TRASFORMATA DI LAPLACE

Def Se $f : [0, +\infty) \to \mathbb{C}$ è loc. sommabile, si dice <u>trasformata di Laplace di f</u> la funzione

$$\mathcal{L}(f) : \Omega_f \subseteq \mathbb{C} \longrightarrow \mathbb{C}$$

definita da

$$\mathcal{L}(f)(s) := \int_0^{+\infty} f(t) e^{-st} \, dt$$

dove

$s \in \Omega_f = \{ s \in \mathbb{C} : f(t) e^{-st} \text{ è sommabile in } t \in [0, +\infty) \}$.

f si dice $\mathcal{L}$-trasformabile se $\Omega_f \neq \emptyset$.

OSS Spesso si scrive $\mathcal{L}(H(t) f(t))$ invece che $\mathcal{L}(f(t))$

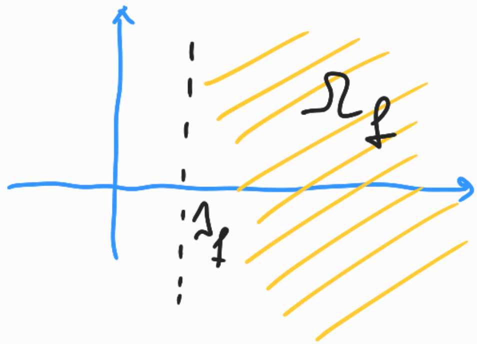
PROP 4 OPZIONI:

(i) $\Omega_f = \mathbb{C}$

(ii) $\Omega_f = \emptyset$

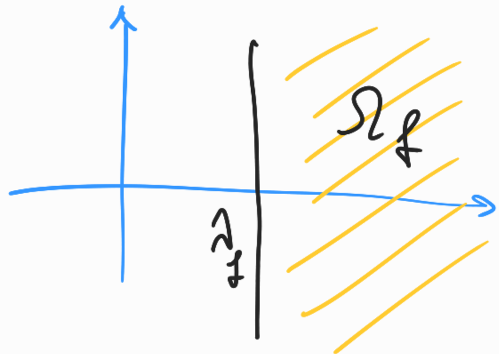
(iii) $\exists\, \lambda_f \in \mathbb{R}$ t.c.

$$\Omega_f = \{ s \in \mathbb{C} : \operatorname{Re} s > \lambda_f \}$$



(iv) $\exists\, \lambda_f \in \mathbb{R}$ t.c.

$$\Omega_f = \{ s \in \mathbb{C} : \operatorname{Re} s \geq \lambda_f \}$$



(CHE SI DIMOSTRA ESISTERE)

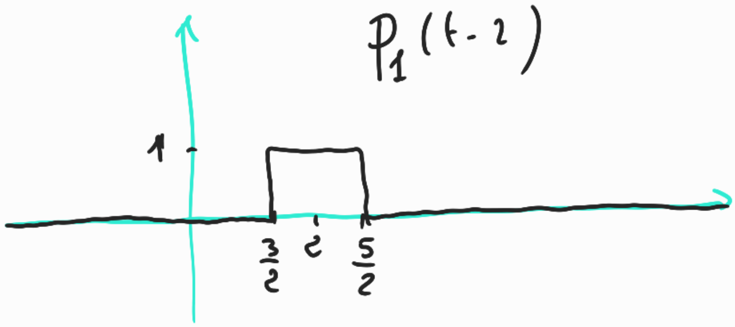
λ_f si dice <u>ascissa di $\mathcal{L}$-trasformabilità</u>

<u>NO DIM</u>

VALGONO LE PROP. DEL FORMULARIO
(ANALOGHE A QUELLE VISTE IN ANALISI 2)

