GSERM 2021
Analyzing Panel Data

June 15, 2021

One- and Two-Way "Unit Effects"

Two-way variation:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \gamma V_i + \delta W_t + u_{it}$$

$\longrightarrow$ "two-way" effects:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + \eta_t + u_{it}$$

Also: "One-way" effects:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \eta_t + u_{it} \quad \text{(time)}$$

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + u_{it} \quad \text{(units)}$$

"Brute force" model:

$$\begin{aligned} Y_{it} &= \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + u_{it} \\ &= \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_1 I(i=1)_i + \alpha_2 I(i=2)_i + ... + u_{it} \end{aligned}$$

Alternatively, decompose:

$$\bar{\mathbf{X}}_i = \frac{\sum_{N_i} \mathbf{X}_{it}}{N_i}$$

and

$$\tilde{\mathbf{X}}_{it} = \mathbf{X}_{it} - \bar{\mathbf{X}}_i.$$

Yields:

$$Y_{it} = \bar{\mathbf{X}}_i\boldsymbol{\beta}_B + \tilde{\mathbf{X}}_{it}\boldsymbol{\beta}_W + \alpha_i + u_{it}$$

Means that:

$$
\begin{aligned}
Y_{it}^{*} &= Y_{it} - \bar{Y}_i \\
\mathbf{X}_{it}^{*} &= \mathbf{X}_{it} - \bar{\mathbf{X}}_i
\end{aligned}
$$

$$Y_{it}^{*} = \boldsymbol{\beta}_{FE}\mathbf{X}_{it}^{*} + u_{it}.$$

$\rightarrow$ a "Fixed Effects" Model is actually a "Within-Effects" Model.

“Fixed” Effects: Test(s)

Standard F-test for

$$H_0 : \alpha_i = \alpha_j \forall i \neq j$$

versus

$$H_A : \alpha_i \neq \alpha_j \text{ for some } i \neq j$$

is $\sim F_{N-1, NT-(N-1)}$.

Example Data: Demonstrations, 1945-2014

Data:

- Data are (a subset of) Banks (2019)
- $N = 180$ countries, $T = 70$ years [1945-2014]
- Variables:
  - `Demonstrations`: Number of social/political demonstrations in that country in that year
  - `POLITY`: The country's POLITY IV score that year (-10 = fully autocratic; 10 = fully democratic)
  - `POLITY`^{2}: POLITY IV squared (expected curvilinear relationship)
  - `GDP`: The per capita GDP (PPP, in constant $US) for that country / year
  - `Monarch`: Whether (=1) or not (=0) that country was a monarchy in that year
  - `ColdWar`: Indicator variable, coded 1 for the period 1945-1989, 0 otherwise

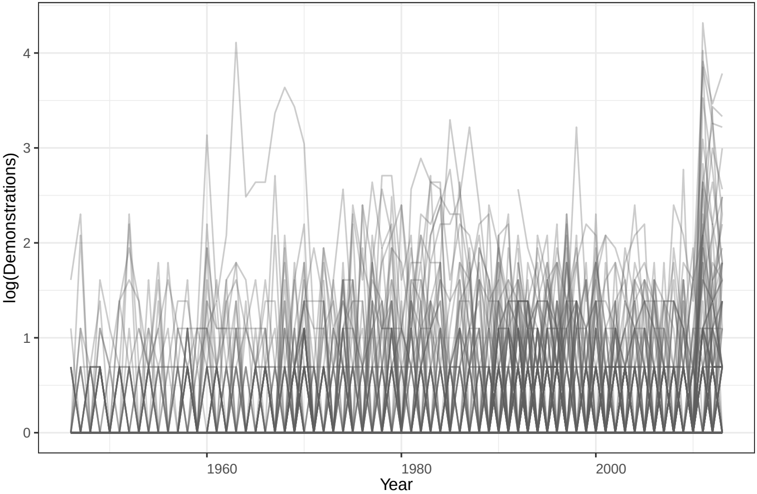
Regression model:

$$
\begin{aligned}
\mathit{ln}(\text{Demonstrations} + 1)_{it} &= \beta_0 + \beta_1 \text{POLITY}_{it} + \beta_2 \text{POLITY}^2_{it} + \beta_3 \ln(\text{GDP})_{it} + \\
&\quad \beta_4 \text{Monarch}_{it} + \beta_5 \text{Cold War}_{it} + u_{it}
\end{aligned}
$$

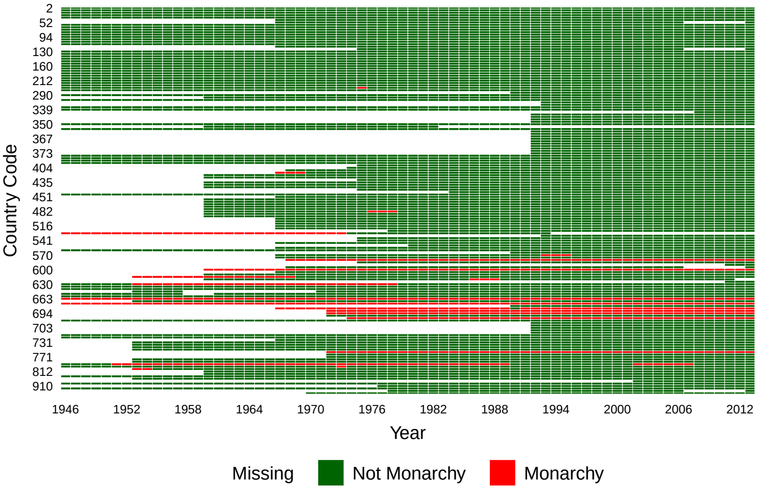
Data Summary

```
> summary(Demos)
     ccode            Year           POLITY              GDP        
 Min.   :  2   Min.   :1945   Min.   :-10.00   Min.   :   185  
 1st Qu.:235   1st Qu.:1969   1st Qu.: -7.00   1st Qu.:  1580  
 Median :451   Median :1985   Median :  0.00   Median :  4002  
 Mean   :456   Mean   :1984   Mean   :  0.63   Mean   :  8120  
 3rd Qu.:663   3rd Qu.:2000   3rd Qu.:  8.00   3rd Qu.: 10365  
 Max.   :950   Max.   :2014   Max.   : 10.00   Max.   :134040  
                              NA's   :111      NA's   :2348    
    Monarch          lnDemons         ColdWar          lnGDP     
 Min.   :0.0     Min.   :0.0     Min.   :0.000   Min.   : 5.2  
 1st Qu.:0.0     1st Qu.:0.0     1st Qu.:0.000   1st Qu.: 7.4  
 Median :0.0     Median :0.0     Median :1.000   Median : 8.3  
 Mean   :0.1     Mean   :0.3     Mean   :0.563   Mean   : 8.3  
 3rd Qu.:0.0     3rd Qu.:0.0     3rd Qu.:1.000   3rd Qu.: 9.2  
 Max.   :1.0     Max.   :4.3     Max.   :1.000   Max.   :11.8  
 NA's   :1198    NA's   :1149                    NA's   :2348  
```

