Sean Z, Z' variables aleatorias tales que nos interesa estimar la cantidad

$$\mathbb{E}[Z]\mathbb{E}[Z']$$

mediante la simulación de $Z_1, \ldots, Z_n \overset{iid}{\sim} \mathcal{L}(Z)$ y $Z'_1, \ldots, Z'_n \overset{iid}{\sim} \mathcal{L}(Z')$, tales que $(Z_1, \ldots, Z_n) \perp (Z'_1, \ldots, Z'_n)$,

y el uso de estimadores

$$\underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)}_{\to \mathbb{E}[Z]} \underbrace{\left(\frac{1}{n}\sum_{i=1}^{n} Z'_i\right)}_{\to \mathbb{E}[Z']} \quad \text{y} \quad \underbrace{\frac{1}{n}\sum_{i=1}^{n} Z_i Z'_i}_{\to \mathbb{E}[ZZ'] = \mathbb{E}[Z]\mathbb{E}[Z']}$$

Calculemos las varianzas de estos estimadores y comparémoslas (¿Cuál es más grande?)

Lema: Dados X, Y v.a. t. $X \perp Y$, tenemos que

$$Var(XY) = Var(X)Var(Y) + Var(X)\mathbb{E}^2[Y] + Var(Y)\mathbb{E}^2[X]$$

Dem:

$$Var(XY) = \mathbb{E}[(XY)^2] - \mathbb{E}^2[XY]$$

$$= \mathbb{E}[X^2]\mathbb{E}[Y^2] - \mathbb{E}^2[X]\mathbb{E}^2[Y]$$

$$= \left(Var(X) + \mathbb{E}^2[X]\right)\left(Var(Y) + \mathbb{E}^2[Y]\right) - \mathbb{E}^2[X]\mathbb{E}^2[Y]$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}^2[X]$$

$$\Updownarrow$$

$$\mathbb{E}[X^2] = Var(X) + \mathbb{E}^2[X]$$

$$\left(\mathrm{Var}(x)+\mathbb{E}^2[x]\right)\left(\mathrm{Var}(y)+\mathbb{E}^2[y]\right)-\mathbb{E}^2[x]\,\mathbb{E}^2[y]$$

$$=\mathrm{Var}(x)\mathrm{Var}(y)+\mathrm{Var}(x)\,\mathbb{E}^2[y]+\mathbb{E}^2[x]\,\mathrm{Var}(y)+\mathbb{E}^2[x]\,\mathbb{E}^2[y]-\mathbb{E}^2[x]\,\mathbb{E}^2[y]$$

$$=\mathrm{Var}(x)\mathrm{Var}(y)+\mathrm{Var}(x)\,\mathbb{E}^2[y]+\mathrm{Var}(y)\,\mathbb{E}^2[x]$$

$$\mathrm{Var}\left(\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} Z_i'\right)\right) \quad \text{y} \quad \frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'$$

$$\bar{Z} = \frac{1}{n}\sum_{i=1}^{n} Z_i, \quad \bar{Z}' = \frac{1}{n}\sum_{i=1}^{n} Z_i'$$

$$\mathrm{Var}(\bar{Z}\bar{Z}') = \mathrm{Var}(\bar{Z})\mathrm{Var}(\bar{Z}') + \mathrm{Var}(\bar{Z})\mathbb{E}^2[\bar{Z}'] + \mathrm{Var}(\bar{Z}')\mathbb{E}^2[\bar{Z}]$$

$$= \frac{\mathrm{Var}(Z)}{n}\frac{\mathrm{Var}(Z')}{n} + \frac{\mathrm{Var}(Z)}{n}\mathbb{E}^2[Z'] + \frac{\mathrm{Var}(Z')}{n}\mathbb{E}^2[\bar{Z}]$$

$$\frac{\mathrm{Var}(Z)\mathrm{Var}(Z')}{n^2} + \frac{\mathrm{Var}(Z)}{n}\mathbb{E}^2[Z'] + \frac{\mathrm{Var}(Z')}{n}\mathbb{E}^2[\bar{Z}]$$

$$\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right)\left(\frac{1}{n}\sum_{i=1}^{n} Z_i\right) \quad \text{y} \quad \frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'$$

$$\mathrm{Var}\left(\frac{1}{n}\sum_{i=1}^{n} Z_i Z_i'\right) = \frac{1}{n^2}\mathrm{Var}\left(\sum_{i=1}^{n} Z_i Z_i'\right)$$

$$= \frac{n\,\mathrm{Var}(ZZ')}{n^2}$$

$$= \frac{\mathrm{Var}(ZZ')}{n}$$

$$= \frac{\mathrm{Var}(Z)\mathrm{Var}(Z') + \mathrm{Var}(Z)\mathbb{E}^2[Z'] + \mathrm{Var}(Z')\mathbb{E}^2[Z]}{n}$$

