Computational Linear Algebra for Large Scale Problems: Homework 2

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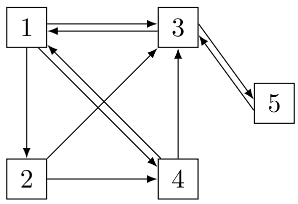
2 Second Section

Exercise 1

QUESTION: *Suppose the people who own page 3 in the web of Figure 1 are infuriated by the fact that its importance score, computed using formula (2.1), is lower than the score of page 1. In an attempt to boost page 3's score, they create a page 5 that links to page 3; page 3 also links to page 5. Does this boost page 3's score above that of page 1?*:

SOLUTION:



Before the node 3 generated $n_3 = 1$ backlinks. This means that we could calculate a relationship between the scores of the pages 1 and 3 as follows:

$$\begin{cases} x_1 = \frac{x_3}{1} + \frac{x_4}{2} \\ x_2 = \frac{x_1}{3} \\ x_4 = \frac{x_1}{3} + \frac{x_2}{2} \end{cases} \implies \begin{cases} x_1 = x_3 + \frac{x_4}{2} \\ x_4 = \frac{x_1}{3} + \frac{x_1}{6} = \frac{x_1}{2} \end{cases} \implies x_1 = x_3 + \frac{x_1}{4} \implies x_1 - \frac{x_1}{4} = x_3 \implies x_1 = \frac{4}{3}x_3$$

This was indeed confirmed with the vector $[12, 4, 9, 6]^T$ found after the computation of the eigenvectors of the link matrix A of the graph.

With the new graph the page 3 generates one more link, connecting to the page 5, meaning we have $n_3 = 2$. If we repeat the same steps done above we find a different relationship between x_1 and x_3

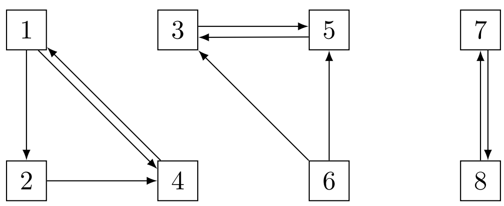
$$x_1 = \frac{x_3}{2} + \frac{x_1}{4} \implies x_1 - \frac{x_1}{4} = \frac{x_3}{2} \implies x_1 = \frac{2}{3}x_3$$

On page 2 and 4, since they are not backlinked to page 3, they continue to contribute the same percentage (i.e. $\frac{x_1}{4}$) to the rank of page 1. However, due to the different value of n_3, page 1 ends up with a lower score than page 3.

Exercise 2

QUESTION: *Construct a web consisting of three or more subwebs and verify that dim(V1(A)) equals (or exceeds) the number of the components in the web.*

SOLUTION:



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Let's consider the above graph and its link matrix $A \in R^n$. We have that the dimension of $V_1(A)$, the eigenspace of the eigenvalue $\lambda = 1$, is given by the dimension of the nullspace of $A - 1I$, that can be found by using the formula:

$$dim(Ker(A - 1I)) = n - rank(A - 1I)$$

To obtain the rank we compute $A - 1I$ and we find its *Reduced Row-Echelon Form* (RREF), applying the following transformations:

$$
\text{A} - 1\text{I} = \begin{bmatrix} -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 1 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \rightarrow \begin{array}{l} -R_1 \rightarrow R_1 \\ \frac{1}{2}R_1 - R_2 \rightarrow R_2 \\ R_4 - \frac{1}{2}R_1 - R_2 \rightarrow R_4 \\ R - R_3 \rightarrow R_3 \\ R_5 - R_3 \rightarrow R_5 \\ R_3 + \frac{1}{2}R_5 \rightarrow R_3 \\ R_6 + R_5 \rightarrow R_6 \\ R_7 + R_8 \rightarrow R_7 \\ \textit{swap } R_5 \textit{ with } R_4 \\ \textit{swap } R_8 \textit{ with } R_5 \end{array} \rightarrow \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

The number of *pivots* (the first non vanishing number of a row) equal 1 of the reduced matrix is 5. This means that $rank(A - 1I) = 5$ and then:

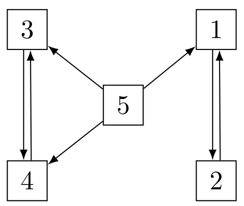
$$dim(V_1(A)) = dim(Ker(A - 1I) = 8 - 5 = 3$$

We have verified that the dimension of the eigenspace is $\geq$ r, the number of subwebs, that in this case is 3.

Exercise 3

QUESTION: *Add a link from page 5 to page 1 in the web of Figure 2. The resulting web, considered as an undirected graph, is connected. What is the dimension of $V_1(A)$?*

SOLUTION:



$$
\text{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \frac{1}{3} \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 1 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

Similar to the Exercise 2 we compute $dim(Ker(A - 1I)$

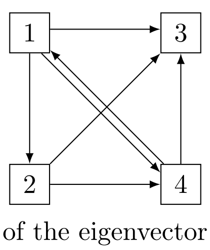
$$
\text{A} - 1\text{I} = \begin{bmatrix} -1 & 1 & 0 & 0 & \frac{1}{3} \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & \frac{1}{3} \\ 0 & 0 & 1 & -1 & \frac{1}{3} \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \rightarrow \begin{array}{l} -R_5 \rightarrow R_5 \\ R_1 + R_2 - \frac{1}{3}R_5 \rightarrow R_1 \\ R_3 + R_4 - \frac{2}{3}R_5 \rightarrow R_3 \\ = R_4 - \frac{1}{3}R_5 \rightarrow R_4 \\ \textit{swap } R_1 \textit{ with } R_2 \\ \textit{swap } R_2 \textit{ with } R_1 \\ \textit{swap } R_3 \textit{ with } R_5 \end{array} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
$$

This means that: $dim(V_1(A)) = dim(Ker(A - 1I) = 5 - 3 = 2$

Exercise 4

QUESTION: *In the web of Figure 2.1, remove the link from page 3 to page 1. In the resulting web page 3 is now a dangling node. Set up the corresponding substochastic matrix and find its largest positive (Perron) eigenvalue. Find a non-negative Perron eigenvector for this eigenvalue, and scale the vector so that components sum to one. Does the resulting ranking seem reasonable?*

