Ahora, busquemos simular una normal N(0,1) con el método de aceptación-rechazo con distribución candidata g una distribución doble exponencial L(alpha) con densidad

$\longrightarrow f$

$$g(x|\alpha) = \left(\frac{\alpha}{2}\right) \exp(-\alpha|x|), \quad x \in \mathbb{R}$$

$$\frac{f(x)}{g(x)} = \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}}{\left(\frac{\alpha}{2}\right)\exp(-\alpha|x|)} = \frac{\sqrt{2}}{\sqrt{\pi}} \alpha^{-1} \exp\left(-\left(\frac{x^2}{2} - \alpha|x|\right)\right)$$

$$l(x) = \ln\left(\frac{\sqrt{2}}{\sqrt{\pi}}\right) - \ln(\alpha) - \left(\frac{x^2}{2} - \alpha|x|\right)$$

Si $x \geq 0$,

$$f(x) = \ln\left(\frac{\sqrt{2}}{\sqrt{\pi}}\right) - \ln(\alpha) - \left(\frac{x^2}{2} - \alpha x\right)$$

$$f'(x) = -(x - \alpha)$$

$$f'(x) = 0 \iff x = \alpha$$

Si $x \leq 0$,

$$f(x) = \ln\left(\frac{\sqrt{2}}{\sqrt{\pi}}\right) - \ln(\alpha) - \left(\frac{x^2}{2} + \alpha x\right)$$

$$f'(x) = -(x+\alpha)$$

$$f'(x) = 0 \iff x = -\alpha$$

$$f''(x) = -1 < 0$$

$$\max_{x \in \mathbb{R}} \frac{f(x)}{g(x)} = \frac{f(\alpha)}{g(\alpha)} = \frac{f(-\alpha)}{g(-\alpha)}$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \alpha^{-1} \exp\left(-\left(\frac{\alpha^2}{2} - \alpha^2\right)\right)$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \alpha^{-1} \exp\left(\frac{\alpha^2}{2}\right)$$

$$= M(\alpha)$$

$$M(\alpha) = \frac{\sqrt{2}}{\sqrt{\pi}} \alpha^{-1} \exp\left(\frac{\alpha^2}{2}\right)$$

$$l(M(\alpha)) = \ln\left(\frac{\sqrt{2}}{\sqrt{\pi}}\right) - \ln(\alpha) + \frac{\alpha^2}{2}$$

$$l'(M(\alpha)) = -\frac{1}{\alpha} + \alpha$$

$$l'(M(\alpha)) = 0 \iff \alpha = \frac{1}{\alpha}$$

$$\iff \alpha^2 = 1$$

$$\iff \alpha = \pm 1$$

$$l''(M(\alpha)) = \frac{1}{\alpha^2} + 1 > 0$$

$\therefore$ $M(\alpha)$ es la mínima cota superior, con $\alpha = 1$

Muestreador de Gibbs de dos etapas

Sean X, Y variables aleatorias con densidad conjunta $f(x, y)$ y condicionales $f_{Y|X}$, $f_{X|Y}$. El muestreador de Gibbs de dos etapas genera la cadena de Markov (X_t, Y_t) de la siguiente forma:

Toma $X_0 = x_0$.

Dato $t \in \mathbb{N}$, genera

1) $Y_t \sim f_{Y|X}(\cdot|x_{t-1})$

2) $X_t \sim f_{X|Y}(\cdot|y_t)$

$f_{Y|X}$

Ejemplo 1: Consideremos el modelo normal bivariado

$$(x,y) \sim N_2\left(0, \underbrace{\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}}_{=\Sigma}\right)$$

$$f(x,y) = (2\pi)^{-\frac{k}{2}} \det(\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}(\bar{x}-\mu)^T \Sigma^{-1} (\bar{x}-\mu)}, \quad k=2$$

$$\mu = 0$$

$$\left(\begin{array}{cc|cc} 1 & \rho & 1 & 0 \\ \rho & 1 & 0 & 1 \end{array}\right) \Rightarrow \left(\begin{array}{cc|cc} 1 & \rho & 1 & 0 \\ 0 & 1-\rho^2 & -\rho & 1 \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & \rho & 1 & 0 \\ 0 & 1 & \frac{-\rho}{(1-\rho^2)} & \frac{1}{(1-\rho^2)} \end{array}\right)$$

$$\Rightarrow \left(\begin{array}{cc|cc} 1 & 0 & 1+\frac{\rho^2}{(1-\rho^2)} & \frac{-\rho}{(1-\rho^2)} \\ 0 & 1 & \frac{-\rho}{(1-\rho^2)} & \frac{1}{(1-\rho^2)} \end{array}\right)$$

$$\therefore \quad \Sigma^{-1} = \frac{1}{(1-\rho^2)} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix}$$

