

# **GSERM - St. Gallen 2022**

## Analyzing Panel Data

June 10, 2022

Start with:

$$Y_i^* = \mathbf{X}_i\beta + u_i$$

$$Y_i = 0 \text{ if } Y_i^* < 0$$

$$Y_i = 1 \text{ if } Y_i^* \geq 0$$

So:

$$\begin{aligned} \Pr(Y_i = 1) &= \Pr(Y_i^* \geq 0) \\ &= \Pr(\mathbf{X}_i\beta + u_i \geq 0) \\ &= \Pr(u_i \geq -\mathbf{X}_i\beta) \\ &= \Pr(u_i \leq \mathbf{X}_i\beta) \\ &= \int_{-\infty}^{\mathbf{X}_i\beta} f(u) du \end{aligned}$$

“Standard logistic” PDF:

$$\Pr(u) \equiv \lambda(u) = \frac{\exp(u)}{[1 + \exp(u)]^2}$$

CDF:

$$\begin{aligned}\Lambda(u) &= \int \lambda(u) du \\ &= \frac{\exp(u)}{1 + \exp(u)} \\ &= \frac{1}{1 + \exp(-u)}\end{aligned}$$

## Logistic $\rightarrow$ “Logit”

$$\begin{aligned}\Pr(Y_i = 1) &= \Pr(Y_i^* > 0) \\ &= \Pr(u_i \leq \mathbf{X}_i\boldsymbol{\beta}) \\ &= \Lambda(\mathbf{X}_i\boldsymbol{\beta}) \\ &= \frac{\exp(\mathbf{X}_i\boldsymbol{\beta})}{1 + \exp(\mathbf{X}_i\boldsymbol{\beta})}\end{aligned}$$

$$\text{(equivalently)} = \frac{1}{1 + \exp(-\mathbf{X}_i\boldsymbol{\beta})}$$

$$L_i = \left( \frac{\exp(\mathbf{X}_i\beta)}{1 + \exp(\mathbf{X}_i\beta)} \right)^{Y_i} \left[ 1 - \left( \frac{\exp(\mathbf{X}_i\beta)}{1 + \exp(\mathbf{X}_i\beta)} \right) \right]^{1-Y_i}$$

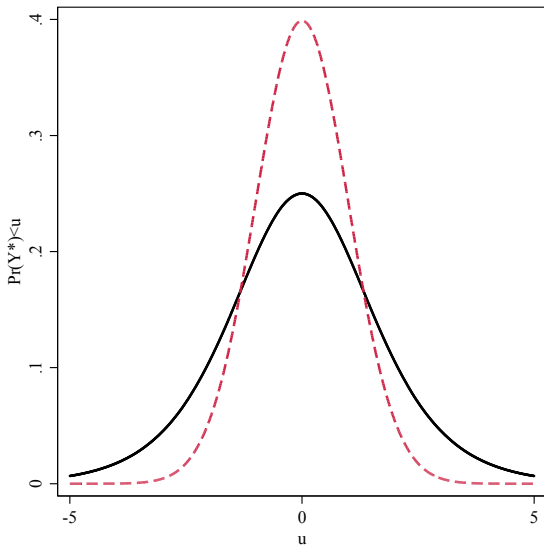
$$L = \prod_{i=1}^N \left( \frac{\exp(\mathbf{X}_i\beta)}{1 + \exp(\mathbf{X}_i\beta)} \right)^{Y_i} \left[ 1 - \left( \frac{\exp(\mathbf{X}_i\beta)}{1 + \exp(\mathbf{X}_i\beta)} \right) \right]^{1-Y_i}$$

$$\begin{aligned} \ln L &= \sum_{i=1}^N Y_i \ln \left( \frac{\exp(\mathbf{X}_i\beta)}{1 + \exp(\mathbf{X}_i\beta)} \right) + \\ &\quad (1 - Y_i) \ln \left[ 1 - \left( \frac{\exp(\mathbf{X}_i\beta)}{1 + \exp(\mathbf{X}_i\beta)} \right) \right] \end{aligned}$$

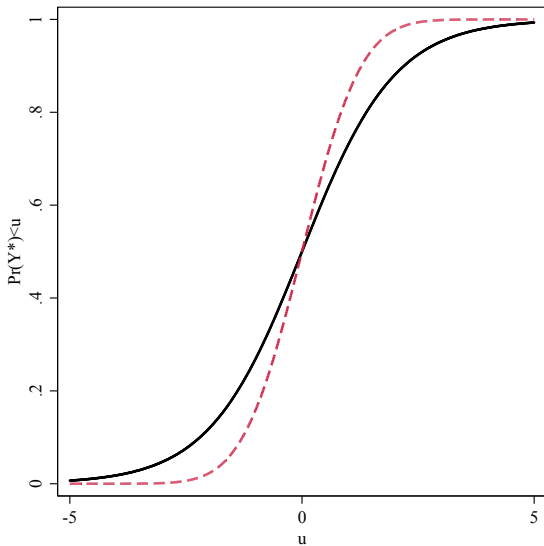
$$\Pr(u) \equiv \phi(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right)$$

$$\Phi(u) = \int_{-\infty}^u \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

# Standard Normal and Logistic PDFs



# Standard Normal and Logistic CDFs





$$\begin{aligned}\Pr(Y_i = 1) &= \Phi(\mathbf{X}_i\beta) \\ &= \int_{-\infty}^{\mathbf{X}_i\beta} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(\mathbf{X}_i\beta)^2}{2}\right) d\mathbf{X}_i\beta\end{aligned}$$

$$L = \prod_{i=1}^N [\Phi(\mathbf{X}_i\beta)]^{Y_i} [1 - \Phi(\mathbf{X}_i\beta)]^{(1-Y_i)}$$

$$\ln L = \sum_{i=1}^N Y_i \ln \Phi(\mathbf{X}_i\beta) + (1 - Y_i) \ln [1 - \Phi(\mathbf{X}_i\beta)]$$

# Panel / TSCS: What Can Go Wrong?

Suppose:

$$\begin{aligned}X_{it} &= \rho_X \mathbf{X}_{it-1} + \nu_{it} \\ u_{it} &= \rho_u u_{it-1} + \epsilon_{it}\end{aligned}$$

For high values of  $\rho$ , logit/probit:

- $\hat{\beta}$ s are consistent, but s.e.s are biased, inefficient (Poirier and Ruud 1988);
- $\rightarrow$  underestimate  $\text{Var}(\beta)$  by up to 50 percent (Beck and Katz 1997).

One-way unit effects:

$$Y_{it} = f(\mathbf{X}_{it}\beta + \alpha_i + u_{it})$$

for logit only, so:

$$\Pr(Y_{it} = 1) = \frac{\exp(\mathbf{X}_{it}\beta + \alpha_i)}{1 + \exp(\mathbf{X}_{it}\beta + \alpha_i)} \equiv \Lambda(\mathbf{X}_{it}\beta + \alpha_i)$$

Incidental Parameters:

- Nonlinearity  $\rightarrow$  inconsistency in both  $\hat{\alpha}$ s and  $\hat{\beta}$ .
- Anderson:

$$L^U = \prod_{i=1}^N \prod_{t=1}^T \Lambda(\mathbf{X}_{it} + \alpha_i)^{Y_{it}} [1 - \Lambda(\mathbf{X}_{it} + \alpha_i)]^{1-Y_{it}}$$

- Chamberlain:

$$L^C = \prod_{i=1}^N \Pr \left( Y_{i1} = y_{i1}, Y_{i2} = y_{i2}, \dots, Y_{iT} = y_{iT} \mid \sum_{t=1}^T Y_{it} \right)$$

## Fixed-Effects (continued)

Intuition: Suppose we have  $T = 2$ . That means that:

- $\Pr(Y_{i1} = 0 \text{ and } Y_{i2} = 0 \mid \sum_T Y_{it} = 0) = 1.0$
- $\Pr(Y_{i1} = 1 \text{ and } Y_{i2} = 1 \mid \sum_T Y_{it} = 2) = 1.0$

and:

$$\Pr\left(Y_{i1} = 0 \text{ and } Y_{i2} = 1 \mid \sum_T Y_{it} = 1\right) = \frac{\Pr(0, 1)}{\Pr(0, 1) + \Pr(1, 0)}$$

with a similar statement for  $\Pr(Y_{i1} = 1 \text{ and } Y_{i2} = 0 \mid \sum_T Y_{it} = 1)$ .