SOLUTION:



$$
\text{A} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 \end{bmatrix}
$$

From the definition of the eigenvector $\mathbf{x}$ corresponding to the eigenvalue λ we have:

$$A\mathbf{x} = \lambda\mathbf{x} \implies A\mathbf{x} - \lambda\mathbf{x} = (A - \lambda I)\mathbf{x} = 0$$

The equation has a nonzero solution $\iff det(A - \lambda I) = 0$

$$
det(A - \lambda I) = \begin{vmatrix} -\lambda & 0 & 0 & \frac{1}{2} \\ \frac{1}{3} & -\lambda & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -\lambda & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{2} & 0 & -\lambda \end{vmatrix} \underset{\substack{\downarrow \\ \text{Leibniz Formula:} \\ \det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}}}{=} = \lambda^4 - \frac{1}{6}\lambda^2 - \frac{1}{12}\lambda = \frac{1}{12}\lambda\left(12\lambda^3 - 2\lambda - 1\right) = 0
$$

To find the Perron eigenvalue, let's rewrite the cubic equation $12\lambda^3 - 2\lambda - 1 = 0$ to $\lambda^3 - \frac{1}{6}\lambda - \frac{1}{12} = 0$ and, after defining the variables $p = -\frac{1}{6}$ and $q = -\frac{1}{12}$, we can compute the discriminant as follows:

$$\Delta = -4p^3 - 27q^2 \approx -0.17$$

The discriminant is negative, meaning the polynomial has one real root and two non-real complex conjugate roots. This implies that the only real root has to be the Perron eigenvalue we are searching for. By applying the Cardano's formula we can compute it:

$$\lambda_p = \sqrt[3]{u_1} + \sqrt[3]{u_2} \quad \text{where } u_1 = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \approx 0.081 \text{ and } u_2 = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \approx 0.002 \implies \lambda_p \approx 0.561$$

Now let's find the eigenvector of λ_p by applying gaussian elimination to solve the system $(A - \lambda_p I)\,\mathrm{x}_p = 0$

$$\left[\begin{array}{cccc|c} -0.561 & 0 & 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & -0.561 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & -0.561 & \frac{1}{2} & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & -0.561 & 0 \end{array}\right] \to \begin{array}{c} R_1/-0.561 \to R_1 \\ R_2 - \frac{1}{3}R_1 \to R_2 \\ R_3 - \frac{1}{3}R_1 \to R_3 \\ R_4 - \frac{1}{3}R_1 \to R_4 \end{array} \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -0.891 & 0 \\ 0 & -0.561 & 0 & 0.297 & 0 \\ 0 & \frac{1}{2} & -0.561 & 0.797 & 0 \\ 0 & \frac{1}{2} & 0 & -0.264 & 0 \end{array}\right] \to$$

$$\to \begin{array}{c} R_2/-0.561 \to R_2 \\ R_3 - \frac{1}{2}R_2 \to R_3 \\ R_4 - \frac{1}{2}R_2 \to R_4 \\ R_3/-0.561 \to R_3 \end{array} \to \left[\begin{array}{cccc|c} 1 & 0 & 0 & -0.891 & 0 \\ 0 & 1 & 0 & -0.529 & 0 \\ 0 & 0 & 1 & -1.891 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \implies \text{The general solution of the system is: } \mathrm{x} = \begin{bmatrix} 0.891\,x_4 \\ 0.529\,x_4 \\ 1.891\,x_4 \\ x_4 \end{bmatrix}$$

This means that by choosing $x_4 = 1$ we have that a non-negative eigenvector for λ_p is the vector

$$\mathrm{x}_p \approx \begin{bmatrix} 0.891 \\ 0.529 \\ 1.891 \\ 1 \end{bmatrix} \quad \text{that after a scaling becomes: } \tilde{\mathrm{x}}_p \approx \begin{bmatrix} 0.207 \\ 0.123 \\ 0.438 \\ 0.232 \end{bmatrix}$$

With the change in the graph, the loss of the crucial backlink from node 3 is expected to reduce the importance score of node 1. This effect is reflected in the importance scores of the graph with dangling nodes. Node 1's rank position drops from first place to third, ranking only above node 2, whose importance score is also slightly lower due to its dependence on node 1. For these reasons, we can assert that the implemented ranking system appears logical.

Exercise 5

QUESTION: *Prove that in any web the importance score of a page with no backlinks is zero*

SOLUTION: Given a graph with n nodes, let "i" be the index of the node with no backlinks. By definition, the link matrix A will have its i-th row consisting entirely of zeros. If by contradiction we assume that the eigenvector $\mathbf{x}$, of the Perron eigenvalue λ, has its i-th element $\mathbf{x}_i \neq 0$ then we will have that:

$$A\mathbf{x} = \begin{bmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} A_{1,1}x_1 + \cdots + A_{1,i}x_i + \cdots + A_{1,n}x_n \\ \vdots \\ \vdots \\ 0x_1 + \cdots + 0x_i + \cdots + 0x_n \\ \vdots \\ A_{1,1}x_1 + \cdots + A_{1,i}x_i + \cdots + A_{1,n}x_n \end{bmatrix}$$

i-th row

Moreover, by definition of eigenvector, it's true that $A\mathbf{x} = \lambda\mathbf{x}$, meaning: $0 = \lambda\mathbf{x}_i$. But this is impossible because $\mathbf{x}_i \neq 0$ by hypothesis, and by definition the Perron eigenvalue has to be positive. So we have found a contradiction and the importance score of the node i has to be equal to 0

Exercise 6

QUESTION: *Implicit in our analysis up to this point is the assertion that the manner in which the pages of a web W are indexed has no effect on the importance score assigned to any given page. Prove this, as follows: Let W contains n pages, each page assigned an index 1 through n, and let A be the resulting link matrix. Suppose we then transpose the indices of pages i and j (so page i is now page j and vice-versa). Let $\tilde{A}$ be the link matrix for the relabelled web.*