Visualization (using `panelView`)

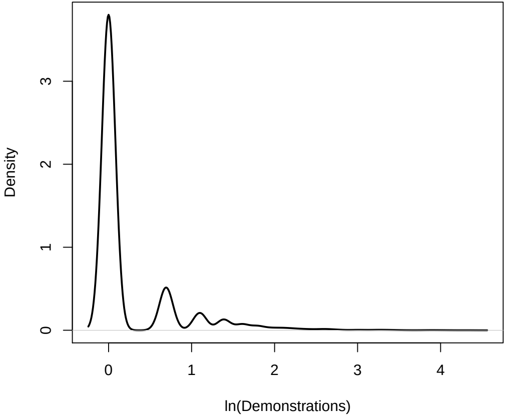


Categorical Variable Visualization



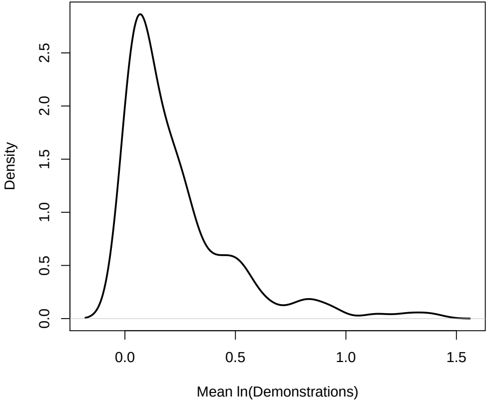
Demonstrations: Total Variation

```
> with(Demos, describe(lnDemons)) # all variation
   vars    n mean   sd median trimmed mad min  max range skew kurtosis   se
X1    1 8219 0.25 0.55      0    0.11   0   0 4.32  4.32 2.54      7.3 0.01
```



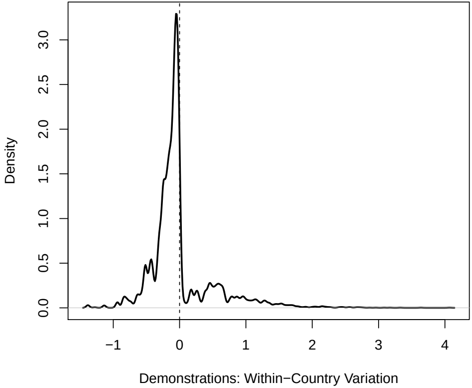
Demonstrations: "Between" Variation

```
> DemonsMeans <- ddply(Demos,.(ccode),summarise,
+                      DemonsMean = mean(lnDemons,na.rm=TRUE))
>
> with(DemonsMeans, describe(DemonsMean)) # "between" variation
   vars   n mean   sd median trimmed  mad min  max range skew kurtosis   se
X1    1 160 0.23 0.25   0.15    0.18 0.15   0 1.38  1.38 1.99     4.55 0.02
```



Demonstrations: “Within” Variation

```
> Demos <- ddply(Demos, .(ccode), mutate,
+                DemonsMean = mean(lnDemons,na.rm=TRUE))
> Demos$DemonsWithin <- with(Demos, lnDemons-DemonsMean)
>
> with(Demos, describe(DemonsWithin))
   vars    n mean   sd median trimmed  mad   min  max range skew kurtosis   se
X1    1 8219    0 0.48  -0.09   -0.07 0.16 -1.38 4.07  5.45 1.96     6.41 0.01
```



Regression: Pooled OLS

```
> OLS<-plm(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+ColdWar,
+         data=PDF,model="pooling")

> summary(OLS)
Pooling Model

Call:
plm(formula = lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch +
    ColdWar, data = PDF, model = "pooling")

Unbalanced Panel: n = 145, T = 1-62, N = 6505

Residuals:
   Min. 1st Qu.  Median 3rd Qu.    Max.
-0.4501 -0.2930 -0.2176 -0.0754  4.1073

Coefficients:
             Estimate Std. Error t-value Pr(>|t|)
(Intercept) -0.124639   0.058208   -2.14    0.032 *
POLITY       0.006296   0.001179    5.34  9.5e-08 ***
I(POLITY^2) -0.002267   0.000255   -8.90  < 2e-16 ***
lnGDP        0.057679   0.007513    7.68  1.9e-14 ***
Monarch     -0.046393   0.028572   -1.62    0.104
ColdWar      0.027883   0.013961    2.00    0.046 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Total Sum of Squares:    1850
Residual Sum of Squares: 1800
R-Squared:      0.0261
Adj. R-Squared: 0.0253
F-statistic: 34.8228 on 5 and 6499 DF, p-value: <2e-16
```

“Fixed” (Within) Effects

```
> FE<-plm(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+ColdWar,
+             data=PDF, effect="individual",model="within")

> summary(FE)
Oneway (individual) effect Within Model

Call:
plm(formula = lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch +
    ColdWar, data = PDF, effect = "individual", model = "within")

Unbalanced Panel: n = 145, T = 1-62, N = 6505

Residuals:
   Min. 1st Qu.  Median 3rd Qu.    Max.
-1.3556 -0.2120 -0.0768  0.0193  4.0496

Coefficients:
             Estimate Std. Error t-value Pr(>|t|)
POLITY       0.001526   0.001553    0.98  0.32604
I(POLITY^2) -0.001942   0.000296   -6.55  6.1e-11 ***
lnGDP        0.054586   0.015200    3.59  0.00033 ***
Monarch      0.047976   0.068071    0.70  0.48097
ColdWar     -0.035487   0.016235   -2.19  0.02887 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Total Sum of Squares:    1410
Residual Sum of Squares: 1400
R-Squared:      0.013
Adj. R-Squared: -0.0102
F-statistic: 16.7177 on 5 and 6355 DF, p-value: <2e-16
```

A Nicer Table

Table: Models of Demonstrations

	OLS	FE
POLITY	0.006***	0.002
	(0.001)	(0.002)
POLITY Squared	−0.002***	−0.002***
	(0.0003)	(0.0003)
ln(GDP)	0.058***	0.055***
	(0.008)	(0.015)
Monarch	−0.046	0.048
	(0.029)	(0.068)
Cold War	0.028**	−0.035**
	(0.014)	(0.016)
Constant	−0.125**	
	(0.058)	
Observations	6,505	6,505
R^2	0.026	0.013
Adjusted R^2	0.025	-0.010
F Statistic	34.820*** (df = 5; 6499)	16.720*** (df = 5; 6355)