Points:

- Fixed effects = no estimates for  $\beta_b$
- Interpretation: per logit, but  $\mid \hat{\alpha}_i$ .
- BTSCS in IR: Green et al. (2001) v. B&K (2001).

Model is:

$$\begin{aligned} Y_{it}^* &= \mathbf{X}_{it}\beta + u_{it} \\ Y_{it} &= 0 \text{ if } Y_{it}^* \leq 0 ; \\ &= 1 \text{ if } Y_{it}^* > 0 \end{aligned}$$

with:

$$u_{it} = \alpha_i + \eta_{it}$$

with  $\eta_{it} \sim \text{i.i.d. } N(0,1)$ , and  $\alpha_i \sim N(0, \sigma_\alpha^2)$ . This implies:

$$\text{Var}(u_{it}) = 1 + \sigma_\alpha^2$$

and so:

$$\text{Corr}(u_{it}, u_{is}, t \neq s) \equiv \rho = \frac{\sigma_\alpha^2}{1 + \sigma_\alpha^2}$$

which means that we can write  $\sigma_\alpha^2 = \left(\frac{\rho}{1-\rho}\right)$ .

Probit:

$$\begin{aligned} L_i &= \text{Prob}(Y_{i1} = y_{i1}, Y_{i2} = y_{i2}, \dots Y_{iT} = y_{iT}) \\ &= \int_{-\infty}^{X_{i1}\beta} \int_{-\infty}^{X_{i2}\beta} \dots \int_{-\infty}^{X_{iT}\beta} \phi(u_{i1}, u_{i2} \dots u_{iT}) du_{iT} \dots du_{i2} du_{i1} \end{aligned}$$

Logit:

$$\begin{aligned} L_i &= \text{Prob}(Y_{i1} = y_{i1}, Y_{i2} = y_{i2}, \dots Y_{iT} = y_{iT}) \\ &= \int_{-\infty}^{X_{i1}\beta} \int_{-\infty}^{X_{i2}\beta} \dots \int_{-\infty}^{X_{iT}\beta} \lambda(u_{i1}, u_{i2} \dots u_{iT}) du_{iT} \dots du_{i2} du_{i1} \end{aligned}$$

Solution?

$$\phi(u_{i1}, u_{i2}, \dots u_{iT}) = \int_{-\infty}^{\infty} \phi(u_{i1}, u_{i2}, \dots u_{iT} \mid \alpha_i) \phi(\alpha_i) d\alpha_i$$

- $\hat{\rho}$  = proportion of the variance due to the  $\alpha_i$ s.
- Implementation: Gauss-Hermite quadrature or MCMC.
- Best with  $N$  large and  $T$  small.
- Critically requires  $\text{Cov}(\mathbf{X}, \alpha) = 0$  (see notes re: Chamberlain's CRE Estimator).

# Unit Effects in Practice - Some Simulations

Start with:

$$\begin{aligned} Y_{it}^* &= 0 + (1 \times X_{it}) + (1 \times D_{it}) + (1 \times \alpha_i) + u_{it} \\ Y_{it} \in \{0, 1\} &= f(Y_{it}^*) \end{aligned}$$

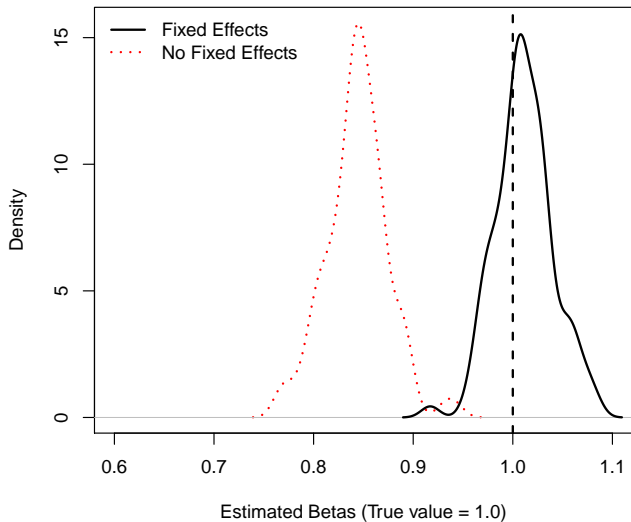
where:

- $\alpha_i \sim N(0, 1)$
- $X_{it} \sim N(0, \sigma_X^2)$
- $D_{it} \in \{0, 1\}$
- $\text{Cov}(X_{it}, \alpha_i) = \{0, 0.69\}$
- $\text{Cov}(D_{it}, \alpha_i) = 0$
- $f(\cdot) = \{\text{logit, probit}\}$  (as appropriate)

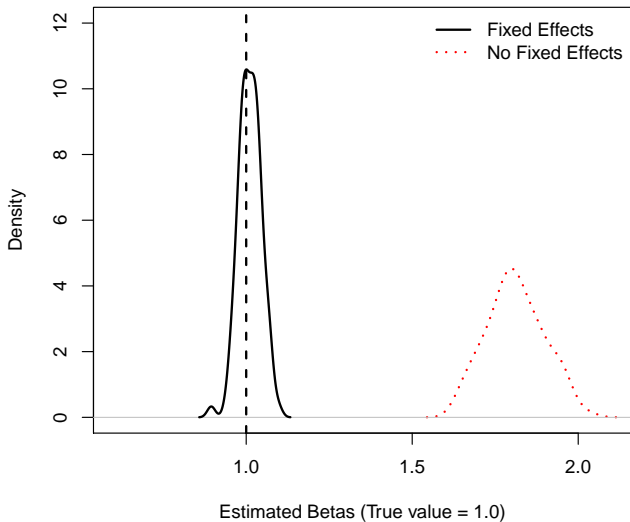
and  $N = T = 100$ .



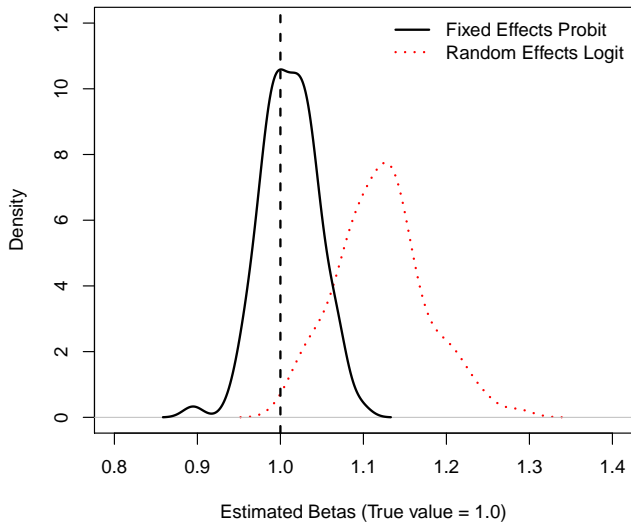
Logit  $\hat{\beta}_X$ s for  $\text{Cov}(X_{it}, \alpha_i) = 0$



Logit  $\hat{\beta}_{Xs}$  for  $\text{Cov}(X_{it}, \alpha_i) \approx 0.69$



Logit  $\hat{\beta}_{Xs}$  for  $\text{Cov}(X_{it}, \alpha_i) \approx 0.69$



## R

- `pglm` (panel GLMs) (maximum likelihood + quadrature)
- `bife` (fixed-effects logit / probit only)
- `glmer` (general mixed-effects models, including RE)
- `glmmML` (via Gauss-Hermite quadrature)
- `MCMCpack` (`MCMChlogit`)
- Various user-generated functions (e.g., [here](#)).

