Numerical Analysis - Lec. 2

8.10.2020

$$y = F(x) \qquad F : \mathcal{X} \longrightarrow \mathcal{Y}$$
$$x \longrightarrow y = F(x)$$

Functional Evaluation

$\mathcal{X} : \mathbb{R}$, $\mathcal{Y} : \mathbb{R}$ F: continuous func.

$$y = F(x)$$

Sum of two numbers:

$\mathcal{X} : \mathbb{R}^2$, $\mathcal{Y} : \mathbb{R}$; F: sum of numbers.

$x = (a, b)$ $\quad F(x) = F((a,b)) = a+b$

Computation of derivative

$\mathcal{X} : C^1([a,b])$ $\quad \mathcal{Y} : \mathbb{R}$

F: evaluation of first derivative in $x_0 \in [a,b]$

$$y = F(x) := \underbrace{x}_{\text{function}}{}'(x_0)$$

Only well posed problems

1) $\forall x \in \mathcal{X}$, $\exists!\, y \in \mathcal{Y}$ s.t. $F(x) = y$

2) $\exists\, K$ (condition number) s.t. $\forall x, \hat{x} \in \mathcal{X}$

$$\| F(x) - F(\hat{x}) \|_{\mathcal{Y}} \leq K \, \| x - \hat{x} \|_{\mathcal{X}}$$

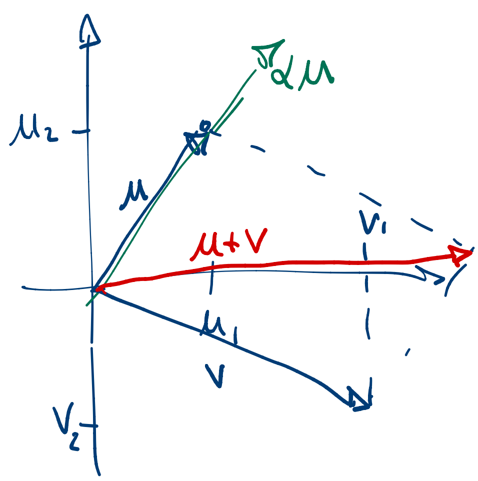
$\mathcal{X}, \mathcal{Y}$ Real Vector Spaces (Normed)

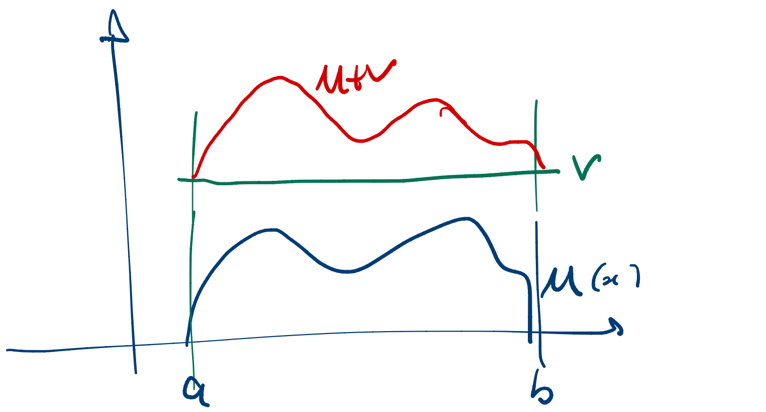
RVS: "V" collection of objects for which it makes sense to "sum" and "scale"

$$+ : V \times V \longrightarrow V$$
$$(u,v) \longrightarrow w = u+v$$
$$\cdot : V \times \mathbb{R} \longrightarrow V$$

$$\forall u,v \in V, \quad \forall \alpha, \beta \in \mathbb{R}$$
$$\alpha u + \beta v = w \in V$$

$\mathbb{R}^2$	$C^0([a,b])$
$u = (u_1, u_2)$	$u : [a,b] \longrightarrow \mathbb{R}$
$v = (v_1, v_2)$	$v : [a,b] \longrightarrow \mathbb{R}$
$u+v = (u_1+v_1, u_2+v_2)$	$w = u+v : [a,b] \longrightarrow \mathbb{R}$
$\alpha u = (\alpha u_1, \alpha u_2)$	$x \longrightarrow u(x)+v(x)$





