Structured LISTA for Multidimensional Harmonic Retrieval

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Outline

- ❏ MHR: INSTA, LISTA and LISTA-Toeplitz
- ❏ Numerical Simulation Results
- ❏ Final Comments

MHR: INSTA, LISTA and LISTA-Toeplitz

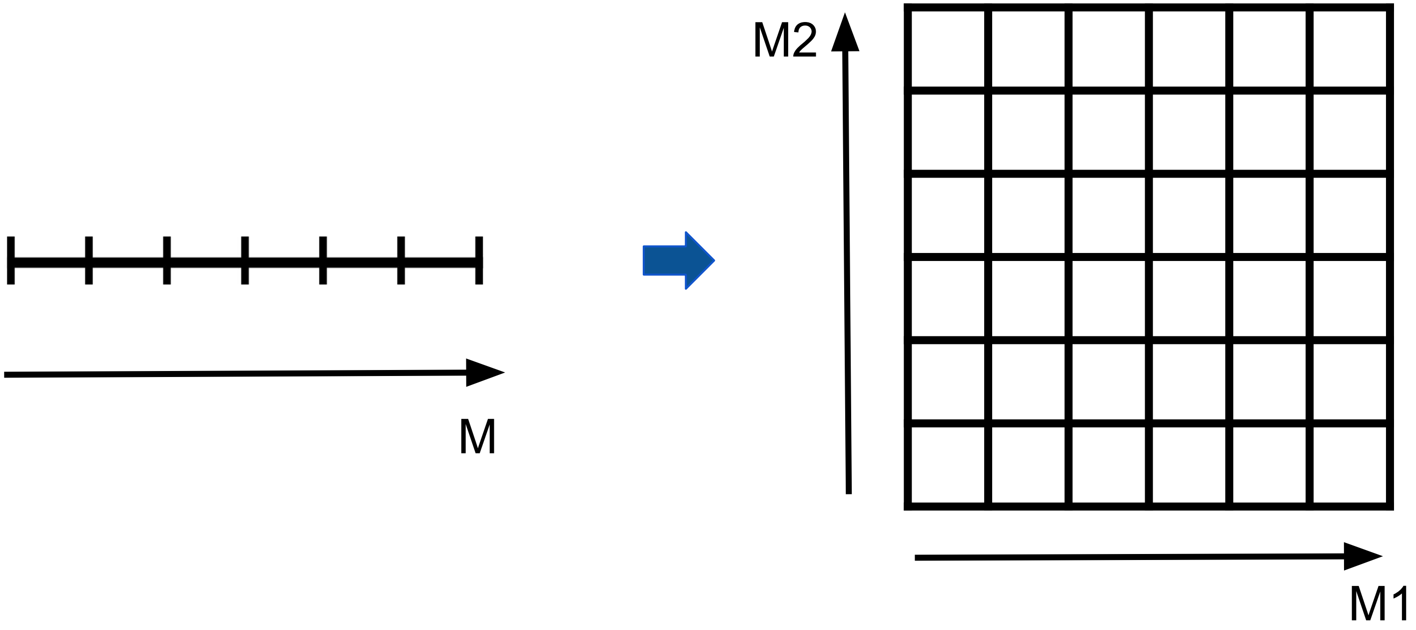
1D Harmonic Retrieval

CS and Sparse Recovery $\Rightarrow$ Linear decoding problem

$$\boxed{\boldsymbol{y} = \boldsymbol{\Phi}\boldsymbol{x} + \boldsymbol{w}}$$

$$\boldsymbol{\Phi} = \boldsymbol{R}\boldsymbol{\Psi} \qquad \boldsymbol{\Phi} \in \mathbb{C}^{N\times M} \qquad \boldsymbol{y} \in \mathbb{C}^{N} \qquad \boldsymbol{x} \in \mathbb{C}^{M}$$