*p<0.1; **p<0.05; ***p<0.01

Time-Period Fixed Effects

The model is:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \eta_t + u_{it}$$

which is estimated via:

$$\begin{aligned} Y_{it}^{**} &= Y_{it} - \bar{Y}_t \\ \mathbf{X}_{it}^{**} &= \mathbf{X}_{it} - \bar{\mathbf{X}}_t \end{aligned}$$

$$Y_{it}^{**} = \boldsymbol{\beta}_{FE}\mathbf{X}_{it}^{**} + u_{it}.$$

Comparison: Unit vs. Time Fixed Effects

Table: FE Models of Demonstrations

	FE.Units	FE.Time
POLITY	0.002	0.007***
	(0.002)	(0.001)
POLITY Squared	−0.002***	−0.002***
	(0.0003)	(0.0003)
ln(GDP)	0.055***	0.058***
	(0.015)	(0.008)
Monarch	0.048	−0.038
	(0.068)	(0.028)
Cold War	−0.035**	
	(0.016)	
Observations	6,505	6,505
R^2	0.013	0.028
Adjusted R^2	-0.010	0.018
F Statistic	16.720*** (df = 5; 6355)	46.270*** (df = 4; 6439)

*p<0.1; **p<0.05; ***p<0.01

Fixed Effects: Testing

The specification:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + u_{it}$$

...suggests that we can use an F-test to examine the hypothesis:

$$H_0 : \alpha_i = 0 \; \forall \; i$$

(and a similar test for $\eta_t = 0$ in the time-centered case).

Arguably better are Lagrange multiplier-based tests:

- Breusch-Pagan (1980)
- King and Wu (1997)
- See (e.g.) Croissant and Millo (2018, §4.1) for details

FE Model Tests

```
> pFtest(FE,OLS)

F test for individual effects

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
F = 13, df1 = 144, df2 = 6355, p-value <2e-16
alternative hypothesis: significant effects


> plmtest(FE,effect=c("individual"),type=c("bp"))

Lagrange Multiplier Test - (Breusch-Pagan) for unbalanced panels

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
chisq = 8016, df = 1, p-value <2e-16
alternative hypothesis: significant effects


> plmtest(FE,effect=c("individual"),type=c("kw"))

Lagrange Multiplier Test - (King and Wu) for unbalanced panels

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
normal = 90, p-value <2e-16
alternative hypothesis: significant effects
```

Same For Time Effects

```
> pFtest(FE.Time,OLS)

F test for time effects

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
F = 3, df1 = 60, df2 = 6439, p-value = 1e-13
alternative hypothesis: significant effects


> plmtest(FE.Time,effect=c("time"),type=c("bp"))

Lagrange Multiplier Test - time effects (Breusch-Pagan) for unbalanced
panels

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
chisq = 144, df = 1, p-value <2e-16
alternative hypothesis: significant effects


> plmtest(FE.Time,effect=c("time"),type=c("kw"))

Lagrange Multiplier Test - time effects (King and Wu) for unbalanced panels

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
normal = 12, p-value <2e-16
alternative hypothesis: significant effects
```

FE models are *subject-specific* models, that rely entirely on *within-unit* variability to estimate variable effects.

- This means that:

$$\hat{\beta}_k = \frac{\partial E(Y|\hat{\alpha})}{\partial X_k}$$

- That is, $\hat{\beta}_k$ is *the expected change in E(Y) associated with a one-unit increase in observation i's value of* X_k
- Key: *within-unit* changes in **X** are associated with *within-unit* expected changes in Y.
- In a linear model, the value of $\hat{\alpha}$ doesn't affect the value of that partial derivative...

Fixed Effects: Interpretation

Mummolo and Peterson (2018) note that:

> *"...because the within-unit variation is always smaller (or at least, no larger) than the overall variation in the independent variable, researchers should use within-unit variation to motivate counterfactuals when discussing the substantive impact of a treatment" (2018, 829).*

Significance:

- Predictors $\mathbf{X}$ in FE models typically have both cross-sectional and temporal variation
- FE models only consider *within-unit* variation in $\mathbf{X}$ and Y
- As a result, the degree of actual/observed (within-unit) variation in predictors is almost always less – and sometimes significantly less – than if cross-sectional variation were also considered

Interpretation Example: Monarchy

`Monarchy` – All Variation:

```
> with(Demos, sd(Monarch,na.rm=TRUE))
[1] 0.2601
```

`Monarchy` – "Within" Variation:

```
> Demos <- ddply(Demos, .(ccode), mutate,
+                MonMean = mean(Monarch,na.rm=TRUE))
> Demos$MonarchWithin <- with(Demos, Monarch-MonMean)

> with(Demos, sd(MonarchWithin,na.rm=TRUE)) # "within" variation
[1] 0.09418
```

> *"While the overall variation in the independent variable may be large, the within-unit variation used to estimate β may be much smaller" (M & P 2018, 830).*

Pros and Cons of "Fixed" Effects

Pros:

- Mitigates (Some) Specification Bias
- Simple + Intuitive
- Widely Used/Understood

Cons (see e.g. Collischon and Eberl 2020):

- Can't Estimate $\boldsymbol{\beta}_B$
- Slowly-Changing **X**s
- (In)Efficiency / Inconsistency (Incidental Parameters)
- Cannot Control for (e.g.) Time-Varying Heterogeneity
- Sensitivity to Measurement Error

“Between” Effects

From the equation above:

$$Y_{it} = \bar{\mathbf{X}}_i \boldsymbol{\beta}_B + \tilde{\mathbf{X}}_{it} \boldsymbol{\beta}_W + \alpha_i + u_{it}$$

...we can derive a “Between Effects” model:

$$\bar{Y}_i = \bar{\mathbf{X}}_i \boldsymbol{\beta}_B + u_{it}$$

This model:

- is essentially cross-sectional,
- is based on N observations,
- considers *only* between-unit (average) differences
- Interpretation:

  $\hat{\beta}_k$ is the expected difference in Y between two units whose values on $\bar{X}_k$ differ by a value of 1.0.

