Es # 3, Esame 03/07/2017 (VERSIONE C)

Sia $T_n = \frac{2n^2}{1-n}\left(\delta_{\frac{2}{n^2}} - \delta_{\frac{1}{n}}\right)$ per $n > 1$.

Calcolare, se esiste, il limite di T_n in $\mathcal{D}'(\mathbb{R})$

SOL

Sia $\varphi \in \mathcal{D}(\mathbb{R})$, si ha:

$$\langle T_n, \varphi\rangle = \frac{2n^2}{1-n}\left(\langle \delta_{\frac{2}{n^2}}, \varphi\rangle - \langle \delta_{\frac{1}{n}}, \varphi\rangle\right)$$

$$= \underbrace{\boxed{\frac{2n^2}{1-n}}}_{\to -\infty}\underbrace{\boxed{\left(\varphi\left(\tfrac{2}{n^2}\right) - \varphi\left(\tfrac{1}{n}\right)\right)}}_{\to 0} \qquad \text{"}\infty\cdot 0\text{"}$$

$$= \frac{\varphi\left(\frac{2}{n^2}\right) - \varphi\left(\frac{1}{n}\right)}{\frac{1-n}{2n^2}} \qquad \text{"}\frac{0}{0}\text{"}$$

$$= \frac{\varphi\left(\frac{2}{n^2}\right) - \varphi\left(\frac{1}{n}\right)}{\frac{1}{2n^2} - \frac{1}{2n}} \qquad \text{"}\frac{0}{0}\text{"}$$

Calcolo ora il limite

$$\lim_{x\to 0} \frac{\varphi(2x^2) - \varphi(x)}{\frac{x^2}{2} - \frac{x}{2}} \overset{\text{(De L'Hôpital)}}{=} \lim_{x\to 0} \frac{\overbrace{\boxed{\varphi'(2x^2)4x}}^{\to\varphi'(0)\cdot 0} - \overbrace{\varphi'(x)}^{\to\varphi'(0)}}{\underbrace{x - \frac{1}{2}}_{\to -\frac{1}{2}}} = 2\varphi'(0)$$

quindi il limite sarà lo stesso lungo la successione $a_n = \frac{1}{n} \longrightarrow 0$:

$$\lim_{n\to\infty} \frac{\varphi(2a_n^2)-\varphi(a_n)}{\frac{a_n^2}{2}-\frac{a_n}{2}} = \lim_{n\to\infty} \frac{\varphi\left(2\left(\frac{1}{n}\right)^2\right)-\varphi\left(\frac{1}{n}\right)}{\frac{1}{2}\left(\frac{1}{n}\right)^2-\frac{1}{2}\left(\frac{1}{n}\right)} = 2\varphi'(0), \quad \text{perciò}$$

$$\langle T_n, \varphi\rangle \xrightarrow{n\to\infty} 2\varphi'(0) = 2\langle \delta_0, \varphi'\rangle =$$

$$= 2\langle -\delta_0', \varphi\rangle = \langle -2\delta_0', \varphi\rangle$$

$$\Rightarrow \quad T_n \to -2\delta_0' \quad \text{in } \mathcal{D}'(\mathbb{R})$$

— o —

il limite precedente si può anche calcolare così:
per il teor. del valor medio esiste $x_n \in \left(\frac{2}{n^2}, \frac{1}{n}\right)$, con $\lim_{n\to\infty} x_n = 0$, t.c.

$$\frac{2n^2}{1-n}\left(\varphi\left(\frac{2}{n^2}\right)-\varphi\left(\frac{1}{n}\right)\right) = \frac{2n^2}{1-n}\left[\varphi'(x_n)\left(\frac{1}{n}-\frac{2}{n^2}\right)\right]$$

$$= \frac{2n^2}{1-n}\varphi'(x_n)\frac{n-2}{n^2} = \frac{2(n-2)}{1-n}\varphi'(x_n) \xrightarrow{n\to\infty} -2\varphi'(0)$$

[N.B. $\frac{2}{n^2} < \frac{1}{n}$ per n sufficientemente grande $(n>2)$]