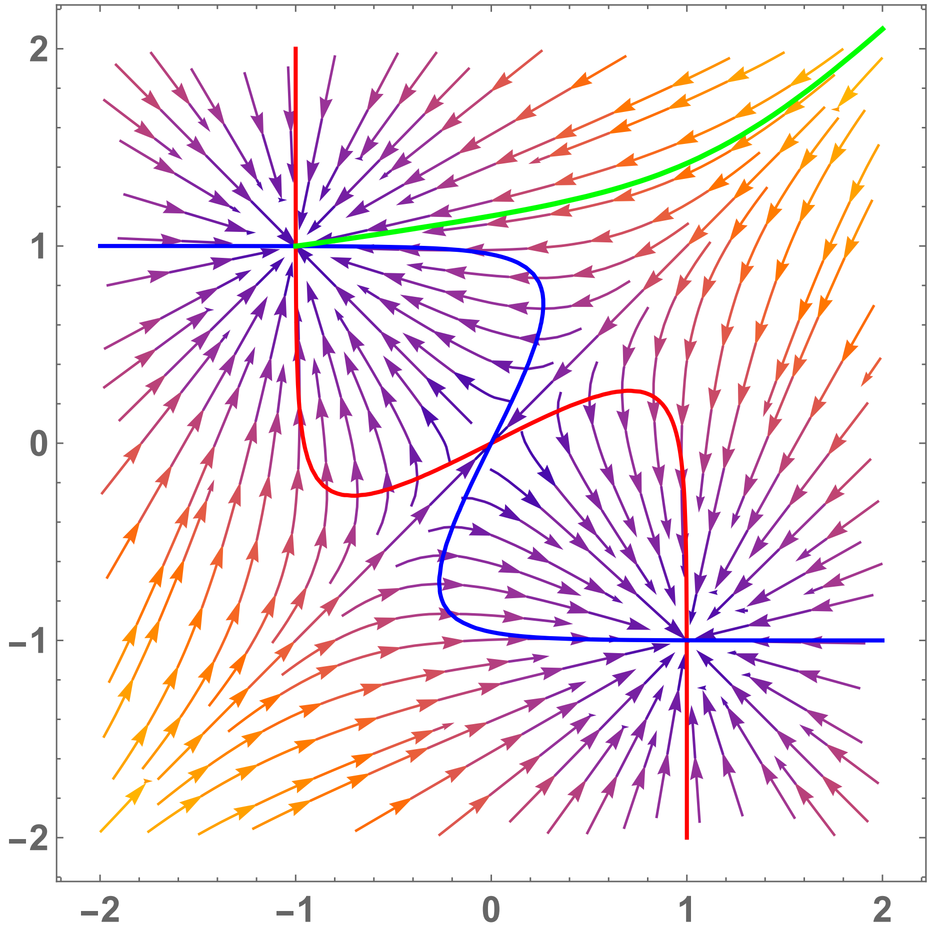


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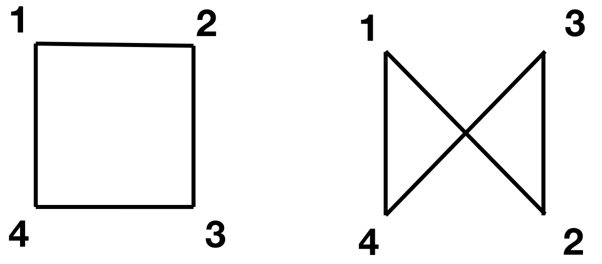

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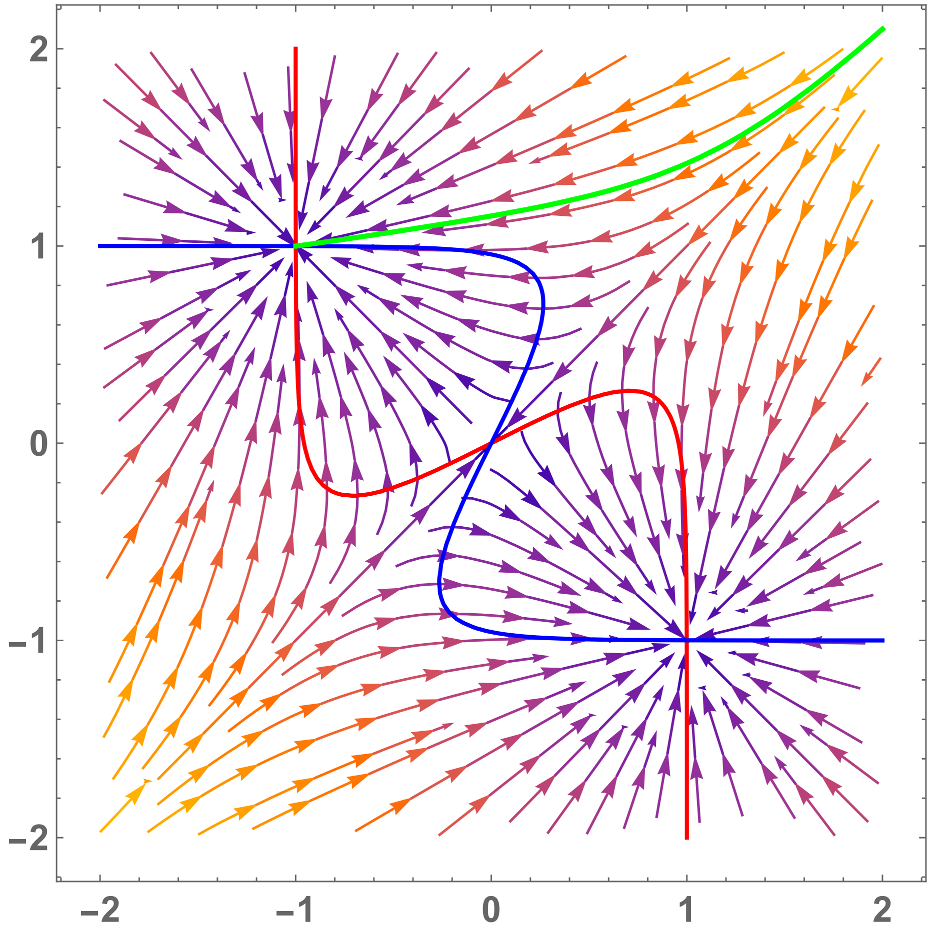


