Thermal model: a conductive bar example

Marco Mirabilio

Contents

1 Problem setup

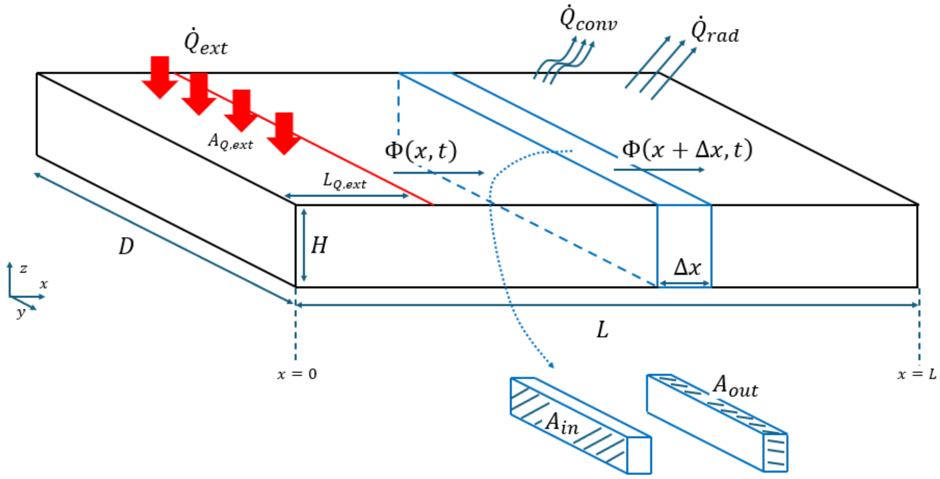

Figure 1: Problem setup

The purpose of this document is the derivation of a 1D thermal model (one-dimensional, or one-degree-of-freedom) describing the phenomena inside a conductive bar subjected to a heat flux in input as illustrated in Figure 1. For the purpose of this document, the following definitions are introduced.

- L, D, H: respectively, the length, depth, and height of the conductive bar ($[m]$)
- T, T_a: respectively, the temperature of the bar (at a certain point) and the ambient temperature ($[K]$)
- $\dot{q}_{ext}$: the transfer rate per unit of area of the heat flow delivered to the bar by an external source ($[W/m^2]$); we assume that the heat is transferred only on one side of the bar
- $A_{Q,ext} = L_{Q,ext} \cdot D$: the portion of surface subjected to the external heat source $\dot{q}_{ext}$ ($[m^2]$)
- $A_{in} = H \cdot D$: the cross-sectional area of the bar
- $A_{out} = 2\Delta x(H + D)$: the surface extension of a slice of length Δx
- $\dot{q}_{rad} = \sigma\epsilon(T^4 - T_a^4)$: the transfer rate per unit of area of heat flowing away by radiation ($[W/m^2]$)
- $\dot{q}_{conv} = h(T - T_a)$: the transfer rate per unit of area of heat flowing away by convection ($[W/m^2]$)
- $\varphi(x,t)$: the amount of thermal energy per unit time flowing from the left to the right of the bar per unit of area ($[W/m^2]$)
- $e(x,t)$: density of thermal energy ($[J/m^2]$)
- $\sigma = 5.670373 \cdot 10^{-8}$: the Stefan-Boltzmann constant ($[W \cdot m^{-2} \cdot K^{-4}]$)
- $\epsilon \in [0,1]$: the emissivity (1 for black surfaces)
- h: the convective heat transfer coefficient ($[W \cdot m^{-2} \cdot K^{-1}]$)
- c: bar specific heat ($[J \cdot kg^{-1} \cdot K^{-1}]$)
- ρ: density of the bar material ($[kg \cdot m^{-3}]$)
- k: thermal conductivity ($[W \cdot m^{-1} \cdot K^{-1}]$)