1. *Argue that $\tilde{A} = PAP$, where P is the elementary matrix obtained by transposing rows i and j of the $n \times n$ identity matrix. Note that the operation $A \to PA$ has the effect of swapping rows i and j of A, while $A \to AP$ swaps columns i and j. Also, $P^2 = I$, the identity matrix.*
2. *Suppose that x is an eigenvector for A, so $A\mathbf{x} = \lambda\mathbf{x}$ for some λ. Show that $\mathbf{y} = P\mathbf{x}$ is an eigenvector for $\tilde{A}$ with eigenvalue λ.*
3. *Explain why this shows that transposing the indices of any two pages leaves the importance scores unchanged, and use this result to argue that any permutation of the page indices leaves the importance scores unchanged*

SOLUTION:
1. To prove that $\tilde{A} = PAP$ let's assume that $i < j$, the proof is similar in the other case. We have that

$$A = \begin{bmatrix} A_{1,1} & \cdots & A_{1,i} & \cdots & A_{1,j} & \cdots & A_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{i,1} & \cdots & A_{i,i} & \cdots & A_{i,j} & \cdots & A_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{j,1} & \cdots & A_{j,i} & \cdots & A_{j,j} & \cdots & A_{j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n,1} & \cdots & A_{n,i} & \cdots & A_{n,j} & \cdots & A_{n,n} \end{bmatrix} \to PA = \begin{bmatrix} A_{1,1} & \cdots & A_{1,i} & \cdots & A_{1,j} & \cdots & A_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{j,1} & \cdots & A_{j,i} & \cdots & A_{j,j} & \cdots & A_{j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{i,1} & \cdots & A_{i,i} & \cdots & A_{i,j} & \cdots & A_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n,1} & \cdots & A_{n,i} & \cdots & A_{n,j} & \cdots & A_{n,n} \end{bmatrix} \to$$

$$\to PAP = \begin{bmatrix} A_{1,1} & \cdots & A_{1,j} & \cdots & A_{1,i} & \cdots & A_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{j,1} & \cdots & A_{j,j} & \cdots & A_{j,i} & \cdots & A_{j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{i,1} & \cdots & A_{i,j} & \cdots & A_{i,i} & \cdots & A_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{n,1} & \cdots & A_{n,j} & \cdots & A_{n,i} & \cdots & A_{n,n} \end{bmatrix} \quad \tilde{A} = \begin{bmatrix} \tilde{A}_{1,1} & \cdots & \tilde{A}_{1,j} & \cdots & \tilde{A}_{1,i} & \cdots & \tilde{A}_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{A}_{j,1} & \cdots & \tilde{A}_{j,j} & \cdots & \tilde{A}_{j,i} & \cdots & \tilde{A}_{j,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{A}_{i,1} & \cdots & \tilde{A}_{i,j} & \cdots & \tilde{A}_{i,i} & \cdots & \tilde{A}_{i,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \tilde{A}_{n,1} & \cdots & \tilde{A}_{n,j} & \cdots & \tilde{A}_{n,i} & \cdots & \tilde{A}_{n,n} \end{bmatrix}$$

As can be seen from above, for PAP and $\tilde{A}$ to be equal, needs to be true that: $A_{k_1,k_2} = \tilde{A}_{k_1,k_2} \forall\ 1 \le k_1 \le k_2 \le n$. This is true because swapping the indices j and i order doesn't change the number of backlinks a page generates.

2. $$A\mathbf{x} = \lambda\mathbf{x} \implies PA\mathbf{x} = P\lambda\mathbf{x} \implies PA\mathbf{x} = \lambda P\mathbf{x} \underset{P^2 = I}{\implies} PAPP\mathbf{x} = \lambda P\mathbf{x} \underset{\substack{\tilde{A} = PAP, \\ \mathbf{y} = P\mathbf{x}}}{\implies} \tilde{A}\mathbf{y} = \lambda\mathbf{y}$$

3. In the initial graph the importance score of the nodes i and j is respectively $\mathbf{x}_i$ and $\mathbf{x}_j$, the i-th and j-th components of the Perron eigenvector $\mathbf{x}$. After the swap of indices the old pages i and j will have their new importance score respectively in $\mathbf{y}_j$ and $\mathbf{y}_i$ the j-th and i-th components of the Perron eigenvector $\mathbf{y}$ But by Hp: $\mathbf{y} = P\mathbf{x}$ meaning that $\mathbf{y}_j = \mathbf{x}_i$ and $\mathbf{y}_i = \mathbf{x}_j$. So the ranking stays the same.

Let's consider a generic permutation of indices $\sigma \in S_n$. The permutation admits a decomposition into a product of disjoint 2-cycles s.t. $\sigma_{h,k}$ where:

$$\sigma(j) = \begin{cases} j_h & \text{if } j = j_k, \\ j_k & \text{if } j = j_h, \\ j & \text{otherwise.} \end{cases} \quad \text{i.e.} \quad \sigma = \begin{pmatrix} j_1 & \cdots & j_h & \cdots & j_k & \cdots & j_n \\ j_1 & \cdots & j_k & \cdots & j_h & \cdots & j_n \end{pmatrix}.$$

Each of these 2-cycles can be connected to a matrix: $P_{h,k}$ that given a matrix A will transform it into the matrix:

$$\tilde{A}_{h,k} = P_{h,k}AP_{h,k} \text{ with both the rows } h \text{ and } k \text{ and the columns } h \text{ and } k \text{ swapped.}$$

For what we proved before, the matrix $\tilde{A}_{h,k}$ produce a Perron eigenvector that preserves the importance scores of the pages. Moreover a product of two of these 2-cycles σ_{h_1,k_1} and σ_{h_2,k_2} can be seen as the matrix:

$$\tilde{A}_{h_2,k_2} = P_{h_2,k_2}\tilde{A}_{h_1,k_1}P_{h_2,k_2} = P_{h_2,k_2}P_{h_1,k_1}AP_{h_1,k_1}P_{h_2,k_2} \text{ that still preserve the importance scores}$$