## Stata

- `xtprobit`, `xtlogit`, `xtcloglog`
- Plus `xttrans` (transition probabilities), `quadchk` (quadrature checking), `xtrho` / `xtrhoi` (estimation of within-unit covariances)

# Example: WDI “Plus”

## Data from the WDI plus POLITY and the UCDP:

- IS03 - The country's International Standards Organization (ISO) three-letter identification code.
- Year - The year that row of data applies to.
- CivilWar - Civil conflict indicator: 1 if there was a civil conflict in that country in that year; 0 otherwise. From UCDP.
- OnsetCount - The sum of new conflict episodes in that country / year. From UCDP.
- LandArea - Land area (sq. km).
- PopMillions - Population (in millions).
- PopGrowth - Population Growth (percent).
- UrbanPopulation - Urban Population (percent of total).
- GDPPerCapita - GDP per capita (constant 2010 \$US).
- GDPPerCapGrowth - GDP Per Capita Growth (percent annual).
- PostColdWar - 1 if Year > 1989, 0 otherwise.
- POLITY - The POLITY score of democracy/autocracy. Scaled so that 0 = most autocratic, 10 = most democratic.

$N = 216$ ,  $\bar{T} = 61$ ,  $NT$  varies (due to missingness).

```
> describe(DF,skew=FALSE)
```

	vars	n	mean	sd	min	max	range	se
IS03*	1	13392	108.50	62.36	1.00	216.0	215.00	0.54
Year*	2	13392	31.50	17.90	1.00	62.0	61.00	0.15
country*	3	13330	108.00	62.07	1.00	215.0	214.00	0.54
CivilWar	4	9052	0.13	0.34	0.00	1.0	1.00	0.00
OnsetCount	5	9394	0.05	0.24	0.00	4.0	4.00	0.00
LandArea	6	12906	613525.38	1766486.19	2.03	16389950.0	16389947.97	15549.43
PopMillions	7	13073	24.64	103.13	0.00	1410.9	1410.93	0.90
UrbanPopulation	8	13045	51.39	25.74	2.08	100.0	97.92	0.23
GDPPerCapita	9	9582	11685.74	18675.05	144.20	181709.3	181565.14	190.78
GDPPerCapGrowth	10	9598	1.89	6.21	-64.99	140.4	205.36	0.06
PostColdWar	11	13330	0.52	0.50	0.00	1.0	1.00	0.00
POLITY	12	8279	5.55	3.71	0.00	10.0	10.00	0.04
POLITYSquared	13	8279	44.57	40.24	0.00	100.0	100.00	0.44

# Pooled Logit

```
> Logit<-glm(CivilWar~log(LandArea)+log(PopMillions)+
+           UrbanPopulation+log(GDPPerCapita)+
+           GDPPerCapGrowth+PostColdWar+POLITY+
+           POLITYSquared,data=DF,family="binomial")

> summary(Logit)

Call:
glm(formula = CivilWar ~ log(LandArea) + log(PopMillions) + UrbanPopulation +
    log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar + POLITY +
    POLITYSquared, family = "binomial", data = DF)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)   -1.03275    0.52731   -1.96  0.05017 .
log(LandArea)    0.01085    0.03246    0.33  0.73815
log(PopMillions) 0.66364    0.03696   17.96 < 2e-16 ***
UrbanPopulation  0.01090    0.00335    3.26  0.00113 **
log(GDPPerCapita) -0.50128    0.06128   -8.18  2.8e-16 ***
GDPPerCapGrowth -0.04029    0.00644   -6.26  3.9e-10 ***
PostColdWar     -0.31102    0.08588   -3.62  0.00029 ***
POLITY          0.67438    0.06122   11.02 < 2e-16 ***
POLITYSquared   -0.06526    0.00579  -11.27 < 2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1 1

(Dispersion parameter for binomial family taken to be 1)

    Null deviance: 5843.6  on 6996  degrees of freedom
Residual deviance: 4624.8  on 6988  degrees of freedom
(6395 observations deleted due to missingness)
AIC: 4643
```

# Fixed Effects

```
> FELogit<-bife(CivilWar~log(LandArea)+log(PopMillions)+
+               UrbanPopulation+log(GDPPerCapita)+
+               GDPPerCapGrowth+PostColdWar+POLITY+
+               POLITYSquared|ISO3,data=DF,model="logit")

> summary(FELogit)
binomial - logit link

CivilWar ~ log(LandArea) + log(PopMillions) + UrbanPopulation +
  log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar + POLITY +
  POLITYSquared | ISO3

Estimates:
              Estimate Std. error z value Pr(> |z|)
log(LandArea)   -4.00079    6.80808   -0.59  0.5568
log(PopMillions)  0.79303    0.29847    2.66  0.0079 **
UrbanPopulation  0.01179    0.01228    0.96  0.3368
log(GDPPerCapita) -0.33859    0.17226   -1.97  0.0493 *
GDPPerCapGrowth  -0.04960    0.00833   -5.96  2.6e-09 ***
PostColdWar      -0.21475    0.17822   -1.20  0.2282
POLITY           0.70692    0.09365    7.55  4.4e-14 ***
POLITYSquared    -0.07382    0.00890   -8.29  < 2e-16 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

residual deviance= 2846,
null deviance= 4422,
nT= 3971, N= 83

( 6395 observation(s) deleted due to missingness )
( 3026 observation(s) deleted due to perfect classification )

Number of Fisher Scoring Iterations: 6

Average individual fixed effect= 48.24
```



# Random Effects

```
> RELogit<-pglm(CivilWar~log(LandArea)+log(PopMillions)+
+               UrbanPopulation+log(GDPPerCapita)+
+               GDPPerCapGrowth+PostColdWar+POLITY+
+               POLITYSquared|ISO3,data=DF,family=binomial,
+               effect="individual",model="random")

> summary(RELogit)
-----
Maximum Likelihood estimation
Newton-Raphson maximisation, 18 iterations
Return code 2: successive function values within tolerance limit (tol)
Log-Likelihood: -1634
10 free parameters
Estimates:
      Estimate Std. error t value Pr(> t)
(Intercept)  -4.08609    1.02028   -4.00 6.2e-05 ***
log(LandArea)   0.15120    0.05920    2.55 0.01065 *
log(PopMillions) 1.20067    0.08537   14.06 < 2e-16 ***
UrbanPopulation  0.01973    0.00598    3.30 0.00097 ***
log(GDPPerCapita) -0.61681    0.11732   -5.26 1.5e-07 ***
GDPPerCapGrowth -0.04979    0.00816   -6.10 1.1e-09 ***
PostColdWar     -0.38811    0.12189   -3.18 0.00145 **
POLITY          0.68171    0.08400    8.12 4.9e-16 ***
POLITYSquared   -0.07368    0.00811   -9.08 < 2e-16 ***
sigma           2.29777    0.11784   19.50 < 2e-16 ***
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1 1
-----
```

Models of Civil War

	Logit	FE Logit	RE Logit
Intercept	-1.03 (0.53)		-4.09* (1.02)
ln(Land Area)	0.01 (0.03)	-4.00 (6.81)	0.15* (0.06)
ln(Population)	0.66* (0.04)	0.79* (0.30)	1.20* (0.09)
Urban Population	0.01* (0.00)	0.01 (0.01)	0.02* (0.01)
ln(GDP Per Capita)	-0.50* (0.06)	-0.34* (0.17)	-0.62* (0.12)
GDP Growth	-0.04* (0.01)	-0.05* (0.01)	-0.05* (0.01)
Post-Cold War	-0.31* (0.09)	-0.21 (0.18)	-0.39* (0.12)
POLITY	0.67* (0.06)	0.71* (0.09)	0.68* (0.08)
POLITY Squared	-0.07* (0.01)	-0.07* (0.01)	-0.07* (0.01)
Estimated Sigma			2.30* (0.12)
AIC	4642.76		3287.00
BIC	4704.44		
Log Likelihood	-2312.38	-1422.95	-1633.50
Deviance	4624.76	2845.89	
Num. obs.	6997	3971	

\* $p < 0.05$

# Censoring and Event Counts

“Lower” censored  $Y$ :