Note that for simplicity, the thermal coefficients introduced here are supposed to be constant in time, in space, and with respect to temperature. Also, the choice of 1D thermal model corresponds to the implicit assumption of homogeneous heat transfer through the y and z axes. Given the above definitions, we derive the following "macroscopic" quantities:

- $\dot{Q}_{ext} = A_{Q,ext} \cdot \dot{q}_{ext}$: the transfer rate of the heat flowing into the bar through the surface $A_{Q,out}$
- $\dot{Q}_{rad} = A_{out} \cdot \dot{q}_{rad}$: the transfer rate of the heat flowing away by radiation from the surface of the slice with length Δx
- $\dot{q}_{conv} = A_{out} \cdot \dot{q}_{conv}$: the transfer rate of the heat flowing away by convection from the surface of the slice with length Δx
- $\Phi(x,t) = A_{in} \cdot \varphi(x,t)$: amount of thermal energy flowing from the left to the right of the cross-sectional area of the bar
- $E(x,t) = A_{in}\Delta x e(x,t)$: the thermal energy of the slice with length Δx

2 Derivation of PDE

Assuming the continuity of energy $e(x,t)$ and heat, we get:

$$\frac{\partial E(x,t)}{\partial t} = \Phi(x,t) - \Phi(x+\Delta x,t) + A_{Q,ext}^{slice} \cdot \dot{q}_{ext} - \dot{Q}_{rad} - \dot{Q}_{conv} \tag{1}$$

where

$$A_{Q,ext}^{slice} = \begin{cases} \Delta x \cdot D, & \text{if slice in } [0, L_{Q,ext}] \\ 0, & \text{otherwise} \end{cases} \tag{2}$$

then,

$$A_{in}\Delta x \frac{\partial e(x,t)}{\partial t} = A_{in} \cdot (\varphi(x,t) - \varphi(x+\Delta x,t)) + A_{Q,ext}^{slice} \cdot \dot{q}_{ext} - A_{out} \cdot (\dot{q}_{rad} + \dot{q}_{conv}) \tag{3}$$

By rearranging the terms, and considering without loss of generality the case of slice subjected to the external heat source, we get:

$$\begin{aligned} \frac{\partial e(x,t)}{\partial t} &= \frac{(\varphi(x,t) - \varphi(x+\Delta x,t))}{\Delta x} + \frac{A_{Q,ext}^{slice}}{A_{in}\Delta x} \cdot \dot{q}_{ext} - \frac{A_{out}}{A_{in}\Delta x} \cdot (\dot{q}_{rad} + \dot{q}_{conv}) \\ &= \frac{(\varphi(x,t) - \varphi(x+\Delta x,t))}{\Delta x} + \frac{\Delta x D}{HD\Delta x} \cdot \dot{q}_{ext} - \frac{2\Delta x(H+D)}{HD\Delta x} \cdot (\dot{q}_{rad} + \dot{q}_{conv}) \\ &= \frac{(\varphi(x,t) - \varphi(x+\Delta x,t))}{\Delta x} + \frac{1}{H}\dot{q}_{ext} - \frac{2(H+D)}{HD}(\dot{q}_{rad} + \dot{q}_{conv}) \end{aligned} \tag{4}$$

Taking $\Delta x \to 0$:

$$\frac{\partial e(x,t)}{\partial t} = -\frac{\partial \varphi(x,t)}{\partial x} + \frac{1}{H}\dot{q}_{ext} - \frac{2(H+D)}{HD}(\sigma\epsilon(T^4 - T_a^4) + h(T - T_a)) \tag{5}$$

By Fourier's law of heat exchange

$$\varphi(x,t) = -k\frac{\partial T(x,t)}{\partial x} \tag{6}$$

and since

$$e(x,t) = x\rho T(x,t) \tag{7}$$

we get:

$$c\rho\frac{\partial T(x,t)}{\partial t} = k\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{1}{H}\dot{q}_{ext} - \frac{2(H+D)}{HD}(\sigma\epsilon(T^4 - T_a^4) + h(T - T_a)) \tag{8}$$