This process can be extended to the full product of the 2-cycles decomposition, showing that the permutation of indices doesn't affect the ranking of the pages

3 Third Section

Exercise 7

QUESTION: *Prove that if A is an $n \times n$ column-stochastic matrix and $0 \leq m \leq 1$, then $M = (1-m)A + mS$ is also a column-stochastic matrix.*

SOLUTION: A square matrix is called a column-stochastic matrix if all of its entries are non-negative and the entries in each column sum to 1. The matrix M is the sum of two column-stochastic matrices: A and S, multiplied respectively by two positive numbers: (1-m) and m, meaning its entries are non-negative. Moreover, let $M_{i,j}$ be the element of the matrix M at row i and column j, we have that:

$$\sum_{i=1}^{n} M_{i,j} = \sum_{i=1}^{n} [(1-m)A_{i,j} + mS_{i,j}] \underset{\substack{\downarrow \\ \text{distributive} \\ \text{property}}}{\implies} \sum_{i=1}^{n} M_{i,j} = (1-m)\sum_{i=1}^{n} A_{i,j} + m\sum_{i=1}^{n} S_{i,j} \underset{\substack{\downarrow \\ \text{A, S column} \\ \text{stochastic}}}{=} (1-m)\cdot 1 + m \cdot 1 = 1$$

This indicates that the sum of the entries in each column of the matrix M is equal to 1, making it column-stochastic.

Exercise 8

QUESTION: *Show that the product of two column-stochastic matrices is also column-stochastic*

SOLUTION: Let A and B be column-stochastic matrices, we have that their product is the matrix C s.t:

$$C_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}$$

This means that its entries are a product of non-negative numbers, due to A and B being column-stochastic, so they are non-negative. Moreover, we have that:

$$\sum_{i=1}^{n} C_{i,j} = \sum_{i=1}^{n}\sum_{k=1}^{n} A_{i,k} B_{k,j} \underset{\substack{\downarrow \\ \text{we can change the} \\ \text{order of summation}}}{=} \sum_{k=1}^{n} B_{k,j} \sum_{i=1}^{n} A_{i,k} \underset{\substack{\downarrow \\ \text{A column} \\ \text{stochastic}}}{=} \sum_{k=1}^{n} B_{k,j} \cdot 1 \underset{\substack{\downarrow \\ \text{B column} \\ \text{stochastic}}}{=} 1$$

This indicates that the sum of the entries in each column of the matrix C is equal to 1, making it column-stochastic

Exercise 9

QUESTION: *Show that a page with no backlinks is given importance score ${}^{m}\!/_{n}$ by formula (3.2)*

SOLUTION: Given a graph with n nodes, let "i" be the index of the node with no backlinks. By definition, the link matrix A will have its i-th row consisting entirely of zeros. This means that:

$$M_{i,.} = (1-m)A_{i,.} + mS_{i,.} = (1-m)\underbrace{[0,\ldots,0]}_{\text{n times}} + m\underbrace{[{}^{1}\!/_{n},\ldots,{}^{1}\!/_{n}]}_{\text{n times}} = \underbrace{[{}^{m}\!/_{n},\ldots,{}^{m}\!/_{n}]}_{\text{n times}}$$

Let $\mathbf{x}$ be the Perron eigenvector and let's assume by contradiction that node i's importance score is $\mathbf{x}_i \neq {}^{m}\!/_{n}$ then:

$$M_{i,.}\mathbf{x} = [{}^{m}\!/_{n},\ldots,{}^{m}\!/_{n}] \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_i \\ \vdots \\ \mathbf{x}_n \end{bmatrix} = \frac{m}{n}\sum_{i=1}^{n} \mathbf{x}_i \underset{\substack{\downarrow \\ \text{by definition of} \\ \text{Perron eigenvector}}}{=} = \frac{m}{n} \cdot 1 = \frac{m}{n}$$

But, by definition of Perron eigenvector is true that:

$$M\mathbf{x} = \mathbf{x} \implies M_{i,.}\mathbf{x} = \mathbf{x}_i \implies x_i = {}^{m}\!/_{n}$$

We have found a contradiction, meaning that the importance score of the node i has to be equal ${}^{m}\!/_{n}$

Exercise 10

QUESTION: *Suppose that A is the link matrix for a strongly connected web of n pages (any page can be reached from any other page by following a finite number of links). Show that $dim(V_1(A)) = 1$ as follows. Let $(A^k)_{ij}$ denote the (i, j)-entry of A^k*

1. *Note that page i can be reached from page j in one step if and only $A_{ij} > 0$ (since $A_{ij} > 0$ means there's a link from j to i!) Show that $(A^2)_{ij} > 0$ if and only if page i can be reached from page j in exactly two steps. Hint: $(A^2)_{ij} = \sum_k A_{ik}A_{kj}$; all A_{ij} are non-negative, so $(A^2)_{ij} > 0$ implies that for some k both A_{ik} and A_{kj} are positive.*
2. *Show more generally that $(A^p)_{ij} > 0$ if and only if page i can be reached from page j in EXACTLY p steps.*
3. *Argue that $(I + A + A^2 + \cdots + A^p)_{ij} > 0$ if and only if page i can be reached from page j in p or fewer steps (note $p = 0$ is a legitimate choice-any page can be reached from itself in zero steps!)*
4. *Explain why $I + A + A^2 + \cdots + A^{n-1}$ is a positive matrix if the web is strongly connected.*
5. *Use the last part (and Exercise 8) so show that $B = \frac{1}{n}(I+A+A^2+\cdots+A^{n-1})$ is positive and column-stochastic (and hence by Lemma 3.2, $dim(V_1(B)) = 1$)*
6. *Show that if $x \in V_1(A)$ then $x \in V_1(B)$. Why does this imply that $dim(V_1(A)) = 1$?*

SOLUTION:

1. The sentence: "page i can be reached from page j in exactly two steps" means that there exists a page K that backlinks to page i, and that is reached from page j. This can alternatively be expressed as follow:

$$\exists \text{ a page } K \text{ s.t. } K \neq i,\ K \neq j \text{ and } A_{i,K} > 0,\ A_{K,j} > 0$$