V. FORMULARIO
O DISPENSE

Es $\mathcal{L}(p_1(t-2))(s) = ?$



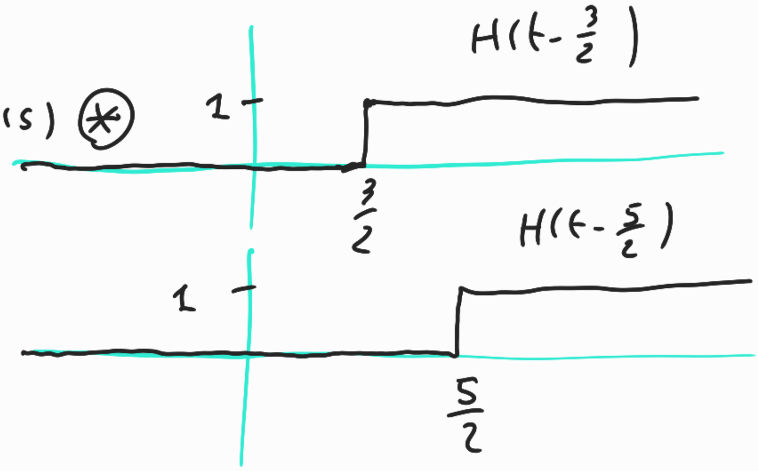
Sol

utilizziamo

$$\mathcal{L}(t^K H(t))(s) = \frac{K!}{s^{K+1}}$$ con $K \in \mathbb{N}$, vale per $\operatorname{Re} s > 0$

$$\mathcal{L}(p_1(t-2))(s) = \mathcal{L}\left(H\left(t-\tfrac{3}{2}\right) - H\left(t-\tfrac{5}{2}\right)\right)(s)$$

$$= \mathcal{L}\left(H\left(t-\tfrac{3}{2}\right)\right)(s) - \mathcal{L}\left(H\left(t-\tfrac{5}{2}\right)\right)(s) \; (*)$$



ORA VOGLIO USARE

$$\mathcal{L}(f(t-t_0)H(t-t_0))(s) = e^{-t_0 s}\mathcal{L}(f(t)) \quad \forall s \in \Omega_f$$

con $t_0 = \frac{3}{2}$ e $t_0 = \frac{5}{2}$

e con $f = H$

$(*)$
$$= \mathcal{L}\left(H\left(t-\tfrac{3}{2}\right)H\left(t-\tfrac{3}{2}\right)\right)(s) - \mathcal{L}\left(H\left(t-\tfrac{5}{2}\right)H\left(t-\tfrac{5}{2}\right)\right)(s)$$

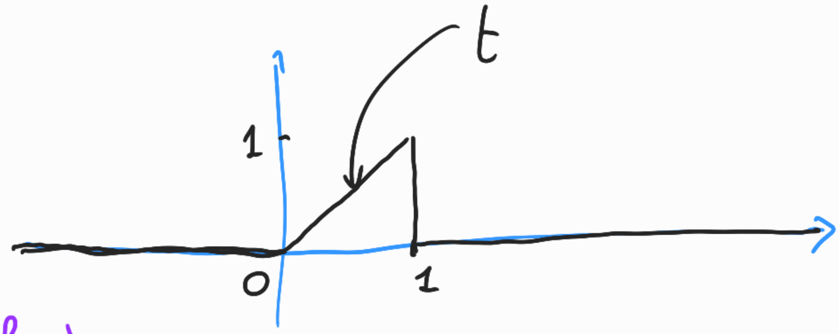
$$= e^{-\frac{3}{2}s}\mathcal{L}(H(t))(s) - e^{-\frac{5}{2}s}\mathcal{L}(H(t))(s)$$

$$= e^{-\frac{3}{2}s}\frac{1}{s} - e^{-\frac{5}{2}s}\frac{1}{s}$$

$$= \frac{e^{-\frac{3}{2}s} - e^{-\frac{5}{2}s}}{s}$$ per Rosso

Es

$$\mathcal{L}\left(t\, p_1\left(t-\tfrac{1}{2}\right)\right) = ?$$



Sol

$$\left[\mathcal{L}(f(t))\right]'(s) = -\mathcal{L}(t f(t))(s)$$

$$\mathcal{L}\left(t\, p_1\left(t-\tfrac{1}{2}\right)\right)(s) = -\left[\mathcal{L}\left(p_1\left(t-\tfrac{1}{2}\right)\right)\right]'(s)$$