“Between” Effects

```
> BE<-plm(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+ColdWar,
+           data=PDF, effect="individual",model="between")
>
> summary(BE)
Oneway (individual) effect Between Model

Call:
plm(formula = lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch +
    ColdWar, data = PDF, effect = "individual", model = "between")

Unbalanced Panel: n = 145, T = 1-62, N = 6505
Observations used in estimation: 145

Coefficients:
            Estimate Std. Error t-value Pr(>|t|)
(Intercept) -0.30601    0.20837   -1.47   0.1442
POLITY       0.00597    0.00489    1.22   0.2244
I(POLITY^2) -0.00302    0.00112   -2.69   0.0079 **
lnGDP        0.06883    0.02734    2.52   0.0130 *
Monarch     -0.04966    0.10320   -0.48   0.6312
ColdWar      0.25872    0.08482    3.05   0.0027 **
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Total Sum of Squares:    8.75
Residual Sum of Squares: 7.64
R-Squared:      0.127
Adj. R-Squared: 0.0961
F-statistic: 4.06164 on 5 and 139 DF, p-value: 0.0018
```

A Nicer Table (Again)

Table: Models of Demonstrations

	OLS	FE	BE
POLITY	0.006***	0.002	0.006
	(0.001)	(0.002)	(0.005)
POLITY Squared	−0.002***	−0.002***	−0.003***
	(0.0003)	(0.0003)	(0.001)
ln(GDP)	0.058***	0.055***	0.069**
	(0.008)	(0.015)	(0.027)
Monarch	−0.046	0.048	−0.050
	(0.029)	(0.068)	(0.103)
Cold War	0.028**	−0.035**	0.259***
	(0.014)	(0.016)	(0.085)
Constant	−0.125**		−0.306
	(0.058)		(0.208)
Observations	6,505	6,505	145
R^2	0.026	0.013	0.127
Adjusted R^2	0.025	-0.010	0.096
F Statistic	34.820*** (df = 5; 6499)	16.720*** (df = 5; 6355)	4.062*** (df = 5; 139)

*p<0.1; **p<0.05; ***p<0.01

“Random” Effects

Model:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + u_{it}$$

with:

$$u_{it} = \alpha_i + \lambda_t + \eta_{it}$$

and

$$\begin{aligned}
E(\alpha_i) = E(\lambda_t) = E(\eta_{it}) &= 0, \\
E(\alpha_i\lambda_t) = E(\alpha_i\eta_{it}) = E(\lambda_t\eta_{it}) &= 0, \\
E(\alpha_i\alpha_j) &= \sigma^2_\alpha \text{ if } i = j,\ 0 \text{ otherwise}, \\
E(\lambda_t\lambda_s) &= \sigma^2_\lambda \text{ if } t = s,\ 0 \text{ otherwise}, \\
E(\eta_{it}\eta_{js}) &= \sigma^2_\eta \text{ if } i = j,\ t = s,\ 0 \text{ otherwise}, \\
E(\alpha_i\mathbf{X}_{it}) = E(\lambda_t\mathbf{X}_{it}) = E(\eta_{it}\mathbf{X}_{it}) &= 0.
\end{aligned}$$

If those assumptions are met, we can consider the "two-way variance components" model where:

$$\begin{aligned} Var(u_{it}) &= Var(Y_{it}|\mathbf{X}_{it}) \\ &= \sigma^2_\alpha + \sigma^2_\lambda + \sigma^2_\eta \end{aligned}$$

If we assume $\lambda_t = 0$, then we get a model like:

$$Y_{it} = \mathbf{X}_{it}\beta + \alpha_i + \eta_{it}$$

with total error variance:

$$\sigma^2_u = \sigma^2_\alpha + \sigma^2_\eta.$$

"Random" Effects: Estimation

The model in 1 will violate the standard OLS assumptions of uncorrelated errors, because the (compound) "errors" within each unit share a common component α_i.

Consider the within-i variance-covariance matrix of the errors $\mathbf{u}$:

$$\begin{aligned} E(\mathbf{u}_i\mathbf{u}_i') \equiv \mathbf{\Sigma}_i &= \sigma_\eta^2 \mathbf{I}_T + \sigma_\alpha^2 \mathbf{i}\mathbf{i}' \\ &= \begin{pmatrix} \sigma_\eta^2 + \sigma_\alpha^2 & \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \sigma_\alpha^2 & \sigma_\eta^2 + \sigma_\alpha^2 & \cdots & \sigma_\alpha^2 \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_\alpha^2 & \sigma_\alpha^2 & \cdots & \sigma_\eta^2 + \sigma_\alpha^2 \end{pmatrix} \end{aligned}$$

Assuming conditional independence across units, we then have:

$$\mathrm{Var}(\mathbf{u}) \equiv \mathbf{\Omega} = \begin{pmatrix} \mathbf{\Sigma}_1 & 0 & \cdots & 0 \\ 0 & \mathbf{\Sigma}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{\Sigma}_N \end{pmatrix}$$

"Random" Effects: Estimation

We can then show that:

$$\mathbf{\Sigma}^{-1/2} = \frac{1}{\sigma_\eta}\left[\mathbf{I}_T - \left(\frac{\theta}{T}\mathbf{ii}'\right)\right]$$

where

$$\theta = 1 - \sqrt{\frac{\sigma_\eta^2}{T\sigma_\alpha^2 + \sigma_\eta^2}}$$

is an unknown quantity to be estimated.

Starting with an estimate of $\hat{\theta}$, calculate:

$$\begin{aligned} Y_{it}^* &= Y_{it} - \hat{\theta}\bar{Y}_i \\ X_{it}^* &= X_{it} - \hat{\theta}\bar{X}_i, \end{aligned}$$

then estimate:

$$Y_{it}^* = (1-\hat{\theta})\alpha + X_{it}^*\beta_{RE} + [(1-\hat{\theta})\alpha_i + (\eta_{it} - \hat{\theta}\bar{\eta}_i)]$$

and iterate between the two processes until convergence.

"Random" Effects: An Alternative View

```
> RE<-plm(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+ColdWar,
+                data=PDF, effect="individual", model="random")

> summary(RE)
Oneway (individual) effect Random Effect Model
   (Swamy-Arora's transformation)

Call:
plm(formula = lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch +
    ColdWar, data = PDF, effect = "individual", model = "random")

Unbalanced Panel: n = 145, T = 1-62, N = 6505

Effects:
                 var std.dev share
idiosyncratic 0.2197  0.4687   0.8
individual    0.0563  0.2373   0.2
theta:
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  0.108   0.708   0.736   0.724   0.757   0.757

Coefficients:
             Estimate Std. Error z-value Pr(>|z|)
(Intercept) -0.131708   0.104987   -1.25    0.210
POLITY       0.002574   0.001450    1.78    0.076 .
I(POLITY^2) -0.001953   0.000287   -6.81  9.6e-12 ***
lnGDP        0.057117   0.012443    4.59  4.4e-06 ***
Monarch     -0.006937   0.053291   -0.13    0.896
ColdWar     -0.023580   0.015008   -1.57    0.116
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Total Sum of Squares:    1440
Residual Sum of Squares: 1430
R-Squared:      0.0124
Adj. R-Squared: 0.0117
Chisq: 81.0788 on 5 DF, p-value: 4.99e-16
```

A Nicer Table (Yet Again)