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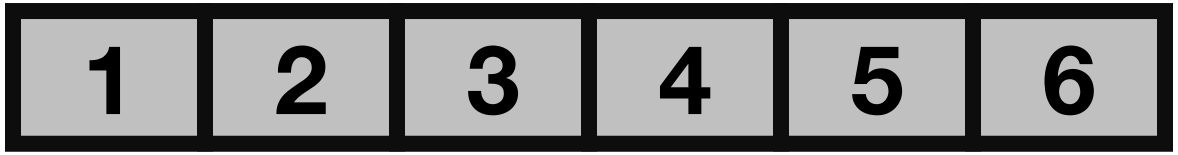
**Symmetric systems can have non–symmetric solutions: this is symmetry breaking**

# Part 6
# The ring attractor



# Ring attractor

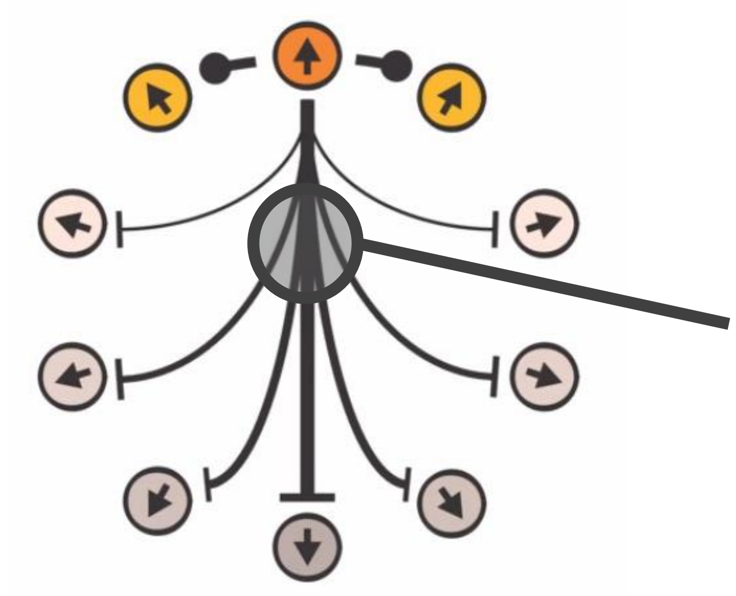
Direction

Interaction's kernel

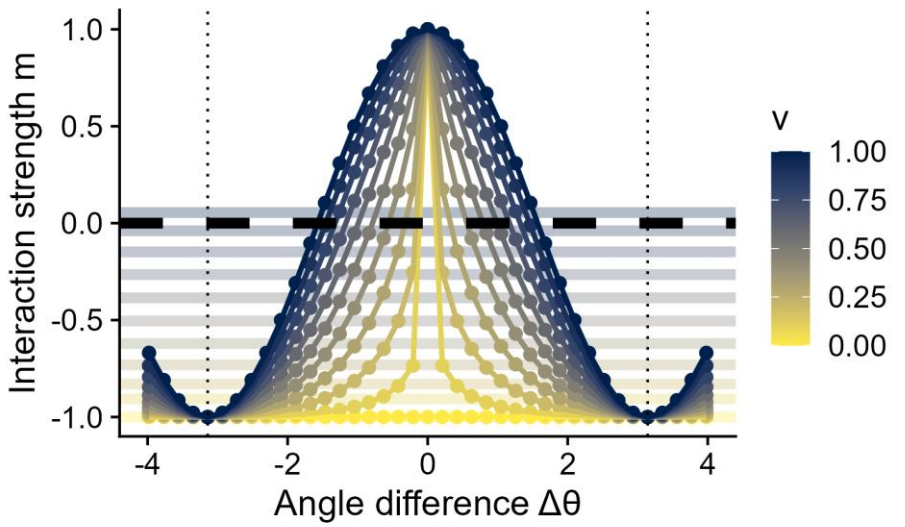
$$\dot{z} = -z + tanh.(u\, \boldsymbol{M}\, z + \boldsymbol{b})$$