Multidimensional Harmonic Retrieval



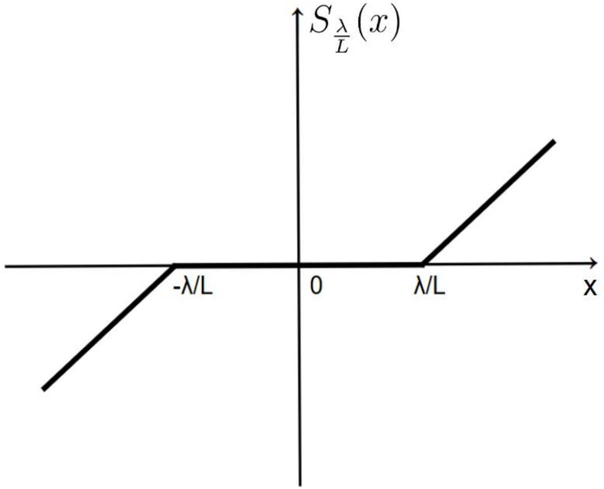
1D **2D**

Iterative Shrinkage Thresholding Algorithm (ISTA)

$$\min_{\boldsymbol{x}} \frac{1}{2}\|\boldsymbol{y} - \boldsymbol{\Phi}\boldsymbol{x}\|_2^2 + \lambda\|\boldsymbol{x}\|_1$$

Proximal gradient descent method

$$\boldsymbol{x}^{(t+1)} = S_{\lambda/L}\left(\boldsymbol{x}^{(t)} + \frac{1}{L}\Phi^H(\boldsymbol{y} - \Phi\boldsymbol{x}^{(t)})\right)$$



Iterative Shrinkage Thresholding Algorithm (ISTA)

SLOW TO CONVERGE $O(1/\varepsilon)$

HIGH COMPUTATIONAL COST

☹

Learned ISTA (LISTA)

$$\frac{1}{L}\boldsymbol{\Phi}^H \Rightarrow \boldsymbol{W_e}^{(t)} \in \mathbb{C}^{M\times N}$$

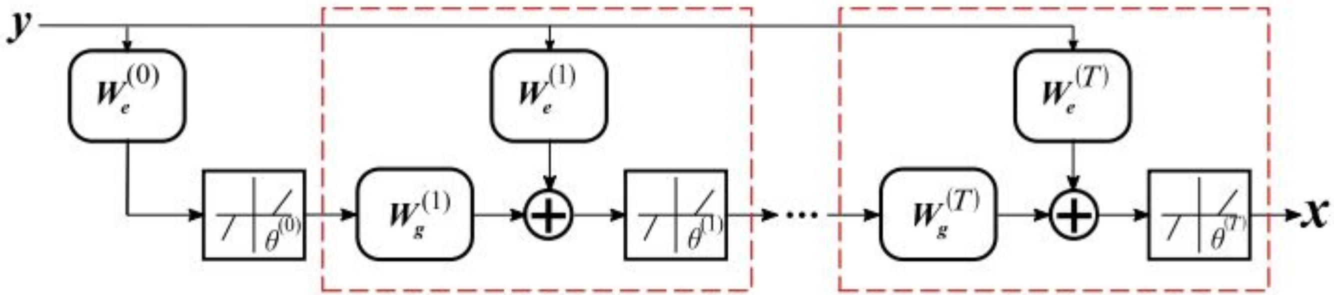
$$\boldsymbol{I} - \frac{1}{L}\boldsymbol{\Phi}^H\boldsymbol{\Phi} \Rightarrow \boldsymbol{W_g}^{(t)} \in \mathbb{C}^{M\times M}$$

$$\lambda/L \Rightarrow \boldsymbol{\theta}^{(t)}$$

$$\boxed{\boldsymbol{x}^{(t+1)} = S_{\theta^{(t)}}\left(\boldsymbol{W_e}^{(t)}\boldsymbol{y} + \boldsymbol{W_g}^{(t)}\boldsymbol{x}^{(t)}\right)}$$

$\{\boldsymbol{W_e}^{(t)}, \boldsymbol{W_g}^{(t)}, \boldsymbol{\theta}^{(t)}\}$ Parameters to learn

LISTA: Unfolded Version of ISTA



Source: [1]

☺

Same accuracy of ISTA

Fewer iteration

BUT

Many parameters to learn

Wg is big (M>>N)