Table: Models of Demonstrations

	OLS	FE	BE	RE
POLITY	0.006^{***}	0.002	0.006	0.003^{*}
	(0.001)	(0.002)	(0.005)	(0.001)
POLITY Squared	-0.002^{***}	-0.002^{***}	-0.003^{***}	-0.002^{***}
	(0.0003)	(0.0003)	(0.001)	(0.0003)
ln(GDP)	0.058^{***}	0.055^{***}	0.069^{**}	0.057^{***}
	(0.008)	(0.015)	(0.027)	(0.012)
Monarch	-0.046	0.048	-0.050	-0.007
	(0.029)	(0.068)	(0.103)	(0.053)
Cold War	0.028^{**}	-0.035^{**}	0.259^{***}	-0.024
	(0.014)	(0.016)	(0.085)	(0.015)
Constant	-0.125^{**}		-0.306	-0.132
	(0.058)		(0.208)	(0.105)
Observations	6,505	6,505	145	6,505
R^2	0.026	0.013	0.127	0.012
Adjusted R^2	0.025	-0.010	0.096	0.012
F Statistic	34.820^{***} (df = 5; 6499)	16.720^{***} (df = 5; 6355)	4.062^{***} (df = 5; 139)	81.080^{***}

$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

"Random" Effects: Testing

Hausman test (FE vs. RE):

$$\hat{W} = (\hat{\boldsymbol{\beta}}_{\mathbf{FE}} - \hat{\boldsymbol{\beta}}_{\mathbf{RE}})'(\hat{\mathbf{V}}_{\mathbf{FE}} - \hat{\mathbf{V}}_{\mathbf{RE}})^{-1}(\hat{\boldsymbol{\beta}}_{\mathbf{FE}} - \hat{\boldsymbol{\beta}}_{\mathbf{RE}})$$

$$W \sim \chi^2_k$$

Issues:

- Asymptotic
- No guarantee $(\hat{\mathbf{V}}_{\mathbf{FE}} - \hat{\mathbf{V}}_{\mathbf{RE}})^{-1}$ is positive definite
- A <u>general</u> specification test...

Hausman Test: Intuition

If the data-generating process is the result of "random" effects [that is, if $\text{Cov}(\mathbf{X}_{it}, \alpha_i = 0)$]:

- the random-effects estimate $\hat{\boldsymbol{\beta}}_{RE}$ will be consistent *and* (more) efficient
- the fixed-effects / within-unit estimator $\hat{\boldsymbol{\beta}}_{FE}$ will be consistent but *inefficient*

BUT...

If the data-generating process is in fact the result of "fixed" effects [that is, if $\text{Cov}(\mathbf{X}_{it}, \alpha_i \neq 0)$]:

- the fixed-effects / within-unit estimator $\hat{\boldsymbol{\beta}}_{FE}$ will be consistent and efficient
- the random-effects estimate $\hat{\boldsymbol{\beta}}_{RE}$ will be *inconsistent*

Hausman Test Results

Hausman test (FE vs. RE):

```
> phtest(FE, RE)

Hausman Test

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
chisq = 11, df = 5, p-value = 0.05
alternative hypothesis: one model is inconsistent
```

Practical “Fixed” vs. “Random” Effects

Factors to consider:

- “Panel” vs. “TSCS” Data
- Nature of the Data-Generating Processes
- Importance of Non-Time-Varying Covariate Effects
- Dimension/Location of the Variance in the Data (N vs. T)

Connections: Hierarchical Linear Models

Begin by considering a two-level "nested" data structure, with:

$$\begin{aligned} i &\in \{1, 2, ...N\} \text{ indexing first-level units, and} \\ j &\in \{1, 2, ...J\} \text{ indexing second-level groups.} \end{aligned}$$

A general two-level HLM is an equation of the form:

$$Y_{ij} = \beta_{0j} + \mathbf{X}_{ij}\boldsymbol{\beta}_j + u_{ij} \tag{1}$$

where β_{0j} is a "constant" term, $\mathbf{X}_{ij}$ is a $NJ \times K$ matrix of K covariates, $\boldsymbol{\beta}_j$ is a $K \times 1$ vector of parameters, and $u_{ij} \sim$ i.i.d. $N(0, \sigma_u^2)$ is the usual random-disturbance assumption.

Each of the $K+1$ "level-one" parameters is then allowed to vary across Q "level-two" variables $\mathbf{Z}_j$, so that:

$$\beta_{0j} = \gamma_{00} + \mathbf{Z}_j\gamma_0 + \varepsilon_{0j} \tag{2}$$

for the "intercept" and

$$\beta_{kj} = \gamma_{k0} + \mathbf{Z}_j\gamma_k + \varepsilon_{kj} \tag{3}$$

for the "slopes" of $\mathbf{X}$. The εs are typically assumed to be distributed multivariate Normal, with parameters for the variances and covariances selected by the analyst. Substitution of (2) and (3) into (1) yields:

$$Y_{ij} = \gamma_{00} + \mathbf{Z}_j\gamma_0 + \mathbf{X}_{ij}\gamma_{k0} + \mathbf{X}_{ij}\mathbf{Z}_j\gamma_k + \mathbf{X}_{ij}\varepsilon_{kj} + \varepsilon_{0j} + u_{ij} \tag{4}$$

The form is essentially a model with "saturated" interaction effects across the various levels, as well as "errors" which are multivariate Normal.

HLM Details

Model Assumptions

- Linearity / Additivity
- Normality of us
- Homoscedasticity
- Residual Independence:
  - $\text{Cov}(\varepsilon_{\cdot j}, u_{ij}) = 0$
  - $\text{Cov}(u_{ij}, u_{i\ell}) = 0$

Model Fitting

- MLE
- "Restricted" MLE ("RMLE")
- Choosing:
  - MLE is biased in small samples, especially for estimating variances
  - RMLE is not, but prevents use of LR tests when the models do not have identical fixed effects
  - In general: RMLE is better with small sample sizes, but MLE is fine in larger ones

HLMs: Attributes

Note that if we specify:

$$\beta_{0j} = \gamma_{00} + \varepsilon_{0j}$$

and

$$\beta_{kj} = \gamma_{k0}$$

we get:

$$Y_{ij} = \gamma_{00} + \mathbf{X}_{ij}\gamma_{k0} + \varepsilon_{0j} + u_{ij}$$

which is the RE model above (formally, a "one-level random-intercept" HLM).

In addition:

- HLMs can be expanded to 3- and 4- and higher-level models
- One can include cross-level interactions...