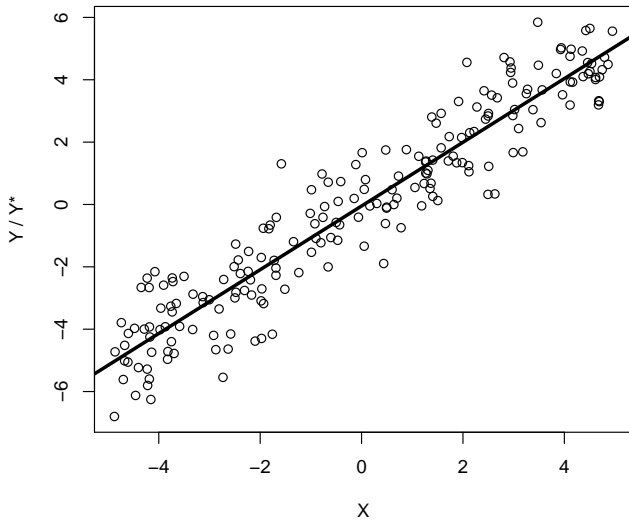
$$\begin{aligned} Y_i &= Y_i^* \text{ if } Y_i^* > L \\ &= L \text{ if } Y_i^* \leq L \end{aligned}$$

“Upper-censored”:

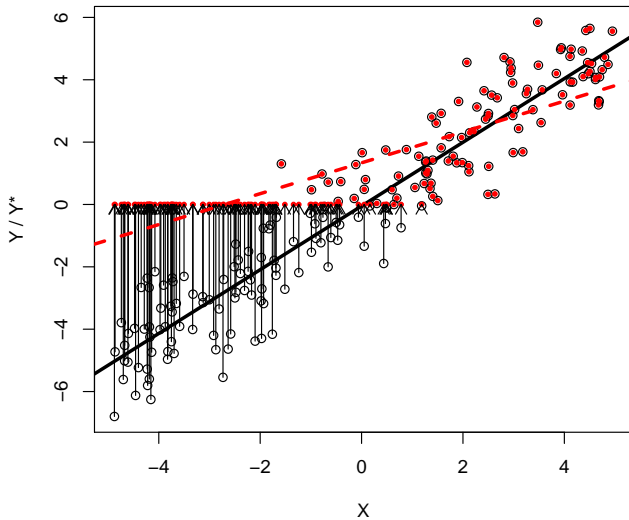
$$\begin{aligned} Y_i &= Y_i^* \text{ if } Y_i^* < L \\ &= U \text{ if } Y_i^* \geq L \end{aligned}$$

→ bias in  $\hat{\beta}$  (toward zero) + inconsistency...

# Censoring Bias



# Censoring Bias



In the lower-censoring case, for  $Y^* > L$ , we have:

$$\mathbf{L}_1(\beta, \sigma^2 | Y, L) = \prod_{Y_i > L} \phi(Y_i^* | \mathbf{X}_i, \beta, \sigma^2).$$

and for  $Y^* \leq L$ :

$$\begin{aligned} \Pr(Y_i = L) &= \Pr(Y_i^* \leq L) \\ &= \int_{-\infty}^L \phi(Y_i^* | \mathbf{X}_i, \beta, \sigma^2) dY^* \\ &= \Phi(L | \mathbf{X}_i, \beta, \sigma^2). \end{aligned}$$

which implies:

$$\mathbf{L}_2(\beta, \sigma^2 | Y, L) = \prod_{Y_i = L} \Phi(L | \mathbf{X}_i, \beta, \sigma^2).$$

Combined likelihood:

$$\mathbf{L}(\beta, \sigma^2 | Y, L) = \prod_{Y_i > L} \phi(Y_i^* | \mathbf{X}_i, \beta, \sigma^2) \prod_{Y_i = L} \Phi(L | \mathbf{X}_i, \beta, \sigma^2).$$

One-way unit effects:

$$Y_i^* = \mathbf{X}_i\boldsymbol{\beta} + \alpha_i + u_i$$

Models:

- No fixed-effects conditioning (a la logit)  $\rightarrow$  inconsistency.
- Generally use random effects (via `survival` or `xttobit`).



## Properties:

- Discrete / integer-values
- Non-negative
- “Cumulative”

## Motivation:

$$\text{Arrival Rate} = \lambda$$

$$\Pr(\text{Event})_{t,t+h} = \lambda h$$

$$\Pr(\text{No Event})_{t,t+h} = 1 - \lambda h$$

$$\begin{aligned}\Pr(Y_t = y) &= \frac{\exp(-\lambda h) \lambda h^y}{y!} \\ &= \frac{\exp(-\lambda) \lambda^y}{y!}\end{aligned}$$

# Poisson: Assumptions and Motivations

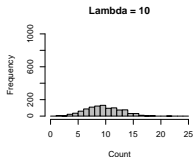
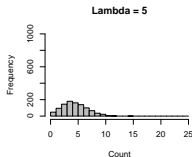
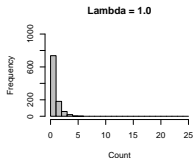
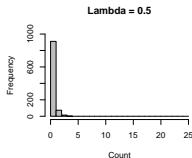
- No Simultaneous Events
- Constant Arrival Rate
- Independent Event Arrivals

Another motivation: For  $M$  independent Bernoulli trials with (sufficiently small) probability of success  $\pi$  and where  $M\pi \equiv \lambda > 0$ ,

$$\begin{aligned}\Pr(Y_i = y) &= \lim_{M \rightarrow \infty} \left[ \binom{M}{y} \left(\frac{\lambda}{M}\right)^y \left(1 - \frac{\lambda}{M}\right)^{M-y} \right] \\ &= \frac{\lambda^y \exp(-\lambda)}{y!}\end{aligned}$$

# Poisson: Characteristics

- Discrete
- $E(Y) = \text{Var}(Y) = \lambda$
- Is not preserved under affine transformations...
- For  $X \sim \text{Poisson}(\lambda_X)$  and  $Y \sim \text{Poisson}(\lambda_Y)$ ,  $Z = X + Y \sim \text{Poisson}(\lambda_{X+Y})$   
iff  $X$  and  $Y$  are independent but
- ...same is not true for differences.
- $\lambda \rightarrow \infty \iff Y \sim N$



Suppose

$$E(Y_i) \equiv \lambda_i = \exp(\mathbf{X}_i\beta)$$

then

$$\Pr(Y_i = y | \mathbf{X}_i, \beta) = \frac{\exp[-\exp(\mathbf{X}_i\beta)][\exp(\mathbf{X}_i\beta)]^y}{y!}$$

with likelihood:

$$L = \prod_{i=1}^N \frac{\exp[-\exp(\mathbf{X}_i\beta)][\exp(\mathbf{X}_i\beta)]^{Y_i}}{Y_i!}$$

and log-likelihood:

$$\ln L = \sum_{i=1}^N [-\exp(\mathbf{X}_i\beta) + Y_i\mathbf{X}_i\beta - \ln(Y_i!)]$$

# Event Counts: Unit Effects

$$Y_{it} \sim \text{Poisson}(\mu_{it} = \alpha_i \lambda_{it})$$

with  $\lambda_{it} = \exp(\mathbf{X}_{it}\beta)$  implies:

$$\begin{aligned} E(Y_{it} \mid \mathbf{X}_{it}, \alpha_i) &= \mu_{it} \\ &= \alpha_i \exp(\mathbf{X}_{it}\beta) \\ &= \exp(\delta_i + \mathbf{X}_{it}\beta) \end{aligned}$$

where  $\delta_i = \ln(\alpha_i)$ .