☹

LISTA-Toeplitz

$\mathbf{\Phi}^H \mathbf{\Phi}$ Toeplitz structure

$\Downarrow$

$\boldsymbol{W_g}$ Toeplitz structure

Use **h**(1D), **H**(2D) instead of **Wg**

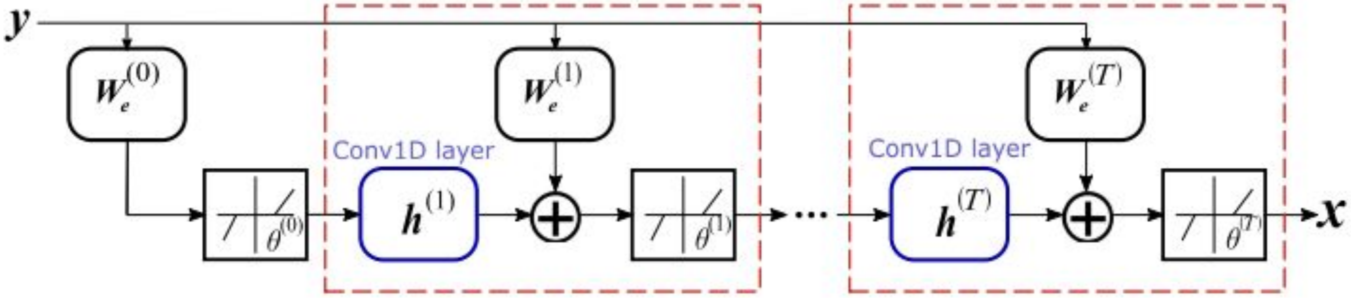
$$\boldsymbol{x}^{(t+1)} = S_{\theta(t)}\left(\boldsymbol{W_e}^{(t)}\boldsymbol{y} + \boldsymbol{h}^{(t)} * \boldsymbol{x}^{(t)}\right)$$

1D

$$\boldsymbol{x}^{(t+1)} = S_{\theta(t)}\left(\boldsymbol{W_e}^{(t)}\boldsymbol{y} + vec(\boldsymbol{H}^{(t)} * \boldsymbol{X}^{(t)})\right)$$

2D

LISTA-Toeplitz



Source: [1]

1D: $\boldsymbol{W_g}^{(t)} \in \mathbb{C}^{M \times M}$ ➡ $\boldsymbol{h^{(t)}} \in \mathbb{C}^{(2M-1)}$

2D: $\boldsymbol{W_g}^{(t)} \in \mathbb{C}^{(M_1 M_2) \times (M_1 M_2)}$ ➡ $\boldsymbol{H}^{(t)} \in \mathbb{C}^{(2M_1-1) \times (2M_2-1)}$

LISTA-Toeplitz Advantages

Huge **reduction of the parameters to learn** wrt LISTA,

from $O(M^2)$ to $O(M)$(1D), from $O(M^4)$to $O(M^2)$(2D)

Convolutions can be efficiently computed via **FFT**

Same advantages of LISTA: same accuracy but fewer iterations wrt ISTA

Numerical Simulation Results

Signal Model Parameters:

- ❑ M : Number of samples of the sparse signal
- ❑ N : Number of observation samples
- ❑ k : Sparsity level
- ❑ σ^2 : Noise power

Network Hyperparameters:

- ❑ T : Number of layers/iterations
- ❑ λ : Regularization parameter
- ❑ N_{tr}, N_{test}: Number of training/test samples
- ❑ Learning rate
- ❑ Number of epochs
- ❑ Batch size

Learnable Parameters and Initialization

LISTA ➡ $\{\boldsymbol{W_e}^{(t)}, \boldsymbol{W_g}^{(t)}, \boldsymbol{\theta}^{(t)}\}$

LISTA Toeplitz ➡ $\{\boldsymbol{W_e}^{(t)}, \boldsymbol{h}^{(t)}, \boldsymbol{\theta}^{(t)}\}$