Given eq. 8 and introducing the Robin boundary conditions, the complete thermal model is:

$$\begin{cases} c\rho\frac{\partial T(x,t)}{\partial t} = k\frac{\partial^2 T(x,t)}{\partial x^2} + \frac{\mathbb{1}_x}{H}\dot{q}_{ext} - \frac{2(H+D)}{HD}(\sigma\epsilon(T^4 - T_a^4) + h(T - T_a)), \quad \mathbb{1}_x = 1 \text{ if } x \in [0, L_{Q,ext}] \\ k\frac{\partial T(0,t)}{\partial x} = -h(T(0,t) - T_a) \\ k\frac{\partial T(L,t)}{\partial x} = h(T(L,t) - T_a) \end{cases} \tag{9}$$

To interpret the boundary conditions, let us consider the presence of fictitious slices with temperature T_a at both ends of the bar, and

- $\partial T(0,t) \approx T(-\partial x, t) - T(0,t) = T_a - T(0,t)$
- $\partial T(L,t) \approx T(L,t) - T(L+\partial x, t) = T(L,t) - T_a$

then, if $T(0,t) > T_a$ or $T(L,t) > T_a$ we should have

- $\partial T(0,t) < 0$, that is satisfied by $-h(T(0,t) - T_a)$
- $\partial T(L,t) > 0$, that is satisfied by $h(T(L,t) - T_a)$

on the contrary, if $T(0,t) < T_a$ or $T(L,t) < T_a$ we should have

- $\partial T(0,t) > 0$, that is satisfied by $-h(T(0,t) - T_a)$
- $\partial T(L,t) < 0$, that is satisfied by $h(T(L,t) - T_a)$

3 Application of Finite Difference Method (FDM)

FDE is a methodology to discretize PDEs and then calculating a numerical solution of the dynamical equations. As illustrated in Figure 2, we consider the bar as composed by N slices of length Δx (here we recycle the symbol for convenience). Then, we approximate the differential equations with finite differences.

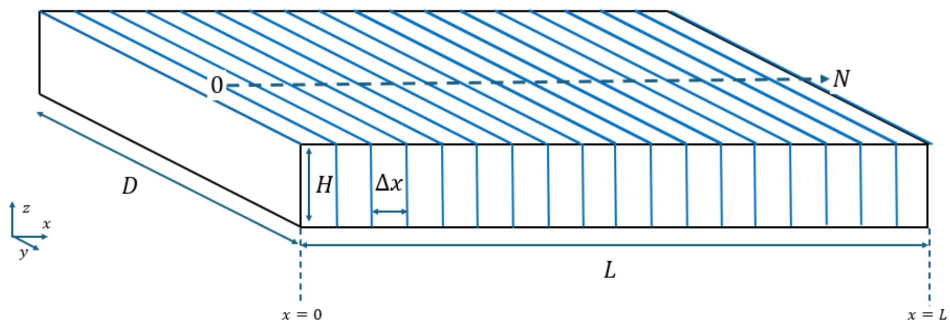

Figure 2: FDM

By indicating with i the i-th slice, we get a lumped model for each partition of the bar:

$$\begin{cases} c\rho \dot{T}_i = \frac{k}{\Delta x^2}(T_{i+1} - 2T_i + T_{i-1}) + \frac{\mathbb{1}_x}{H}\dot{q}_{ext,i} - \frac{2(H+D)}{HD}(\sigma\epsilon(T_i^4 - T_a^4) + h(T_i - T_a)), \quad i \in 0, \dots, N \\ k\frac{T_1 - T_{-1}}{2\Delta x} = -h(T_0 - T_a) \\ k\frac{T_{N+1} - T_{N-1}}{2\Delta x} = h(T_N - T_a) \end{cases} \tag{10}$$

