Def: Sea $\mathbb{P}$ una dist. de probabilidad asociada a una v. a. Z. Ent., $\mathbb{P}$ es invariante bajo el mapeo f si se cumple que

$$\mathbb{P}(Z \leq x) = \mathbb{P}(f(Z) \leq x)$$

Sea $U \sim U(0,1)$, y sea $f(x) = 1-x$

Ent. $\mathbb{P}(f(U) \leq x) = \mathbb{P}(1-U \leq x)$

$= \mathbb{P}(1-x \leq U) = 1 - \mathbb{P}(U \leq 1-x)$

$= 1-(1-x) = x$

$= \mathbb{P}(U \leq x)$

Sea $r \in (0,4]$, ent. definimos al mapeo logístico como

$$f(x) = rx(1-x)$$

Sea Z v.a. t.q. $Z \sim \mathcal{L}(g)$ y es invariante bajo f

a) $F(x) = \mathbb{P}(Z \leq x)$

$$= \mathbb{P}\left[X \leq \frac{1}{2}\left(1 - \sqrt{1 - \frac{4x}{r}}\right)\right]$$

$$+ \mathbb{P}\left[X \geq \frac{1}{2}\left(1 + \sqrt{1 - \frac{4x}{r}}\right)\right]$$

$$\mathbb{P}(Z \le x) = \mathbb{P}(f(Z) \le x) =$$

$$\mathbb{P}(rZ(1-Z) \le x)$$

$$= \mathbb{P}\left((Z - Z^2) \le \frac{x}{r} \right)$$

$$= \mathbb{P}\left(Z^2 - Z \ge -\frac{x}{r} \right)$$

$$= \mathbb{P}\left(Z^2 - Z + \frac{1}{4} \ge \frac{1}{4} - \frac{x}{r} \right)$$

$$= P\left(\left(Z - \frac{1}{2}\right)^2 \geq \frac{1}{4} - \frac{x}{r} \right)$$

$$= P\left(\left|Z - \frac{1}{2}\right| \geq \sqrt{\frac{1}{4} - \frac{x}{r}} \right)$$

$$= P\left[\left[Z \leq \frac{1}{2}\left(1 - \sqrt{\frac{1}{4} - \frac{x}{r}}\right) \right] \cup \left[Z \geq \frac{1}{2}\left(1 + \sqrt{\frac{1}{4} - \frac{x}{r}}\right) \right] \right]$$

$$= P\left(Z \le \frac{1}{2}\left(1-\sqrt{1-\frac{4x}{r}}\right)\right)$$

$$+ P\left(Z \ge \frac{1}{2}\left(1+\sqrt{1-\frac{4x}{r}}\right)\right)$$

$$\therefore F(x) = P(Z \le x) = P(f(Z) \le x) = P\left(Z \le \frac{1}{2}\left(1-\sqrt{1-\frac{4x}{r}}\right)\right)$$

$$+ P\left(Z \ge \frac{1}{2}\left(1+\sqrt{1-\frac{4x}{r}}\right)\right)$$

$$= F\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4x}{r}}\right)\right)$$

$$+ 1 - F\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{r}}\right)\right)$$

P.D. $g(x) = \frac{1}{r\sqrt{1-\frac{4x}{r}}}\left[g\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4x}{r}}\right)\right)+g\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{r}}\right)\right)\right]$

Dem. Dado a)

$$g(x) = F'(x) = F'\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4x}{r}}\right)\right)\left(-\frac{1}{4}\frac{-\frac{4}{r}}{\sqrt{1-\frac{4x}{r}}}\right)$$

$$-F'\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{r}}\right)\right)\left(-\frac{1}{4}\frac{\frac{4}{r}}{\sqrt{1-\frac{4x}{r}}}\right)$$

$$= \frac{4}{4r\sqrt{1-\frac{4x}{r}}}\left[g\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4x}{r}}\right)\right)+g\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{r}}\right)\right)\right]$$

$$= \frac{1}{r\sqrt{1-\frac{4x}{r}}}\left[g\left(\frac{1}{2}\left(1-\sqrt{1-\frac{4x}{r}}\right)\right)+g\left(\frac{1}{2}\left(1+\sqrt{1-\frac{4x}{r}}\right)\right)\right]$$

Si $r=4$, ent

$$g(x) = \frac{1}{4\sqrt{1-x}}\left[g\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right) + g\left(\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)\right]$$

Ahora, si $X \sim Beta\left(\frac{1}{2}, \frac{1}{2}\right)$, la densidad de X es

$$g(x) = \frac{\Gamma(1)}{\Gamma\left(\frac{1}{2}\right)^2} x^{\frac{1}{2}-1} (1-x)^{\frac{1}{2}-1}$$

$$= \frac{1}{\pi\sqrt{x(1-x)}} \qquad 0 \le x \le 1$$

$$g\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)$$

$$= \frac{1}{\pi}\left(\left[\frac{1}{2}\left(1-\sqrt{1-x}\right)\right]\left[\frac{1}{2}\left(1+\sqrt{1-x}\right)\right]\right)^{-1/2}$$

$$= \frac{1}{\pi}\,\frac{2}{\sqrt{\left(1-\sqrt{1-x}\right)\left(1+\sqrt{1-x}\right)}}$$

$$= \frac{1}{\pi}\,\frac{2}{\sqrt{x}}$$

De manera análoga,

$$g\left(\frac{1}{2}\left(1+\sqrt{1-x}\right)\right) = \frac{1}{\pi}\,\frac{2}{\sqrt{x}}$$

Reemplazando en la fórmula,

$$\frac{1}{4\sqrt{1-x}}\left[g\left(\frac{1}{2}\left(1-\sqrt{1-x}\right)\right)+g\left(\frac{1}{2}\left(1+\sqrt{1-x}\right)\right)\right]$$

$$= \frac{1}{\pi\sqrt{x(1-x)}} = g(x)$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} h(f^k(x_0)) = \mathbb{E}[h(X)]$$

$f^k(x_0)$ = iterar f k veces sobre x_0