HLMs are widely used in education, psychology, ecology, etc. (less so in economics, political science). Also, there are many, many excellent books, websites, etc. that address HLMs

Random Effects Remix (using `lmer`)

```
> library(lme4)

> AltRE<-lmer(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+ColdWar+
+                  (1|ccode), data=Demos)
>
> summary(AltRE)
Linear mixed model fit by REML ['lmerMod']
Formula: lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar +
    (1 | ccode)
   Data: Demos

REML criterion at convergence: 9005

Random effects:
 Groups   Name        Variance Std.Dev.
 ccode    (Intercept) 0.0536   0.232
 Residual             0.2200   0.469
Number of obs: 6507, groups:  ccode, 145

Fixed effects:
             Estimate Std. Error t value
(Intercept) -0.133634   0.104246   -1.28
POLITY       0.002623   0.001447    1.81
I(POLITY^2) -0.001972   0.000287   -6.88
lnGDP        0.057513   0.012371    4.65
Monarch     -0.015175   0.052863   -0.29
ColdWar     -0.022225   0.014986   -1.48

Correlation of Fixed Effects:
            (Intr) POLITY I(POLI lnGDP  Monrch
POLITY       0.109
I(POLITY^2)  0.134 -0.135
lnGDP       -0.968 -0.140 -0.270
Monarch      0.004  0.172 -0.163 -0.022
ColdWar     -0.391  0.387 -0.210  0.351  0.014
```

Q: Are They The Same? [A: Yes]

Table: RE Models of Demonstrations

	RE	AltRE
POLITY	0.003*	0.003*
	(0.001)	(0.001)
POLITY Squared	−0.002***	−0.002***
	(0.0003)	(0.0003)
ln(GDP)	0.057***	0.057***
	(0.012)	(0.012)
Monarch	−0.007	−0.008
	(0.053)	(0.053)
Cold War	−0.024	−0.023
	(0.015)	(0.015)
Constant	−0.132	−0.132
	(0.105)	(0.104)
Observations	6,505	6,505
R^2	0.012	
Adjusted R^2	0.012	
Log Likelihood		-4,496.000
Akaike Inf. Crit.		9,009.000
Bayesian Inf. Crit.		9,063.000
F Statistic	81.080***	

*p<0.1; **p<0.05; ***p<0.01

HLM with Country-Level Random βs for `lnGDP`

```
> HLM1<-lmer(lnDemons~POLITY+I(POLITY^2)+lnGDP+(lnGDP|ccode)+
+              Monarch+ColdWar, data=Demos,
+              control=lmerControl(optimizer="bobyqa"))

> summary(HLM1)
Linear mixed model fit by REML ['lmerMod']
Formula: lnDemons ~ POLITY + I(POLITY^2) + lnGDP + (lnGDP | ccode) + Monarch +
    ColdWar
   Data: Demos
Control: lmerControl(optimizer = "bobyqa")

Scaled residuals:
   Min     1Q Median     3Q    Max
-2.821 -0.442 -0.175 -0.005  8.530

Random effects:
 Groups   Name        Variance Std.Dev. Corr
 ccode    (Intercept) 1.377    1.174
          lnGDP       0.021    0.145    -0.98
 Residual             0.214    0.462
Number of obs: 6505, groups:  ccode, 145

Fixed effects:
            Estimate Std. Error t value
(Intercept) -0.21563    0.15416   -1.40
POLITY       0.00198    0.00149    1.33
I(POLITY^2) -0.00206    0.00030   -6.88
lnGDP        0.06938    0.01894    3.66
Monarch      0.02635    0.05681    0.46
ColdWar     -0.01394    0.01542   -0.90

Correlation of Fixed Effects:
            (Intr) POLITY I(POLI lnGDP  Monrch
POLITY       0.093
I(POLITY^2)  0.127 -0.096
lnGDP       -0.984 -0.114 -0.214
Monarch     -0.015  0.136 -0.167  0.006
ColdWar     -0.286  0.368 -0.215  0.246  0.019
```

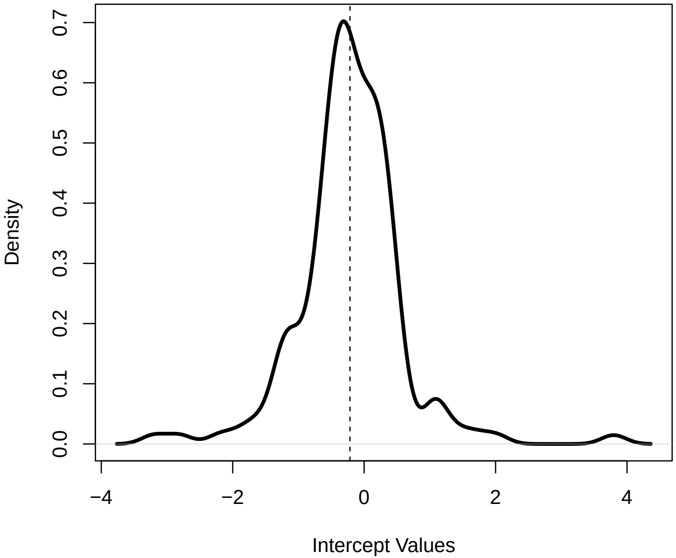
```
> anova(AltRE,HLM1)

refitting model(s) with ML (instead of REML)
Data: Demos
Models:
AltRE: lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar +
AltRE:     (1 | ccode)
HLM1: lnDemons ~ POLITY + I(POLITY^2) + lnGDP + (lnGDP | ccode) + Monarch +
HLM1:     ColdWar
      npar  AIC  BIC logLik deviance Chisq Df Pr(>Chisq)
AltRE    8 8959 9013  -4471     8943
HLM1    10 8887 8955  -4434     8867  75.6  2     <2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1
```

```
> VarCorr(HLM1)
 Groups   Name        Std.Dev. Corr
 ccode    (Intercept) 1.174
          lnGDP       0.145    -0.98
 Residual             0.462
```

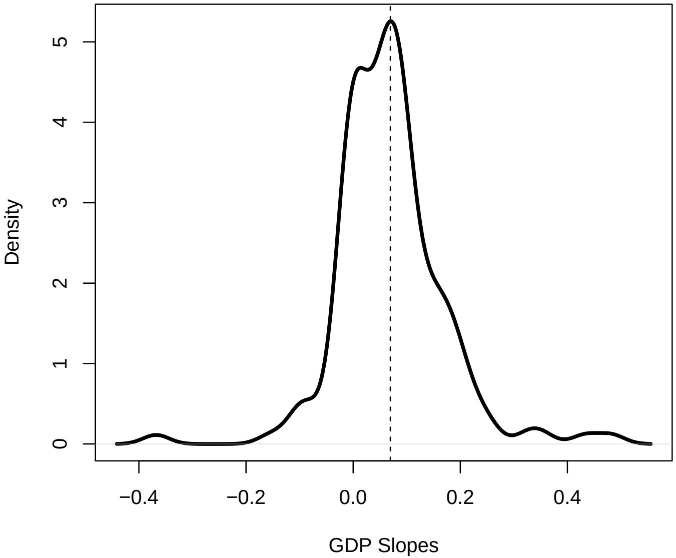
Random Coefficients

```
> Bs<-data.frame(coef(HLM1)[1])

> head(Bs)
   ccode..Intercept. ccode.POLITY ccode.I.POLITY.2. ccode.lnGDP ccode.Monarch ccode.ColdWar
2            0.45231     0.001981         -0.002065     0.09778       0.02635      -0.01394
20           0.12457     0.001981         -0.002065     0.02329       0.02635      -0.01394
42          -0.45766     0.001981         -0.002065     0.10707       0.02635      -0.01394
51          -0.04898     0.001981         -0.002065     0.05320       0.02635      -0.01394
52           0.44058     0.001981         -0.002065    -0.02591       0.02635      -0.01394
70          -1.34265     0.001981         -0.002065     0.20813       0.02635      -0.01394


> mean(Bs$ccode..Intercept.)
[1] -0.2156


> mean(Bs$ccode.lnGDP)
[1] 0.06938
```