For simplicity, we suppose that $L_{Q,ext} = n \cdot \Delta x$, then $\mathbb{1}_x = 1$ if $i \in [0, 1, \dots, n]$. Also, we assume

$$\dot{q}_{ext,i} = \dot{Q}_{ext}/n \tag{11}$$

For the sake of completeness, system of equations in eq. 10 is rewritten in its full nonlinear form:

$$\begin{cases} c\rho\dot{T}_0 = -\alpha_r T_0^4 + \left[2\alpha_{cd}\left(\frac{h\Delta x}{k} - 1\right) - \alpha_{cv}\right] T_0 + 2\alpha_{cd}T_1 + \alpha_r T_a^4 + \left(\alpha_{cv} - \frac{2h}{\Delta x}\right) T_a + \alpha_{q,0}\dot{q}_{ext} \\ c\rho\dot{T}_i = -\alpha_r T_i^4 - (2\alpha_{cd} + \alpha_{cv})T_i + \alpha_{cd}(T_{i+1} + T_{i-1}) + \alpha_r T_a^4 + \alpha_{cv}T_a + \alpha_{q,i}\dot{q}_{ext} \\ c\rho\dot{T}_N = -\alpha_r T_N^4 + \left[2\alpha_{cd}\left(\frac{h\Delta x}{k} - 1\right) - \alpha_{cv}\right] T_N + 2\alpha_{cd}T_{N-1} + \alpha_r T_a^4 + \left(\alpha_{cv} - \frac{2h}{\Delta x}\right) T_a + \alpha_{q,N}\dot{q}_{ext} \end{cases} \tag{12}$$

where

- $\alpha_r = 2\frac{\sigma\epsilon(H+D)}{HD}$
- $\alpha_{cv} = 2h\frac{H+D}{HD}$
- $\alpha_{cd} = \frac{k}{\Delta x^2}$

- $\alpha_{q,i} = \left\{\frac{1}{H} \text{ if } i \in \{0,\dots,n\}\right\}$ or $\{0 \text{ if } i > n\}$

By forward calculations, the above system of equations can be rewritten in a more compact form as:

$$C\dot{T} = -A_{rad}T^4 + (A_{cnd} - A_{cnv})T + A_{rad}T_a^4 + \tilde{A}_{cnv}T_a + B_{ext}\dot{q}_{ext} \tag{13}$$

where:

$$C = c\rho I_N \tag{14}$$

$$T^4 = [T_0^4, T_1^4, \dots, T_N^4]^T \tag{15}$$

$$A_{rad} = \alpha_r I_N \tag{16}$$

$$A_{cnv} = \alpha_{cv} I_N \tag{17}$$

$$\tilde{A}_{cnv} = A_{cnv} - diag\left(\frac{2h}{\Delta x}, 0, \dots, 0, \frac{2h}{\Delta x}\right) \tag{18}$$

$$A_{cnd} = \alpha_{cd} \begin{bmatrix} 2(\frac{h\Delta x}{k} - 1) & 2 & 0 & \cdots & & & 0 \\ 1 & -2 & 1 & 0 & \cdots & & \\ 0 & 1 & -2 & 1 & 0 & \cdots & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 1 \\ 0 & \cdots & \cdots & \cdots & \cdots & 2 & 2(\frac{h\Delta x}{k} - 1) \end{bmatrix} \tag{19}$$

$$B_{ext} = \frac{1}{H}[1, 1, \dots, 1, 0, \dots, 0]^T \tag{20}$$

4 Backward Euler method

Let us consider an ordinary differential equation:

$$\begin{cases} \frac{dx}{dt} = f(t, x) \\ x(t_0) = x_0 \end{cases} \tag{21}$$

Given an integration step $\Delta t > 0$, the backward Euler (BE) method is a numerical integration approach for the computation of a sequence $x_k = x(t_0 + k\Delta t)$ approximating the solution of eq. 21. Since the BE method is an implicit method, the new approximation x_{k+1} appears on both sides of the integration scheme:

$$x_{k+1} = x_k + \Delta t f(t_{k+1}, x_{k+1}) \tag{22}$$

Then, an algebraic equation for the unknown variable x_{k+1} has to be solved to calculate the new value of the solution sequence.