## Fixed-Effects Poisson:

- ...has no “incidental parameters” problem (see e.g. Cameron and Trivedi, pp. 281-2)
- This means “brute force” approach works
- Fitted via `glmml` in R, `xtpoisson` (and `xtnbreg`) in Stata

$$\begin{aligned}\Pr(Y_{i1} = y_{i1}, \dots, Y_{iT} = y_{iT}) &= \int_0^\infty \Pr(Y_{i1} = y_{i1}, \dots, Y_{iT} = y_{iT}) f(\alpha_i) d\alpha_i \\ &= \int_0^\infty \left[ \prod_{t=1}^T \Pr(Y_{it} | \alpha_i) \right] f(\alpha_i) d\alpha_i\end{aligned}$$

- Simplest to assume  $\alpha_i \sim \Gamma(\theta)$
- Yields a model with  $E(Y_{it}) = \lambda_{it}$  and  $\text{Var}(Y_{it}) = \lambda_{it} + \frac{\lambda_{it}^2}{\theta}$
- Via `glmmML` or `glmer` in R, or `xtpois`, `re` in Stata
- $\exists$  random effects negative binomial too...

R:

- Tobit = `censReg` (in **`censReg`**)
- Poisson (random effects) = `glmmML` in **`glmmML`** or `glmer` in **`lme4`**
- Poisson (fixed effects) = `glmmML` or “brute force”

Stata:

- Tobit = `xttobit` (re only)
- Poisson / negative binomial = `xtpoisson`, `xtnbreg` (both with `fe`, `re` options)

# Conflict Onsets: Pooled Poisson

```
> xtabs(~DF$OnsetCount)

DF$OnsetCount
  0    1    2    3    4
8981 375  30   7   1

> Poisson<-glm(OnsetCount~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+             GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared,data=DF,family="poisson")

> summary(Poisson)

Call:
glm(formula = OnsetCount ~ log(LandArea) + log(PopMillions) +
    UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar +
    POLITY + POLITYSquared, family = "poisson", data = DF)

Coefficients:
              Estimate Std. Error z value Pr(>|z|)
(Intercept)  -2.38261    0.72320  -3.29   0.00099 ***
log(LandArea)   0.06936    0.04693   1.48   0.13941
log(PopMillions) 0.42571    0.04569   9.32 < 2e-16 ***
UrbanPopulation  0.00603    0.00472   1.28   0.20106
log(GDPPerCapita) -0.42991    0.08086  -5.32 0.00000011 ***
GDPPerCapGrowth -0.03595    0.00641  -5.61 0.00000002 ***
PostColdWar     0.27202    0.12002   2.27   0.02343 *
POLITY          0.32968    0.08289   3.98 0.00006961 ***
POLITYSquared   -0.03636    0.00793  -4.59 0.00000449 ***
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

(Dispersion parameter for poisson family taken to be 1)

Null deviance: 2390.6 on 6996 degrees of freedom
Residual deviance: 1949.8 on 6988 degrees of freedom
(6395 observations deleted due to missingness)
AIC: 2704

Number of Fisher Scoring iterations: 6
```



# Fixed Effects Poisson

```
> FEPoisson<-pglm(OnsetCount~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+               GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared,data=DF,family="poisson",
+               effect="individual",model="within")
```

```
> summary(FEPoisson)
```

```
-----
Maximum Likelihood estimation
```

```
Newton-Raphson maximisation, 3 iterations
```

```
Return code 8: successive function values within relative tolerance limit (reltol)
```

```
Log-Likelihood: -1021
```

```
8 free parameters
```

```
Estimates:
```

	Estimate	Std. error	t value	Pr(> t)
log(LandArea)	-1.67100	2.83168	-0.59	0.55512
log(PopMillions)	0.61473	0.32126	1.91	0.05568 .
UrbanPopulation	-0.04603	0.01335	-3.45	0.00056 ***
log(GDPPerCapita)	-0.09145	0.14421	-0.63	0.52600
GDPPerCapGrowth	-0.02637	0.00654	-4.03	0.00005499 ***
PostColdWar	0.48566	0.19617	2.48	0.01330 *
POLITY	0.52507	0.10791	4.87	0.00000114 ***
POLITYSquared	-0.05379	0.01060	-5.07	0.00000039 ***

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

# Alternative Fixed Effects Poisson (using feglm)

```
> FEPoisson2<-feglm(OnsetCount~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+                 GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared|IS03,data=DF,family="poisson")
```

```
NOTES: 6,395 observations removed because of NA values (LHS: 3,998, RHS: 6,395).
       67 fixed-effects (2,499 observations) removed because of only 0 outcomes.
```

```
> summary(FEPoisson2,cluster="IS03")
```

```
GLM estimation, family = poisson, Dep. Var.: OnsetCount
```

```
Observations: 4,498
```

```
Fixed-effects: IS03: 93
```

```
Standard-errors: Clustered (IS03)
```

	Estimate	Std. Error	t value	Pr(> t )
log(LandArea)	-1.67100	2.159264	-0.7739	0.4390039115
log(PopMillions)	0.61473	0.340011	1.8080	0.0706106957 .
UrbanPopulation	-0.04603	0.019252	-2.3911	0.0167991301 *
log(GDPPerCapita)	-0.09145	0.151293	-0.6045	0.5455437492
GDPPerCapGrowth	-0.02637	0.006008	-4.3900	0.0000113372 ***
PostColdWar	0.48566	0.293791	1.6531	0.0983179526 .
POLITY	0.52507	0.112045	4.6862	0.0000027826 ***
POLITYSquared	-0.05379	0.011709	-4.5937	0.0000043554 ***

```
---
```

```
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Log-Likelihood: -1,156.1    Adj. Pseudo R2: 0.094671
```

```
BIC: 3,163.5    Squared Cor.: 0.162849
```

# Random Effects Poisson

```
> REPoisson<-glmer(OnsetCount~log(LandArea)+log(PopMillions)+UrbanPopulation+log(GDPPerCapita)+
+                 GDPPerCapGrowth+PostColdWar+POLITY+POLITYSquared+(1|ISO3),data=DF,family="poisson")
```

```
> summary(REPoisson)
```

```
Generalized linear mixed model fit by maximum likelihood (Laplace Approximation) ['glmerMod']
```

```
Family: poisson (log)
```

```
Formula: OnsetCount ~ log(LandArea) + log(PopMillions) + UrbanPopulation +
log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar + POLITY +
POLITYSquared + (1 | ISO3)
```

```
Data: DF
```

AIC	BIC	logLik	deviance	df.resid
2602	2670	-1291	2582	6987

```
Scaled residuals:
```

Min	1Q	Median	3Q	Max
-0.945	-0.227	-0.144	-0.086	17.093

```
Random effects:
```

Groups	Name	Variance	Std.Dev.
ISO3	(Intercept)	0.588	0.767

Number of obs: 6997, groups: ISO3, 160

```
Fixed effects:
```

	Estimate	Std. Error	z value	Pr(> z )
(Intercept)	-4.33127	1.09253	-3.96	0.0000735687 ***
log(LandArea)	0.07661	0.07524	1.02	0.309
log(PopMillions)	0.42058	0.08230	5.11	0.0000003215 ***
UrbanPopulation	-0.00756	0.00649	-1.16	0.244
log(GDPPerCapita)	-0.16788	0.10506	-1.60	0.110
GDPPerCapGrowth	-0.03182	0.00660	-4.82	0.0000014481 ***
PostColdWar	0.29773	0.12970	2.30	0.022 *
POLITY	0.49337	0.09700	5.09	0.0000003649 ***
POLITYSquared	-0.05419	0.00942	-5.75	0.0000000089 ***

```
---
```

```
Signif. codes: 0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

```
Correlation of Fixed Effects:
```

	(Intr)	lg(LA)	lg(PM)	UrbanPp	1(GDPP	GDPPCG	PostC1W	POLITY
log(LandAr)	-0.774							
lg(PpMlms)	0.395	-0.656						
UrbanPopltn	0.364	-0.043	-0.033					
lg(GDPPPrCp)	-0.589	0.020	0.022	-0.737				
GDPPPrCpGrwt	0.041	0.066	-0.106	0.126	-0.165			
PostColdWar	-0.112	0.186	-0.245	-0.218	0.035	-0.053		
POLITY	-0.278	0.006	-0.001	-0.075	0.214	0.066	-0.255	
POLITYSqurd	0.261	0.028	-0.038	0.052	-0.241	-0.065	0.208	-0.968

optimizer (Nelder\_Mead) convergence code: 0 (OK)

Model failed to converge with max|grad| = 0.116002 (tol = 0.002, component 1)

Model is nearly unidentifiable: very large eigenvalue

- Rescale variables?

# Alternative RE Poisson (using pg1m)