Norms on Real Vector Spaces

a function $\|\cdot\| : V \longrightarrow \mathbb{R}_0^+$

1) $\|u\| \geq 0 \qquad \forall u \in V$

2) $\|u+v\| \leq \|u\| + \|v\|$ Triangle Inequality

3) $\|\alpha u\| = |\alpha| \, \|u\|$

optional 4) $\|u\| = 0 \iff u = 0$

without 4 it is a semi-norm

ℓ_p norms on $\mathbb{R}^n$

L_p norms on $\Omega \subset \mathbb{R}^d$

$\|\cdot\|_*$ operatorial norms induced by $\|\cdot\|_V$

$\mathbb{R}^n$: ℓ_p norm $\quad \|u\|_p := \left(\sum_{i=1}^n |u_i|^p \right)^{\frac{1}{p}}$

$$\|u\|_\infty := \max_i |u_i|$$

$\Omega \in \mathbb{R}^d$ $\quad L_p$ norms $\quad \|u\|_p := \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}$

$$\|u\|_\infty := (\mathrm{ess}) \sup_{x \in \Omega} |u(x)|$$

$\|\cdot\|_*$ induced by vector space norm $\|\cdot\|_V$ on V and $\|\cdot\|_W$

$A : V \longrightarrow W \qquad \|A\|_* := \sup_{0 \neq x \in V} \frac{\|A(x)\|_W}{\|x\|_V}.$

ℓ_p norm of Matrices in $\mathbb{R}^{n\times m}$

$$A^{n\times m}\cdot x$$

$$A : \mathbb{R}^m \longrightarrow \mathbb{R}^n \qquad A\cdot \overset{x\in\mathbb{R}^m}{x} = \underline{\underline{y}} \in \mathbb{R}^n$$

$$\|A\|_p := \sup_{0\neq x\in\mathbb{R}^n} \frac{\|Ax\|_p}{\|x\|_p} = \|A\|_* \text{ induced by } \|\cdot\|_p \text{ on } \mathbb{R}^n$$

$$A\in\mathbb{R}^{n\times m} \qquad A(x)=y \qquad \sum_{j=1}^{m} A_{ij}x_j = y_i$$

$$A\in\mathcal{L}(\mathbb{R}^m,\mathbb{R}^n)$$

$$\|\cdot\|_* \text{ norm on } \mathcal{L}(\mathbb{R}^m,\mathbb{R}^n) \underset{=}{\equiv} \mathbb{R}^{n\times m}$$

well posed problems

1) $\forall x\in\underline{\mathcal{X}}$, $\exists!\, y\in\underline{\underline{\mathcal{Y}}}$ s.t. $F(x)=y$

Normed Vector Space (under $\mathcal{X}$) — Normed Vector Space (under $\mathcal{Y}$)

2) $\exists\, K$ (condition number) s.t. $\forall x,\hat{x}\in\mathcal{X}$

$$\|F(x)-F(\hat{x})\|_{\underline{\underline{\mathcal{Y}}}} \leq K_{Abs}\|x-\hat{x}\|_{\underline{\underline{\mathcal{X}}}}$$

3) Relative cond. number (only if $x\neq 0$, $F(x)\neq 0$)

K_{rel} s.t. $\forall x,\hat{x}$

$$\frac{\| F(x) - F(\tilde{x}) \|_{\mathcal{Y}}}{\| F(x) \|_{\mathcal{Y}}} \leq K_{rel} \frac{\| x - \tilde{x} \|_{\mathcal{X}}}{\| x \|_{\mathcal{X}}}$$

A problem is well posed if $K_{rel/abs}$ is "small"

Example : sum of two numbers : (ℓ_1 norm)

$\mathcal{X} : \mathbb{R}^2 \qquad \mathcal{Y} : \mathbb{R}$

$x \in \mathbb{R}^2 \to \| x \|_1 := |x_1| + |x_2| \qquad \| y \|_1 = |y|$

$$\frac{\| F(x) - F(\hat{x}) \|_1}{\| F(x) \|_1} = \frac{|x_1 - \hat{x}_1 + x_2 - \hat{x}_2|}{|x_1 + x_2|}$$

$$\frac{\| x - \hat{x} \|_1}{\| x \|_1} = \frac{|x_1 - \hat{x}_1| + |x_2 - \hat{x}_2|}{|x_1| + |x_2|}$$

$\Delta y , \quad \Delta x \qquad \Delta x = x - \hat{x} \qquad \Delta y = F(x) - F(\hat{x})$

Find k_{abs}: $\exists k_{abs} \mid \forall x_1, x_2, \quad |\Delta y| \leq k \| \Delta x \|$?

$$|\Delta y| \leq k \| \Delta x \| \leq k \left(|\Delta x_1| + |\Delta x_2| \right)$$