Since the BE method applies for both linear and non-linear cases, it can be directly applied to compute the solution of the thermal model in eq. 13. Then, by approximating the time derivative as

$$\frac{dx}{dt} \approx \frac{x_{k+1} - x_k}{\Delta t} \tag{23}$$

the dynamical system derived in the previous section can be rewritten as a non-linear system of N equations in N variables as follows:

$$C\frac{T_{k+1} - T_k}{\Delta t} = -A_{rad}T_{k+1}^4 + (A_{cnd} - A_{cnv})T_{k+1} + A_{rad}T_{a,k}^4 + \tilde{A}_{cnv}T_{a,k} + B_{ext}\dot{q}_{ext,k}$$

then,

$$\begin{aligned} \Delta t A_{rad}T_{k+1}^4 + (C - \Delta t(A_{cnd} - A_{cnv}))\, T_{k+1} = \\ CT_k + \Delta t\left(A_{rad}T_{a,k}^4 + \tilde{A}_{cnv}T_{a,k} + B_{ext}\dot{q}_{ext,k}\right) \end{aligned} \tag{24}$$

where the term on the right side is known. Note that we consider $T_a(k)$ and $\dot{q}_{ext}(k)$ since we suppose the use of a Zero-Order Holder to sample the input data (then the value remains constant for the entire sampling time from k to $k+1$). To solve the above non-linear system, we need dedicated solvers for non-linear problems. However, a possible approach to solve eq. 24 is to move the radiative term T^4 to the side of known terms by using the values of temperatures calculated in the previous step (thus, T_k^4). In this case, we derive a linear system of the form

$$AT_{k+1} - b = 0 \tag{25}$$

that can be solved by several numbers of approaches. In the case in which we are interested in maintaining as much as possible the information of the non-linear dynamics inside the solution of the problem, we can choose another path: first, we linearize the system, then we integrate it via a discretization method as the BE. In the next section, it is illustrated the linearization and the discretization via BE method of a non-linear system.

5 Linearization and BE discretization of a non-linear system

Let us consider a dynamic system described by

$$\dot{x}(t) = f(x(t), u(t)) \tag{26}$$

where x denotes the state of the system, u the input term, and $f(\cdot,\cdot)$ is a non-linear function describing the evolution of the state. Our purpose is to linearize the non-linear system in a neighborhood of the linearization point $(\bar{x}(t), \bar{u}(t))$. This task is accomplished by truncating the Taylor series at the first-order derivative:

$$f(x(t), u(t)) \approx f(\bar{x}(t), \bar{u}(t)) + \left.\frac{\partial f}{\partial x}\right|_{\bar{x}(t),\bar{u}(t)} \cdot (x(t) - \bar{x}(t)) + \left.\frac{\partial f}{\partial u}\right|_{\bar{x}(t),\bar{u}(t)} \cdot (u(t) - \bar{u}(t)) \tag{27}$$

By defining:

- $A_c(t) = \left.\frac{\partial f}{\partial x}\right|_{\bar{x}(t),\bar{u}(t)}$
- $B_c(t) = \left.\frac{\partial f}{\partial u}\right|_{\bar{x}(t),\bar{u}(t)}$

we get

$$\dot{x}(t) \approx= f(\bar{x}(t), \bar{u}(t)) + A_c(t)(x(t) - \bar{x}(t)) + B_c(t)(u(t) - \bar{u}(t)) \tag{28}$$

We assume $t - \Delta t = k\Delta t$, and as linearization point:

- $\bar{x}(t) = x(t - \Delta t) = x(k\Delta t) = x(k)$
- $\bar{u}(t) = u(t - \Delta t) = u(k\Delta t) = u(k)$

given that Δt is small enough. Then, we get

$$\dot{x}(t) \approx f(x(k), u(k)) + A_c(k)(x(t) - x(k)) + B_c(k)(u(t) - u(k)) \tag{29}$$