Initialization:

$$\boldsymbol{W_e} = \frac{1}{L}\hat{\boldsymbol{\Phi}}^H$$

where:
$$\hat{\boldsymbol{\Phi}} = \boldsymbol{Y}(\boldsymbol{X}^H\boldsymbol{X})^{-1}\boldsymbol{X}^H$$
$$\boldsymbol{Y} = [\boldsymbol{y}_1, \boldsymbol{y}_2, \ldots, \boldsymbol{y}_{N_{tr}}]$$
$$\boldsymbol{X} = [\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{N_{tr}}]$$

$$\boldsymbol{\theta} = \frac{\lambda}{L}$$

$$\boldsymbol{W_g} = 0 \qquad \boldsymbol{h} = 0$$

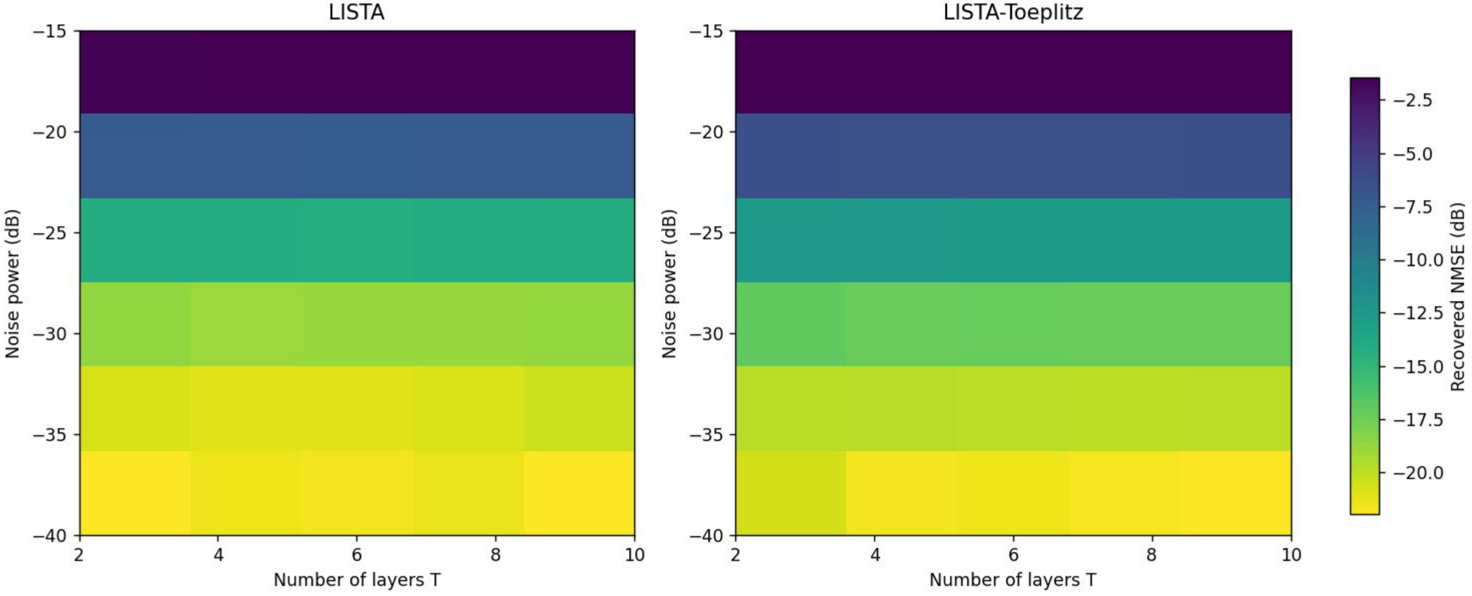
otherwise: $\boldsymbol{W_g} = I - \frac{1}{L}\hat{\boldsymbol{\Phi}}^H\hat{\boldsymbol{\Phi}}$