So we have to prove that: $(A^2)_{i,j} > 0 \iff \exists$ a page K s.t. $K \neq i$, $K \neq j$ and $A_{i,K} > 0,\ A_{K,j} > 0$

$\boxed{\Leftarrow}$: $(A^2)_{i,j} = \sum_k A_{ik}A_{kj} \underset{\substack{\downarrow \\ \text{sum bigger then} \\ \text{single element of sum}}}{>} A_{iK}A_{Kj} \underset{\substack{\downarrow \\ \text{product of numbers that} \\ \text{are positive by Hp}}}{\geq} 0$

$\boxed{\Rightarrow}$: $(A^2)_{i,j} = \sum_k A_{ik}A_{kj} > 0 \implies \exists$ a page K s.t. $A_{i,K} > 0,\ A_{K,j} > 0$ Let's assume by contradiction that K is equal to i or j, because if that's not the case then $\boxed{\Rightarrow}$ would be proved. We have that:

$\boxed{K = i}$ Then: $A_{i,i} > 0$, but that is not possible by how A is constructed $\implies$ contradiction

$\boxed{K = j}$ Then: $A_{j,j} > 0$, but that is not possible by how A is constructed $\implies$ contradiction

2. The sentence: "page i can be reached from page j in exactly p steps" can alternatively be expressed as follows:

$$\exists \text{ a set of indices } K = \{K_1, K_2, \ldots, K_{p-1}\} \text{ s.t. } i, j \notin K \text{ and } A_{i,K_1}, A_{K_1,K_2}, \ldots, A_{K_{p-2},K_{p-1}}, A_{K_{p-1},j} > 0$$

The proof can be done in a similar way to point 1. by using instead that:

$$(A^p)_{i,j} = \sum_{l_1=1}^{n}\sum_{l_2=1}^{n}\cdots\sum_{l_{p-1}=1}^{n} A_{i,l_1}A_{l_1,l_2}\cdots A_{l_{p-2},l_{p-1}}A_{p-1,j}$$

3. Because $A_{i,j} \geq 0$, $\forall\ i, j$ it means that $(I + A + A^2 + \cdots + A^p)_{ij}$ is a sum of non negative components meaning it will be > 0 if at least one of its element is > 0. This means that:

$\boxed{\Leftarrow}$: If page i can be reached from page j in P steps, with $0 \leq P \leq p$, then by 2. it is true that $(A^P)_{i,j} > 0$. Meaning $(I + A + A^2 + \cdots + A^p)_{i,j} \geq (A^P)_{i,j} > 0$

$\boxed{\Rightarrow}$: $(I+A+A^2+\cdots+A^p)_{i,j} > 0 \implies \exists\ 0 \leq P \leq p$ s.t $(A^P)_{i,j} > 0 \underset{\substack{\downarrow \\ \text{by 2.}}}{\implies}$ page i can be reached from page j in P steps

4. Web strongly connected means that any page i can be reached from any page j in a finite number of steps, (at worst it will require $n-1$ steps). So, for what was proven in 3. , it is true that

$$(I + A + A^2 + \cdots + A^{n-1})_{i,j} > 0,\ \forall\ i, j \implies (I + A + A^2 + \cdots + A^{n-1})_{i,j} \text{ is a positive matrix}$$

5. A matrix B is column-stochastic if its entries are not negative, (this is true for 4.), and the entries in each column sum to 1. In the exercise 8 was proven that the product of column-stochastic matrices is column-stochastic. Meaning the matrix $(I + A + A^2 + \cdots + A^{n-1})$ is a sum of column stochastic matrices and for any column j is true that

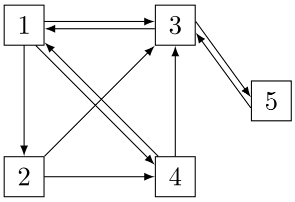
$$\sum_{i=1}^{n}(I + A + A^2 + \cdots + A^{n-1})_{i,j} = 1 + 1 + \cdots + 1 = n \implies \sum_{i=1}^{n} B_{i,j} = \frac{1}{n}n = 1 \text{ , } \forall\ j \in \{1, \ldots, n\}$$

6. $\mathbf{x} \in V_1(A) \implies A\mathbf{x} = \mathbf{x} \implies B\mathbf{x} = \frac{1}{n}(I + A + A^2 + \cdots + A^p)\mathbf{x} \underset{\substack{\downarrow \\ A^n\mathbf{x} = A^{n-1}(A\mathbf{x}) = A^{n-1}\mathbf{x} = \cdots = A\mathbf{x} = \mathbf{x}}}{=} \frac{1}{n}(\mathbf{x} + \mathbf{x} + \ldots \mathbf{x}) = \frac{1}{n}n\mathbf{x} = \mathbf{x} \implies \mathbf{x} \in V_1(B)$

Exercise 11

QUESTION: *Consider again the web in Figure 2.1, with the addition of a page 5 that links to page 3, where page 3 also links to page 5. Calculate the new ranking by finding the eigenvector of M (corresponding to $\lambda = 1$) that has positive components summing to one. Use $m = 0.15$*

SOLUTION:



$$A = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$M = (1{-}m)A{+}mS = \begin{bmatrix} 0.030 & 0.030 & 0.455 & 0.455 & 0.030 \\ 0.313 & 0.030 & 0.030 & 0.030 & 0.030 \\ 0.313 & 0.455 & 0.030 & 0.455 & 0.880 \\ 0.313 & 0.455 & 0.030 & 0.030 & 0.030 \\ 0.030 & 0.030 & 0.455 & 0.030 & 0.030 \end{bmatrix}$$