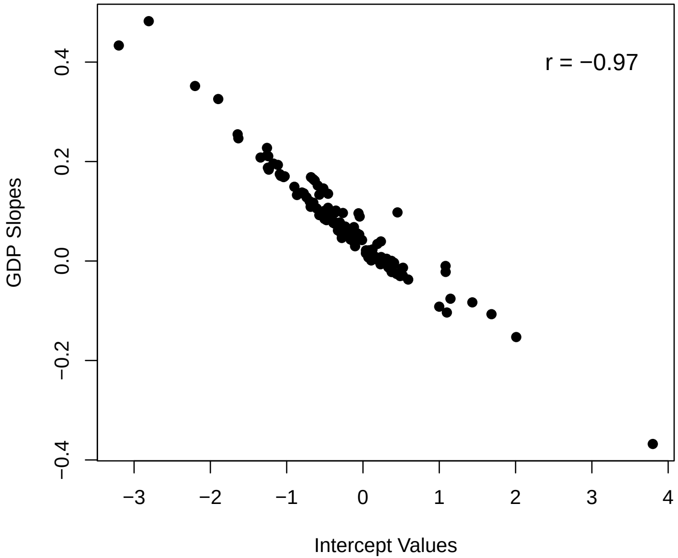
Random Intercepts (Plotted)



Random Slopes for `lnGDP` (Plotted)



Scatterplot: Random Intercepts and Slopes



Separating Within and Between Effects

Recall that we can decompose the variance of our regression model as:

$$Y_{it} = \bar{\mathbf{X}}_i \boldsymbol{\beta}_B + (\mathbf{X}_{it} - \bar{\mathbf{X}}_i)\boldsymbol{\beta}_W + u_{it} \tag{5}$$

This raises the possibility of fitting the model in (5) directly...

- Simple to estimate (via OLS or other)
- Relatively easy interpretation
- Easy to test $\hat{\boldsymbol{\beta}}_B = \hat{\boldsymbol{\beta}}_W$

Example data: Separate effects for within- and between-country wealth (GDP)...

Combining Within- and Between-Effects

Table: BE + WE Model of Demonstrations

	WEBE.OLS
POLITY	0.007^{***}
	(0.001)
POLITY Squared	-0.002^{***}
	(0.0003)
Within-Country ln(GDP)	0.082^{***}
	(0.016)
Between-Country ln(GDP)	0.053^{***}
	(0.008)
Monarch	-0.042
	(0.029)
Cold War	0.040^{**}
	(0.016)
Constant	-0.091
	(0.062)
Observations	6,505
R^2	0.027
Adjusted R^2	0.026
Residual Std. Error	0.526 (df = 6498)
F Statistic	29.510^{***} (df = 6; 6498)

$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Two-Way Unit Effects

Our original decomposition considered "two-way" effects:

$$Y_{it} = \mathbf{X}_{it}\boldsymbol{\beta} + \alpha_i + \eta_t + u_{it}$$

This implies that we can use (e.g.) an F-test to examine the hypothesis:

$$H_0 : \alpha_i = \eta_t = 0 \ \forall \ i, \ t$$

...that is, whether adding the (two-way) effects improves the model's fit.

We can also consider the partial hypotheses:

$$H_0 : \alpha_i = 0 \ \forall \ i$$

and

$$H_0 : \eta_t = 0 \ \forall \ t$$

separately.

Two-Way Effects: Good & Bad

The Good:

- Addresses both time-invariant, unit-level heterogeneity and cross-sectionally-invariant, time-point-specific heterogeneity
- May be "fixed" or "random"...
- Two-way FE is equivalent to differences-in-differences when $N = T = 2$ (more on that later)

The (Potentially) Bad:

- Ability to control for unobserved heterogeneity depends on (probably invalid) functional form assumptions (Imai and Kim 2020)
- FE *requires* predictors that vary both within and between both units and time points
- RE requires the (additional) assumption that $\text{Cov}(\mathbf{X}_{it}, \eta_t) = Cov(\alpha_i, \eta_t) = 0$
- Two-way effects models ask a *lot* of your data (effectively fits $N + T + k$ parameters using NT observations)

Example: Two-Way Fixed Effects

```
> TwoWayFE<-plm(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+ColdWar,
+          data=PDF,effect="twoway",model="within")
>
> summary(TwoWayFE)

Twoways effects Within Model

Call:
plm(formula = lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch +
    ColdWar, data = PDF, effect = "twoway", model = "within")

Unbalanced Panel: n = 145, T = 1-62, N = 6505

Residuals:
   Min. 1st Qu.  Median 3rd Qu.    Max.
-1.4299 -0.2154 -0.0790  0.0674  3.7220

Coefficients:
             Estimate Std. Error t-value Pr(>|t|)
POLITY       0.001864   0.001552    1.20    0.230
I(POLITY^2) -0.002135   0.000293   -7.29  3.4e-13 ***
lnGDP       -0.013235   0.018399   -0.72    0.472
Monarch      0.112024   0.067260    1.67    0.096 .
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Total Sum of Squares:    1350
Residual Sum of Squares: 1330
R-Squared:      0.009
Adj. R-Squared: -0.0239
F-statistic: 14.2917 on 4 and 6295 DF, p-value: 1.29e-11
```