Figures of Merit

- **Training and Validation Loss**
- **Recovered NMSE**
- **Hit rate**

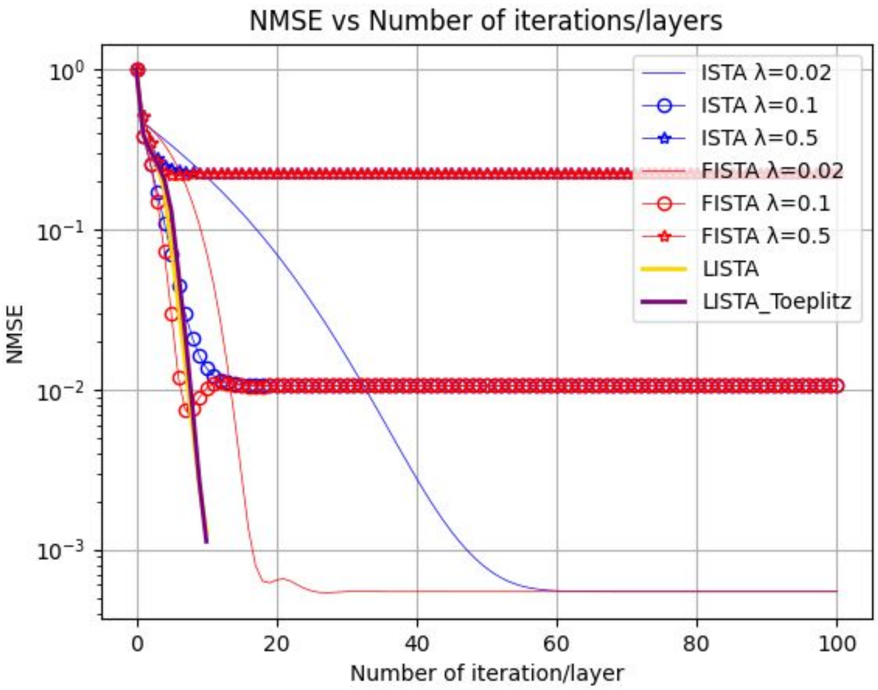
1D MHR Numerical Results

Recovered NMSE of LISTA and LISTA-Toeplitz



$$M = 128, N = 64, k = 4, N_{tr} = 3000, N_{test} = 100, N_{epoch} = 50, T = 2 \div 10, \sigma^2 = (-40 \div -15)dB, \lambda = 0.02$$