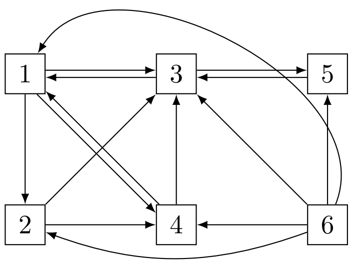
We can find the Perron eigenvector $\mathbf{x}$ by solving the linear system $(M - I)\mathbf{x} = 0$. This can be done by applying the Gaussian Elimination Algorithm to the matrix $M - I$, to transform it into its row echelon form, and then by applying the backward substitution method to find $\mathbf{x}$. The algorithm is implemented in Matlab, and returns:

$$M = \begin{bmatrix} -0.9700 & 0.0300 & 0.4550 & 0.4550 & 0.0300 \\ 0 & -0.9603 & 0.1770 & 0.1770 & 0.0397 \\ 0 & 0 & -0.7374 & 0.6876 & 0.9089 \\ 0 & 0 & 0 & -0.4925 & 0.3826 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{\downarrow \\ \text{By choosing} \\ x_5 = 1}}{\rightarrow} \quad \mathbf{x} = \begin{bmatrix} 1.3302 \\ 0.5452 \\ 1.9570 \\ 0.7768 \\ 1 \end{bmatrix}$$

Exercise 12

QUESTION: *Add a sixth page that links to every page of the web in the previous exercise, but to which no other page links. Rank the pages using A, then using M with $m = 0.15$, and compare the results.*

SOLUTION:



$$A = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & \frac{1}{5} \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 1 & \frac{1}{5} \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M = (1{-}m)A{+}mS = \begin{bmatrix} 0.0250 & 0.0250 & 0.4500 & 0.4500 & 0.0250 & 0.1950 \\ 0.3083 & 0.0250 & 0.0250 & 0.0250 & 0.0250 & 0.1950 \\ 0.3083 & 0.4500 & 0.0250 & 0.4500 & 0.8750 & 0.1950 \\ 0.3083 & 0.4500 & 0.0250 & 0.0250 & 0.0250 & 0.1950 \\ 0.0250 & 0.0250 & 0.4500 & 0.0250 & 0.0250 & 0.1950 \\ 0.0250 & 0.0250 & 0.0250 & 0.0250 & 0.0250 & 0.0250 \end{bmatrix}$$

We can find the Perron eigenvector $\mathbf{x}$ by solving the linear system $(M - I)\mathbf{x} = 0$. This can be done by applying the Gaussian Elimination Algorithm to the matrix $M - I$, to transform it into its row echelon form, and then by applying the backward substitution method to find $\mathbf{x}$.

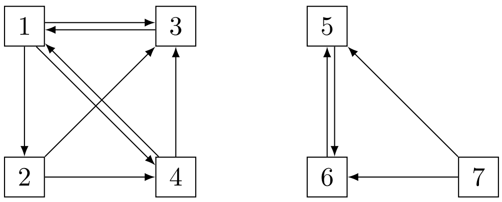
The algorithm is implemented in Matlab, and returns:

$$M = \begin{bmatrix} -0.9750 & 0.0250 & 0.4500 & 0.4500 & 0.0250 & 0.1950 \\ 0 & -0.9671 & 0.1673 & 0.1673 & 0.0329 & 0.2567 \\ 0 & 0 & -0.7535 & 0.6715 & 0.8985 & 0.3782 \\ 0 & 0 & 0 & -0.5338 & 0.3425 & 0.5019 \\ 0 & 0 & 0 & 0 & 0.1251 & -0.8698 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{\text{By choosing} \\ x_6 = 1}}{\overset{\rightarrow}{\downarrow}} \quad \mathbf{x} = \begin{bmatrix} 9.2485 \\ 3.7904 \\ 13.6069 \\ 5.4013 \\ 6.9529 \\ 1 \end{bmatrix}$$

Exercise 13

QUESTION: *Construct a web consisting of two or more subwebs and determine the ranking given by formula (3.1).*

SOLUTION:



$$A = \begin{bmatrix} 0 & 0 & 1 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M = (1-m)A + mS = \begin{bmatrix} 0.0214 & 0.0214 & 0.8714 & 0.4464 & 0.0214 & 0.0214 & 0.0214 \\ 0.3048 & 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.0214 \\ 0.3048 & 0.4464 & 0.0214 & 0.4464 & 0.0214 & 0.0214 & 0.0214 \\ 0.3048 & 0.4464 & 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.0214 \\ 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.8714 & 0.4464 \\ 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.8714 & 0.0214 & 0.4464 \\ 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.0214 & 0.0214 \end{bmatrix}$$

We can find the Perron eigenvector $\mathbf{x}$ by solving the linear system $(M - I)\mathbf{x} = 0$. This can be done by applying the Gaussian Elimination Algorithm to the matrix $M - I$, to transform it into its row echelon form, and then by applying the backward substitution method to find $\mathbf{x}$. The algorithm is implemented in Matlab, and returns:

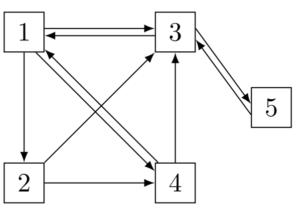
$$M = \begin{bmatrix} -0.9786 & 0.0214 & 0.8714 & 0.4464 & 0.0214 & 0.0214 & 0.0214 \\ 0 & -0.9719 & 0.2928 & 0.1605 & 0.0281 & 0.0281 & 0.0281 \\ 0 & 0 & -0.5707 & 0.6603 & 0.0412 & 0.0412 & 0.0412 \\ 0 & 0 & 0 & -0.2680 & 0.0722 & 0.0722 & 0.0722 \\ 0 & 0 & 0 & 0 & -0.9500 & 0.9000 & 0.4750 \\ 0 & 0 & 0 & 0 & 0 & 0.0974 & -0.9250 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \underset{\substack{\text{By choosing} \\ x_6 = 1}}{\overset{\rightarrow}{\downarrow}} \quad \mathbf{x} = \begin{bmatrix} 9.8174 \\ 3.7816 \\ 7.6790 \\ 5.3888 \\ 9.5000 \\ 9.5000 \\ 1 \end{bmatrix}$$