$\bar{x} = (x, y)$

$$(\bar{x} - \mu)^T \Sigma^{-1} (\bar{x} - \mu)$$

$$= (x, y) \frac{1}{(1-\rho^2)} \begin{pmatrix} 1 & -\rho \\ -\rho & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{(1-\rho^2)} \left((x - y\rho,\ y - x\rho) \begin{pmatrix} x \\ y \end{pmatrix} \right)$$

$$= \frac{1}{(1-\rho^2)} \left(x^2 - xy\rho + y^2 - xy\rho \right)$$

$$= \frac{1}{(1-\rho^2)} \left(x^2 - 2xy\rho + y^2 \right)$$

• Si x es constante:

$$(x^2 - 2xy\rho + y^2) = x^2 - 2xy\rho + y^2 + x^2\rho^2 - x^2\rho^2$$

$$= (y - x\rho)^2 + (1-\rho^2)x^2$$

• Si y es constante:

$$\left(x^2 - 2xy\rho + y^2\right) = x^2 - 2xy\rho + y^2 + y^2\rho^2 - y^2\rho^2$$

$$= (x - y\rho)^2 + (1-\rho^2)y^2$$

$$= \frac{1}{(1-\rho^2)}\left(x^2 - 2xy\rho + y^2\right)$$

• Si x es constante:

$$f(x,y) \propto e^{-\frac{1}{2}\bar{x}^T \Sigma^{-1} \bar{x}}$$

$$= e^{-\frac{1}{2(1-\rho^2)}\left[(y-\rho x)^2 + (1-\rho^2)x^2\right]}$$

$$\propto e^{-\frac{(y-\rho x)^2}{2(1-\rho^2)}}$$

$$\therefore \quad Y|X \sim N(\rho x, 1-\rho^2)$$

• Si y es constante:

$$f(x,y) \propto e^{-\frac{1}{2} \bar{x}^T \Sigma^{-1} \bar{x}}$$

$$= e^{-\frac{1}{2(1-\rho^2)}\left[(x-\rho y)^2 + (1-\rho^2) y^2\right]}$$

$$\propto e^{-\frac{(x-\rho y)^2}{2(1-\rho^2)}}$$

$$\therefore \quad Y|X \sim N(\rho y, 1-\rho^2)$$

Ejemplo 2: Consideremos el par de distribuciones

$$X|\theta \sim Bin(n, \theta)$$

$$\theta \sim Beta(a, b)$$

$$f(x,\theta) = f_{X|\theta}(x|\theta)\, f(\theta)$$

$$= \left[\binom{n}{x} \theta^{x} (1-\theta)^{n-x}\right] \left[\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1} (1-\theta)^{b-1}\right]$$

$$= \binom{n}{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{x+a-1} (1-\theta)^{n-x+b-1}$$

La distribución condicional correspondiente a X | theta ya está dada, mientras que theta | X lo podemos calcular de la siguiente forma: $\theta \mid x$

$$f(x,\theta) \propto \theta^{\overbrace{x+a-1}^{}} (1-\theta)^{\overbrace{n-x+b-1}^{}}$$

$$\therefore \quad \theta \mid x \sim \text{Beta}(x+a,\ n-x+b)$$

Finalmente, calculemos la distribución X para saber a
dónde converge la respectiva cadena

$$f_X(x) = \int f(x,\theta)\, d\theta$$

$$= \int \binom{n}{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{x+a-1} (1-\theta)^{n-x+b-1}\, d\theta$$

$$= \binom{n}{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int \theta^{x+a-1} (1-\theta)^{n-x+b-1}\, d\theta$$

$$= \binom{n}{x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \frac{\Gamma(x+a)\Gamma(n-x+b)}{\Gamma(a+b+n)} \underbrace{\int \frac{\Gamma(a+b+n)}{\Gamma(x+a)\Gamma(n-x+b)} \theta^{x+a-1} (1-\theta)^{n-x+b-1}\, d\theta}_{1}$$