$$= -\left[\mathcal{L}\left(H(t)-H(t-1)\right)\right]'(s)$$

$$= -\left[\mathcal{L}(H(t))\right]'(s) + \left[\mathcal{L}\left(H(t-1)H(t-1)\right)\right]'(s)$$

$$= -\frac{d}{ds}\left(\frac{1}{s}\right) + \frac{d}{ds}\left[e^{-s}\mathcal{L}(H(t))\right](s)$$

$$= \frac{1}{s^2} + \frac{d}{ds}\left(\frac{e^{-s}}{s}\right) = \frac{1}{s^2} + \frac{-e^{-s}s - e^{-s}}{s^2}$$

$$= \frac{1 - se^{-s} - e^{-s}}{s^2} \qquad \text{per } \operatorname{Re} s > 0$$

Es $\mathcal{L}\left(H(t)e^{t-1} + H(t-2)\cos t\right) = ?$

$$\left\{ \begin{array}{l} \mathcal{L}(e^{s_0 t} f(t))(s) \\ \quad \| \\ \mathcal{L}(f(t))(s-s_0) \\ \text{con } s_0 \in \mathbb{C} \\ \text{e con} \\ s \in \Omega_f + s_0 := \\ := \{ s+s_0 : s \in \Omega_f \} \end{array} \right.$$

Sol

$$? = \mathcal{L}\left(H(t)\overbrace{e^{t}}^{s_0=1} e^{-1}\right)(s) + \mathcal{L}\left(H(t-2)\,\frac{e^{it}+e^{-it}}{2}\right)(s)$$

$$= e^{-1}\mathcal{L}(H(t))(s-1) + \frac{1}{2}\mathcal{L}(H(t-2)e^{it})(s) + \frac{1}{2}\mathcal{L}(H(t-2)e^{-it})$$

$$= e^{-1}\underbrace{\frac{1}{s-1}}_{\text{per } \operatorname{Re}s>1} + \frac{1}{2}\mathcal{L}(H(t-2)e^{i(t-2)}e^{2i})(s) + \frac{1}{2}\mathcal{L}(H(t-2)e^{-i(t-2)}e^{-2i})(s)$$

$$= \frac{e^{-1}}{s-1} + \frac{e^{2i}}{2}\,\mathcal{L}(H(t-2)e^{i(t-2)}H(t-2))(s) + \frac{e^{-2i}}{2}\,\mathcal{L}(H(t-2)e^{-i(t-2)}H(t-2))(s)$$

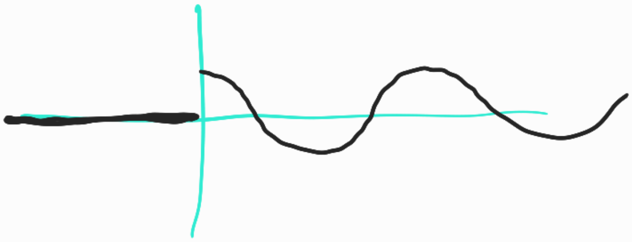
$$= \frac{e^{-1}}{s-1} + \frac{e^{2i}}{2}e^{-2s}\mathcal{L}(H(t)e^{+it})(s) + \frac{e^{-2i}}{2}e^{-2s}\mathcal{L}(H(t)e^{-it})(s)$$

[usiamo ora $\mathcal{L}(e^{s_0 t}H(t))(s) = \frac{1}{s-s_0}$ per $\operatorname{Re}s > \operatorname{Re}s_0$]

$$= \underbrace{\frac{e^{-1}}{s-1}}_{\text{per } \operatorname{Re}s>1} + \frac{e^{2i}}{2}e^{-2s}\underbrace{\frac{1}{s-i}}_{\text{per } \operatorname{Re}s>\operatorname{Re}(i)=0} + \frac{e^{-2i}}{2}e^{-2s}\underbrace{\frac{1}{s+i}}_{\text{per } \operatorname{Re}s>\operatorname{Re}(-i)=0} \quad \text{per } \operatorname{Re}s > 1$$