4 Fourth Section

Exercise 14

QUESTION: *For the web in Exercise 11, compute the values of* $\left\|M^k \mathbf{x}_0 - \mathbf{q}\right\|_1$ *and* $\frac{\left\|M^{k-1}\mathbf{x}_0 - \mathbf{q}\right\|_1}{\left\|M^k \mathbf{x}_0 - \mathbf{q}\right\|_1}$ *for* $k = 1, 5, 10, 50$, *using an initial guess* $\mathbf{x}_0$ *not too close to the actual eigenvector* $\mathbf{q}$ *(so that you can watch the convergence). Determine* $c = \max_{1 \le j \le n} \left|1 - 2 \min_{1 \le j \le n} M_{ij}\right|$ *and the absolute value of the second largest eigenvalue of M.*

SOLUTION:



$$A = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 & \frac{1}{2} & 1 \\ \frac{1}{3} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

$$M = (1-m)A + mS = \begin{bmatrix} 0.030 & 0.030 & 0.455 & 0.455 & 0.030 \\ 0.313 & 0.030 & 0.030 & 0.030 & 0.030 \\ 0.313 & 0.455 & 0.030 & 0.455 & 0.880 \\ 0.313 & 0.455 & 0.030 & 0.030 & 0.030 \\ 0.030 & 0.030 & 0.455 & 0.030 & 0.030 \end{bmatrix}$$

In the Exercise 11 was found that an eigenvector of the eigenvalue $\lambda = 1$ is the vector $\mathbf{x} = [1.3302,\ 0.5452,\ 1.9570,\ 0.7768,\ 1]^{\mathrm{T}}$ that if normalized to unit norm becomes: $\mathbf{q} = [0.2371, 0.0972, 0.3489, 0.1385, 0.1783]^{\mathrm{T}}$
As initial guess the vector $\mathbf{x}_0 = [0.05, 0.3, 0.05, 0.35, 0.25]^{\mathrm{T}}$ is chosen, and by using Matlab, we obtain the table:

k	Norm	Ratio
0	0.97207	0
1	0.43441	0.44689
5	0.083082	0.57533
10	0.0066536	0.61354
50	2.6335e-05	1

Moreover, the value of $c = \max\limits_{1\le j\le n} |1 - 2 \min\limits_{1\le i\le n} M_{i,j}| = 0.94$

We can now use the deflation method to find the matrix $M_2 \in \mathbb{R}^{(n-1)\times(n-1)}$, that has the same eigenvalues of $M_1 = M$, except $\lambda_1 = 1$, and then we can apply the power method to M_2 so that we can find λ_2 the second largest eigenvalue of M. By implementing the method in Matlab we find that:

$$\lambda_2 = 0.61127, \text{ with } \mathbf{v} = [-0.6022, 0.2791, 0.7811, 0.0851,\ -0.5431]^{\mathrm{T}} \text{ as an eigenvector}$$

Exercise 15

QUESTION: *To see why the second largest eigenvalue plays a role in bounding* $\left\|M^k\mathbf{x}_0 - \mathbf{q}\right\|_1$ *and* $\frac{\left\|M^{k-1}\mathbf{x}_0-\mathbf{q}\right\|_1}{\left\|M^k\mathbf{x}_0-\mathbf{q}\right\|_1}$, *consider an $n\times n$ positive column-stochastic matrix M that is diagonalizable. Let $\mathbf{x}_0$ be any vector with non-negative components that sum to one. Since M is diagonalizable, we can create a basis of eigenvectors $\{\mathbf{q}, \mathbf{v}_1, \dots, \mathbf{v}_{n-1}\}$, where $\mathbf{q}$ is the steady state vector, and then write $\mathbf{x}_0 = a\mathbf{q} + \sum_{k=1}^{n-1} b_k\mathbf{v}_k$ Determine $M^k\mathbf{x}_0$ and then show that $a = 1$ and the sum of the components of each $\mathbf{v_k}$ must equal 0. Next apply Proposition 4 to prove that, except for the non-repeated eigenvalue $\lambda = 1$, the other eigenvalues are all strictly less than one in absolute value. Use this to evaluate* $\lim_{k\to\infty} \frac{\left\|M^{k-1}\mathbf{x}_0-\mathbf{q}\right\|_1}{\left\|M^k\mathbf{x}_0-\mathbf{q}\right\|_1}$

SOLUTION:
$\boxed{1}$ We start by determining $M^k\mathbf{x}_0$:

$$M^k\mathbf{x}_0 = aM^k\mathbf{q} + M^k\sum_{i=1}^{n-1} b_i\mathbf{v}_i \underset{\substack{M\mathbf{q}=\mathbf{q} \implies M^k\mathbf{q} = M^{k-1}M\mathbf{q} = \\ = M^{k-1}\mathbf{q} = \cdots = M\mathbf{q} = \mathbf{q}}}{\overset{=}{\downarrow}} =$$

$$= a\mathbf{q} + M^k\sum_{i=1}^{n-1} b_i\mathbf{v}_i \underset{\substack{M \text{ diagonalizable} \implies \exists\, P \text{ such that:} \\ M = PDP^{-1} = \mathrm{diag}\{\lambda_i,\, i=1:n\} \implies \\ \implies M^k = (PDP^{-1})^k = PD(P^{-1}P)D(P^{-1}P)\cdots(P^{-1}P)DP^{-1} = PD^kP^{-1}}}{\overset{=}{\downarrow}} = a\mathbf{q} + \sum_{i=1}^{n-1} b_i\lambda_i^k\mathbf{v}_i$$

$\boxed{2}$ Let's now prove that the sum of each component of $\mathbf{v}_k$ is equal 0 :

$$\mathbf{v}_k \text{ eigenvector} \implies \exists\, \lambda_k \text{ s.t. } M\mathbf{v}_k = \lambda_k\mathbf{v} \implies \sum_{i=1}^{n}(M\mathbf{v}_k)_i = \lambda_k\sum_{i=1}^{n}(\mathbf{v}_k)_i \implies$$

$$\implies \sum_{i=1}^{n}(M\mathbf{v}_k)_i \underset{\substack{\text{product of matrix} \\ \text{times vector}}}{\overset{=}{\downarrow}} \sum_{i=1}^{n}\sum_{j=1}^{n} M_{i,j}(\mathbf{v}_k)_j = \sum_{j=1}^{n}(\mathbf{v}_k)_j\sum_{i=1}^{n} M_{i,j} \underset{\substack{\text{M column} \\ \text{stochastic}}}{\overset{=}{\downarrow}} \sum_{j=1}^{n}(\mathbf{v}_k)_j\cdot 1 = \lambda_k\sum_{i=1}^{n}(\mathbf{v}_k)_i$$