Two-Way Effects: Testing

```
> # Two-way effects:
>
> pFtest(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+ColdWar,
+        data=PDF,effect="twoway",model="within")

F test for twoways effects

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
F = 11, df1 = 204, df2 = 6295, p-value <2e-16
alternative hypothesis: significant effects

> plmtest(TwoWayFE,c("twoways"),type=("kw"))

Lagrange Multiplier Test - two-ways effects (King and Wu) for unbalanced
panels

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
normal = 59, p-value <2e-16
alternative hypothesis: significant effects


> # One-way effects in the two-way model:
>
> plmtest(TwoWayFE,c("individual"),type=("kw"))

Lagrange Multiplier Test - (King and Wu) for unbalanced panels

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
normal = 90, p-value <2e-16
alternative hypothesis: significant effects

> plmtest(TwoWayFE,c("time"),type=("kw"))

Lagrange Multiplier Test - time effects (King and Wu) for unbalanced panels

data:  lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch + ColdWar
normal = 12, p-value <2e-16
alternative hypothesis: significant effects
```

Two-Way Fixed Effects via `lm`

```
> TwoWayFE.BF<-lm(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+
+                    factor(ccode)+factor(Year),data=PDF)

> summary(TwoWayFE.BF)

Call:
lm(formula = lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch +
    factor(ccode) + factor(Year), data = PDF)

Coefficients:
                  Estimate Std. Error t value Pr(>|t|)
(Intercept)       1.353540   0.195636    6.92  5.0e-12 ***
POLITY            0.001864   0.001552    1.20  0.22984
I(POLITY^2)      -0.002135   0.000293   -7.29  3.4e-13 ***
lnGDP            -0.013235   0.018399   -0.72  0.47198
Monarch           0.112024   0.067260    1.67  0.09586 .
factor(ccode)20  -1.150899   0.082756  -13.91  < 2e-16 ***
factor(ccode)42  -1.111681   0.091558  -12.14  < 2e-16 ***
factor(ccode)51  -1.172966   0.096647  -12.14  < 2e-16 ***
.
.
.
factor(Year)2000  0.361636   0.088496    4.09  4.4e-05 ***
factor(Year)2001  0.249294   0.088601    2.81  0.00491 **
 [ reached getOption("max.print") -- omitted 10 rows ]
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Residual standard error: 0.46 on 6295 degrees of freedom
  (2863 observations deleted due to missingness)
Multiple R-squared:  0.278,Adjusted R-squared:  0.254
F-statistic: 11.6 on 209 and 6295 DF,  p-value: <2e-16
```

Example: Two-Way Random Effects

```
> TwoWayRE<-plm(lnDemons~POLITY+I(POLITY^2)+lnGDP+Monarch+ColdWar,
+              data=PDF,effect="twoway",model="random")
>
> summary(TwoWayRE)

Twoways effects Random Effect Model
   (Swamy-Arora's transformation)

Call:
plm(formula = lnDemons ~ POLITY + I(POLITY^2) + lnGDP + Monarch +
    ColdWar, data = PDF, effect = "twoway", model = "random")

Unbalanced Panel: n = 145, T = 1-62, N = 6505

Effects:
                  var std.dev share
idiosyncratic 0.21186 0.46028  0.77
individual    0.05647 0.23763  0.21
time          0.00527 0.07258  0.02
theta:
         Min. 1st Qu. Median   Mean 3rd Qu.   Max.
id    0.11143  0.7136 0.7406 0.7289  0.7611 0.7611
time  0.27832  0.4773 0.4942 0.4840  0.5263 0.5289
total 0.09787  0.4485 0.4720 0.4571  0.4943 0.5081

Residuals:
   Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
  -0.41   -0.25   -0.18    0.03   -0.07    4.09

Coefficients:
             Estimate Std. Error z-value Pr(>|z|)
(Intercept) -0.003886   0.244584   -0.02  0.98732
POLITY       0.003325   0.003132    1.06  0.28847
I(POLITY^2) -0.002048   0.000615   -3.33  0.00088 ***
lnGDP        0.042258   0.028693    1.47  0.14082
Monarch      0.021130   0.114872    0.18  0.85405
ColdWar     -0.045119   0.053826   -0.84  0.40190
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1   1

Total Sum of Squares:    1850
Residual Sum of Squares: 1820
R-Squared:      0.0181
Adj. R-Squared: 0.0174
Chisq: 92.4819 on 5 DF, p-value: <2e-16
```

A Prettier Table

Table: Models of Demonstrations

	OLS	FE	BE	RE	TwoWayFE	TwoWayRE
POLITY	0.006^{***}	0.002	0.006	0.003^{*}	0.002	0.003
	(0.001)	(0.002)	(0.005)	(0.001)	(0.002)	(0.003)
POLITY Squared	-0.002^{***}	-0.002^{***}	-0.003^{***}	-0.002^{***}	-0.002^{***}	-0.002^{***}
	(0.0003)	(0.0003)	(0.001)	(0.0003)	(0.0003)	(0.001)
ln(GDP)	0.058^{***}	0.055^{***}	0.069^{**}	0.057^{***}	-0.013	0.042
	(0.008)	(0.015)	(0.027)	(0.012)	(0.018)	(0.029)
Monarch	-0.046	0.048	-0.050	-0.007	0.112^{*}	0.021
	(0.029)	(0.068)	(0.103)	(0.053)	(0.067)	(0.115)
Cold War	0.028^{**}	-0.035^{**}	0.259^{***}	-0.024		-0.045
	(0.014)	(0.016)	(0.085)	(0.015)		(0.054)
Constant	-0.125^{**}		-0.306	-0.132		-0.004
	(0.058)		(0.208)	(0.105)		(0.245)
Observations	6,505	6,505	145	6,505	6,505	6,505
R^2	0.026	0.013	0.127	0.012	0.009	0.018
Adjusted R^2	0.025	-0.010	0.096	0.012	-0.024	0.017
Residual Std. Error	0.526 (df = 6499)					

$^{*}p<0.1$; $^{**}p<0.05$; $^{***}p<0.01$

Other Variations: FEIS

"Fixed Effects Individual Slope" models

- Cite: Bruederl, Josef, and Volker Ludwig. 2015. "Fixed-Effects Panel Regression." In *The Sage Handbook of Regression Analysis and Causal Inference*, Eds. Henning Best and Christof Wolf. Los Angeles: Sage, pp. 327-357.
- FE + unit-level slopes for (some / all) predictor variables
- Equivalent to including $N - 1$ interactions between a predictor $\mathbf{X}$ and each of the α_is
- Also can test for homogeneity of estimated slopes (Hausman-like test)
- See the `feisr` R package, and its accompanying vignette, or `xtfeis` in Stata

Unit Effects Models: Software

R :

- the `plm` package; `plm` command
  - Fits one- and two-way FE, BE, RE models
  - Also fits first difference (FD) and instrumental variable (IV) models
- the `fixest` package; fast/scalable FE estimation for OLS and GLMs
- the `lme4` package; command is `lmer`
- the `nlme` package; command `lme`
- the `Paneldata` package

Stata : `xtreg`

- option `re` (the default) = random effects
- option `fe` = fixed (within) effects
- option `be` = between-effects