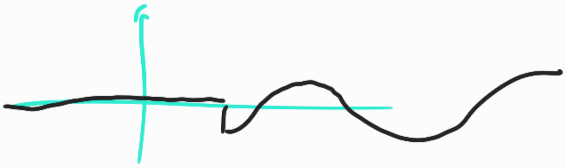
N.B. Sul formulario c'è

$$\mathcal{L}(H(t)\cos t)$$



ma noi dovevamo calcolare

$$\mathcal{L}(H(t-2)\cos t)$$



DIVERSE! $\Longrightarrow$ utilizziamo

$$\cos t = \frac{e^{it}+e^{-it}}{2}$$

Es

$$\mathcal{L}\left(e^{-it}t^7 H(t)\right)(s) = ?$$

$$? = \mathcal{L}\left(t^7 H(t)\right)(s+i)$$

$$= \frac{7!}{(s+i)^8} \qquad \text{per } \operatorname{Re} s > 0$$

Es $\mathcal{L}((t-4)H(t-5))(s) = ?$

$$? = \mathcal{L}((t-5+1)H(t-5))(s)$$

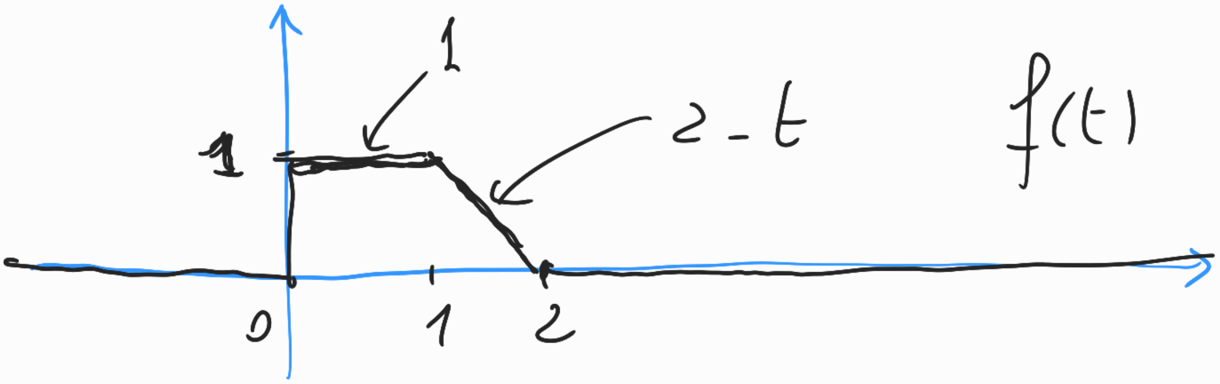
$$= \mathcal{L}((t-5)H(t-5))(s) + \mathcal{L}(H(t-5))(s)$$

$$= e^{-5s}\mathcal{L}(tH(t))(s) + e^{-5s}\mathcal{L}(H(t))(s)$$

$$= e^{-5s}\frac{1}{s^2} + e^{-5s}\frac{1}{s} = \frac{e^{-5s} + s e^{-5s}}{s^2}$$ per $\operatorname{Re} s > 0$

Es

$$f(t)=\begin{cases}1 & \text{se } 0\le t\le 1\\ 2-t & \text{se } 1<t<2\\ 0 & \text{altrimenti}\end{cases}\quad ;\ \mathcal{L}(f)=?$$