```
> REPoisson2<-pglm(OnsetCount~log(LandArea)+log(PopMillions)+
+               UrbanPopulation+log(GDPPerCapita)+
+               GDPPerCapGrowth+PostColdWar+POLITY+
+               POLITYSquared,data=DF,family="poisson",
+               effect="individual",model="random")

> summary(REPoisson2)

-----
Maximum Likelihood estimation
Newton-Raphson maximisation, 4 iterations
Return code 8: successive function values within relative tolerance limit (reltol)
Log-Likelihood: -1292
10 free parameters

Estimates:

```

	Estimate	Std. error	t value	Pr(>  t )
(Intercept)	-3.67347	1.05113	-3.49	0.00047 ***
log(LandArea)	0.05547	0.07325	0.76	0.44888
log(PopMillions)	0.44374	0.08003	5.54	0.000000030 ***
UrbanPopulation	-0.00613	0.00637	-0.96	0.33518
log(GDPPerCapita)	-0.19283	0.10268	-1.88	0.06038 .
GDPPerCapGrowth	-0.03201	0.00655	-4.88	0.000001044 ***
PostColdWar	0.29663	0.12891	2.30	0.02139 *
POLITY	0.47529	0.09584	4.96	0.000000708 ***
POLITYSquared	-0.05274	0.00929	-5.68	0.000000014 ***
sigma	1.70087	0.41233	4.12	0.000037074 ***