NMSE of Different Algorithms and Networks in Each Iteration/Layer

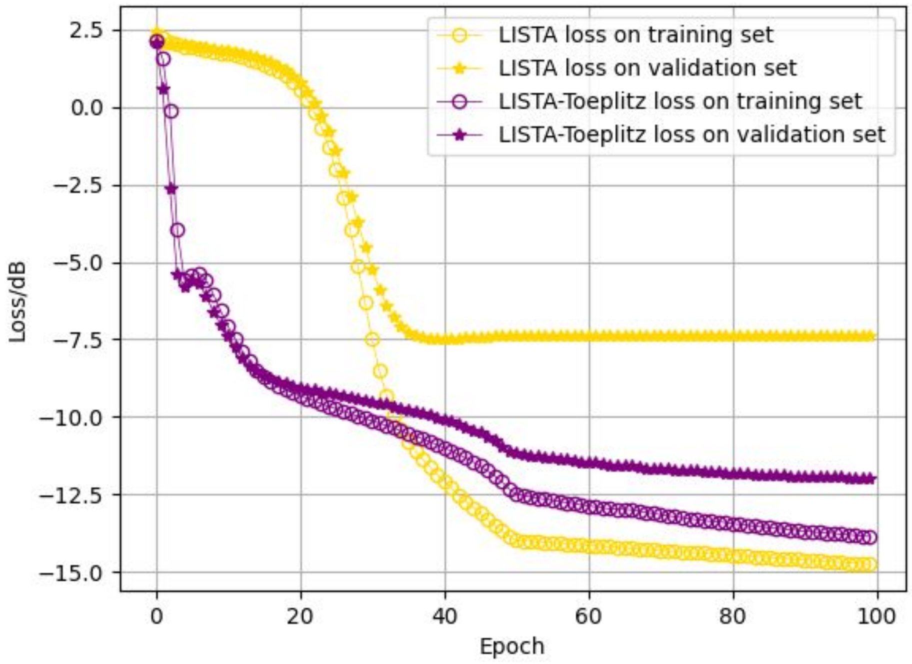


$M = 128, N = 64, k = 4, \sigma^2 = -40dB$

$N_{tr} = 3000, N_{test} = 100, N_{epoch} = 50$

$T_{LISTA} = 5$

Limited Training Samples



$M = 128, N = 64, k = 4, \sigma^2 = -40dB$

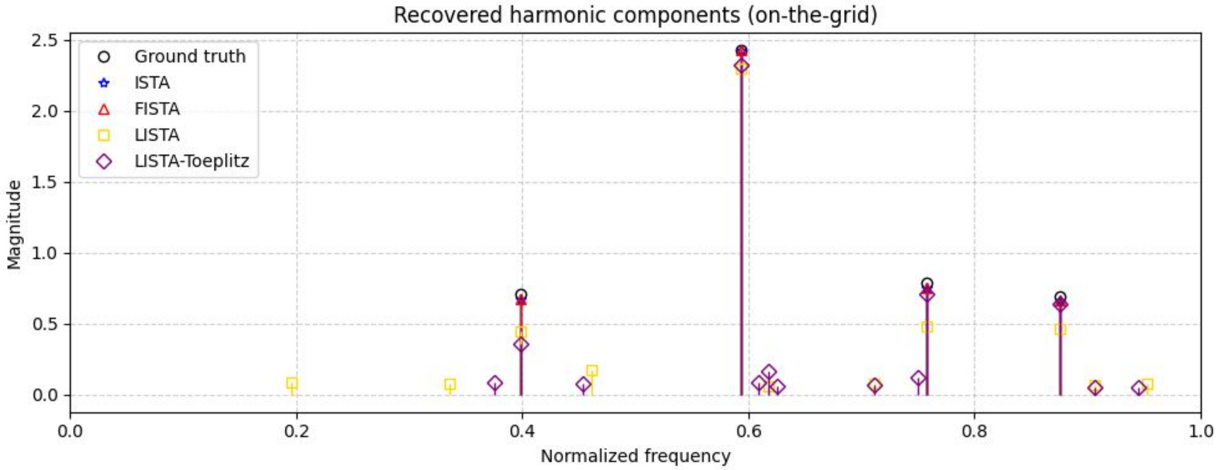
$N_{tr} = 1500, N_{test} = 100$

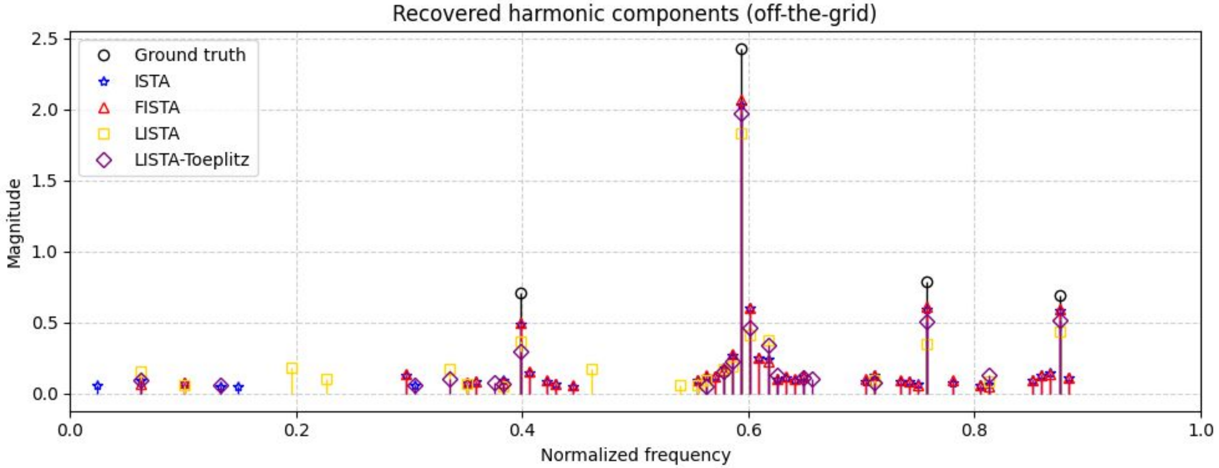
$T_{LISTA} = 5$