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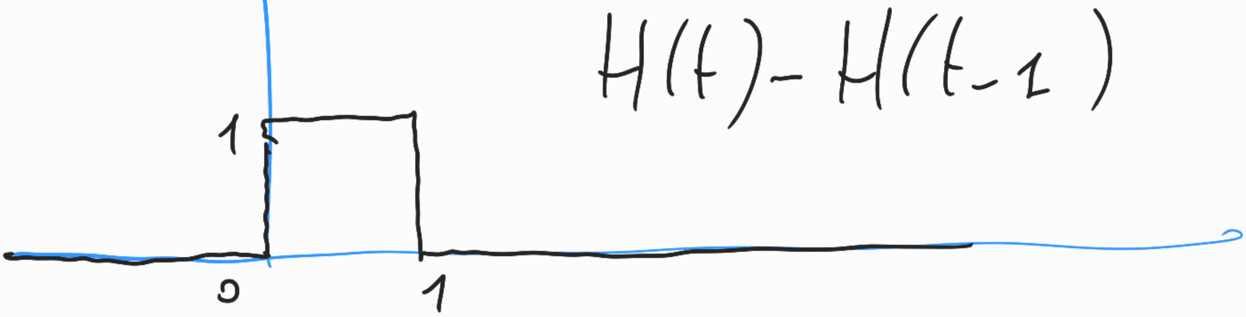

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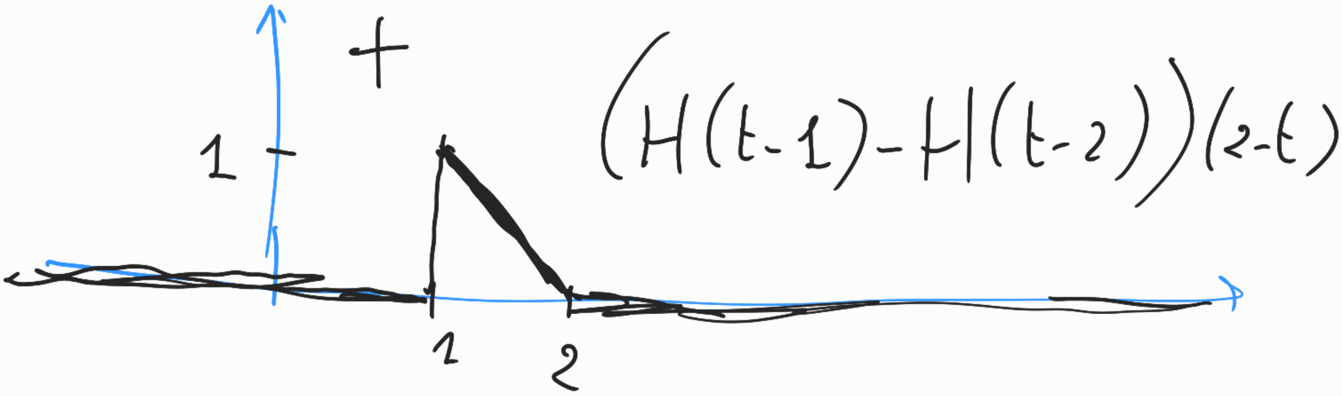

$$f(t)=\big(H(t)-H(t-1)\big)+\big(H(t-1)-H(t-2)\big)(2-t)$$

$$=H(t)-H(t-1)+(2-t)H(t-1)+(t-2)H(t-2)$$

$$=H(t)+(1-t)H(t-1)+(t-2)H(t-2)$$

$$=H(t)-(t-1)H(t-1)+(t-2)H(t-2)$$

$$\Rightarrow \mathcal{L}(f)(s) = \mathcal{L}(H(t))(s)$$

$$-\mathcal{L}((t-1)H(t-1))(s) + \mathcal{L}((t-2)H(t-2))$$

$$= \frac{1}{s} - e^{-s}\mathcal{L}(tH(t))(s) + e^{-2s}\mathcal{L}(tH(t))(s)$$

$$= \frac{1}{s} - \frac{e^{-s}}{s^2} + \frac{e^{-2s}}{s^2} \quad \text{per } \operatorname{Re} s > 0$$

IN elettrotecnica

$$u(t) := H(t)$$