But $\lambda_k \neq 1$, meaning that: $\sum_{i=1}^{n}(v_k)_i = 0$

$\boxed{3}$ Then we continue by proving that $a = 1$:

$$1 = \|\mathbf{x}_0\|_1 \underset{\substack{\downarrow \\ \text{components of} \\ \mathbf{x}_0 \text{ are positive}}}{=} \sum_{i=1}^{n} (\mathbf{x}_0)_i = a \sum_{i=1}^{n} \mathbf{q}_i + \sum_{k=1}^{n-1} b_k \sum_{i=1}^{n} (\mathbf{v}_k)_i = a \cdot 1 + \sum_{i=1}^{n-1} b_k \cdot 0 \implies a = 1$$

$\boxed{4}$ Let's now consider V the subspace of $\mathbb{R}^n$ consisting of vectors $\mathbf{v}$ s.t $\sum_{i=1}^{n} \mathbf{v}_i = 0$. Then the eigenvectors v_k of the basis are inside V for what was proven in $\boxed{2}$, meaning by proposition 4 is true that:

$$\|M\mathbf{v}_k\|_1 < c\|\mathbf{v}_k\|_1 \text{ with } 0 < c < 1$$

Then we have that:

$$M\mathbf{v}_k = \lambda_k \mathbf{v}_k \implies \|M\mathbf{v}_k\|_1 = |\lambda_k| \|\mathbf{v}_k\|_1 < c\|\mathbf{v}_k\|_1 \underset{\substack{\downarrow \\ \text{dividing both} \\ \text{side by } \|\mathbf{v}_k\|}}{\implies} |\lambda_k| < c < 1$$

Finally let λ_K be the eigenvalue of the matrix M s.t $\lambda_K < 1$ and $|\lambda_K| > |\lambda_k| \ \forall \ k \in \{1, \dots, n-1\}/\{K\}$ then:

$$M^k \mathbf{x}_0 \underset{\substack{\downarrow \\ a = 1 \text{ for } \boxed{3} \\ \text{and } \boxed{1}}}{=} \mathbf{q} + \sum_{k=1}^{n-1} b_k \lambda_k^k \mathbf{v}_k \implies \lim_{k \to \infty} \frac{\|M^k \mathbf{x}_0 - \mathbf{q}\|_1}{\|M^{k-1} \mathbf{x}_0 - \mathbf{q}\|_1} = \lim_{k \to \infty} \frac{\|\sum_{i=1}^{n-1} b_i \lambda_i^{k-1} \mathbf{v}_i\|_1}{\|\sum_{i=1}^{n-1} b_i \lambda_i^{k} \mathbf{v}_i\|_1} =$$

Exercise 16

QUESTION: *Consider the link matrix*

$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \\ 1 & \frac{1}{2} & 0 \end{bmatrix}$$

Show that $M = (1-m)A + mS$ *(all* $S_{ij} = 1/3$*) is not diagonalizable for* $0 \le m < 1$.

SOLUTION:

$$M - \lambda I = \begin{bmatrix} \frac{m}{3} - \lambda & \frac{1}{2}(1 - \frac{m}{3}) & \frac{1}{2}(1 - \frac{m}{3}) \\ \frac{m}{3} & \frac{m}{3} - \lambda & \frac{1}{2}(1 - \frac{m}{3}) \\ \frac{1}{2}(1 - \frac{m}{3}) & \frac{1}{2}(1 - \frac{m}{3}) & \frac{m}{3} - \lambda \end{bmatrix} \to det(M - \lambda I) = -\lambda^3 + \lambda^2 m - \frac{1}{3}\lambda m^2 - \frac{1}{6}\lambda m + \frac{1}{2}\lambda + \frac{1}{24} m^3 + \frac{1}{24} m^2 - \frac{5}{24} m + \frac{1}{8}$$

By proposition 5, M column-stochastic $\implies$ M has 1 as eigenvalues. Moreover, it is true that:

$$tr(M) := \sum_{i=1}^{n} M_{i,i} = \sum_{i=1}^{n} \lambda_i \text{ where } \lambda_1, \ \dots, \ \lambda_n \text{ are the eigenvalues of M}$$

This means that if we assume by contradiction that M is diagonalizable then it has to be true that:

$$\frac{tr(M) - 1}{2} = \frac{m-1}{2} \text{ can't be an eigenvalue of M}$$

If that were the case, it would imply that M is a matrix with an eigenvalue of multiplicity 2, and therefore, it would not be diagonalizable. By substituting $\lambda = \frac{m-1}{2}$ in the characteristic polynomial, we can find that $p(\lambda) = 0$. This proves that M has not 3 distinct eigenvalues, meaning it is not diagonalizable.

Exercise 17

QUESTION: *How should the value of m be chosen? How does this choice affect the rankings and the computation time?*

SOLUTION: The power method converge asymptotically according to $\|M\mathbf{x}_k - \mathbf{q}\|_1 \approx |\lambda_2| \|\mathbf{x} - \mathbf{q}\|_1$ with $|\lambda_2| \le 1 - m$. This means that higher values of m will produce a faster convergence. But, as suggested in the paper [9] the choice of a $m = 0.15$ carries intuitive weight. It implies that roughly five-sixths of the time a web surfer randomly clicks on hyperlinks, while one-sixth of the time this web surfer will go to the URL line and type the address of a new page to "teleport" to. A lower value of m not only slows the convergence of the power method, but also places much greater emphasis on the hyperlink structure of the web and much less on the teleportation tendencies of surfers. Perhaps it gives a "truer" PageRanking, but this might not be worth it. So in conclusion, since PageRank is trying to take advantage of the underlying link structure, it is more desirable (at least in this respect) to choose m close to 0, However, if m is too close to 0, then PageRanks will be unstable, and the convergence rate slows.