Es. 5c, cap. 1 Metodi-Analisi

Risolvere in $\mathbb{C}$ l'equazione $z|z| - 2z + i = 0$

Sol Posto $z = x+iy$, $x, y \in \mathbb{R}$, si ha:

$$z|z| - 2z + i = 0 \iff (x+iy)\sqrt{x^2+y^2} - 2(x+iy) + i = 0$$

$$\iff x\left(\sqrt{x^2+y^2} - 2\right) + i\left[y\left(\sqrt{x^2+y^2} - 2\right) + 1\right] = 0$$

$$\iff \begin{cases} x\left(\sqrt{x^2+y^2} - 2\right) = 0 \\ y\left(\sqrt{x^2+y^2} - 2\right) + 1 = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y\left(\sqrt{x^2+y^2} - 2\right) + 1 = 0 \end{cases} \cup \begin{cases} \sqrt{x^2+y^2} - 2 = 0 \\ y\left(\sqrt{x^2+y^2} - 2\right) + 1 = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y(|y| - 2) + 1 = 0 \end{cases} \cup \begin{cases} \sqrt{x^2+y^2} - 2 = 0 \\ 1 = 0 \end{cases}$$

$$\iff \begin{cases} x = 0 \\ y|y| - 2y + 1 = 0 \end{cases} \cup \emptyset$$

$$\iff \begin{cases} x = 0 \\ y \geq 0 \\ y^2 - 2y + 1 = 0 \end{cases} \cup \begin{cases} x = 0 \\ y < 0 \\ -y^2 - 2y + 1 = 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} x=0 \\ y\geq 0 \\ (y-1)^2=0 \end{cases} \cup \begin{cases} x=0 \\ y<0 \\ y=-1-\sqrt{2} \cup y=-1+\sqrt{2} \end{cases}$$

$$\Leftrightarrow \begin{cases} x=0 \\ y\geq 0 \\ y=1 \end{cases} \cup \begin{cases} x=0 \\ y<0 \\ y=-1-\sqrt{2} \end{cases} \cup \boxed{\begin{cases} x=0 \\ y<0 \\ y=-1+\sqrt{2}\ (>0) \end{cases}} \ \emptyset$$

$$\Leftrightarrow \begin{cases} x=0 \\ y=1 \end{cases} \cup \begin{cases} x=0 \\ y=-1-\sqrt{2} \end{cases}$$

$$\Leftrightarrow \quad z=i \quad \cup \quad z=-i(1+\sqrt{2})$$

Volendo si può risolvere l'esercizio anche in coordinate polari. Se $z=\rho e^{i\vartheta}$, con $\rho\geq 0$, $\vartheta\in\mathbb{R}$, si ha

$$z|z|-2z+i=0 \Leftrightarrow \rho^2 e^{i\vartheta}-2\rho e^{i\vartheta}+i=0 \Leftrightarrow$$

$$(\rho^2-2\rho)\, e^{i\vartheta}=-i \Leftrightarrow (\rho^2-2\rho)\, e^{i\vartheta}=e^{-i\frac{\pi}{2}}$$

bisogna distinguere i due casi $(\rho^2-2\rho)\geq 0$ e $(\rho^2-2\rho)<0$:

(i) se $(\rho^2-2\rho) \geq 0$ si ha

$$\begin{cases} (\rho^2-2\rho)e^{i\vartheta} = e^{-i\frac{\pi}{2}} \\ \rho^2-2\rho \geq 0 \end{cases} \iff \begin{cases} \rho^2-2\rho = 1 \\ \vartheta = -\frac{\pi}{2}+2k\pi, \ k\in\mathbb{Z} \\ \rho(\rho-2)\geq 0 \end{cases}$$

$$\iff \begin{cases} \rho = \overbrace{1+\sqrt{2}}^{\overset{2}{\wedge}} > 0 \\ \vartheta = -\frac{\pi}{2} \\ \rho \geq 2 \end{cases} \iff \begin{cases} \rho = 1+\sqrt{2} \\ \vartheta = -\frac{\pi}{2} \end{cases}$$

$\rho \leq 0$ non è ammesso

$$\iff z = (1+\sqrt{2})e^{-i\frac{\pi}{2}} = -i(1+\sqrt{2})$$

(ii) se $(\rho^2-2\rho) < 0$ si ha

$$\begin{cases} (\rho^2-2\rho)e^{i\vartheta} = e^{-i\frac{\pi}{2}} \\ \rho^2-2\rho < 0 \end{cases} \iff \begin{cases} (-1)(2\rho-\rho^2)e^{i\vartheta} = e^{-i\frac{\pi}{2}} \\ \rho(\rho-2) < 0 \end{cases}$$

$$\iff \begin{cases} \overbrace{(2\rho-\rho^2)}^{\overset{0}{\wedge}} e^{-i\pi}e^{i\vartheta} = e^{-i\frac{\pi}{2}} \\ 0<\rho<2 \end{cases} \iff$$

$$\Longleftrightarrow \begin{cases} \overbrace{(2\rho-\rho^2)}^{1^0} e^{(\theta-\pi)i} = e^{-i\frac{\pi}{2}} \\ 0<\rho<2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} 2\rho-\rho^2 = 1 \\ \theta-\pi = -\frac{\pi}{2}+2K\pi, \ K\in\mathbb{Z} \\ 0<\rho<2 \end{cases}$$

$$\Longleftrightarrow \begin{cases} \rho = 1 \\ \theta = \frac{\pi}{2}+2K\pi, \ K\in\mathbb{Z} \\ 0<\rho<2 \end{cases} \Longleftrightarrow \begin{cases} \rho = 1 \\ \theta = \frac{\pi}{2} \end{cases}$$

$$\Longleftrightarrow \quad z = e^{i\frac{\pi}{2}} \quad \Longleftrightarrow \quad z = i$$

Riassumendo (i) e (ii) si trova che le soluzioni sono i e $-i(1+\sqrt{2})$