# Numerical Analysis – Lec. 3 – LH 13.10.2020

1) $\exists !$ solution

2) pb. is stable ( well posed, finite cond. Num.)

$\mathcal{X}$, $\mathcal{Y}$ Normed Vector Spaces

$F : \mathcal{X} \longrightarrow \mathcal{Y}$

1) $K_{abs} \in \mathbb{R}^+ \cup \{+\infty\}$

s.t. $x, \hat{x} \in \mathcal{X}$ $\quad \|F(x) - F(\hat{x})\|_{\mathcal{Y}} \leq K_{abs} \|x - \hat{x}\|_{\mathcal{X}}$

$$K_{abs} \geq \frac{\|F(x) - F(\hat{x})\|_{\mathcal{Y}}}{\|x - \hat{x}\|_{\mathcal{X}}} \Rightarrow K_{abs} : \sup_{\substack{x, \hat{x} \in \mathcal{X} \\ x - \hat{x} \neq 0}} \frac{\|F(x) - F(\hat{x})\|_{\mathcal{Y}}}{\|x - \hat{x}\|_{\mathcal{X}}}$$

1) $K_{rel} \in \mathbb{R}^+ \cup \{+\infty\}$

s.t. $x, \hat{x} \in \mathcal{X}$ $\quad \frac{\|F(x) - F(\hat{x})\|_{\mathcal{Y}}}{\|F(x)\|_{\mathcal{Y}}} \leq K_{rel} \frac{\|x - \hat{x}\|_{\mathcal{X}}}{\|x\|_{\mathcal{X}}}$

$$K_{rel} : \sup_{\substack{x, \hat{x} \in \mathcal{X} \\ x - \hat{x} \neq 0 \\ F(x) \neq 0}} \frac{\|F(x) - F(\hat{x})\|_{\mathcal{Y}}}{\|x - \hat{x}\|_{\mathcal{X}}} \frac{\|x\|_{\mathcal{X}}}{\|F(x)\|_{\mathcal{Y}}}$$

$$\leq K_{abs} \sup_{\substack{x, \hat{x} \in \mathcal{X} \\ x - \hat{x} \neq 0 \\ F(x) = 0}} \frac{\|F^{-1}(F(x))\|_{\mathcal{X}}}{\|F(x)\|_{\mathcal{Y}}} = K_{abs} \|F^{-1}\|_{*}$$

$\delta x = x - \hat{x}$

$\delta y = y - \hat{y}$

Example: $Ax = y$ $\quad y = F(x)$ $\quad y = F(x)$

Absolute stability: $A\hat{x} = \hat{y}$ $\quad \hat{y} = F(\hat{x})$ $\longrightarrow$ $A \delta x = \delta y$

$$\|Ax - A\hat{x}\| = \|A(x - \hat{x})\| = \|\delta y\|$$

$$\|\delta y\| \leq \|A\|_* \; \|\delta x\|$$

$$\frac{\|\delta y\|}{\|y\|} \leq \underbrace{\|A\|_* \, \|A^{-1}\|_*}_{\text{Condition number of } A} \; \frac{\|\delta x\|}{\|x\|}$$

A problem is (Absolutely) Relatively well posed if ($K_{abs}$) $K_{rel}$ is $< +\infty$.

What is an approximation?

Sequence of problems $F_n$ $(n \in \mathbb{N})$ applied to a sequence of "approximated data" $x_n$ (implied approximated sequence of domains $\mathcal{X}_n$)

$$\lim_{n \to \infty} \left| F_n(x_n) - F(x) \right| = 0$$

Note 1 : " $\mathcal{X}_n \longrightarrow \mathcal{X}$ " simple case: $\mathcal{X}_n \subset \mathcal{X}$

2) Note 2 : " $F_n \longrightarrow F$ " consistency

Assume that $\mathcal{X}_n \equiv \mathcal{X}$ $\forall n$

2) $\longrightarrow \lim_{n \to \infty} \left| F_n(x) - F(x) \right| \longrightarrow 0$

LAX-RICHTMEYR THEOREM:

For consistent pb $\left( \lim_{n \to \infty} |F_n(x) - F(x)| = 0 \right)$

CONVERGENCE $\Longleftrightarrow$ STABILITY

$$x \xrightarrow{\;F\;} y$$

$$F^{-1}(y_n) = \hat{x} \xrightarrow{\;F\;} y_n$$

$$x_n \xrightarrow{\;F_n\;} y_n$$

Convergence $\lim_{n\to\infty} |F_n(x_n) - F(x)| \to 0$

$$\Big|\underbrace{F_n(x_n) - F(x_n)}_{\substack{\text{consistency} \\ \to 0}} + \underbrace{F(x_n) - F(\hat{x})}_{\substack{\text{stability} \\ \leq \\ K|x_n - \hat{x}_n|}} + \underbrace{F(\hat{x}) - F(x)}_{K|\hat{x} - x|}\Big|$$

- stability errors (of continuous pb)
- " " (of discrete pb)
- consistency error

Example: Interpolation (Polynomial)

One dimension. $\mathcal{X} := C^0([a,b])$

Pb: Find a polynomial of order $n$ that coincides with the input function at (given) $n+1$ interpolation points

1. in $\mathcal{X}$ we use the $L^\infty$ norm:

$$\|u\|_\infty := \max_{x \in [a,b]} |u(x)|$$

2. The space $\mathcal{Y}$ is $\mathbb{P}^n([a,b])$ we use again $\|\cdot\|_\infty$ as a norm

Note 1: $\mathcal{Y}$ is finite dimensional (with dimension $n+1$), and we can write it as $\mathcal{Y} = \mathbb{P}^n([a,b]) = \operatorname{span}\{v_i\}_{i=0}^n$

$\forall p \in \mathcal{Y} \equiv \mathbb{P}^n([a,b]) \quad \exists! \{p^i\}_{i=0}^n \in \mathbb{R}^{n+1}$ s.t.

$$p(x) = \sum_{i=0}^n p^i \, v_i(x)$$

$$p^i = 0 \quad i = 0, \dots, n \iff p(x) = 0 \quad \forall x \in [a,b]$$

Polynomial Interpolation: (Given $n+1$ points $\{x_i\}_{i=0}^n$)

$$F : \mathcal{X} \equiv C^0([a,b]) \longrightarrow \mathcal{Y} \equiv \mathbb{P}^n([a,b])$$

$$u \longrightarrow p$$

where $\underbrace{p(x_i) = u(x_i)}_{n+1 \text{ conditions}} \quad i = 0, \dots, n$

$$\sum_{J=0}^n p^J \, v_J(x_i) = u(x_i)$$

$$\iff$$

$$\sum_{J=0}^n V_{iJ} \, p^J = u_i \qquad V \underline{p} = \underline{u}$$

$$V_{iJ} = v_J(x_i)$$

Vandermond Matrix (when $v_i(x) = \mathrm{pow}(x, i)$)

Einstein notation: $V_{ij}\, p^j \equiv \sum_{j=0}^{n} V_{ij}\, p^j$

$$\left(V_{ij}\right)^{-1} = V^{ij}$$

$$\boxed{V^{ij} V_{jk} = \delta^i_k = \begin{cases} 1 & \text{if } i = k \\ 0 & \text{if } i \neq k \end{cases}}$$

$$V_{ij}\, p^j = u_i$$

$$p^j = V^{ji} u_i \iff \underline{p} = V^{-1} \underline{u}$$

"Reduced problem": $\mathcal{X} \equiv \mathbb{R}^n \longrightarrow \mathcal{Y} \equiv \mathbb{R}^n$

$\mathcal{X}$ := set of values of $u$ in $x_i$

$\mathcal{Y}$ := coefficients of the polynomial

with norm $\|\underline{u}\|_2 := \left(\sum_{i=0}^{n} u_i^2\right)^{\frac{1}{2}}$

Condition number of F: $\underline{u} \longrightarrow V^{-1}\underline{u} = \underline{p}$

input $\{u_i\}_{i=0}^{n} \longrightarrow$ output $\{p^j\}_{j=0}^{n}$

$$\hat{p} = V^{-1} u$$

$$\|p - \hat{p}\|_2 \leq K_{abs} \|u - \hat{u}\|_2$$

$$\|p - \hat{p}\|_2 \leq \|V^{-1}\|_* \|u - \hat{u}\|_2$$

eigenvalues of $V^{-1}$

$$\|V^{-1}\|_* := |\rho(V^{-1})| = \max_i |\lambda_i(V^{-1})|$$

To make the problem better conditioned, we need to minimize $K_{abs} = \|V^{-1}\|_2 \leadsto V \equiv I$