Comb $\quad \dfrac{Var(z)Var(z')}{n^2} \le \dfrac{Var(z)Var(z')}{n}$

Ent. $\quad Var(\bar{z}\bar{z}') \le Var\left(\frac{1}{n}\sum_{i=1}^{n} z_i z_i'\right)$

(2) $\quad \dfrac{\hat{\sigma}_{Anth}}{R} = \dfrac{\hat{\sigma}_{CMC}(1+\rho)}{R}$

$\Rightarrow \quad \hat{\sigma}_{Anth} = \hat{\sigma}_{CMC}(1+\rho)$

$R = 2M$

$(z_1, z_2) \ldots (z_{R-1}, z_R)$

En integración Monte Carlo, una selección estándar es considerar $Z_1 = g(U)$, $Z_2 = g(1-U)$ para estimar $Z = \int_0^1 g$

Si $g(x) = x$, ent. tenemos que

$$Z_1 + Z_2 = U + (1-U) = 1,$$ por lo que el estimador antitético tiene varianza cero (es una cte).

Si $g(x) = x^2$, ent. tenemos

$$\rho = \frac{\mathrm{Cov}(U^2, (1-U)^2)}{\sqrt{\mathrm{Var}(U^2)\,\mathrm{Var}((1-U)^2)}}$$

$$= \frac{E[U^2(1-U)^2] - E[U^2]\,E[(1-U)^2]}{\sqrt{\mathrm{Var}(U^2)\,\mathrm{Var}(U^2)}}$$

$$= \frac{\mathbb{E}[u^2(1-u)^2] - \mathbb{E}[u^2]\,\mathbb{E}[(1-u)^2]}{\sqrt{Var(u^2)}\sqrt{Var(u^2)}}$$

$$= \frac{\mathbb{E}[u^2(1-2u+u^2)] - \mathbb{E}^2[u^2]}{Var(u^2)}$$

$$= \frac{\mathbb{E}[u^2] - 2\,\mathbb{E}[u^3] + \mathbb{E}[u^4] - \mathbb{E}^2[u^2]}{Var(u^2)}$$

$$Var(u^2) = \mathbb{E}[u^4] - \mathbb{E}^2[u^2]$$

$$= 1 + \frac{\mathbb{E}[U^2] - 2\mathbb{E}[U^3]}{Var(U^2)}$$

$U(a,b)$ Si $a=0$, $a^0 = 1$

$$m_k = \frac{1}{k+1}\sum_{i=0}^{k} a^i b^{k-i}$$

$$= 1 + \frac{\frac{1}{3} - \frac{2}{4}}{\frac{1}{5} - \left(\frac{1}{3}\right)^2}$$

$$= 1 + \frac{\frac{4-9}{12}}{\frac{9-5}{45}} \approx -\frac{7}{8}$$

Se deja al lector

$$\hat{\sigma}_{Anth} = \hat{\sigma}_{CMC}(1+\rho)$$

$$= \hat{\sigma}_{CMC}\left(1-\frac{7}{8}\right)$$

$$= \frac{1}{8}\hat{\sigma}_{CMC}$$

3 $\quad \mathrm{Var}(\mathbb{E}[X|Y]) \leq \mathrm{Var}(X)$

Sean X, τ v.a. con las sig. distribuciones.

$$X|\tau \sim \mathrm{Normal}(0, 1/\tau)$$

$$\tau \sim \mathrm{Gamma}(\alpha, \beta)$$

¿Cual es la densidad marginal de X?