Network coupling u

Targets' inputs



Short-range excitation and long-range inhibition



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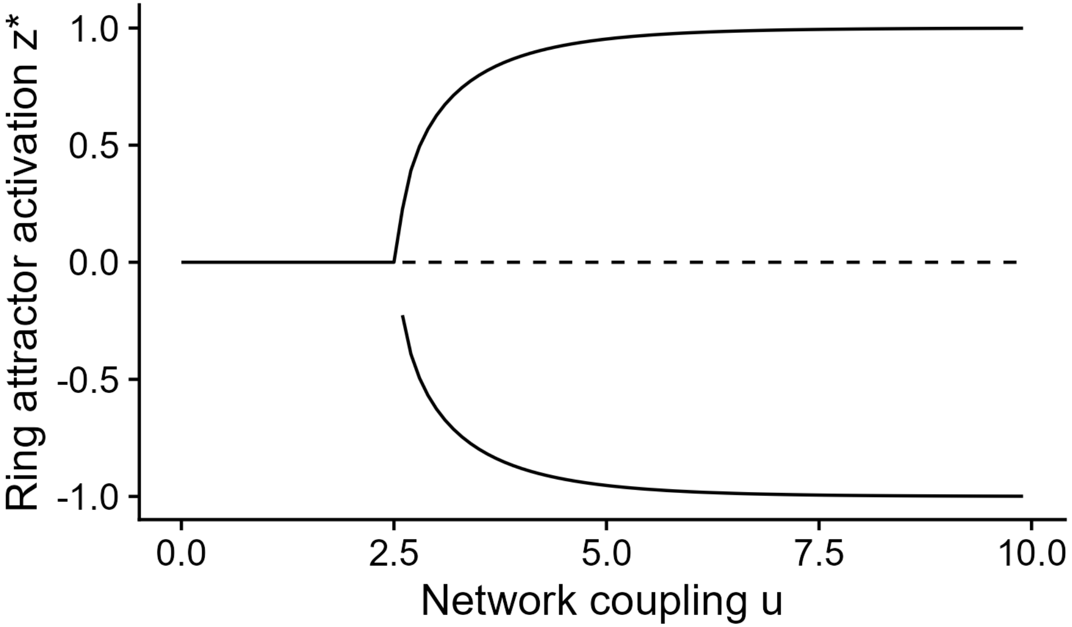
Network coupling u

