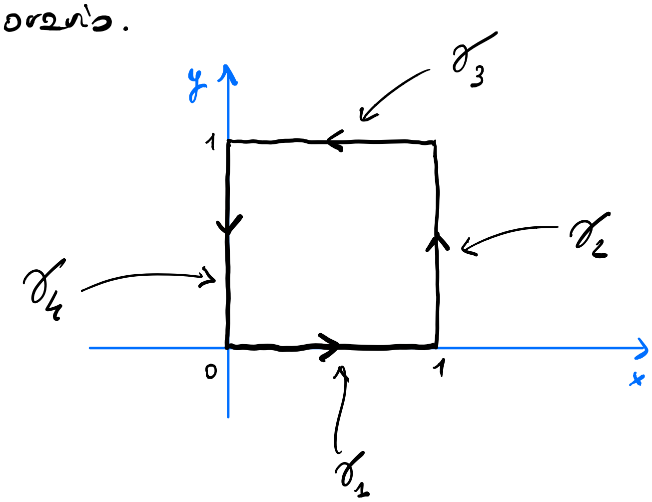
Es. #2, Cap. 3 Dispense

Calcolare $\displaystyle\int_C e^{\pi \bar{z}}\, dz$

dove C è il quadrato di vertici 0, 1, 1+i, i percorso una volta in senso antiorario.

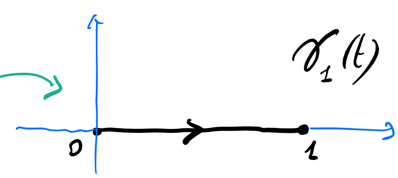
Sol

Si ha



$$\int_C e^{\pi\bar{z}}\,dz = \int_{\gamma_1} e^{\pi\bar{z}}\,dz + \int_{\gamma_2} e^{\pi\bar{z}}\,dz + \int_{\gamma_3} e^{\pi\bar{z}}\,dz + \int_{\gamma_4} e^{\pi\bar{z}}\,dz$$

(1) $\displaystyle\int_{\gamma_1} e^{\pi\bar{z}}\,dz = ?$



x varia da 0 a 1

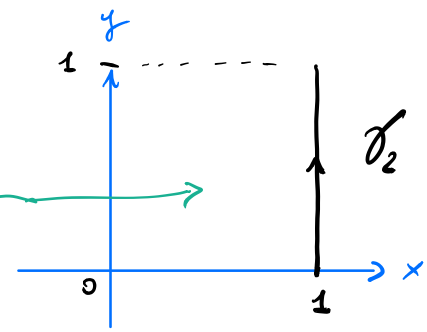
y = 0 fissa

$\gamma_1(t) = (t, 0) = t + i0 = t \qquad \forall t \in [0, 1],$

$\Rightarrow \gamma_1'(t) = 1 \qquad \forall t \in [0\,1]$

$$\Rightarrow \int_{\gamma_1} e^{\pi\bar{z}}\,dz = \int_0^1 e^{\pi\overline{\gamma_1(t)}}\,\gamma_1'(t)\,dt = \int_0^1 e^{\pi\bar{t}}\,1\,dt = \int_0^1 e^{\pi t}\,dt$$

$$= \left[\frac{e^{\pi t}}{\pi} \right]_{t=0}^{t=1} = \frac{e^{\pi}}{\pi} - \frac{1}{\pi}$$



(2) $\displaystyle\int_{\gamma_2} e^{\pi \bar{z}}\, dz = ?$

x = 1 fissa

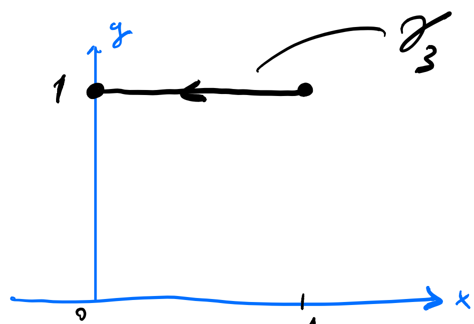
y varia da 0 a 1

$$\gamma_2(t) = (1, t) = 1 + it \qquad \forall t \in [0,1]$$

$$\Rightarrow \gamma_2'(t) = i \qquad \forall t \in [0,1]$$

$$\Rightarrow \int_{\gamma_2} e^{\pi \bar{z}}\, dz = \int_0^1 e^{\pi \overline{\gamma_2(t)}}\, \gamma_2'(t)\, dt = \int_0^1 e^{\pi(1-it)}\, i\, dt$$