On-the-grid

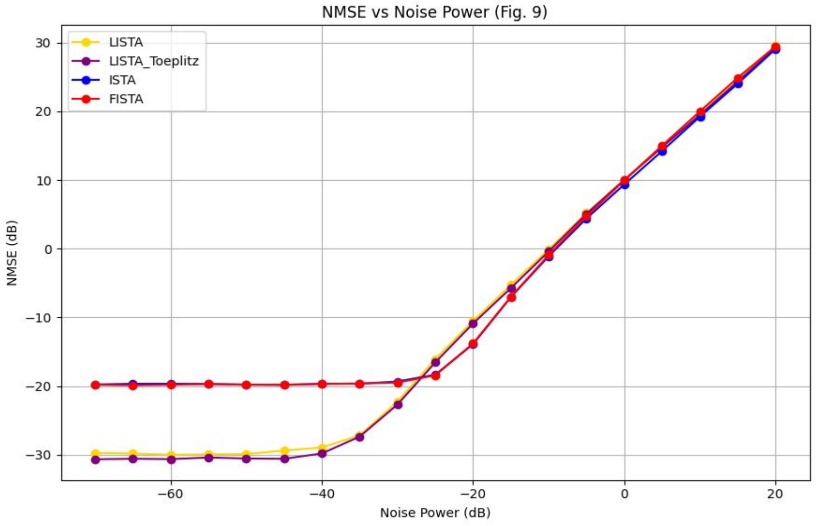
$M = 128, N = 64, k = 4, \sigma^2 = -40dB$
$N_{tr} = 3000, N_{test} = 100, N_{epoch} = 50$
$T_{LISTA} = 5$

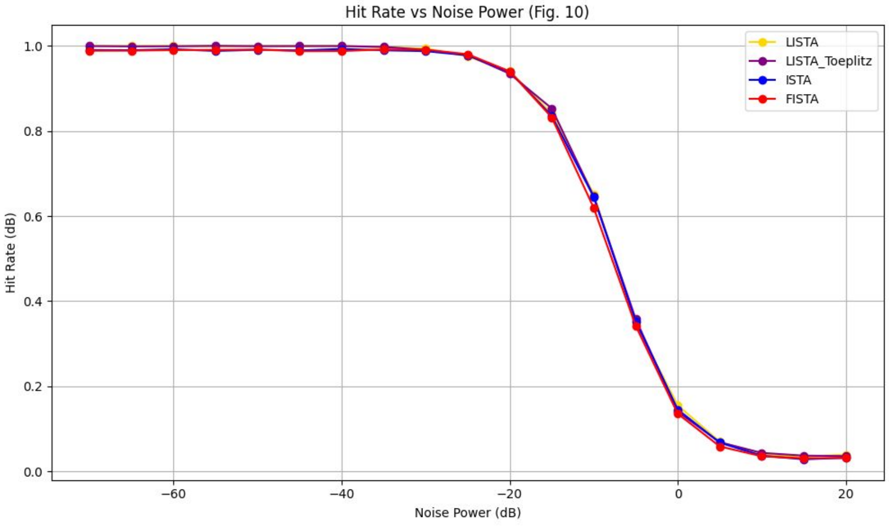
Off-the-grid





NMSE and Hit Rate vs Noise Power

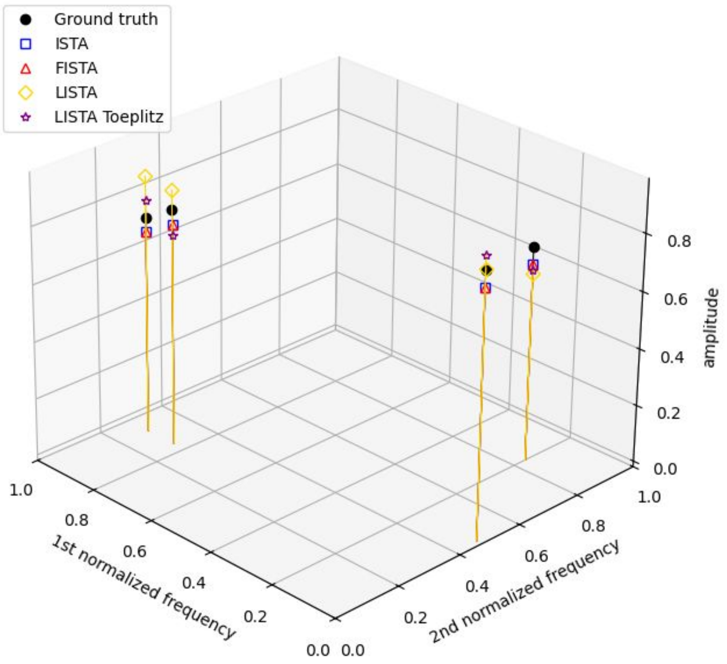




$$M = 128, N = 64, k = 4, N_{tr} = 3000, N_{test} = 100, N_{epoch} = 50, T_{LISTA} = 5, \sigma^2 = -40dB, \lambda = 0.1$$