Assume that $\Delta x \neq 0$

$$\frac{|\Delta y|}{|\Delta x|} \le \kappa$$

$$\frac{|\Delta x_1 + \Delta x_2|}{|\Delta x_1| + |\Delta x_2|} \le \frac{|\Delta x_1| + |\Delta x_2|}{|\Delta x_1| + |\Delta x_2|} \le 1$$

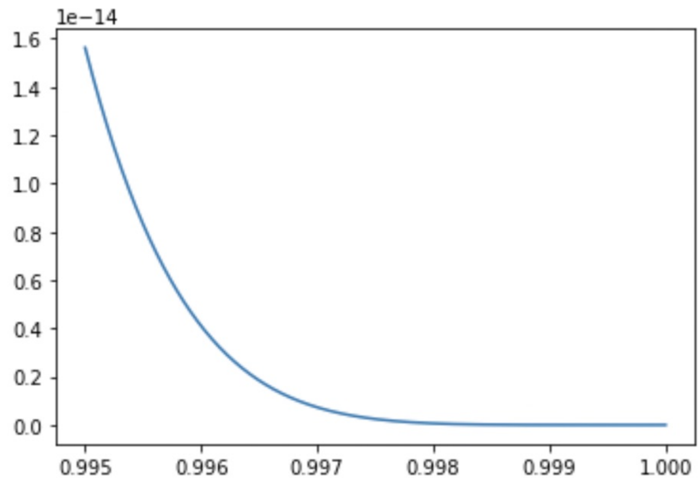
$\kappa_{abs} = 1$

$\cancel{\exists}\ \kappa_{rel}$ s.t.

$$\underbrace{\frac{|\Delta y|}{|\Delta x|}}_{\le 1} \cdot \frac{|x|}{|y|} \le \frac{|x|}{|y|} \le \kappa_{rel}$$

because $\frac{|x|}{|y|} = \frac{|x_1| + |x_2|}{|x_1 + x_2|}$ can be as large as we want

Example $\underline{(1-x)^6}$ in $[1-\epsilon, 1]$ ϵ is small.

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x = linspace(.995, 1,1029)
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y = (x-1)**6
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y = x**6 - 6*x**5 + 15*x**4 - 20*x**3 + 15*x**2 - 6*x + 1
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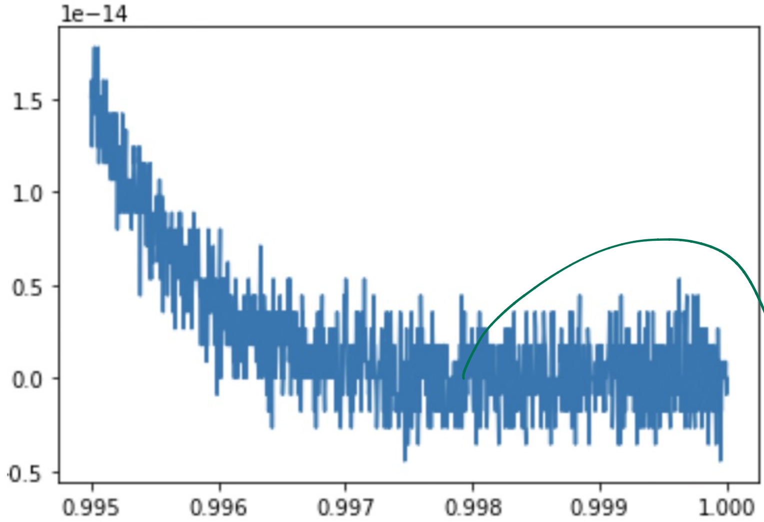

$x - \hat{x}$ = floating point errors (rounding errors)

Δy

If $x_1 = -(x_2 + \varepsilon)$ with ε "small"

$$\rightarrow \frac{|x|}{|y|} = \frac{|x_2+\varepsilon| + |x_2|}{|\cancel{-x_2} - \varepsilon + \cancel{x_2}|} \le \frac{2|x_2|+|\varepsilon|}{|\varepsilon|}$$

if x_2 not close to zero then

$$\frac{|x|}{|y|} \le \frac{2|x_2|}{|\varepsilon|} + 1$$

$$\frac{|\Delta y|}{|y|} \cdot \frac{|x|}{|\cancel{\Delta x}|} \le K_{rel}$$

$$\frac{|\Delta y|}{|\Delta x|} \frac{|x|}{|y|} \le \underline{K_{rel}}$$