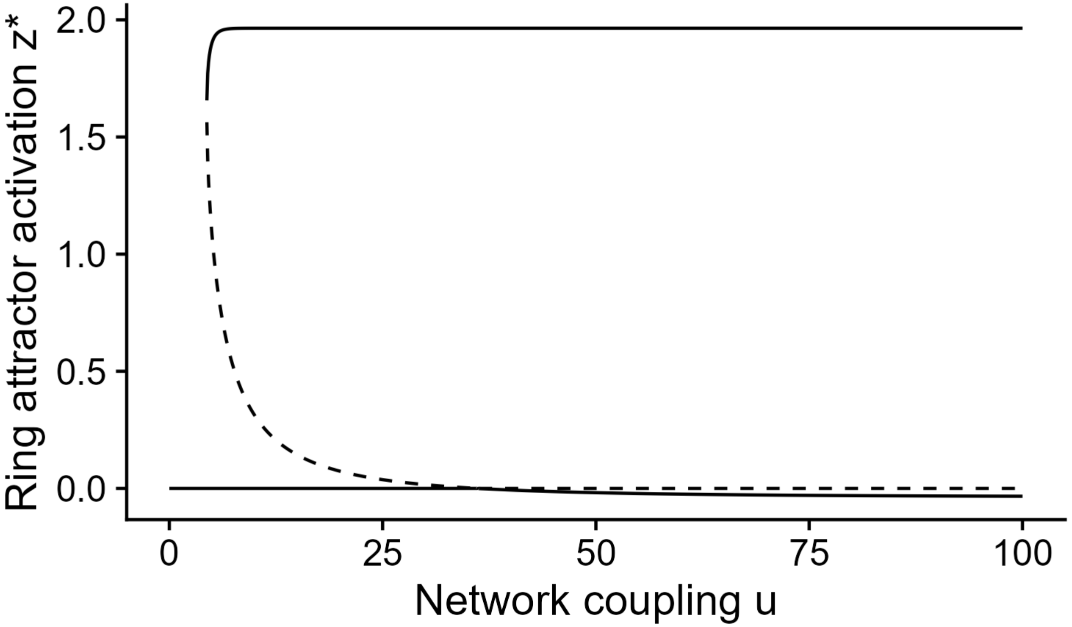
Targets' inputs

**Multi-option decision-making**

**Circular symmetry $C_n$**

# Reduced model dynamic



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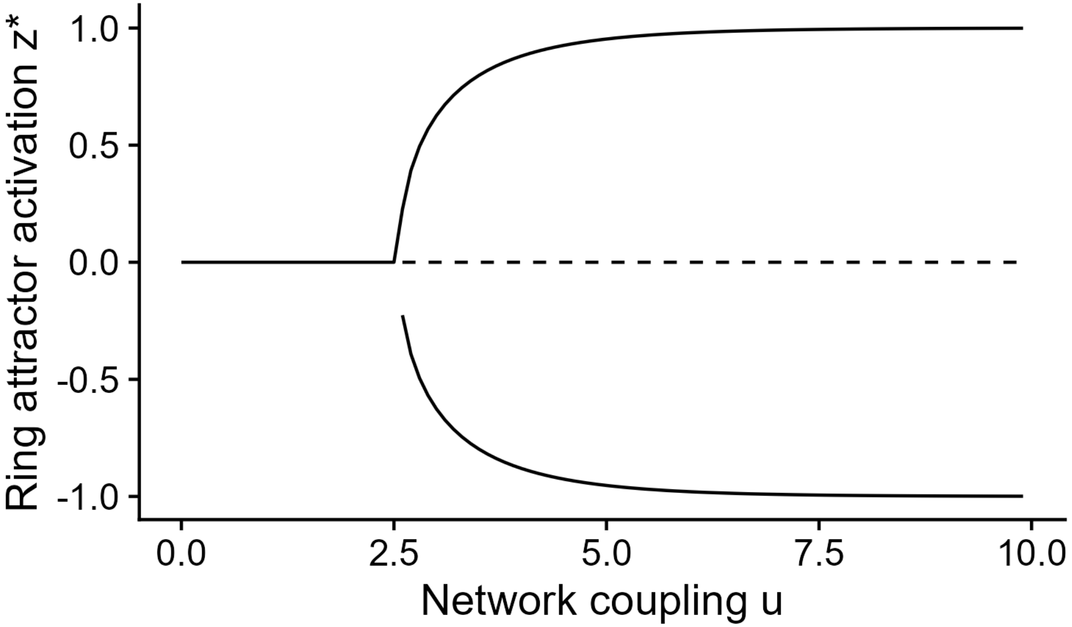
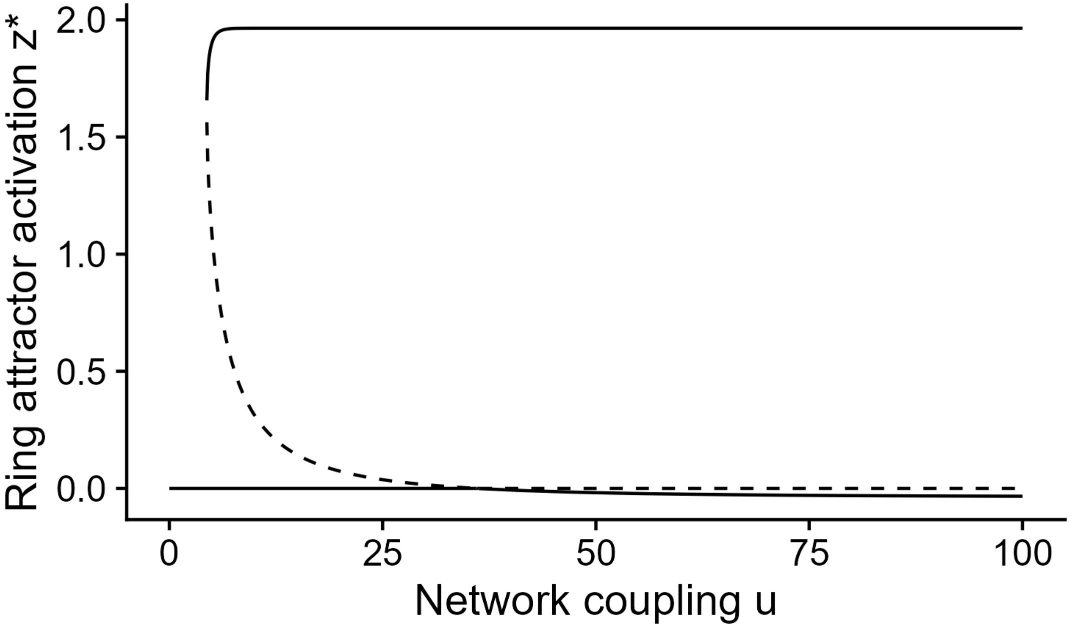
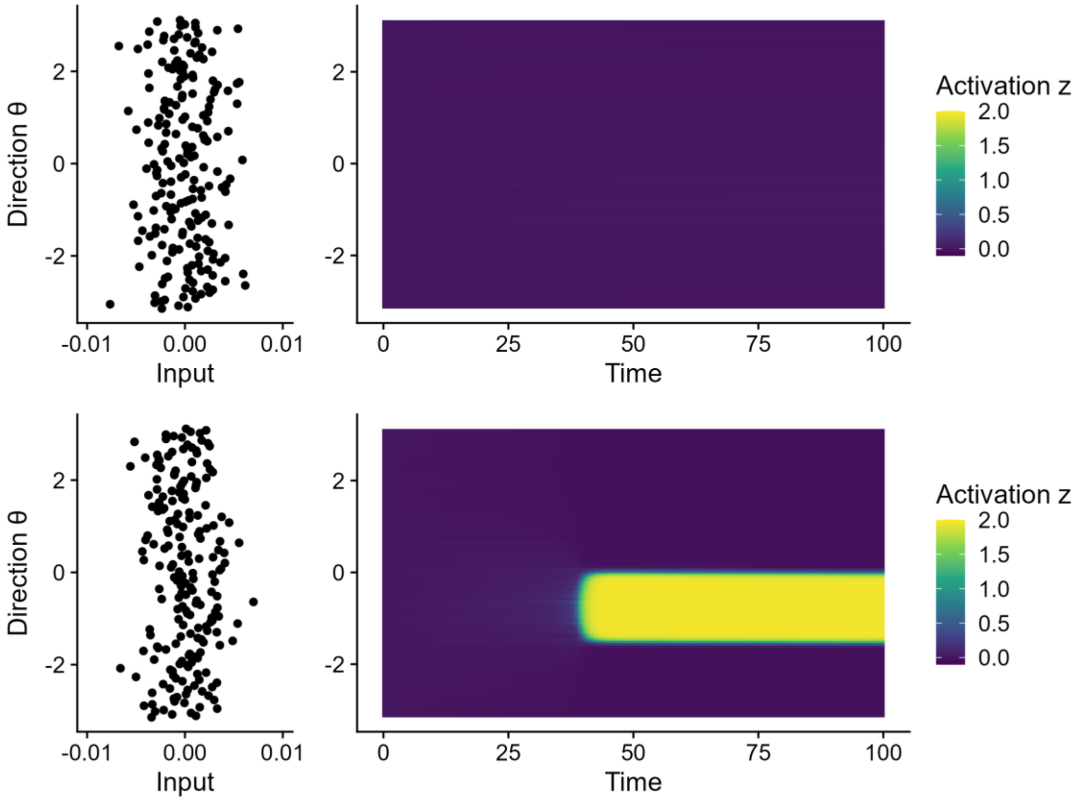