```
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

-----
```

Panel Event Count Models

	Poisson	FE Poisson	RE Poisson	Neg. Bin.	FE N.B.	RE N.B.
Intercept	-2.38* (0.72)		-4.33* (1.09)	-2.41* (0.74)	-62.39	-4.32* (1.09)
ln(Land Area)	0.07 (0.05)	-1.67 (2.83)	0.08 (0.08)	0.07 (0.05)	6.56	0.08 (0.08)
ln(Population)	0.43* (0.05)	0.61 (0.32)	0.42* (0.08)	0.42* (0.05)	1.25 (1.46)	0.42* (0.08)
Urban Population	0.01 (0.00)	-0.05* (0.01)	-0.01 (0.01)	0.01 (0.00)	-0.10 (0.08)	-0.01 (0.01)
ln(GDP Per Capita)	-0.43* (0.08)	-0.09 (0.14)	-0.17 (0.11)	-0.42* (0.08)	3.26* (1.25)	-0.17 (0.11)
GDP Growth	-0.04* (0.01)	-0.03* (0.01)	-0.03* (0.01)	-0.04* (0.01)	-0.07* (0.03)	-0.03* (0.01)
Post-Cold War	0.27* (0.12)	0.49* (0.20)	0.30* (0.13)	0.27* (0.12)	-0.57 (1.15)	0.30* (0.13)
POLITY	0.33* (0.08)	0.53* (0.11)	0.49* (0.10)	0.32* (0.09)	1.29* (0.59)	0.49* (0.10)
POLITY Squared	-0.04* (0.01)	-0.05* (0.01)	-0.05* (0.01)	-0.04* (0.01)	-0.10* (0.05)	-0.05* (0.01)
Estimated Sigma				0.06 (0.03)		
AIC	2704.01	2057.19	2601.46	2699.78	-1271.03	2603.46
BIC	2765.69		2670.00			2678.84
Log Likelihood	-1343.01	-1020.59	-1290.73	-1339.89	644.51	-1290.73
Deviance	1949.83					
Num. obs.	6997		6997			6997
Num. groups: ISO3			160			160
Var: ISO3 (Intercept)			0.59			0.59

\*  $p < 0.05$

# Wrap-Up: Some Useful Packages

- `pglm`
  - Workhorse package for panel (FE, RE, BE) GLMs
  - Binary + ordered logit/probit, Poisson / negative binomial
  - Discussed + used extensively in Croissant and Millo (2018) *Panel Data Econometrics with R*
  - The one thing it won't (apparently) do is fixed-effects, binary-response models...
- `fixest`
  - Fast / efficient fitting of FE models
  - Fits linear models, logit, Poisson, and negative binomial
  - Includes easy coefficient plots & tables; simple multi-threading; built-in "robust" S.E.s
- `alpaca`
  - Fast / efficient fitting of GLMs with high-dimensional fixed effects
  - *Includes bias correction for incidental parameters after binary-response models*
  - Also includes useful panel data simulation routines + average partial effects

# Generalized Estimating Equations

Linear-normal model is:

$$Y_i = \mu_i + u_i$$

with:

$$\mu_i = \mathbf{X}_i\boldsymbol{\beta}.$$

Generalize:

$$g(\mu_i) = \mathbf{X}_i\boldsymbol{\beta}$$

and:

$$Y_i \sim \text{i.i.d. } F[\mu_i, \mathbf{V}_i].$$



“Score” equations:

$$\mathbf{U}(\beta) = \sum_{i=1}^N \mathbf{D}_i' \mathbf{V}_i^{-1} [Y_i - \mu_i] = \mathbf{0}.$$

with:

- $\mathbf{D}_i = \frac{\partial \mu_i}{\partial \beta}$ ,
- $\mathbf{V}_i = \frac{h(\mu_i)}{\phi}$ , and
- $(Y_i - \mu_i) \approx$  a “residual.”
- Known as “quasi-likelihood” (e.g. Wedderburn 1974 *Biometrika*).

Now suppose:

$$Y_{it} = \mu_{it} + u_{it}$$

where

- $i \in \{1, \dots, N\}$  are i.i.d. “units,”
- $t \in \{1, \dots, T\}$ ,  $T > 1$  are “time points,”
- we want  $g(\mu_{it}) = \mathbf{X}_{it}\beta$ .

**Key issue:** Accounting for (conditional) dependence in  $Y$  over time.

Full joint distributions over  $T$  are hard. But...

Define:

$$\mathbf{R}_i(\boldsymbol{\alpha})_{T \times T} = \begin{pmatrix} 1.0 & \alpha_{12} & \cdots & \alpha_{1,T} \\ \alpha_{21} & 1.0 & \cdots & \alpha_{2,T} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{T,1} & \cdots & \alpha_{T,T-1} & 1.0 \end{pmatrix},$$

→ “working correlation” matrix.

- Completely defined by  $\boldsymbol{\alpha}$ ,
- Structure specified by the analyst.

Liang and Zeger (1986): We can decompose the variance of  $Y_{it}$  as:

$$\mathbf{V}_i = \text{diag}(\mathbf{V}_i^{\frac{1}{2}}) \mathbf{R}_i(\boldsymbol{\alpha}) \text{diag}(\mathbf{V}_i^{\frac{1}{2}})$$

With a standard GLM assumption about the mean and variance, this is:

$$\mathbf{V}_i = \frac{(\mathbf{A}_i^{\frac{1}{2}}) \mathbf{R}_i(\boldsymbol{\alpha}) (\mathbf{A}_i^{\frac{1}{2}})}{\phi}$$

where

$$\mathbf{A}_i = \begin{pmatrix} h(\mu_{i1}) & 0 & \cdots & 0 \\ 0 & h(\mu_{i2}) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & h(\mu_{iT}) \end{pmatrix}$$

$\mathbf{V}_i = \text{Var}(Y_{it} | \mathbf{X}_{it}, \beta)$  has two parts:

- $\mathbf{A}_i = \text{unit-level variation}$ ,
- $\mathbf{R}_i(\alpha) = \text{within-unit temporal variation}$ .

## Specifying $\mathbf{R}_i(\alpha)$

*Independent:*

$$\mathbf{R}_i(\alpha) = \begin{pmatrix} 1.0 & 0 & \dots & 0 \\ 0 & 1.0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 1.0 \end{pmatrix}$$

- Assumes no within-unit temporal correlation.
- Equivalent to GLM on pooled data.

*Exchangeable:*

$$\mathbf{R}_i(\alpha) = \begin{pmatrix} 1.0 & \alpha & \dots & \alpha \\ \alpha & 1.0 & \dots & \alpha \\ \vdots & \vdots & \ddots & \vdots \\ \alpha & \dots & \alpha & 1.0 \end{pmatrix}$$

- One free parameter in  $\mathbf{R}_i(\alpha)$  ( $\alpha_{ts} = \alpha \forall t \neq s$ )
- Temporal correlation within units is constant across time points.
- Akin (in some respects) to a random-effects model...

## Specifying $\mathbf{R}_i(\alpha)$

$AR(p)$  (e.g.,  $AR(1)$ ): 
$$\mathbf{R}_i(\alpha) = \begin{pmatrix} 1.0 & \alpha & \alpha^2 & \cdots & \alpha^{T-1} \\ \alpha & 1.0 & \alpha & \cdots & \alpha^{T-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha^{T-1} & \cdots & \alpha^2 & \alpha & 1.0 \end{pmatrix}$$

- One free parameter in  $\mathbf{R}_i(\alpha)$  ( $\alpha_{ts} = \alpha^{|t-s|} \forall t \neq s$ ).
- Conditional within-unit correlation an exponential function of the lag.

$Stationary(p)$ : 
$$\mathbf{R}_i(\alpha) = \begin{pmatrix} 1.0 & \alpha_1 & \cdots & \alpha_p & 0 & \cdots & 0 \\ \alpha_1 & 1.0 & \alpha_1 & \cdots & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & \alpha_p & \cdots & \alpha_1 & 1.0 \end{pmatrix}$$

- AKA “banded,” or “ $p$ -dependent.”
- $p \leq T - 1$  free parameters in  $\mathbf{R}_i(\alpha)$ .
- Conditional within-unit correlation an exponential function of the lag, up to lag  $p$ , and zero thereafter.

*Unstructured:*  $\mathbf{R}_i(\alpha) = \begin{pmatrix} 1.0 & \alpha_{12} & \alpha_{13} & \cdots & \alpha_{1,T-1} \\ \alpha_{12} & 1.0 & \alpha_{23} & \cdots & \alpha_{2,T-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \alpha_{1,T-1} & \alpha_{2,T-1} & \cdots & \alpha_{T-1,T-1} & 1.0 \end{pmatrix}$

- $\frac{T(T-1)}{2}$  free parameters in  $\mathbf{R}_i(\alpha)$ .
- Conditional within-unit correlation is completely data-dependent.



Score equations:

$$\mathbf{U}_{GEE}(\boldsymbol{\beta}_{GEE}) = \sum_{i=1}^N \mathbf{D}'_i \left[ \frac{(\mathbf{A}_i^{\frac{1}{2}}) \mathbf{R}_i(\boldsymbol{\alpha}) (\mathbf{A}_i^{\frac{1}{2}})}{\phi} \right]^{-1} [Y_i - \mu_i] = \mathbf{0}$$

Two-step estimation:

- For fixed values of  $\boldsymbol{\alpha}_s$  and  $\phi_s$  at iteration  $s$ , use Newton scoring to estimate  $\hat{\boldsymbol{\beta}}_s$ ,
- Use  $\hat{\boldsymbol{\beta}}_s$  to calculate standardized residuals  $(Y_i - \hat{\mu}_i)_s$ , from which consistent estimates of  $\boldsymbol{\alpha}_{s+1}$  and  $\phi_{s+1}$  can be estimated.

Liang & Zeger (1986):

$$\hat{\beta}_{GEE} \underset{N \rightarrow \infty}{\sim} \mathbf{N}(\beta, \Sigma).$$

For  $\hat{\Sigma}$ , two options:

$$\hat{\Sigma}_{\text{Model}} = N \left( \sum_{i=1}^N \hat{\mathbf{D}}_i' \hat{\mathbf{V}}_i^{-1} \hat{\mathbf{D}}_i \right)$$

$$\hat{\Sigma}_{\text{Robust}} = N \left( \sum_{i=1}^N \hat{\mathbf{D}}_i' \hat{\mathbf{V}}_i^{-1} \hat{\mathbf{D}}_i \right)^{-1} \left( \sum_{i=1}^N \hat{\mathbf{D}}_i' \hat{\mathbf{V}}_i^{-1} \hat{\mathbf{S}}_i \hat{\mathbf{V}}_i^{-1} \hat{\mathbf{D}}_i \right) \left( \sum_{i=1}^N \hat{\mathbf{D}}_i' \hat{\mathbf{V}}_i^{-1} \hat{\mathbf{D}}_i \right)^{-1}$$

where  $\hat{\mathbf{S}}_i = (Y_i - \hat{\mu}_i)(Y_i - \hat{\mu}_i)'$ .

# Inference (aka, magic!)

- $\hat{\Sigma}_{\text{Model}}$ 
  - Requires that  $\mathbf{R}_i(\alpha)$  be “correct” for consistency.
  - Is slightly more efficient than  $\hat{\Sigma}_{\text{Robust}}$  if so.
- $\hat{\Sigma}_{\text{Robust}}$ 
  - Is consistent *even if*  $\mathbf{R}_i(\alpha)$  is misspecified.
  - Is slightly less efficient than  $\hat{\Sigma}_{\text{Model}}$  if  $\mathbf{R}_i(\alpha)$  is correct.

**Moral: Use  $\hat{\Sigma}_{\text{Robust}}$ .**

## GEEs:

- Are a straightforward variation on GLMs, and so
- Can be applied to a range of data types (continuous, binary, count, proportions, etc.),
- Yield robustly consistent point estimates of  $\beta$ s,
- Account for within-unit correlation in an informed way, but also
- Yield consistent inferences even if that correlation is misspecified.

## Practical Issues: Model Interpretation

- In general, GEEs = GLMs.
- GEEs are *marginal* models, so:
  - $\hat{\beta}$ s have an interpretation as average / total effects.
  - Estimates / effect sizes generally be smaller than conditional (e.g. fixed/random) effects models.
  - E.g., for logit,  $\hat{\beta}_M \approx \frac{\hat{\beta}_C}{\sqrt{1+0.35\sigma_\eta^2}}$ , where  $\sigma_\eta^2 > 0$  is the variance of the unit effects.

# Practical Issues: Specifying $\mathbf{R}_i(\alpha)$

- Has been called “more art than science.”
- Pointers:
  - Choose based on *substance* of the problem.
  - Remember that  $\mathbf{R}_i(\alpha)$  is conditional on  $\mathbf{X}$ ,  $\hat{\beta}$ .
  - Consider unstructured when  $T$  is small and  $N$  large.
  - Try different ones, and compare.
- In general, it shouldn't matter terribly much...

Software	Command(s)/Package(s)
R	<code>gee / geepack / geeM / multgeeB / orth / repolr</code>
Stata	<code>xtgee / xtlogit / xtprobit / xtpois / etc.</code>
SAS	<code>genmod (w/ repeated)</code>

- Generally follow GLMs (specify “family” + “link”)
- Certain combinations not possible/recommended
- Estimation: Fisher scoring, MLE, etc. (MCMC?)

From the geepack manual:

**Warning**

Use "unstructured" correlation structure only with great care. (It may cause R to crash).



# Civil War Redux... GEE: Independence

```
> GEE.ind<-geeglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+
+               log(GDPPerCapita)+GDPPerCapGrowth+PostColdWar+POLITY+
+               POLITYSquared,data=DF,id=ISO3,family="binomial",
+               corstr="independence")
```

```
> summary(GEE.ind)
```

Call:

```
geeglm(formula = CivilWar ~ log(LandArea) + log(PopMillions) +
        UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar +
        POLITY + POLITYSquared, family = "binomial", data = DF, id = ISO3,
        corstr = "independence")
```

Coefficients:

	Estimate	Std.err	Wald	Pr(> W )
(Intercept)	-1.0327	1.9726	0.27	0.60059
log(LandArea)	0.0109	0.1234	0.01	0.92992
log(PopMillions)	0.6636	0.1568	17.90	0.000023 ***
UrbanPopulation	0.0109	0.0137	0.64	0.42538
log(GDPPerCapita)	-0.5013	0.2454	4.17	0.04106 *
GDPPerCapGrowth	-0.0403	0.0128	9.88	0.00167 **
PostColdWar	-0.3110	0.2594	1.44	0.23049
POLITY	0.6744	0.2105	10.26	0.00136 **
POLITYSquared	-0.0653	0.0194	11.34	0.00076 ***

---

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

Correlation structure = independence

Estimated Scale Parameters:

	Estimate	Std.err
(Intercept)	0.803	0.291

Number of clusters: 160 Maximum cluster size: 57

# GEE: Exchangeable

```
> GEE.exc<-geeglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+
+               log(GDPPerCapita)+GDPPerCapGrowth+PostColdWar+POLITY+
+               POLITYSquared,data=DF,id=IS03,family="binomial",
+               corstr="exchangeable")
```

```
> summary(GEE.exc)
```

Call:

```
geeglm(formula = CivilWar ~ log(LandArea) + log(PopMillions) +
        UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar +
        POLITY + POLITYSquared, family = "binomial", data = DF, id = IS03,
        corstr = "exchangeable")
```

Coefficients:

	Estimate	Std.err	Wald	Pr(> W )
(Intercept)	-2.91574	2.05337	2.02	0.15561
log(LandArea)	0.05297	0.15494	0.12	0.73245
log(PopMillions)	0.55323	0.16035	11.90	0.00056 ***
UrbanPopulation	0.00533	0.01165	0.21	0.64714
log(GDPPerCapita)	-0.21791	0.17470	1.56	0.21229
GDPPerCapGrowth	-0.03530	0.00904	15.23	0.000095 ***
PostColdWar	-0.14044	0.23285	0.36	0.54641
POLITY	0.54979	0.17023	10.43	0.00124 **
POLITYSquared	-0.05610	0.01664	11.36	0.00075 ***

---

Signif. codes: 0 \*\*\* 0.001 \*\* 0.01 \* 0.05 . 0.1 1

Correlation structure = exchangeable

Estimated Scale Parameters:

	Estimate	Std.err
(Intercept)	0.725	0.185

Link = identity

Estimated Correlation Parameters:

	Estimate	Std.err
alpha	0.34	0.112

Number of clusters: 160 Maximum cluster size: 57

# GEE: AR(1)

```
> GEE.ar1<-geeglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+
+                 log(GDPPerCapita)+GDPPerCapGrowth+PostColdWar+POLITY+
+                 POLITYSquared,data=DF,id=ISO3,family="binomial",
+                 corstr="ar1")
```

```
> summary(GEE.ar1)
```

Call:

```
geeglm(formula = CivilWar ~ log(LandArea) + log(PopMillions) +
        UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + PostColdWar +
        POLITY + POLITYSquared, family = "binomial", data = DF, id = ISO3,
        corstr = "ar1")
```

Coefficients:

	Estimate	Std.err	Wald	Pr(> W )
(Intercept)	-2.11808	2.41377	0.77	0.380
log(LandArea)	0.17430	0.18542	0.88	0.347
log(PopMillions)	0.32266	0.19145	2.84	0.092 .
UrbanPopulation	0.00279	0.01595	0.03	0.861
log(GDPPerCapita)	-0.39669	0.23482	2.85	0.091 .
GDPPerCapGrowth	-0.01526	0.00728	4.40	0.036 *
PostColdWar	0.19787	0.24491	0.65	0.419
POLITY	0.18284	0.12351	2.19	0.139
POLITYSquared	-0.02066	0.01320	2.45	0.117

```
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
```

Correlation structure = ar1

Estimated Scale Parameters:

	Estimate	Std.err
(Intercept)	0.825	0.352

Link = identity

Estimated Correlation Parameters:

	Estimate	Std.err
alpha	0.92	0.0404

Number of clusters: 160 Maximum cluster size: 57

# GEE: Unstructured (2013-2017)

```
> GEE.unstr<-geeglm(CivilWar~log(LandArea)+log(PopMillions)+UrbanPopulation+
+ log(GDPPerCapita)+GDPPerCapGrowth+POLITY+
+ POLITYSquared,data=DF5,id=IS03,family="binomial",
+ corstr="unstructured")
```

```
> summary(GEE.unstr)
```

Call:

```
geeglm(formula = CivilWar ~ log(LandArea) + log(PopMillions) +
  UrbanPopulation + log(GDPPerCapita) + GDPPerCapGrowth + POLITY +
  POLITYSquared, family = "binomial", data = DF5, id = IS03,
  corstr = "unstructured")
```

Coefficients:

	Estimate	Std.err	Wald	Pr(> W )
(Intercept)	-2.38896	3.25077	0.54	0.46241
log(LandArea)	0.16453	0.19119	0.74	0.38949
log(PopMillions)	0.85836	0.24080	12.71	0.00036 ***
UrbanPopulation	0.03406	0.01715	3.95	0.04699 *
log(GDPPerCapita)	-0.81577	0.31150	6.86	0.00882 **
GDPPerCapGrowth	-0.00896	0.03066	0.09	0.77000
POLITY	0.53049	0.43746	1.47	0.22526
POLITYSquared	-0.06053	0.03800	2.54	0.11119

---  
Signif. codes: 0 '\*\*\*' 0.001 '\*\*' 0.01 '\*' 0.05 '.' 0.1 ' ' 1

Correlation structure = unstructured

Estimated Scale Parameters:

	Estimate	Std.err
(Intercept)	0.658	0.783

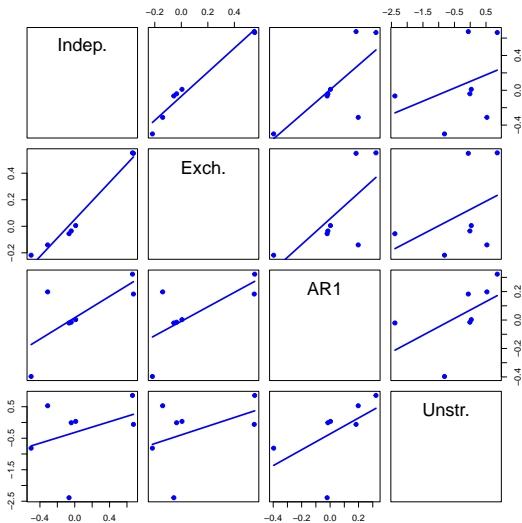
Link = identity

Estimated Correlation Parameters:

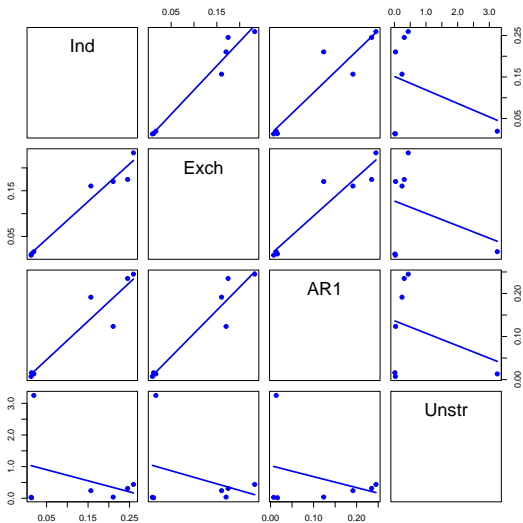
	Estimate	Std.err
alpha.1:2	0.380	0.471
alpha.1:3	0.393	0.489
alpha.1:4	0.356	0.447
alpha.1:5	0.296	0.372
alpha.2:3	0.748	0.851
alpha.2:4	0.289	0.369
alpha.2:5	0.466	0.541
alpha.3:4	0.407	0.517
alpha.3:5	0.677	0.795
alpha.4:5	0.446	0.558

Number of clusters: 159 Maximum cluster size: 5

# Comparing $\hat{\beta}$ s



# Comparing $\widehat{s.e.s}$



GEEs are:

- Robust
- Flexible
- Extensible beyond panel/TSCS context