2D MHR Numerical Results

Recovered 2D Plane of Multiple Harmonic Components

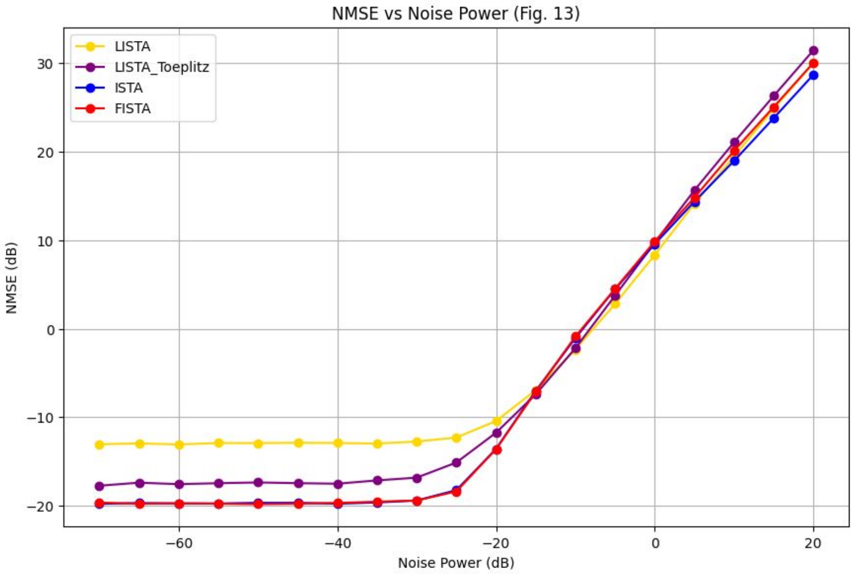


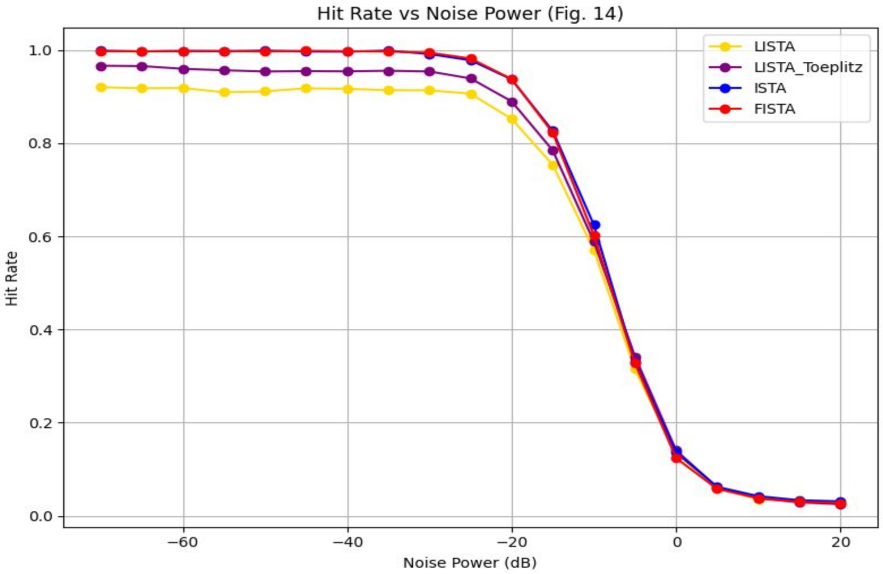
$$M_1 = 12, M_2 = 12, N = 64, k = 4$$

$$N_{tr} = 4000, N_{test} = 100, N_{epoch} = 50$$

$$T = 5, \lambda = 0.02, \sigma^2 = -40dB$$

NMSE and Hit Rate vs Noise Power





$$M_1 = 12, M_2 = 12, N = 64, k = 4, N_{tr} = 3000, N_{test} = 100, N_{epoch} = 50, T_{LISTA} = 5, \sigma^2 = -40dB, \lambda = 0.1$$

Final Comments

Bibliography

[1] Tianyao Huang Rong Fu Yimin Liu and Yonina C. Eldar. “Structured LISTA for Multidi mensional Harmonic Retrieval”. In: IEEE TRANSACTIONS ON SIGNAL PROCESSING 69 (2021), pp. 3459–3472.