$$g_{X|T}(x|\tau) = \sqrt{\frac{\tau}{2\pi}} \exp\left(-\frac{\tau x^2}{2}\right)$$

$$g_T(\tau) = \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} \exp(-\beta\tau)$$

$$f_X(x) = \int g_{X|T}(x|\tau)\, g_T(\tau)\, d\tau$$

$$g_{X|T}(x|\tau) = \sqrt{\frac{\tau}{2\pi}}\, \exp\left(-\frac{\tau x^2}{2}\right)$$

$$g_T(\tau) = \frac{\beta^\alpha}{\Gamma(\alpha)}\, \tau^{\alpha-1} \exp(-\beta\tau)$$

$$= \int \frac{\tau^{1/2}}{\sqrt{2\pi}} \exp\left(-\frac{\tau x^2}{2}\right) \frac{\beta^\alpha}{\Gamma(\alpha)}\, \tau^{\alpha-1} \exp(-\beta\tau)$$

$$= \frac{\beta^\alpha}{\sqrt{2\pi}\,\Gamma(\alpha)} \int \tau^{\overbrace{(\alpha+\frac{1}{2})}^{\alpha'}-1} \exp\left(-\tau \underbrace{\left(\frac{x^2}{2}+\beta\right)}_{\beta'}\right)$$

$$= \frac{\beta^\alpha}{\sqrt{2\pi}\,\Gamma(\alpha)} \frac{\Gamma(\alpha')}{\beta'^{\alpha'}} \underbrace{\int \frac{\beta'^{\alpha'}}{\Gamma(\alpha')}\, \tau^{\alpha'-1} \exp(-\tau\beta')}_{\to 1}$$

$$= \frac{\beta^\alpha}{\sqrt{2\pi}\,\Gamma(\alpha)}\, \Gamma\left(\alpha+\frac{1}{2}\right) \left(\frac{x^2}{2}+\beta\right)^{-(\alpha+\frac{1}{2})}$$

$$= \frac{\Gamma\left(\frac{2\alpha+1}{2}\right)}{\sqrt{2\pi}\, \Gamma\left(\frac{2\alpha}{2}\right)}\, \cancel{\beta^\alpha}\, \cancel{\beta}^{-\alpha-\frac{1}{2}} \left(\frac{x^2}{2\beta}+1\right)^{-(\alpha+\frac{1}{2})}$$

$\nu = 2\alpha$

$$= \frac{\Gamma\left(\frac{2\alpha+1}{2}\right)}{\sqrt{2\pi}\,\sqrt{\beta}\; \Gamma\left(\frac{2\alpha}{2}\right)} \left(\frac{x^2}{2\beta}+1\right)^{-\left(\frac{2\alpha+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{2\pi}\,\sqrt{\beta}\; \Gamma\left(\frac{\nu}{2}\right)} \left(\frac{x^2}{2\alpha\frac{\beta}{\alpha}}+1\right)^{-\left(\frac{\nu+1}{2}\right)}$$

$\hat{\sigma} = \sqrt{\frac{\beta}{\alpha}}$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\frac{\sqrt{2\alpha}}{\sqrt{\alpha}}\,\sqrt{\beta}\; \Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{\nu}\frac{x^2}{\left(\frac{\beta}{\alpha}\right)}+1\right)^{-\left(\frac{\nu+1}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi\alpha}\,\hat{\sigma}\;\Gamma\left(\frac{\nu}{2}\right)} \left(\frac{1}{\nu}\left(\frac{x}{\hat{\sigma}}\right)^2 + 1\right)^{-\left(\frac{\nu+1}{2}\right)}$$

$$x \sim t_{\left(\nu = 2\alpha,\ \mu = 0,\ \sigma = \sqrt{\frac{\beta}{\alpha}}\right)}$$

Si $\alpha = \beta = 1$

$$x \sim t_{(\nu = 2,\ \mu = 0,\ \sigma = 1)}$$

$$\sim t_{(2)}$$

$\therefore \; \mathbb{E}[x] = 0$

Ahora, si nos fijamos en $\mathbb{E}[x \mid T]$,

$$\mathbb{E}[x \mid T] = 0$$

Dado $T = \tau$, $X \mid T = \tau \sim N(0, 1/\tau)$

$$x \mid T \sim N(0, 1/T)$$

$$\mathbb{E}[x \mid T] = 0$$