# What bifurcations are these?

# Selective ultra-sensitivity

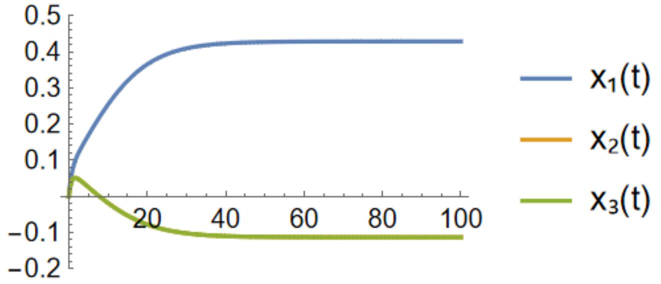


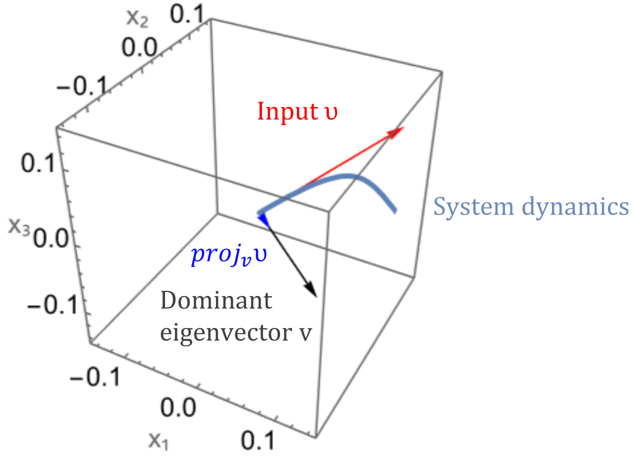

**A ring attractor showing off his Fourier glasses**

Mathematica notebook for designing input in frequency domain

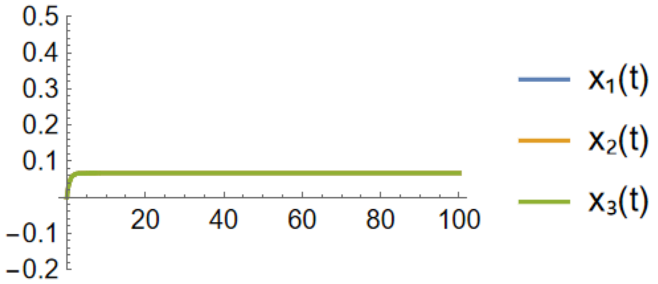
# Selective ultra-sensitivity

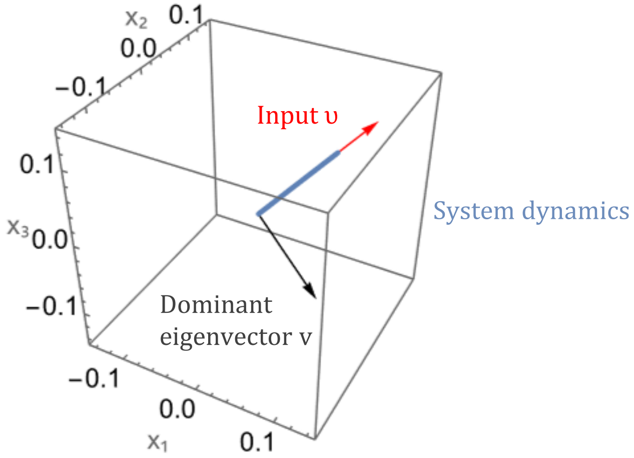
## Partially aligned input




## Misaligned input

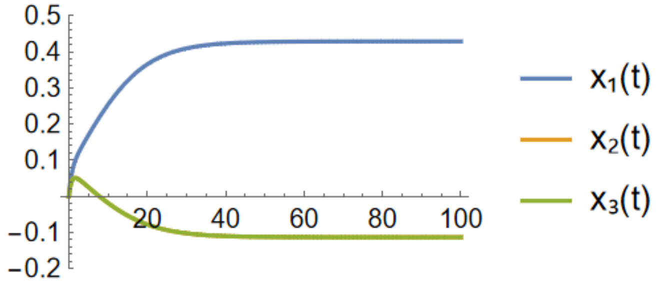


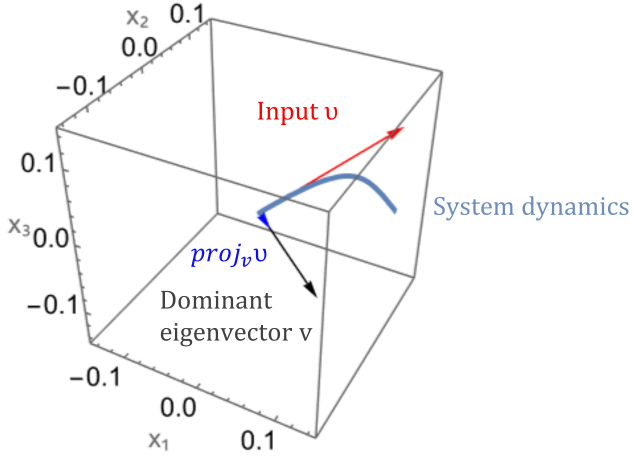




- **At the critical point, a decision-maker responds to inputs that have a non-zero component when projected along the center manifold (one of the dynamics' dominant eigenvector)**

# Selective ultra-sensitivity

### Partially aligned input

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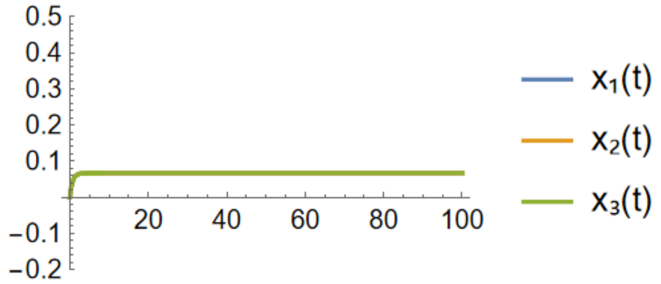