At this point, considering $t = (k+1)\Delta t$ and that the time derivative is approximated by $(x(k+1) - x(k))/\Delta t$, we get

$$\frac{x(k+1) - x(k)}{\Delta t} = f(x(k), u(k)) + A_c(k)(x - x(k)) + B_c(k)(u(k) - u(k)) \tag{30}$$

where

- x depends on the chosen integration method, for BE it is $x(k+1)$
- $u(t) = u(k)$ since we assume the use of a zero-order-holder sampler

Since the purpose is to implement the BE method, we get

$$x(k+1) = x(k) + \Delta t(I - \Delta t A_c(k))^{-1} f(x(k), u(k)) \tag{31}$$

It is worth highlighting that despite the fact that the input-related linear term B_c disappears, the effects of the input are still considered by the function $f(\cdot,\cdot)$ calculated with respect to the state and input at the previous time step k.

5.1 Application to the thermal model

For simplicity, we rewrite the system in eq. 13 with respect to the i-th row:

$$C_i\dot{T}_i = -A_{rad,i}T_i^4 + (A_{cnd,i} - A_{cnv,i})T_i + A_{rad,i}T_{a,i}^4 + \tilde{A}_{cnv,i}T_{a,i} + B_{ext,i}\dot{q}_{ext,i}, \quad \forall i = 1,\dots,N \tag{32}$$

For each row of the system we get:

$$A_{c,i}(t) = -4A_{rad,i}T_i(t)^3 + A_{cnd,i} - A_{cnv,i}, \quad B_{c,i}(t) = [4A_{rad,i}T_{a,i}^3 + \tilde{A}_{cnv,i} \quad B_{ext,i}]^T \tag{33}$$

Then, the resultant linearized and discretized system is:

$$\begin{aligned} C\frac{T_i(k+1) - T_i(k)}{\Delta t} =& (-4A_{rad,i}T_i(k)^3 + A_{cnd,i} - A_{cnv,i})(T_i(k+1) - T_i(k)) \\ & - A_{rad,i}T_i(k)^4 + (A_{cnd,i} - A_{cnv,i})T_i(k) \\ & + A_{rad,i}T_{a,i}(k)^4 + \tilde{A}_{cnv,i}T_{a,i}(k) + B_{ext,i}\dot{q}_{ext,i}(k) \end{aligned} \tag{34}$$

by straightforward calculations, we get

$$\begin{aligned} &\left(C_i + \Delta t(4A_{rad,i}T_i(k)^3 - A_{cnd,i} + A_{cnv,i})\right) T_i(k+1) = \\ &\qquad \left(C_i + \Delta t(4A_{rad,i}T_i(k)^3 - A_{cnd,i} + A_{cnv,i})\right) T_i(k) \\ &\qquad + \Delta t(-A_{rad,i}T_i(k)^4 + (A_{cnd,i} - A_{cnv,i})T_i(k) + A_{rad,i}T_{a,i}(k)^4 \\ &\qquad + \tilde{A}_{cnv,i}T_{a,i}(k) + B_{ext,i}\dot{q}_{ext,i}(k)) \end{aligned} \tag{35}$$

Setting $A_i = C_i + \Delta t(4A_{rad,i}T_i(k)^3 - A_{cnd,i} + A_{cnv,i})$, we now have two options:

1. compute $b_i = A_iT_i(k) + \Delta t f(T(k), u(k))$ and solve the linear system $AT(k+1) = b$ without calculating the inverse of A
2. compute the inverse of A and integrate the system $T(k+1) = T(k) + A^{-1}\Delta t f(T(k), u(k))$

where for simplicity

- $u = [T_a \quad \dot{q}_{ext}]^T$
- $f(T, u) = -A_{rad}T^4 + (A_{cnd} - A_{cnv})T + A_{rad}T_a^4 + \tilde{A}_{cnv}T_a + B_{ext}\dot{q}_{ext}$