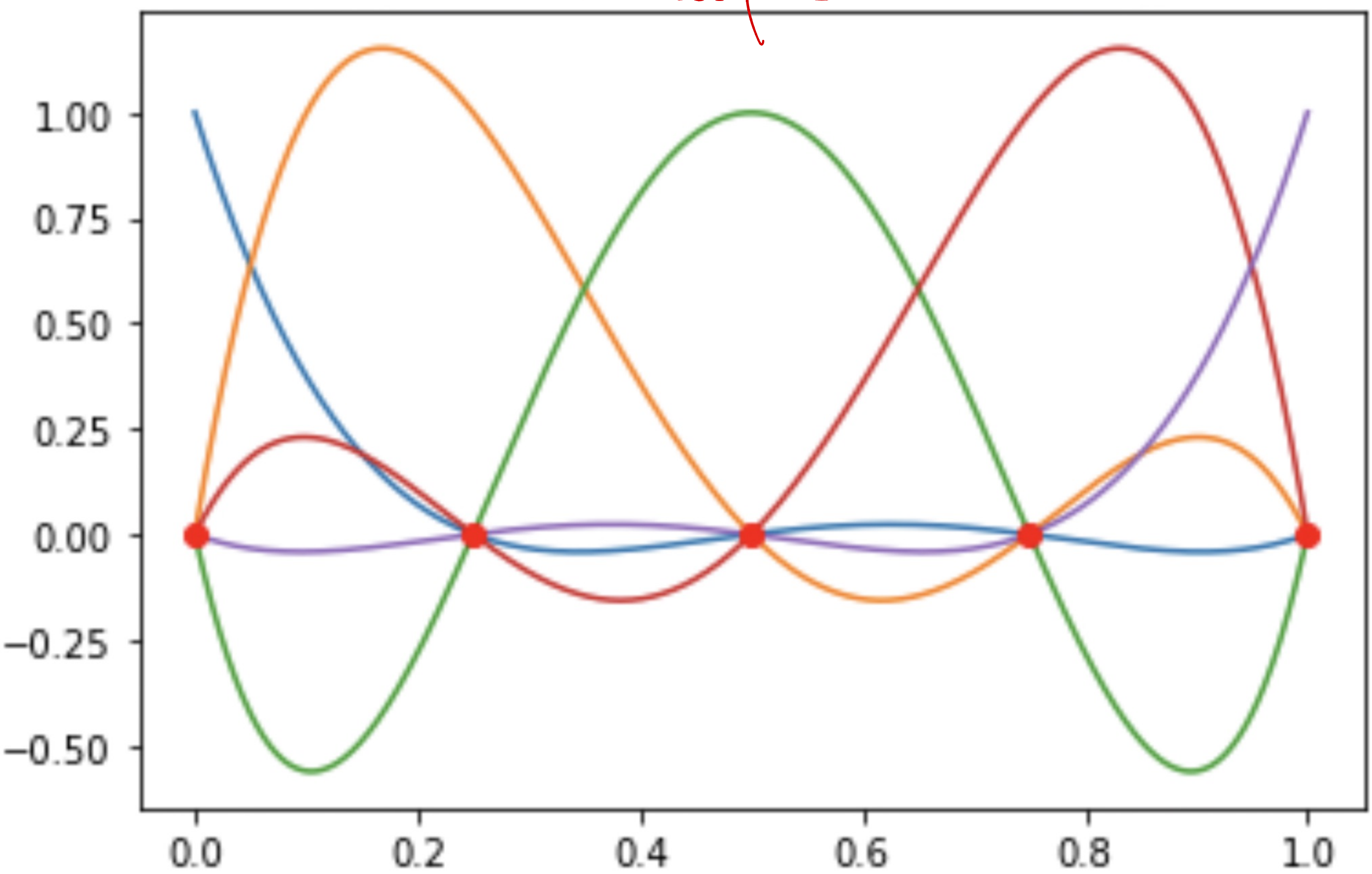
$\mathbb{V} = \mathbb{I} \implies v_J(x_i) = \delta_{Ji}$

Polynomial that is zero on every point $x_i$ except on $x_J$ where it is $1$.

$$e_i := \prod_{\substack{i \neq J \\ J=0}}^{n} \frac{(x - x_J)}{(x_i - x_J)}$$

Lagrange Basis

example with $n = 5$




$$\mathbb{V} = \mathbb{I} \rightsquigarrow p(x) = \sum_{J=0}^{n} u(x_J)\, v_J(x)$$

$$p^J = u(x_J)$$

because $\mathbb{V}^{-1} = \mathbb{I}$