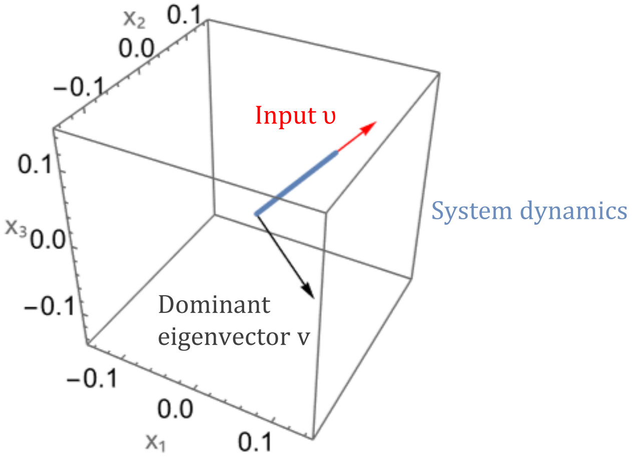


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- **This is linked to infinite susceptibility**
- **For random inputs, dimensionality decreases probability of alignment**

# Selective ultra-sensitivity

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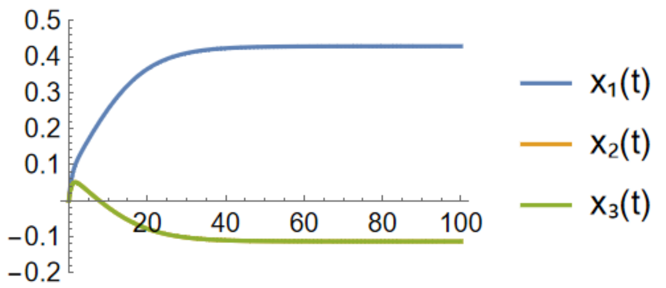


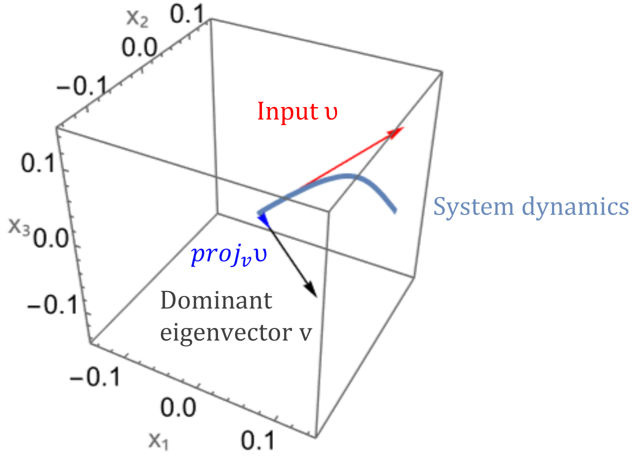


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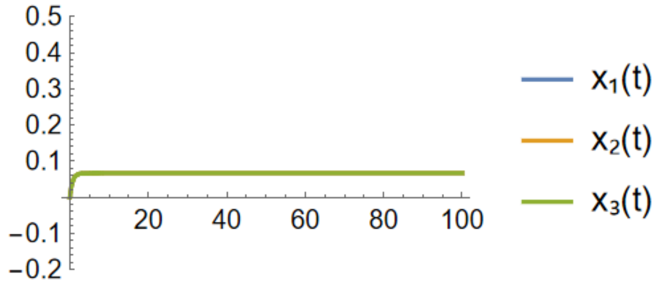


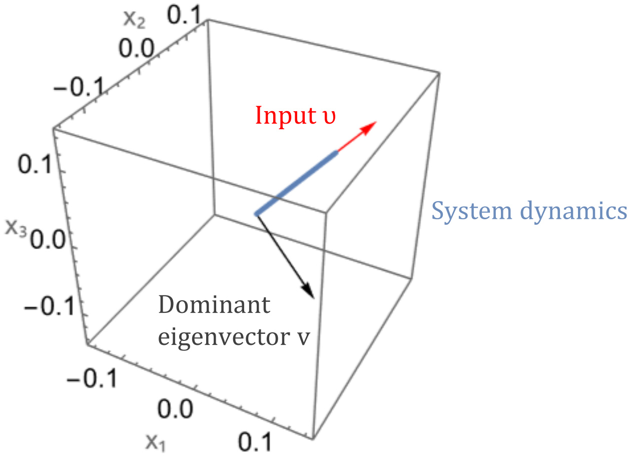


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- **This is linked to infinite susceptibility**
- **For random inputs, dimensionality decreases probability of alignment**
- **For all circulant unimodal kernels, the center manifold is spanned by the Fourier basis of the first harmonic, which modes are the dominant eigenvalues. Every linear combination of the Fourier basis of a given harmonic is also one of its corresponding eigenvector (which preserves circulant symmetry of opinion formation by encoding phase). The center manifold for the ring attractor is a plane (2-d)**