$$= \left[-\frac{e^{\pi(1-it)}}{\pi} \right]_{t=0}^{t=1} = -\frac{e^{\pi(1-i)}}{\pi} + \frac{e^{\pi}}{\pi}$$



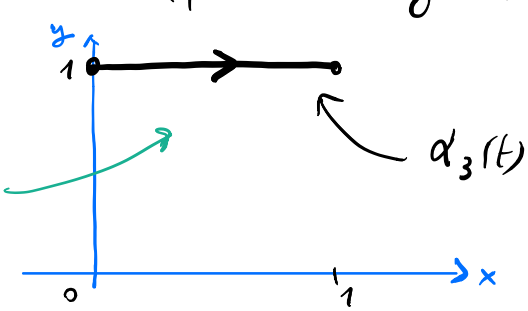
(3) $\displaystyle\int_{\gamma_3} e^{\pi \bar{z}}\, dz = ?$

è forse più semplice parametrizzare la curva di integrazione percorsa al contrario (poi cambierò segno):

$$\alpha_3(t) = (t, 1) = t + i \qquad \forall t \in [0,1]$$

y = 1 fissa

x varia da 0 a 1



$$\Rightarrow \alpha_3'(t) = 1$$

$$\Rightarrow \int_{\gamma_3} e^{\pi\bar{z}}\,dz = -\int_{\alpha_3} e^{\pi\bar{z}}\,dz = -\int_{\alpha_3} e^{\pi\overline{\alpha_3(t)}}\,\alpha_3'(t)\,dt$$

$$= -\int_0^1 e^{\pi(t-i)}\,1\,dt = -\left[\frac{e^{\pi(t-i)}}{\pi}\right]_{t=0}^{t=1}$$

$$= -\left(\frac{e^{\pi(1-i)}}{\pi} - \frac{e^{-\pi i}}{\pi}\right) = \frac{1}{\pi}\left(e^{-\pi i} - e^{\pi(1-i)}\right)$$

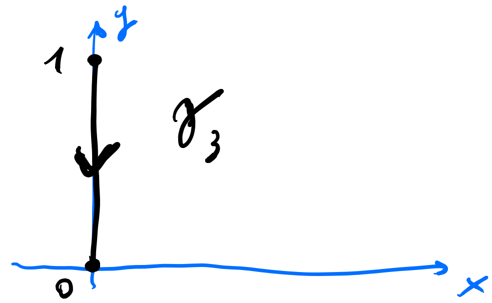
<u>oss</u> γ_3 si sarebbe potuta parametrizzare, volendo, come segmento con primo estremo $z_1 = 1+i$ e secondo estremo $z_2 = i$, quindi $\gamma_3 : [0,1] \to \mathbb{C}$ def. da

$$\begin{aligned}\gamma_3(t) &= z_1 + t(z_2 - z_1)\\ &= (1+i) + t(1+i-i)\\ &= (1+i) + t\\ &= (1+t) + i \qquad \forall t \in [0,1]\end{aligned}$$

$$\Rightarrow \gamma_3'(t) = 1 \quad \forall t \in [0,1]$$

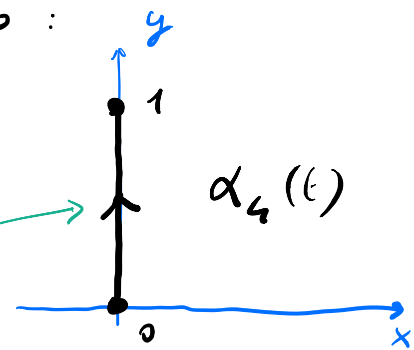
$$\Rightarrow \int_{\gamma_3} e^{\pi\bar{z}}\,dz = \int_0^1 e^{\pi[(1+t)-i]}\,1\,dt = \dots$$