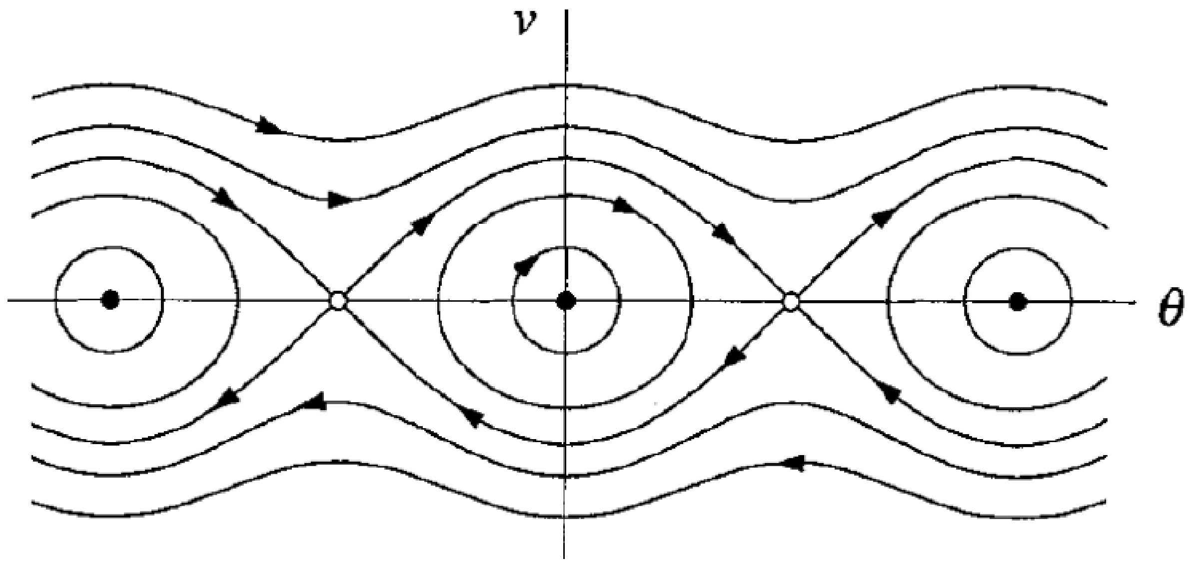
# Caution

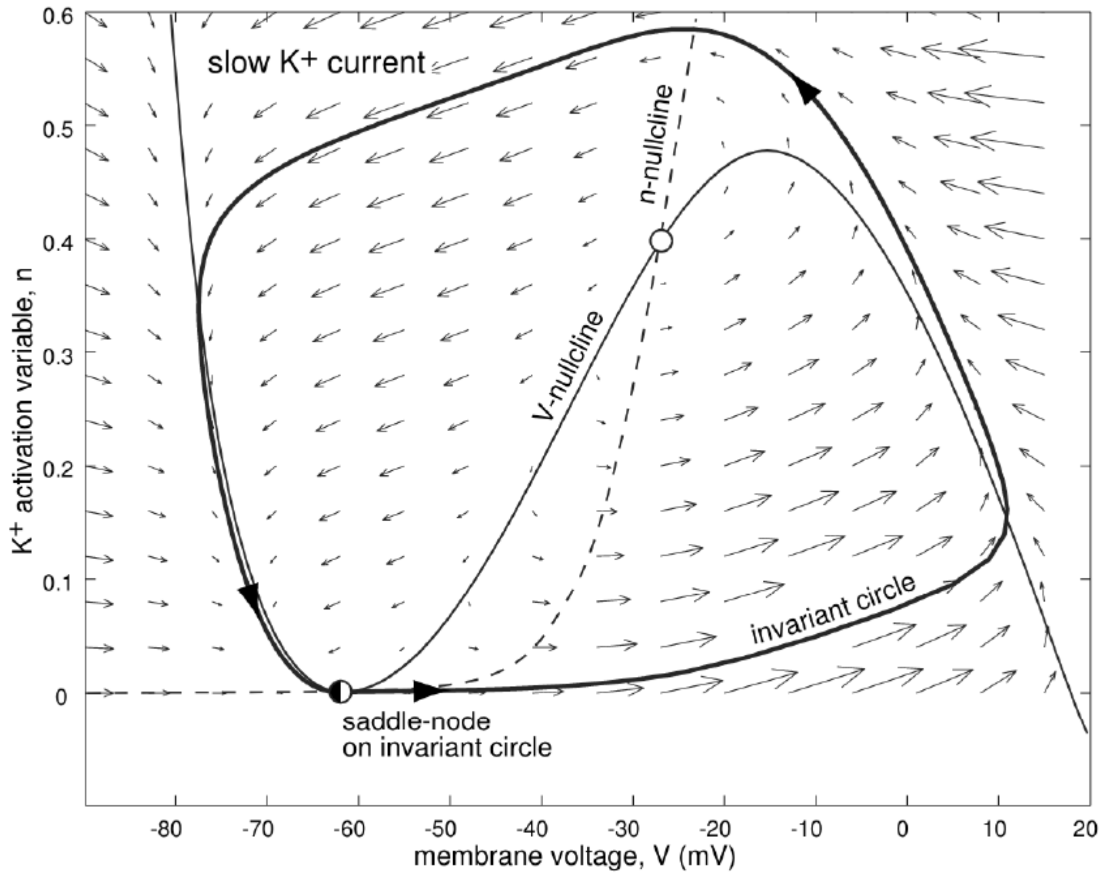
When linearization fails $\mathbf{nullity}(J(\boldsymbol{x}^*, \boldsymbol{p}^*)) \neq 0$, different definitions of stability (Lyapunov) can be used and strange things can happen: a point can be *stable but not attractive* (e.g., Lotka-Volterra predator-prey, undampened pendulum) or *attractive but unstable* (e.g., in excitable systems). See end if curious

Lyapunov stable: if for every distance ε >0 from the equilibrium there exists a starting condition within δ>0 for which the trajectory remains within ε, the equilibrium is Lyapunov stable

**Why is δ needed in the definition?**

**Which points are stable but not attractive and attractive but not stable?**



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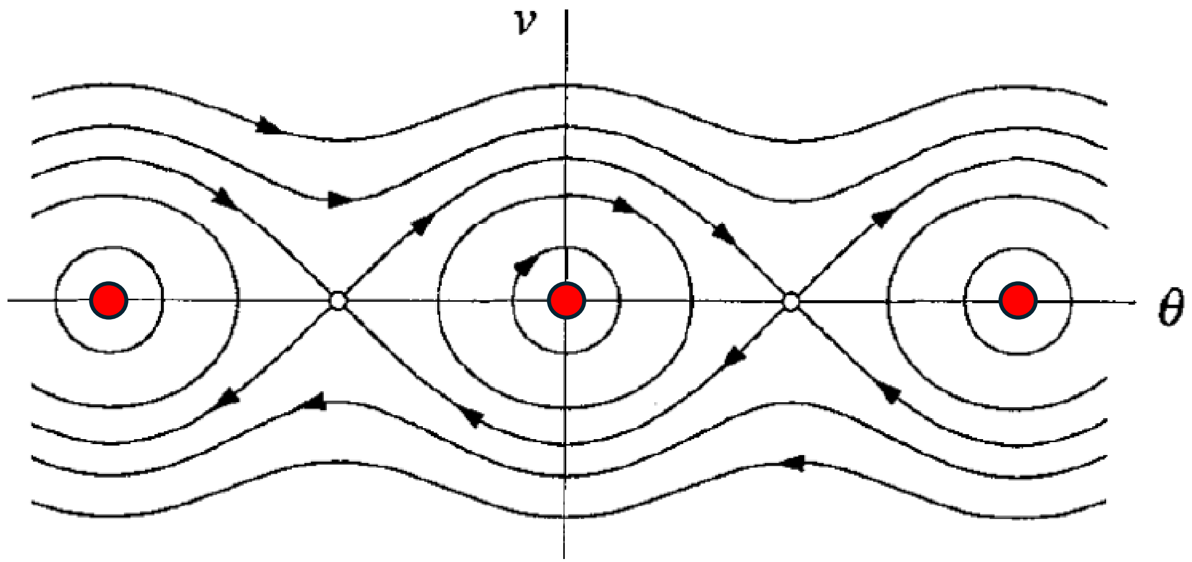
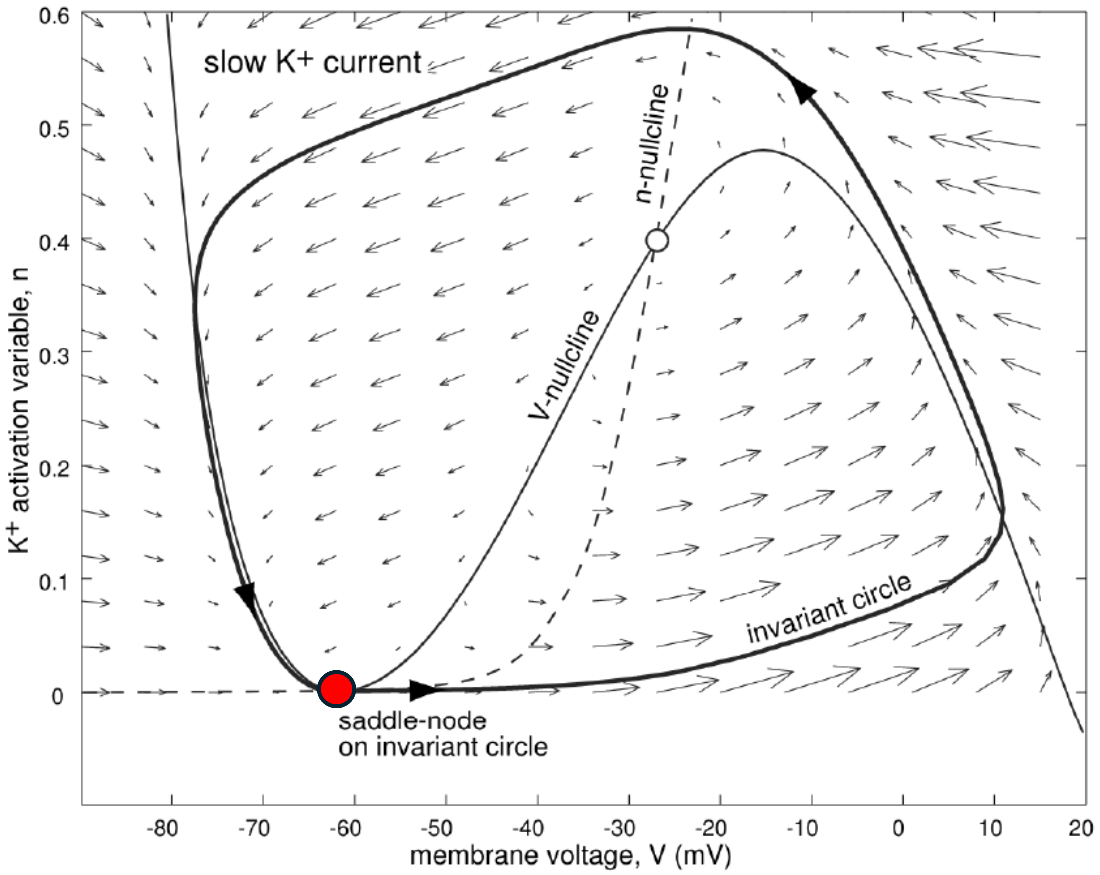



**Lyapunov unstable but attractive**



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