(4) $\displaystyle\int_{\gamma_4} e^{\pi\bar z}\,dz$



è forse più semplice parametrizzare la curva di integrazione percorsa al contrario :

$$\alpha_4(t) = (\underbrace{0}_{x=0 \text{ fissa}}, \underbrace{t}_{y \text{ varia da } 0 \text{ a } 1}) = 0+it = it \quad \forall t\in[0,1]$$



$$\Rightarrow \alpha_4'(t) = i \quad \forall t \in [0,1]$$

$$\Rightarrow \int_{\gamma_4} e^{\pi\bar z}\,dz = -\int_{\alpha_4} e^{\pi\bar z}\,dz = -\int_0^1 e^{\pi\overline{\alpha_4(t)}}\,\alpha_4'(t)\,dt$$

$$= -\int_0^1 e^{\pi(-it)}\,i\,dt = -\left[-\frac{e^{-\pi i t}}{\pi}\right]_{t=0}^{t=1}$$

$$= -\left[-\frac{e^{-\pi i}}{\pi} + \frac{1}{\pi}\right] = \frac{e^{-\pi i}}{\pi} - \frac{1}{\pi}$$

oss γ_4 si sarebbe potuta parametrizzare, volendo, come segmento con primo estremo $z_A = i$ e secondo estremo $z_B = 0$, quindi $\gamma_4 : [0, 1] \to \mathbb{C}$ def. da

$$\gamma_4(t) = z_A + t(z_B - z_A)$$
$$= i + t(0 - i)$$
$$= i - ti$$
$$= (1-t)i \qquad \forall t \in [0, 1]$$

$$\Rightarrow \gamma_4'(t) = 1 \qquad \forall t \in [0, 1]$$

$$\Rightarrow \int_{\gamma_4} e^{\pi \bar{z}} dz = \int_0^1 e^{\pi[-(1-t)i]} 1 \, dt = \ldots$$

Mettendo insieme (1), (2), (3) e (4) si trova

$$\int_C e^{\pi \bar{z}} dz = \int_{\gamma_1} e^{\pi \bar{z}} dz + \int_{\gamma_2} e^{\pi \bar{z}} dz + \int_{\gamma_3} e^{\pi \bar{z}} dz + \int_{\gamma_4} e^{\pi \bar{z}} dz$$

$$= \left(\frac{e^\pi}{\pi} - \frac{1}{\pi} \right) + \left(-\frac{e^{\pi(1-i)}}{\pi} + \frac{e^\pi}{\pi} \right) + \left(\frac{1}{\pi} \left(e^{-\pi i} - e^{\pi(1-i)} \right) \right)$$
$$+ \left(\frac{e^{-\pi i}}{\pi} - \frac{1}{\pi} \right)$$

$$= \frac{1}{\pi}\left[e^{\pi} - 1 - e^{\pi - \pi i} + e^{\pi} + e^{-\pi i} - e^{\pi - \pi i} + e^{-\pi i} - 1\right]$$

$$= \frac{1}{\pi}\left[e^{\pi} - 1 - e^{\pi} e^{-\pi i} + e^{\pi} + e^{-\pi i} - e^{\pi} e^{-\pi i} + e^{-\pi i} - 1\right]$$

$$= \frac{1}{\pi}\left[e^{\pi} - 1 + e^{\pi} e^{-\pi i} + e^{\pi} + e^{-\pi i} - e^{\pi} e^{-\pi i} + e^{-\pi i} - 1\right]$$

$$= \frac{1}{\pi}\left[e^{\pi} - 1 + e^{\pi} + e^{\pi} - 1 + e^{\pi} - 1 - 1\right]$$

$$= \frac{1}{\pi}\left[4e^{\pi} - 4\right] = \frac{4}{\pi}\left(e^{\pi} - 1\right)$$